Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education

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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-second annual Conference on Research in Undergraduate Mathematics Education in Oklahoma City, Oklahoma from February 28 - March 2, 2019.

The program included plenary addresses by Dr. Dan Battey, Dr. Vilma Mesa, and Dr. Mike Oehrtman, and the presentation of 134 contributed, preliminary, and theoretical research reports and 77 posters.

The conference was organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in the field of undergraduate mathematics education, each of which underwent a rigorous review by two or more reviewers:

- Contributed Research Reports describe completed research studies

- Preliminary Research Reports describe ongoing research projects in early stages of analysis

- Theoretical Research Reports describe new theoretical perspectives for research

- Posters are 1-page summaries of work that was presented in poster format

The conference was hosted by Oklahoma State University and the University of Oklahoma.

Many members of the RUME community volunteered to review submissions before the conference and during the review of the conference papers. We sincerely appreciate all of their hard work.

We wish to acknowledge the conference program committee for their substantial contributions to RUME and our institutions. Without their support, the conference would not exist.

Finally, we wish to express our deep appreciation for Dr. William "Bus" Jaco and Mathematics Learning by Inquiry for their support in organizing and funding the conference. Their support enabled us to have our conference and support our community.

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Executive Summary of the Ad Hoc Committee for the Advancement of Lesbian, Gay, Bisexual, Transgender, Queer, Intersex, and Asexual (LGBTQIA+) Inclusion in the RUME Community

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The SIGMAA on Research in Undergraduate Mathematics Education (RUME) community recently had to grapple with issues encountered as a result of California state law, which states that, "California must take action to avoid supporting or financing discrimination against lesbian, gay, bisexual, and transgender people" (California Assembly Bill No. 1887, 2016). In effect, this law prohibits California state-funded travel to other states which have religious freedom laws that are viewed as discriminatory to lesbian, gay, bisexual and transgender (LGBT) individuals. On June 22, 2018 Oklahoma was added to the list of states where travel was prohibited, due to a newly enacted law SB 1140. This law states that child-placement agencies will not be required to place a child in adoption or foster care in situations that "violate the agency's written religious or moral convictions or policies" (Oklahoma Senate Bill No.1140, 2018). The enactment of SB 1140 and the prohibited state-funded travel meant that some members of the RUME community would be unable or unwilling to travel to the 2019 and 2020 RUME conferences, which were slated to be held in Oklahoma.

The SIGMAA RUME Executive Committee decided to respond to this issue while attempting to uphold the SIGMAA on RUME's principles of equity and mentorship, to address previously contracted financial obligations, and to instill fairness and transparency within the RUME community. As a result of conversation with the Oklahoma organizing committee, the RUME Executive Committee decided to host the 2019 Conference in Oklahoma but to relocate the 2020 SIGMAA RUME Conference. On October 13, 2018 the Executive Committee communicated this decision in an email sent to the RUME listserv and read in part:

> The SIGMAA RUME EC has had several difficult discussions among ourselves, as well as with the Oklahoma planning committee regarding how this impacts our community especially given our equity statement, which explicitly states that as an organization we will respect our LGBQTA+ members. As such, have decided that, in the context of the California travel ban, having our conference in Oklahoma in 2020 would violate this statement. We very much want to also honor our equity statement and strongly believe that having the 2020 SIGMAA RUME conference in Oklahoma would send the wrong message to our LGBQTA+ SIGMAA RUME members. Given we very much want to support all of our members, we have decided to not have the 2020 SIGMAA RUME conference in Oklahoma. At the same time we want to honor our initial commitment to our Oklahoma members who have been working hard to plan the 2019 SIGMAA RUME conference – as such the 2019 conference will still be in Oklahoma.

This initial communication sparked a rapid dialogue, resulting in 23 posts to the listserv in less than two days (as well as numerous non-listserv communications) before posting was suspended for a short duration. In an effort to address the concerns via the listserv and foster a positive and affirming SIGMAA RUME community, the Executive Committee created an Ad Hoc Committee for the Advancement of Lesbian, Gay, Bisexual, Transgender, Queer, Intersex, and Asexual (LGBTQIA+) Inclusion in the RUME Community. The committee was formed via nominations of willing and interested researchers in the community, representing both individuals within and outside the LGBTQIA+ community. The authors of this executive summary comprised the members of this committee, and they were charged with creating a proposal for incorporating activities and/or sessions into the 2019 conference that would promote education about—and discussion of—issues related to the participation of LGBTQIA+ colleagues and students in our research community in particular and in our society in general.

The ad hoc committee met on several occasions to determine the focus and intent of the activities, to draft a list of recommendations, and to plan for their implementation. One of the initial considerations addressed was how broadly to implement issues of inclusivity at the conference. For instance, the committee considered focusing on inclusivity and marginalization generally, to include activities on how these may be experienced by women, scholars of color, those in the religious minority, etc. Yet, for this conference, the committee decided to keep it focused on LGBTQIA+ issues because it was directly related to the charge of the committee, helped ground conversations in particular experiences, and provided a common thread throughout the activities.

LGBQTIA+ Activities and Sessions

In this paper, we showcase the recommendations proposed by the committee, the rationale behind such efforts, and a discussion of how they were implemented at the conference. Our hope is that by sharing these efforts others can learn and implement similar practices at other conferences, in departments, at their institution, etc.

Opening Session to Address LGBTQIA+ Issues

The SIGMAA RUME conference opened with a session that included a panel who specifically addressed LGBTQIA+ issues. Incorporating this panel into the well-attended opening session set a tone concerning these issues and opened a dialogue for the remainder of the conference. Keeping the SIGMAA RUME equity statement in mind, the opening session was framed around this statement. SIGMAA RUME's Position Statement on Equity "reflects the commitments and perspectives of the community in advancing equity in undergraduate mathematics education with respect to: 1) participation within the community; 2) teaching practices; and 3) research. For purposes of this document, equity is defined as a state in which all participants are enabled to fully participate and become successful in a community of practice" (Committee on Equity, 2018, p. 1). Therefore, the panel of testimonials or narratives was geared towards equity researchers, RUME faculty, and students. The overarching questions addressed by the panel included:

- How does identifying within the LGBTQIA+ community impact your experience in RUME?
- How does identifying within the LGBTQIA+ community impact your experience in mathematics classes?
- How does your research agenda impact the LGBTQIA+ community and your pursuits as a scholar?

As such, the panel included both in-person sharing and written testimonials that were submitted prior to the conference and projected for conference attendees to see and read in silence. Both senior and junior members of the community shared testimonials about their experience as LGBTQIA+ RUME members. These testimonials of such participants included comments on making career decisions based on safety and livelihood, on monitoring or tracking their feelings of inclusion within the community, and on the emotional impact of discriminatory laws in both the U.S. and abroad. One of the panelists shared that as a first-time conference attendee it was important for them to see the efforts to promote inclusivity at the conference, since their first exposure to the community was through the email listserv exchange. Another panelist shared that although they didn't share about their personal life, it was important to them to serve on the panel in order to share how they constantly monitor/assess ways in which they feel safe and included in situations and ways in which they feel marginalized and at-risk. Another panelist shared the emotional toll of not seeing members of the community at the conference because of the travel ban and expressed their struggle with how we as a community should address this topic.

Figure 1. Opening session participants and Committee members

Finally, a set of testimonials from others within the community, students and equity researchers were projected on the screen. The ad hoc committee decided not to read aloud the testimonials of those who submitted written responses, as we believed that it would be inappropriate to voice their words and experiences when they are not our own. Rather, these testimonials were projected and read quietly by audience members. Those who could not view the testimonials were welcomed to stand and come forward for a closer view. To use the old idiom, you could have heard a pin drop; the silence was all-encompassing. The session was attended by well over 200 people and ended in a standing ovation. The remaining conference activities capitalized on this energy, providing a space for attendees to engage in deeper conversations about LGBTQIA+ issues and inclusivity in general.

Wall of Identity

As mentioned, the opening session with testimonials provided a window into the vulnerability and the human endeavor of research; this seemed to help others connect and share their journey with the RUME community. In order to allow all conference participants to share such experiences, a "wall of identity" was created to feature the printed versions of the written testimonials and block paper for others to respond to the following prompt: "Please feel free to share how your identity (e.g., who you are) has impacted your experiences with the RUME community or your interactions at this conference." This prompt allowed what are sometimes less visible or public experiences to become part of the communal dialogue, and the wall was actively contributed to throughout the conference.

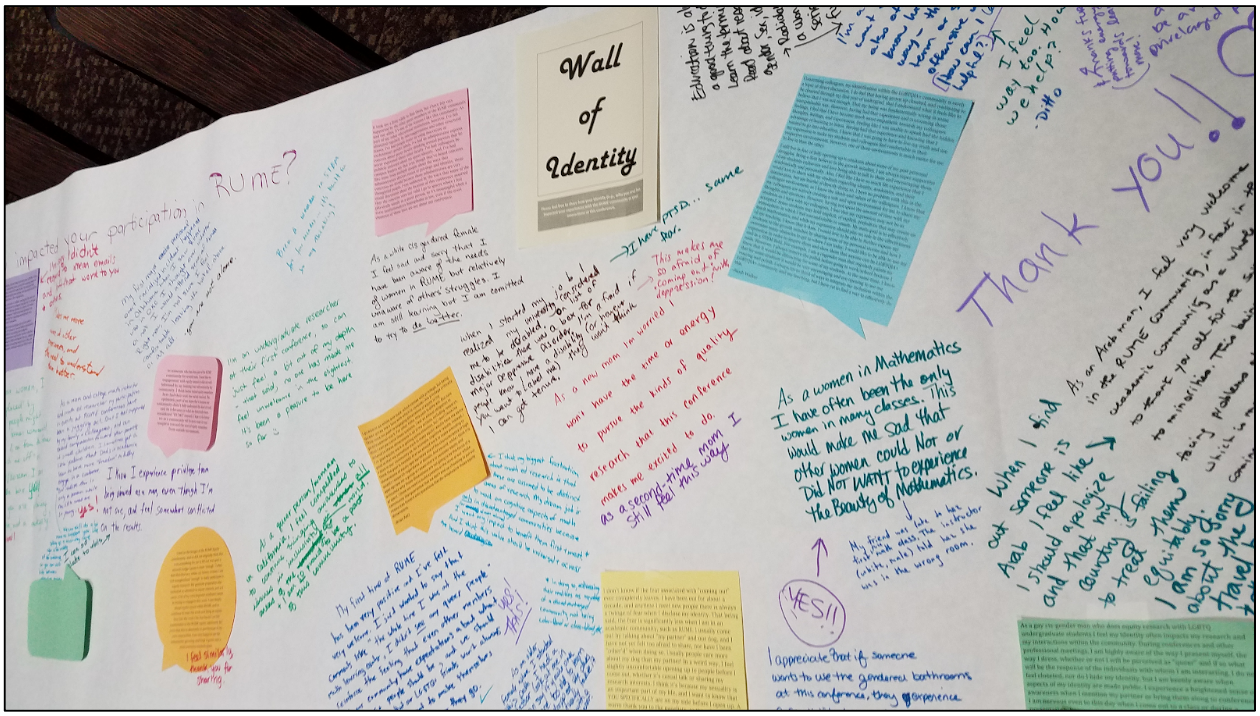

Figure 2. Wall of Identity

On the Wall of Identity, over 50 people described their experiences in RUME, shared how their identities impact their participation in the RUME community, and responded to others' concerns. Many of these experiences shared on the Wall of Identity resonated with others and led to a chain of people commenting. Some of the themes that were shared on the wall related to feelings of imposter syndrome, to feelings of isolation, and to the lack of representation Individuals shared the ways in which their identity led to racialized experiences in RUME, gendered experiences in RUME, and differential involvement as a parent. Additionally, several members shared that they felt like peripheral members of the community because of a research focus in community college, developmental mathematics, or equity. There were also general positive experiences in RUME expressed such as admiration for the RUME community and feeling welcomed at the conference. Finally, the wall included statements from an individual recognizing their privilege as a straight white cisgender man in RUME.

Coffee Break Dialogue Sessions

In addition to the wall of identity, a series of targeted questions were posted at each of the conference breaks along with poster paper to allow participants to express and expand on their views, understandings, and knowledge base of LGBTQIA+ issues. These questions were designed to align with the ongoing activities at the conference and to allow individuals to continue the conversation about LGBTQIA+ issues throughout the entire conference. The set of questions included the following:

Table 1. Coffee break dialogue prompts

Break Session	Question Prompt
One	What ideas resonated with you, or what insights did you gather as a result of attending the panel discussion about LGBTQ+ issues?
Two	What might be potential challenges as it relates to critically engaging with and/or discussing LGBTQ+ issues? What has been or could be the most helpful mechanism to assist you with supporting LGBTQ+ mathematics students?
Three	What were your takeaways from the Faculty LGBTQ+ Ally critical discussion? What resources or support systems are needed in your community/institutional/departmental space to truly advocate for LGBTQ+ inclusivity?
Four	How might you redesign one of your mathematics lessons, examples, or projects to (further) engage with LGBTQ+ issues? How might your research projects, tasks, etc. (better) attend to LGBTQ+ issues?
Five	How might these critical conversations about LGBTQ+ inclusivity benefit the RUME community? What are potential next steps for advancing LGBTQ+ inclusivity in RUME?

The questions posted at each of the breaks helped to keep transformative conversations about LGBTQIA+ issues occurring throughout the conference. For example, an instructor shared that one way to support LGBTQIA+ students was to humanize the subject of mathematics to promote relational interactions with students. This discussion shows the synergy of having these coffee break questions woven throughout the conference since the questions about LGBTQIA+ inclusion linked well with one of the plenary sessions on equity “for all” and relational interactions.

Name Badges and Pronouns

Beginning with the 2017 RUME conference in San Diego, space was added on name badges to allow for participants to enter in their pronouns (e.g., they/them/theirs, he/him/his, she/her/her). This practice (GLSEN, n.d.) helps support inclusive spaces at the conference by allowing individuals to be referred to by their self-selected pronouns and conveys that the conference organizers are open and accepting of non-binary or non-traditional pronouns.

Gender Inclusive Restrooms

Often times there are not gender-inclusive restrooms at conference locations, or these are not easily accessible or clearly indicated for participants to find. This can present a barrier and challenge for transgender and non-binary individuals who are uncomfortable using gendered restrooms (e.g., men's, women's). To address this issue, gender-inclusive restrooms were created by re-labeling the gendered restrooms to be all gender restrooms This proved to be especially impactful because the inclusive restrooms were central to the conference activities and they were the primary restrooms available. One participant during the coffee break discussion pointed out that the gendered restrooms were inconveniently located further from the main conference rooms, and so those wishing to use such restrooms were inconvenienced in a way that many members of the LGBTQIA+ community feel every day by not being welcomed to use the restroom of their choice.

Figure 3. Gender Inclusive Restrooms

Having gender-inclusive restrooms generated several discussions throughout the conference regarding restroom availability for transgender individuals, feelings of unsafety for women, and the privilege experienced by many cisgender individuals. In a post-conference survey, 11 out of 52 open-ended responses to the LGBTQIA+ activities focused on the gender-inclusive restrooms. Over half of these expressed concern with how the restrooms were announced and updated (e.g., first the women's restroom was updated, and then a few hours later the men's restroom was updated) and concern with availability of gender-specific restrooms. One participant expressed concern that re-labeling the men's restrooms as gender-inclusive is problematic since they also had urinals present, which they expressed as problematic for fear of being exposed to a colleague's genitalia. This participant suggested that the urinals could be marked off as "out of order" during the conference. Another suggestion was to clearly label what is inside the restroom (e.g. 5 urinals and 6 stalls) so individuals can choose which restroom they desire. Another participant expressed concern using a restroom with a man present, because a man in the women's restroom could be a potential sexual assault. The ad hoc committee does not have definitive answers as to what would be the best practice for restroom access for future conferences; however, we hold steadfast to the necessity of having gender-inclusive restrooms available and easily accessible for conference participants.

Safe-Space Training

In order to promote education, our intention was to partner with the local LGBT and Gender Center to offer safe-space training during the conference. Due to scheduling conflicts, this was not able to be offered in its entirety and instead there was a lunch session that promoted a critical discussion on being an ally for LGBTQIA+ colleagues and students. This session was attended by 10-15 participants and provided a space for unpacking the series of events at the conference as well as discussing issues of teaching and mentoring LGBTQIA+ students. Participants in attendance expressed that having a space to learn about issues, unpack conference events, discuss personal struggles, and get practical advice was a helpful. They also appreciated the space to have such conversation that were separate from the research focus of other sessions. Therefore, the ad hoc committee posit that there is a need to have sessions and programming efforts that offer education as well as informal spaces to discuss our practices and lived experience at future SIGMAA RUME meetings.

Anonymous Feedback Platform

An anonymous feedback platform was created for participants to ask questions. The purpose behind having this anonymous platform was to allow individuals to ask questions that they may feel uncomfortable asking in a group setting. That way, the ad hoc committee could respond to or post their feedback for other conference attendees to see. With this platform, a total of 13 responses were received throughout the conference ranging from general positive affirmations of the activities, questions about how to address gendered language when calling on people (e.g., yes, ma'am?), and suggestions for revising the equity statement to address the inclusion of developmental mathematics constituents and K-12 practitioners.

Other Efforts

There were several other efforts that were undertaken to promote inclusion at the conference. These included having a letter writing campaign to the state legislature, having a social hour at a local LGBT bar, and connecting with local LGBT organizations and student clubs such as Out in STEM (oSTEM).

Participant Reactions

In an effort to assess the impact and effectiveness of the previously mentioned activities, several survey questions were included in a post-conference survey distributed to all participants. There were a total of 155 survey responses, with 150 that included responses to questions about the LGBTQIA+ activities and sessions. Asked to what extent they appreciated the inclusion of the LGBTQIA+ activities at RUME out of the 151 responses to this question, 54% of respondents supported including all of them and 23% supported including some of the activities but neutral to others (see Figure 4). Only 3% of respondents did not support including most or all of the activities and sessions. Of the remaining respondents, 13% were neutral about most of the activities and 8% supported some activities but did not support others. Given the listserv catalyst for the creation of this committee, these results help provide context that a majority of RUME participants are supportive of including activities that address issues of identity and inclusion at the conference. In fact, 15 of the 52 open-ended responses discussed a desire for including activities at the next RUME conference that addressed issues of identity and marginalization of other groups of individuals (e.g., people of color, undocumented students, people with a disability, women).

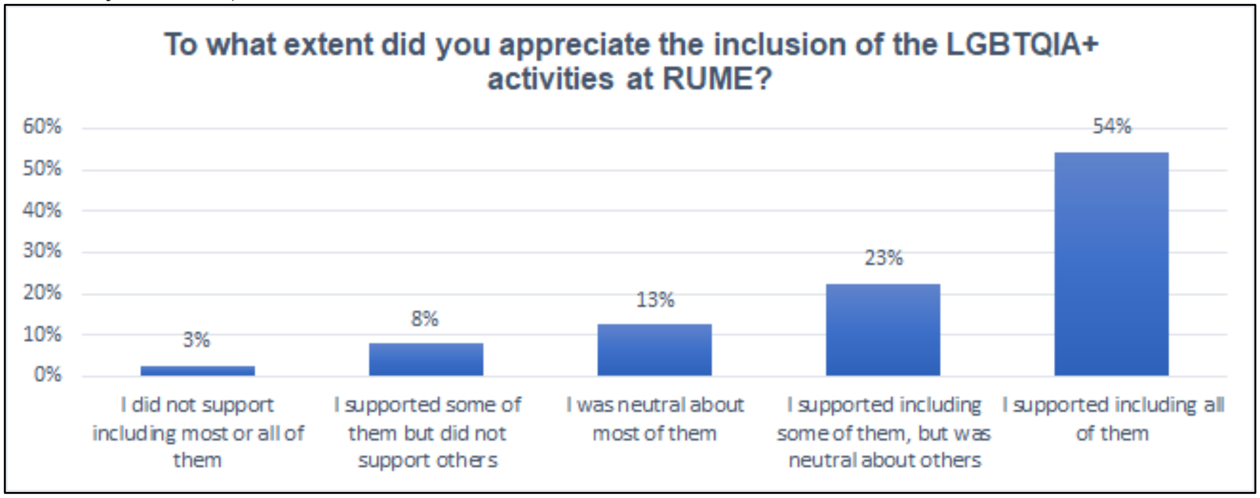

Figure 4. RUME conference participants response (n=151) indication appreciation of LGBTQIA+ activities. Due to rounding the percentages total greater than 100%.

Asked which of the activities or sessions participants (n=129) found the most helpful for facilitating learning and discussion about LGBTQIA+ issues (see Figure 5), the most indicated sessions were the introductory panel of testimonials (100), the wall of identity (62), the pronouns on name tags (56) and the gender inclusive restrooms (54).

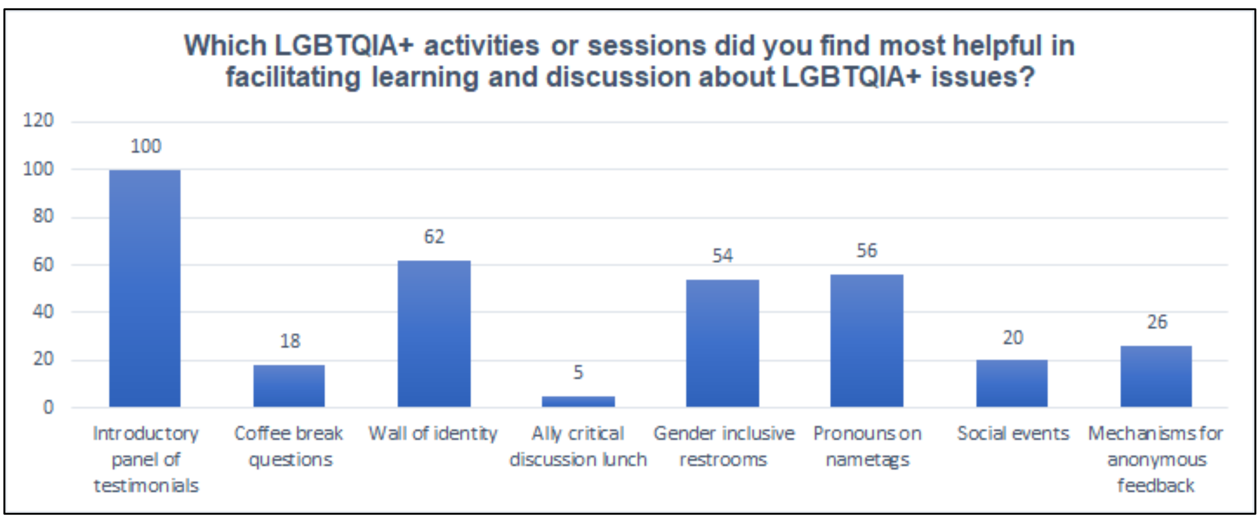

Figure 5. Response counts from 129 participants to the activities and sessions that were most helpful in facilitating learning and discussion about LGBTQIA+ issues.

Participants (n=151) were also asked how the activities and sessions impacted their *understanding* of LGBTQIA+ issues and experiences, resulting in 25% expressing a lot, 57% a little, and 18% not at all. The fact that a plurality of participants grew in their understanding of LGBTQIA+ issues suggests that the educative goals of this committee were supported by conference activities. Participants (n=150) were also asked how the sessions created opportunities for *discussion* about LGBTQIA+ participation, resulting in 45% expressing a lot, 46% a little, and 9% not at all. These results indicate that the vast majority of participants saw opportunities in the activities to discuss with other participants about LGBTQIA+ participation in RUME. Based on these results, a majority of participants found the activities and sessions encouraging discussion and promoting understanding of LGBTQIA+ issues and experiences, which was the charge of the committee.

Reflections on the Initiatives

The activities and sessions mentioned in this article are not an exhaustive list of the ways in which inclusivity within the RUME community can be promoted, but they are a start in recognizing the humanity and dignity of our colleagues and friends. The activities implemented at RUME 2019 included:

- The opening session to address LGBTQIA+ issues,
- The wall of identity,
- Coffee break questions and dialogue,
- Pronouns on name badges,
- Gender-inclusive restrooms,
- A safe-space training,
- An anonymous online feedback platform, and
- Other local informal efforts.

Additionally, all of these efforts helped to promote discussions within and outside the formal conference program, with participants discussing how the issues shared related to their identity within the field. Our hope is that by engaging in these discussions, experiencing vulnerability and empowerment, we can support each other to allow all members to engage fully within the SIGMAA RUME community.

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Understanding and Enacting Organizational Change: An Approach in Four Frames

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This paper reports on an instance of change in a university mathematics department which revitalized and improved their precalculus/calculus program by implementing a series of strategies, techniques, and programs which are supported by educational research. Using the Four Frames perspective for organizational culture (Bolman & Deal, 2008; Reinholz & Apkarian, 2018), we explore how the dimensions of structures, symbols, people, and power support a rich understanding of how the department's culture supported and constrained the change initiative. We do so both generally speaking, for the entire initiative, and more in depth, regarding the development of a course coordination system. Furthermore, this case study suggests the utility of these four frames for change agents elsewhere as a tool to support the design and enactment of successful and sustainable change towards the improvement of, specifically, undergraduate mathematics education.

Key words: Institutional change, departmental culture, course coordination

Objectives & Purpose

For decades, education researchers, professional societies, and government agencies have called for educational reform in introductory undergraduate STEM courses. Many of these calls point specifically to the implementation of research based instructional strategies and programs (National Research Council, 1999, 2013; Saxe & Braddy, 2015). Although there are numerous examples of improvement initiatives, they have not had a sustained impact at the desired scale. Commonly cited reasons for the lack of success are inadequate attention to theories of change and local cultural context (Borrego & Henderson, 2014; Elrod & Kezar, 2016; Henderson, Beach, & Finkelstein, 2011; Kezar, 2014).

This paper has two main goals: (1) to help introduce the RUME community to research from organizational change, and (2) expand such research by further contextualizing it to the discipline of mathematics, which is not a typical research area for organizational change. Through these goals, we hope to contribute to a conversation with the RUME community on how to sustainably improve undergraduate mathematics education. To achieve these goals, we present a case study of change in a university mathematics department which implemented research-based programs to support student success over a period of several years. We present a story of this change which attends to the cultural aspects of the department which supported and constrained the initiative using the four frames model (Bolman & Deal, 2008). Our results suggest strategies for enacting sustainable changes in undergraduate mathematics.

Theoretical Framing & Perspective

The *Four Frames* perspective originated in organizational science (Bolman & Deal, 2008) and was adapted for undergraduate STEM department contexts by Reinholz and Apkarian (2018). In this perspective, culture is defined as "a historical and evolving set of *structures* and *symbols*, and the resulting *power* relationships between *people*" (Reinholz & Apkarian, 2018, p. 3). This definition highlights four interrelated dimensions of institutional culture as well as acknowledging that culture is historical and ever-evolving. Table 1 summarizes the definitions of each dimension and how they can relate to both the products and process of change – analytically and for design purposes. In general, *structures* refer to observable mechanisms which determine how members of a community interact (e.g., meeting structures, teaching assignments, committees). *Symbols* include espoused beliefs, underlying assumptions, and shared values (e.g., mathematics is the purest discipline; precalculus is taken by non-majors) which are generally used by community members to guide their reasoning and give purpose to structures. The *people* frame focuses on the importance of recognizing individuals within a community, who bring their own lens, goals, needs, and identities to bear on their interactions with others in the community. The *power* frame brings to the fore ideas of how explicit hierarchies and implicit status or positioning influence community interactions and decision making.

Table 1. Definitions and aspects of the products and process of change according to the four frames perspective. Adapted from Reinholz and Apkarian (2018, p. 6).

	Description	Aspect of product	Aspect of process
Structures	Roles, responsibilities, routines, etc. which organize how people interact	A new *thing* that addresses an issue in an ongoing and sustainable way	Create incentives and support for individuals to engage in the change process and new things
Symbols	Cultural artifacts, language, myths, and values that community members use to guide their reasoning	Attitudes and beliefs that support a proposed change so that it is optimally taken up	Use language, data, and evidence that align with present ways of thinking
People	Individuals within the community and their individual needs, goals, and identities	Solutions that embody a shared vision which attends to the needs goals, and identities of many within the community	Afford individuals agency and ownership of the direction of the change initiatives
Power	Status, control, position, control, and political coalitions which mediate interactions between people	Leadership structures that promote equity by attending to the needs of diverse stakeholders and participants	Use concrete signs of success to develop and maintain the sanction of key stakeholders

In this study, we use the four frames to understand the products and process of change in a single department during a major improvement initiative. This allows for a robust story which addresses many interrelated aspects of change and culture, and how various aspects of departmental culture supported or constrained the efforts of change agents. Our experience suggests that the four frames perspective is valuable for change agents when planning and evaluating their own initiatives to increase the likelihood of sustained success.

Methodology

This three-year study took place in a mathematics department at a large public university (LPU) while the department enacted a major change initiative to align their precalculus and calculus courses with the findings of a national study of successful programs in college calculus (Bressoud, Mesa, & Rasmussen, 2015; Rasmussen, Ellis, Zazkis, & Bressoud, 2014). This paper reports on one part of a deep case study of the change initiative. Data for this paper comes from 30 interviews with 22 members of the department and university at large, several of whom were interviewed at yearly intervals. These were semi-structured interviews, consisting of a core set of questions related to each participant's role in and perception of the mathematics department, introductory mathematics sequence, and ongoing change initiative. Observations of departmental meetings and online surveys served to contextualize each interview.

Interviews were analyzed using thematic analysis (Braun & Clarke, 2006; Miles & Huberman, 1994). The first phase of data familiarization, informed by the four frames perspective, provided a starting list of codes. Iterative rounds of tagging and coding data served to revise the coding scheme by developing new codes as ideas emerged across interviews, then to merge and combine these codes into a refined set. Once code clusters were developed, the data was re-examined to identify themes. The validity of these themes was examined using multiple qualitative validity testing procedures. This included triangulation with other data sources from the study, member-checking with a subset of the study's participants, peer debriefing for sensibleness of interpretation, and searching through the interviews for confirming and disconfirming evidence (Creswell & Miller, 2000; Lincoln & Guba, 1985; Miles & Huberman, 1994). These themes were then turned into thick descriptions, rich narratives of the themes which include quotations and context supporting each of the major ideas. The emerging themes were also considered in light of the four frames perspective, and a condensed version of these framed narratives is presented next. We also include a more detailed review of the implementation of a course coordination system, a complex undertaking which connected to many different structures.

Results – General Overview

The change initiative at LPU aimed to implement new *structures* to better support students. These included a new course coordination system, more systematic review and use of local data, the development of a GTA teaching preparation program, the implementation of active learning in GTA-led recitation and lab sessions, a new adaptive computer system for placing students into appropriate courses, and the development of a new and more dedicated tutoring center specifically for mathematics. These structures were successfully implemented. The pre-existing structures at LPU included high enrollment precalculus/calculus courses, which were taught primarily by lecturers with some tenure-track faculty involvement. During the change, each course was assigned a dedicated coordinator who holds a tenured or tenure-track position in the department of mathematics and teaches the course each term, alongside lecturers and other faculty. Change agents at LPU took advantage of the registrar's regulations to re-define the course as a lab course, which provided an extra contact hour a week in addition to the weekly recitation section without increasing the credit load of the course, so as not to interfere with credit limits that affect tuition. The implementation of a coordination system, consisting of uniform course elements (e.g., textbook, homework, exams) is one of the central changes to the ways in which the department functions in relation to the precalculus/calculus course sequence at LPU. The new structures have been implemented as a system, and the interlocking pieces of the system amplify the effectiveness of each program for supporting student success. The interlocking nature of the new system also increases the likelihood of sustainability, as the pieces depend on each other so discontinuing any one feature will affect the others.

There was no explicit attempt to change what Reinholz and Apkarian (2018) consider to be *symbols*, though some department members indicated that they hoped some shifts might occur organically. A pervasive belief that students in mathematics courses at LPU were unprepared at every level supported the implementation of a change initiative, as generally everyone in the department agreed that something needed to be done. Attitudes toward the calculus sequence are that it is a service sequence, primarily taken by non-mathematics majors and offered as support for other STEM departments. Some department members take this up as a duty to support applied science students, while others see their duty as “weeding out” those who will not succeed in rigorous scientific programs. This attitude means that department members do not have as much interest in the details of how precalculus/calculus are taught as compared to graduate courses or upper division courses taken by majors, which limited the intensity of pushback to the initiatives. Another major aspect of the department culture from the symbols perspective is a strong belief in pedagogical autonomy and instructor independence, which made the implementation of a course coordination system more challenging. However, this was mitigated by the large number of lecturers teaching in the course sequence, and by strategic teaching assignments which moved resistors to other courses. This belief about pedagogical autonomy impacted the nature of the coordination system as well, in that the system primarily focused on uniform course elements while instructional change was pushed to graduate students teaching recitation sections.

The *power* frame highlights hierarchies within LPU and the effect of these on the change initiative. Change agents positioned their intentions in line with the university’s strategic plan, and thus leveraged institutional power to gather resources and support from stakeholders in the administration. Within the department, contingent faculty have less power and respect than tenured and tenure-track faculty. One effect of this power dynamic is that, although some lecturers frequently teach multiple sections of precalculus and calculus, they were not included in the initial discussions nor planning phases of the change initiative. That these faculty bore the brunt of the coordination system reduced pushback from tenured and tenure-track faculty about the coordination, and in fact the few tenured faculty who taught in the new coordination system were the most difficult for the coordinator to keep in line. Graduate students are at the bottom of the teaching hierarchy, and they have been a major part of the change initiative – perhaps in part because they are the most pliable due to their roles in the department. The new coordination system has added to the positional leadership hierarchy, as they control over many aspects of the teaching of precalculus/calculus courses and their input on teaching assignments is taken under consideration by the department chair.

Finally, the *people* frame brings into focus the roles of individuals within the collective department community. Pre-tenure faculty have increased pressure and expectation to publish, and it is generally agreed that they will spend less time working on instruction or teaching professional development, particularly in regards to lower-division undergraduate courses. Contingent faculty at LPU are primarily part-time, and have external pressures as many of them work at other jobs (e.g., local two-year colleges). Additionally, they do not have service expectations at LPU. These contribute to their identity as not being LPU-centric, and they are less likely to participate in decision-making or committee service. There are also idiosyncratic power issues. For example, one coordinator feels strong ownership of the course he coordinates, responsibility to the students, and works tirelessly to achieve and share successes with the new initiatives. Another coordinator feels this is simply another service assignment, has little belief that the changes will make a significant difference, and does not dedicate as much time to the role. This has affected the perspectives of other faculty teaching the respective coordinated courses, and highlights the importance of clear and dedicated leadership. In light of the wide variety of people and opinions within the department, the change initiative was first outlined by a group of faculty with diverse research interests and attitudes about students, teaching, and learning. This task force negotiated many details of the planned initiatives before a departmental vote, and in doing so avoided some of the pitfalls which might have led to a shutdown. This included the initial scope of the coordination system, and who would take on the bulk of the new strategies for instruction.

Course Coordination

The implementation of a course coordination system at LPU was a major feature of the change initiative under study, and provides a rich context for exploration using the four frames. Course coordination is also of particular relevance to mathematics departments across the country, with many universities expressing interest and recommendations for increased coherence from research and policy documents (Apkarian, Kirin, Vroom, & Gehrtz, under review; National Research Council, 2013; Rasmussen et al., in press, 2014; Saxe & Braddy, 2015). Prior to this change initiative, the P2C2 courses at LPU were entirely under the purview of individual instructors, to the extent that when multiple instructors taught a particular course in a single term they might each select a different textbook. Therefore, the implementation of a coordination system including uniform textbooks, common assignments, and common exams was a major change to the status quo for instructors. In terms of *structure*, the coordination system changed how people in the department interacted around P2C2 courses; cultural *symbols* affected how and how quickly this system could be implemented; individual *people* and their personal histories were leveraged in the design and roll-out of the system; and the leveraging of *power*, both formal (in the case of the chair) and informal (in terms of relative status).

When the idea of coordinating the P2C2 courses was first floated in the department, it was met with heavy resistance. Some of this resistance came from a widespread and entrenched belief in the importance of instructor autonomy. There were also individuals in the department whose personal identity and individual experiences impacted their ideas about coordination – some were open to the idea of insisting that others use their materials, but were unwilling to use others'. There were also a variety of opinions related to change in general. The department generally agreed that students were entering and exiting the P2C2 courses without the desired conceptual understandings and procedural skills, so were somewhat open to the general idea that some improvement was needed. Individuals within the department viewed the problems through their own idiosyncratic lenses, leading to a range of proposed strategies for improving outcomes. The department chair organized a calculus task force, composed of faculty representatives of several viewpoints, which counted as departmental committee service for those involved. This group considered various suggestions and concerns related to each proposed new structure, including the implementation of course coordinators, uniform course elements, and regular meetings for instructors in the P2C2 courses. As a group, they rejected, accepted, and adapted these ideas to find something palatable to all. The development of a shared conception, agreed upon by so many already, smoothed the path to a wider departmental vote in favor of coordinating Precalculus, Calculus 1, and Calculus 2. The lower status of lecturers also contributed to the implementation of course coordination, as it was suggested that faculty coordinators would make decisions that lecturers needed to follow, rather than faculty telling other faculty what to do. It seems that this contributed to the passage of the faculty vote. A final contributing factor to the task force, and then the department, agreeing to course coordination courses was the general view of the P2C2 sequence as a set of service courses taken primarily by non-mathematics majors. As the major impact would be to non-majors, faculty in the mathematics department were less concerned about what topics needed to be covered and how than they might be for courses which directly lead students into upper-division mathematics electives. Thus, the design of a new *structure* (coordination system) involved the development of a shared vision across many *people*, leveraging an existing *structure* (department committees). It came to be, in part, because of the existing *power* dynamics (relative status of lecturers) and beliefs and values that are highlighted by the *symbols* view (P2C2 as service courses, frustration with existing course outcomes).

In the first implementations of P2C2 course coordination at LPU, there were areas for improvement. For example, during the first term, the course coordinators did not yet have a robust system for communicating with the other instructors about the content and format of the exams they were writing, which resulted in drastic differences in scores from section to section. As a result, instructors refused to abide by a common grading scheme and made adjustments to their students' scores to reflect the variation in what had been covered and how in their respective courses. The lines of communication between coordinators, instructors, and GTAs allowed this issue to be brought up quickly and discussed openly, leading to a new protocol for sharing and collaborating on the writing of midterm and final exams. Communication was a general challenge during the roll-out of the changes at LPU, one that is increasingly being addressed. Aside from the course coordinators, all of whom were tenured faculty, only three faculty members taught P2C2 courses in the first two years of coordination (in total, there were three coordinators, three other faculty, and nine lecturers). Two of these three were unhappy with their lack of control, and refused to go along with all the coordinators' decisions. In response, the department chair (who promoted the change initiative) has made efforts to assign those faculty to other courses in the foreseeable future, leveraging his official powers in the department.

Given the previous discussion of concerns, one might assume that the effort to implement course coordination would fail. To date, however, the coordination is in place. Certain aspects of the department and institution's culture were strong enough to overcome the concerns of a few. The aforementioned belief that the pre-existing P2C2 courses were not sufficiently preparing students was part of this, with many faculty members willing to test the new system thoroughly, especially as one of the coordinators repeatedly voiced his belief that students were doing better, on harder exams, than they did previously. His insistent presentations to the department of early wins and markers of success, including better attendance, fewer instances of cheating, and increased performance helped the initiative maintain its course. The course coordination system also supported the work of GTAs and the tutoring center, as all students in a given course were grappling with the same material at the same time. This did not go unnoticed by those working with the GTAs or at the tutoring center. Additionally, instructors noted that the consistency of the P2C2 courses made teaching any course with P2C2 prerequisites more straightforward, as they could be assured that students had seen certain material presented in a certain way, and using the same textbook. Administrators, who had been supportive and secured some of the necessary funding for the larger initiative, have also continued to support the coordination of courses for a variety of reasons – all of which contribute to the department-at-large's interest in maintaining course coordination. Over the next few years, the effects of the overall initiative, and course coordination, on long-term metrics such as persistence, completion rates, and time-to-degree will be measured and use to more appropriately gauge the successfulness of these changes.

Discussion & Significance

This brief paper provides an example of how the four frames can be used to capture the complexity of department-level change over a number of years. The change initiative we studied was largely successful, making numerous and, as yet, lasting changes in the department. The change process began with a taskforce that helped create a common vision for the department.

This vision was enacted through a variety of structures and the creation of an integration coordination structure. Rather than quick fixes, these large structural changes are new aspects of the department that modify its basic operation. The frames also highlight that department members paid less attention to symbolic aspects of change, such as focusing on the beliefs about the purpose of teaching (i.e. supporting students vs. weeding out students). There was also no explicit focus on equity as far as the role of lecturers. The four frames draw attention to these areas of symbols and power as key areas of focus for sustainable change and future efforts at LPU. These particular beliefs and power issues are a part of academia generally, and specifically mathematics (i.e., in terms of a weed-out culture). Thus, we see that the four frames theory can draw attention to the types of things one should attend to in a change initiative, but deep contextual knowledge also helps support how the theories are applied to mathematics in specific. Conversations with change agents about this study's findings suggest that they will make efforts to address these aspects of the department as they continue to move forward and improve the LPU mathematics department. This shows how the four frames theory can help change agents attend to areas of focus that they otherwise may have not considered, which is a tool to support holistic, sustainable change.

Here we have provided an example of how the four frames are a useful tool from organizational change that can be adapted to the context of educational change. These frames help organize an understanding of what has happened at LPU, how that process has played out, and the impact of existing and evolving departmental culture on the products, enactment, and process of change. Crucially, the frames also reveal gaps in the change initiative, areas for growth and cultural factors which potentially affect the enactment and sustainability of the new system. The four frames, therefore, are a tool for other researchers, change agents, or university administrators to increase the likelihood of implementing sustained changes. The four frames can be used by internal and external members of a community to identify supports (e.g., stakeholders, institutional goals, champions) and constraints (e.g., weed-out mentality, power differentials) to better navigate the pathways of change. Leveraging this framework to identify aspects of departmental and institutional culture that can be used to individualize and personalize the generic products and processes of change found in the literature, thus addressing one of the primary obstacles to sustainable improvement initiatives.

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Calculus II Students' Understanding of the Univalence Requirement of Function

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A robust conceptual understanding of function is essential for students studying calculus and higher levels of mathematics as they continue to pursue the learning of mathematics. In this study, we investigated the ways in which students in a Calculus II course understand functions by examining student engagement with a vending machine applet. Specifically, we considered how these students made sense of the univalence requirement of functions in the context of a vending machine in which a single input produces an output of two cans. We identify and discuss in detail several themes that emerged in students' categorization of machines as functions or non-functions when encountering this two-can scenario.

Keywords: Functions, Calculus, Univalence

Functional relationships are an essential construct in undergraduate students' mathematical learning (Cooney, Beckmann, & Lloyd, 2010; Dubinsky & Harel, 1992; Leinhardt, Zaslavsky, & Stein, 1990). However, research has shown that undergraduate students often display incomplete conceptions regarding the concept of function (e.g., Oehrtman, Carlson, & Thompson, 2008), including an incomplete conceptual understanding of domain and range (Dorko & Weber, 2014). These conceptions or other difficulties that students have may be due to a lack of understanding of the nature of connections between the different representations of functions (e.g., Clement, 2001; Stylianou, 2011), the abstract nature of the function concept (Steele, Hiller, Smith, & 2013), or lack of a fully developed definition of function (Clement, 2001). Additionally, without a robust understanding of function, students may struggle with the function concept when moving from two to three dimensions in multivariable calculus (Dorko & Weber, 2014).

One's concept of function depends on his or her previous experiences with function, including the definitions to which they have been introduced (Thompson & Carlson, 2017). The most commonly used definition of function in schools is a variation of Dirichlet's definition (e.g., a function is a relation between two sets in which every element in the domain is mapped to exactly one element in the range) (Cooney & Wilson, 1993; Thompson & Carlson, 2017; Vinner & Dreyfus, 1989). This definition attends primarily to the relationship between two sets of elements (i.e., domain and range). As a result, students' difficulties are often related to the univalence requirement of the definition of function (Dubinsky & Wilson, 2013). To address this issue, we designed an applet in the form of a vending machine to problematize univalence. We used this applet to examine how Calculus II students make sense of the univalence requirement of functions situated in a vending machine context in which an output of two cans is produced by a single input. In this study, we seek to identify the themes that arose through this two-can scenario. We attempt to answer the question: In what ways do Calculus II students make sense of a vending machine applet that produces two cans from a single input?

Background Literature

Much of the research on student understanding of function has occurred in the context of college algebra, precalculus, or calculus classes. Through these studies there has been a careful identification of common understandings that students develop related to the concept of function. Common student understandings include that functions are defined by an algebraic formula or two expressions separated by an equal sign, and that functions are represented by graphs (that pass the vertical line test) (Carlson, 1998; Clement, 2001; Breidenbach et al., 1992; Thompson & Carlson, 2017). All of these conceptions are limited and can be problematic when distinguishing functions from non-functions, especially in non-algebraic settings (Steele et al., 2013).

An important aspect of the identification of functions is the univalence requirement (i.e., a function maps each element in the domain to exactly one element in the range). When using a graphical view of function, students often satisfy the univalence condition by using the vertical line test; however, the arbitrary nature of what a function can represent is lost within this narrow view (Clement, 2001; Steele et al., 2013). In addition, research has shown that a common incomplete conception regarding the univalence requirement is believing that it is synonymous to saying that the function has a one-to-one correspondence (Dubinsky & Wilson, 2013).

Due to the concern that calculus students may have developed a weak understanding of the concept of function (Moore, Carlson, & Oehrtman, 2009), researchers have suggested that students be engaged in activities that require using various representations (Zeytun, Cetinkaya, & Erbas, 2010; Moore et al., 2009). One way in which this can be accomplished is by using interactive applets that do not make use of any type of algebraic representations. The use of technology in this way can cause a cognitive conflict and require students to reflect and reassess their current understanding of function (Pea, 1987). For instance, a student whose understanding of function is only related to input-output relationships or reliance on the vertical line test may have trouble when encountering non-algebraic functions in a novel context (Steele et al., 2013). Sherman, Lovett, McCulloch, Edgington, Dick, and Casey (2018) found that the use of an online applet in the context of a vending machine, designed to support calculus students' opportunities to consider functions in a novel environment, improved student understanding of the definition of function and strengthened their ability to distinguish functions from non-functions. Through analysis of 105 undergraduate students' pre- and post-definitions, they found that students' interaction with the applet resulted in improved attention to the univalence requirement in their stated post-definitions (pre-definitions 36.6%; post-definitions 85.3%). However, by attending only to the students' pre- and post-definitions, student thinking in regard to univalence in the context of differentiating between function and non-function relationships remains unclear.

Theoretical Perspective/Conceptual Framework

In considering undergraduate students' learning related to function, we adopted a theoretical lens of transformation theory (Mezirow, 2009). Transformation theory is consistent with constructivist assumptions, specifically in that meaning resides within each person and is constructed through experiences (Confrey, 1990). Mezirow (2009) describes four forms of learning that lie at the heart of this theory: elaborating upon existing meaning schemes, learning new meaning schemes, transforming meaning schemes, and transforming meaning perspectives. Meaning schemes are the specific expectations, knowledge, beliefs, attitudes or feelings that are used to interpret experiences (Cranton, 2006; Peters, 2014).

Learning by transforming meaning schemes often begins with a *disorienting dilemma*. This stimulus requires one to question his or her current understandings that have been formed from previous experiences (Mezirow, 2009). It is this type of learning experience that we are particularly interested. Given the evidence that undergraduates often have a view of function that is limited to algebraic expressions and the associated graphs (e.g., Carlson, 1998; Even, 1990) and that such understandings typically result in a "vertical line test" related definition of function (e.g., Carlson, 1998), we designed an experience that would problematize these understandings, thereby creating a stimulus for transformation.

One strategy that has been suggested for diminishing common misunderstandings related to function is the use of a function machine as a cognitive root. The idea of a *cognitive root* was introduced by Tall, McGowen, and DeMarois (2000) as an "anchoring concept which the learner finds easy to comprehend yet forms a basis on which a theory may be built" (p.497). As an example of a cognitive root for the function concept, Tall et al. (2000) suggest the use of a *function machine*, typically referring to a type of "guess my rule" activity in which inputs and associated outputs are provided, challenging students to determine the pattern (i.e., identify the function rule). The use of such machines proved quite promising as a cognitive root for function, yet some students still struggled with connecting representations and determining what is and is not a function (McGowen, DeMarois, & Tall, 2000). Given the potential of using a machine metaphor as a cognitive root for function, as well as our desire to present a disorienting dilemma for undergraduate students, we designed an applet to provide students with a learning experience.

Context of this Study: Vending Machine Applet

The vending machine applet (McCulloch, Lovett, & Edgington, 2017) was designed to provide an opportunity for students to reexamine their definition of function by interacting with a non-algebraic representation. The applet was built using a GeoGebra workbook and uses a vending machine metaphor to represent functions and non-functions. The first three pages of the applet each contain two soda vending machines (Machines A-F), each with buttons for Red Cola, Diet Blue, Silver Mist, and Green Dew. When the user clicks a button (input), one or more cans (red, blue, silver, and/or green) appear in the bottom of the machine (output). To remove the can(s) the user clicks the "take can" button. Students are asked to compare the different machines and determine which of the two represent a function. The non-function machines have at least one button that produces at least one random can when clicked (i.e., the resulting can is not predictable based upon the button that is pressed). The fourth page of the applet contains an additional six vending machines (Machines G-L); students are asked to consider whether or not each machine could represent a function. Student work on Machines D, E, I, and K (see Table 1) are the focus of this study, as these are the machines that result in a two can output.

Table 1. Machine output for each button clicked

	Button Clicked			
	Red Cola	Diet Blue	Silver Mist	Green Dew
Machine D	random pair	blue can	silver can	green can
Machine E	red can	blue & random can	silver can	green can
Machine I	two silver cans	green can	red can	blue can
Machine K	red can	blue can	silver can	red & green can

Research Methods

The purpose of this qualitative study was to investigate the ways in which Calculus II students make sense of functions in a vending machine context. We attempt to answer the question: In what ways do Calculus II students make sense of a vending machine applet that produces two cans from a single input?

Participants and Data Collection

A total of 40 students from one centrally located U.S. university participated in the study. At the time of data collection, all participants were enrolled in a Calculus II class. Each student recorded a screencast of themselves working through the applet while noting decisions leading to the classification of each machine as a function or non-function. Students also wrote their rationale for each decision on an accompanying worksheet. Students were asked to use a *think aloud* protocol to explain their reasoning while interacting with the applet. The data used for this study include the video-recorded screencasts and the accompanying worksheets. Upon review of the data, four students were eliminated from analysis as their screencasts lacked audio or were incomplete. A total of 36 students' data were analyzed.

Analysis

The first phase of data analysis consisted of creating descriptions of students' engagement with the applet that included timestamps with direct student quotes. Next, we coded for articulated dilemmas and the triggers for those dilemmas. This study is focused on one trigger, one input mapped to an output of two cans, as such descriptive transcriptions were created for the portions of the video in which this specific situation occurred. While transcribing the screencasts, we created memos related to student thinking about the two-can scenario.

A preliminary codebook was developed based on themes that emerged in the memos related to the ways in which students made sense of the two can dilemma. The researchers then used this codebook along with open coding and a constant comparative method (Strauss & Corbin, 1998) to develop a final set of codes. For example, the final codebook included: function - consistent, function - corresponding color, non-function - two outputs, and non-function - random. Once the codebook was finalized and inter-rater reliability achieved, all remaining data was double coded with any differences discussed until agreement was reached. Finally, the researchers looked within the data for each code to identify themes across and within machines.

Results

Analyses illuminated several themes that directly address the purpose of this study – namely, to examine the ways in which students identify functions and non-functions when faced with the dilemma of an output consisting of two elements (see Table 2). Looking across all machines that produced a two can output, nearly a quarter of the students (8 of 36) indicated that a machine with an output of two cans is never a function regardless of the consistency of the output. For example, one student commented "Machine K is consistent with what it's giving out, but it's giving out two cans, and I feel like it shouldn't be able to do that. Like a function shouldn't allow it to have two outcomes for one input." This student's decision was based exclusively on the number of cans produced in the output. Students who decided that machines producing two cans automatically qualified as a non-function often attempted to make sense of the applet by viewing the cans as numbers or coordinates. One student stated, "Oh, it's not a function, because the Green Dew produces two y-values." This statement indicates that the student was considering the idea of univalence and viewing the two green cans as two separate outputs. Many of these students incorrectly used the language "one to one" to refer to univalence while relying on procedural knowledge to make a decision.

While some students focused on the number of cans, others made decisions based upon predictability of the output or lack thereof. Many students (56%) commented on consistency or randomness for at least one of the two can machines, however only 22% *always* used the idea of consistency or randomness to decide whether the machine was or was not a function. Students who decided that both Machines I (Red Cola → two silver) and K (Green Dew → red and green) were functions tended to focus on a consistent two can output. For example, one student remarked, "So even though the green button dispenses two different cans, it does generate the same outcome each time, as well as the other button, so Machine K is also a function." This student was unconcerned with the output quantity and was attending to the predictability of the machine. The remaining 78% of students did not reliably consider consistency or randomness. The following sections detail the emerging themes that arose from the two can dilemmas.

Table 2. Overarching Themes regarding the Two-Can Scenario

Overarching Themes	
Rationale	Percentage of Students (N=36)
Based decision on number of cans every time	22% (8)
Based decision on consistency and randomness of output every time	22% (8)
Based decisions on both consistency/randomness and number of cans	22% (8)
Various other reasons	34% (12)

Dilemma: Two cans with at least one being random

Machines D (Red Cola → random pair) and E (Diet Blue → blue and random) not only had a two can output but also included an element of randomness in the output. When confronted with both randomness and a two can output, 22% of the students described the lack of consistency of the output in their justifications. For example, when one student clicked the Red Cola button on Machine D the first time, two red cans were given as the output. When Red Cola was clicked a second time, two blue cans were dispensed. The student commented,

> Okay so it looks like the Red Cola is different, see it's moving between the different colors for the two cans; so, it was red, now it's blue. So, because of that, I feel that Machine D is not a function because, because it's creating different outputs for the Red Cola and it's not consistent.

Similarly, another student used randomness to justify a decision,

> However, red is the odd one out here as it is a different, it's giving off two of a random color drink. Because Red Cola has a random, has a random effect. Every time, there's no rhyme or reason as to why it does it, it's just, possibly, random number generator.

Both students commented on the random or inconsistent output of the Red Cola button and decided that the lack of predictability make Machine D a non-function. When assessing Machine E, another student stated "The blue always does random, while the other ones keep clicking their same color can. So, I think in this case, F is a function because blue is always a constant silver. While in E blue is a random." The predictability of Machine F and the randomness of Machine E seemed to inform this student's decision.

In contrast, eight of the 36 students attended to both randomness and the number of cans in the output when justifying that Machines D and E were non-functions. For example, one student commented that "Machine E, however, can't be, uh, multiple outputs, especially multiple different outputs. So, E should not be a function." This student noticed that Machine E both produces two cans (“multiple”) and produces random colored output (“different outputs”). In the cases of these students who attended to both of these factors, it was unclear in both the screencasts and worksheets which reason predominated their decision-making process.

Dilemma: Two Consistent Cans

Interacting with Machines I (Red Cola → two silver) and K (Green Dew → red and green) presented the students with the situation of a button having a consistent output of two cans, yet eight of the 36 students treated these machines differently from one another, labeling one as a function and the other as a non-function (see Table 3).

Table 3. Machine I and K Inconsistencies

Rationale	Inconsistency	
	I is a function; K is not a function (N=6)	K is a function; I is not a function (N=2)
Lack of Coherent Explanation	1	
Machine I produces 2 cans of same color; Machine K produces 2 different colors	5	
Machine K: Cola button *does* match output can color; Machine I: Cola button *does not* match output can color		2

The majority of the students who classified I and K differently were attending to the color of the two can output instead of the consistency of those outputs. The attention to color manifested in two ways: students either identified the machine as a function 1) if the two cans produced were the same color (Machine I: Red Cola → two silver), or 2) if the button color matched at least one of the output cans (Machine K: Green Dew → red and green). The two students who labeled Machine K as a function and Machine I as a non-function were looking for colors of the output cans to correspond with the color of the pressed button. For example, one student said, “Machine K is a function because for all of the buttons I do get what I want, but even though I click Green Dew I get something else, I still get what I want right.” This student’s explanation included that the Green Dew button output both a green can (“I still get what I want”) and a red can (“something else”). Students are specifically examining whether the button colors correspond to the color of the output can(s). This reasoning was unique to these two students.

The remaining five students who gave a coherent explanation regarding their attention to color labeled Machine I as a function and Machine K as a non-function. These students identified Machine I as a function because the output created two cans of *the same color*. For example, one student commented that “I guess it’s still a function, but if they had two separate color cans, then I think that would imply different y, values for y. So I’m gonna say that Machine I is definitely a function.” These students also indicated that Machine K was not a function since the output consisted of two cans of *different colors*. For example, one student wrote on the worksheet, “Although the 'Green Dew' button always gives the same outcome it releases two different cans unlike all the other buttons on this machine.” Some students elaborated further and commented that different buttons produce the same color can, “It appears that multiple input buttons, like green and red, both produce red cans as their output which makes them not a function.” In an attempt to make sense of this problem, one student tried to connect the cans to numbers and make use of the vertical line test,

> Um, I think that it is not a function because every, every input should only have one output, and this one, it has two. So, I just, I picture it on a graph and I don't think that would pass a vertical line test and I think that is something, um, that a function needs to pass, so I don't think Machine K is a function.

This student is viewing the two cans as two separate outputs because they are different colors. Lastly, one student did not provide a clear enough think aloud or written rationale to ascertain why Machine I was labelled as a function and Machine K as a non-function.

Discussion and Conclusion

Sherman et al. (2018) found that students' definitions of function showed increased attention to univalence after engaging with the vending machine applet. This study builds on that work by attending to students' engagement with the applet as it relates to the two-can scenario. Our results revealed that while many students justified their decisions by referencing consistency or randomness, it was uncommon for students to do so reliably. One concern is that 78% of students in this study steadfastly focused on irrelevant elements or unreliable rationale when presented with machines producing a two can output. For example, some students focused on whether the button color matched the output can color (irrelevant elements). Other students switched reasoning from machine to machine focusing on predictability one time and on the number of cans the next time (unreliable rationale). This may be due to students lacking a fully developed definition of a function (Moore et al., 2009), or that students' understandings of function are too narrow or include erroneous assumptions (Clement, 2001). This suggests that Calculus II instructors need to help students develop a strong definition of function which can be applied to a variety of representations.

The attention that some students placed on attempting to connect the vending machine context to numbers or coordinates confirms the known difficulties students have with univalence (Dubinsky & Wilson, 2013) and their over reliance on procedures (Steele et al., 2013). It was evident from the screencasts that these students were linking the two can output to two y-values and confusing univalence with one-to-one correspondence. The prevalence of this confusion and the incorrect use of language regarding one-to-one correspondence suggests that some students do not understand that one-to-one correspondence is a special case and not a requirement. This is an area that warrants further research.

One limitation of our study was that our analysis only included transcriptions of videos of students' interactions with the applet, in that we did not utilize students' personal definition of function in tandem with their interactions. Future studies with this applet should analyze the reliability of student rationale in conjunction with their definitions to determine if weak or narrow function definitions are related to the inconsistent classification of machines. Moreover, as this study focused on only one dilemma trigger, future studies should explore other triggers.

Calculus II students have had many experiences with functions, yet the analysis of their interactions with the vending machine applet revealed the possibility that students have underdeveloped definitions of function or consider functions too narrowly. Further research is needed to explain the unreliable rationales when determining function from non-function in non-algebraic settings to better understand why students have difficulties with univalence. With the concept of function permeating mathematics past Calculus II, the results of this study demonstrate the need to allow students to reexamine their conceptual understanding of function in advanced classes, where these topics are not necessarily in the scope of the class.

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Analyzing Topology Students' Schema Qualities

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A schema is a mental structure of concepts that are connected together and allows for the efficient functioning of director systems. Skemp (1979) discusses various qualities that this study used to look at students' schemas. This case study focuses on a pair of Topology students and their work on a problem involving the product topology on $X \times Y$. There were many positive qualities that the students demonstrated, but there were also difficulties with particular connections between concepts.

Keywords: Topology, schema, director system

Theoretical Background

Topology is an important course for advancement in mathematics. Many graduate students need to take topology to continue in their mathematics degrees. Regrettably, research on pedagogy of topology is still in its infancy. In a study by Berger and Stewart (2018), the analysis of the data revealed that the majority of undergraduate students were in the beginning stages of schema development, even though they were completing a final examination at the end of their semester. Cheshire (2017) looked at axiomatic structures of Topology and how students' schemas undergo accommodation to understand these structures.

Understanding mathematical concepts at an advanced level is an enormous undertaking for many students. The word 'understanding' reminds us of the well referred work by Skemp (1976) on *relational understanding* and *instrumental understanding*. Some years later, Skemp (1979) developed a model of intelligence in which its focus was the construct of the idea of *schema*. In Skemp's (1979) notion:

> A schema is a structure of connected concepts. The idea of a cognitive map is a useful introduction, a simple particular example of a schema at one level of abstraction only, having concepts with little or no interiority, and representing actuality as it has been experienced. A schema in its general form contains many levels of abstraction, concepts with interiority, and represents possible states (conceivable states) as well as actual states. (p. 190)

A schema is what allows for the efficient functioning of a *director system,* which is a central focus to Skemps' model. A person or object has a present state that they are currently in, and a goal state that they would like to be in. "That which is changed from one state to another and kept there" (p. 41) is what Skemp calls the *operand.* The *operator* is "that which actually does the work of changing the state of the operand." (p. 41) Finally, a director system is "that which directs the way in which the energy of the operator system is applied to the operand so as to take it to the required state and keep it there." (p. 41-42)

Skemp (1979) communicated his theoretical ideas through many everyday examples. He referred to the temperature of an oven in many instances. Say an oven is at room temperature and needs to heat to 400 degrees Fahrenheit. The present state is the current temperature and the goal state is to reach 400 degrees Fahrenheit. The operand is the interior of the oven and the operator is the temperature of the oven. The thermostat in the oven is the director system. If there is a new goal state of 350 degrees Fahrenheit, the same director system (the thermostat) will be used.

A schema is what gives a director system this flexibility when states change. "The greatest adaptability of behavior is made possible by the possession of an appropriate schema, from which a great variety of paths can be derived, connecting any particular present location to any required goal location." (Skemp, 1979, p. 169) Skemp's work with director systems was also used by Olive and Steff (2002, p. 106) to build "a theoretical model of children's constructive activity in the context of learning about fractions."

Skemp (1979) believed that "a schema is a highly abstract concept" (p. 167). Some of his qualities of schema and the definitions of certain words that are used are shown in Table 1.

Table 1. Certain qualities of a schema.

Qualities of a schema	Definitions
(ii) "Relevance of content to the task in hand (rather obviously, but not always met)." (p. 190)	
(iii) "The extent of its domain." (p. 190)	Domain: "The set of states within which (and only within which) a director system can function, i.e., can take the operand to its goal state and keep it there, provided that the operators are capable." (p. 312)
(iv) "The accuracy with which it represents actuality." (p. 190)	
(v) "The completeness with which it represents actuality within this domain." (p. 190)	
(vi) "The quality of organization which makes it possible to use the concepts of lower or higher order as required, and to interchange concepts and schemas. (The vari-focal part of the model, linked with the idea of interiority.)" (p. 190)	Vari-focal: "A way of describing the different ways in which the same concept or schema can be viewed, from a simple entity to a complex and detailed structure." (p. 316)
(vii) "By a high-order schema, we mean one containing high-order concepts…This determines its generality…" (p. 190)	
(viii) "The strength of the connections." (p. 190)	
(ix) "The quality of the connections, whether associative or conceptual." (p. 190)	
(x) "The content of ready-to-hand plans…" (p. 191)	Plan: "A path from a present state to a goal state, together with a way of applying the energies available to the operators in such a way as to take the operand along this path. A plan is thus one essential part of the director system." (p. 314)

The notion of schema has also been explored by others in the literature. For example, a definition of schema is embedded in APOS Theory (Dubinsky & McDonald, 2001). They claimed that "a schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept." (p. 277)

In this paper we will examine students' development of their schemas and their qualities based on Skemp's (1979) model. The research question to guide this study was: What qualities of schema do Topology students demonstrate?

Methods

In this case study, we examined students' schemas for a basis for a topology. This is part of a larger study for the first author's dissertation. The participants were first year graduate students who were enrolled in a graduate topology course at a Southwestern University in the U.S. The four participants were divided into two pairs. Each pair did two task-based interviews together, the first about a month into the semester, and the second during finals week of the same semester. Pairs were used to try to get the participants to demonstrate their ideas and discuss how they think about the tasks to their partner, making their thoughts more observable. No data was collected regarding what took place in the classroom before, after, or between these interviews.

The participants were given a task sheet and a definition sheet. Each interview began with a period of time where the participants could look through and work on the tasks individually. After that, they worked on the tasks as a pair, explaining their thoughts to each other and coming to a consensus for each problem. In the final part of each interview, the pairs were asked follow-up questions about what they thought was needed to complete each task. They were also asked about their background with Topology. In the interviews that occurred during finals week, they were additionally given their work from earlier in the semester and asked to discuss their progress between then and finals. Each interview was video recorded and then transcribed. If the participants utilized the white boards, their written work was also transcribed. In the transcriptions, a scribble indicates that the pair erased something on the board.

For this study, we focused on the third task only (see Figure 1). We chose to analyze the data from the third task for a few reasons. First, this problem is one that frequently shows up on homework assignments and exams when the product topology is covered in class. As such, we have been able to collect data in the past involving this same problem, regardless of what Topology course the data was collected from. This problem also requires a higher-order schema for a basis and, because of this, we hypothesize that the data will be more informative about certain qualities of schema needed. After transcribing the data from this task, we established some themes from Skemp's model based on the qualities and created Table 2 as a framework. After creating an ideal proof with ideal qualities, we examined our data against it.

3. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.
   (b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$ is an open map.

Figure 1. Task 3. Show that the projection map is an open map.

Results and Discussion

The step by step proof of part (b) has been illustrated in Table 2. The lower-order concepts needed for this proof, as well as the schema qualities ideal for completing each portion of the proof, are shown. We acknowledge that there are more qualities of schema that can be applied in each step, however, the listed qualities are what we focused on based on our experience with this problem and our data set. After being transcribed, the data was divided up by what portion of the proof it aligned with. In each portion, we provide some explanation of the data in terms of Skemp's qualities of schema. Ideally, learners' schemas become more structured as they go throughout the course, but this study is not solely focused on comparing the early and final interviews. Instead, we are looking for changes in students' schemas and what qualities are involved in those changes.

Table 2. Qualities of each portion of task 3.

Portion of proof	Proof for part (b)	Explanation of Proof Step	Lower-order Concepts Needed	Qualities of Schema
b1	Consider $p_X(W)$ where $W \subset X \times Y$ is an arbitrary open set of $X \times Y$.	Start with an arbitrary open set of $X \times Y$ and see where p_X sends it.	-Open map	-Plan -Domain -Relevance
b2	Now $p_X(W) = p_X(\cup_\alpha (U_\alpha \times V_\alpha))$ where $U_\alpha \times V_\alpha \in \mathcal{T}$ are basis elements.	By the definition of a basis, W can be written as a union of basis elements.	-Topology generated by a basis -Equality of sets	-Strength -Quality -Domain
b3	Note $p_X(\cup_\alpha (U_\alpha \times V_\alpha)) = \cup_\alpha p_X(U_\alpha \times V_\alpha)$.	The projection of a union is a union of projections.	-Projection map -Equality of sets	-Generality -Domain -Accuracy -Strength -Quality
b4	Now $\cup_\alpha p_X(U_\alpha \times V_\alpha) = \cup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{T}_X$.	The projection map sends basis elements to open sets of X.	-Projection map -Definition of the product topology on $X \times Y$	-Accuracy -Completeness
b5	Since $\cup_\alpha U_\alpha \in \mathcal{T}_X$, $p_X(W) \in \mathcal{T}_X$ and p_X is an open map.	The union of open sets of X is also open in X.	-Topology -Open map	-Strength -Vari-focal

Initial Interview

For the purposes of this paper, we focus on only one of the pairs of participants: Brandon and Kyle (pseudonyms). Brandon had not completed a Topology course before the initial interview and Kyle had previously taken an introductory Topology course, so their experience with the subject matter was limited. Additionally, this pair was more interactive with each other and took their time discussing each task.

In their initial interview, Brandon and Kyle defined the product topology without using a basis and then had a short proof based on their incorrect definition. This was the typical mistake that was found in previous work with undergraduate students (Berger & Stewart, 2018). Specifically, in defining the product topology in part (a), the pair incorrectly stated that the product topology consisted of all sets of the form $U \times V$, not the unions of such sets. Therefore, they set themselves up for a fairly trivial proof for part (b). They both took some time wrapping their minds around the problem, Kyle in more of a verbal manner. For b1, they took their arbitrary open set to be exactly what is expected based on their response to part (a), which is $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ (see Figure 2). Their director system functioned appropriately, but their previous knowledge led them to start part (b) at the wrong present state.

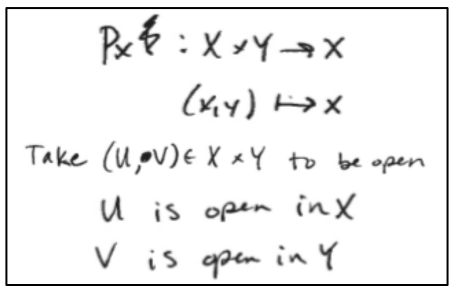

Figure 2. Brandon and Kyle's attempt early in the semester.

From there, they followed the definition of the projection map and immediately got what they needed in order to show that the map is open (see Figure 3). Since this step was a fairly straightforward computation for the pair, their ideas were accurate and complete, but only relative to their incorrect definition in part (a). With regards to their schemas for a basis, we cannot make any claims since their work here provided no evidence regarding a basis.

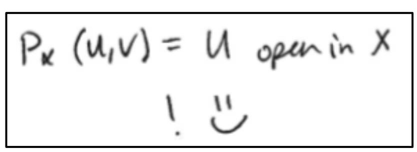

Figure 3. The end of Brandon and Kyle's proof.

Final Interview

Now we will discuss what Brandon and Kyle did during finals week. For part (a), they correctly defined the product topology by generating it with a basis. For b1, Brandon quickly wrote an arbitrary open set, W, on the board, but neither Brandon nor Kyle discussed it (see Figure 4). Brandon had a plan and executed it without discussing it with Kyle. Mathematically, this was relevant and fit within the domain of the problem.

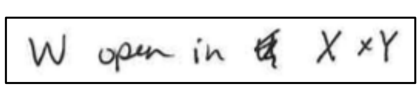

Figure 4. The start of Brandon and Kyle's proof late in the semester.

For b2, Brandon began writing W as a union of basis elements, but Kyle seemed unsatisfied and wanted to make the notation clearer. Brandon asked how they wanted to proceed in showing the union. Kyle took over writing on the board, erased the union that Brandon had written, and came up with the beta notation shown in Figure 5. After this, he was still unsatisfied with his notation and the usage of too many b's, but Brandon said it was fine, so they moved on. Note that Kyle immediately knew that they were wanting to write a union of basis elements, but the pair spent their time here struggling to denote it. Towards the beginning of their discussion on b2, Kyle stated "W is the union of elements of...some arbitrary union of elements of that B thing [referring to the basis they wrote in part (a)]." Kyle's statement demonstrates a strong conceptual connection between open sets and a basis. Again, this was appropriate within the domain of the problem.

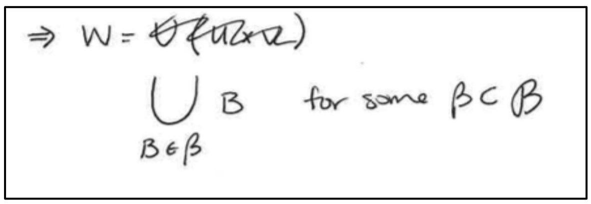

Figure 5. Where Brandon and Kyle had notational difficulties.

In moving on to b3, they started off quickly saying that the projection gives you a union of open sets, but then Kyle starts thinking about if they could get "weirder things". This launched the pair into a discussion about what they showed in class regarding the projection map. Brandon finally says something that takes them back to what they originally (and correctly) said, which prompted Kyle to read the task again and agree that they had been on the right track. Although their discussion may have been helpful for them in checking their ideas, it ended up not affecting the proof that they wrote down as the discussion was entirely verbal and ended with the same conclusions that they began with. They discuss what exactly the projection does, and this prompts Kyle to suddenly write up the proof seen in Figure 6. They verbally acknowledge that they get a union of open sets of X from the basis elements and use this as the end of their proof.

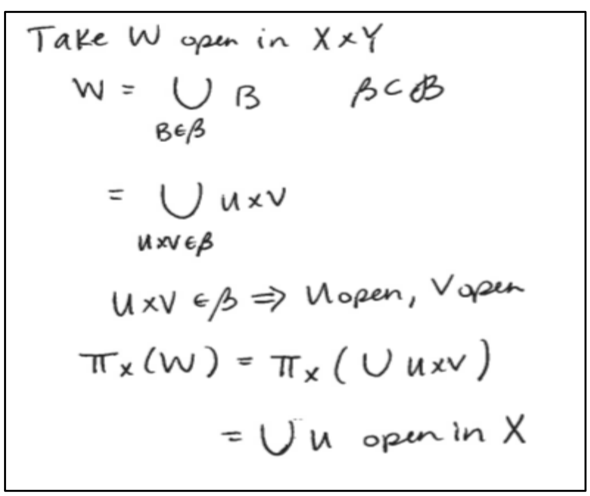

Figure 6. Final part of Brandon and Kyle's proof.

The part where the pair got off topic demonstrates a loss of relevance to the task and not as strong of connections between the projection map and unions. At the beginning and end of b3, however, they did demonstrate accuracy in their statements about what the projection map does. They combined b4 and b5 of the proof with their b3 statements and did not make any conclusion statements for their proof. Their final statements from b3 are accurate, but do not demonstrate complete ideas. We cannot say anything about what qualities they demonstrated for b4 or b5 since they did not say or write any concluding remarks.

Towards the end of the final interview, the pair was asked to reflect on their work from the initial interview. After reviewing their previous work, Brandon and Kyle quickly confirmed that they had not considered a basis in the first interview. When asked about what could have aided in correcting that mistake, Brandon responded that what corrected it for him was getting feedback on his homework "…with that specific thing being torn apart on it." Kyle admitted that he didn't "get the basis topology stuff at all when [they] were going over it in class" but later realized the importance when reading through the textbook. Brandon commented on their earlier work on the problem with,

> In terms of when we first did this, I guess, um, it's easy to just jump straight into just choosing U or something because of the way we defined the basis of just being an open set cross an open set where each of those are coming from the individual…so…it just seems natural to just go to one thing instead of considering the most general thing, which is a union of those things.

Brandon's reflection suggests that the generality and relevance of a schema are not necessarily intuitive.

Concluding Remarks

In this study we saw that accuracy and completeness are not typically a difficulty, but rather the generality of a schema, strength of connections, and the relevance of a schema can be difficult to navigate. Both Brandon and Kyle demonstrated strong, conceptual connections between lower-order concepts and a basis in the final interview, but not in the early interview. In the final interview, the pair demonstrated a weaker connection between the projection map and unions.

Employing Skemp's model in order to develop a more general theoretical framework to examine learner's schemas are among our next steps. Future work will also involve analyzing the other pair of participants, as well as analyzing the remainder of the tasks. Additionally, we will be interviewing graduate students whose research area is in Topology, as well as postdoctoral fellows to examine their schema qualities.

Based on this work, some teaching recommendations could include emphasizing scenarios where arbitrary objects and generality are necessary in higher-order topics, focusing on giving good feedback for students regarding what qualities are missing in their work, and explicitly discussing conceptual connections between concepts frequently.

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Construction and Application Perspective: A Review of Research on Teacher Knowledge Relevant to Student-Teacher Interaction

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This paper is a review of research that either explicitly or implicitly examines the interplay between teacher knowledge and teaching practices sensitive to students' mathematical thinking. I use radical constructivism as a lens to analyze how the researchers conceptualize the role of teacher knowledge in student-teacher interaction. My analysis reveals that some researchers attribute teachers' observable actions to what knowledge teachers possess (i.e., application perspective) while some others focus on what knowledge teachers construct in-the-moment (i.e., construction perspective). I conclude the paper by discussing the potential causes and consequences of these differences as well as the affordances and limitations of each perspective.

Keywords: Teacher Knowledge, Student-Teacher Interaction, Constructivism, Literature Review

Student-teacher interaction is a common and critical activity in teaching practices. However, some researchers have suggested that, in mathematical teaching and learning, there is a discrepancy between the mathematical meanings teachers intend to teach and the mathematical meanings students actually develop (e.g., Bauersfeld, 1980; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016; Thompson, 2013; Thompson & Thompson, 1994). This miscommunication between teachers and students creates a fundamental dysfunction in mathematics education (P. W. Thompson, 2013), which highlights the need for mathematics teachers to systematically bring forth the mathematics of students and make instructional decisions sensitive to the mathematics of students (e.g., Steffe & Thompson, 2000b; Teuscher, Moore, & Carlson, 2016). In extant literature, researchers have found that teachers' development and enactment of teaching practices sensitive to student thinking is associated with their knowledge of all kinds (e.g., Hill et al., 2008; Johnson & Larsen, 2012; Seymour & Lehrer, 2006; A. G. Thompson & Thompson, 1996). The goal of this review is to identify the researchers' various ways of operationalizing teacher knowledge in these studies.

Teacher Knowledge and Teaching Practices Sensitive to Student Thinking

Mathematics educators continue to highlight the importance of teachers eliciting and using student thinking in teaching practices. For example, National Council of Teachers of Mathematics (2014) announced eight Mathematics Teaching Practices, among which four are relevant to making instructional actions based on student thinking (i.e., facilitate meaningful mathematical discourse, pose purposeful questions, support productive struggle in learning mathematics, and elicit and use evidence of student thinking). Many researchers have studied how teachers learn and implement these practices, which resulted in a growing body of literature on teacher discourse moves, teacher noticing, teacher decentering and instruction in general. Findings of these studies indicate that most teachers are not used to attending to and interpreting student thinking (e.g., Franke, Carpenter, Levi, & Fennema, 2001; Jacobs, Lamb, & Philipp, 2010). Even when teachers do attend to student thinking, it is especially difficult for teachers to interpret what students are thinking and the sources of the students' difficulties (e.g., Johnson & Larsen, 2012; Maher & Davis, 1990; Speer & Wagner, 2009).

This issue motivates researchers to investigate what contributes to teachers' varying abilities to notice, listen to, model, and act on student thinking. Some researchers have suggested that teachers' knowledge is associated with teachers' abilities to notice student thinking (e.g., Lee,

2017) and instruction quality (e.g., Charalambous, Hill, & Mitchell, 2012; Hill et al., 2008). These researchers typically assessed teacher knowledge by using assessment items grounded in existing teacher knowledge frameworks (e.g., Ball, Thames, & Phelps, 2008) and analyze the teachers' actions independent of the knowledge assessments. Some other researchers have attempted to infer teachers' knowledge in the moment of interaction with students and, at times, combined this with retrospective teacher interviews to support their inferences (e.g., Seymour & Lehrer, 2006; Teuscher et al., 2016). Given these different approaches to teacher knowledge relevant to student-teacher interaction, I conjecture that researchers might have different views on teacher knowledge and affect their methodological designs and interpretation of their data. In this paper, I review select literature that examines the interplay between teacher knowledge and teaching practices sensitive to students' mathematical thinking in order to answer the following research questions: What are the researchers' different conceptualizations of teacher knowledge and what are the affordances and limitations of these conceptualizations?

Radical Constructivism, Knowledge, and Social Interaction

The epistemological stance of radical constructivism (von Glasersfeld, 1995) informs my analysis of the literature. I consider knowledge as actively constructed by a knower through interaction with the environment. Knowledge is not a representation of an objective ontological "reality"; rather, it functions and organizes viably within a knower's experience and is idiosyncratic to the knower. We thus have no access to anyone else's knowledge nor an objective environment; the best we can do is to construct hypothetical models of others' knowledge that viably explain our observation of their behaviors (Steffe & Thompson, 2000b). As it relates to mathematical teaching and learning, students' mathematical knowledge consists of their *ways of understandings* (Harel, 2008) of mathematics that are product of their mental actions constructed from their experience *including* interactions with their teachers; accordingly, teachers' *mathematical knowledge for teaching* (MKT; see Silverman & Thompson (2008)) is grounded in their *ways of understandings* of mathematics constructed from their experience *including* their interactions with students. In order to transform these personal understandings so that they have pedagogical power, teachers need to try to model their students' perspectives and consider how to foster their constructing similar understandings (Silverman & Thompson, 2008).

The radical constructivist view of knowledge and knowing is also useful for operationalizing social interaction among students and teachers. As Steffe & Thompson (2000a) stated, "interaction enters radical constructivism at its very core" (p. 192). Student-teacher interaction (or human communication in general) involves each individual engaged in a conversation interpreting others' meanings, anticipating others' responses, and adjusting her models of the others' meanings in order to decide how to act and what to expect in future conversations. As teachers and students communicate, they reciprocally construct (sometimes with intention) knowledge about the other in the moment of interacting through *assimilating* and *accommodating* (von Glasersfeld, 1995) the language and observable actions of the other (Steffe & Thompson, 2000a) (see the four blue arrows in Figure 1a). A teacher's constructed knowledge potentially perturbs and constrains their personal mathematical knowledge and affect their following actions (see the "enact" arrows on the teacher side). A teacher can also refine and construct knowledge of students' mathematical thinking through reflecting on their own ways of interacting along with the mathematical and pedagogical consequences of these ways of interacting in-the-moment and retrospectively (see the "reflect" arrow on the teacher side).

Related to the current review, I aim at examining select literature by focusing on the extent to which researchers capture this assimilation and accommodation aspects of teacher knowledge in their conceptualization of teacher knowledge and interaction (see my distinction between the *construction* and *application perspective* in a later section).

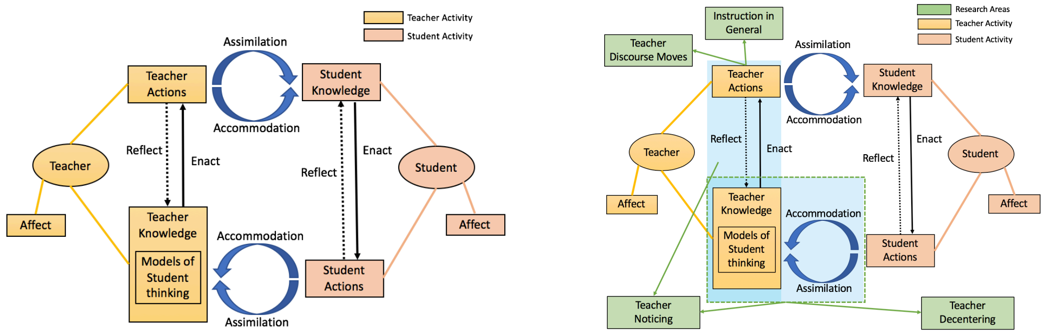

Figure 1. (a) A proposed framework of student-teacher interaction from a radical constructivist perspective with (b) four research areas situated in the framework (the blue area indicates activity relevant to the current review).

Methods

I first situated the literature base in the above framework and located four research areas relevant to teaching practices sensitive to student thinking, which included teacher discourse moves (i.e., teacher talks for eliciting and using evidence of student thinking), teacher noticing (i.e., teachers' ability to observe and recognize student thinking), teacher decentering (i.e., teachers' attempt to set aside her own thinking and model student thinking), and instructional actions in general (see green boxes in Figure 1b).

I selected literature through the consultation of the university library Multi-Search that simultaneously searched more than 130 databases. I elected to include only English articles that had been published in academic journals to guarantee that I included research of a scholarly and authoritative nature. I used keyword combinations of "mathematics", "teacher knowledge", and "decentering" or "discourse" or "listening" or "teacher moves" or "analytic scaffolding" or "teacher noticing" or "student thinking" to search only in abstracts. I then ranked the results of each search by "relevance" and reviewed the abstracts of the first 50 articles to filter 26 articles that satisfied the following four criteria: (1) empirical studies, (2) either explicitly or implicitly touches on teacher knowledge, (3) involves the activity of student-teacher interaction (e.g., research on teacher noticing during analyzing students' written work was excluded), and (4) touches on student thinking. Eighteen of these articles constituted the literature base for the current review, which included two studies on teacher decentering, four on teacher discourse moves, two on teacher noticing, and ten being relevant to teacher knowledge of student thinking.

I followed Galvan and Galvan's (2017) guidelines to conduct this review. I first conducted a vertical analysis (Miles & Huberman, 1994) of each of the 18 articles in my first pass of reading. I used an EXCEL sheet to organize the information of each article in six aspects. In this paper, I only focus on reporting my analysis on "researchers' conceptualization of (teacher) knowledge". I drew on two sources to interpret researchers' perspectives on teacher knowledge. First, I examined the theoretical framework section of the paper to infer the authors' conceptualizations of teacher knowledge or their interpretations and adaptations of the existing teacher knowledge frameworks. In addition, I drew attention to the teacher knowledge claims researchers made for explaining teacher actions to infer what kinds of knowledge they considered to be critical in generating these explanations. I then conducted a horizontal analysis (Miles & Huberman, 1994) to identify similarities and differences within this aspect across the literature. I developed a code for each emerged theme and assigned codes to each article during the second pass of reading. I continually searched for examples that the generated themes could not account for, and I modified my definition of the existing themes or created new themes.

Results

I identify four themes regarding researchers' conceptualizations of teacher knowledge and summarized the literature by themes in Table 1.

Table 1. Summary of literature by emerged themes (literature categorized as more than one theme is underlined).

Theme	Teaching Practices					Number of Literature
	Noticing	Listening	Discourse	Decentering	Instruction in General	
Application Perspective	Kersting (2008)	Johnson and Larsen (2012)	Bray (2011); Speer and Wagner (2009)		Charalambous et al. (2012); Hill et al. (2008); Park and Oliver (2008); Wilkie (2016)	8
Construction Perspective	Lee (2014)	Jenkins (2010); Johnson and Larsen (2012)	Seymour and Lehrer (2006)	Teuscher et al. (2016); Walters (2017)	Franke et al. (2001); Park and Oliver (2008); M. G. Sherin (2002); Wilkie (2016); Wilson et al. (2013)	11 (3 duplicated)
Action/Skill Perspective	Lee (2017)					1
Lack of Perspective			Jacobson and Lehrer (2000)			1

Application Perspective

To take an *application perspective*, a researcher conceives teacher knowledge as possessed knowledge enacted in contexts and being ready for researchers' evaluation. The researcher potentially presumes teachers' observable behaviors as driven by existing knowledge and aims at understanding what knowledge the teachers possess that enables them to act in particular ways. I identified 8 out of 18 studies in which researchers used this perspective. For example, Johnson and Larsen (2012) investigated the role of a teachers' mathematical knowledge for teaching in supporting her ability to listen to students. Although the authors did not elaborate on their conceptualization of teacher knowledge, I inferred their perspective based on their interpretation of the data:

> The authors demonstrated a transcript in which a researcher was explaining a student's thinking to the teacher, Dr. Bond, in a post-class interview and Dr. Bond responded with a new realization of what the student was thinking] Provided with this extra piece of information, that Adam [the student] was thinking about multiplying symmetries as a left to right sequential procedure, Dr. Bond was able to make sense of his concern…we argue that she was constrained by a limitation in her knowledge of content and students…Dr. Bond was constrained by a lack of knowledge about how her students might have been thinking about the operation of composing symmetries. (p. 122)

Here, the authors claimed that the teacher's lack of knowledge of content and students led to her difficulty with understanding the student's struggles and constrained her from interacting with Adam in ways that were sensitive to his mathematical thinking. They further argued that, if the teacher was armed with the knowledge of students' conceptions of binary operation, she would be able to apply such knowledge in-the-moment to address the student's concern. Because the authors attributed a teacher's thinking and actions to what knowledge she possessed or lacked, I infer that they held a view of knowledge that was consistent with an application perspective.

As another example, Speer & Wagner (2009) provided the following statement about *pedagogical content knowledge* (PCK):

> We use *recognize* in our description of the component practices to denote situations in which teachers are already familiar with the ways that students think about and come to understand

> the mathematics. In other words, their existing PCK may include knowledge of how students think about the specific ideas at hand and/or typical students' difficulties with the topic…At other times, even if teachers are not familiar with the particular ways of reasoning that students offer, they may be able to "figure out" what the students are suggesting and thinking. Therefore, recognizing draws heavily on a teacher's PCK, whereas figuring out requires that a teacher do some mathematical work in the moment. (p. 536-537)

First, the authors claimed that teachers might hold some existing PCK of students' mathematical thinking that enabled them to "recognize" similar students thinking in certain situations. This view of PCK is consistent with the application perspective. I also drew attention to the authors' awareness of situations where teachers' existing knowledge of student thinking might not include all possible ways of thinking they could observe and thus the teachers need to "figure out" student thinking in-the-moment ("do some mathematical work in the moment"). However, it was unclear as to whether the authors considered "figure out" to be a result of a teacher's application of existing knowledge (i.e., an application perspective) or as a process of constructing new knowledge (i.e., a construction perspective). In later sections, I found multiple pieces of evidence consistent with the former case—the authors made claims that the teacher's inability to "figure out" student thinking in-the-moment was due to her lack of knowledge (e.g., "Had Gage's SCK enabled him to figure out the mathematical ideas the students were suggesting…he might have been able to provide different kinds of guidance for the students…" [p. 553].).

Construction Perspective

The above example opens up an alternative interpretation of the "figure out." That is, the presence of novel student thinking may offer teachers an opportunity to *construct* new knowledge as they interpret what the student may be thinking in-the-moment. This interpretation is consistent with my definition of "construction perspective." To take a *construction perspective*, a researcher believes that teacher knowledge is generative, dynamic, evolving, and co-emerging from on-going interaction with students (including their reflection on their own interaction). The researcher explains teachers' observable actions in their teaching or reflection by inferring what knowledge about the students the teachers construct in-the-moment.

The construction perspective applies to 11 out of 18 studies. For example, Seymour and Lehrer (2006) conceived the growth in PCK as "an interactional achievement" in a sense that teachers develop PCK by engaging students in conversations to make sense of their mathematical thinking that is different from their own and by reflecting on students' thinking; meanwhile, students engage in understanding teachers' verbal meanings and actions. The authors also discussed their conception of orchestration as a site for developing PCK, stating "Orchestration and PCK are essentially coconstituted…PCK cannot emerge all at once, but rather evolves during the course of a protracted series of attempts to orchestrate classroom conversations" (ibid, p. 550-553). Transitions in a teacher's PCK can be characterized as "the emergence, stabilization, and adaptation of couplings" between student Discourse and teacher Discourse (ibid, p. 554). I interpret that the authors conceived teacher knowledge as being constructed and developed through sustained negotiation between knowledge of distinct perspectives, which aligns with a construction perspective.

Mixed Perspectives

I should note that making the distinction between an application and a construction perspective does not imply that they contradict each other. A teacher's possessed knowledge can be a result of her construction during *prior* experiences and can inform the construction of knowledge in *future* interactions. I consider it possible for a researcher to take both perspectives simultaneously in one study since the researcher can conceive teachers' observable actions as results of their application of possessed knowledge and is also aware that they can construct new knowledge or modify existing knowledge through interaction with students. I identified three studies that used a combination of the two perspectives (underlined in Table 1). For example, Park and Oliver (2008) characterized that,

> PCK as knowledge-in-action became salient in situations where a teacher encountered an unexpectedly challenging moment…In order to transform the challenging moment into a teachable moment, the teacher had to integrate all components of PCK accessible at that moment and apply them to students through an appropriate instructional response. In this respect, the development and enactment of PCK is an active and dynamic process. (p. 268)

The authors illustrated with an example that a teacher learned about a student's misconception of a concept in her teaching and integrated this knowledge with her knowledge of subject matter and curriculum to confront student misconceptions through instructional strategies in-the-moment. Characterizing what knowledge the teacher constructed in terms of the student's misconceptions implied that the authors took a construction perspective. Meanwhile, they considered the teacher's instructional decisions as results of her integrating and applying multiple sources of knowledge, which implied an application perspective.

Action/Skill Perspective and Lack of Perspective

In Lee's (2017) conceptualization of teacher knowledge, she conflated teacher knowledge (i.e., what teachers know) and teacher actions or skills (i.e., what teachers do), stating that "PCK for preschool mathematics can be conceptualized as a set of three interrelated skills (p. 233)" that included noticing, interpreting, and enhancing student thinking. I also identified a literature in which the authors did not discuss any theoretical orientations of teacher knowledge and assumed teachers who participated in additional professional development program had more knowledge than those who did not (Jacobson & Lehrer, 2000).

Discussions

My intention of this review is to identify researchers' different approaches to teacher knowledge relevant to student-teacher interaction, which included the different teacher knowledge frameworks they used and their ways of operationalizing these frameworks to make claims about teacher knowledge when explaining teachers' observable actions. First, my analysis unsurprisingly suggests that researchers hold different perspectives on teacher knowledge when conducting research in relation to teacher knowledge and student-teacher interaction. This phenomenon is not necessarily due to the diversity of frameworks prevalent in mathematics education. Researchers who used the same framework might interpret and use the framework differently. For example, researchers of 9 (out of 18) studies used Ball et al.'s (2008) MKT taxonomies but with some of them using it as a construction perspective, some using it as an application perspective, and one using it as an action/skill perspective (see a summary in Table 2). Because Ball et al. (2008) developed the MKT framework with an intention of identifying teacher knowledge demanded by the work teachers do, it made sense that this framework widely applied to the literature on teacher knowledge relevant to student-teacher interaction. However, this conceptualization of teacher knowledge emphasizes specific types of teaching practices that signal teacher knowledge rather than the cognitive content and nature of teacher knowledge itself. As a result, it is not surprising that many researchers used this framework either from an action/skill perspective to conflate knowledge and actions or an application perspective to focus on the function of knowledge in the form of teacher actions. A limitation of these views of teacher knowledge is that they do not allow researchers to gain insights into the cognitive content and structure of the teachers' knowledge. We are left wondering: what mathematical meanings the teacher construct from their students and how are those meanings organized in the teachers' minds so that they can enact those meanings when interacting with students?

Table 2. Summary of literature using Ball et al.'s (2008) MKT framework.

Theme	Literature	# of literature/Total # of literature in each theme
Application Perspective	Bray (2011); Charalambous et al. (2012); Hill et al. (2008); Johnson & Larsen (2012); Kersting (2008); Speer & Wagner (2009); Wilkie (2016)	7/8
Construction Perspective	Johnson & Larsen (2012); Wilkie (2016); Wilson, Mojica, & Confrey (2013)	3/11
Action/Skill	Lee (2017)	1/1

Second, researchers' views of teacher knowledge reflect what source of knowledge they consider as critical in explaining teachers' observable actions. Researchers who hold a construction perspective attribute teacher actions to what knowledge teachers construct in the moment of interacting with students and make claims about what constitutes the teachers' constructed knowledge. Researchers who hold an application perspective explain teacher actions in terms of what knowledge teachers possess or lack. At times, the researchers make claims about what teachers *cannot* do due to the absence of certain types of knowledge, emphasizing the deficit of teacher knowledge instead of the affordances. While this perspective allows researchers to understand what knowledge enable teachers to notice, listen, and act on student thinking, it may constrain researchers from understanding how noticing, listening, and teaching provides teachers sustained opportunities to *develop* their knowledge of their students.

I am not arguing that a construction perspective is always preferable over an application perspective. I understand that researchers may have different research goals and thus focus on different aspects of teacher knowledge. However, I do believe that researchers who are oriented to radical constructivism should consider taking *both* perspectives to explain teacher actions. A radical constructivist view of social interaction as ongoing assimilation and accommodation of meanings requires researchers to simultaneously take into account how teachers *apply* and *modify* knowledge in the moment of interacting with students. Applying existing knowledge to interpret a current situation implies a mental process of assimilation; namely, the teachers are treating what they perceive about the students' activity as fitting into their existing conceptual structure. In comparison, modifying knowledge from teaching implies that the teachers are experiencing accommodation—the teachers are modifying their existing conceptual structures to account for what they perceive in-the-moment. This echoes some scholars' call that researchers should focus on the knowledge construction process to capture the dynamic and constructive nature of teacher knowledge, as opposed to identifying particular knowledge needed for effective teaching (e.g., Bauersfeld, 1980; Mason & Spence, 1999; Silverman & Thompson, 2008).

A final observation is that some researchers did not provide sufficient descriptions of their views of teacher knowledge in the theoretical framework section (e.g., Bray, 2011; Jacobson & Lehrer, 2000; Lee, 2014)—some of them summarized the existing frameworks without elaborating on how they interpreted and used the framework in their studies. It is important for researchers to carefully consider their uses of teacher knowledge frameworks and to be aware of how they conceive the role of teacher knowledge in student-teacher interaction. As suggested in my analysis, such consideration may not only help situate the research in the literature in a more rigorous way but also impact researchers' ways of interpreting and explaining their data.

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This is Us: An Analysis of the Social Groups Within a Mathematics Learning Center

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The mathematics learning center (MLC) of a university may influence more aspects of a student's life than the targeted mathematics learning. In this study we examined an MLC from the perspective of the undergraduate peer tutors employed there seeking to understand the space as a figured world. Differential use of pronouns emerged during analysis of collected stimulated recall data from the participating undergraduate mathematics peer tutors. My examination of which individuals, groups, or subgroups were included in "we" and "us" statements by the participants revealed social patterns within the MLC where both academic and non-academic behaviors indicated belonging or the potential to belong. The personal narratives of the participant tutors expanded on these ideas of coming to belong within the MLC and the implications of that belonging for their developing mathematics and STEM identities.

Keywords: peer tutoring, figured worlds, discourse analysis

Increasing STEM retention is an important contemporary topic in mathematics education and beyond (Bressoud, Mesa, & Rasmussen, 2015; Daempfle, 2003; Ellis, Fosdick, & Rasmussen, 2016; Shin, Levy, & London, 2016). One proposed mechanism for students' decisions to persist or leave a STEM major is a sense of belonging in mathematics or more broadly in STEM (Ellis et al., 2016; Estrada-Hollenbeck, Woodcock, Hernandez, & Schultz, 2011; Wilson et al., 2015). A mathematics learning center (MLC) is a possible space for students to gain a sense of belonging in mathematics.

The MLC and mathematics tutoring have become almost ubiquitous on university campuses with a recent national survey suggesting that virtually all calculus-offering institutions offer mathematics tutoring in some form (Bressoud et al., 2015). A study in the UK concluded that an MLC may be seen by mathematics students as qualitatively different than other physical spaces in which doing mathematics and social interactions can provide a space where students feel both ownership and belonging in a way that they do not in the classroom or traditional office hours (Solomon, Croft, & Lawson, 2010). However, there is very little research on tutoring compared to other subfields within mathematics education, particularly when considering tutoring at the undergraduate level within the US, even though approximately 40% of calculus students in the US report utilizing on-campus mathematics tutoring at least once (Bressoud et al., 2015; Mills, Tallman, & Rickard, 2017).

This research report is part of a broader study in which undergraduate mathematics peer tutors (UMPTs) were recruited as case study participants. We explored their views of their tutoring and of the MLC where they were employed. The results presented here focus on the community building affordances of the MLC and what it provided for tutors and STEM majors to gain a sense of belonging in mathematics. We take up the framework *figured worlds* (Holland, Lachicotte, Skinner, & Cain, 1998) to explore the complex, reflexive, and fluid nature of tutoring and extra-tutoring interactions within the MLC. Our primary data source was a series of video-recorded stimulated recall interviews with four participants around their tutoring enactments and a final semi-structured interview where the participants provided information on their background and experiences (Dempsey, 2010; Ginsburg, 1997). In our results presented here we focus on the emerging patterns observed in pronoun usage within the stimulated recall transcripts along with participants' explanations of their perceptions of the MLC and their place within it.

Background and Setting

The setting for this study was a large, public university in the southwestern US. The university is a Hispanic-serving institution and has a diverse student population. A recent increased interest in improving mathematical outcomes for students in mathematics courses lead to the development of the MLC in its current form and location. The MLC has the stated goal of improve the DWF rate in Precalculus, Calculus I, and Calculus II, while also offering tutoring in other undergraduate mathematics and statistics courses. The MLC is well-utilized with over 5000 student-tutor contacts recorded during the past year. There were 28 UMPTs employed at the MLC in the semester when this data was collected. The mathematics department has an expectation that graduate teaching assistants (TAs) will hold some or all of their office hours in the MLC as well, creating an additional pool of tutors.

Theoretical Framework

Figured worlds as a theoretical framework couches identity – being a certain "kind of person" – into a sociocultural space where some kinds of person are allowed and others disallowed, where some roles are well-defined and others ill-defined, and some are more valued than others (Holland et al., 1998). Figured worlds are spaces where identities are produced and individuals can develop new identities through interactions within these spaces. The culmination of experiences and interactions influence who a person believes that they are and can be in the future and thus how they can and should act in a situation. The tutoring interaction creates at least one such figured world, but as Colvin (2007) found, it is not a world in which every tutor or student knows who they are expected to be and what enactments to engage in to reach their goal in that space. Identities must be negotiated and who you are and can come to be within the figured worlds is largely dependent on the willingness of others to view you as having that identity. An individual's perception that certain identities are granted greater power in certain spaces can alter their sense of mathematical belonging (Solomon et al., 2010). The explicit focus of this report is on the identity work that the tutors engaged in as revealed by their use of pronouns. We sought to answer the research question: Who is "us" and what does it mean to be "us" within the figured world of the MLC?

Data Collection

As part of the broader study, a web-based survey was distributed by the director of the MLC to all tutors in the MLC. A subset of seven survey respondents indicated their willingness to talk further with a researcher. Of these, two declined further participation when contacted. Four participants of the remaining five were selected with the goal of a gender-balanced sample and a representation of multiple undergraduate majors. No Hispanic/Latin@s volunteered and thus were not represented in this sample but they do represent a significant minority of the UMPTs employed in the MLC. The characteristics of the participants are summarized in Table 1.

The case studies consisted of repeated observations and video recording of tutoring interactions in the MLC. Each observation coincided with a full tutoring work shift for each participant and lasted 2 or 3 hours. Participants were observed four times and each observation was followed with a stimulated recall interview (Dempsey, 2010; El Chidiac, 2017; Lyle, 2010), with two exceptions. Lily was observed five times but only completed four stimulated recalls, and Eric had an observed tutoring shift where only one very short (<10 minute) tutoring interaction occurred and so only participated in three stimulated recalls. After the final stimulated recall each participant also participated in a semi-structured interview of approximately an hour with the same initial guiding questions (Ginsburg, 1997).

Table 1. Case study participants key characteristics

Case Study Participants*	Year	Major	Ethnicity	Time Tutoring in the MLC
Danielle	Junior	Mathematics	White/Caucasian	1-2 years
Eric	Senior	Mathematics	Chinese-Filipino	1-2 years
Jake	Sophomore	Engineering	White/Caucasian	<1 year
Lily	Junior	Mathematics & Physics	Cambodian	1-2 years

**Gender-preserving pseudonyms chosen by the participants*

The observing researcher created a log of key features or incidents in real time field notes while video-recording tutoring interactions. These field notes were used to direct the selection of interactions during the stimulated recall. A second set of notes were made by rewatching the recorded tutoring interactions at a later time and both sets of notes were utilized to help interpret the stimulated recall data (Emerson, Fretz, & Shaw, 1995). The stimulated recall and final interview were video recorded and transcribed. In their interpretations of their interactions, we can learn about UMPT's view of their own identity and that of others within the figured world.

Analysis

For the purpose of analysis the two sets of notes were combined with the stimulated recall transcripts and the primary unit of analysis were the 71 tutoring interactions discussed in stimulated recall (Dempsey, 2010; Emerson et al., 1995). Data was analyzed using grounded theory to code the resulting transcripts (Strauss & Corbin, 1994). While coding this data set we noted pronoun useage as a meaningful emergent category. Participants differed between the pronouns they would use to reference students they had worked with, for example one participant had a pattern of "then I told *them*" pronoun usage when talking about tutoring Precalculus but "then *we* worked on" when talking about tutoring upper division courses. A more robust analysis was produced classifying every use of first-person plural pronouns based on whom the pronoun was referencing. This allowed us to explore who was afforded similar social standing to an UMPT within the figured world. For example, when Lily was asked " 'we' meaning…?" and responded with "The MLC." during a stimulated recall interview it was taken as evidence that she felt a sense of belonging within the group of UMPTs at the MLC.

Final semi-structured interviews were also transcribed and coded separately. Pronoun usage was not directly analyzed in the same fashion for this data as it was a qualitatively different dialog than the stimulated recalls. Rather, the final interview data served to answer broader questions about the tutors' views of the figured world of MLC and their reflexive identity work within it. Grounded theory was again utilized (Strauss & Corbin, 1994). Of most interest for the present report were the answers regarding the social aspect of the MLC and the tutors' personal narratives regarding how they came to be a tutor there.

Results

The three most prominent groups identified by UMPTs in the MLC were students seeking tutoring (SSTs), upper-division STEM majors (SMs), which here we will use to refer only to non-tutors who are not seeking tutoring, and the undergraduate tutors (UMPTs). These groups blended into one another with the UMPTs talking about socializing with the SMs, particularly after hours when they would retain access to the MLC as a study space. The participants introduced the language about this group of people as "the club" who "practically live here." SSTs were seen by the UMPTs to varying degrees as potential UMPTs and potential members of the club – Danielle's story of becoming a tutor is one of beginning as an SST, and Jake talked about perceiving one of his SSTs to be someone who would make a good tutor and encouraging them to apply. Interestingly, the graduate teaching assistants (TAs) were not seen by the UMPTs as belonging to any of these groups. First person plural pronouns never referred to TAs. The first-person plural pronoun usages referring to each group are summarized in table 2.

Table 2. First-person plural pronoun use in stimulated recall interviews

Group Referenced	Number of First Person Plural Pronouns	Percentage of First Person Plural Pronouns
UMPTs	218	54%
STTs	104	26%
"The MLC"	40	10%
SMs	10	2%
TAs	0	0%
Other*	31	8%
Total:	403	

**this category included tutors using pronouns as-if quoting what someone else said – i.e. "the student was like 'we don't learn anything by going to class.'" These cases were excluded as the use of the pronoun was not inclusive of the tutor.*

The Club

The MLC has become an important social space for UMPTs beyond their tutoring duties and hours. Jake described to us how the MLC was the place where he first met other SMs and found peer-mentors and friendships in his major. The other participants also talked about the social aspect both as part of what makes it work as an educational space, as Lily put it:

> "You get close to the tutors, at least the ones who stay here [when off-duty or after hours], that's most of us. But usually the people that we hang out with. And we become best friends which gives us a friendly environment. Which we think is a good thing." *(from a stimulated recall)*

The participants spoke of the MLC being their primary social as well as professional outlet.

> *Researcher:* So this place has become more than, more than a job to most tutors.
> *Eric:* We usually also live here, also, yeah…yup it is the place to be not just for work but for work not like work but homework and things to do that are mathish.
> *R:* It's the nerd clubhouse.
> *E:* Pretty much, yeah. *(from final interview)*

Beyond the tutors, "the club" as a social unit also extended to some upper division STEM students, specifically but not exclusively mathematics majors, and a group of them could always be found in one corner of the MLC.

> "It's [the MLC] a nice place to be for math students… we're here all the time, and there are whiteboards and things like that…It's kind of fun when it closes at the end of the day. We close down and then people will stay and study and stuff [Tutor] and I have done that a few times. Then it's really cool. We feel really included, like it's this little club. Because we're like "we get to stay after hours."" – *(Danielle, stimulated recall)*

First-person plural pronouns explicitly included SMs in ten cases our of 403 analyzed in the stimulated recalls, such as the first usage in Danielle's quote "math students… we're here all the time," this a small percentage but any mention is a bit surprising as the SMs were not part of the interactions being discussed. There is high degree of overlap between upper division STEM majors and the UMPTs, not only because the UMPTs are recruited and hired primarily from the SMs but also, interestingly, the SMs are sometimes called upon to act as if they were tutors employed in the MLC when it gets busy.

> "So, you have, one group that's your regular, upper division math students who all are working together on their classes and they're always just hanging out in here. And they'll help people, too, if people have questions they will intervene even if they're not on." – (*Jake, final interview)*

Jake is reporting that it is normalized within the MLC to ask an SM for help if you are stuck on a student's question. In my observations asking for assistance was common between UMPTs, even when they were off duty, but we observed SMs being asked as well. The club seemed to function as the social core of the MLC and in pronoun usage while discussing the social aspects of the MLC it was common for "us" to perhaps implicitly include its non-employee as well as UMPT members, as when Danielle said "we get to stay after hours" a privilege extended explicitly to employees of the MLC but practically also to SMs around at closing time so long as at least one employee was there.

The UMPTs included non-tutors in their core social unit but did not seem to consider all students as members of the club. A pattern was noted in Jake's responses while analyzing the pronouns he used for students in the stimulated recall interviews. Jake did not use "us/we" pronouns for a student a single time while tutoring Precalculus or Business Calculus (nine interactions), but did use it when tutoring upper division mathematics courses (above Calculus II) in five out of ten interactions, and once when tutoring Calculus we (out of four interactions). A similar pattern could not be traced in other participants' due to a lack of data – only three total other upper division mathematics tutoring interactions were recorded across the other participants. However, Jake's pronoun selections of what students are part of "we/us" may reveal how more advanced students are closer to being considered peers. In the finer-grained analysis the first-person plural pronoun analysis also reflected that UMPTs included students in 104 of 403 cases – half the number including other UMPTs but a non-trivial percentage of the time.

Becoming a Tutor

Becoming a member of the club seemed to refer to individuals who repeatedly showed up in the MLC and were in upper division courses with other SMs. The process of becoming an UMPT employed at the MLC is naturally more formal, but may begin with the process of joining the club. One of my participants, Danielle, described how she became an UMPT through first being a regular student in the MLC:

> "I was coming in here [to the MLC] because we needed help with Calc II… we started seeing kids that were around my age like being tutors and so we would ask friends, or like kids we knew, how did you become a tutor? Like I'd ask [tutor] how did you become a tutor here? And he'd say like, "oh, we don't know we just volunteered to tutor here and we was here all the time and we wanted the job and we just asked [MLC director]." and we was like "OK." So then we think we emailed her over the summer asking if we could work here and she sent out an application…" *(final interview)*

Danielle's story of becoming a tutor was unique among my four case study participants. We found evidence beyond her story that her trajectory was a normalized way to become an UMPT. During one of our stimulated recall sessions, Jake commented about a particular student that he was trying to recruit the student to become a tutor,

Researcher: So this becomes almost like, *Jake:* Advice, mentoring – it also can be a recruiting process for me, too… For our 'club', [Student], he seemed like a very good candidate… I could even see him as a tutor. Just kind of giving him advice like, if you want to tutor, here's what you do. I can see that you would be good at this subject, which we lack. And then so like, I'm going to stay in contact with him now because he's a very good potential candidate. He actually cares which is sometimes hard to find in students. So that was really just cool.

R: But this is the kind person that you want to be like, [MLC director] hire this person.

J: Yeah, exactly, that was exactly it. *(stimulated recall)*

The use of " 'our' club" is an example of a potentially ambiguous case which we counted as only explicitly referencing the UMPTs due to the context speaking directly about tutors, though 'the club' to Jake in other contexts appears to include SMs and he clarifies in his next sentence that beyond the club this student could 'even' be a tutor perhaps indicating that the earlier 'our' could be taken to include SMs. His discussion of recruiting of a student gives evidence that Danielle's experience was not unique, but rather that in the figured world of the MLC, it is both possible and desirable for a student seeking tutoring to become part of the social life of the MLC and to be hired as an UMPTs, thus authoring an identity of being a person who belongs in mathematics.

TA vs. Tutor

Another key group in the MLC are TAs who are graduate degree seeking students in the mathematics department who lead break-out sections for students in large lecture courses of Calculus we and II. In addition, they are expected to hold office hours by tutoring in the MLC. According to the UMPTs the TAs form a separate social unit and are perceived to fulfill a different role despite also having a tutoring function. Of the 403 instances of plural first-person pronoun usage in the stimulated recall sessions not one could be determined to unambiguously include the TAs. In fact, when asked if a "we" statement included the TAs, participants would indicate that it did not. While no direct conflicts were observed, UMPTs did complain that the TAs did not follow the same rules (such as always wearing the required vests and nametags). According to Lily, "the TAs almost view us as some kind of undergrad nuisance" and Jake stated that there was little to no social interaction between UMPTs and TAs:

> "With a lot of them we don't socially as tutors talk to them as much because they're always just at their table with their group of students not really roaming around like "hey, can you help them." The tutors kind of fill in for everybody when there isn't a TA for a specific class and the TAs just come here work with the students and then go. That's how we usually see it." *(final interview)*

This dynamic was interesting to observe as the other major groups in the MLC – the UMPTs, the SMs, the SSTs – seemed to form almost a single, fluid social group where roles shifted based on taking more advanced courses or an individual's prowess in a subject area, and where students were actively recruited to join the club. TAs were seen as something apart from this social order and participants never expressed to us the possibility of becoming a TA even when several of them talked about going to graduate school.

Conclusion

The MLC is a figured world with several perceived possible roles for undergraduate STEM majors. These roles seem to fall along a continuum from a lower-division SST to an UMPT member of the club. This study has sought to examine the nature and boundaries of the perceived social groups within the MLC in order to better understand how those dynamics may play into an individuals' development of a mathematical identity, and thus, positive student outcomes. The UMPTs who participated in this study spoke of social groups including ‘the club’ and their students some of who may become tutors and members of the club at a future time. This reveals a figured world where SSTs may author roles as club members and UMPTs. Meanwhile, the UMPTs consistently positioned themselves as a role and social group ‘other’ than the TAs. We found no evidence of an imagined path where a current UMPT would become a TA in the future.

The ability of any student, even formerly struggling students like Danielle, to become tutors and part of the club speaks to the MLC as a possible source of a sense of belonging for mathematics and other STEM majors – a figured world where identity work affirming of mathematical struggle and peer collaboration regularly takes place. This is a significant finding as the majority of research to date on undergraduate mathematics tutoring has emphasized the academic gains of students who are tutored and not attended as strongly to affective factors like belonging or the sociocultural factors that would support it. Researchers like Solomon et al. (2010) in the UK have explored how having a neutral space for peer interactions around mathematics can be beneficial to feelings of community and belonging in a mathematics department. In the US, it is of note that students cited tutoring centers and other opportunities for social interactions with peers outside of class as being important to their mathematical development and experiences (Bressoud et al., 2015). The UMPTs in this study emphasized their role as a *peer* in the space as being particularly valuable for students. As Danielle put it:

> “Oh, it's so helpful to students that have questions in math or are struggling in math. It's really hard, it's hard to find someone to go to who isn't your professor or your TA… They may not be able to help you even though it's the same class. So it's a lot easier to come here to the MLC because there are a lot of other people who are here who have taken that class with that professor and they know what you should be working on and they know what you should be helping with… I feel like this is the easiest – it is the easiest and most relaxed way to get help without feeling the pressure of going to a TA or a professor because that can be really nerve-wracking.” *(final interview)*

As researchers, we were surprised to see the social distance between two groups (TAs and UMPTs) that we initially perceived as fulfilling a similar role in the space. Our initial response to this social gap was one of concern – surely it would be beneficial for students if all the educators in the space collaborated as peers? We wonder if the distance might serve a social function after hearing the UMPTs talk about social mobility and the narratives of students coming into the club and being recognized as mathematical peers. Social distance between the TAs with institutional authority and the near-peer UMPTs may in fact be positive in some cases in allowing students currently enrolled in TA-utilizing courses access to a mathematical community that does not include individuals with institutional authority over their coursework and grades.

Other MLCs at other institutions will have different social groups and roles emerge as their figured world develops in interactions between students, tutors, and other key players. This study is an example of some of the types of identity work that can arise within a university’s MLC and highlights the ways that social as well as mathematical factors play a role in identity work such as coming to belong. This direction of research extending beyond an increase in academic gains should not be neglected in future studies and should perhaps garner greater attention during formal and informal assessments of MLCs.

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Empowering Students in Learning Proof: Leveraging the Instructor's Authority

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When students are coming to understand how to construct proofs, as well as how mathematicians use proofs in their work, the role of the instructor cannot be overstated. In this paper, we present an investigation into how an instructor uses her authority to empower students as legitimate proof producers and learners of mathematics. We view this empowerment and student learning through a situated lens, accounting for relationships of disciplinary authority and student agency. In our investigation, we analyzed the transcripts from three classroom episodes in an inquiry-based transition-to-proofs course. We identified instances when the instructor leveraged her institutional authority as well as her mathematics expertise authority to support students' engagement in the dance of agency, asserting their own creative ideas as learners of mathematics while still adhering to the norms and standards of the discipline.

Keywords: Authority, Agency, Proving, Proof, Empowerment

Mathematical proof has a special place in the discipline of mathematics, and it plays an important role in undergraduate mathematics education (Stylianides, Stylianides, and Weber, 2017). When students learn to write proofs, they must find a balance between the norms of proof writing established by the discipline of mathematics and their own creativity. Such a balance may be considered a dance of agency between the authority of the discipline and the agency of the student (Boaler, 2002; Pickering, 1995). The instructor plays an important role in setting the stage for this dance, with his or her position to both represent the discipline and support students' agency. In mathematics education literature, the problem of striking a balance between the norms of the discipline and the ideas of the students is often framed as a problem of establishing shared authority between student and teacher in the classroom (e.g. Webel, 2010). So how can shared authority be achieved? Gerson and Bateman (2010) suggested one potential answer--that undergraduate instructors should limit both their *institutional authority* (based on their position as instructor of the course) and their *mathematical expertise authority* (based on proven mathematical expertise) in order to promote shared authority within a classroom community. But, just as students must find a balance between the authority of the discipline and their own agency, we contend that instructors must find a balance between exerting and limiting their own authority in order to facilitate productive learning. In fact, we believe the field could benefit from questioning the often-cited claim that teachers should *limit* their authority, and instead explore how instructor authority can usefully be employed to support students as mathematics learners. In this study, we use data from an inquiry-based undergraduate transition-to-proof course in order to understand how an instructor may use her institutional and mathematical expertise authority to empower students as legitimate practitioners of mathematics.

Theoretical Framework

Situated Learning

We adopt a situated view of learning and consider learning to occur when students engage as legitimate peripheral participants in the mathematics community of practice (Lave & Wenger, 1991; Wenger, 1998). "Viewing learning as legitimate peripheral participation means that learning is not merely a condition for membership, but is itself an evolving form of membership" (Lave & Wenger, 1991, p. 53). So in our case, to learn mathematics is to be in an evolving state of membership within the mathematics community. Students, as "newcomers" to the mathematics community, gradually develop their agency and shape the discipline itself as they become more accustomed to the norms practiced by the "oldtimers" of the discipline (Lave & Wenger, 1991). Through this process, a student gains fuller membership into the community of mathematicians. Thus, understanding how to facilitate the balance between the authority of the discipline and the agency of the learner is critical to supporting students as learners of proof, and hence legitimate practitioners of mathematics.

Agency and Authority

Recall that in this study, we seek to understand how instructors may use institutional and mathematical expertise authority to empower students as legitimate practitioners of mathematics. Empowerment, as we conceive it, is linked to a student's ability to engage in the *dance of agency* (Boaler, 2002; Pickering, 1995), balancing his or her own creative ideas with the ideas and norms of the discipline. Although Boaler (2002) and Pickering (1995) use the term *agency* to describe both the individual and the discipline, we find it more useful to follow Amit and Fried (2005) and speak of the *agency* of the student and the *authority* of the discipline. A few definitions will be helpful in this regard. By *agency*, we mean "a dynamic competence of human beings to act independently and to make choices" (Andersson and Norén, 2011, p. 1). Given this definition, agency is most appropriate when describing a human being in action. We identify instances of student agency when students debate, ask questions, and propose their own ideas or proving strategies.

Alternatively, by *authority*, we refer to "situations in which a person or group, fulfilling some purpose, project or need, requires guidance or direction from a source outside himself or itself…. The individual or group grants obedience to another person or group (or to a rule, a set of rules, a way of coping, or a method) which claims effectiveness in mediating the field of conduct or belief as a condition of receiving assistance" (Benne, 1970, pp. 392-393). In the undergraduate setting, students experience the authority of the discipline in textbooks and communications from their instructor. Established theorems, standards for rigor, symbolic conventions, and accepted proof formats are instantiations of the authority of the discipline of mathematics. We contend that the discipline possesses authority in terms of its norms and standards for rigor and proving.

The instructor plays an important role in not only promoting student agency (e.g., by asking probing questions, providing opportunities for discussion) but also in helping students learn about what is established in the discipline of mathematics. In this sense, the instructor serves as a broker between the community of students in the classroom and the community of mathematicians in the discipline (Wenger, 1998). If we consider a continuum to represent students' learning of proof, as in Figure 1, in which the right side of the continuum represents students exerting their agency in deciding what should count as mathematical proof, and the left side represents obedience to disciplinary authority, we recognize the limitations of each extreme. If students are strictly taught to obey the authority of the discipline in regards to learning mathematical proof, then they may focus on form rather than reasoning. They may fail to see proof as a sense-making endeavor and feel isolated from the process. On the other hand, if students are given complete agency to decide what counts as proof, their proofs may not meet the standards of the discipline. This makes the instructor's role as broker between the classroom community and mathematics community central to empowering students in the dance of agency.

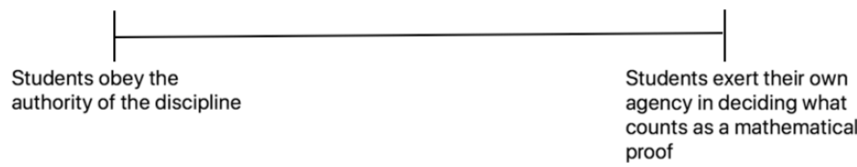

Figure 1. Continuum of Agency and Authority when Learning Proof

Methodology

We adopted Gerson and Batemen's (2010) classification of authority, because to date they have provided the most extensive framework for identifying authority roles within mathematics classrooms that "give way to the realization of autonomy with interdependence" (p. 195), a goal we see as aligned to our theoretical perspective, in which we view student empowerment as a successful balance of student agency and the authority of the discipline of mathematics. As a way to describe the interactions amongst students and the instructor in an inquiry-based calculus class, Gerson and Bateman distinguished between four authority types (hierarchal, mathematical, expertise, and performative) encompassing seven sub-types. Two of these subtypes are particularly important to our study: *institutional authority* which is "held by instructors based on the position as instructor of the course" (p. 201), and mathematics expertise authority, "based on the proven mathematical expertise of the bearer" (p. 201). Recall, Gerson and Bateman suggested that instructors should limit these two sub-types of authority in order to promote shared authority in the classroom. We, however, sought to explore if and how an instructor's use of these two authority sub-types may support student learning, highlighting the teacher's role as a more nuanced set of actions and decisions. Hence, we pose the following research questions: Can an instructor's *institutional authority* be used to empower students as legitimate practitioners of mathematics? If so, how? Can an instructor's *mathematics expertise authority* be used to empower students as legitimate practitioners of mathematics? If so, how?

The data for this study were collected as part of a larger study on the nature of mathematics (Pair, 2017). For the purpose of this study, we transcribed audio recordings of three classroom episodes depicting whole-class discussions that took place within an undergraduate transition-to-proof course at a large Southeastern University. Two mathematics education scholars, also authors of this manuscript, co-taught the course utilizing collaborative, inquiry-based pedagogy. Twenty-three students, including mathematics majors and minors, agreed to participate.

Three researchers independently read and coded the selected transcripts, taking one speaker's (student or instructor) turn as a unit of analysis. Each time a new speaker contributed to the whole-class discussion, the researchers assigned as a code any of the seven authority sub-types from Gerson and Bateman's framework that seemed to represent the authority relationship demonstrated within that speaker's contribution. Sometimes a researcher assigned more than one authority code to a turn, and sometimes a researcher assigned no codes to a turn. Also, researchers wrote memos and questions related to their coding in the margins of the transcript. After independent coding of each transcript, the three researchers came together in group meetings to (a) discuss and negotiate the types of authority relationships evident in the transcript, and (b) identify key instances within the transcript where *institutional authority* and/or *mathematics expertise authority* were used by the instructor in a way that empowered students as legitimate practitioners of mathematics.

Results

To begin our presentation of results, we first reiterate what we mean by *empowered learning* with respect to students as learners of proof. Recall, we view empowerment as related to a student's ability to exert his or her own agency/ideas while also considering and respecting the norms and understandings of the discipline of mathematics. To clarify this notion, we present two student quotations from an early class session in the transition-to-proofs course. Students were asked to respond to the following question, "Based on your past learning experience with mathematical proof (either high school or college), *how did you learn* about what makes a good mathematical proof?" We highlight two responses below, one from Eddie and one from Josiah.

> And as far as my personal struggles with them [proofs]... what helped me the most honestly was continually being wrong. I would be wrong and the teacher would be like this doesn't work. Logically this doesn't work, that math is wrong there. And so the more I was wrong and the more I thought about why I was wrong and how to fix it, I got better at it. (Eddie)

> I feel like I haven't really formally learned about how to do proofs. In the lower classes it was something that was just sort of tacked in like in the textbook, as like oh here is a thing, you understand this right? Or let's look at how to do integration with this infinite series, and everyone is like WHAOOO??? And by the time you get to the higher classes they just assume you know how do to it. (Josiah)

We argue that the first student, Eddie, describes an empowering learning experience because he is actively engaged in the dance of agency. Eddie exerted his ideas (agency), and was often incorrect. He would receive feedback from his teacher (representing disciplinary authority) and then take the feedback to revise and reflect on how to modify his proofs, in an empowering cycle. Alternatively, Josiah describes a disempowering learning experience as he did not have the opportunity to engage in the dance of agency. He was instead simply given the instantiation of authority from the discipline (in the textbook), or assumed to understand the knowledge and norms of the discipline, with little opportunity for inserting his own agency.

With this conceptualization of empowerment in mind, we turn to our research questions: (1) Can an instructor's *institutional authority* be used to empower students as legitimate practitioners of mathematics? If so, how?, and (2) Can an instructor's *mathematics expertise authority* be used to empower students as legitimate practitioners of mathematics? If so, how?

Leveraging Institutional Authority

Early on in the semester, students sought direct guidance from the instructors regarding what constitutes mathematical proof, asking questions including, "Is there a best way of doing proofs? Something that works stronger than all other ways?", "What is official/professional proof supposed to look like? What are the requirements?," and "When will we know for sure we are writing proofs correctly?" Instead of providing a direct answer about what makes a proof, the instructors engaged the students in a two-day activity in which students debated the criteria for a valid argument by analyzing and critiquing one another's written arguments for a given problem.

The quotation below comes from the lead instructor, Dr. BB, as she responded to students' questions regarding the best way of writing proofs:

> We are looking for some more guidance right? About what is proof. You all were working last week, working on a few problems and thinking about how to prove things. Today what we are going to do, is try to as a class, as a community, come up with some criteria that would help us describe what a proof should be. Okay? And we [the instructors] think that can come from you all, that it doesn't necessarily need to come from us. That based on logic and based on the understanding of mathematics that you have so far, that you all are very capable of creating some criteria that would help you think about what should count as proof and what maybe shouldn't count as proof. Okay? So that is what we are going to do today.

Creating a class criterion for proof writing represents students interacting at the rightmost end of our continuum (Figure 1); students exerted their own agency on what counts as correctness in proof. This exercise was a novel experience for most students, as the majority of students described learning mathematical proof in ways similar to Josiah, where proofs were given as examples to be mimicked or memorized. We noticed that in these early-semester encounters, the instructors often leveraged their *institutional authority* in order to drive the norms of the classroom, asserting that students could and should insert their voice into the classroom conversation, and implementing a classroom activity that allowed for active student contributions. Given students' initial inclination to live on the left side of the continuum (obeying disciplinary authority), we argue that the instructors' use of institutional authority here was empowering as it allowed students the opportunity to offer their own thoughts and ideas to the classroom discourse. This is highlighted again in the following exchange, as Jackson contrasted his classmates' ideas and his own thoughts on proof:

> *Jackson:* I'm having a debate about ... examples about where they belong in proofs. I guess I may not agree with the [class] consensus that examples belong in proofs. I certainly do examples for myself to support the veracity of what I am working on. But I don't think a thousand examples prove anything. So I don't know if they belong in there or not. I think one example that disproves has a lot of value. But I don't know if putting half a dozen examples in a proof really supports the proof or not.
>
> *Dr. BB:* Mhhm. Great. And [that's] another thing that I would like all of us to be thinking about today, okay? So keep raising that question Jackson, when we get back to revising our criteria later maybe you could bring it up again based on what you discuss in your group today.

This instance can be interpreted both as a leveraging of institutional authority and as a limiting of mathematics expertise authority. Dr. BB leveraged her institutional authority (as the instructor of the course) by encouraging Jackson and the other students in the class to continue to think about the role of examples in proofs. She simultaneously limited her mathematics expertise authority, as Gerson and Bateman (2010) suggest, by not offering her own thoughts (as a disciplinary expert) on the role of examples in proofs. Dr. BB's actions continued to emphasize the norm that student contributions were valuable to this classroom community, and these actions provided opportunities for empowered learning by moving students toward the right side of the continuum. Note that even though students were exerting their agency, they were coming to conclusions that were aligned to the disciplinary norms of the mathematics community. While a mathematician may use examples to generate ideas for a proof (de Villiers, 2004) or check the claims of a proof (Weber, Inglis, & Mejia-Ramos, 2014), examples are not part of formal deductive argumentation. Eventually, the class was able to come to this conclusion on their own through whole-class negotiation. We observed several similar instances early in the semester when Dr. BB simultaneously exerted her institutional authority and limited her mathematics expertise authority in ways that prompted students to attribute value to their class contributions, while also considering legitimate practices of the mathematics discipline. It was not until later in the semester that we were able to identify instances where the instructor's *use* of mathematics expertise authority could be perceived as empowering (Research Question 2). We now highlight such a case.

Leveraging Mathematics Expertise Authority

It was about one month into the semester and the Blue Team had just finished their whole-class presentation of a direct proof for the claim, "If l and m are odd integers, then $l+m$ is even." After their presentation, Jayden (a member of the team) suggested, "Also, we could prove it by contrapositive: By showing if $l+m$ is odd, then l and m are even." We believe that Jayden demonstrated agency by offering an alternative proof method unprovoked. He also offered a specific disciplinary technique (contrapositive) that he saw as part of his growing knowledge base. Note however that while Jayden offered a viable alternative proof approach (i.e. contrapositive), his structuring of the contrapositive statement was incorrect. Dr. BB took this as an opportunity to highlight Jayden's suggestion for an alternative proof approach, but to also ensure that the students in the class had an understanding of the correct form for this particular contrapositive statement.

Dr. BB: Let's take one minute, if we were to prove this by contrapositive, what would we need to prove, what would that be?

Natalie: If $l+m$ is odd, then l and m are even integers.

Sofia: I think it is 'l *or* m is even,' because the negation of *and* is *or*.

Dr. BB: The negation of an *and* statement will end up being *or*. So essentially this is like 'l is odd and m is odd.' Then taking the negation of that, this is really important, we need 'l is even or m is even.' Which law is that?

Students: DeMorgan's law.

Dr. BB: Yes. So we can use contrapositive, but make sure we can negate this piece correctly. Okay take a minute to talk in your groups.

In this exchange we see Dr. BB exerting her mathematics expertise authority in two ways. First, Dr. BB recognized Jayden's mistake in the statement of the contrapositive and decided to focus the class's attention on the formation of that statement by pausing the group presentation and asking a probing question. Second, Dr. BB reiterated Sofia's point, summarizing the correct approach to forming the contrapositive by negating the component statements, alluding to DeMorgan's law. Student agency is valued, because of the student-centered nature of the ideas in the discussion and the ability of students to negotiate the correct format of the contrapositive. However, the instructor also honors the discipline by explicitly confirming the correct approach and then having students pause to reason about the proof by contrapositive format as applied to a conjunction statement. In this instance of Dr. BB's exertion of mathematics expertise authority, we see an empowering balance between student agency and students' growing understanding of the norms and truths of the discipline of mathematics.

Discussion and Conclusions

In the results above, we presented brief vignettes of instructor/student interactions that we claim to illustrate the following two situations: (1) An instructor's use of institutional authority that empowered students as legitimate practitioners of mathematics (i.e., Dr. BB's use of institutional authority to set norms within the classroom community that student contributions would be valued and to engage students in an activity that offered them opportunities to consider legitimate disciplinary practices regarding proof), and (2) An instructor's use of mathematics expertise authority that empowered students as legitimate practitioners of mathematics (i.e., Dr. BB's use of mathematics expertise authority to identify a student's mathematical error and to probe students in the class to explore and explain the mathematical error).

These vignettes highlight an instructor's use of authority that contradicts the suggestion by Gerson and Bateman (2010) that "an ideal instructional environment to promote shared authority would limit the instructor's institutional and mathematics expertise authorities" (p. 206). Instead, we offer a more nuanced view of an instructor's use of authority in the teaching and learning of proof, where institutional and mathematics expertise authorities may be used to empower students as legitimate practitioners of mathematics. Further exploration of how different authority types may be used in empowering or disempowering ways would benefit the field.

As we conducted this analysis of classroom transcripts, we noted some interesting connections between institutional authority and mathematics expertise authority. First, as discussed above, we noticed that when the instructor exerted her *institutional authority* in an attempt to set norms that valued student agency, she often simultaneously limited her *mathematics expertise authority.* This may be what Gerson and Bateman (2010) meant in their suggestion to limit mathematics expertise authority, as a way to provide space for students to assert their own mathematical ideas rather than adhere only to the mathematical ideas of the instructor. This led us to wonder, is it possible to exert mathematics expertise authority and simultaneously limit institutional authority? And if so, what would this look like? Institutional authority is an authority type that is at play in the classroom no matter how the instructor decides to act (Amit & Fried, 2005). In fact, when an instructor removes him or herself from contributing to a discussion, they are making use of their institutional authority. So, what would it look like to *limit* institutional authority in an empowering way, or is that possible at all?

We also noted that there were instances within the classroom transcripts where, as researchers, we could clearly distinguish between the instructor's exertion of institutional and mathematics expertise authorities. Our clarity in distinguishing these types of authority is likely due to our advanced knowledge of disciplinary norms in mathematics together with our professional knowledge as instructors. However, we hypothesize that it would not have been as straightforward for the students in the class to differentiate between these authority types. After an instructor makes an authoritative statement, the students may be left to wonder whether they should adhere to the instructor's statement because it would be beneficial for their participation in this classroom community (institutional authority) or whether the instructor is speaking on behalf of the broader mathematics community (mathematics expertise authority). We believe that future research could explore if and when it is important for students to discern between these authority types, and how such discernment aids in their empowerment as legitimate practitioners of mathematics.

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Student Reasoning about Span and Linear Independence:
A Comparative Analysis of Outcomes of Inquiry-Oriented Instruction

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In this report we examine the performance and reasoning of span and linear independence of 126 linear algebra students who learned through a particular inquiry-oriented (IO) instructional approach compared to 129 students who did not. Students who received IO instruction outperformed Non-IO students on questions focused on span, but not on questions focused on linear independence. Our open-ended coding additionally suggested that IO students' concept images of span and linear independence were more aligned with corresponding concept definitions than those of Non-IO students.

Keywords: inquiry-oriented instruction, student reasoning, span, linear independence, learning outcomes

A growing body of research documents improved student learning outcomes in undergraduate science, technology, engineering, and mathematics courses that use active approaches to learning (Freeman et al, 2014). However, there is limited work that documents differences in *how* students reason about particular disciplinary ideas under particular instructional approaches. In this paper, we analyze differences in performance and reasoning about span and linear independence of students whose instructors received instructional supports to teach linear algebra in an inquiry-oriented way from those who did not. In inquiry-oriented (IO) approaches to mathematics teaching, students inquire into mathematics by working on carefully designed sequences of open-ended problems, and instructors inquire into students' thinking and use their ideas to drive the development and formalization of mathematical ideas to align with language and notation more conventionally used among the broader mathematical community (Rasmussen & Kwon, 2007).

Throughout this report, we will refer to students who learned through an IO approach as IO students, and we will refer to students learned through other approaches as Non-IO students. Our analysis uses data from an assessment developed to assess student performance and reasoning around core concepts in linear algebra (Haider et al., 2016). This report will focus on students' responses to two multi-part questions that offer insights into students' understanding of span and linear independence. In this study we will try to analyze two research questions: (1) How did IO and Non-IO students reason differently about span? (2) How did IO and Non-IO students reason differently about linear independence?

Literature & Theoretical Framing

Many works that examined how students reasoned about span and linear independence discuss findings related to the categories of algebraic and geometric interpretations. Students tend to be more comfortable with algebraic than geometric approaches, and often do not use geometric intuition when solving problems about span and linear independence (Bogomolny, 2007; Aydin, 2014, Ertekin, Erhan, Solak, & Yazici, 2010, Stewart & Thomas 2010).

Students often think of linear dependence in a variety of algebraic ways: in terms of free variables, pivot positions, or rows of 0's in the reduced row-echelon form (RREF); they often think of linear independence as meaning there are no free variables, or that vectors are not multiples of each other (Bogomolny, 2007; Aydin, 2014). A common theme in this literature is that many students treat linear independence as a process; some think of it in terms of the row reduction procedure and some connect it to the homogeneous linear system $Ax = 0$.

Stewart & Thomas (2010) also found that students tended to rely on algebraic approaches when solving problems involving span. Bogomolny (2007) found that for some students geometric and algebraic representations seemed completely detached. This was seen in students' attempts to provide a geometric interpretation of the span of the set of column vectors of a matrix; instead of giving a geometric representation of the span of the columns of the matrix A, some students found a geometric representation of the solution set of the homogeneous system $Ax = 0$. By definition, span does not require linear independence, but by involving this concept students successfully interpreted span as a subspace of certain dimension (Wawro, Sweeney, and Rabin, 2011).

In this work, rather than focusing on distinctions between algebraic and geometric interpretations for analyzing student reasoning about span and linear independence, we draw on a helpful theoretical distinction made by Tall and Vinner (1981) which offers language for differentiating the way individuals *think* about particular mathematical ideas (concept image) from formal mathematical definitions for particular mathematical ideas that are more conventionally accepted by the broader mathematical community (concept definition).

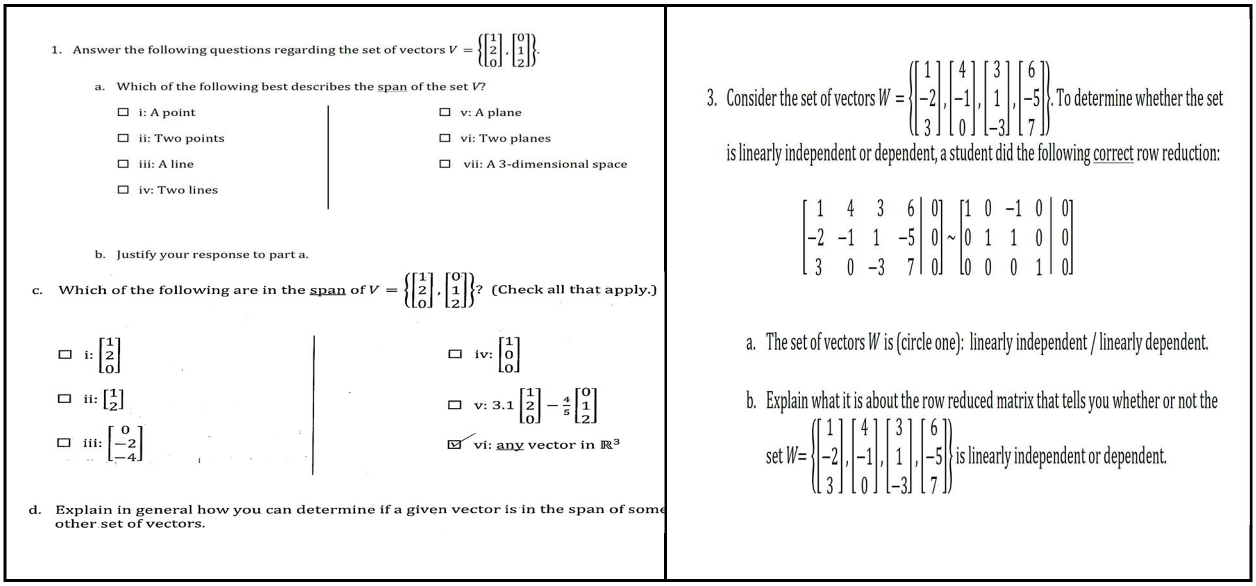
Data Sources & Study Context

Data for this analysis is drawn from a broader study (NSF #1431595/1431641/1431393) of instructors who received a set of three instructional supports to teach linear algebra in inquiry-oriented ways. These instructional supports were: curricular support materials (consisting of task sequences, learning goals, descriptions of common student approaches to tasks, and implementation notes and suggestions), a 16-hour summer workshop, and facilitated online work groups that met for one hour per week during the semester instructors implemented the curricular support materials.

For this study, we have analyzed a total of 255 assessments where 126 assessments were collected from students in IO classes and 129 were from students in comparable Non-IO classes. The linear algebra assessment is a paper-pencil based test and was administered as a post-test in IO and Non-IO classes. All students were given up to 1 hour to complete the test. The assessment carries 9 questions, which are combinations of multiple-choice and open-ended items, and the focus of the assessment is to capture students' conceptual understanding of linear algebra concepts. The assessment was designed in way that a calculator was not required to answer any question on the test. In this study, we focused on an in-depth analysis of students' reasoning on the assessment questions related to span (question 1) and linear independence (question 3; see Figure 1).

Questions Q1a and Q1b offer insight into how students interpret the span of a set of vectors as a geometric object; Q1c and Q1d offer insight into how students identify when particular vectors are part of the span of a set of vectors. The multiple choices for these items provide systematic insights on these students' concept images of span, whereas their open-ended responses have the potential to provide insights into connections to the concept definition. Question 3b explicitly asks students to justify their response to whether a given set of vectors are linearly independent by connecting the result of a procedure (row reduction) – which we also think will offer insights into the links between students' concept image and the concept definition of linear independence.

Instructors using the IO approach used a 4-task sequence developed to support students' reinvention of the concepts of span and linear (in)dependence (Wawro, Rasmusen, Zandieh, Sweeney, & Larson 2012). In task 1, students have two modes of transportation whose movement is restricted to correspond with two particular vectors in R^2 to try to arrive at a particular given location. In task 2, students explore whether it is possible to "get anywhere" in the plane using the



1. Answer the following questions regarding the set of vectors $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

   a. Which of the following best describes the span of the set V?

   ☐ i: A point
   ☐ ii: Two points
   ☐ iii: A line
   ☐ iv: Two lines
   ☐ v: A plane
   ☐ vi: Two planes
   ☐ vii: A 3-dimensional space

   b. Justify your response to part a.

   c. Which of the following are in the span of $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$? (Check all that apply.)

   ☐ i: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
   ☐ ii: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
   ☐ iii: $\begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}$
   ☐ iv: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
   ☐ v: $3.1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$
   ☑ vi: any vector in $\mathbb{R}^3$

   d. Explain in general how you can determine if a given vector is in the span of some other set of vectors.

3. Consider the set of vectors $W = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} \right\}$. To determine whether the set is linearly independent or dependent, a student did the following correct row reduction:

$$\left[\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 0 \\ -2 & -1 & 1 & -5 & 0 \\ 3 & 0 & -3 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

   a. The set of vectors W is (circle one): linearly independent / linearly dependent.

   b. Explain what it is about the row reduced matrix that tells you whether or not the set $W = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} \right\}$ is linearly independent or dependent.

FIGURE 1. Assessment items related to span and linear independence

same two vectors; after students work on this task, the instructor formalizes the definition of span of a set of vectors as the set of all possible linear combinations of the vectors in the set. In the third task, students are given three modes of transportation in R^3 and explore whether it is possible to take a non-trivial journey using those vectors that starts and ends at home. Sets of vectors that allow such non-trivial journeys are linearly dependent – an idea the instructor leverages following task 3 to formalize the definition that a set of vectors is linearly dependent when the corresponding homogeneous vector equation has a non-trivial solution. In the final task, students work to try to generate examples of sets 2 and 3 of vectors in R^2 and R^3 that are linearly dependent and independent; students form and justify generalizations based on this example-generating activity.

Methods of Analysis

To identify differences between IO and Non-IO student' performance and reasoning about span, we first look quantitatively at response patterns to multiple choice questions to Q1a and Q1c, and then look qualitatively at open ended responses to Q1b and Q1d to better understand the nature of student reasoning and differences between IO and Non-IO students. For linear independence, we did a similar quantitative and qualitative analysis to Q3a and Q3b. Quantitative comparisons of response patterns between IO and Non-IO students on multiple choice items were made using z-tests to see if there were statistically significant differences in the proportion of choices that IO and Non-IO students picked for every item. To qualitatively see how IO and Non-IO students reasoned, we engaged in open coding by first examining a subset of student responses to identify the variety of mathematically distinct ways students reasoned about each open-ended response question; we continued analyzing additional responses, refining categories as we did so, until our categories were saturated. This process led to 7 main categories of students' reasoning about Q1b, 2 categories about Q1d and 6 about Q3b (see Table 1). Items that did not fall into the categories described in Table 1 were labelled as "other" or marked if they were left blank. Student responses could be coded in multiple categories. During the coding we also paid attention to these reasonings if they align with the definitions or not and assign them as correct reasonings, otherwise they were incorrect reasoning (For example, students who reasoned in terms of linear independence did so correctly if they wrote something like 'the two vectors are linearly independent (or not scalar multiples of each other or not parallel …) so they make a plane.') We also use z-test to compare the proportion of codes assigned to the responses in both groups.

Table 1. Codes for Q1b, Q1d and Q3b and their descriptions

Questions	Code	Description
Q1b (Span)	Linear Independence	Response indicates that the two vectors are linearly independent or are not (scalar) multiples of each other.
	Linear Combination	Response refers to a linear combination of the two vectors (either directly in words, by giving the formula $xv_1 + yv_2 = w$, or stating something like 'getting anywhere')
	Different Directions	Response indicates that the two vectors point in different directions.
	Row Reduction	Student row reduces a matrix comprised of the given vectors.
	Dimensionality	Response makes explicit reference to the number of vectors, entries, or pivots; or claims that the two vectors are a basis
	Vector as Point/Line/Plane	Student identifies each vector individually as corresponding to either a point, line, or plane
	Geometric/ Graphical representation	Response includes a drawing showing a geometric representation as a response or part of it.
Q1d (Span)	Augmented Matrix/Row Reduction	Student row reduces the matrix comprised of the given vectors and concludes the vector is/is not in the span if the result is consistent/inconsistent or there is / is not a solution.
	Linear Combination	Same description as in Q1b.
Q3b (Lin. Ind.)	Compares RREF to Identity Matrix	Response indicates whether row reduction leads to identity matrix, especially comparing number of rows/columns
	Pivots	Response indicates if there are missing pivots in one or more columns/rows, if there is a pivot in every column/row, or explicitly references number of pivots
	Linear Combination	Explicitly or implicitly observes that one of the columns is a linear combination of other columns
	Solving Ax = 0	Response refers to solutions to the equation Ax = 0, e.g. non-trivial or infinitely many solutions
	# columns > # rows, or # vectors > dim(R^3):	Response indicates the number of columns or vectors is bigger than the number of rows or the dimension of R^3, or that the matrix M is not square
	Free Variable	Response explicitly indicates there is a free variable

Findings

Our quantitative analysis of the multiple-choice questions showed that IO students outperformed Non-IO students on span questions, but not on linear independence questions. Our open-ended coding additionally suggested that IO students' concept images of span and linear independence were more aligned with corresponding concept definitions than those of Non-IO students. Additional details about trends in student reasoning follow.

Differences in IO and Non-IO Student Performance and Reasoning about Span

When asked to identify which best describes the span of a given set of two (linearly independent) vectors in R^3 on Q1a, almost twice as many IO as Non-IO students correctly selected "A Plane" (see Table 2). This difference was statistically significant ($p < .001$). All other choices (which are incorrect answers to the given problem) were picked at higher rates by Non-IO students than IO students; in the case of choices Two Points, A Line, and Two Planes this difference was also statistically significant.

TABLE 2. Popularity of choices of Q1a Picked by IO and Non-IO Students

Choices		IO (n=126)	Percentage (IO)	Non-IO (n=129)	Percentage (Non-IO)	Significance* (z-test)
i.	A point	1	.79	1	.77	p=.984
ii.	Two points	0	00	5	3.9	**p=.026**
iii.	A line	4	3.2	12	9.3	**p=.043**
iv.	Two lines	6	4.8	8	6.2	p=.617
v.	A plane	94	74.6	53	41.1	**p<.001**
vi.	Two planes	5	4	17	13.2	**p=.009**
vii.	A 3-D space	12	9.5	14	10.9	p=.726

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

When comparing the reasoning of IO and Non-IO students, we note two key trends. First, IO students were significantly more likely to reason about span in terms of linear independence, dimensionality, or row reduction than Non-IO students, and they employed these forms of reasoning *correctly* at much higher rates. Non-IO students on the other hand, were significantly more likely to interpret the span of a set of vectors by interpreting each vector individually as a geometric object. (This is consistent, for example, with significantly more Non-IO students selecting "Two points" and "Two planes" on Q1a.) Table 3 summarizes the coding of justifications students gave for their choices on Q1a; the p-values provided regard the comparison of the number of IO and non-IO students who used an approach (not the number using it correctly).

TABLE 3. Codes for IO and Non-IO Students' Approaches to Q1b

Codes	IO students |# used correctly (n=126)	Non-IO students |# used correctly(n=129)	Significance* (z-test)
Linear independence	53(42%) | 51(40%)	28(20%) | 27(21%)	**p<.001**
Linear Combination	22(17%) | 19(15%)	18(14%) | 16(12%)	p=.441

Different Directions	7(6%) | 7(6%)	3(2%) | 3(2%)	p=.183
Row Reduction	10(8%) | 6(5%)	0(0%) | 0(0%)	**p=.001**
Vector as Point/Line/Plane	21(17%) | 10(8%)	36(28%) | 6(5%)	**p=.032**
Geometric/Graphical	23(18%) | 15(12%)	17(13%) | 9(7%)	p=.267
Dimensionality	51(40%) | 42(33%)	32(25%) | 21(16%)	**p=.008**

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

When asked to identify whether or not particular vectors lie in the span of a set of vectors, IO students were significantly more likely to select choices that were a scalar multiple of one of the vectors in the set (iii) or explicitly expressed as a linear combination of vectors in the set (v), (see Table 4.) On the other hand, Non-IO students were significantly more likely to incorrectly select the choice that indicates any vector in R^3 is in the span of the given set of two vectors.

TABLE 4. Popularity of Choices of Q1c Picked by IO and Non-IO Students

Choices		IO (n=126)	Percentage (IO)	Non-IO (n=129)	Percentage (Non-IO)	Significance* (z-test)
i.	$[1,2,0]$	107	85%	110	85%	p=.936
ii.	$[1,2]$	19	15%	24	19%	p=.453
iii.	$[0,-2,-4]$	101	80%	78	60%	**p<.001**
iv.	$[1,0,0]$	13	10%	22	17%	p=.119
v.	$3.1[1,2,0]-\frac{4}{5}[0,1,2]$	90	71%	77	60%	**p=.049**
vi.	Any Vector in R^3	10	8%	23	18%	**p=.019**

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

Table 5. Codes for IO and Non-IO Students' Approaches to Q1d

Codes	IO Students | # used correctly (n=126)	Non-IO Students | # used correctly (n=129)	Significance (z-test)
Linear Combination	99(79%) | 93(74%)	97(75%) | 81(63%)	p=.522
Augmented Matrix (RR)	27(21%) | 12(9.5%)	10(8%) | 5(3.9%)	**p=.002**
Other	5(4%) | 0(0%)	18(14%) | 0(0%)	**p=.005**
Empty	5(4%)	6(5%)	p=.787

We noted above, Q1a and Q1b provide insight into students' geometric interpretations and justifications. Q1c and Q1d provide insight into how students interpret span in terms of individual elements, i.e. how students decide if individual vectors are in the span of a set of vectors, as opposed to describing the entire span of that same set of vectors as a geometric object. Looking across these two questions, we note one key interesting story: in Qc, IO students pick correct choices, (especially scalar multiple and linear combination of vectors in the set are in the span of the set) at higher rates, suggesting they have a better sense of how to identify vectors in the span than Non-IO students. In Q1d, we see IO and Non-IO students use linear combination reasoning at similar rates, though IO students did so correctly more than Non-IO students. Based on results from Q1c, IO students have a more robust concept image of span (e.g. they have a better sense of the variety of forms this can take; scalar multiples and linear combinations). See table 5.

Differences in IO and Non-IO student performance and reasoning about linear independence.

When asked whether a given set of 4 vectors in R^3 is linearly independent or dependent (and given the correct RREF of the augmented matrix comprised of those column vectors), there was no statistically significant difference in the portion of IO and Non-IO students who correctly said the set was linearly dependent (see Table 6). However, there were some differences in reasoning of IO and Non-IO students.

Table 6. Choices Selected by IO and Non-IO Students on Q3a

Choices	IO Students (n=126)	Percentage (IO)	Non-IO Students (n=129)	Percentage (Non-IO)	Significance* (z-test)
Linear Dependence	101	80%	97	75%	p=.343
Linear Independence	19	15%	31	24%	p=.072

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

When justifying their responses about whether the set in Q3a was linearly independent, IO students were more likely to reason by comparing the number of rows/columns in the RREF (in comparison to the identity matrix), or in terms of the solution to $Ax = 0$, and more IO students reasoned correctly using those approaches. This suggests for IO students, there may be better alignment between their concept image and concept definition.

Table 7. Codes for Various Students' Approaches to Q3b

Codes	IO Students (n=126)	Used Correctly (IO)	Non-IO Students (n=129)	Used Correctly (Non-IO)	Significance* (z-test)
Compare RREF to I	20 (16%)	19 (15%)	6 (5%)	6 (5%)	**p=.003**
Pivots	31 (25%)	21 (17%)	42 (33%)	31 (24%)	p=.159
Lin. Comb	25 (20%)	22 (17%)	28 (22%)	20 (16%)	p=.711
Solving $A\bar{x} = 0$	31 (25%)	26 (21%)	17 (13%)	10 (8%)	**p=.020**
#Col > #Rows OR #Vectors > $\dim(R^3)$	14 (11%)	14 (11%)	19 (15%)	18 (14%)	p=.390
Free Variable	32 (25%)	30 (24%)	31 (24%)	27 (21%)	p=.802

* Difference between percentages of IO and Non-IO students for each choice based on z-scores

Discussion

We found IO students outperformed Non-IO students on span questions, exhibiting a wider range of appropriate concept images of span. While IO students did not outperform Non-IO students on the linear independence question, our data suggests IO students' interpretations were more explicitly linked to the concept definition. Future work will further explore this issue.

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A Framework for the Natures of Negativity in Introductory Physics

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Positive and negative quantities are ubiquitous in physics, and the sign carries important and varied meanings. Unlike physics experts, novices struggle to understand the many roles signed numbers can play in physics contexts, and recent evidence shows that unresolved struggle carries over to subsequent physics courses. The mathematics education research literature documents the cognitive challenge of conceptualizing negative numbers as mathematical objects. We contribute to the growing body of research that focuses on student reasoning in a physics context about signed quantities and the role of the negative sign. This paper contributes a framework for categorizing the natures of the negative sign in physics contexts, inspired by the research into the natures of negativity in algebra. Using the framework, we analyze several published studies associated with reasoning about negativity drawn from the physics education and mathematics education research communities. We provide implications for mathematics and physics instruction and further research.

Keywords: Negative, Quantity, Negativity, Physics, Signed

Introduction

Development of mathematical reasoning skills is an important goal in introductory physics courses, particularly those geared toward students majoring in physics and engineering fields. Positive and negative quantities are ubiquitous in physics, and sign carries important and varied meanings. Unlike physics experts, novices struggle to understand the many roles sign plays in physics contexts.

Negative pure numbers represent a more cognitively difficult mathematical object than positive pure numbers do for pre-college students (Bishop et al., 2014). Mathematics education researchers have isolated a variety of natures of negativity fundamental to algebraic reasoning in the context of high school algebra that go beyond a 'position on a number line' nature (Gallardo & Rojano, 1994; Nunes, 1993; Thompson & Dreyfus, 1988). These various natures of negativity form the foundation for scientific quantification, where the mathematical properties of negative numbers are a good representation of natural processes and quantities. Physics education researchers report that the majority of calculus-based physics students struggle to make meaning of positive and negative quantities in spite of successfully passing Calculus I and beyond in mathematics (Brahmia & Boudreaux, 2016, 2017). Developing flexibility with negative numbers is a known challenge in mathematics education, and there is mounting evidence that reasoning about negative quantity poses a significant hurtle for physics students at the introductory level and beyond.

Few published studies have focused on negativity in the context of the mathematics used in physics courses. Studies conducted in the context of upper division physics courses reveal robust student difficulties (Hayes & Wittmann, 2010; Huynh & Sayre, 2018). Brahmia and Boudreaux constructed physics assessment items based on the natures of negativity from mathematics education research (Vlassis, 2004) and administered them to introductory physics students in the introductory sequence of courses (Brahmia, 2017; Brahmia & Boudreaux, 2016, 2017). The authors

Table 1: A map of the different uses of the negative sign in elementary algebra (Vlassis, 2004)

Unary (Struct. signifier)	Symmetrical (Oper. signifier)	Binary (Oper. signifier)
Subtrahend	Taking opposite of,	Completing
Relative number	or inverting the operation	Taking away
Isolated number		Difference between numbers
Formal concept of neg. number		Movement on number line

report that students struggle to reason about signed quantity in the contexts of negativity typically found in the curriculum (e.g., negative work, negative acceleration in one dimension, negative direction of electric field), and they concluded that science contexts may overwhelm some students' conceptual facility with negativity. In addition, they observed that students struggled to interpret the meaning of either a positive or negative signed quantity—it is the existence of a sign that causes difficulty (Brahmia & Boudreaux, 2017). These studies reveal that signed quantities, and their various meanings in introductory physics, present cognitive difficulties for students that many don't reconcile before completing the introductory sequence.

The current study contributes to this body of research by introducing a framework for categorizing the natures of negativity in introductory physics (NoNIP), analogous to the natures of negativity developed in the context of algebra. The intention is to provide a framework that can help researchers characterize and address the mathematical conceptualization of signed quantity in introductory physics. We conclude that the natures of negativity should be explicitly addressed in the context of introductory physics and calculus. We provide recommendations that can support the use of the NoNIP framework in the context of these courses.

A Model of the Natures of Negativity

The first generation of the natures of negativity for introductory physics was based on the natures of negativity described by Vlassis (Vlassis, 2004). We developed survey items to help map the algebra natures to a physics context—one survey question for each of the three natures in two contexts: mechanics quantities and E&M. The first survey item probes student understanding of the unary nature of the negative sign, the second probes the symmetrical nature, and the third, the binary nature (see Table 1). Table 2 presents all three mechanics items for reference.

We found that most uses of the negative sign typically found in introductory physics courses could be categorized using the map summarized by Vlassis. By using a mathematics-based sorting theme, however, we found we often lost the nuances of the physics described by the math; for example, we found that both scalars and vectors might be placed in the same broad category, despite the importance in physics of distinguishing between vector and scalar quantities. Because our intent was to encode both physical and mathematical meanings of the negative sign, we started from scratch keeping the physics as our primary guide.

Because of the importance of the difference between scalar and vector quantities in physics, our first attempt at mapping the natures of negativity in introductory physics began with a broad categorization based on whether quantities were vector or scalar. Some vector relationships are exclusively opposite in nature, such as Newton's Third-Law pairs, and the relationship between force and potential, $\vec{F} = -\vec{\nabla}U$. It was determined that this was the only possible categorization for 'complete' vector quantities, rather than vector components; in this case, 'opposite' indicates

Table 2: Questions representing different algebraic natures of negativity in introductory mechanics

Unary *structural signifier*	Symmetrical *operational signifier*	Binary *operational signifier*
Direction of a vector component	**Signifies work results in decreasing the system, energy, not increasing it**	**Position relative to an origin**
An object moves along the x-axis, and the acceleration is measured to be $a_x = -8$ m/s^2. Describe in your own words the meaning of the negative sign in the mathematical statement "$a_x = -8$ m/s^2".	A hand exerts a horizontal force on a block as the block moves on a frictionless horizontal surface. For a particular interval of the motion, the work W done by the hand is $W = -2.7$ J. Describe in your own words the meaning of the negative sign in the mathematical statement "$W = -2.7$ J".	A cart is moving along the x-axis. At a specific instant, the cart is at position $x = -7$ m. Describe in your own words the meaning of the negative sign in the mathematical statement "$x = -7$ m".

that the vectors in the relationship point in opposite directions (i.e., they are anti-parallel). Another vector-related category was for vector component quantities, and had two sub-categories: quantities for which the negative sign indicates the direction of the component relative to a coordinate system (such as v_x, F_x, E_x, or Δp_x), and one-dimensional relationships similar to the 'opposite' category for vector quantities described above.

Scalar quantities were subdivided into four categories: a) Amount; b) Opposite/opposing; c) Difference/change; and d) Label. The subcategory Amount is reserved for quantities for which we consider a negative amount of a thing. Such quantities are rare, and are only derived (not base) quantities. Total and potential energy, as well as scalar product quantities such as work and electric or magnetic flux were categorized in this way. Scalars in the opposite/opposing category include charge (as positive and negative charge are opposite types of charge) and relationships such as Faraday's Law, and $\Delta V = W/q$. The Difference/change category was used for time rates of change of scalar quantities, where the sign of the quantity indicates an increase or decrease, and for changes in a system such as energy change, ΔE, or temperature change, ΔT. Finally, the Label category was used only for charge; the sign of a charge tells us the type of charge, while the charge of an object tells us the type of charge in excess on the object.

Although this categorization did allow for the differentiation of vector and scalar quantities, we found it unsatisfactory overall. There seemed to be more variation within categories than between them, and we found that it placed quantities with similar physical characteristics (such as relationships that fell into the "opposite" categories for both scalars and vectors) into different categories. We also found that this categorization scheme did not allow for differentiation between uses of the negative sign as an operator. Moreover, quantities for which the negative sign has multiple interpretations (e.g., mechanical work as a scalar product and as measure of system energy change) were poorly represented by this categorization. Because our focus was on physics quantities rather than relationships between quantities, it was difficult to categorize models for which a negative sign is not an explicit part of the relationship. Finally, we recognized that there were quantities such as the product $f(x)dx$ that were not well-represented in this scheme. Physics and mathematics education researcher had indicated that products of integrands and differentials pose challenges for students when one or both of the factors are negative (Bajracharya, Wemyss, & Thompson, 2012; Sealey & Thompson, 2016).

The first two authors employed a modified card-sorting task for a second attempt at creating an expert version of the natures of negativity in physics, in which we again brainstormed and sorted physics quantities *and* relationships typically introduced in introductory physics. Categories were created based on the overarching similarities without first dividing quantities based on whether they were vector or scalar in nature. We created several sub-categories for each main category, largely to account for nuances in physical meaning. We determined three basic categories: *Direction (D), Opposition (O),* and *Change (Ch)*. A fourth category, *Compound (Co)* was added for instances when multiple meanings are assigned to the negative sign in a single expression or concept. Table 3 shows the results of this effort to create a map of the natures of negativity in introductory physics. We have surveyed introductory physics textbooks, checking that described signed quantities can be categorized satisfactorily with our scheme. We conducted expert interviews with physics instructors to ensure that this map of natures of negativity is valid for describing a majority of signed quantities in introductory physics and proposes a categorization that makes sense in the introductory physics context. A number of small changes were made based on these interviews, resulting in the form included in this paper. Additionally, we conducted expert interviews with mathematics faculty who were familiar with the physics contexts, including one math education researcher, to ensure that mathematical validity of this categorization; a repeating theme from these interviews with math experts was the importance of the meaning of 'zero' or 'origin' in each of these cases. This also indicated to us that reasoning about the sign of every quantity (not just reasoning about negativity) was important for more complete understanding of physics quantities.

We note that the *Direction* and *Opposition* categories are supported by the categories isolated by mathematics education researcher Chiu. In their study, they identified three categories of metaphorical reasoning that both middle school students and undergraduate and graduate mathematics and engineering majors used during problem-solving interviews—motion, manipulation of objects/opposing objects, and social transaction (associated with the experiences of giving and exchanging) (Chiu, 2001). While these are metaphors in mathematics, they are in fact *contexts* in physics in which a conceptual mathematical understanding is essential for learning the physics. The entire content of mechanics is focused on actual motion in space (not motion along a number line). Phenomena that arise due to the parallel or antiparallel orientations of two quantities are ubiquitous throughout physics (i.e. speeding up/slowing down, friction and air resistance, electromagnetic induction). Direction and Opposition are central natures of signed quantities in physics.

The *Direction* category is used largely for components of vector quantities. We differentiate between **1. Location** (for which the sign tells us the position relative to an origin), **2. Direction of motion** (typically used for a vector component, where sign indicates direction of motion relative to a coordinate system), and **3. Other vector quantities** (where the sign of a vector component tells us the direction of that component relative to a coordinate system, but when motion is not an intrinsic quality of the vector quantity). We consider subcategories **2** and **3** separately, as direction of motion is readily apparent and observable. Finally, we consider **4. Above/below reference** for *scalar* quantities such as electric potential and temperature, for which the zero of the quantity is an arbitrary reference point.

For the category *Opposition*, we consider quantities for which a negative sign implies opposite direction or relationship. **1. Opposite type**, as positive and negative charge are "opposite" types

Table 3: The Natures of Negativity in Introductory Physics, a map of the different uses of the negative sign in introductory physics

(D) Direction	(O) Opposition	(Ch) Change	(Co) Compound
1. Location	**1. Opposite type**	**1. Removal (operator)**	**1. Scalar rates of change**
x	*Q (charge)*	$0-(-5\mu C)$	$\frac{d\Phi}{dt}$
2. Direction of motion	**2. Opposes**	**2. Difference (operator)**	**2. Base + change**
$v_x, \Delta x$	$\vec{F}_{12} = -\vec{F}_{21}$	$E_f - E_i$	$\phi + \frac{d\phi}{dt}t$
p_x	$\vec{F} = -\vec{\nabla}U$	$\vec{p}_f - \vec{p}_i$	$\vec{v} + \vec{a}t$
3. Other vec. quant. comp.	$\mathscr{E} = -\frac{d\Phi_B}{dt}$	**3. System scalar quantities**	**3. Products** $f(x)dx$
E_x, B_x	$\vec{F} = -k\vec{r}$	$\Delta K, \Delta E$	$E(r)dr$
F_x, L_z	**3. Scalar products**	ΔS	$P(V)dV$
a_x	$W = \vec{F} \cdot \Delta\vec{x}$	**4. Scalar, vector change**	**4. Models**
$\Delta p_x, \Delta v_x$	$\Phi = \vec{B} \cdot \vec{A}$	$\Delta E = E_f - E_i, \Delta V = V_f - V_i$	$W_{net,ext} = \Delta E$
4. Above/below reference		$\vec{\Delta p} = \vec{p}_f - \vec{p}_i$	$\vec{F}_{net} = m\vec{a}$
T (temperature)			$\Delta U = Q - W$
V (electric potential)			

of charge, and obey the mathematical relationship of $+q+(-q)=0$ (i.e., adding equal amounts of opposite types of charge leads to a system with no net charge). For the the subcategory **2. Opposes**, we consider scalar and vector relationships between quantities that indicate that the quantities oppose each other in direction or change, such as members of a Newton's Third Law force-pair.

The category *Change* encompasses both the meaning of the sign of the change of a quantity as well as the negative sign as an operator that signifies a change in a quantity. (**1. Removal (operator)**). We may also use the negative sign to signify that we are taking a difference between two quantities (as in determining the change of a quantity), as described by **2. Difference (operator)**. Subcategory **3. System scalar quantities** describes quantities that characterize change in a system, such as changes in energy or entropy. For **4. Scalar, Vector change**, when students are asked to calculate a change in a quantity such as energy or momentum, they must first account for the signs of the initial and final quantities, then successfully subtract one from the other and make sense of the result.

Finally, the *Compound* category covers instances when the negative sign spans more than one category, or that require one to 'keep track' of several signs when making sense of a quantity or relationship. **1. Scalar rates of change**, **2. Base + change** (base quantities that are increased or decreased by the addition of a change; the concept of accumulated change is ubiquitous in physics), and **3. Products** $f(x)dx$ (products of integrands and differentials). We also include in this category **4. Models**, to account for models that require sensemaking of a negative sign. The Work-Energy Theorem, where the sign of $W_{net,ext}$ indicates whether a system gains or loses mechanical energy, is an example of such a model.

Applying the framework

In this section, we use the NoNIP as an analytical lens through which to view recently published studies in physics and calculus and that mostly involve advanced physics or math students.

Bajracharya, Wemyss, and Thompson (2012) investigated upper-division student understanding of integration in the context of definite integrals commonly found in introductory physics, but with physics context stripped from the representation: the variables typically used in physics contexts were replaced with x and $f(x)$ (Bajracharya et al., 2012). Their results suggest difficulties with the criteria that determine the sign of a definite integral. Students struggle with the concept of a negative area-under-the-curve, and in particular negative directions of single-variable integration. Sealey and Thompson (2016) interviewed math majors to uncover how they made sense of a negative definite integral. Undergraduate (beyond introductory) and graduate mathematics students had difficulty to make meaning of a negative differential in the context of integration (Sealey & Thompson, 2016). The struggles these researchers described can be seen through the lens of NoNIP as struggle with the product of the integrand, $f(x)$, and the differential, dx (Co.4 in NoNIP). The negativity of the integrand (D in NoNIP) was less of a struggle for the students in these studies than was the notion of a negative differential (Ch in NoNIP), which has application throughout physics.

Hayes and Wittmann (2010) report on an investigation in a junior-level mechanics course of negative signs and quantities associated with the equation of motion of an object thrown downward, with non-negligible air resistance (Hayes & Wittmann, 2010). The equation of motion is $ma = mg - bv$, or $m\frac{d^2x}{dt^2} = mg - b\frac{dx}{dt}$, where the initial velocity exceeds the terminal velocity so the object is thrown downward and slows down—the velocity and the acceleration oppose each other initially. The student interviewed struggles with treating one-dimensional acceleration as a signed quantity, and feels there should be an additional negative sign included to indicate that the acceleration is opposing the motion. The authors conclude that the multiple natures of the negative sign are a source of cognitive conflict that the student can't resolve. Mathematics education researchers have found that younger students tend to assign only natural numbers to literal symbols or to treat expressions such as $-x$ as if they represent solely negative quantities (Christou & Vosniadou, 2012; Lamb et al., 2012). Although the students in the Hayes and Wittman study are well beyond Calculus II, it appears they revert to a more primitive treatment of vector quantities when they encounter a challenging context that calls on multiple meanings of the negative sign. Seen through the lens of NoNIP, minus is an operator, and negative signs are used to represent many mathematical objects in physics. In this context the student struggles with D.3 and D.2 in the contexts of one-dimensional acceleration and velocity. The negative sign that modifies the bv term is used as O.2, to indicate that the force is in the opposite direction to the velocity. Combining terms, the students struggle to make sense of the equation of motion. The cognitive load associated with the individual terms contribute to a higher-level struggle of making physical sense (Co.5).

In their study of negativity in junior level Electricity and Magnetism, Huynh and Sayre (2018) describe the in-the-moment thinking of a student solving for the direction only of a positive and negative charge distributed along a line symmetrically about the origin (Huynh & Sayre, 2018). The solution involves an algebraic superposition of the field due to each charge individually. In their study the authors focus on the student reasoning about the sign of the the electric field vector component along the axis of symmetry in three regions of space—to the left of one charge, between the two charges and to the right of the other charge. The authors detail the students' development of an increasingly blended approach that is situated in a mental space informed by both mathematical and physical concepts. The student starts reasoning about the direction of the field by (unintentionally) combining multiple natures of negativity into one, using the canceling procedure that two negatives make a positive, without considering the source of each negative sign. In Coulomb's law, signs come in associated with the charges, the unit vector and the electric field vector direction. Collapsing the signs using arithmetic rules is a common approach first tried by the students in this study, which focuses on the multiplicative rules of signed numbers rather than the physics of the meaning of the signs. Next the student rarefies his approach as he considers more carefully the natures of negativity in the context of the problem. Seen through the lens of NoNIP we can see evidence of the student first conflating the natures superficially; the authors describe, "...he decides to absorb the destructive meaning...into the opposite meaning...and changes the second negative sign to a plus sign...however he didn't consider the...relative direction...leading to...the opposite sign of the correct answer." Then as he slows his thinking, first recognizing D.2 and D.3, the unit and electric field vectors and as sources of negative signs, the student says, "...I should have figured it out...which direction it is. This is exactly what is changing signs." After reconciling the basic level, then he struggles with O.2, the authors describe that the student "has successfully affiliated the sign's meaning to the relative direction...electric fields and x-hat." The authors conclude, and we agree, that the most sophisticated challenge occurs when these natures are combined in which three natures of the negative sign must be made sense of in the context of a single equation, Co.5. This example illustrates the challenges associated with reasoning about the natures of negativity even for strong majors, and reveals a hierarchy that lends plausibility the NoNIP model being representative of emergent expert-like reasoning.

Implications for instruction

Student difficulties are embedded in natures of negativity that can be, and we argue should be, explicitly addressed in the context of introductory physics and calculus. We suggest that instructors familiarize themselves with the many jobs that the negative sign does in introductory physics courses, and help students recognize the varied natures of signed quantities. The NoNIP framework can help. We offer two suggestions as a start:

1. In problems associated with motion, aligning the positive coordinate axis with the direction of motion eliminates the need for signed quantities when discussing velocity. This choice, however, could be a missed opportunity to distinguish between orientation and sense. The opposite coordinate choice can prime students to consider the signed nature of position, velocity, and subsequent vectors quantities they encounter.

2. Applications that involve quantities that are inherently signed quantities should be prefaced with a negative sign when the quantity is negative, and a positive sign when positive. Priming students in a math course to expect that real-world quantities have signs that carry meaning, and that 'no sign' is a different kind of quantity than a positively-signed quantity, will help better prepare students. These quantities in physics include, but aren't limited to: position, displacement, velocity, acceleration, force, and work.

In addition to enriching subsequent physics learning, a focus on natures of negativity in physics contexts can also enrich the corequisite mathematics learning. Sealey and Thompson report on a context in which physics helps math students make sense of negativity in calculus. The researchers observed that invoking a physics example of a stretched spring helped catalyze sense making—the physics helped them to make sense of an abstract binary nature of the negative sign (Sealey & Thompson, 2016). We suggest that there is a symbiotic cognition possible in which both mathematics and physics learning can be enriched by conceptualization of the other. We present NoNIP as a representation of signed quantity providing a step in that direction.

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Opportunities to Engage Secondary Students in Proof Generated by Pre-service Teachers

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For reasoning and proving to become a reality in mathematics classrooms, pre-service teachers (PSTs) must develop knowledge and skills for creating lessons that engage students in proof-related activities. Supporting PSTs in this process was among the goals of a capstone course: Mathematical Reasoning and Proving for Secondary Teachers. During the course, the PSTs designed and implemented in local schools four lessons that integrated within the regular secondary curriculum one of the four proof themes discussed in the course: quantification and the role of examples in proving, conditional statements, direct proof and argument evaluation, and indirect reasoning. In this paper we report on the analysis of 60 PSTs' lesson plans in terms of opportunities for students to learn about the proof themes, pedagogical features of the lessons and cognitive demand of the proof-related tasks.

Keywords: Reasoning and Proving, Preservice Secondary Teachers, Lesson Plans

Despite persistent calls to make reasoning and proof an integral part of everyday teaching and learning mathematics, the reality in secondary schools is far from what mathematics educators and policy leaders had in mind (NCTM, 2009, 2014, 2018; CCSSI, 2010). Studies consistently show that proof and proving are "notoriously difficult for students to learn and for teachers to teach," and that making proof a reality in mathematics classrooms requires systemic change in classroom culture (Nardi & Knuth, 2017, p. 267). Since teachers are instrumental to any instructional change (NCTM, 2014), pre-service teachers (PSTs) need to develop knowledge and skills to successfully implement reasoning and proving in their future classrooms.

To address this goal, we developed a capstone course *Mathematical Reasoning and Proving for Secondary Teachers,* which is part of an NSF-funded, 3-year design-based research project. The course comprised four modules, each focused on one proof theme: quantified statements and the role of examples in proving, conditional statements, direct proof / argument evaluation, and indirect reasoning. The themes were chosen because they are known in the literature as central to proof production and comprehension, but challenging for students and teachers alike (e.g., Antonini & Mariotti, 2008; Durand-Guerrier, 2003; Weber, 2010). Our primary goal was to support future teachers in integrating reasoning and proving in their classroom instruction, regardless of the content or grade level, with the specific focus on these proof themes. Thus, the course activities aimed (a) to increase PSTs' awareness of the importance of the logical aspects of proof, and student difficulties with proving, (b) to teach PSTs to identify *within regular school curricula* opportunities to integrate proof-related tasks, and (c) to equip PSTs with pedagogical tools and ideas on how to create or modify mathematical tasks to integrate proof within them.

In Buchbinder and McCrone (in press) we describe the theoretical foundations of the course design and detail its structure and activities. Here, we focus on one critical component of the course: having the PSTs design and implement, in local schools, lessons that integrate mathematical topics with one of the proof themes. In this paper, we analyze the PSTs' lesson plans, focusing on the opportunities that PSTs engineered for secondary students to learn about the four proof themes. Our analysis addressed two overarching questions:

1. What opportunities to learn about reasoning and proving, specifically about the four proof-themes, did PSTs integrate in their lesson plans?

2. How were these learning opportunities realized in the lesson plans?

While the most intriguing question might be: "what did the secondary students learn from such lessons?", our ability to answer this question is limited. First, the focus of the study was on the PSTs' ability to engage students in proving. Second, most PSTs taught multiple different groups of students throughout the semester, and although all lessons were video-taped, we can at most assess student engagement with the lesson rather than their learning from a single lesson.

Background and Theoretical Perspectives

For the purpose of our study, we adopt a definition of proof that is appropriate for the secondary school context: "a mathematical argument for or against a mathematical claim that is both mathematically sound and conceptually accessible to the members of the local community where the argument is offered" (Stylianides & Stylianides, 2017, p. 212). By reasoning and proving, we refer to a wide range of processes such as conjecturing, generalizing and making valid arguments on the basis of mathematical deductions rather than authority or empirical evidence (Ellis, Bieda & Knuth, 2012; Stylianides, 2008). This definition and these processes were used in the analysis of the PSTs' lesson plans.

Stein, Remilard and Smith (2007) distinguish between *written curriculum*, which is written artifacts that teachers and students use, *intended curriculum*, which is the teacher's lesson plan, and *enacted curriculum* that is the lesson as it unfolds in the classroom. A lesson plan contains information on the mathematical content of the lesson, the types of tasks, how students will be engaged in them and the goals the teacher seeks to achieve. All these aspects shape the quality of students' mathematical experiences. For example, mathematical tasks of high vs. low cognitive demand determine whether students will be engaged in meaningful mathematical processes such as exploring and justifying, or simply applying standard procedures and recalling facts (Smith et al., 2004). Pedagogical features of the lessons provide information on how it will be enacted and on the organizational aspects of the lesson that "have potential to generate opportunities for students to develop or display mathematical understanding" (Silver et al., 2009, p. 511). In this paper, we analyze PSTs' lesson plans and focus on the *opportunities* to learn about reasoning and proof embedded in them and *how* the PSTs intended to enact these opportunities.

Methods

Fifteen PSTs participating in the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* took part in this study. The PSTs (4 middle-school, and 11 high-school track; 6 males and 9 females) were in their senior year, meaning that they have completed most of their content courses and two courses on methods of teaching mathematics.

During the course, the PSTs designed four lesson plans integrating a particular proof theme with a mathematical topic from the secondary curriculum, based on information from cooperating teachers from the local schools. Due to the course structure, the PSTs were required to address particular proof themes while the current classroom mathematical topic might have been more conducive to a different proof theme. The PSTs were encouraged to include in their lessons high cognitive demand tasks, and to use pedagogical tools that were illustrated and discussed in the course, among them proof task models, such as, *Who is right?*, *True-or-False?*, *Always-Sometimes-Never?* and *Is it a coincidence?*. These task models have been shown to elicit rich student engagement with the logical aspects of proof and can be modified for various mathematical topics, while maintaining their original structure and goals (Buchbinder & Zaslavsky, 2013). However, turning these pedagogical devices into a lesson plan was up to the

PSTs; we did not offer lesson plan templates that were specific to the proof themes, and PSTs were not directly told how to integrate these themes into the content of their lessons.

The lessons were 50 minutes long, designed for small groups of 4-8 students. The PSTs then taught each lesson and videotaped their teaching. The lesson plans followed a particular format that included: (1) general information on grade level, subject area, topic of the lesson, student prior knowledge, content and process objectives; (2) outline of the lesson explaining what the teacher and the students will be doing, description of anticipated student difficulties and ways to address them; and (3) student worksheets with solutions. These lesson plans, 60 in total, comprise the main corpus of data for this paper. Supplementary data sources supporting our analysis were the PSTs' reflections on each lesson and on the course overall, and video records of the course sessions and of the PSTs' classroom teaching.

In our analysis we relied on the frameworks developed by Silver et al., (2009) who analyzed lessons submitted by teachers seeking national board certification. The analysis proceeded in several stages. First, we mapped out the grade level, mathematical content and pedagogical features of each lesson plan. Second, since each lesson intended to integrate some proof theme, we assigned each lesson plan, as a whole, a rating (high, medium or low) reflecting the prevalence of the proof theme in it. We illustrate this coding and its outcomes in the results section below. Next, we examined the level of cognitive demand of the tasks designed by the PSTs. In each lesson we identified *proof-related tasks*, that is, tasks in which students had to develop/evaluate an argument, justify, explain, or compare their own mathematical work with that of others. Regardless of whether or not the tasks were focused on the proof theme, we coded them as high or low-demand using Silver's et al (2009) framework. Note that the attributes of proof-related tasks are often associated with high-cognitive demand (Stein, et al., 2000), however our analysis showed that these two characteristics are not identical.

The coding procedures were carried out as follows: the two researchers coded each lesson plan independently, and then compared and discussed their coding until agreement was reached.

Results

Mathematical Topics and Pedagogical Features of the Lesson Plans

Table 1 summarizes the mathematical content and pedagogical features of the lesson plans.

Table 1. Mathematical Content and Pedagogical Features

	8th Grade Mathematics Pre-Algebra (HS) 22 lessons	Algebra 1 College-Prep Alg. 1 18 lessons	Geometry College-Prep Geometry 20 lessons
Mathematical Content	• Rules of exponents • Scientific notation • Order of operations • Problem solving • Variable expressions • Distributive property	• Proportions • Order of operations • Combining "like" terms • Solving equations • Linear functions/graphs	• Quadrilaterals • Parallel lines • Vertical angles • Line and angle proofs • Pythagorean theorem • Simplifying square roots
Pedagogical Features of Lessons	• Manipulatives (e.g., dice and playing cards) • Matching activities • Logic riddles	• Manipulatives (e.g., algebra tiles)	• Card sorting tasks • Exploration and conjecturing
	• Real-world context; • Assessing sample student work; • Using task models (e.g., Is this a Coincidence?); • Games (e.g., Jeopardy, Math Baseball)		

We were encouraged to see the PSTs' efforts to creatively incorporate multiple pedagogical techniques for addressing a range of mathematical content at various grade levels. Other common features of the lesson plans, not reflected in Table 1, were due to the special nature of this teaching experience. One such feature is the use of PST-developed worksheets to reduce reliance on students' textbooks to which the PSTs often had no access. Second, since the lessons were designed for small groups of students, all plans embedded opportunities for students to work with their peers and share ideas. In the next section we describe how the PSTs used these and other features to focus on the proof themes.

Focus on the Proof Themes

There was substantial variation in how focused the lesson plans were on the proof themes for the four modules of the capstone course. Since a proof theme could appear in multiple parts of the lesson e.g., exposition, warm-up, some or all student tasks, we took the whole lesson plan as a unit of analysis. Based on how prevalent a proof theme was in the lesson plan, we broadly categorized each plan as having high, medium or low focus on a given proof theme. For example, for the lesson in which PSTs were asked to integrate the proof theme of quantification and the role of examples in proving, Ellen's (all names are pseudonyms) Geometry lesson contained several two-column proofs about parallel lines and vertical angles, but nothing related to the proof theme; thus, it was coded as having *low* proof theme focus.

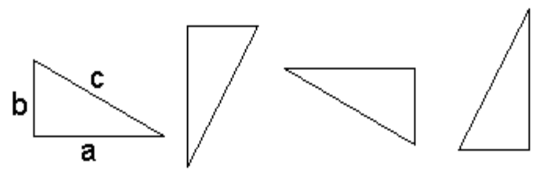
Nate's lesson plan aimed to integrate this proof theme with the topic of proportions and unit conversion in Algebra 1. Nate used a real-world context to create a problem about two investors buying land in the United States and Europe; the solution required area and money conversion to decide who got a better deal. The lesson also contained four sample arguments, each claiming that another investor got a better deal. The task for students was to evaluate these arguments, decide whether they were correct or not and justify their decisions. Nate wrote that he intended to use these explanations as counterexamples to the claims made by the imaginary students in the problem. That is, if an imaginary student made a claim that one investor got a better deal, but the students in class could refute this argument by showing that the second investor got a better deal, this would illustrate that a counterexample disproves a claim. Although it might be possible to use Nate's problem in this way, we felt unconvinced that the lesson plan was sufficiently explicit in positioning the problem in this light, hence we coded it as having *medium* focus on the proof theme.

On the contrary, Rebecca's lesson plan was categorized as *highly* focused on the proof theme. It started with exposition on what a universal statement is, and used examples outside mathematics, such as "A man who is wearing a suit and tie is attending a funeral," to explain that one needs a general proof to prove a universal statement, and a counterexample to disprove it. Next, Rebecca had students explore and develop a conjecture about types of quadrilaterals created by connecting the midpoints of the sides of another quadrilateral. The students were not required to prove their conjectures, but only to consider what information may be needed to prove or disprove it. This lesson constitutes creative and high integration of the proof theme.

Overall, 28 lessons were coded as having high focus on the proof theme, 13 as medium and 19 as low (see Table 2 below). Table 2 shows that the highest focus on proof themes occurred in lessons on conditional statements (11 out of 28) and on direct proof/argument evaluation (10 out of 28). Most lesson plans on these proof themes contained tasks engaging students in evaluating the mathematical work or arguments of imagined students, providing justification for why these arguments are true or finding and correcting mistakes in them. Another frequently used feature was the task model *Is this a coincidence?* In this type of task students are given a description of a mathematical exploration along with one or two related examples generated by an imaginary student, and an observation that he/she made based on these examples. A set of prompts, including: "Is this a coincidence?", invite students to formulate a conjecture, explore it and then prove or disprove it. Figure 2 shows Angela's task of this type.

A student said: I took four congruent triangles, with side lengths 3in, 4in, and 5in, and found that I could rearrange them in a square. I tried to do the same thing with four triangles of side lengths 6in, 7in, and 8in and I couldn't make a square.

Is this a coincidence?



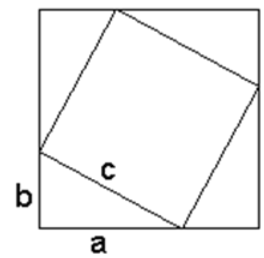


Figure 2: Angela's task using the model of "Is this a coincidence?"

The two proof themes in which the majority of lessons were coded as having low focus on the intended theme were: (a) quantification and the role of examples in proving and (b) indirect reasoning (Table 2). Despite the attempt to integrate the proof theme with the ongoing mathematical topic, in reality these lesson plans were only tangentially related to the proof themes. However, some PSTs found creative ways to incorporate proof themes in their lessons, cf. Rebecca's lesson on quantification. Another strong example is Logan's lesson on indirect reasoning. Logan created six problems on applications of Pythagorean theorem, each asking students to explain why certain measures of triangle sides cannot be true (see Figure 3 for one problem). Indirect reasoning would come into play by assuming that the ramp is 9 feet long, and using the Pythagorean theorem to calculate the length of the ramp to arrive at a contradiction.

You're working as an independent contractor and your latest client needs a ramp built at one of their properties. The client knows that the ramp must come to an elevation of three feet and that they only have enough room for the ramp to come out six feet from the wall. The client mentions that the length of the ramp's surface will be 9 feet. *Explain to the client why the length of the ramp cannot be nine feet*. Also include what the correct third measurement is for the ramp.

Figure 3: Logan's task on indirect reasoning. Emphasis added.

Although conditional statements and direct proofs are relatively common in the high school geometry curriculum, it was reassuring to see the PSTs implementing such lessons within algebra and prealgebra. We turn now to describing a cognitive demand of the proof-related tasks.

Cognitive Demand

In each lesson plan, we identified proof-related tasks and examined how cognitively demanding they were, using the framework of Silver et al. (2009, p. 511). The tasks coded as high-demand asked students to: (a) explain, describe, justify, compare or assess; (b) make decisions or choices, formulate questions or problems, (c) work with multiple representations; (d) read, comprehend or complete proofs. Tasks coded as low-demand: (a) required application of routine procedures, (b) lowered expectations or provided too much guidance making a potentially high-demand task into a routine one, (c) targeted non-challenging issues (e.g., required explanation of standard procedures). If a plan contained more than one proof-related task, the lesson plan was assigned the score of the task with the highest demand. Table 2 shows, for each proof theme, the number of the lesson plans with high, medium and low focus on that proof theme, and the cognitive demand of proof-related tasks.

Table 2. Focus on proof themes vs. cognitive demand of proof-related tasks.

Proof theme	Focus of the lesson of a proof theme			Cognitive demand of proof-related tasks	
	High	Medium	Low	High-demand	Low demand
Quantification and the role of examples	3	4	8	5	10
Conditional statements	11	1	3	5	10
Direct proof, argument evaluation	10	5	0	14	1
Indirect reasoning	4	3	8	7	8

Although we only coded *proof-related* tasks, i.e., tasks that require developing / evaluating arguments, justifying, explaining, or comparing one's mathematical work to that of others, not all tasks were highly demanding. In fact, in all proof themes, except for direct proof and argument evaluation, the number of low-demand tasks exceeded the number of high-demand. This often happened when PSTs lowered the cognitive demand of a proof-related task. For example, Audrey created a worksheet with several problems that called for identifying and correcting student mistakes. One item was: "Carly thinks that $(x^2)^4 = x^6$. Is she correct? Explain why or why not". The answer key showed that Audrey expected students to respond: "Carly is not correct because you do not add the exponents", an answer that relies on rule memorization characteristic of low-demand tasks, rather than mathematical reasoning.

Our analysis suggests that the relationship between the lesson's focus on proof themes and cognitive demand of proof-related tasks was not straightforward. While 19 of highly-demanding tasks occurred in lessons with high focus on a proof theme, and 17 of low-demanding lessons appeared in the lessons with a low proof theme focus, other combinations were also present in the data. For example, Nate's problem on unit conversion was proof-related and highly demanding, but it had only medium focus on the proof theme for which it was designed, namely, quantification and the role of examples in proving.

Discussion and Implications for Education

Our study focused on two overarching research questions:

1. What opportunities to learn about reasoning and proving, specifically about the four proof-themes, did PSTs integrate in their lesson plans?
2. How were these learning opportunities realized in the lesson plans?

We operationalized the first question by examining the ways PSTs integrated the four proof themes in their lesson plans and noting the prevalence of these proof themes in the plans. We addressed the second question by examining the pedagogical features of the lessons as a whole and the cognitive demand of the proof-related tasks.

The lesson plans encompassed a variety of mathematical topics and embedded multiple pedagogical features demonstrating that a wide range of topics can provide opportunities for introducing reasoning and proof across the grades, and that PSTs were capable of identifying and capitalizing on these opportunities in their lesson plans. The variation in the level of focus on the proof themes stems from several factors, some beyond the PSTs' control (e.g., responding to a cooperating teacher's request to devote time to exam review). Data from other sources, such as course classroom video and the PSTs' course reflections, suggest that two main reasons for low or moderate focus on proof themes were: (a) the PSTs' own doubts about feasibility of integrating proof themes in secondary mathematics, and (b) lack of experience with proof-related tasks at the secondary level. The quotes by Ethan and Laura illustrate these points, with Ethan sharing what he saw as challenging and Laura explaining how she addressed the challenge:

> It was definitely easier to implement certain proof topics compared to others. I found implementing two themes the role of examples in proving and evaluating arguments to be rather easy/less challenging and beneficial to the students. On the other hand, I found conditional statements and proof by contradiction to be challenging to teach middle school students, even if it was at the most basic level. These topics can be very difficult to grasp so finding a way to relate them to exponents or linear equations I found to be challenging. (Ethan)

> At the start of this class, I believed that proofs were only appropriate in geometry classrooms, or in proving Calculus theorems. However, I was tasked with teaching a geometry class, a pre-algebra class, and two Algebra 1 classes. I found that if you focus on the kinds of reasoning involved in different proof-themes, and if you don't overwhelm students by attempting formal proof right away, the four proof-themes could easily be applied to any mathematics topic. (Laura)

As instructors, we invested a considerable amount of course time and efforts to get the PSTs on board with the idea that all students are capable of doing proof-related tasks and can benefit from them. Some of this included providing examples of pedagogical features, such as assessing sample student work or proof task models to inspire PSTs' creativity. Our data suggest that PSTs could benefit from greater exposure to examples of successful integration of proof themes with mathematics instruction. We plan to use the current sample of lesson plans as a pool of examples on how this can be achieved. Another critical point that came up in our data is the cognitive demand of proof-related tasks. We found it somewhat surprising that inclusion of a proof theme in a lesson plan did not automatically translate to highly demanding proof-related task. We intend to address this in the next iteration of the course by having the PSTs assess cognitive demand of their own tasks and the tasks of their peers, to increase their awareness of different learning opportunities in tasks with high vs. low cognitive demand.

We conclude this paper by noting that there is a long way between creating lesson plans that integrate reasoning and proving in secondary mathematics as a course assignment and being able to identify opportunities to integrate proving in mathematics instruction as a part of one's regular teaching practice. We hope that our course helped the PSTs to make an important step in this direction, as the following quote from Angela's reflection suggests:

> So while the task of incorporating the proof themes into our lessons was challenging, it was also very eye-opening into the multitude of ways that higher-level mathematics topics can be brought into lower level subjects and it is something I want to continue to try and do in my own practice.

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Investigating STEM Students' Measurement Schemes with a Units Coordination Lens

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Measurement is a foundational concept in all STEM fields. Difficulties with measurement and converting between units of measure have been documented in medical students, chemistry students, and mathematics students at varieties of educational levels. However, less is known about why this topic is so difficult and what mental operations are entailed in mastering it. Steffe (2012) argued that students must assimilate situations with three levels of units to understand measurement conversions so we attend to students' units coordination schemes while remaining open to other factors impacting students' responses to measurement tasks. We found that the STEM majors in our sample who assimilated tasks with two levels of units had more difficulty with measurement tasks than those who assimilated tasks with three levels of units.

Keywords: Measurement, Units Coordination, Quantitative Reasoning, STEM Majors

Research on quantitative reasoning is an important area of Research in Undergraduate Mathematics Education (Thompson, 2012). Thompson (2012) defined quantification as the mental process of conceiving of some aspect of an object as measurable and understanding that the measure of the object is some multiple of the chosen unit of measure. Steffe presented a conceptual analysis of the cognitive foundations of quantitative reasoning and measurement by building on his research into children's coordination of partitions and iterations of multiple units (Steffe, 2013). We use Steffe's units coordination constructs to understand students' thinking about measurement. Given students' difficulties with measurement it is important to understand the conceptual roots of the issues. Thus our research question is:

How are STEM majors' units coordination structures related to their understanding of measurement?

Literature Review

Measurement and conversions that are fundamental in many STEM courses (e.g., DeLorenzo, 1994; Saitta et al., 2011; Scott, 2012). However, there is evidence that these ideas are poorly understood. Large samples of university calculus students and secondary mathematics teachers found it difficult to convert between liters and gallons given a conversion factor (Thompson, Carlson, Byerley & Hatfield, 2013; Byerley & Thompson, 2017; Byerley, 2016). Difficulties with measurement are also common in doctors with medical degrees. For example, in one study there were 55 medication errors per 100 patients admitted with 28% of those errors related to prescribing appropriate doses of medicine (Kaushal et. al., 2001). Chemistry students struggle to interpret what it means to perform dimensional analysis. This has driven many to investigate more effective methods of teaching this technique; for instance, by including descriptive words with calculations (DeLorenzo, 1994), having students work collaboratively with manipulatives (Saitta, Gittings, & Geiger, 2011), and using interactive software that shows the sizes of units (Ellis, 2013). Chemistry educators debate teaching dimensional analysis as rote procedure vs deliberately scaffolded logic and reason (DeLorenzo, 1994).

Less is known about why measurement is so difficult. One hypothesis is that many students do not assimilate situations with three levels of units when they are asked to make sense of measurement in elementary school (Steffe, 2013). Smith and Barrett (2017) conjecture that part of the difficulty might be the way measurement is taught, and the lack of focus on the underlying structures of various measurement situations.

> [We] found it striking how often the same conceptual principles and associated learning challenges appear in the measurement of different quantities… Despite the clear focus in research on equipartitioning, units and their iteration, units and subunits… curricula (and arguably most classroom teaching) focus students' attention on particular quantities and the correct use of tools, as if each was a new topic and challenge (p. 377).

Our study investigates STEM majors' units coordination schemes and their measurement schemes to describe the conceptual structures needed to understand measurement.

Theoretical Perspective

Steffe (2013) posited that students need to be able to assimilate situations with *three levels of units* to make sense of measurement situations where one quantity is measured with more than one unit. He also explained how the ability to assimilate situations with *two levels of units* is related to being able to construct a measure of one quantity. Steffe and colleagues came to these conclusions based on teaching experiments with mostly K-8 students (Steffe & Olive, 2010) and their hypotheses have not been investigated with undergraduate students.

Units Coordinating

Units coordinating is described as "students' ability to create units and maintain their relationships with other units they contain or constitute" (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015). Units coordinating is foundational for the construction of early whole-number concepts, such as *n* 1s being equivalent to one *n* (Steffe & Cobb, 1988). Units coordinating is a useful construct for understanding students' fractions conceptions. To understand the fraction m/n as a number, one must understand m/n as commensurate with m $1/n$ths, n of which are commensurate with 1 (Hackenberg, 2010). In the case of $m > n$, the meaning of $1/n$ must transform from thinking of $1/n$ as one out of n total pieces to thinking of $1/n$ as an amount that could be *iterated* more than n times without changing its relationship with the size of 1 (Steffe & Olive, 2010; Tzur, 1999). This involves coordinating three levels of nested units: 7/3 is 7 times (1/3), 1=3/3 is 3 times (1/3), thus a 7/3 unit contains both a unit of 1 and a unit of 1/3 within 1 (see Figure 1). Students thinking this way about fractions are said to have constructed an *iterative fraction scheme* (Steffe & Olive, 2010).

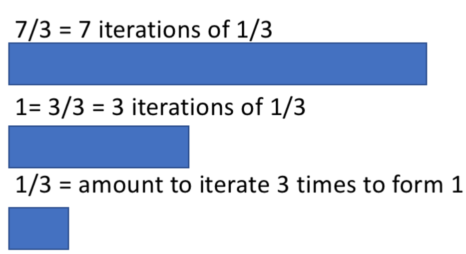

Figure 1. Three level structure for 7/3

Students coordinating with fewer levels of fractional units may construct measurement conceptions of fractions limited to proper fractions (i.e., *partitive* fraction schemes) or be limited to conceptions of fractions disconnected from measurement (i.e., *part-whole* fraction schemes; Steffe & Olive, 2010). Students who can assimilate with two levels of units can often correctly solve tasks that have a three-part unit structure if they are able to use manipulatives or images. We say these students can coordinate three levels of units in activity, but do not assimilate tasks to a three-part structure that they have already constructed mentally. In other words, although the two-level students do not simultaneously keep track of three units and their relationships in their mind they can cope with this three-part structure using tools and correctly solve many problems.

Reciprocal Reasoning

Construction of an iterative fraction scheme is necessary for *reciprocal reasoning,* which has connections to students' reasoning in school algebra (Hackenberg & Lee, 2015) as well as measurement. To construct reciprocal reasoning, students must abstract a structure for their coordination of three levels of fractional units that can apply more generally to *unknown* units (Hackenberg and Lee, 2015, p. 226). For instance, consider the equation y = 7/3 x. A student employing reciprocal reasoning may reverse the multiplicative relationship, to obtain x = 3/7 y, by understanding that each ⅓ of x is 1/7 of y, so 3/3 of x is 3/7 of y (Hackenberg, 2010). Reciprocal reasoning is one form of reversible multiplicative reasoning - a student may instead reverse a multiplicative relationship by reasoning about reversing whole number arithmetic operations (e.g., by multiplying 3 and dividing by 7). This ostensibly yields the same result, but it is disconnected from the multiplicative relationship between the *x* and *y.*

Methods

We recruited eight calculus students from two universities by visiting calculus courses and asking for volunteers. We interviewed all students who volunteered. Six students were enrolled in Calculus II at one university and two students were enrolled in Calculus I for Biologists at the second university. Each student was interviewed individually for approximately one hour by one of the authors and answered units coordination and measurement tasks. We report on three students whose reasoning illustrates trends we saw in all interviews.

The interview protocol included seven units coordination items developed and validated by Norton et al. (2015) for assessment of middle school students' reasoning. We chose these tasks because there was guidance from prior research on how to use them to diagnose units coordination structures. The most difficult task in the assessment is shown in Figure 2.

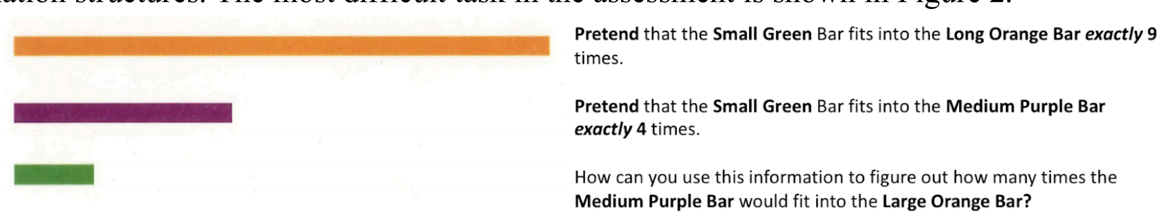

Figure 2. The task "Measuring Bars" from Norton et.al. 2015.

The liters to gallons conversion task was developed for secondary mathematics teachers (Byerley & Thompson, 2017). We knew this task was very difficult to solve correctly based on prior research, but did not know what made the task so challenging for people with math degrees.

A container has a volume of m liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Figure 3: The task "Liters to Gallons" from Byerley and Thompson, 2017. © Arizona Board of Regents 2015.

The other measurement items came from assessments and worksheets in the first author's Calculus for Biologists course. These included drawing a ruler with both centimeters and inches on it, determining the number of square centimeters in one square inch, and doing unit conversions with fictional units given a conversion factor: A Mump is 7/3 times as large as a Tog. We chose these tasks because we knew they were difficult but did not know why.

Our research team watched video recordings of each interview and made initial notes about how students responded to the units coordination and measurement tasks. After independently making notes summarizing each interview, we individually wrote descriptions of the students' responses to units coordination and measurement tasks. If we all determined a student assimilated tasks with three levels of units independently we felt more confident in our model of that students' thinking. We shared our notes and discussed differences in our interpretations, using the discussion as a chance to identify and test multiple conjectures that could explain the students' activities.

Results

We will compare and contrast our interpretations of three students' units coordination and measurement schemes. Students 1 and 3 were independently categorized by all team members as assimilating situations with three levels of units. Student 2 was categorized by all team members as assimilating situations with two levels of units.

Students' Responses to *Measuring Bars*

The research team used students' responses to seven units coordination tasks to decide how many levels of units the student assimilated with. We will discuss the evidence from the most difficult task "Measuring Bars." It is the most difficult because unlike the other tasks the answer is not a whole number. Table 1 summarizes features of each students' response.

Table 1. Summary of three students' responses to the Measuring Bars Task

Student	Answer or Answers	Time to Giving Correct Answer	Needs an image to produce answer?	Number of levels assimilated?
Student 1	9/4	42 sec	No.	Three
Student 2	2 1/9 then 2 1/4	4 min 15 sec	Yes.	Two
Student 3	9/4	49 sec	No.	Three

Student 1. Student 1 correctly answered all of the tasks on Norton et. al.'s (2015) instrument without needing supporting images, which is evidence he assimilated the situations with three levels of units. The short amount of time he took to solve the *Measuring Bars* Task suggests he was able to assimilate the task to his existing three level unit structure. We also considered other evidence of his use of a three-level unit structure in his strategies for partitioning bars. For example, when partitioning a bar into 6 equally sized pieces (the fourth bar task) the student first partitioned the bar into three equally sized pieces, then partitioned each of

those pieces into two equally sized pieces (he used a similar strategy to make 12 inches on the ruler task: split the ruler in half, each half in half, each quarter into thirds).

Student 2. Student two was able to coordinate three levels of units in activity with the aid of pictures and repeated prompting but did not assimilate tasks with three levels of units. When faced with tasks involving improper fractions, she expressed a preference of converting them to decimals.

Unlike Student 1, Student 2 identified that she could not solve the *Measuring Bars* task (Figure 2) without drawing a picture. Even with the support of the picture she did not keep in mind relationships between three quantities. Student 2 answered "two and one out of nine." She was fairly confident in her answer of 2 1/9 but also considered "two and one out of four" before choosing 2 1/9. The interviewer told her that one green bar is one ninth of an orange bar and asked her what fraction one green is of a purple bar. Student 2 determined correctly the green bar is one fourth of the purple bar but then reconfirmed "so I think my answer should be 2 and one ninth." The conversation continued:

I: How did you decide you should write that fraction in terms of the size of the orange versus the size of a green or a purple?"

S: Like you said, it got me thinking, that makes sense, because this whole one is a green one, and when we look at it in terms of orange it is just one ninth of an orange, the question is asking to answer in terms of the long orange bar so I decided it would be one ninth.

I: Does this to you also refer to long orange bars. [points to the 2 in the answer 2 1/9.]

S: That refers to how many purple fits into the long orange bar. So it would be two purples and an extra of the green. [student laughs]

I: Okay. And the extra green is one fourth of one purple.

S: Oh. [sense of realization]

I: So this answer is correct in the sense that you mean two purples and one…[gets cut off]

S: one ninth of a green.

I: [corrects student] one ninth of an orange.

The conversation continued until Student 2 decided to change her answer to 2 ¼ (the intended answer). Student 2 had difficulty keeping track of three units in her mind as evidenced by calling a green square both one ninth of a green and one ninth of an orange. She also does not remember her measure of two is in terms of the purple unit when she determines the size of the leftover green piece. Steffe hypothesized that constructing an iterative fraction scheme to understand nine fourths requires assimilating the situation with three levels of units. In this case understanding that the green is one fourth of the purple while at the same time thinking of the orange as nine copies of the green.

The interviewer asked the student if the answer of 2 1/4 was related to the nine and four given in the problem statement. She replied:

> Ummm... I think it is related to the nine? [questioning tone.] Ummm... I would usually check my work using like a calculator because I'm not really good with fractions. I don't usually do fractions, I would put it into decimals. So I guess like two point two five would fit into nine... [pause of six seconds to compute.] like four times. So that would be four times two point one four to get the nine.

This excerpt provides evidence that Student 2 does not have an iterative fraction scheme. Student 2 was not aware that 9/4 was the same number as 2 ¼, as indicated by the multiple pauses and computations the student made when asked how 9 and 4 were related to her answer of 2 ¼.

Student 3. This student was able to answer units coordination tasks correctly before drawing any pictures, but sometimes would make units-related errors when discussing his reasoning (e.g. mixing up number of purple and green bars). His response to *Measuring Bars* was distinctly different than Student 2's and demonstrates the student likely had a meaning for division as producing a measure of two quantities and is comfortable with fractions like 9/4. His ability to answer *Measuring Bars* quickly without an image suggests he assimilated the task to an existing three-unit mental structure in his mind. He explained his answer of 9/4:

> Basically, the small green bar into purple is four, uh, the green bar into the full thing is nine, so if I take the full thing and I want to know how many of these there is. I'm basically just using green as units, so it's like the full bar of greens is 9, the purple's size is 4, 9 divided by 4, basically using it as the smallest unit. [Points to the greens on his diagram.] These are the fourths because they are the greens and there are 9 of them.

Student 3's language describing the orange as nine copies of the fourths is consistent with an iterative fraction scheme which students typically construct after assimilating tasks with three levels of units (Steffe & Olive, 2010).

Students' Responses to Measurement Tasks.

Student 1. Student 1 had the strongest measurement schemes in the group of eight students interviewed this summer. He told the interviewer he had not previously seen many of the measurement questions but was able to figure them out fluently without help from the interviewer. For example, Student 1 did not know that there are 2.54 cm in one inch, but given that information by the interviewer he was able to draw an essentially flawless representation of a ruler using a straightedge. He attended to making sure that the 2 inch mark was lined up with 5.08 cm mark and that the 12 inch mark was lined up with the 30.48. He was the only student of eight to correctly answer the *Liters to Gallons* conversion task, which is known to be hard for secondary math teachers (Byerley & Thompson, 2017). He utilized reciprocal reasoning to express x liters as 50/189 x gallons, but keeping track of the distinction between the number of liters (x) and the size of a liter (some agreed upon amount of volume) was non-trivial for him. He reread the prompt four times to make sense of it and spent a few minutes contemplating his answer before feeling confident.

Student 2. Student 2 expressed apprehension about drawing a ruler with centimeters and inches on it despite having memorized that 2.54 centimeters equals one inch. She first drew a picture of a ruler with inches on it. Unlike the students in the interviews who assimilated tasks with three levels of units, she did not partition a partition to form twelve equal parts, and none of her inches ended up the same size. The following excerpt shows that despite the interviewer's attempts to orientate her and help her understand the question she did not come up with a plan for drawing the centimeter side of the ruler.

S: So that would be 2.54.[marks 2.54 cm across ruler from 1 inch]. I don't know. I don't know how like proportionate it should be....

I: By proportionate do you mean that the lines don't usually line up?

S: Yeah. But the big lines do, but like there is like small lines in between that don't. I don't really know what the cm side should look like as I never really use that side. We use the conversion but don't use a ruler to look at it.

I: If you were trying to fit… I'm going to draw it bigger so it is easier to look at….if this is one inch and this is two inches...and then you were trying to put centimeters over here would you be able to? And I agree at one inch you get 2.54 cm, but usually on rulers what they do is that they put whole number values of centimeters. They put whole number values they don't put decimals. Does that make sense?

S: Yeah. I don't think I'll be able to draw the centimeter side in whole numbers, I just know the conversion.

On the *Liters to Gallons* task Student 2 knew that there should be more liters than gallons in a given container but struggled to respond to the question for a variety of reasons. One of her repeated difficulties was choosing between a meaning of x as a number of liters and a meaning of x as the size of one liter. It might be that differentiating between the number of copies of a unit and the size of a unit while also attending to a second unit of measure requires assimilating the task with three level of units. She did not express awareness of the reciprocal relationship between the relative size of units of measure and the measurement of a container. Thus she knew there should be fewer gallons in the container, but did not know how to find the number of gallons precisely. This is consistent with prior observations that assimilating situations with three levels of units is important for development of reciprocal reasoning.

Student 3. Student 3 drew a ruler correctly, and his ruler drawing and other work showed that he understood that if a quantity was measured with a larger unit of measurement, the resulting measure was smaller. However, Student 3 answered the *Liters to Gallons* task incorrectly with the expression (189/50) x. He did not distinguish between the number of liters (x) and the size of a liter in his response. Based on his answers to other questions he seemed to have the unit structures he needed to answer the task, but he did not consider the meaning of x in his expressions and so did not notice it stood for two different ideas. When asked how many mumps were in a tog, Student 3 correctly answered 3/7, but he questioned himself, stating, "You're asking how many mumps are in the tog, so how many big are in the small. So, it'll be a fraction. I'm saying 3/7." There is evidence of reversible multiplicative reasoning in Student 3's immediately attributing the reciprocal of 7/3 to how many mumps are in a tog. But Student 3's pausing and consideration of generic relative sizes ("how many big are in the small") before settling on 3/7, together with his incorrect response to the *Liters to Gallons* task, suggests his reciprocal reasoning involving unknown quantities was not well-established.

Conclusions

This evidence suggests that development of fraction schemes and units coordination structures described by Steffe and colleagues to model children's reasoning is useful for understanding adults' measurement schemes. As his theory predicted, the calculus students who assimilated situations with two levels of units had not constructed productive measurement schemes. Student 2 had developed many strategies (such as dimensional analysis) for understanding problems without needing to assimilate them with three levels of units. However, some of her strategies, such as converting fractions to decimals, made it more difficult for her to make useful observations about reciprocal relationships and were detrimental to her conceptual understanding of unit conversions. It is much easier to see the reciprocal relationship between 50/189 and 189/50 when the numbers are left as fractions. When student 2 converted all fractions to decimals it often made it much harder for her to generalize important aspects of the problem. The example of Student 3 shows that assimilating tasks with three levels of units is not enough to make sense of *Liters to Gallons* without help. Even students with strong units coordination schemes and strong measurement schemes, like Student 1, may still find *Liters to Gallons* difficult. Across our sample of eight, students' units coordination structures are related to their ability to reason about measurement in non-routine ways. Assimilating tasks with three levels of units appears to be necessary, but not sufficient to understand a variety of measurement tasks.

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Defining the Varied Structures of Tutoring Centers: Laying a Foundation for Future Research

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The creators and leaders of mathematics tutoring centers at universities make many choices about the organizational structure of their centers. Some of those choices include the location of the center, the education level of the tutors, the method of tutor training, the number of hours tutoring is available, and the way tutoring is provided (i.e. drop in or scheduled). Our group's long-term goal is to provide research-based evidence to help faculty and administrators choose effective structures for centers. This paper documents similarities and differences between centers to provide a descriptive foundation for future hypothesis generation and testing.

Keywords: tutoring centers, organizational structure, definition of constructs

While most would recognize that mathematics tutoring centers (henceforth called math centers or centers) are units associated with post-secondary institutions whose purposes are to aid students studying mathematics, little is actually known about their general characteristics. Whetten, Felin, and King (2009) insist that organizational research should begin with context sensitivity that acknowledges and accounts for the relevant conditions of the entities to be studied. For that reason, we began this study with a series of questions: What are the structural similarities and differences of math centers? What are useful ways to define organizational structures of math centers?

Our group's long-term goal is to generate and test hypotheses about the effectiveness of competing organizational structures. The three stages of this research are:

1. Describe the organizational structures of various tutoring centers.
2. Compare qualitative and quantitative measures of success from various centers and develop testable hypotheses about the choices that impacted the success of these centers.
3. Design research methods to specifically test hypotheses generated in stage two.

This paper addresses the first stage. We draw on our experiences as tutor center leaders to identify, define, and document significant organizational structures of various tutoring centers.

Theoretical Perspective

Research on organizational identity suggests that identity is a construct formed from comparisons. Gioa, Patvardhan, Hamilton, and Corley (2013) suggest that some of the first stages of organizational identity formation process should involve considering contrasts and converging on a consensual identity. Albert and Whetten (1985) put forth that "organizational identity is formed by a process of ordered inter-organizational comparisons and reflections upon them over time" (p. 273). While theories such as these are more common in management fields, for educational researchers this might be reminiscent of variation theory, which suggests a concept is understood when its critical features are acknowledged, and the means to deem features as critical is discerned only through experienced variation (Runesson, 2006). To determine the organizational identity that is associated with math centers we look at the central, stable features of math centers that make them distinctive (Gioa, Patvardhan, Hamilton, & Corley, 2013). We looked outside of mathematics education for theoretical guidance on how to study organizational structures because this is not a common topic in research in undergraduate mathematics education. We narrowed in on the idea of defining identity by making comparisons between organizations for the first stage of the research.

Literature Review

Empirical investigation of mathematics tutoring and tutoring centers at the undergraduate level is still in the beginning stages. *Characteristics of Successful Programs in College Calculus* reported that undergraduate tutoring is commonplace: 97% of the 105 institutions surveyed had a tutoring center for students to receive help for calculus, and 89% of the institutions offered tutoring by undergraduate students (Bressoud, Mesa, & Rasmussen, 2015). Several quantitative studies indicate that visiting mathematics tutoring centers is correlated with higher final grades (Byerley & Rickard, 2018; Rickard & Mills, 2018; Xu, Hartman, Uribe & Menke, 2014). Each of these studies focused on a single institution, and the metrics for success are limited to final grades, rather than other indicators of success such as persistence through a STEM major. Matthews, Croft, Lawson, and Waller (2013) reviewed the literature concerning tutoring center effectiveness and found a wide diversity in the metrics that determine success, such as grades, retention rates, frequency of repeated visitors, and student reports of confidence and motivation. They recommend further investigation into "what constitutes effective delivery of mathematics support" (p. 23). Due to the small number of studies on tutoring centers we feel confident that no studies have compared centers with contrasting organizational structures to make hypotheses about structures of effective centers.

To explain the success of tutoring, a number of studies have examined the intricacies of tutor-student interactions. Research indicates effective tutoring commonly includes active inquiry and self-explanations on the part of the student (Chi, 1996; Lepper and Wolverton, 2002; Topping, 2005; Van Lehn, 2011), and appropriate questioning and responsive scaffolding on the part of the tutor (Graesser et al., 2011; Roscoe & Chi, 2007; Topping, 1996). While these findings are helpful in guiding productive tutoring, they were not conducted in a mathematics tutoring context, nor do they offer significant insight into the structural organization of a tutoring center. Solomon, Croft, & Lawson (2010) is one of the few studies to describe the impact of a physical space on the climate of a math center.

Methods

We will compare structures that differ between tutoring centers by drawing on our experiences with tutoring centers. All six authors of this paper are actively involved in their university's math center, attend a national conference for tutor center leaders, participate in weekly or monthly online meetings with other tutor center directors, and lead or attend tutoring center working groups at the RUME conference. Our understanding of tutoring center structures is built on our frequent interaction with our universities' tutoring centers, notes from conferences and online meetings, and a shared digital resource library.

Multiple center leaders wrote descriptions of various aspects of their centers such as tutor selection, tutor training, center hours, classes tutored, numbers of students served, description of physical space, and description of relationship with the mathematics department. We analyzed these documents to create and refine definitions of organizational structures. Of course, there are other differences between our centers that did not emerge in our conversations and writing. This methodology relies on the expertise of tutor center leaders to choose what they believe are the most important contrasts between centers. We make no claims that the structures we define will turn out to be the most important once more formal study is conducted.

Descriptions of Different Tutoring Center Structures

In the following section, we describe six significant dimensions of undergraduate mathematics tutoring centers: (1) Specialist versus Generalist Math Tutor Models, (2) Strength of Relationship between Center and Math Instructors, (3) Type and Extent of Tutor Training, (4) Types of Tutoring Services, (5) Physical Layout and Location, and (6) Tutoring Capacity.

Specialist versus Generalist Math Tutor Models

A specialist math tutor is assigned to tutor for one course. This tutor helps numerous students with the same course and becomes familiar with the homework problems, student mistakes, homework solutions, the syllabus, and expectations for testing. Ideally the tutor would communicate with at least one instructor of the course to give feedback on students' experiences and to ask for any clarification needed. Sometimes specialized tutors also serve as Learning Assistants in the course (see Goretzen et. al., 2011 for definition of Learning Assistant). Often Learning Assistants attend a course and assist faculty with in class group work and hold mentor groups outside of class. Typically, specialist tutors are not available during all times the center is open and students must attend the center when tutors are available for their course.

A generalist math tutor is someone who tutors for many or all of the courses the center serves that the tutor understands. When students come to the center while it is open they have a reasonable expectation that someone will be there to help them. A generalist tutor should be able to respond to questions about all or most of the courses served by the center. Students typically ask tutors some of the harder questions in the homework and it is difficult even for experienced tutors with advanced degrees to answer questions on the spot in courses they took semesters ago. Compared to specialist tutors generalist tutors will spend more time solving the problem and be more likely to use a textbook or other resources. We are not claiming that tutors needing to use resources to solve the problem is negative. Perhaps seeing tutors model how to solve unknown problems is better for students that seeing tutors who know the assignment and answers intimately. Generalist tutors are less likely to understand the scope of the course or the particular procedures the instructor is assessing, especially for courses like College Algebra the tutors typically took in high school.

Strength of Relationship between Center and Mathematics Instructors

There are a variety of characteristics of tutoring centers we identified as having strong relationships with the mathematics instructors at their university. It is unknown if the effectiveness of a tutoring center is related to the strength of its connection with the mathematics instructors. It might be that having strong connections with student services or centers for teaching and learning are more important predictors of effectiveness.

Course Coordinators Collaborate with Tutor Center Leaders. At some centers the leaders interact frequently with course coordinators. The center leaders might be math faculty who are also course coordinators or instructors. An example of collaboration is a course coordinator that offers extra credit to students for completing a task at the center and the center records this information. Other center directors do not teach math or communicate frequently with the course coordinators or instructors. If a center relied on the mathematics faculty occasionally to provide recommendations for tutors or to provide course syllabi the center could still be categorized as having minimal collaboration with course coordinators.

Instructors of Courses Hold Office Hours in the Center. Some department chairs request that instructors hold office hours in the center. Other centers are only staffed by undergraduates or a mixture of undergraduate and graduate students. We believe that when instructors are tutors in the center there is more potential for dialogue between instructors and other tutors.

Tutors Interact Frequently with Course Instructors. In some of the universities with specialist tutoring models, the tutors attend the courses they tutor for as Learning Assistants who help with group work. Generalist tutors might also interact frequently with course instructors if the instructors also tutor in the center or the tutor center leaders are also course instructors.

Type and Extent of Tutor Training

In our centers, undergraduate tutors are the most likely to receive training and graduate student tutors are the second most likely. In our centers, faculty do not receive tutoring training.

Content Training. Content training is focused on refreshing and deepening the tutors' knowledge of the content of the classes they are responsible for. Examples of content training that exist at our centers are asking tutors to read the book according to the posted schedule or asking tutors to complete homework problems focused on relevant material.

Pedagogical Training. Pedagogical topics include how to help the student use resources to solve a problem, how to report students in crisis, how to ask good questions, how to motivate students, how to teach study strategies, how to respond to complaints about instructors, etc. Pedagogical training varies between centers because of the variations in the philosophy of tutoring between center leaders.

Mathematical Knowledge for Tutoring. We suspect that effective tutors draw upon more than content knowledge and pedagogical knowledge and have developed additional insight into learning mathematics. We speculate that the construct mathematical knowledge for tutoring is not identical to the construct mathematical knowledge for teaching (Thompson A. and Thompson P., 1996; Hill, Ball, & Schilling, 2008) but has some similarities. There are no known programs to develop mathematical knowledge for tutoring, but tutor center leaders report that they try to help tutors understand this issue sporadically. For example, some center leaders analyze student work with tutors and help them generate hypothesis about student thinking.

Time Spent on Training. Training time includes meetings between tutors and center leaders focused on improving content or pedagogical knowledge. Most centers who provide training do more training in the first semester of the tutor's job. Although tutors might learn from experiences such as attending class to facilitate group work, we do not count this as training.

Types of Tutoring Services

Some centers focus on a particular type of mathematics, such as calculus, and only serve a few courses and are typically housed in smaller locations. Other centers serve upwards of twenty different courses ranging from developmental mathematics to linear algebra and are typically housed in much larger spaces. One advantage of having large centers that serve the majority of courses is that the university can put one person in charge of managing the center. If smaller centers serve restricted clumps of courses, the university might need more people to manage the centers. We wonder if smaller centers develop different cultures than larger centers serving many courses. Additionally, some centers offer drop in tutoring, others offer scheduled one-on-one tutoring, and others offer a combination of services. A potential benefit of drop in tutoring is that some students work together and make study friends at the center. A downside of drop in tutoring is that some students complain of waiting too long for help and not having enough time with a tutor.

Physical Layout and Location

Oklahoma State's center is housed in beautiful rooms with huge windows, ample natural light and expansive views of campus while other tutoring centers have no windows. Some centers do not have enough chairs for students during busy times, and students choose to either sit on the floor or leave the center after evaluating the crowd. The ceiling height and ventilation differs at centers leading some students to complain of stuffiness or smell. In addition to wide variations in the quality of the centers' spaces, there are variations in the center's location on campus and how far the students typically must travel to attend the center. Some centers offer tutoring services in other locations. For example, University of Oklahoma offers tutoring in one of the largest dorms in the evenings before a coordinated exam.

Tutoring Capacity

Our centers have wide variation in the number of tutor hours available per eligible student. We propose multiple metrics to evaluate the availability of tutors. First, we define *tutor hours* to mean the sum of all the hours tutors are employed. One metric is the number of tutor hours per student eligible to use the center. We consider a student eligible to use the center if they are enrolled in a course the center serves at the end of the semester. Another metric is the number of tutor hours per student visit. This metric takes into account the wide variation in the percentage of eligible students who use a center at a particular university. Some universities have multiple options for tutoring and so a particular center needs fewer tutors to satisfy demand. A third metric that is harder to track, but available at some universities, is the number of tutor hours per student hour spent at the center. Although these metrics are relatively easy to compute they do not capture the number of tutors per student at peak hours before tests and before homework is due. A potential solution is to use electronic queueing systems and record the time between when a student asked for help and when the tutor responded to their request.

Structural Organization of Selected Math Centers

Table 1 compares two distinct centers that serve students at large state schools. In Fall 2017 Oklahoma State has an average of 6.9 visits for each eligible student and Ohio State has an average of 1.6 visits per eligible student. There are so many variations between the two centers and student bodies it is difficult to hypothesize why one center is used more frequently. Is the quality of the space, a connection to math, or something else?

We are in the process of describing approximately 14 centers using definitions offered here and then looking for patterns in measures of effectiveness that might be related to structural choices. We plan to use Table 1 to define the organizational structures of each center. Some aspects of the table were suggested by the literature. For example, usage is one commonly reported measure of effectiveness of a center (Matthews et. al., 2013). The strength of correlation between the number of visits to a center and the student's grade is another measure of effectiveness (Rickard & Mills, 2018). By comparing data from many centers we hope to create hypothesis about shared components of the most effective centers. After creating hypotheses we can do targeted data collection and surveys designed to evaluate the most important features of successful centers

Table 1. Characteristics of Tutoring Centers at Two Universities

	Oklahoma State	**Ohio State**
<u>Tutoring Services</u>		
Generalist or Specialist	Generalist	Tutors begin as specialists then become generalists
Drop in or Scheduled	Drop in	Drop in
Number of Courses Served	12 Math	19 Math (8 Stats)
<u>Physical Space</u>		
Location	Fifth Floor of Library	Basement and first floor of building near math dept.
Windows	Large and Plentiful	Few
Square Footage	8000	7000
Number of Chairs	266	360
Ventilation	No Complaints	Temp Regulation Issues
Computers available	130	13 but starting this year all freshman receive ipads
<u>Relationship With Instructors</u>		
Course Coordinators Collaborate with Center.	Yes	Minimal
Instructors Tutor in Center.	Yes	No
Tutors Interact with Faculty.	Yes	No
<u>Tutor Training</u>		
Content Training (UG)	5 hours per semester	3 hours per semester
Pedagogy Training (UG)	3 hours per semester	10 in first semester as tutor
Content Training (G)	0 hours	0 hours
Pedagogy Training(G)	0 hours	.25 hours

Table 2 compares measures of Tutoring Capacity at multiple centers. These measures can be used to describe capacity and will be used in the future to investigate relationships between tutoring capacity and the effectiveness of a center. We suspect that once the ratio of tutor hours per student visit becomes too small that complaints about availability of tutors will become common on the evaluation surveys. Colorado State, which has a ratio of 0.19 tutor hours per student visit, finds that approximately one third of students complain about tutor availability on their evaluation surveys. In our discussions we realized that some of the numbers are not easy to compare across universities. For example, at Colorado State a separate campus organization provides evening and weekend tutoring so it would not make sense to open the center much more than 36 hours a week. Further, Colorado State has a relatively high number of student visits per eligible student but all instructors' office hours are held at the center. At other institutions the students who seek help from instructors would not be counted as visiting the center. Multiple dimensions must be considered simultaneously and it is not possible to say that one center is more effective than another based on one line of the table.

Table 2. Measures of Tutoring Capacity at Various Centers. Data refers to Fall, 2017.

Undergraduate Institution	Students Eligible to Use Center	Location of Center	Total Student Visits	Hours per Week Center Open
Colorado State	1,148	Math Dept	7,330	36 hours
U of Arkansas	6,021	Math Dept	10,175	55 hours
Oklahoma State	4,523	Library	31,411	64 hours
U of Oklahoma	5,515	By math	22,031	33 hours
U of Portland	1,124	Commons	1,139	29 hours
Ohio State U	8,632	Math Dept	14,096	39 hours

Undergraduate Institution	Average visits per eligible student	Tutor hours per eligible student per week	Type of Tutor (Grad, UG, Faculty)	Tutor hours per student visit
Colorado State	6.38	.08	UG, G, F	.19
U of Arkansas	1.69	.03	G, F	.29
Oklahoma State	6.9	.11	UG, G, F	.24
U of Oklahoma	4	.08	UG, G	.30
U of Portland	1.01	.04	UG	.56
Ohio State U	1.632	.06	UG, G	.53

Limitations and Conclusions

We believe our collective experiences are adequate to offer definitions of many tutoring center structures in use in the United States. This paper contributes to the growing work on tutoring centers by offering shared definitions that researchers can adopt in their work. This paper does not list all of the differences in tutoring centers. For example, each center has a different budget and different restrictions on how the money can be used. Further we recognize that the metrics identified vary for many reasons. Some reasons are connected to the organizational structure of the center and some reasons are beyond the control of center leaders. For example, some universities with low numbers of visits per student have many other tutoring options for students. On the other hand, it seems logical that well-advertised and helpful centers might have higher number of visits per eligible students than centers with less effective organizational structures. As we proceed in identifying ways to measure effectiveness and ways to define centers we will have to continue to grapple with these issues. It will take a lot of reflective consideration to identify effective organizational structures without inappropriately concluding that a lower score on a metric is caused by a structural decision made at the center.

The creation and testing of hypothesis about the effectiveness of various structures will happen later and will involve the analysis of data about students visits to the center, the students' grades, and the students' demographic information. Data collection will also include student surveys about their experiences at the center. The survey questions will be designed to test hypothesis coming out of exploratory data analysis. We welcome participation in our project from other tutor center leaders, and offer these definitions as a starting point for those seeking to define their centers' identity. Please feel free to contact the authors to become involved.

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The Interaction Between a Teacher's Mathematical Conceptions and Instructional Practices

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This study raises questions about a common assumption that an advanced degree in mathematics is sufficient for teaching courses in undergraduate mathematics meaningfully. The study reports results from 24 mathematics PhD students' solutions to a precalculus level problem requiring quantitative reasoning. We also describe the PhD students' conceptions of what knowledge is needed to produce a meaningful solution to this task. These graduate students' problem solving approaches and images of the reasoning abilities needed to solve the problem were classified as having either a static calculational orientation or a dynamic conceptual orientation. We share how these two orientations are exhibited in the context of teaching precalculus students. We further illustrate ways in which a teacher's actions to support her students in conceptualizing and relating quantities led to her engaging her students in more dynamic conceptually oriented discussions.

Keywords: quantitative reasoning, mathematical knowledge for teaching, teaching practice

Mathematics departments across the nation assign incoming mathematics PhD students teaching assignments in precalculus and beginning calculus. In US mathematics departments these assignments are often made based on the students' prior coursework in mathematics and their ability to communicate clearly in English. Some mathematics departments provide teaching workshops for their incoming PhD students in mathematics. However, it is common for these workshops to focus primarily on the mechanics of teaching, with little or no focus on what the mathematics education research literature has revealed about the processes of learning or teaching ideas in the courses they are assigned to teach (Ellis, 2015). It has also been reported that graduate students in mathematics sometimes have weak understandings of fundamental ideas they are expected to teach (Musgrave & Carlson, 2016). These PhD students are typically offered little support in considering what is involved in understanding or learning the key ideas of courses they are assigned to teach; nor are they supported in determining how or whether to engage students during class, what to include in a lecture, how to assess student learning. Given the background and experiences of these new mathematics instructors it is likely that their instructional decisions and actions will be based on such things as their current conceptions of the mathematics they teach and their experiences in learning these ideas as students (Stigler & Hiebert, 1999).

In this study we investigated the mathematical approaches that incoming PhD students in mathematics used when completing a standard applied problem (see *Figure 1*) in a course in precalculus. We also probed their view of the knowledge they used to complete the problem. Subsequently two of these 24 students participated in weekly professional development aimed at supporting precalculus teachers in engaging their students in developing stronger meanings of the ideas of precalculus and improved ability to access these ideas when confronting novel problems. The analyses of these two teachers' classroom videos reveal stark differences in the teachers' images of how students' understandings develop. They also highlight instructional practices that led to students' constructing stronger meanings. The results of this study may also provide new directions for preparing mathematics PhD students for teaching.

Literature Overview and Theoretical Framing

Over 30 years ago Shulman (1986) encouraged research to pay more attention to the knowledge base that teachers need to carry out the practice of teaching. He called for increased attention on what he called pedagogical content knowledge, "the ways of representing and formulating the subject that make it comprehensible to others" and greater understanding of what makes student learning of specific topics difficult (Shulman, 1986, p. 9). Even and Tirosh (1995) further called for teachers to develop understandings of student ways of thinking and suggested that this knowledge should inform the activities they use to engage students. One such reconceptualization was the introduction of the construct of mathematical knowledge for teaching (MKT) in which pedagogical and mathematical knowledge were combined into one category (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Hill et al., 2008). Silverman and Thompson's (2008) study of teaching placed greater focus on the mathematical understandings, how they are connected, and how a teacher might spontaneously leverage these understandings when teaching. They also call for teachers to ponder how these understandings might develop in the minds of students (Silverman & Thompson, 2008, p. 500). Silverman and Thompson (2008) later proposed that developing MKT involves transforming a teacher's personal understandings of a mathematical concept to an understanding of how this understanding might be useful for students' learning of related ideas. They call for teachers to be supported in developing their images of the kinds of activities and conversations that might support another person in developing an understanding of an idea. They advocate that teachers try to envision learning the concept as a student and keep this in mind when developing activities to use with his students. By imagining scenarios from the viewpoint of a student, a teacher is better prepared to guide and direct conversations with his students.

Other lines of inquiry into teaching have considered what teachers pay attention to and how they respond to student utterances in the context of teaching. Some of these constructs include calculational orientation, teacher noticing, and decentering. Thompson, Philipp, Thompson, and Boyd (1994) characterized and contrasted a teacher exhibiting a calculational orientation when interacting with students with one who exhibited a conceptual orientation when conversing with students about their approach to working an applied problem in a 7^{th} grade classroom. They illustrate questions posed by a teacher that is oriented more toward helping students understand why an approach works (e.g., can you explain why that calculation makes sense?), and contrasted these to questions (e.g., what do you do next?) asked by a teacher that was focused on students' completing the calculations to get the correct answer.

Jacobs, Lamb, and Philipp (2010) studied teaching by examining children's strategies, interpreting their understanding, and deciding how to respond on the basis of children's understanding. They call these integrated abilities "professional noticing" and claim that they enable teachers to make appropriate instructional decisions based on student thinking. Thompson (2000), Steffe (1990), and Moore and Carlson (2012) leveraged Piaget's (1955) idea of decentering to characterize the quality of teacher-student discourse. In Piaget's work on children's cognitive development, he introduced the idea of decentering to describe a child's transition from his or her egocentric thought to the capability of adopting the perspective of another. As a teacher shifts to consider a student's perspective and expressed meanings she is said to be attempting to decenter (Thompson, 2000).

Conceptualizing Quantities in a Problem Context

In recent years many researchers have found Thompson's (1990, 1994, 2011) idea of quantitative reasoning to be fundamental to working applied problems in precalculus and calculus (e.g., Engelke, 2007; Moore & Carlson, 2012). Thompson claims that when students process the words in an applied problem they should be conceptualizing the measureable attributes of objects that are described in the problem context. According to Thompson a quantity does not exist in the world; rather a quantity is constructed in the mind of an individual when she imagines measuring some quality of an object, such as a person's height or the person's distance from home as she drives to work (Thompson, 2011). A quantity's value is the numerical measurement that a quantity assumes. When the value of a quantity is static it is called a constant or fixed quantity. If the value of a quantity changes throughout a situation it is referred to as a varying quantity. A quantitative operation occurs in the mind of an individual when conceptualizing a new quantity in relation to one or more already-conceived quantities (Thompson, 2011). When one conceives of three quantities related by means of a quantitative operation, he has conceptualized a quantitative relationship. One is said to be engaging in quantitative reasoning when he is actively engaged in constructing a network of quantities and quantitative relationships (Thompson, 1988, 1990, 2011).

Context of the Study

The *Pathways to Transforming Undergraduate Mathematics Education* project supports future mathematicians (PhD students in mathematics) to develop as reflective teachers who leverage research on student learning and formative data to adapt their instructional practices. The PhD students in the program attend a 3 day workshop prior to teaching with research based instructional materials, and then attend a weekly seminar during each semester that they teach a course using these materials. The materials include cognitively scaffolded in-class investigations that engage students in quantitative and covariational reasoning as cross-cutting ways of thinking that lead to students' understanding and using the course's ideas. Detailed instructor notes and solutions illustrate both productive and unproductive student thinking relative to specific ideas.

Method

The data presented in this study is from a larger study that followed 2 PhD students from a pool of 24 incoming mathematics PhD students over the first two years of their teaching precalculus at a large public university. Upon their entering the program they and 22 other students completed 5 mathematics problems to assess their conceptions of fundamental ideas of precalculus. Two of the 24 PhD students who were assigned to teach pre-calculus were subsequently video-taped when teaching during their first 3 semesters of teaching precalculus in the context of using a research based curriculum and attending weekly professional development meeting based in research on student learning, and designed to foster growth in the instructor's mathematical conceptions of precalculus ideas and how they are learned. The written responses of 24 incoming PhD students were analyzed relative to their: (a) conceptualization of the quantities in the problem context; (b) their usage and meanings for variables, terms and expressions; (c) their image of the transformation of the box; (d) the degree to which the box's transformation influenced their image of the constrained covariation of the two varying quantities to be related. We analyzed classroom video data of two teachers during their third semester in the Pathways TUME program. The lessons analyzed for this report had a focus on conceptualizing quantities in the context of the familiar box problem (see *Figure 2*). This video data was analyzed relative to the same four criteria used to analyze the written responses. In addition we analyzed the teachers' actions (utterances, drawings, questions, etc.) to glean insights about their approaches for supporting their students in engaging in quantitative reasoning, and their conceptions of how students might acquire the ideas that were central to the lesson.

Toy Chest Problem

An 8-foot by 4-foot piece of plywood is being used to build an open-top toy chest. The chest is formed by making equal-sized square cutouts from two corners of the plywood (see *Figure 1*). We remove these squares and make three folds (illustrated as dashed lines on the figure) to form three sides of a box. We then attach the three-sided box to the wall, so that we get an open top toy chest. Define a function f that determines the volume of the toy chest (in cubic feet) in terms of the length (number of feet) of one side of the square cutout, x.

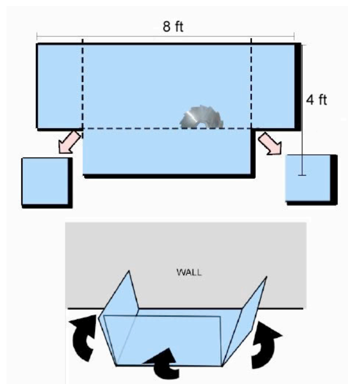

Figure 1. The toy chest problem

Results

The toy chest problem was one of the five problems that 24 PhD students in mathematics completed during an initial teaching workshop that took place during the summer prior their beginning their graduate studies. This problem asked student to define a function to determine the volume of a toy chest given the side-length of equal-sized squares that are cut from two corners of a plywood board. The problem was illustrated in a drawing with the dimensions of the plywood labeled and dashed lines indicating where the cuts could be made (see *Figure 1*).

Analysis of the responses of the 24 PhD students responding to this task revealed that only 13 of 24 of these students produced a correct response of $f(x)=(8-2x)(4-x)(x)$. The majority (7 of the 11) who produced an incorrect answer responded by writing $f(x) = (8)(4)(x)$. This response suggests that these mathematics graduate students were not imagining the sides of the toy chest varying with x, the length of the sides of the squares cut from the two corners. Instead they appeared to imagine a fixed length and width for the box, and failed to recognize how the box's length and width would vary as the value of the side-length of the squares varied. Other incorrect responses included, $f(x) = (8 - x)(4 - x)$ and $f(x) = (8 - 2x)(4 - 2x)(x)$, also suggesting that the symbols they produced were not based in an accurate image of the quantitative relationships described in the problem context.

In a follow up prompt these same PhD students were asked to describe how they would explain what it means to solve the equation $f(x) = 9$, and how they would support students in understanding what it means to evaluate $f(3)$ and solve the equation $f(x) = 9$. The PhD students' responses included: (a) one is finding x and the other is finding $f(x)$ so I would show them how to calculate these values when the other value is known; (b) solving $f(x) = 9$ using algebra might be too hard for them, but they should have no problem finding the point that has a y-value of 9; (c) when evaluating $f(3)$ you are putting 3 in for x and finding a value for the box's volume. When solving $f(x) = 9$ you are putting in a value for the box's volume and finding a value for x. The first 2 responses (typical of over half to the 24 subjects) focus on what students should do to answer the questions, with no mention of the quantities represented by the symbols or what it means to evaluate a function or solve an equation. In contrast, the third response includes references to the quantities and describes what the process of "evaluating" and "solving for" produces in terms of the quantities in the situation. A stronger response (not provided by any of the incoming PhD students) might also convey that evaluating a function for a particular value of the input quantity involves using the function rule or process to determine the corresponding value of the output quantity. Solving $f(x) = 9$ would then be described as producing a value of the input quantity x as an instance of reversing the process of f, or determining a value for the square's side-length, x, when the box's volume is known. This data provides evidence of weaknesses in these graduate students' conceptions of a function, also suggesting that the majority of these PhD students viewed a function formula as a tool for determining values.

The Teaching of Jack and Gloria

The video excerpts of Jack and Gloria are presented to contrast two teachers' conceptions of a mathematics lesson that required their students to use quantitative reasoning to relate two varying quantities. Recall that this data was collected during the third semester in which Jack and Gloria were teaching precalculus in the context of the Pathways TUME project.

Jack's conceptions operationalized during teaching. Jack began his lesson with a picture of an 8.5" by 11" sheet of paper with squares 2 inches on each side removed from each of the four corners (see *Figure 2*). He had labeled one side of one of the four squares with the label 2".

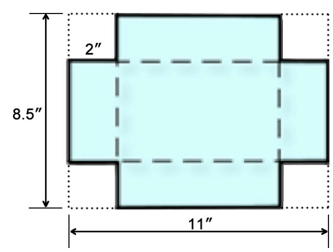

Figure 2. Jack's illustration of the box

He began his discussion of this problem by saying, "What I have drawn out on the board is an 8 and ½ by 11 inch piece of paper, out of which we have cut 2 inch squares." (Jack assumed the students understood that the squares removed from all four corners all had side lengths of 2 inches). He followed by saying, "We are going to fold the paper along these dashed lines." (The students are expected to imagine a paper being folded). He then said, "What we're interested in is the volume of this box when a 2 inch square is removed from each corner." He goes on to tell students that the volume of the box is length times width times height, but then follows by saying that "We're not going to worry why this formula works." but invites them to think about this on their own time. The lesson continues with him doing almost all of the talking while focusing on the calculations needed to determine the length, width, and height of the box that he has drawn.

Jack labeled the fixed quantities of 11" and 8.5" on his drawing (see *Figure 2*) and placed a 2 above one of the squares. It is noteworthy that he failed to make clear whether he was speaking about the square's area or the square's side length when referencing the 2. He followed by asking students what in the picture represented the length of the box. A student who appeared confused raised her hand and said in an inquiring way, "So the length would not be 11." The teacher followed by saying, “That is correct, the length will not be 11, but we may want to use 11 to determine the box’s length later on; just hold that thought.” This response suggests that the teacher was not interested in how the student was conceptualizing the quantities in the situation, rather he seemed more focused on what he wanted to say next. Jack then moved on to ask the same student what the width would be. She responded similarly in an inquiring tone, “So the width wouldn’t be 8.5 (pause), would it be 2 something?” The teacher did not respond to her question, but again points to the box’s width in his illustration on the board. He followed by calculating each dimension of the box while writing $(11-(2)(2))(8.5-2(2))(2)$ and concluded the discussion by saying, “We could use the same method to find a box with a different square. Right?”

This exchange suggests that Jack was not interested in how the student was conceptualizing his drawing. When the student asked why the box’s length was not 11” Jack took no action to support her in conceptualizing the box’s length; nor did he pose questions to support this student in visualizing how the box’s length varied with changes in the side of the square. His description of how to calculate the box’s length and width suggests that he believed that writing and saying these calculations conveyed an image of how the box’s width and the square’s side length are related. He did not appear to be interested in how he was being interpreted and did not show interest in his student’s thinking. His questions direct students’ attention to a static image of the paper and were focused more on what calculations to use to compute a volume.

Gloria’s conceptions operationalized during teaching. Gloria’s discussion of the box problem began with her providing pairs of students with an 8.5” by 11” sheet of paper, scissors, and tape. On the overhead projector were instructions to build a box by cutting four equal sized squares from the corners, and folding up the sides. As she circulated from table to table she challenged students to build a box that would hold only a small amount of popcorn and others to build a box that would hold the largest amount of popcorn possible. After the students had built their boxes she held up four boxes and asked her students to vote on which box would hold the most popcorn. Her choice to have students build the box suggests that Gloria recognized the need for students to take time to initially conceptualize quantities in the problem context and to consider how they are related. After the students had built their boxes, Gloria asked students to discuss what quantity in the situation determined each box’s shape. After a few minutes of discussion, students expressed a consensus that each box’s shape depended on the side length of the squares cut from the corners. Gloria followed by displaying a Geogebra animation she had developed prior to class (This applet allowed her to vary the side length of the squares while displaying how both the paper and box’s dimensions were transformed). Gloria began her discussion around the applet by saying, “Since we decided that the box’s shape and dimensions depend on the side length of the squares cut from the corners, let’s see what happens when we vary this quantity.” As she varied the side length steadily from 0 to its maximum value (4.25”) she asked students to describe how the box’s volume was changing. She interjected a prompt for students to explain what they were visualizing when thinking about the box’s volume. Students who responded conveyed they were visualizing such things as the amount of space inside the box and how much popcorn the box holds. She then asked her class what units they might use for measuring the box’s volume. After they discussed this with one another, she used a second applet that allowed her to vary the shape of the box, while displaying a varying number of cubes 1 inch on each side that would fit into the displayed box. As Gloria continued to vary the side length of the square cutout she prompted students to move their index finger upward from their desk to represent the box’s volume increasing and downward to represent the box’s volume decreasing. Students' first attempt to represent how the volume of the box was varying resulted in many students moving their finger upward only. Gloria called on one student to explain why she was moving her finger the way she did, and she replied that she was visualizing the height of the box getting taller and taller. Gloria asked the student to describe what attribute she was looking at when she was thinking about the box's volume. The student quickly recognized that she was paying attention to the wrong attribute of the box. Gloria again moved the side length continuously form 0 to 4.25, while all students moved their fingers upward to a point, and as the box became taller and narrower, they began to move their fingers downward until the paper folded onto itself. Gloria called on particular students to verbalize what they were imagining as they moved their fingers upward and then downward. She also asked particular students to describe the minimum and maximum values for the side length of the square that could be cut from the paper.

Gloria then had students work in their groups to complete a table to determine the value of the box's width, length, height and volume, given 4 values for the square's side length. While they were working she walked around the classroom as students completed the calculations and asked specific students to describe what their calculations represented in the context of the box's dimensions. Gloria posed a final question for students to determine an expression to represent the box's volume in terms of the side-length of the squares cut from the box's corners. While students were working she circulated around the class to ask students how they defined the independent variable and what quantity their expressions represented in the context of the box. In one case a student had written 2x – 11 for the box's side length, instead of $11 - 2x$. Gloria asked this student to point to what x represented in the context of the box, what $2x$ represented in the context of the box and what 11 represented. Once the student had done this, the student noticed that his answer did not represent anything in the context of the box and he changed his answer to $11 - 2x$, explaining that 2 side-lengths are subtracted from 11 to get the box length. Gloria's persistent attention on her students' conceptions of the quantities in the problem context suggests that she believed that quantitative reasoning would enable her students to visualize what variables, expressions and formulas represent, and to see these symbols as representing how the box's volume varied with (or was related to) the length of the side of the square x that was cut from the box's corners.

Conclusions and Discussion

The data collected from the 24 incoming PhD students suggests that these highly successful mathematics students may have some of the same impoverished meanings and ways of approaching contextual problems as what has been reported in the literature about undergraduate students in mathematics. This finding suggests that even PhD students in mathematics might benefit from professional development focused on what is involved in understanding and learning ideas that are the focus of their instruction. The teaching episodes of Jack and Gloria contrast two conceptions of what is involved in supporting students in engaging in quantitative reasoning as a means for constructing formulas that represent how quantities in a problem vary together. Jack displayed a strong tendency to focus on static relationships and computations, while Gloria focused more on understanding ideas and visualizing quantities as they varied. Jack's interactions further reveal that he had little interest in understanding the meanings his students were constructing, while Gloria was regularly concerned with how students were conceptualizing a situation or representing a quantity. Her strong orientation toward her students' thinking and her actions to support students in constructing productive meanings led to many instances in which her interactions with her students led to advancements in their thinking.

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Revisiting Graduate Teaching Assistant Instructor Expertise and Algebra Performance of College Students

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This longitudinal study revisits a decade old study about the relationship between level of Graduate Teaching Assistant (GTA) instructional expertise, amount of GTA teaching experience, and academic performance of their college algebra students measured by course grades. The questions posed then remain relevant today. In the present study, college algebra grades for all students in classes taught by GTAs since the original experiment were analyzed. That is, data from twelve years (AY2006 – AY2017) and 168 sections (n = 6675) were examined. Noteworthy is the fact that success in lowering the drop rate in the treatment group held true for 15 years since the treatment was initiated. Included is a look at what has changed and what has remained the same since the original study.

Keywords: Graduate teaching assistants, professional development, program efficacy

"Mathematics education, unlike mathematics itself, is not an exact science; it is much more empirical and inherently multidisciplinary. Its aims are not intellectual closure but helping other human beings, with all of the uncertainty and tentativeness that that entails" (Bass, 1997, p. 21). Professional development programs for Graduate Teaching Assistants (GTAs) are becoming more common across the United States, with 81% of PhD granting institutions and 45% of Masters granting institutions reporting having some kind of department-run professional development for their GTAs (Rasmussen et al., 2016). However, it is not clear what the results of these various professional development programs are, both on teaching efficacy and student achievement.

Teachers of mathematics need both knowledge of content and knowledge of the best way to teach that content to students. Pedagogical content knowledge or subject-specific pedagogical knowledge consists of how to represent specific topics and issues in ways that are appropriate to the diverse abilities and interests of learners (Ball, Thames, & Phelps, 2008). Brown and Borko (1992) said this requires making the transition from a personal orientation to a discipline to thinking about how to organize and represent the content of the discipline to facilitate student understanding. Naturally, GTAs need support and guidance in making this transition from learner to learning to teach.

In this particular study, GTAs were given professional development to help support them in their teaching of college algebra. Results from the past 12 years since the program was implemented of the change in withdrawal rates are given. In the talk, we will further discuss changes in the grade distributions for the course.

Background of GTA Professional Development Programs

Researchers in higher education have suggested that for decades universities and colleges gave little regard to the impact of GTAs on undergraduate education (Boyer, 1990; Sykes, 1988). Sykes (1988) said that the professoriate, in pursuit of research, grants, and academic politicking, has left undergraduate students in the care of under-prepared and under-paid GTAs. During the late 1980's, in response to the sharp criticism about the practice of using graduate students as inexpensive labor, many GTA orientation and training programs were started at colleges and universities across the United States (Bartlett, 2003). Now, as the focus is returning to the teaching of mathematics at the undergraduate level, there is renewed interest in professional development programs for GTAs. For instance, one of the seven recommendations from the Mathematical Association of America study of successful Calculus programs was to improve the professional development offered to the GTAs (Bressoud, Mesa, & Rasmussen, 2015).

Research about college student learning and development clearly shows that student learning is "unmistakably linked to effective teaching" (Pascarella & Terenzini, 1992, p.182). Furthermore, there is research to support that "good teaching" has a positive effect on the change in students' attitudes towards mathematics (Mesa, Burn, & White, 2015), with "good teaching" referring to three components: classroom interactions that acknowledge students, encouraging and available faculty, and fair assessments. However, this same study has shown that students are still citing their experiences in college mathematics as a top reason for why they are switching out of a STEM (science, technology, engineering, and mathematics) major (Rasmussen, Ellis, & Bressoud, 2015). So, more work needs to be done on how the experience in undergraduate mathematics courses can be improved.

Within the various studies done on the range of professional development programs available for GTAs, most studies can be described by three main themes: temporal, structural, and topical. In temporal studies, researchers describe the duration of the professional development and how it varies across the nation (e.g. Belnap & Allred, 2009). In structural studies, the focus is on the various ways the programs for professional development of GTAs are structured (e.g. Ellis, 2015; Palmer, 2011). In topical studies, there is an effort to create a list of standard topics and teaching practices on which the professional development programs are focused (e.g. McDaniels, 2010). Finally, outside of the three topics described above, there are a group of studies on the efficacy of particular professional development programs (e.g. Griffith, O'Loughlin, Kearns, Braun, & Heacock, 2010).

The research base on the state of professional development of GTAs is still relatively small. There have been only a handful of studies done exclusively on the state of professional development of GTAs across the nation (Belnap & Allred, 2009; Kalish et al., 2011; Palmer, 2011; Robinson, 2011). Additionally, there have been a few meta-studies conducted over the years on the state of research in the teaching of undergraduate mathematics (Speer, Gutmann, & Murphy, 2005; Speer, Smith, & Horvath, 2010). Outside of the national studies, there are also a handful of articles on particular programs at specific institutions, with a focus on the structure of the program or the efficacy of the program (e.g. Griffith et al., 2010; Marbach-Ad, Shields, Kent, Higgins, & Thompson, 2010).

During the Research in Undergraduate Mathematics Education (RUME) conference in 2017, there were five different studies presented that involved examining what GTAs learned from a particular professional development program. The study done by Pascoe and Stockero (2017) focused on the results of an intervention in which the GTAs learn about a noticing framework and how to use it while watching videos of teaching. Reinholz (2017) and Wakefield and colleagues (2017) focused on the use of reflections in the development of teaching in GTAs, with Reinholz also looking into the role of peer feedback. Each of these studies focused on a cognitive approach to learning.

Furthermore, Speer, Deshler, and Ellis (2017) presented results from a study done on the ways departments are evaluating the undergraduate student outcomes from their GTA professional development programs. With this greater focus on GTA professional development programs, ways to evaluate their efficacy is an important aspect that has not been widely studied.

Their results showed that many departments are relying on student evaluations to evaluate the teaching of their GTA's, which has been shown to be an ineffective measure of teaching (Krautmann & Sander, 1999).

The purpose of the present longitudinal study was to revisit a decade old study about the relationship between instructor participation in a GTA professional development program and academic performance of college algebra students measured by course grades (Childs, 2008). Furthermore, the relationship between algebra performance of college students in courses taught by first year GTAs and second year GTAs was reexamined.

Methods

Participants

All of the participants in the present study were enrolled in sections of MATH 113, College Algebra, taught by GTAs at a midsized Midwestern University during the spring and fall semesters over the AY 2006-17. The University remains a traditional college campus with average class size of 18 students. Approximately 7,000 students annually were enrolled in more than 200 academic programs and emphasis areas in four colleges.

College Algebra is one of three choices for all baccalaureate students to satisfy the Mathematics Area under the General Education Degree Requirements as stated in the university catalog. Enrollment for the course is approximately 700 students each academic year. College Algebra courses offered during the summer term are not taught by GTAs and not considered in this study. The students were males and females, freshman, sophomores, juniors, and seniors between the ages of 17 and 65. Participants for this study were enrolled in this course, as well as other courses, with the assistance of an academic advisor. Quantitative data was gathered from this purposive sample to examine the relationship between algebra performance among college students and instructor expertise.

Procedure

In the current study as well as the original study, there are important common components of Math 113, College Algebra, during the control and treatment years. They include course syllabi, Basic Skills Exams, final exams, and GTA instructors. These standardized conditions of college algebra during the years under investigation help control for potential group differences and allow for investigation of the treatment variable with more reliability.

Course Syllabi. All students in the participating sections of college algebra are exposed to the same set of course topics during the semester. All college algebra classes have a common day-by-day schedule and a common syllabus of topics and skills outlined by the State Board of Regents. The Core Competency Committee, called by the State Board of Regents, determined minimum core competencies for common courses under its jurisdiction. Mathematics instructors and professors from all of the State institutions comprised the committee to develop the mathematics syllabi. To ensure this set of minimum core competencies and department approved learning goals and objectives are taught uniformly in all college algebra courses within the mathematics department, course syllabi are scrutinized by either the GTA supervisor or the department chair.

Basic Skills Exam. The Basic Skills Exam is an important formal assessment tool used in college algebra at this university. The math department requires a Basic Skills Exam for college algebra in which students must get 9 out of 11 problems completely correct in order to successfully exit the course. Students start taking this exam at the beginning of week 9 of the semester. If a student fails the exam, he or she works one on one with the instructor and tutors and may continue to repeat versions of the exam until week 11 of the semester. If the student still does not pass the exam after week 11, he or she must repeat the course.

Every student from every section of college algebra had to demonstrate mastery of these basic algebra skills to the same high degree of accuracy by passing the standardized Basic Skills Exam during the semesters under study.

Final exam. Students in college algebra take a common comprehensive final exam that is prepared by the full-time instructor who coordinates the college algebra sections. All students in all sections of college algebra take this comprehensive final exam on the same date and at the same time. The contents of final exams during the years under investigation were analyzed for concepts tested, number of questions, and number of questions per concept. Two mathematics instructors participated in this analysis to provide inter-rater reliability and determine if there were any significant differences among the years being studied.

Each of the final exams for the 24 semesters being examined contained questions in six categories: Basics, Algebraic Operations, Solving Equations and Inequalities, Functions, Graphing, and Matrices. Just as in the original study, the contents of final exams during the years under investigation were analyzed for concepts tested, number of questions, and number of questions per concept. Two instructors participated in this analysis to provide inter-rater reliability and determine if there were any significant differences among the years being studied. The results suggest that no mean differences exist between the number of questions in each of the six categories during the control years and treatment years. In addition, there was not a significant difference in the total number of questions on the finals in the control group, AY 1999 – 2001, ($M_C = 36.6$) and the treatment group AY 2002-17 ($M_T = 37.2$). The results of the chi-square test substantiated there were no differences between the groups by content area on the final exams ($\chi^2 = 0.198$, $df = 6$, $p > .95$). The P-Value is 0.99985. The result is *not* significant at $p < 0.05$.

Treatment Procedures

Beginning in the fall 2002 semester and continuing to the present, the mathematics department implemented a coordinated program of support and professional development for its GTAs. Release time was given to a tenure-track faculty member for this assignment. Also a new course, MATH 871 Teaching Mathematics, for one-credit-hour was added and required of all GTAs.

Prior to the fall semester 2002, GTAs teaching mathematics attended a fall orientation to cover the department handbook but did not receive any further training. Under the new program, since the fall semester 2002, graduate students teaching in the mathematics department meet for a half day of professional development training before the fall semester begins and then for a one hour class each week throughout the semester. The curriculum for MATH 871 Teaching Mathematics was designed specifically to assist GTAs in their role as educators and to address the unique professional challenges and limitations they face.

The following sections describe the program and procedures of the GTA training model used in this study. They comprise information about what has changed and what has remained the same about the treatment from the original study to the current study.

Treatment that continued from the original study.

In all treatment years, pre-service training for GTAs in the mathematics department was held prior to the start of the fall semester. During this time GTAs were given their assignments, a day by day schedule of textbook sections to teach along with unit test dates. They were provided with a copy of the textbook and ancillary materials to be used for teaching. The typical semester assignment for full-time GTAs consisted of complete responsibility for two, 3-credit-hour sections of college algebra. Both first and second-year GTAs participated in the orientation.

MATH 871 Teaching Mathematics Course Description. The course was designed to promote guidance, direction, and support for GTAs. From inception, the course goal was to encourage excellence in teaching through a program of sharing ideas, concerns, problems, and information on an ongoing basis with GTAs in the mathematics department. No one model was followed in course development. Instead, the aim was to build a unique model that drew from the research on best practices in GTA training and effective programs that fit the needs of mathematics GTAs at this university.

All GTAs attended an hour-long class once a week with the GTA supervisor. During the entire time period of the study, the researcher served as the GTA supervisor. Both new and returning GTAs participated in class activities with second-year GTAs acting as mentors for new GTAs.

Treatment new to this study.

Peer Observations. Peer observations were instituted starting in Fall 2012. GTAs were given opportunities to provide feedback about teaching, not just receive it. It was hypothesized that they may learn as much from providing feedback as receiving feedback. The process started with peer conferences. In this meeting GTAs were encouraged to discuss specific behaviors that they were interested in receiving feedback on. Following the peer observation, a second conference allowed students to discuss their feedback and analyses. Peer observations provided an additional learning opportunity for the GTAs, beyond only receiving feedback from a supervisor.

Journals. During the AY 2003-05 journal entries were required and submitted weekly. During the years of the current study, AY 2006-17, GTA's had a choice of a weekly face-to-face conference with the supervisor or a weekly journal submission. Both options were used as a way for the GTA supervisor to have continuing dialogue with individual GTAs and as a vehicle for GTAs to reflect on their own experiences and growth. From time to time a specific prompt activity was assigned. Regardless of the means, graduate teaching assistants were encouraged to regularly reflect intelligently on the work they were doing.

Teacher Noticing. A lesson about Teacher Noticing was added to the content of the GTA training course in AY 2014. GTA's were assigned research articles to read about this relatively new field in education and a GTA class meeting was devoted to discussion and question/answer dialogue about Noticing. The goal of introducing these student-centered pedagogies was to help GTAs to attend to and respond to student thinking in their classrooms.

Portfolios. All GTAs during the treatment years of the original study maintained teaching portfolios that documented their accomplishments during the semester. Portfolios during the treatment years of the current study were recommended by not required. This change was made help alleviate the many demands for their time.

Results

To assess the effectiveness of GTA training and the influence of GTA experience, course grades in college algebra were used as the dependent variable in this analysis. Students who finished the course were assigned grades of A, B, C, D, or F by their instructors. For the purpose of analysis, these grades were assigned numeric values (e.g., an “A” was assigned a value of 4; a “B” was assigned a value of 3, etc.). Students who withdrew from the course were assigned a grade of W. Of the 6675 participants, 4826 (73.4%) completed college algebra and 1849 (26.6%) withdrew from the course.

Table 1: Percentage of students who received each grade in the course broken down by semester.

AY	Semester	A	B	C	D	F	W
06	SP	30.4%	15.6%	8.0%	8.0%	6.3%	31.6%
	WF	21.5%	21.5%	16.8%	4.4%	7.7%	27.9%
07	SP	29.6%	21.3%	19.2%	4.5%	3.8%	21.6%
	WF	24.7%	18.8%	15.3%	9.0%	5.9%	26.4%
08	SP	40.2%	14.5%	14.1%	3.9%	4.3%	23.0%
	WF	24.2%	23.9%	13.8%	8.0%	3.1%	27.0%
09	SP	25.0%	17.3%	16.5%	5.0%	6.9%	29.2%
	WF	24.2%	15.9%	13.6%	5.5%	7.1%	33.8%
10	SP	19.9%	18.8%	10.8%	8.3%	7.2%	35.0%
	WF	18.8%	19.6%	19.0%	9.7%	6.3%	26.7%
11	SP	22.1%	17.2%	13.1%	4.5%	7.4%	35.7%
	WF	25.4%	20.6%	9.8%	6.0%	5.8%	32.4%
12	SP	23.5%	20.5%	14.0%	9.2%	5.1%	27.6%
	WF	46.8%	16.4%	8.8%	3.2%	4.5%	20.3%
13	SP	47.1%	15.8%	12.9%	4.0%	3.7%	16.5%
	WF	30.8%	17.8%	14.9%	5.5%	6.3%	24.6%
14	SP	25.3%	16.6%	18.4%	3.7%	8.8%	27.2%
	WF	24.1%	22.2%	13.8%	6.8%	4.6%	28.6%
15	SP	18.5%	19.8%	11.3%	7.2%	7.7%	35.6%
	WF	32.8%	25.5%	12.1%	6.6%	5.3%	17.7%
16	SP	15.2%	20.6%	15.5%	8.3%	8.3%	32.1%
	WF	34.6%	22.6%	12.1%	4.2%	6.8%	19.7%
17	SP	31.3%	18.8%	11.6%	5.8%	6.7%	25.9%
	WF	34.2%	22.7%	12.3%	5.1%	6.4%	19.3%

Examination of Grades of Students Who Completed the Course

Of the 6675 participants, 4826 (73.4%) completed college algebra with an average grade of 2.73 (approximately C+). The course grade data for students who finished the course were entered into an analysis of covariance with Math ACT scores as the covariate. Covariates are influential variables that affect the dependent variable but do not interact with any of the other factors being tested at the time. Therefore, since prior mathematics knowledge was present during the study, using Math ACT scores as a covariate in the analysis allowed for control of its influence.

The results of the analysis of covariance revealed that there were no main effects or interactions involving Year of Teaching. The only main effect was that of Math ACT, $F(1,4826)$

= 166.72, $p < .0001$. The results of this analysis indicated that students' math abilities (MACT) explained the variability in course grades rather than the GTAs' experience.

Examination of Withdrawals Only

Of the 6675 participants in the current study, 1849 (26.6%) withdrew from college algebra. In the original study, of the 2,198 participants, 670 (30.5%) withdrew from college algebra and for those who withdrew, 60.5% withdrew from classes taught by GTAs who were not trained and 39.6% withdrew from classes taught by GTAs who were trained. Upon examination of the frequency of withdraws by year of teaching, there were slightly fewer withdraws (48.2%) from classes taught by GTAs who had two years of teaching experience than had one year of teaching experience (51.8%). This result is in contrast to the finding from the original study where the results indicated that GTAs who had been trained and were in their second year of teaching had significantly fewer withdraws from their courses.

Discussion

A noteworthy finding is the fact that success in lowering the drop rate in the treatment group held true for 15 years. Of the 6675 participants in the current study, 1849 (26.6%) withdrew from college algebra. In the original study, of the 2198 participants, 670 (30.5%) withdrew from college algebra.

Furthermore, the results of this analysis indicated that students' math abilities (MACT) explained the variability in course grades rather than the GTAs' experience. This result corroborates with the findings in the original study.

Upon examination of the frequency of withdraws by year of teaching, there were only slightly fewer withdraws (48.2%) from classes taught by GTAs who had two years of teaching experience than had one year of teaching experience (51.8%). This result is in contrast to the finding from the original study where the results indicated that GTAs who had been trained and were in their second year of teaching had significantly fewer withdraws from their courses. A possible reason for this difference is the new treatments within the GTA professional development program. The result that the withdrawal rates are no longer significantly different based on the number of years the GTA has been teaching provides evidence that the new treatments may be helping to reduce the withdrawal rates starting in their first year of teaching.

Finally, the pass rates (receiving an A, B, or C in the course) for the students since the change in the professional development program for the GTAs are 58.7% on average. This average pass rate is higher than that of the national average for college algebra, which is 50% (Saxe & Braddy, 2015). So, there is some evidence that shows the students in these college algebra courses with GTAs who have had additional support may be doing better in the course than the national average.

With the increase in professional development programs for graduate teaching assistants across the nation, large data sets are needed to gain an understanding of the impact the support may have on student success. This study provides evidence of the impact a professional development program can have on student pass rates in college algebra and adds to the literature base on the efficacy of professional development programs.

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Determining Significant Factors for Relating Beliefs to Lecture

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When trying to examine instructors' instructional practices, specifically lecturing, qualitative studies have indicated the necessity to consider their beliefs. However, there is a dearth of quantitative belief measures specific to instructors of undergraduate mathematics courses. No one specific instrument captures the relationship between beliefs and lecturing. This paper, therefore, attempts to establish a foundation of significant factors for researchers to consider when developing belief measures to predict lecturing. We use pre-existing data from Calculus and Abstract Algebra courses to conduct factor analyses and develop composite variables. We then use multiple regression to examine composites with significant effects on time spent lecturing. Results suggest that beliefs related to a focus on skills and content, knowledge facilitation authority, expectations of student success, and the importance of particular concepts are of particular importance.

Keywords: lecture, beliefs, factor analysis, regression

Within mathematics education research, there has been extensive work focusing on improving mathematics instruction. Much of this research has shown that the type of instructional strategies instructors employ depends on their teaching philosophy, how they feel students should learn the material, along with other factors such as attitudes and content knowledge (Mesa, Celis, & Lande, 2014; Remillard, 2005; Weber, 2004; White & Mesa, 2014; Wilkins, 2008). Research has shown there is an interaction between content knowledge, attitudes, beliefs, and instructional practices (Remillard, 2005; Wilkins, 2008). For example, Wilkins (2008) found that content knowledge had a negative effect on both beliefs and instructional practice concerning inquiry-based instruction, indicating that teachers with more content knowledge had lower beliefs and were less likely to use inquiry-based practices; attitudes had a positive effect on beliefs and instructional practices; and beliefs had a positive effect on instructional practices. Remillard (2005) also found that the type of curriculum instructors implement in the classroom was centered on their teaching beliefs and attitudes (Remillard, 2005). Similar results have been found in post-secondary settings in undergraduate mathematics classrooms (Johnson, Keller, & Fukawa-Connelly, 2017; Mesa et al., 2014; Weber, 2004; White & Mesa, 2014).

By understanding beliefs, researchers are able to gain insight on how to modify instruction. Johnson et al. (2017) took this charge and examined instructors' beliefs and the "nature of instruction" to help explain "why there has been little change" (p. 259) concerning instructional practices. They found that some instructors identified as lecturers but used more student-centered instructional practices and instructors who identified as non-lecturers reported lecturing sometimes during class (Johnson et al., 2017). This suggests that an instructor's instructional practices are a complex system made up of both internal and external factors. These factors may very well be in conflict with one another, causing the instructor to sacrifice one belief for another. This calls for the need of models that can help describe instructors' beliefs and offer more insight into conflicting beliefs. These models can also better explain why instructors may choose certain instructional practices over others. However, this can be a taxing job since beliefs are hard to capture in a way that is predictive. As a result, more research is needed to investigate instructors' beliefs to gain better insight for improving instruction. Therefore, the purpose of this paper is to use existing data to establish a foundation of important factors for others considering developing belief measures. Specifically we ask: what belief factors can be used to predict undergraduate mathematics instruction?

Literature Review

Background of Beliefs and Teaching Instruments

We conducted an extensive literature review, searching for literature that focused exclusively on quantitative analysis of beliefs and practices in the STEM or higher education field. What we found was a dearth of instruments used to capture beliefs regarding teaching. These instruments range in disciplines, focusing on general teaching beliefs to more content specific such as Science and Statistics. However, none of the instruments we examined were specific to the mathematics context. The majority of instruments were general, focusing on teaching style preference (Heimlich, 1990), approaches to teaching (Trigwell & Prosser, 2004), or teaching self-efficacy (DeChenne, Enochs, & Needham, 2012; Tschannen-Moran & Hoy, 2001; Thadani, Breland, & Dewar, 2010). Although some of these instruments were newly developed by the researcher (e.g., Heimlich, 1990; Sampson & Grooms, 2013; Trigwell & Prosser, 2004; Zieffler et al., 2012), most often these instruments were developed by adopting previous instruments (e.g., DeChenne et al., 2012; Justice, Zieffler, & Garfield, 2017; Thadani et al., 2010) or expanding them from the K-12 setting to higher education (e.g., Sunal et al., 2001).

The theme from the results of these studies showed that beliefs are directly linked to instruction, and also are predictors of instructional changes (e.g., Sampson & Grooms, 2013; Trigwell & Prosser, 2004; Thadani et al., 2015). Thadani et al. (2015) used four instruments to measure instructors' beliefs: Implicit theories about teaching, Teaching self-efficacy, Implicit theories of intelligence, and Beliefs about students' learning needs. They found that an instructor's belief that teaching skills cannot change subsequently hinders their willingness to improve (Thadani et al., 2015). Sampson and Grooms (2013), as well as Pelch and McConnell (2016), used the Beliefs about Reformed Science Teaching and Learning instrument to investigate instructor's beliefs about science teaching and learning in relation to reformed-based teaching strategies. Results from both studies showed that instructors typically fell on a continuum, ranging from traditional to reform aligned. They also found that by using those beliefs and offering specific training, instructors were able to change beliefs, and that the greatest changes occurred on items related to situational classroom factors (Pelch & McConnell, 2016). Further examining instructors' reform-based beliefs and instructional practices, Borrego, Froyd, Henderson, Cutler, and Prince (2013) used the Research-Based Instructional Strategies survey and found that the instructional practices employed in class aligned with the instructors' beliefs about how students best learn in a limited amount of time. This study identified a "direct link between instructor beliefs and classroom activities specific to engineering courses which rely heavily on problem-solving" (p. 1468). The researchers also claim that this study provides evidence that instructors resistant change due to time constraints.

One concern regarding all the studies we examined was that none of the instruments used were specific to undergraduate mathematics. Although some, such as the STEM GTA-Teaching Self-Efficacy Scale (DeChenne et al., 2012), were specific to STEM, the instruments were not tailored to the field of mathematics specifically. Research has shown that mathematics is a unique content to teach, as there are many beliefs concerning the teaching and learning of it (Johnson et al., 2017; Weber, 2004). For example, Johnson et al. (2017) note that there is a large debate over whether lecture or reformed-based pedagogy is best for the teaching and learning of mathematics. They also note how it is argued that instructors employ instructional practices simply out of habit or because of their beliefs. Due to this debate, there needs to be an instrument designed specifically for mathematics that captures instructors' beliefs and how that might predict instruction.

Building a New Instrument/Model

Prior research has identified numerous belief factors that may influence instructional practices. As was noted above however, very few of the studies we found were specific to undergraduate mathematics instruction. Without such research, those attempting to capture beliefs as they relate to undergraduate mathematics instruction may face confusion over what sets of beliefs to focus on and how to capture them. This concern becomes especially important if researchers are trying to see what kinds of beliefs may predict openness to instructional change as Johnson, et al. (2017) call for. By knowing what belief factors may relate to instructional practices and how to capture them, the mathematics education community can take steps to use those beliefs as leverage points to examine, predict, and even change instruction to meet the calls for educational reform. The aim of our study then is to provide a baseline for which belief factors to focus on in the undergraduate mathematics context and how to capture them quantitatively.

Method

This report draws on pre-existing data from the MAA's 2010-2012 NSF supported study on the *Characteristics of Successful Programs in College Calculus* (CSPCC) and abstract algebra (AA) instructor surveys. Sonnert and Sadler (2015) identified numerous teaching practices students classified as 'ambitious teaching', with many of these paralleling Saxe and Braddy's (2015) definition of active learning. We looked for parallel questions representing instructors' beliefs in such practices in the CSPCC and AA instructor surveys. Further details of the CSPCC study can be found in Bressoud, Mesa, and Rasmussen (2015) while details on the AA study can be found in Fukawa-Connelly, Johnson, and Keller (2016).

Survey Items and Factor Analyses

There were numerous items of interest relating to instructors' instructional beliefs in the CSPCC (16 initial items) and AA surveys (23 initial items). For use in regression analyses, we wanted to maximize our degrees of freedom and create a more parsimonious model and thus used an exploratory factor analysis to create composite independent variables for each survey separately. Numerous models were run with different number of items while eliminating cross-loaded items. We included 13 and 20 items in our final CSPCC and AA factor analyses respectively. The CSPCC data resulted in a four-factor solution (PROMAX rotated) explaining 54.72% of the variance. The AA data resulted in a five-factor solution (PROMAX rotated) explaining 68.22% of the variance. All items had factor loadings above 0.4. Items that loaded onto the same factor were standardized, with items that loaded negatively being reverse coded. Items were then averaged together to create composite variables representing each factor. The factors and included variables are presented below with their factor loadings in parentheses.

CSPCC data. The variables loading onto the first factor asked teachers to estimate what percentage of their students were prepared for the course (.61), and would pass (-.98), fail (.79), or withdraw (.79). As such, we felt the factor represented *Expectations of student success*. The second factor consisted of the questions: 1) *From your perspective, when students make unsuccessful attempts when solving a Calculus I problems, it is: 0 (a natural part of solving the problem) to 5 (an indication of their weakness in mathematics*; .63*)*, 2) rate on a scale of 0 (*Strongly Disagree*) to 5 (*Strongly Agree*) the statement *Calculus students learn best from lectures, provided they are clear and well-organized* (.78), and 3) rate on a scale of 0 (*Strongly Disagree*) to 5 (*Strongly Agree*) the statement *Understanding ideas in calculus typically comes after achieving procedural fluency* (.55). By examining the descriptive statistics for these items (means of 2.65, 3.77, and 3.76 respectively), we felt these reflected a focus on achieving procedural fluency and covering content before conceptual understanding and thus called the composite *Focus on skills and content*.

The third factor consisted of the questions: 1) *From your perspective, students' success in Calculus I PRIMARILY relies on their ability to: 0 (solve specific kinds of problems) to 5 (make connections and form logical arguments*; .75*)*, 2) *My primary role as a Calculus instructor is to: 0 (work problems so students know how to do them) to 5 (help students learn to reason through problems on their own*; .71*)*, and 3) rate on a scale of 0 (*Strongly Disagree*) to 5 (*Strongly Agree*) the statement *In my teaching of Calculus I, I intend to show students how mathematics is relevant* (.59). We felt these reflected instructors' beliefs about what conceptions they wanted to portray to their students and thus we called the composite *Conceptions of mathematics*.

The fourth factor consisted of: 1) *From your perspective, in solving Calculus I problems, graphing calculators or computers help students to: 0 (understand underlying mathematical ideas) to 5 (find answers to problems*, -.46*)*, 2) rate on a scale of 0 (*Strongly Disagree*) to 5 (*Strongly Agree*) the statement *If I had a choice, I would continue to teach calculus* (.68), and 3) rate on a scale of 0 (*Strongly Disagree*) to 5 (*Strongly Agree*) the statement *Familiarity with the research literature on how students think about ideas in calculus would be useful for teaching* (.76). This factor seemed to reflect instructors' interest in teaching and perceptions of resources to aid in their instruction and as such, we call the composite *Teaching and Learning Focus*.

AA data. The variables loading onto the first and second factors related to topics teachers felt they should: 0 (*would not cover*), 1 (*try to teach*), or 2 (*always teach*). The first factor consisted of rings (.84), fields (.82), field extensions (.66), ring isomorphisms (.88), ring homomorphisms (.90), and polynomial rings (.86). The second factor consisted of groups and subgroups (.69), group isomorphisms (.83), group homomorphisms (.86), quotient groups (.83), Lagrange's theorem (.69), and the fundamental homomorphism theorem (.81). Regardless of instructors' position on these topics, we felt that the loadings of these items together as factors represented a *focus on fields and rings* and a *focus on groups,* respectively.

The third factor consisted of the following statements instructors rated on a 4-point scale of -2 (*Disagree*) to 2 (*Agree*): 1) *I think lecture is the best way to teach* (.63), 2) *I think lecture is the only way to teach that allows me to cover the necessary content* (.62), 3) *I think students learn better when they struggle with the ideas prior to me explaining the material to them* (-.80), and 4) *I think students learn better if I first explain the material to them and then they work to make sense of the ideas for themselves* (.74). Based on the positive and negative loadings of these items, we felt that these questions reflected a focus on who instructors believe should control knowledge facilitation and thus was called the composite *Knowledge facilitation authority*.

The fourth factor consisted of the following statements instructors rated on a 4-point scale of -2 (*Disagree*) to 2 (*Agree*): 1) *I think that all students can learn advanced mathematics* (.94) and 2) *I think all students can learn abstract algebra* (.96). We felt these questions reflected instructors' beliefs about students' learning abilities, paralleling the *Expectations of student* *success* factor in the CSPCC data and thus we similarly called the composite *Expectations of student success*. The fifth factor consisted of items asking instructors to rate how influential instructors' experiences as students (.83) and teachers (.83) were on their teaching on a 3-point scale of 1 (*Not at all*) to 3 (*Very*). These seemed to reflect the personal experiences instructors felt impacted their teaching. Thus, we called the composite *Personal influences on teaching*.

Regression Analysis

For the purposes of this study, we were interested in looking for composites with significant effects on time spent lecturing (as one measure of teaching practice). The dependent variable for our CSPCC analyses had instructors rate on a scale from 0 (*Not at all*) to 5 (*Very often*), the statement *During class time, how frequently did you lecture* (mean= 4.20, SD= 1.16). For the AA analyses, teachers answered on a scale from 0 (*Never*) to 4 (*75-100%*), the question *While teaching, what is the approximate amount of time per class that you are lecturing* (mean= 2.64, SD= 1.09). These are categorical dependent variables (with at least five categories), thus we used multiple regression. For each data set, the dependent variable of the amount of time spent lecturing was regressed on the centered composite independent variables specific to that data set.

In terms of diagnostic tests, the regression analyses resulted in VIF values close to 1 (Table 1), indicating that multicollinearity was not an issue. We tested linearity by fitting a Loess line on the plots of standardized predicted values against standardized residuals and by sequentially entering centered power terms sequentially into separate regression models. We checked homoscedasticity by examining the spread of the plots for irregularities. For the CSPCC data, the spread of the data suggests homoscedasticity was a reasonable assumption while the Loess line and statistically significant quadratic model ($F[4, 424] = 2.61$, $p < .05$) suggests linearity may be an issue. The spread of the AA data suggests homoscedasticity may be a problematic assumption while the curvilinear tests suggest linearity was met. Histograms of residuals and P-P plots indicated normality of residuals was satisfied for the AA data but not for the CSPCC data. We checked for outliers by plotting centered Leverage values against instructor ID, which indicated concerns for the CSPCC data. Taken together, these tests suggest that the results of our regression analyses may be inflated for both data sets and other tests may be more appropriate, particularly for the CSPCC data.

Results

For the CSPCC data, *Expectations of student success, Focus on skills and content, Conceptions of mathematics,* and *Teaching and learning focus* together accounted for 2.8% of the variance in the time spent lecturing and the overall multiple regression was statistically significant ($F[4, 427] = 3.08$, $p < .05$). For the AA data, *Focus on fields and rings, Focus on groups, Knowledge facilitation authority, Expectations of student success, and Personal influences on teaching* together accounted for 37.8% of the variance in the time spent lecturing and the overall multiple regression was statistically significant ($F[5, 161] = 19.58$, $p < .05$). As presented in Table 1, there were statistically significant effects of *Focus on skills and content* on CSPCC instructors' time spent lecturing ($\beta_{focus} = .145$, $t = 2.96$, $p < .05$) as well as statistically significant effects of *Focus on groups, Knowledge facilitation authority*, and *Expectations of student success* on AA instructors' time spent lecturing ($\beta_{groups} = .17$, $t = 2.74$, $p < .05$; $\beta_{authority} = .49$, $t = 7.32$, $p < .05$; $\beta_{expectations} = -.15$, $t = -2.26$, $p < .05$). Thus, the more focused CSPCC instructors were on covering content and imparting basic skills first, the more likely they were to spend time lecturing. For the AA data, the higher expectations AA instructors had for their students, the less likely they were to spend time lecturing. By contrast, the more AA focused on the topic of groups or believed in their role as the driving source for knowledge creation, the more likely they were to lecture.

Table 1. Predictors of Time Spent Lecturing

Variable	b	SE	beta	*t*	Significance level	VIF
CSPCC data (N=432)						
Constant	4.192	0.056		75.119	0.000	
Expectations of student success	-0.117	0.075	-0.075	-1.557	0.120	1.027
Focus on skills and content	0.247	0.084	0.145	2.957	0.003	1.058
Conceptions of mathematics	-0.033	0.087	-0.019	-0.385	0.700	1.050
Teaching and learning focus	-0.070	0.087	-0.039	-0.809	0.419	1.047
AA data (N=167)						
Constant	2.569	0.066		39.064	0.000	
Focus on fields and rings	0.155	0.081	0.120	1.900	0.059	1.025
Focus on groups	0.226	0.082	0.174	2.741	0.007	1.039
Knowledge facilitation authority	0.710	0.097	0.489	7.317	0.000	1.154
Expectations of student success	-0.166	0.074	-0.149	-2.258	0.025	1.120
Personal influences on teaching	0.110	0.086	0.081	1.285	0.201	1.020

Conclusions

The factors that resulted from our EFA may be useful subscales for future work attempting to create surveys of instructors' beliefs . To maintain brevity, we suggest retaining two to three questions per factor. The criterion for choosing items should be based on how strongly the item loads onto a given factor. Specifically, items with loadings of the highest absolute value should be considered representative of the factor they load onto. Taking the AA data for example, if we are to have a subscale on *Knowledge facilitation authority* and want to retain two items, we would retain the questions asking instructors to rate their agreement with the statements: 1) *I think students learn better when they struggle with the ideas prior to me explaining the material to them* and 2) *I think students learn better if I first explain the material to them and then they work to make sense of the ideas for themselves,* as these two had the highest loadings (in absolute value) of all items loading onto that factor (.80 and .74 respectively).

Our regression analyses suggest that the beliefs of particular importance are those related to a focus on skills and content (before conceptual understanding), knowledge facilitation authority, expectations of student success, and the importance of particular concepts. Focusing on these factors can help researchers create more succinct belief assessments. We acknowledge that these factors are only significant in relation to how much instructors lecture. Other factors may be influential in determining other instructional practices and that is an area for future research. Another peculiar finding was the difference in explained variance of instructional practices between the CSPCC and AA data (with more variance explained for the AA data). This may be a result of including items related to topic priority in the AA data but could also result from belief factors having different effects based on context (as can be seen in the difference in beta values for expectations of student success between both data sets). This notion of beliefs varying by context is paralleled in Leatham's (2006) conception of beliefs. Future research should look into how certain belief factors affect instruction differently in different contexts and formulating subconstructs of content specific groupings of concepts (as done with the AA data) which instructors rate on instructional priority.

Our literature review highlighted a dearth of quantitative belief measures specific to undergraduate mathematics instruction. With the results on hand, we have provided some baseline constructs to measure undergraduate mathematics instructors' beliefs in relation to time spent lecturing and other instruction practices.

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Instructors' Pedagogical Decisions and Mathematical Meaning-Related Goals

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There are differing senses of meaning in mathematics education focusing either on mathematical understanding or on relevance. Various pedagogical practices exist in mathematics education, each with its own goals and associated challenges of implementation though the relationship between pedagogical goals and differing senses of meaning has not been explored extensively. Using pre-existing survey data on calculus instructors' pedagogy, we used multiple regression to determine the effect of differing pedagogical decisions aligned with the "meaning of" or "meaning for" mathematics on instructors' perceived pressure to cover course content. The results of our overall test were statistically significant. In particular, we found instructors' focus on the meaning for mathematics had a statistically significant effect on decreasing instructors' stress to cover material. Implications and further areas of study follow.

Keywords: Instruction, Mathematical Meaning, Calculus, Instructor Goals, Pedagogy

Brownell (1947) defines two senses of meaning in mathematics education. The "meaning of mathematics" can be thought of as mathematical understandings while the "meaning for mathematics" can be thought of as the subject's usefulness or relevance for non-mathematical purposes. Thompson (2013, 2015) highlights the importance of the "meaning of" while Jones and Wilkins (2013) identify usefulness ("meaning for") as a key part in motivating student learning. We argue mathematics instructors should thus attend to both senses of meaning. In practice, a focus on the 'meaning of' may manifest in helping students connect concepts while a focus on the 'meaning for' may manifest in tying concepts to everyday experiences.

The senses of meaning instructors adopt relate to broader questions regarding instructors' pedagogical practices. One place this comes up is in the push towards active learning, where teaching for conceptual understanding (meaning of) and including applications and modeling (meaning for) are included in larger instructional reform efforts (Saxe & Braddy, 2015). However, active learning in its various forms poses numerous challenges for teachers such as the lack of breadth and depth of mathematical course content, a threated sense of instructor autonomy, logistic and planning hurdles, issues of departmental support, having the pedagogical knowledge to productively build off student contributions, and difficulty gauging student understanding (Carducci, 2014; Cooper, 2014; Donnay, 2014; Gregson, 2007; Johnson, Keller, & Fukawa-Connelly, 2017; Johnson & Larsen, 2012; Olitsky, 2015; Rousseau, 2004). Of particular note to us was a common thread of pacing and content coverage concerns, which could result from issues with lacking necessary pedagogical knowledge and/or gauging student understanding (Carducci, 2014; Donnay, 2014; Gregson, 2007; Johnson & Larsen, 2012; Wagner et al., 2007). Rousseau (2004) found that the issue of gauging student understanding and pacing could even push teachers towards abandoning pedagogical change. These results parallel the argument made that lecturing is the best way to cover content (Wu, 1999).

A question that arises for us then is how do teachers' struggles with meaning differ according to their pedagogical decisions and how does that affect instruction? Teachers' pressure to cover content may be one area affected as was identified in prior studies. Addressing this question requires considering how to look at instructors' pedagogical decisions and beliefs. We posit that the extent to which instructors are motivated to incorporate differing sense of meaning in their instruction is related to their beliefs about pedagogical goals. Thus, we focus on a particular set of beliefs and goals (related to the extent to which instructors care about the meaning of and/or meaning for mathematics in their instruction) and a particular constraint (coverage concerns) to look for relationships between beliefs about mathematical meaning, pedagogical decisions, and constraints. By doing so, we aim to answer the following research question: how does the effect of pedagogical decisions focused on the 'meaning for' mathematics compare to the effect of pedagogical decisions focused on the 'meaning of' mathematics on instructors' perceived pressure to cover course content? By analyzing this question from survey data on instructors' beliefs and practices, we can see if previous qualitative results generalize quantitatively but also begin elaborating a framework for mathematical meaning which accounts for pedagogical decisions and the diversity of instructors.

Theoretical Framework

We need to first describe what we mean by certain terms. We take pedagogy to mean the methods, assessments and practices instructors adopt to teach, whether these are connected under a larger framework or connected simply by the choice of the instructor to place them together in their instruction. We take pedagogical decisions and beliefs to include the choices and associated beliefs regarding activities or actions the instructor and or students engage in, as determined by the instructor to meet instructional goals. This includes decisions and rationale regarding curriculum, assessment, classroom activities and discourse, and course content delivery. We take instructional goals to mean beliefs about what instruction should emphasize.

As mentioned earlier, Brownell distinguishes two senses of mathematical meaning. The "meaning of" mathematics is synonymous with conceptual understanding in the sense of understanding the connections between and within mathematical concepts (Rittle-Johnson, Schneider, & Star, 2015; Thompson, 2013, 2015). The "meaning for" mathematics can be thought of as understanding mathematics' significance for some non-mathematical purpose such as mathematics' application to everyday life or simply its capacity to endow skills needed for upcoming tests, future coursework, or one's career. To borrow a notion from motivation literature, the 'meaning for' focuses on understanding mathematics for separable outcomes while the 'meaning of' is thought of as understanding mathematics for non-separable outcomes (Ryan & Deci, 2000). Thus, according to the definitions given in prior research (Gregson, 2007; Hadlock, 2013; Leonard, Napp, & Adeleke, 2009; Rozek, Hyde, Svoboda, Hulleman, & Harackiewicz, 2015), pedagogies such as mathematics equity practice, critically relevant pedagogy, service learning, and utility-value interventions would fall under focusing on the "meaning for" mathematics because of their focus on mathematics to attain some separable outcome, like addressing social or community concerns. Inquiry-based learning (Laursen, Hassi, Kogan, & Weston, 2014) would fall under teaching approaches focused on the "meaning of" mathematics because of its focus on mathematical understanding itself rather than for some separable consequence. Problem-based learning however highlights the complicated nature of these constructs since an instructor could focus on either meaning (or both) based on how it is defined in the literature (Dochy, Segers, Van den Bossche, & Gijbels, 2003).

Building on this teacher specific nuance, instruction could focus on conceptual understanding but still fall under focusing on the "meaning for" if the aim was ultimately for a separable outcome. Thus, the senses of meaning adopted by a teacher when they make pedagogical decisions is ultimately dependent on their *goals*. Considering the issues identified between various non-lecture pedagogies and content (Carducci, 2014; Donnay, 2014; Gregson, 2007; Johnson & Larsen, 2012; Wagner et al., 2007) and the possibility that the implementation of such pedagogies may be manifestations of instructors' pedagogical goals related to meaning (as differentiated above), we suspect that differing senses of meaning instructors focus on could lead to content coverage concerns. We further posit that pedagogical decisions in the spirit of these non-lecture pedagogies (teaching for conceptual understanding, focusing on application problems, etc.) are different forms of 'ambitious teaching' (Sonnert & Sadler, 2015) because of the attention such practices have garnered in instructional change efforts (Saxe & Braddy, 2015).

Niss (1996) and Wagner et al. (2007) looked at the goals of mathematics education on different levels, ranging from national policy goals (such as promoting democratic values) to highly specific, localized goals on one instructional task in a class session. We are looking specifically at the goals behind instructors' pedagogical decisions, where we take such goals as being directed towards the "meaning of" or "meaning for" mathematics. Since we take instructional goals as being determined by what teachers *believe* they should emphasize instructionally, we treat goals as a subset of beliefs. One indication of how meaning may direct beliefs is by examining the degree to which one sanctions statements about what they should focus on in instruction. Agreeing with a statement on making math relevant by using real world examples could demonstrate "meaning for"-oriented goals while agreeing with a statement emphasizing that instruction should demonstrate connections between concepts could demonstrate "meaning of"-oriented goals.

Arguing against the notion of inconsistency between beliefs and practice, Speer (2005) points out a flawed methodological assumption that beliefs remain constant across contexts. Hoyles (1992) proposes an alternate conception of beliefs as situated in which beliefs are dialectically constructed products of activity, context, and culture. Beliefs vary by context and as such, researchers' focus on inconsistency is replaced by attention to what factors constrain or scaffold teachers in their practices. Leatham (2006) expands this by viewing teachers as sensible and beliefs as coherently organized within belief structures. Beliefs can vary on three dimensions. Psychological strength describes the relative strength of a belief, ranging from peripheral to central. Psychological strength is determined by how connected beliefs are with other beliefs in a belief system (an often-unconscious sense-making process). Quasi-logical relationships, the 2rd dimension, refers to some beliefs as derivatives of other beliefs (like in an if-then statement in the teachers' mind). The extent to which beliefs cluster in isolation from other beliefs (3rd dimension) is determined by context (department culture for example). Individuals can hold seemingly contradictory beliefs, but the contextual factor simply makes one belief cluster more central in a certain situation. Inconsistencies then are in the eyes of observers who have either misunderstood certain beliefs' implications (second dimension) or failed to account for other beliefs becoming more prominent due to context (first and third dimensions).

Considering the above framework, we hold that teachers' goals behind their pedagogical decisions, and thus the senses of meaning focused on, are situated and vary sensibly according to the three factors. In turn, the senses of meaning a teacher focuses on and to what extent are either scaffolded or constrained by teachers' lived reality. Prior research suggests that stress to cover content can result from pedagogical decisions. We understand these factors can be reflexively related as Chowdhury (2018, February) suggests, but for this study, we are only interested in the one-way relationship identified. By analyzing this relationship, we hope to explore the effect of instructors' differing pedagogical decisions and beliefs (related to the 'meaning of' and/or 'meaning for' mathematics) on instructors' perceived pressure to cover course content.

Method

This quantitative study draws on pre-existing data from the MAA's 2010-2012 NSF supported study on the *Characteristics of Successful Programs in College Calculus* (CSPCC). Pre- and post-survey data was collected from a large sample of students, teachers, and calculus course coordinators across 213 institutions. The focus of this study was on the calculus teachers' responses. Details on the study can be found in Bressoud, Mesa, and Rasmussen (2015).

Based on the relationship between pedagogy and stress to cover content in the literature, we identified two pre-survey questions representing content coverage pressure. On a scale of 1: (*Not at all*) to 6: (*Very often*), instructors rated the statements *When teaching my Calculus I class, I had enough time during class to help students understand difficult ideas* (mean= 4.23, SD= 1.25) and *When teaching my Calculus I class, I felt pressured to go through material quickly to cover all the required topics* (mean= 2.08, SD= 1.38). Responses on both correlated highly with one another ($r = -.6$, $p < .01$) and thus a composite representing content coverage pressure was used as the dependent variable. This composite was formed by reverse coding responses to having time to help students, shifting both variables to start from 0, and averaging.

In relation to our framework on meaning, there were numerous variables of interest relating to pedagogical decisions and instructors' goals. Regarding pedagogical decisions, Sonnert and Sadler (2015) identified numerous teaching practices students identified under the category of 'ambitious teaching'. We looked for parallel questions in the instructors' post-survey responses as representatives of pedagogical decisions. Regarding goals, we looked for questions we felt corresponded to the meaning of or for mathematics. This resulted in 14 items of interest total. However, we wanted to maximize our degrees of freedom and create a more parsimonious model and thus we used an exploratory factor analysis to create composite variables. Numerous models were run with different numbers of items. We settled on including 8 items in our factor analysis. This resulted in a three-factor solution (PROMAX rotated) explaining 38.81 % of the variance. All items had factor loadings above 0.4. The first factor retained four items, while the second and third retained two items each. The factors and included items are presented in Table 1. Composites were created by standardizing items then averaging those that loaded together.

The items loading into the first factor aligned with Sonnert and Sadler's ambitious teaching factor, so we retained their terminology for the composite. The second factor consisted of questions on whether learning mathematics was about solving individual problems or conceptual understanding and drawing connections. Based on descriptives favoring the latter perspective for individual variables (means ranging from 3.97 to 4.87), we felt that these questions reflected a focus on the meaning of mathematics and thus we called the composite *Meaning of*. The third factor consisted of questions related to relevance and application. In our framework, these can be seen as separable outcomes and thus we called the composite *Meaning of*.

Table 1. Results of Factor Analysis of Variables with PROMAX Rotation

Variables	Loading
Factor 1: Ambitious Teaching (rated on a scale from 1-*Not at all* to 6-*Very often*)	
When teaching, I have students work with one another	.71
When teaching, I hold whole class discussions	.63

When teaching, I have students give presentations	.55
When teaching, I ask students to explain their thinking	.52
Factor 2: Meaning of	
Students' success depends on ability to: 0-*solve specific problems* to 5-*make connections*	.59
My primary role is to: 0-*work problems so students know how to do them* to 5-*help students learn to reason through problems on their own*	.59
Factor 3: Meaning for (low values-*strongly disagree/never*, high values-*strongly agree/always*)	
I look for application problems to motivate the idea[a]	.70
I intend to show how mathematics is relevant	.62
Notes: Items had a 6-point scale response unless otherwise noted. a. Item had a 4-point scale	

Analysis

We were interested in the effect of continuous independent variables and any possible interactions between them on a continuous dependent variable. As such, multiple regression was the most appropriate method. The composite for perceived stress to cover content was regressed on the centered composite independent variables (*Meaning of, Meaning for, Ambitious Teaching*). To test if teaching practices were associated with differing senses of meaning, cross-product terms (*of × for, of × Ambitious Teaching, for × Ambitious Teaching, of × for × Ambitious Teaching*) were sequentially added in blocks to test for possible interactions.

In terms of diagnostic tests, the regression analyses resulted in VIF values close to 1, indicating that multicollinearity was not an issue. We tested linearity by computing centered power terms and sequentially entering them into the regression model and by fitting a Loess line on the plot of standardized predicted values against standardized residuals. We checked homoscedasticity by examining the spread of the plot for irregularities. The spread of the data suggested homoscedasticity was met. The Loess line and lack of statistically significant differences from adding power terms indicated linearity was met. A histogram of residuals and P-P plot indicated normality of residuals was satisfied. We checked for outliers by plotting centered leverage values against instructor ID, which indicated only a few cases of concern.

Results

The *Meaning of, Meaning for,* and *Ambitious Teaching* composites together were statistically significant ($F[3, 449] = 2.779$, $p = .041$) and accounted for 1.8% of the variance in instructors' perceived pressure to cover course content. The interactions were not significant ($\Delta R^2 = .004$, $F[4, 445] = .446$, $p = .775$). According to Table 2, there was a statistically significant effect of focusing on the meaning for mathematics on instructors perceived stress to cover content ($\beta = -.103$, $t = -2.077$, $p < .038$). Thus, the more instructors focused on the meaning for mathematics, the less likely they were to feel pressured to cover course content.

Table 2. Predictors of Instructor Pressure to Cover Content

Variable	b	SE	beta	t	Sig. level	VIF
Constant	1.936	0.055		35.314	0.000	
Meaning of	0.079	0.070	0.055	1.140	0.255	1.066
Meaning for	-0.146	0.070	-0.103	-2.077	0.038	1.114
Ambitious Teaching	-0.109	0.080	-0.067	-1.359	0.175	1.115

Discussion

Our analysis suggests that pedagogical decisions focusing on the 'meaning for' mathematics, here entailing the application of problems and conveying relevance, scaffold content coverage goals. This suggests that when our sample teachers focus on the 'meaning for', the notion that they must cover everything is diminished perhaps because the psychological strength of teacher's beliefs on the significance of mathematics as central creates a filter on content to focus on and thus relieves pressure to cover everything.

This highlights Leatham's contextual dimension because one important consideration is the type of mathematics taught. Some fields, like calculus, may be easier to find real world examples to build content off of regardless of what teaching practices instructors enact and thus lead to less pressure (refer to the left representation on Figure 1). In Yoshinobu and Jones' (2012) terms, the mathematical content itself becomes the vehicle for developing mathematical thinkers, but here thinking is specifically about the 'meaning for'. A more theoretical course like abstract algebra, which tends towards the left most quadrants, may be harder to find applied problems to work from and thus instructors may feel pressure to cover content if they are focused on the 'meaning for' mathematics (trying to shift instructional goals towards the right most quadrants as in the right representation in Figure 1). These instructors could then face a question 'meaning of' focused instructors may have faced in our study (to be addressed below): do they choose to cover all content or cut content, and will the former decision shift their 'meaning for' focus to a peripheral position? By contrast, instructors in abstract algebra focused on the 'meaning of' could feel less pressure to cover content since the material lends itself to drawing purely mathematical connections.

Related to the contextual clustering and quasi-logical dimensions is what instructors consider an application. A physics-based mathematics problem may be easy to situate in calculus. A social justice problem however may be harder to accommodate in such a context and could thus anticipate concerns. The data we drew on did not consider what the implications of 'application' and 'significance' entailed. Future studies should investigate these nuances.

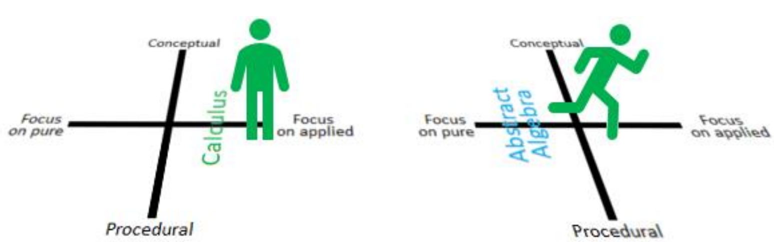

Figure 1. Mathematics and 'meaning for' focused instructors relative to meaning. The horizontal and vertical axes depict the 'meaning for' and 'meaning of' respectively. 'Meaning for' focused instructors may have real world problems to draw on in Calculus. Abstract algebra may have less accessible applied problems and thus 'meaning for' focused instructors may have to go out of their way to meet instructional aims.

Another finding possibly related to the quasi-logical and psychological strength dimensions was that neither ambitious teaching nor a focus on the 'meaning of' affects instructors' perceived pressure to cover course content. Johnson, Ellis, and Rasmussen (2016) found that ambitious teaching practices did not conflict with the centrality of content coverage goals. However, 'meaning of'-focused instructors may be faced with challenges regarding the content learned. Do they, the instructor, choose to cover everything or not? The former choice could lead to beliefs prioritizing content coverage and even shift their 'meaning of' focus to a more peripheral position while imparting procedures becomes central. The latter choice may curtail such concerns and thus scaffold teachers' 'meaning of' focus. This decision could then have quasi-logical implications regarding the roles teachers assign to themselves and their students. Our variables did not capture this distinction however. This may explain the lack of significance associated with a 'meaning of' focus and any subsequent interactions with ambitious teaching since we may have been catching the effects of both decisions.

The previous analyses assumed that both foci ('meaning of' and 'meaning for') are causes of content coverage pressure due to prior research. Our statistical results may suggest however that the differing sense of meaning act on different concerns or factors. Future studies should explore what factors besides content coverage a focus on the 'meaning of' may affect, and if it does affect content coverage, exploring the distinction between instructors who choose to cover everything and those who do not. Answers to the various questions raised can hopefully give insight into how mathematical meanings affect pedagogical decisions.

Conclusions

This study compared the effects of pedagogical decisions aligned with differing senses of meaning on instructors' perceived pressure to cover content. Our analyses suggest instructors' focus on the 'meaning for' could lead to less pressure to cover content whereas a focus on the 'meaning of' did not affect pressure. Our factor analysis suggests that the 'meaning of' mathematics is a separate construct from the 'meaning for' mathematics, but no other items loaded onto either. As a result, we do not have information on how 'meaning of' focused instructors responded to content coverage pressures nor what kinds of applications 'meaning for' focused instructors had in mind. We do know our results are specific to calculus instruction, a course which already serves as a service course for many STEM majors. It may be then that 'meaning for' focused instructors experienced less pressure because the psychological strength of such a focus served as a productive filter of instructional attention and thus scaffolded content coverage goals. This may not hold in other mathematics courses and future research should investigate possible differences and other influencing factors.

As a parting note, the mathematics education community has often focused on 'meaning' or 'meaningful' in accordance with Brownell's "meaning of" sense (Brownell, 1947; Thompson, 2013, 2015; Wawro, Sweeney, & Rabin, 2011, p. 17; Weber & Alcock, 2004, p. 227). Rousseau (2004) found that content coverage pressure could lead instructors to abandon pedagogical change. Our results may suggest that the reverse situation is worth exploring: if instructors feel less pressure to cover content as a result of focusing on the "meaning for" mathematics, would they be more receptive to pedagogical innovations? It may be worth the mathematics education community's time to devote more attention to the "meaning for" mathematics as a possible leverage point in getting instructors on board with pedagogical change initiatives.

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Monster-barring as a Catalyst for Connecting Secondary Algebra to Abstract Algebra

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This proposal reports on a teaching experiment in which a pair of prospective secondary mathematics teachers leverage their knowledge of secondary algebra in order to develop effective understandings of the concepts of zero-divisors and the zero-product property (ZPP) in abstract algebra. A critical step in the learning trajectory involved the outright rejection of the legitimacy of zero-divisors as counterexamples to the ZPP, an activity known as monster-barring *(Lakatos, 1976; Larsen & Zandieh, 2008). This monster-barring activity was then productively repurposed as a meaningful way for the students to distinguish between types of abstract algebraic structures (namely, rings that are integral domains vs. rings that are not). The examples of student activity in this teaching experiment emphasize the importance of identifying, attempting to understand, and leveraging student thinking, even when it initially appears to be counterproductive.*

Keywords: Teaching experiment, Zero-product property, Monster-barring

Introduction

Abstract algebra is seen as an important course in the mathematical preparation of secondary teachers, largely because of its potential to enable students to view the familiar content of secondary algebra through a more advanced lens. For example, it is recommended that prospective teachers come to regard the secondary algebra that they will be teaching as "the algebra of rings and fields" (CBMS, 2012, p. 59). Thus, in light of a significant body of literature reporting that students struggle to view secondary content from such an advanced persepective (e.g., Wasserman, 2016; Wasserman et al., in press; Zazkis & Leikin, 2010), a productive avenue of insight is to investigate student thinking about the algebraic properties that characterize such fundamental structures as rings, integral domains, and fields. To this end, the research question that motivated this study was: how might prospective secondary teachers preparing to take abstract algebra be able to adapt their existing understandings of an algebraic property to be effective in abstract algebra?

To answer this question, I conducted a teaching experiment (Steffe & Thompson, 2000) with a pair of prospective teachers preparing to take an introductory course in abstract algebra. The purpose of the teaching experiment was to investigate how prospective teachers might "assimilate their understanding of secondary mathematics with advanced mathematics" (Wasserman, 2017, p. 199) by focusing on: (i) student thinking related to the zero-product property (ZPP), a tool for solving equations in secondary algebra and the definitive characteristic of integral domains in abstract algebra, and (ii) how such thinking might be leveraged to enable students to develop an effective understanding of the ZPP in abstract algebra.

Literature and Theoretical Framing

With respect to my research question, I employed Thompson's (2008) tools for *conceptual analysis* in order to describe the characteristics of productive understandings of the ZPP in abstract algebra. To this end, a *way of understanding* is a meaning or conception that a student has for a particular mathematical idea (Harel, 1998). A way of understanding might include a system of strategies, analogies, informal descriptions, and examples and non-examples. Harel (1998) proposed that a student holds an *effective* way of understanding a mathematical idea if, in addition to retaining that way of understanding over time, she is able to:

- Criterion I: reformulate and articulate it in her own words,

- Criterion II: think about it in a general way, and
- Criterion III: coordinate it with her ways of understanding other ideas.

These criteria provide an observable way to determine if a student holds an effective way of understanding, but it remains unclear exactly what these criteria mean for zero-divisors and the ZPP in an abstract algebra setting. While criterion I – the student's ability to formulate the concept in her own words – is relatively straightforward, in order to operationalize Harel's criteria it is necessary to specify what it means for a student to think about zero-divisors and the ZPP in a general way (criterion II), and also to incorporate her thinking about other concepts (criterion III).

In order to operationalize[1] criterion II – what it means to think about a concept in a general way – I adopted Alcock and Simpson's (2011) perspective that classification of examples is a fundamental mathematical task. Indeed, a fundamental task for introductory abstract algebra students is to determine if a new example structure is an integral domain, which essentially amounts to determining whether the structure contains zero-divisors. Though the ability to consistently classify examples is rarely the final objective, it can be a useful opportunity for students to gain some initial experience with the underlying concept (e.g. Ross & Makin, 1999). Particularly, students with a way of understanding that is not fully developed will probably be unable to use it to consistently classify examples (e.g. Davis & Vinner, 1986). Thus, I used the ability to consistently classify algebraic structures on the basis of a particular property as evidence that a student was thinking about that property in a general way.

Methods

I adopted the teaching experiment methodology (Steffe & Thompson, 2000) as a means of exploring and refining the conceptual analysis – that is, the characterization of an effective way of understanding the ZPP and my hypothesis about how students might come to achieve such a way of understanding. I conducted the teaching experiment reported here with two undergraduate students, Brian and Julie (both pseudonyms), who were both beginning the first semester of their junior years at a small, public liberal arts college as mathematics education majors and prospective secondary mathematics teachers. Both had completed a course in linear algebra (both earning B's) but had not yet taken an introduction to proof course. This was typical for mathematics education majors at this particular institution, who instead were required to take an 'abstract algebra for future secondary teachers' course that focused more on the relevance of abstract algebra to secondary algebra than on the rigors of proof. Both Brian and Julie were preparing to begin this course when they participated in this study.

The teaching experiment consisted of 4 sessions lasting between 75 and 90 minutes each; I served as the teacher-researcher for all sessions. Each session was recorded with LiveScribe pen technology, which records the students' pen strokes with synchronized audio (called a pencast). I constructed models of Brian and Julie's ways of understanding using *ongoing* and *retrospective* analysis techniques (Steffe & Thompson, 2000). The instructional tasks of the teaching experiment centered on solving equations, a mathematical activity that is familiar to students from school algebra that can serve as a useful means of gaining insight into the algebraic structures – like groups (e.g. Wasserman, 2014) and rings (e.g. Cook, 2014) – that form the foundation of abstract algebra.

[1] Here I will only explicate criterion II, as the excerpts of student activity relevant to criterion III were trimmed to comply with space constraints.

Results

Though it is beyond the scope of this brief proposal to comprehensively document the students' entire learning trajectories, here I will present and analyze the key episode of the teaching experiment in which Brian's outright rejection (i.e. *monster-barring* – see Lakatos, 1976; Larsen & Zandieh, 2008) of zero-divisors was repurposed in order to classify algebraic structures in a way consistent with how experts distinguish between integral domains and rings that are not integral domains.

Monster-Barring Zero-divisors in $\mathbb{Z}_{12}$

At this point in the teaching experiment, Brian and Julie had correctly solved several equations in $\mathbb{R}$, including $4x = 0, 4(x - 5) = 0$, and $(x + 2)(x + 3) = 0$. I encouraged them to solve the same equations in $\mathbb{Z}_{12}$, hoping that they would notice the presence of multiple solutions and ultimately identify the failure of the ZPP as the cause. But, just as in $\mathbb{R}$, they both asserted that $x = 0$ is the only solution to $4x = 0$ and $x = 5$ is the only solution to $4(x - 5) = 0$ in $\mathbb{Z}_{12}$, with Brian specifically mentioning that "the only way for 4 times a number to equal 0 is by multiplying by 0." Similarly, Julie's solution to solving $(x + 2)(x + 3) = 0$ in $\mathbb{Z}_{12}$ employed what appeared to be the ZPP and proceeded almost identically to her response to the same equation in $\mathbb{R}$, the only difference being that her solutions were $x = 9$ and $x = 10$ (instead of $x = -2$ and $x = -3$). Brian's response made it clear that he also did not detect any differences between $\mathbb{R}$ and $\mathbb{Z}_{12}$:

Brian:	Uh … what was the point of that?
Researcher:	What was the point of what?
Brian:	That is literally the exact same as normal math.
Researcher:	OK, so … [laughs]. OK, so I want to break this down. What is, what is that? What are you … what is the same as normal math?
Brian:	The way she solved it with $\mathbb{Z}_{12}$ is the exact same way you solve that in normal factoring.

Simpson and Stehlikova (2006) proposed that, in cases in which students struggle to identify critical aspects of an algebraic structure for themselves, instructors should "explicitly guide attention to, first, those aspects of the structure which will be the basis of later abstraction" (p. 368). As my efforts to guide their attention to zero-divisors implicitly via task design were unsuccessful, I decided to heed these recommendations and explicitly point out an instance of zero-divisors. Specifically, referring to the task in which Brian and Julie had proposed that $x = 5$ was the only solution to $4(x - 5) = 0$ in $\mathbb{Z}_{12}$, I asked them about the possibility that $x - 5 = 3$ (see the excerpt below) so that they might recognize that $4(x - 5) = 4 \cdot 3 = 0$. I phrased my inquiry somewhat unconventionally in terms of the element $x - 5$ (as opposed to simply offering $x = 8$ as an additional solution) because I wanted to maintain focus on the equation's product structure and, potentially, the ZPP. Julie immediately realized (and accepted) that they had overlooked such cases, remarking that she had stopped looking for solutions after identifying $x = 5$ because she had only expected one solution. Brian, on the other hand, rejected the possibility of additional solutions:

Researcher:	What do you think, Brian, you don't look, you've got a skeptical look on your face.
Brian:	I still think that this [motions to $4 \cdot 0 = 0$] is 0, right, but this …
Researcher:	So, can you say what you're pointing to right now?
Brian:	The 4, um, as long as $x = 5$, then that's 0, and I think that's the only way

to 0. This is some type of convoluted plan or a scheme you've come up with. There's no way that this is a 0.

Brian's outright rejection of zero-divisors surprised me – I had predicted that he would react like Julie and reluctantly concede that he had overlooked several solutions (which would then have been an opportunity to encourage them to revisit their rule and whether or not it holds in $\mathbb{Z}_{12}$). Instead, however, I decided to explore Brian's reasons for rejecting (the additional solutions created by) zero-divisors. My first conjecture was that perhaps the clock arithmetic metaphor from the initial task that introduced $\mathbb{Z}_{12}$ was influencing Brian's thinking. Perhaps, for example, he viewed $4 \cdot 3$ as 12, and, as a result, did not identify 12 with 0.

Monster-Barring Zero-divisors in $M_2(\mathbb{R})$

To test this conjecture, I shifted to another example structure, thinking that, if Brian raised no objection to zero-divisors in the new context, then I could conclude that the nature of his previous objection was context-specific to $\mathbb{Z}_{12}$. If, however, he maintained his objections, this would indicate that he was potentially objecting to idea of zero-divisors altogether. I chose $M_2(\mathbb{R})$ as the new example structure because it contains zero-divisors, and it would have been familiar to Brian from linear algebra, thus leaving him with fewer reasons to doubt its legitimacy[2]. I asked if their rule held in $M_2(\mathbb{R})$, and Julie, who seemed relatively unperturbed by the presence and effects of zero-divisors in $\mathbb{Z}_{12}$, drew an analogy with $\mathbb{Z}_{12}$ and seemed to accept the possibility of such elements in $M_2(\mathbb{R})$ (though she was unable to identify any at first), remarking that "when I look back ... there are some other ways to get 0 without multiplying by 0, so I think that maybe there could be a way to multiply two matrices so that you can get the zero matrix." Brian, on the other hand, remained steadfast in his apparent belief that the ZPP was universally inviolable, and responded before even trying to produce a counterexample that "in order to get a zero matrix, you have to multiply by 0." I responded by presenting them with a pair of zero-divisors – specifically, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Brian, after multiplying the two matrices together to obtain the zero matrix, again stood by his original assertion:

Brian:	I don't understand how this example … can count. [sighs]
Researcher:	So why, why wouldn't it count?
Brian:	Because you're still … you still have zeros here. Like you literally just added a 1 somewhere, and said, here you go! It works!
Researcher:	OK, um, when you said 'zeros here,' can … unfortunately, the Livescribe pen can't, uh, can't tell us which ones you're pointing to.
Brian:	OK … these ones [motions to and marks the zeros in $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$]. So there are zeros involved.
Researcher:	There are zeros involved.
Brian:	Yes, so I don't think this should, this should count as an example that we can use. I, I just don't believe that, that this is OK.

Because the nature of Brian's objection in this case was that "there are still zeros involved," I responded by presenting him with a zero-divisor pair that did not involve any 0 entries: $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$. This time, after verifying for himself that the product of these two

[2] It is not completely inconceivable that Brian viewed $\mathbb{Z}_{12}$ as a contrivance that I created purely for the purposes of this teaching experiment. There would be no such concerns with $M_2(\mathbb{R})$.

matrices was indeed the zero matrix, he maintained his skepticism, this time on the grounds that 0 was *not* involved:

Brian:	Um, I'm still skeptical because I still think you need zeros to get zeros, and … you're not multiplying A times, er … you're not multiplying A and B together to get 0, um, because A and B have to be 0.

Brian's refusal to accept zero-divisors in both $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$ suggests that his reasons for doing so were not context-specific and that he was indeed objecting to counterexamples to the ZPP in a more general way.

Brian's rejection of zero-divisors across algebraic contexts is an example of *monster-barring*. In his seminal text *Proofs and Refutations*, Lakatos (1976) defined monster-barring as the outright rejection of a counterexample on the grounds that it is "a pathological case" (p. 14). Similarly, Larsen and Zandieh (2008), who repurposed Lakatos's methods for mathematical discovery as design heuristics for RME, characterized monster-barring as "any response in which the counterexample is rejected on the grounds that it is not a true instance of the relevant concept" (p. 208). This includes cases in which students summarily reject a counterexample without an apparent reason. Indeed, several of Brian's comments support the assertion that he viewed zero-divisors as pathological and, as a result, he refused to consider them as counterexamples to the ZPP.

Although monster-barring might, at first, seem to be counterproductive and in need of correction via direct instruction, Lakatos (1976) suggested there was potential for such activity to be productively repurposed, commenting that mathematical ideas "are frequently proposed and argued about when counterexamples emerge" (p. 16). Accordingly, Larsen and Zandieh (2008) proposed that having students consider and render judgments about the validity of proposed counterexamples and underlying definitions is a form of informal mathematical thinking that can be leveraged to support the development of more formal mathematical concepts.

Leveraging Monster-barring activity to sort algebraic structures

During this new line of inquiry, I asked Brian to identify exactly which products he objected to in the multiplication table for $\mathbb{Z}_{12}$. He and Julie responded by turning to their multiplication table and circling entries.

Brian:	So 6 times 2, 6 times 4 …
Julie:	6 times 6, 6 times 8, 6 times 10.
Researcher:	So you're just going down …
Brian:	We're just finding the places that … it doesn't look like a 0 needs to be there. Like it's awkward, like it shouldn't be on the multiplication table. So, numbers that multiply … don't look like they multiply together would equal 0, we'll find they do.
Julie:	6 times 4, there's a 0.
Researcher:	OK.
Julie:	So, like, the same thing with, like, 8 times 3.
Researcher:	And that's, so, Brian, that's what you're calling an awkward …
Brian:	Yes.
Researcher:	Like zero showing up in an awkward place?
Brian:	Yes.
Researcher:	Where, where does … what are the non-awkward appearances of 0?
Brian:	The places where 0, the top row and the first column in the table show that every one of those numbers is multiplied by 0 to get 0. Those are the normal ways … to get zero.

Researcher:	Are there, so are there any normal ways that are not in the first row or the first column?
Brian:	No.

This was an important exchange for several reasons. First, Brian used the phrase "awkward ways to make zero" to refer to combinations of elements in which "it doesn't look like a zero needs to be there ... numbers that ... don't look like they multiply together would equal 0." Similarly, "normal ways to get zero" are those involving multiplication by 0. This mirrors the distinction between the ZPP (which is equivalent to the absence of zero-divisors in a ring) and its converse (which always holds in a ring). Second, Julie, who was relatively unperturbed by zero-divisors, was able to quickly operationalize Brian's distinction, as evidenced by her immediate engagement in the task. I interpreted this as a sign that Brian's criteria could be a meaningful way for Brian (and even Julie) to engage with zero-divisors and use them to make distinctions between algebraic structures. This hypothesis shaped my instructional decisions and analysis in the remaining sessions of the teaching experiment, which involved Brian using his 'awkward' distinction as a means of distinguishing between structures with and without zero-divisors.

To further elicit Brian and Julie's thinking about awkward and normal ways to make zero, I designed classification tasks that prompted them to decide if a given structure behaved more like $\mathbb{R}$ or more like $\mathbb{Z}_{12}$ (as they had already concluded that $\mathbb{R}$ contains no awkward ways to make zero, unlike $\mathbb{Z}_{12}$). The first structures they considered were $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$, both of which they had worked with earlier in the teaching experiment. Brian immediately responded that $M_2(\mathbb{R})$ should be classified as "more like $\mathbb{Z}_{12}$."

Brian:	Definitely $\mathbb{Z}_{12}$.
Researcher:	Why? What makes you so sure?
Brian:	Well, earlier we discussed that $\mathbb{Z}_{12}$ has some awkward ways to make zero and we also talked earlier that the matrices have awkward ways to make zero. Real numbers don't have awkward ways to make zero. So they share that comparison.
Julie:	That does make a little bit more sense because I guess in $\mathbb{Z}_{12}$ three times four is zero. So that would be an awkward way to make zero. You would have to multiply by zero in [the] real [numbers].

Brian's classification of $M_2(\mathbb{R})$ as "more like $\mathbb{Z}_{12}$" suggested that this adaptation to his way of understanding the ZPP might also be generalizable to other contexts. Brian's statements that "$\mathbb{Z}_{12}$ has some awkward ways to make zero" and "the real numbers don't have awkward ways to make zero" are comparable to the more conventional "$\mathbb{Z}_{12}$ contains zero-divisors" and "$\mathbb{R}$ does not contain zero-divisors." Notably, it is not difficult to find superficial similarities between $M_2(\mathbb{R})$ and $\mathbb{R}$: both are uncountably infinite and, moreover, $M_2(\mathbb{R})$ can be viewed as having been constructed from $\mathbb{R}$. The use of Brian's characterization of zero-divisors seemed to supersede such considerations.

Up to this point, Brian had only applied this way of understanding to $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$, the contexts from which it had emerged in his reasoning, both of which contain zero-divisors. Subsequently, I asked Brian and Julie to classify $(\mathbb{Z}_5, +_5, \cdot_5)$, a structure that, based upon purely superficial characteristics, might be classified as more similar to $\mathbb{Z}_{12}$. However, $\mathbb{Z}_5$ contains no zero-divisors and is thus more similar in this regard to $\mathbb{R}$. Initially, both Brian and Julie hypothesized that $\mathbb{Z}_5$ was more similar to $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$ because, Brian predicted, "they're [probably] awkward ways to make 0 for $\mathbb{Z}_5$ as well." As they attempted to justify this conjecture by constructing the operation tables, however, they changed their minds:

Julie:	That is more like the real numbers, actually. The only way we ended up getting zero is multiplying by zero. And so that would be more like the real numbers, because in $\mathbb{Z}_{12}$ we could do awkward ways like three times four and get zero. But in the real numbers we have to multiply by zero, and $\mathbb{Z}_5$ also, to get zero.
Researcher:	Do you agree, Brian?
Brian:	I would say it's like the real numbers, yes, after drawing the table out.
Researcher:	And what about the table changed your mind?
Brian:	Looking over, there are no other zeros where other numbers should be, except for where zero is multiplied by another number.
Researcher:	Yeah. I was gonna ask you about that. So I see zeros in the first row and the first column here. Are those not awkward?
Brian:	No. Those are normal ways to get zero. Multiply by zero.

In the above exchange, both students indicated awareness that the 'normal' ways to get zero are the only such ways – for example, Julie mentioned that "we *have* to multiply by zero … to get zero" and Brian noticed that "there are no other zeros where other numbers should be." This is notable because it demonstrates that both Brian and Julie were able to operationalize the awkward/normal distinction to identify a structure without zero-divisors.

Conclusion

This project addresses the issue that prospective teachers do not see the relevance of their abstract algebra coursework to the secondary mathematics they will be teaching. In response, guided by the tools of *conceptual analysis* (Thompson, 2008), I conducted a teaching experiment (Steffe and Thompson, 2000) that investigated how students might be able to adapt their ways of understanding familiar properties from secondary algebra to be effective in abstract algebra. Focusing specifically on the zero-product property (ZPP), my primary research question was: How might beginning abstract algebra students be able to adapt their existing understandings of the ZPP to be effective in abstract algebra? Though I have not presented the learning trajectory in full here, I did describe and analyze its key component: the repurposing of Brian's monster-barring of zero-divisors.

I believe this study has some implications for thinking about pedagogy in mathematics teaching more broadly. Namely, it provides an example for how students' experiences, even if they seem counterproductive and irrelevant at first, can be leveraged effectively to advance their mathematical thinking in productive ways. I see this as a more specific case of a broader phenomenon – an approach to teaching that builds on students' thinking. Much of the mathematics education literature advocates for such an approach. In fact, these findings were brought to light by applying Steffe and Thompson's (2000) methodological principle that researchers – and, indeed, teachers – should assume that students' behavior is rational and that there is great value in attempting to understand and build upon it. This study adhered to this principle by using Brian's thinking as he engaged with the notion of a zero-divisor. However, even more so, this study indicates that such an approach is possible *even when a students' thinking initially appears to be counterproductive*. This suggests two things to me about instruction in abstract algebra for an audience of secondary teachers. First, abstract algebra instruction can model good pedagogical practices. As was done in this study, using student thinking to develop abstract algebra ideas models good pedagogy. For secondary teachers, learning mathematics in ways that mirror good teaching contributes to their development as teachers. Second, not only can we model good pedagogical practices as abstract algebra instructors, we can also be explicit about this modeling. That is, as instructors, we can draw attention to the ways that we are building on students' thinking in our own classrooms. And, as evident from this study, building on student thinking is possible even in extreme cases, when their ideas appears to be unproductive.

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How do Transition to Proof Textbooks Relate Logic, Proof Techniques, and Sets?

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Many mathematics departments have transition to proof (TTP) courses, which prepare undergraduate students for proof-oriented mathematics. Here we discuss how common TTP textbooks treat three topics ubiquitous to such courses: logic, proof techniques and sets. We show that these texts tend to overlook the rich connections sets have to proof techniques and logic. Recent research has shown that student thinking about sets is propitious to novice students' ability to reason about logic and construct valid arguments. We suggest several key connections TTP courses can leverage to better take advantage of their unit(s) on sets.

Keywords: transition to proof, textbook analysis, logic

Introduction

Over the past few decades, many mathematics departments have recognized the need to help students through two major undergraduate transitions: the transition to college mathematics and the transition to proof-oriented mathematics. A recent survey found that the majority of mathematics departments at research universities have attempted to address the latter by creating 'transition to proof' courses specifically aimed at helping students navigate the challenges of proof-oriented mathematics (David & Zazkis, 2017). The content of such transition to proof (TTP) courses can be quite diverse, but they often include a number of topics that are necessary – though perhaps not sufficient – for learning how to read and write proofs in later courses. Specifically, such courses usually address mathematical logic, sets, and basic proof techniques. We consider these topics necessary but not sufficient because understanding them will not guarantee success in later courses, but violating logical laws, misusing set structures, or using invalid proof techniques will almost certainly undermine later success. Mathematical logic, sets, and basic proof techniques are ubiquitous amongst transition to proof courses (David & Zazkis, 2017), and thus we expect and proof-oriented course to draw upon ideas from each of these domains. Each of these topics also corresponds to an entire field of mathematics – formal logic, set theory, proof theory – so that any one topic could fill an entire course. Instructors of such courses must therefore make careful pedagogical choices about what and how much to introduce from each of these domains.

Little is known, however, about the results of these choices – that is, how logic, sets, and proof techniques are presented in transition to proof courses. To gain insight into this issue, we analyzed how these three topics are covered and connected in commonly used transition to proof (TTP) textbooks. Our inquiry was guided by the following research questions: How are basic ideas of logic, sets, and proof techniques introduced and explained? How do TTP books connect these domains? In what order do they appear?

Literature and Theoretical Perspective

In this section we consult relevant literature on student thinking about sets, logic, and proof techniques in order to present the beginnings of a *conceptual analysis* (Thompson, 2008), a theoretical model that describes "ways of knowing that might be propitious for students' mathematical learning" (p. 46). We operationalized our conceptual analysis as a lens through which to investigate and compare the presentation of these topics amongst our textbook sample.

We chose to focus on these three topics because, in addition to their ubiquity in TTP courses (David & Zazkis, 2017), they each provide some necessary contribution to understanding proof-oriented mathematics. Furthermore, there are common elements of mathematical text that simultaneously draw upon all three topics.

Recent studies on student thinking about logic (Dawkins, 2017; Dawkins & Cook, 2017; Hub & Dawkins, 2018) have investigated how students read mathematical statements prior to being taught formal logic; the students' intuitive approaches and interpretations in these studies were compared to the normative ways of interpreting such language. One of the key findings of this series of studies was that students who connected mathematical categories (e.g. "rectangle," "even," "divisible by 4") to the sets of objects in the category were able to adopt expert ways of reading mathematical language much faster than their peers (who focused on examples or properties). They were also better at forming valid arguments for why quantified statements were true. Moreover, building the truth table for logical connectives was insufficient for students to successfully build strategies that mirrored Venn diagrams unless they were conversant in thinking about sets. In other words, adding quantifiers posed a significant challenge to students' ability to verify and falsify statements and to formalize their ideas about logic, even when they understood the truth table for a connective. Based upon these studies, we contend that being able to relate set ideas to logic and proof techniques is key – that is, thinking about sets is propitious to novice students' ability to reason about logic and construct valid arguments.

Consider the following example of how the ability to move flexibly amongst understandings that center on logic, sets, and proof techniques might afford different insights in the context of interpreting the following conditional statement (which, conceptually, amounts to stating that divisibility is a transitive relation): "Let $a, b,$ and c be integers. If $a|b$ and $b|c$, then $a|c$."

1. Logically, we might assert that the theorem is true because we cannot find three numbers such that $a|b,\ b|c$, and $a \nmid c$. In other words there does not exist a case that makes the antecedent true and the consequent false.
2. Set-wise, conditionals always connect to subset relations. In this case, the theorem can be restated as $\{(a, b, c) \in \mathbb{Z}^3 : a|b \text{ and } b|c\} \subseteq \{(a, b, c) \in \mathbb{Z}^3 : a|c\}$. If we pick any triplet in the first set, we know that it will necessarily be in the second set.
3. In terms of proof techniques, we might say that the property $a|c$ can be inferred from the properties $a|b$ and $b|c$. Alternatively, $a \nmid b$ or $b \nmid c$ might be provable from $a \nmid c$. Lastly, it may be that $a|b$, $b|c$, and $a \nmid c$ are inconsistent.

While these may seem like subtle distinctions, they have the potential to provide potentially valuable information. The first view discusses truth-values or what kinds of triplets of integers exist. The second view emphasizes how the predicates in the theorem range over all of $\mathbb{Z}^3$ and each have a truth-set they represent. The relationship between the antecedent and consequent properties can be understood as a relationship between these truth-sets (Hub & Dawkins, 2018). The third view draws our attention to the inferences available from the hypotheses of the theorem, such as creating equations $a = mb$ and $b = nc$ for some $m, n \in \mathbb{Z}$ and using substitution to proceed with the proof. Stated this way, it seems that the first and last interpretation are most mathematically useful for the work of reading and writing proofs. However, we posit that students should understand – and, as a consequence, TTP textbooks should address – logic and sets because it appears fruitless to be able to write a valid proof if one does not understand the second and third interpretations as entailments of that proof.

Methods

Our objective was to obtain a sample of TTP textbooks that accurately reflect those in widespread use in undergraduate classrooms in the United States. To do so, we leveraged the results of a recent study of TTP courses (David & Zazkis, 2017), which analyzed the syllabi from TTP courses at all institutions categorized by Carnegie designations as high research activity and very high research activity in the United States. The study reported which portion of those courses used a textbook and which textbooks were most commonly used. To ensure that our sample was reasonably representative yet still tractable enough to allow for detailed individual analyses, we selected those textbooks in use at a minimum of 6 universities (as reported by David & Zazkis, 2017). We included one more book intended for inquiry-based TTP instruction in order to guarantee our sample was more diverse in terms of instructional approaches. A complete bibliography of the textbooks in our sample is included after the references.

After obtaining copies of all 10 textbooks in our sample, the data collection process initiated with each researcher independently reading the front matter (e.g. preface, notes to the instructor and/or student) of a particular text to gain insight into any global themes and general strategies for content presentation. Notes were recorded about any approaches that seemed to place strong emphasis on one of our three main topics (logic, sets, and proof techniques). Next, each researcher used the table of contents and the index to identify the places in each text where logic, sets, and proof techniques appeared. We recorded excerpts and quotations that we deemed provided insight into connections between logic, sets, and proof techniques – as described in our conceptual analysis in the previous section – in a spreadsheet. Each textbook was reviewed by at least two members of the research team. We used *constant comparison* (Creswell, 2007; 2008) of textbook materials to identify common themes across the data set, including common sequences in which logic, proof techniques, and sets appeared in each text and how that might have influenced their presentation of each.

Results, part I: Overview of Textbook Sample

Four general points emerged from a global analysis of our entire sample[1]. First, sets appeared to be the one element that varied in position most widely across the texts. Collectively, logic (L), quantification[2] (Q), and proof techniques (P) most often appeared in the order L – Q – P (seven texts) or Q – L – P (two texts). Sets almost evenly varied between appearing first (four texts), in the middle (three texts), and last (three texts).

Second, the most common connection that textbooks made among the logic, proof techniques, and sets was to explain or justify proof techniques using truth tables. We will consider some examples of these explanations in a later section.

Third, about half of the texts connected logic and sets in explicit ways. Four textbooks explained set ideas using logical structure. This seems a natural approach since one can translate set operations – $A \cup B$ – into set-membership conditions with logical connectives – $\{x\colon x \in A \text{ } or \text{ } x \in B\}$. Alternatively, a natural way to introduce the notion of set itself is through truth-sets for predicates (e.g. the set of multiples of 4, the set of divisors of 52). While some books used

[1] It is beyond the scope of this brief proposal to present the results of our analysis of each textbook in our sample. Those who are interested to see such details may follow this link (https://www.dropbox.com/s/jhrxo8bajhzon8s/Rume2019proposal-table.pdf?dl=0) to a table that includes (1) the authors and names of the books we analyzed, (2) the order in which logic, proof techniques, quantification, and sets appeared, and a brief summary of how each textbook connected the topics in question.

[2] Quantification is often understood as part of logic, but we found it useful to distinguish it because some books dedicated long sections to only propositional logic (without quantifiers) and others dedicated more time to predicate logic (quantified). Also, quantifiers themselves varied from being treated as logical constants to being phrases in mathematical language. In other words, quantifiers sometimes were treated more logically (in terms of truth-conditions) and other times more linguistically (what do these phrases mean and how do we use them).

this as an introduction (i.e. the first explicit mention of sets), they often shifted to talking about sets as general collections without some underlying predicate. Only one book explicitly built the truth-table for a (quantified) conditional statement by considering the truth of an example statement on various sets of inputs. In this case, the set structure guided the exposition of logic.

Fourth, sets generally played no role in explaining or justifying proof techniques. Rather, the primary examples of connections between proof techniques and sets occurred when sets were discussed last and thus the other topics informed the exposition of sets.

Results, part II: Analysis of Illustrative Excerpts from Textbook Sample

The above summary of our textbook analysis findings suggests that TTP textbooks frequently link logic and proof techniques and with some regularity connect sets to logic. Sets in particular appear the most isolated of the three topics. This forms a simple descriptive account of current TTP curricula. We pursue two goals hereafter. First, we will provide some excerpts from the textbooks that illustrate the nature of the connections between logic, sets, and proof techniques to recognize some qualitative differences that likely matter for student sense making. We shall also note some potential connections that, according to our conceptual analysis, could have been made that were not, specifically with regard to sets.

As stated above, the most common connection TTP books made among logic, proof techniques, and sets was to motivate proof techniques for conditional statements by demonstrating their validity through the use of truth tables. Below we provide some excerpts from the books that illustrate how this was done. Overall, we notice that the books draw upon diverse resources to help students make sense of proof techniques.

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion.

Step	**Reason**
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Figure 1. Rosen's (2012, p. 74) example proof connecting proof techniques to rules of inference.

A **direct proof** of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true. You will find that direct proofs of many results are quite straightforward, with a

Figure 2. Rosen's (2012, p. 82) explanation of direct proof techniques using truth tables.

From an early stage, Rosen (2012, Fig 1) invites students to cite rules of inference (e.g. "contrapositive" and "hypothetical syllogism") as warrants in proofs. The example theorem does not concern mathematics and the author immediately replaces the propositions with logical variables to construct a proof in logical syntax. The text's later examples are mathematical and quantified and Rosen uses predicates to explain proof by universal generalization. However, when the author explains the proof technique (as shown in Fig 2), the language shifts back to propositional variables and "assumption" of the hypothesis rather than selecting an arbitrary element of the truth set of the hypothesis predicate.

The table shows that if P is false, the statement $P \Rightarrow Q$ is automatically true. This means that if we are concerned with showing $P \Rightarrow Q$ is true, we don't have to worry about the situations where P is false (as in the last two lines of the table) because the statement $P \Rightarrow Q$ will be automatically true in those cases. But we must be very careful about the situations where P is true (as in the first two lines of the table). We must show that the condition of P being true forces Q to be true also, for that means the second line of the table cannot happen.

This gives a fundamental outline for proving statements of the form $P \Rightarrow Q$. Begin by assuming that P is true (remember, we don't need to worry about P being false) and show this forces Q to be true. We summarize this as follows.

Outline for Direct Proof

Proposition If P, then Q.

Proof. Suppose P.

$\vdots$

Therefore Q. ■

Figure 3. Hammack's (2013, p. 92) explains direct proof of conditionals using the truth table.

Hammack's (2013, Fig 3) representations, which closely mirrored several others, present general proof frames using propositional variables, though mathematical example proofs appeared nearby for comparison. He explains the initial step "Suppose P" in light of the fact that $P \Rightarrow Q$ is always true when P is false (similar to Rosen). Interestingly, the examples all involved predicates, but Hammack presents the proof techniques using only the proof table and propositional variable.

2.21. Remark. *Logical connectives and membership in sets.* Let $P(x)$ and $Q(x)$ be statements about an element x from a universe U. Often we write a conditional statement $(\forall x \in U)(P(x) \Rightarrow Q(x))$ as $P(x) \Rightarrow Q(x)$ or simply $P \Rightarrow Q$ with an implicit universal quantifier.

The hypothesis $P(x)$ can be interpreted as a universal quantifier in another way. With $A = \{x \in U : P(x) \text{ is true}\}$, the statement $P(x) \Rightarrow Q(x)$ can be written as $(\forall x \in A)Q(x)$.

Another interpretation of $P(x) \Rightarrow Q(x)$ uses set inclusion. With $B = \{x \in U : Q(x) \text{ is true}\}$, the conditional statement has the same meaning as the statement $A \subseteq B$. The converse statement $Q(x) \Rightarrow P(x)$ is equivalent to $B \subseteq A$; thus the biconditional $P \Leftrightarrow Q$ is equivalent to $A = B$.

2.24. Remark. *Elementary methods of proving $P \Rightarrow Q$.* The *direct method* of proving $P \Rightarrow Q$ is to assume that P is true and then to apply mathematical reasoning to deduce that Q is true. When P is "$x \in A$" and Q is "$Q(x)$", the direct method considers an *arbitrary* $x \in A$ and deduces $Q(x)$. This must not be confused with the invalid "proof by example". The proof must apply to every member of A as a possible instance of x, because "$(x \in A) \Rightarrow Q(x)$" is a universally quantified statement.

Remark 2.20f suggests another method. The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. The equivalence between a conditional and its contrapositive allows us to prove $P \Rightarrow Q$ by proving $\neg Q \Rightarrow \neg P$. This is the *contrapositive method*.

Remark 2.20c suggests another method. Negating both sides $(P \Rightarrow Q) \Leftrightarrow \neg[P \wedge (\neg Q)]$. Hence we can prove $P \Rightarrow Q$ by proving that P and $\neg Q$ cannot both be true. We do this by obtaining a contradiction after assuming both P and $\neg Q$. This is the *method of contradiction* or **indirect proof**. We summarize these methods below:

Figure 4. D'Angelo and West's (2000, pp. 34,35) explanation of conditional proof methods with reference to quantification.

D'Angelo & West (2000, Fig 4) directly address how proofs of conditionals verify quantified claims making use of the connections they previously established between sets and logical relations. The explanation uses logical variables, though the authors immediately provide mathematical examples thereafter. D'Angelo and West's explanation seems to provide the most attention to the sets underlying the predicates while still using logical variables for exposition.

Thinking about an example should help. Consider the statement "If x is an even integer, then x^2 is an even integer." I suspect that when you conducted the "thought experiment" you decided that this is true. It is a case in which there are infinitely many values of x that make the hypothesis true. So we will have to assume (in the abstract) that x is even and then show that x^2 has to be even, too.

If we are to get anywhere, we first have to recall what it means to say that an integer is even:

When an implication is proved by assuming that the hypothesis is true and then showing that the conclusion is also, the proof is called a **direct proof.**

Here is the proof that if x is an even integer, then x^2 is an even integer.

Proof. Suppose that x is an even integer. Then by definition of even integer, we know that there must exist an integer y such that $x = 2y$. Now we have to show that there is an integer w so that $x^2 = 2w$. Let $w = 2y^2$. Since the product of integers is an integer, $w = 2y^2$ is an integer. Notice that

$$x^2 = (2y)(2y) = 2(2y^2) = 2w.$$

Thus x^2 is an even integer. ■

This argument works for *any* even number; thus all cases have, in some sense, been checked.

Figure 5. Schumacher's (2001, p. 32-33) explanation of direct proof of a quantified conditional.

Schumacher's (2001, Fig 5) presentation attends more directly to quantification, though the quantifiers themselves stay implicit throughout. Her example theorem is mathematical and she does not rely on logical variables to present the proof. [3] She points out that the hypothesis of the theorem is true for infinitely many values of x, so the proof must work for all such values. Woven throughout the exposition is the assumption that "assuming that the hypothesis is true" is tantamount to selecting (any) even value of x.

Discussion

To summarize, the presentations of proof techniques vary from constructing derivations within a propositional logical calculus (in which every step is validated by a rule of inference) to mathematical proofs (in which familiar mathematical content is written in paragraph format using warrants that would likely be familiar to TTP students). Many of the presentations exist between these poles of operating in a logical calculus and examining actual mathematical proofs. Many books explain patterns or strategies in proof construction using logical variables with varying levels of attention to the quantification structure that is present in most of the mathematical proofs constructed later in each text. We offer two primary observations about how these common intermediate approaches may be problematic for students.

First, these textbooks tend to use propositional variables to explain proof techniques that are almost always applied to situations involving predicates. We are sensitive to this trend in light of our experiences researching how novice students interpret mathematical language. When many students read a phrase such as "x is an even number," they are frequently drawn to select a representative even number (or to think about properties such as the units digit being even). Many students need guidance to understand the way that mathematicians infer that this phrase almost always implicitly refers to any even number (unless x is already a bound variable). By referring to these phrases in proofs as propositions, we worry that these TTP texts might reinforce this limiting trend in student reasoning. Assigning truth-values ("assume P is true") does not help students attend to the underlying set structure ("select an arbitrary x from the set of even numbers"). Similarly, the suppression of quantification is common in mathematical proof writing. Indeed, there are likely many familiar theorems that we have never thought about using the subset interpretation mentioned by D'Angelo and West (2000; Fig 4). Our contention is that texts that teach students how to read and write proofs (maybe for the first time) might need to give students more time to understand the role of quantification and sets in proof techniques before these ideas can be left implicit. This matter becomes especially challenging for students when we consider falsifying statements by counterexample or negating statements.

Second, representing proof techniques using logical variables may preclude students' ability to make sense of the set structure that underlies common proof techniques. What we mean is that when students read a meaningful mathematical predicate such as "x is even," "$a|b$," or "$2n^2 + 3$ is a multiple of 5," there is at least the opportunity for them to reason about the truth set of the predicate. However, when TTP books explain proof techniques using logical variables such as P, we expect students to find thinking about $\{x \in U: P(x)\}$ to add little insight. In contrast, we concur with Schumacher's (2001) effort to draw students' attention directly to the way that proofs written using definitions apply to all objects that satisfy the definition. This is part of what Dawkins (2017) refers to as *reasoning with predicates*, which refers to students' propensity to associate with any mathematical category the set of objects in the category. In our research, we

[3] Earlier in the text she invited readers to prove logical equivalences or differences using truth tables, noting that the logical variables there stood for predicates.

find that students do this much more easily with familiar categories such as even numbers, multiples of a, or factors of b. This seems reasonable since they have had experience with such sets since grade school and can anticipate how those sets would be populated. Students need some guidance and experience thinking about the truth sets of negatively stated predicates ("f is a non-continous function") and unfamiliar categories ("$2n^2 + 3$ is a multiple of 5"). Once again, we acknowledge that experts may often write proofs without thinking explicitly about these underlying sets. We contend, though, that novices often do not find such connections immediate when they are learning to read proofs; reading valid proofs without such understanding leaves something to be desired.

Conclusion

We close by proposing a few goals for TTP instruction. We prioritized these goals because 1) our research leads us to question whether students will make these connections unless they are explicitly accounted for in instruction, and 2) our textbook analysis herein reveals that sets are the most underdeveloped of the three core topics we examined.

1. Recognizing that every predicate entails an underlying truth set and membership in any set can be understood as a predicate. We anticipate that it might be helpful to build up to this generalized relationship by starting with familiar sets (even numbers), before moving to property-based predicates ($\{n \in \mathbb{Z}: 5|2n^2 + 3\}$), before thinking about generalized predicates ($P(x)$ is true if $x \in \{1,5,7\}$).
2. Recognizing the set over which the predicates in a theorem range. Many theorems involve a number of variable elements that each constitute a variable in the theorem's predicates. Helping students attend to the variables and their scope is an important part of understanding what a theorem says and what a proof accomplishes. Indeed, this seems one of the most natural ways to see the importance of Cartesian products of sets.
3. Connecting the various ways to interpret mathematical texts listed above: the statement "Suppose [$P(x)$ is true]" can be thought of as assuming the hypotheses true, selecting an arbitrary x in the scope of the predicate P, as beginning proof by universal generalization, or as providing the assumptions from which we must deduce the theorem's conclusions. Part of the work of the TPP course is to help students understand why all of these are accomplished by the same text.

Overall, our analysis of Transition to Proof texts revealed that textbooks intended for such courses frequently connect logic and proof techniques, and connect logic and sets. However, they infrequently connect sets to proof techniques. Indeed, analyzing the representations used to introduce proof techniques reveals that it would be hard to make sense of the underlying truth sets because hypotheses are so often represented by logical variables. Our research suggests that students need help thinking about the underlying sets and that this can help them reason about logic and argumentation. Accordingly, we argue that TTP courses should help students connect assumptions of truth with arbitrary selections from particular sets. We offer this reflection to encourage instructors to think about and attend to the potential for such connections in TTP courses. Ultimately, we hope such considerations can help more of our students succeed in learning how to read, write, and truly understand mathematical proving, thereby gaining access to its great epistemic power.

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Participation in a Mathematical Modelling Competition as an Avenue for Increasing STEM Majors' Mathematics Self-Efficacy

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Though scholars have long called for applications and modeling to be explicitly added to classroom agenda (Niss, Blum, & Galbraith, 2007), opportunities for undergraduates to engage in modeling in the classroom remain scarce. We report a study of undergraduate STEM majors engaging in authentic, open-ended modeling tasks using differential equations through a modeling competition. In this study, we propose a logic model that captures the relationship between the advantages of mathematical modelling and mathematics self-efficacy (MSE) and investigate the extent to which a mathematical modeling intervention increased STEM majors' Mathematics Self Efficacy (MSE).

Keywords: Mathematical Modeling, Self-Efficacy. Mathematics Self Efficacy

Educators have increasingly turned to mathematical modelling to resolve the relevance paradox of mathematics in the curriculum. The relevance paradox refers to the disparity between the objective relevance of mathematics for society and the subjective irrelevance of mathematics perceived by many students who study it (Niss & Hojgaard, 2011). Perceiving mathematics as relevant to their goals maintains student interest in mathematics (Liebendörfer & Schukajlow, 2017), and leads to persistence (Business-Higher Education Forum [BHEF], 2010). In addition to interest, mathematics proficiency and self-efficacy have an interrelated effect on student persistence in mathematics (BHEF, 2010). Self-efficacy plays a role in all aspects of goal-setting, perusal, persistence, and effort (Bandura, 2006), and it has been specifically implicated as a factor in choosing to take mathematics courses and a STEM career (Betz & Hacket,1983). Without interest to learn mathematics, students are less likely to engage in meaningful learning, build self-efficacy, and ultimately develop mathematical proficiency requisite for persistence in STEM careers.

Mathematical modeling experiences potentially address the relevance paradox because in mathematical modeling students use their mathematical knowledge to solve authentic, real-world problems. However, any instructional innovation targeting interest, including modeling, will play a supportive role only insofar as the experiences help to develop both mathematics proficiency and self-efficacy (Lauremann, Tsai & Eccles, 2017). At the post-secondary level, research has documented some of the positive impacts of mathematical modeling experiences on mathematics proficiency (e.g., Author, year; Kwon, Allen, & Rasmussen, 2005). However, at this level, the impact of modeling experiences on Mathematics Self efficacy (MSE) remains an open question. At the middle grades and secondary level, researchers have documented this link between modeling experiences and self-efficacy (Krawitz & Schukajlow, 2017). We conjecture that an extra-curricular modeling competition replicating instructional features associated with increasing mathematics proficiency, self-efficacy, and interest will similarly promote gains in self-efficacy for post-secondary students.

In this paper, we outline a logic model relating mathematical modeling experiences, self-efficacy, proficiency, and interest. We then present a study of the impact of an extra-curricular mathematical modeling competition on STEM students' Mathematics Self Efficacy (MSE). The competition focused on differential equations, because of its presence in STEM major requirements. Drawing on the logic model, we argue that replicating aspects of mathematical modeling experiences that impact middle grades students' self-efficacy is a promising avenue for building STEM students' MSE.

Empirical Background

Twenty-first century education can be characterized by an ever-increasing need for STEM graduates. Unfortunately STEM majors leave their programs at high rates (some estimate as high as 48%, (Chen & Soldner, 2013). Some factors that contribute to attrition are gender, conceptual understanding, and self-efficacy (Geisinger, 2013). Additionally, women are still vastly underrepresented in mathematics-intensive fields, like engineering and computer science (NSF, 2011). Although women are more likely than men to attend college, they are less likely to pursue mathematics-related careers (Perez-Felkner, McDonald & Schneider,2014), a disparity evident by third grade (Lubiensky et al 2013). Proposed causes for attrition are: perceived relevance, mathematical proficiency, and MSE. STEM students do not see mathematics as relevant is that mathematics classes often do a poor job of showing how course content fits into STEM careers (Jahn & Myers, 2015). Lacking mathematical proficiency hinders their persistence in a STEM field. Engineering students reportedly struggle not only with algebraic skills, but also with abstract concepts and mathematical modeling; "somehow [students] think that when engineering starts mathematics stops" (Varsavsky,1995, pp 344). Likewise, in physics, failure to connect mathematical knowledge to real-world knowledge hinders successful problem solving (Black & Wittmann, 2009). At the same time, STEM majors' mathematics-related SE mediates their interest and persistence in mathematics. It is related not only to proficiency but also to decisions to persist (Chemers, Hu & Garcia, 2001; Estrada et al, 2011).

Existing research informs us that articulating positive self-efficacy beliefs can be impacted through participation in constructive experiences at the post-secondary level (e.g. Shaw & Barbuti, 2010). Chemmers and colleagues (2011) reported that three kinds of experiences may lead to positive self-efficacy beliefs: research experience, community involvement, and instrumental socioemotional mentoring. Research experiences that foster a sustained interest in STEM include: extend and apply lessons from their classrooms to authentic scientific inquiry (Pender et al, 2010), collaborating with peers and connecting with faculty (Eagan et al, 2013), integration into social systems of STEM community activities (e.g., competitions, summer camps, or bridge programs) (Estrada et al, 2011), and extracurricular experiences (VanMeter-Adams et al, 2014). The mechanisms through which community activities work to increase self-efficacy include mentorship, collaboration, writing, hands-on exercises, and targeting mathematics competencies (Findley-Van Nostrand & Pollenz, 2017). Finally, learning environments that include emotional support from the teacher, academic support from peers, and encouragement from the teacher to discuss work as well as focus on mastery and feelings of efficacy, motivate students to persist in difficult activities (Patrick, Ryan, & Kaplan, 2007).

We claim that mathematical modelling has the potential to address the relevance paradox, develop mathematics proficiency, and build MSE in specific contexts. Mathematical modelling means using mathematics to solve non-mathematical problems: transforming a real-world problem into a mathematical problem to solve, solving the mathematical problem, and using the results to address the initial real-world problem. Several studies reflect that teaching with a mathematical modeling approach can positively impact students' learning of differential equations content. For example, Author (year) found that consistent emphasis on mathematical modeling principles, even in lecture, could positively impact engineering students' learning of differential equations. Others have shown that drawing on "experientially real" starting points to instruction can positively impact student learning of content (e.g., Rasmussen & Blumenfeld, 2007). Schukajlow et al. (2012) found that for ninth graders, student-centered instruction using modeling tasks was the most beneficial for increasing student affect and Zbiek and Conner (2006) reported that a modeling approach deepened prospective secondary mathematics teachers' motivation to learn new mathematics content. What remains to unknown is whether and how engaging in mathematical modeling might impact STEM students' MSE.

Mathematical Modelling Competition Intervention

Educational interventions should target proficiency, self-efficacy, and interest because projects that target only one may produce non-optimal results (Lauremann, Tsai & Eccles, 2017, pp 1542). Since STEM students' persistence is tied to their MSE, there is a need to study and document the advantages of modeling for MSE in advanced mathematics. We propose a logic model (Figure 1) that captures this relationship. In the present study, we investigated one aspect: *the extent to which a mathematical modeling intervention increased undergraduate STEM majors' MSE*.

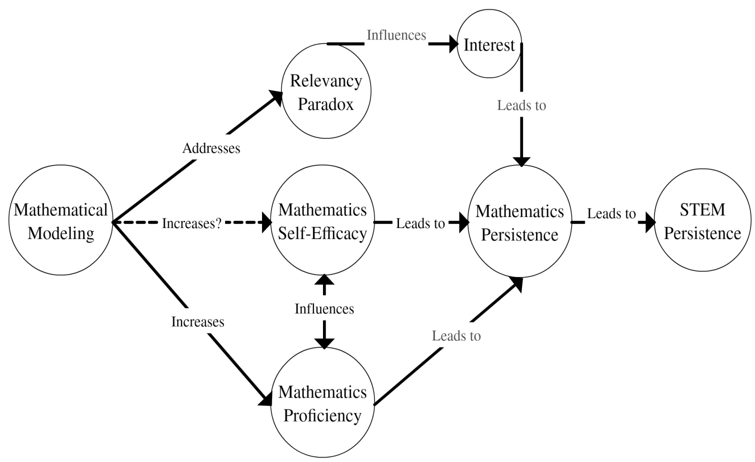

Figure 1 Logic model relating modeling, interest, MSE, and proficiency to persistence

Since mathematical modeling promotes interdisciplinary thinking (Bliss et al., 2016) and fosters mathematical reasoning as a basis for decision making (OECD, 2017), modeling experiences can develop mathematical proficiency in ways that are valued by STEM disciplines. Because undergraduate classrooms typically have little time and instructor support for innovative pedagogical strategies (e.g., Johnson, Keller, & Fukawa-Connelly, 2017), classrooms alone may be unable to replicate optimal learning environments. Extracurricular activities are promising for engaging students in mathematical modeling.

The Systematic Initiative for Modeling Investigations & Opportunities with Differential Equations (SIMIODE) hosted an intervention (the Student Competition Using Differential Equations Modeling [SCUDEM]) that engaged students in authentic modeling problems in differential equations. The competition presented students with a choice of challenging real-world problems, that require genuine inquiry into the mathematical and contextual aspects of the problem. Students from around the US formed teams of three (or two) and selected one of three situations given to them. Each team was led by a faculty member coach from their home school. Teams had one week to work on their chosen problem prior to convening at a local host site. The teams turned in a 2-page executive summary describing their solution to the problem. At the local host site, the faculty coaches met as a panel to discuss the executive summaries, providing constructive feedback, observing weaknesses or inaccuracies in the models, or suggesting directions for improving the models. Each team then had 2.5 hours to address the panel's concerns and put together a 10-minute presentation to communicate their final models to their peers and to the panel to convince their audience that theirs was the best model. In this way, competitors were encouraged to select challenging tasks, record and communicate their thinking, apply or develop mathematical knowledge and contextual knowledge, and engage in the target modeling competencies all while working with teammates and a knowledgeable faculty mentor.

The modeling competition replicates the important aspects of modeling that have been associated with developing interest and proficiency in mathematics. Our goal was to study the modeling competition as an example of an extra-curricular learning environment. We addressed the question: *To what extent does participation in the modeling competition impact student's MSE?*

Theoretical/Conceptual Framework

For the purposes of this study, we operationalized mathematical modelling as producing a conceptual system to describe, interpret, explain, or predict a real-world situation and expressing the conceptual system in conventional mathematical terms. In this cognitive view, mathematical modelling is represented as a cyclic process comprised of distinct activities referred to as *modelling competencies*: formulating a task (identifying important relationships from the real-world situation)), systematization (identifying variables, making assumptions, estimating parameters), mathematization (representing the entities and relationships in mathematical form), mathematical analysis (using mathematical methods to arrive at mathematical conclusions), interpretation of results (re-contextualizing the mathematical result), validation (comparing the model to real-world or known information, establishing limitations of the model's scope), and communicating (sharing the model) (Blomhöj & Jensen, 2003).

We operationalized an individual's self-efficacy about an activity as their perceived capability to carry out that activity. MSE is then an individual's perceived capability to carry out mathematical activities. Since, at the level of differential equations and for STEM majors, mathematical activities include both setting up and solving mathematical problems that arise from real-world problems, we interpret MSE to mean an individual's perceived capability to carry out the interrelated activities that make up mathematical modelling. We asked participants to self-assess their MSE (measuring their confidence in their capability, not their actual capability). We specifically selected the competencies *systematization*, *mathematization*, *validation* and *communication* because they are the most difficult to learn and because *analysis* is usually the focus of mathematics coursework.

Methods

A total of 393 students from 85 schools participated in the competition at 40 host sites around the U.S. Of these, 266 completed the pre-competition survey and 107 completed the post-competition survey. In the analysis, we considered the 90 of 393 students who completed both the pre- and post-competition surveys. Based on the theory of self-efficacy measurement by Bandura (2006), the study used a pre-experimental matched pre and post design to investigate changes in student self-efficacy regarding the following modelling competencies: *systematization* (identifying variables, making assumptions, estimating parameters), *mathematization* (deriving a differential equation), *validation* (comparing the model to real-world or known information, establishing limitations of the model's scope), and communication (sharing conclusions). Participants rated their confidence on 7 statements on a 100 point Likert type scale (see Table 1) The survey also recorded demographic questions such as gender, major, anticipated graduation year, and mathematics courses taken.

A principal component analysis (Abdi & Williams, 2010) was conducted on the full set of pre-test data to explore dimensionality. A pair-wise items correlation was performed and a Cronbach's alpha (Cronbach,1951) was calculated on the set of seven self-efficacy items on the pre and post tests. These analyses suggest that the MSE scale is an internally consistent, unidimensional instrument with high face and construct validity. Therefore, we treated the sum of responses to items on the pre- and post-competition surveys, respectively, as MSE_pre and MSE_post and define gains in modeling self-efficacy as MSE_post – MSE_pre.

We address the following research hypotheses: (1) participating in the modeling competition led to self-efficacy gains, (2) participating in the modeling competition may decrease differences between men and women with regards to their self-efficacy, and additionally the following research question (3) How do different groups, on average, gain in MSE? A matched pair t-test determined there were statistically significant gains in MSE from pre- to post-competition. A comparison of mean MSE_pre and compared mean MSE_post by gender tested hypothesis (2). We answered (3) by statistically model gains in MSE while accounting for the structure of the dataset using hierarchical linear modeling (HLM) techniques (Raudenbush & Bryk, 2001) via SPSS.

Results

Claim 1 - Participating in the competition led to gains in MSE: Across the 90 participants who completed both the pre- and post-competition survey, the mean score on MSE_pre was 444.44 (sd = 134.74) and the mean score on MSE_post was 518.00 (sd = 104.78). The mean individual gain (calculated as MSE_post – MSE_pre-for each individual) was 73.56 (sd 107.92). A paired samples t-test indicated that the individual gain was statistically significant (t(89)=-6.466, p<.001). This gain reflects an effect size of d=0.55, a medium effect size. Mean responses to individual MSE questions are in Table 1.

Table 1. self-efficacy survey, keyed to modeling competencies

Rate your level of confidence by recording a number from 0 to 100 using the scale given below: **0 10 20 30 40 50 60 70 80 90 100** **Cannot do at all Moderately can do Highly certain can do**	**Competency**	**Pre Mean (SD)**	**Post Mean (SD)**
Create a differential equation model for the spread of smart home appliances in the United States during the twenty-first century.	Mathematize	54.67	67.11
In (1) identify the important variables leading to a reasonably accurate prediction.	Identify variables	62.67	74.67
In (1) make simplifying assumptions to reduce the number of important variables.	Make assumptions	59.89	75.67
In (1) consult appropriate resources to check whether your model was reasonable.	Validate	66.36	73.86
In (1) list the real-life and mathematical limitations of your model.	List limitations	67.56	78.89
In (1) create a short presentation to convince a smart appliance manufacturer that they could rely on your model to develop their business plan.	Communicate findings	62.56	74.11
Given a differential equation which describes the rate of formation of material A, $A'(t) = \alpha A(t)^{\beta}$, and a data set of observations for time, t, amount of material A at each	Estimate parameters	71.33	74.56

time t, you could estimate the parameters α and β.

Claim 2 - Gender disparity in MSE decreased after the competition: Among the 58 men, the mean MSE_pre-score was 462.41 (sd = 123.26). Among the 31 women, the mean MSE_pre-score was 409.03 (sd=151.56). An independent samples t-test, assuming equal variance, showed a borderline significant difference between the men and the women (t(87)=1.795, p=0.076) with a small effect size d=0.40. For men, the MSE_post score was 523.45 (sd=102.92) and women's MSE_post score was 503.87 (sd=109.87). An independent samples t-test, assuming equal variance, revealed no statistically significant difference between genders (t(07) =.840, p=.40) and a small effect size d = 0.14. This pair of univariate analyses established a difference between men's and women's MSE as they entered the competition and that the difference decreased after the competition.

Claim 3 – (A) Women experienced more gains in MSE than Men, and (B) Students who did not take differential equation experienced larger MSE gains: Even though women were estimated to enjoy MSE gains of 30.360 beyond equivalent men participating in the competition, gender did not have a statistically significant effect on MSE gains. (See left panel of table 3.) A participant who had not taken differential equations was estimated an MSE gain of 133.24, a large effect size of d=0.99. A participant who had taken differential equations was estimated to have increased 59.8 on the MSE instrument, with a small effect size d=0.44. The difference in gains between the groups was significant (See right panel of table 3). The HLM analysis revealed that participants experienced gains in their MSE from before to after the competition and that these gains were largest for those who had never taken differential equations.

Table 2 and Table 3 present the coefficient estimates for the effects in the HLMs with the final HLM presented in the right-most panel of 3.

Table 2 .Unconditional HLM

Unconditional HLM				
	Coeff	SE	df	p
Intercept	69.835369	14.899584	26.031	0.0000
	Variance	SE		
Level 1 Residual	8646.740100	1592.289984		
Level 2	3172.757189	1836.480544		

Table 3. Conditional HLMs

Conditional HLMs								
	Coeff	SE	df	p	Coeff	SE	df	p
Intercept	131.715661	37.595201	88.084	.001	133.235498	29.892652	82.383	<.001
Gender	-30.359506	21.192516	83.029	.156				
DiffCalc	16.177674	34.131681	75.588	.637				
DiffEQ	15.852225	22.655806	80.821	.486	-73.434996	30.568948	88.071	0.018394
LinAlg	-77.627974	32.711868	88.841	.020				
UsualGrade	4.800528	8.562256	87.750	.576	-29.008721	17.675959	80.914	0.104650
GraduationYear	-31.784291	17.865118	82.786	.079				
	Variance	SE			Variance	SE		
Level 1 Residual	7239.029113	1348.147958			7470.731888	1385.216979		
Level 2	3070.872030	1696.941897			3180.054413	1733.936735		

Discussion

Our exploratory, pre-experimental intervention study drew on theory built up by prior mathematics and STEM education research in order to address one small aspect of the "leaky STEM pipeline." We do not claim that any one characteristic of the competition has directly resulted in gains in MSE, only that the combination of features like communicating one's work, working in teams, working with a faculty mentor, revising reasoning, and working on challenging, authentic problems led to observable gains in MSE.

Our study of the impact of the differential equations modeling competition on STEM majors' self-efficacy used a naturally-occurring pre-experimental design without a treatment group, and so there are standard threats to internal validity. First, participants join the modeling competition of their own volition. It is possible that those who self-select into a modeling competition are already predisposed to engage meaningfully in mathematical activities. If so, then our results may not represent the full potential impact of the competition on the general population of secondary/post-secondary STEM students. Yet, even within this potentially exceptional group we still observed general trends such as gendered disparity in gains that are commonplace in the general population. Future work could follow up by including competition participation as part of regular course requirements to generate a more representative sample.

We have not yet explored the potential impact of coaches on participants' self-efficacy. For example, coaches allowed varying degrees of autonomy to the participants and their own self-efficacy with regards to modeling with differential equation should be accounted for. A larger data set could allow for a 3-level design (as opposed to our 2-level design), nesting participants into coaches into sites. Such a design would take coach background (e.g., experience in coaching competitions) and their own self-efficacy (e.g., teaching differential equations with a modeling approach) into account along with individual-group correlations as covariates to aid in understanding the conditions that affect growth in STEM students' self-efficacy.

In summary, we can conclude the following: (1) Individual MSE increased after participating in the modeling competition, (2) The difference between men and women's MSE decreased after partaking in the competition, and (3) Students who did not take differential equations before experienced greater MSE gains after the competition. In conclusion, our data and the literature review provide further evidence that extra-curricular interventions can be a fertile ground for building undergraduate students' mathematics self-efficacy, perhaps in ways that cannot be achieved in the classroom. Since the extra-curricular environment in this case was designed around principles of mathematical modeling, and many students had no previous coursework in the mathematics being used to model, it provides grounding for claims about the potential of modeling to facilitate STEM majors' persistence through ameliorating their self-efficacy.

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Observing Students' Moment-by-Moment Reading of Mathematical Proof Activity

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This study presents findings from a series of interviews in which we observed undergraduate students' moment-by-moment Reading of Mathematical Proof (ROMP) activity. This methodology is adapted from a validated assessment of narrative reading comprehension developed by cognitive psychologists. We demonstrate the fruitfulness of the method by describing four relatively novel phenomena that we observed in our interviews, and highlight ROMP activities that seemed to distinguish less productive and more productive readers.

Keywords: Proof, Reading, Transition to Proof, Systemic Functional Linguistics

Much of students' apprenticeship in advanced mathematics at the tertiary level involves learning how to read and write mathematical proof. Mathematics educators have studied this transition in terms of students' ability to comprehend proofs after reading (e.g. Mejia-Ramos, Lew, de la Torre, & Weber, 2017), validate proofs (e.g. Alcock & Weber, 2005; Inglis & Alcock, 2012), and write proofs (e.g. Weber, 2001). Fewer studies have investigated the reading of mathematical proof (ROMP) process itself (Weber, 2015). In this paper, we present findings from our adaptation of a moment-by-moment reading assessment method developed by psychologists for studying narrative text reading (Magliano & Millis, 2003; Magliano, Millis, Team, Levinson, & Boonthum, 2011). That methodology of read aloud interview protocols and line-by-line presentation provides different insights into narrative reading than those provided by end-reading comprehension tests. Similarly, we argue here that our method reveals a different set of sense-making activities than has previously been documented. We also contribute to the literature by comparing the ROMP behaviors of novice readers and more experienced readers.

Relevant Research Studies

Our study builds directly on the work of cognitive psychologists Magliano, Millis, and their team, who developed the Reading Strategy Assessment Tool (RSAT) (Magliano et al., 2011). RSAT is a validated measure of reading comprehension. It presents students with single lines of text and asks students to think aloud about each line. The nature of the inferences that students make indicate their relative competence as a reader in the following way: students who connect the given line to previous lines (*bridging inferences*) or to their outside knowledge (*elaborating inferences*) tend to have higher comprehension than students who merely restate lines (*paraphrasing inferences*). The quality of the inferences is less salient to this assessment compared to an end-reading comprehension test. RSAT focuses on qualitative differences in reading behavior rather than post-reading understandings. Forming bridging inferences and elaborating inferences correlates with measures of end-reading comprehension.

Fletcher, Lucas, and Baron (1999) adapted this moment-by-moment reading assessment methodology to ROMP, using secondary geometry proof texts. They directly compared the observed behavior to reading of narrative text. They reported that ROMP was more effortful than reading narrative texts and elicited a different constellation of reading activities. The primary reading activity novel to ROMP was *forward elaboration* in which students anticipate later lines of the text, which was less common in reading narrative text.

Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2012) present a framework for the various kinds of understanding students might develop from ROMP, which built heavily on Yang and Lin's (2008) framework. Those authors successfully adapted their framework into a validated, multiple-choice assessment of end-reading comprehension (Mejia-Ramos et al., 2017). Our study and methodology differ in large part because we seek to investigate moment-by-moment ROMP activities and our analysis focuses more on sense-making activities rather than kinds of understanding to be constructed.

Relatively few undergraduate mathematics education studies focus on reading processes. Shepherd and van de Sande (2014) compared undergraduate student reading of textbooks to faculty reading. They found salient differences regarding the way their subjects articulated equations; experts referred to parts of equations in terms of their meaning or role while novices read the names of the symbols in sequence. A couple of studies compare expert and novice ROMP behaviors using eye-tracking technology (Inglis & Alcock, 2012; Panse, Alcock, & Inglis, 2018). An interesting finding from those studies is that novices attend more commonly to equations in proof texts while experts spend more time examining the connecting statements that state logical inferences. Weber (2015) reports some reading behaviors of very successful undergraduate students by which they made sense of a novel proof text.

Two studies report on interventions aimed at improving student ROMP activity. Hodds, Alcock, and Inglis (2014) adapted self-explanation training from other reading domains to the context of proof and found that self-explanation training was successful in improving student comprehension of proofs they read. Samkoff and Weber (2015) reported findings from trying to train students in the effective reading behaviors reported in Weber (2015). They had modest success, though students needed guidance in using the strategies effectively.

Analytical Framework

Our analysis of the reading process is informed largely by the tradition of Systemic Functional Linguistics (Halliday, 1994; Schleppegrell, 2004). As suggested in its name, SFL emphasizes how language functions to make meaning, either in articulation or interpretation of language. From this standpoint, choice is a key aspect of all language use. In particular, Halliday argues that linguistic choices are made to achieve three metafunctions: *ideational* – what is being talked about, *interpersonal* – who are the interlocutors and how are they positioned, and *textual* – what kind of text is being constructed. In this study, we particularly attend to the first and third metafunctions (though interpersonal metafunctions influenced the observed ROMP activity). The ideational metafunction (which Halliday at times subdivided into experiential and logical) for mathematical proof naturally involves discussion of mathematical objects, properties, and relationships. One of the novel contributions of this study consists in observing how the textual metafunction became salient in students' ability to make meaning of the proof texts they read.

Methodology

Adapted Assessment Method

To select proof texts for students to read, we searched introduction to proof textbooks and asked for ideas from mathematician colleagues. We sought proofs that were at least 10 lines (to increase opportunities to respond), were accessible to novice readers, and that were less likely to appear in common undergraduate courses (to minimize prior exposure). We selected four proof texts, proving the statements listed in Figure 1. Mirroring RSAT, we developed both general and specific response prompts for each line of text. Like RSAT, the two authors began by coding each line of text for all of the connections we expected students might make. This informed our choice of specific response prompts for each line. Students were always asked to think aloud, but the more specific response prompts included:

- "Why is this line justified?" inviting identification of data, definitions, and warrants.
- "What is the purpose of introducing d?" probing student recognition of goals.
- "What does this line accomplish?" assessing achievement of proof goals.
- "What do you expect in the following line[s]?" inviting forward elaboration.

The final prompt was used when we expected that students would be able to elaborate forward based on a proof frame that had been introduced (cases, universal generalization, contradiction, induction, dual inclusion between sets) or because a stated goal was nearly accomplished.

For every primitive Pythagorean triple (a, b, c) there exist some numbers s and t with no common factors such that $s > t \geq 1$ where $a = st$, $b = \frac{s^2-t^2}{2}$, and $c = \frac{s^2+t^2}{2}$.

Euclid's Fifth Postulate implies Playfair's Parallel Postulate.

For any two sets A and B, $A^c \cap B^c = (A \cup B)^c$.

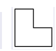
For any positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be covered with L-shaped tiles. 

Figure 1. Statements of the four theorems proven in the texts presented to students.

Study Participants and Interview Methodology

We recruited from courses at one medium and one large public research university in the US. To sample students with varying experience, we recruited from differential equations, introduction to proof, real analysis, and topology courses. We classified our participants in three groups: novice readers who had completed no proof-oriented courses at university (6), experienced readers who had completed at least one such course (9), and graduate students (2). Interviews were conducted outside of class time for 1-2 hours, and students were modestly compensated. All interviews were audio recorded and any student work was retained.

Each proof text reading began by students reading definitions, previous theorems assumed true, and the statement of the theorem to be proven. The interviewers answered questions about mathematical facts and clarified the theorem statement if needed (e.g. L-shaped tiles each covered 3 squares). We generally avoided explanations that would affect the reader's construal of proof text itself. Students could always see all prior lines of the text as they read and had the definitions and theorem statements available on paper. In addition to scripted response prompts, interviewers could ask elaboration questions at their discretion.

Analysis Methods

Interview coding proceeded in three stages. Upon watching the interview recording, the researcher first described the student response to each response prompt for each line, transcribing quotes that seemed significant or relevant. Upon completing these detailed *field notes*, the researcher then compiled a list of notable patterns in each student's *ROMP activity* on each proof. Some organizing categories emerged for this stage of analysis, but these were meant to guide the researcher's noticing more than serving as research constructs. In particular, we always tried to focus on ways students sought to make meaning of the text, regardless of how normative their interpretations were. Initially, both authors completed these first two stages of coding for the same two interviews. Once we compared and reach some agreement about the process, the rest of the interviews were partitioned and each analyzed by only one of the two authors.

The third stage of analysis followed thereafter when we created general categories of ROMP activities that could be assessed on all the tasks by specific indicators. This proved challenging because students did not exhibit particular ROMP activities uniformly throughout each text and students' ability to construe each proof normatively did not appear to correlate with their ROMP experience. The final goal of the analysis is to identify parent *categories of ROMP activities* that can be assessed for each proof text along with *indicator activities* specific to each proof that can be used to represent each student's reading of that text. It is beyond the scope of this report to present these categories and indicators. Rather, in the following section we present some representative reading phenomena observed that demonstrate the fruitfulness of this assessment methodology and the complexity of student ROMP activities. Figure 2 presents the first proof that students interpreted, that will be referenced in the data presented.

1. Let (a, b, c) be a primitive Pythagorean triple.
2. Then $a^2 = c^2 - b^2 = (c+b)(c-b)$.
3. We want to show that $(c+b)$ and $(c-b)$ are both squares and share no common factors.
4. Suppose that d is a common factor of both $(c+b)$ and $(c-b)$.
5. Then d is also a factor of $(c+b)+(c-b) = 2c$ and a factor of $(c+b)-(c-b) = 2b$.
6. So both $2b$ and $2c$ are multiples of d .
7. If d is factor of both b and c , then d is also factor of a.
8. This contradicts (a, b, c) being a primitive Pythagorean triple.
9. Since d is not a factor of both b and c, d divides 2.
10. Hence d is 1 or 2.
11. d divides a, which is odd, thus d cannot be 2.
12. Thus $d = 1$ and $(c+b)$ and $(c-b)$ share no factors greater than 1.
13. This means that $(c+b)(c-b) = a^2$ implies that $(c+b)$ and $(c-b)$ are both perfect squares.
14. So there are some numbers s^2 and t^2 such that$(c+b) = s^2$ and $(c-b) = t^2$.
15. s and t share no common factors as established above.
16. Clearly, $s > t \geq 1$.
17. Finally, $c = \frac{s^2+t^2}{2}$, $b = \frac{s^2-t^2}{2}$, and $a = \sqrt{(c+b)(c-b)} = st$

Figure 2. Proof characterizing primitive Pythagorean triples (adapted from Rotman, 2013).

Results

In this section we exemplify of four ROMP phenomena that we observed in our interviews: 1) computational and inferential orientations, 2) low-level construal of proof claims, 3) ongoing revision of proof construal, and 4) patterns of identifying and stating warrants.

Computational and Inferential Orientations

We observed that some novice readers interpreted the proof texts using what we call a *computational orientation* while more experienced and effective readers exhibited an *inferential orientation*. These two constructs relate to the textual metafunction. That is, they relate to the student's sense of what kind of text is being constructed and what kinds of activities are relevant in such a text. The distinction was most prominent with regard to how students interpreted equations in proofs. We have reported more fully on this distinction elsewhere (Dawkins & Zazkis, 2018), so we shall merely describe this phenomena without extensive data.

The first proof used the equation $a^2 = c^2 - b^2 = (c+b)(c-b)$ in multiple ways. First, it is used to infer that if $(c+b)$ and $(c-b)$ are both multiples of d, then a is also (L7, L11). Later, it was used to infer that since $(c+b)$ and $(c-b)$ had no common factors they are both perfect squares (L13). Students who exhibited a computational orientation saw the equation and the introduction of d as a factor of $(c+b)$ and $(c-b)$ as an opportunity to substitute into the equation and solve for certain variables. They made meaning of the text using practices that were native to the mathematics courses they had thus far completed in college (calculus and differential equations). We understood this as construing the proof as a different kind of mathematical text than was actually being produced. These students often exhibited great perturbation in sense making, and articulated desire to deal with the equations in familiar "plug-and-chug" ways. Students exhibited an inferential orientation when they interpreted the equation as a means of inferring the properties of the various quantities in the equation (as is intended).

Low-Level Construal of Proof Claims

Low-level construal of proof claims refers to the quality of the mental model students build of the information presented in the text. This relates both to the model of the line currently being read and how students' interpretation/recall of previous lines affects their reading of the current line. We report Novice 1's (Nov1) ROMP activity to exemplify this construct.

A number of steps in the primitive Pythagorean triples (PPTs) proof (Fig 2) related to which numbers shared common factors (definition of PPT, L3, L4, L7, L12, L15). This relation thus appears in the proof with reference to at least four sets of numbers: (a, b, c), $((c+b),(c-b))$, $(2b, 2c)$, (s, t). Some relations are assumed by hypothesis (L1), some are assumed toward a contradiction (L7), and others are inferred from other properties (L15). When Nov1 read the definition of PPT, he said, "Run of the mill Pythagorean triple that I've learned since high school." He showed no sign of attending to the word "primitive" or how it modifies the meaning of Pythagorean triple by incorporating an additional no common factors stipulation.

When Nov1 read L4, he correctly noted that d would be used to accomplish the goals stated in L3, likely using proof by contradiction. Nov1 justified L5 by imagining factoring d out of the expressions $(c+b)$ and $(c-b)$ and then factoring again to show that both sides of the equations are multiples of d. He made a similar argument for L7, except applied to the equation in L2. His reasoning suggests his meaning of "factor of" in terms of being able to factor a term out of an expression was productive in helping Nov1 justify certain inferences. He also seemed aware of the goal stated in L3 regarding "no common factors" and how d would be used to accomplish that goal. After reading L7, the interviewer asked what Nov1 expected to follow:

> A little up in the air because of the assumption it would be proof by contradiction because in the assumption of the, it said that "with no common factors," even though that, in the next coming line we are going to be moving towards "d does not work for both b and c." [The interviewer asked him to elaborate.] Because the theorem being proven it says that there are some numbers with no common factors, but then again that's, yeah. But that's for s and t and I just transferred that assumption to a and b, but I don't know. If s and t have no common factors, oh, but b and c already have a common factor of 2 because they are both being divided by 2, or ½ I should say. So the assumption that, from what I derived from the theorem being proven, it's being misassigned to b and c and not necessarily to $\frac{s^2-t^2}{2}$ and $\frac{s^2+t^2}{2}$. So I am excited to see what this next line says.

This marked a shift in Nov1's ability to track the inferences being made. He began trying to interpret L7 in terms of the properties of s and t (part of the theorem's conclusion). He also inferred that b and c are divisible by 2 based on the equations in the theorem's conclusion.

After reading L8, Nov1 questioned his prior claims and decided that L8 was referring to the "no common factors" claim in the definition of PPT. He did not elaborate further on how this revised his interpretation of the proof. When Nov1 read L9, he was able to explain the claim with reference to L6. He exemplified this inference when $2b = 10$ and $2c = 14$. The interviewer asked what the rest of the proof needed to accomplish, and part of Nov1's reply was: "What the last couple of lines have been is giving the evidence and basically proving in a more theoretical way that a, b, and c share no common factors, and so the next part of the proof will be defining b and c in terms of s and t so that they will have no common factors." When Nov1 read L12 that explicitly refers back to the goal in L3, he again concluded that this line verified that a, b, and c share no common factors.

Nov1 read the last part of the proof frequently using the conclusions of the theorem to justify proof claims. He used the equations in the theorem statement to justify L14. Reading L15, Nov1 said that it was self-explanatory because it was stated in the theorem. In his explanation, he referred to factors of b and c, s and t, and $(c + b)$ and $(c - b)$, but he showed no sense of dependence among these claims. Rather, he said this line simply reminded the reader of what had been done, since everything was being redefined in terms of s and t. He similarly noted that L16 was "a statement made in the theorem." After he had read the entire proof, Nov1 reflected, "I would have plugged and chugged would have to worked to get this expression from that expression. But I would have skipped all the $2b$ and $2c$ and the common factor stuff."

To summarize, in Nov1's ROMP activity he was quite successful at using equations to show that if some constituents had a factor of d, then others would also. He used his meaning for factor to connect L9 to L6 using particular examples (c.f., Weber, 2015). He recognized the beginning of proof by contradiction in light of the goals stated in L3. Less productively, it appeared that he only became aware of the "no common factors" stipulation in the definition of PPT when it was used in L8. He initially tried to make sense of that line in terms of the "no common factors" claim in the theorem's statement. In this middle section of the proof, he seemed to lack a clear sense of "no common factor" claims were known and which required justification. As a result, his emerging construal of the proof began to completely reverse the intended relationship between hypotheses and conclusions. Nov1 reached the point of claiming that L12 proved that property held for (a, b, c) rather than for $((c + b), (c - b))$.

We argue that Nov1's weak image of what was taken as hypothesis in the proof influenced the way that he confused the various "no common factor" claims. For lines that clearly stated the hypotheses and conclusions, he produced valid justifications. However, he never developed a clear sense of what the overall proof began assuming and how the set of claims proven grew over the course of the text. This is why we describe this as a *low-level construal of proof claims*. This account of Nov1's sense making of the text helps explain why he ended the reading unable to explain the necessity of the middle section of proof.

We observed other forms of this construct, especially among novices. This often seemed to result from a weak understanding of the underlying concepts. For instance, students who thought about "d is a factor of $(c - b)$" in terms of the process of dividing, rather than being made up of units of d, and students who had trouble thinking of $(c - b)$ as a unit all tended to have trouble building a mental model of what was assumed and what needed to be shown. Like Nov1, such students ended up trying to draw inferences from the equations in the theorem's conclusion because they seemed to provide richer resources for sense making.

Ongoing Revision of Proof Construal

This construct represents a complex form of bridging inference (Magliano et al., 2011). It describes when students revised their existing model of the proof's prior claims in light of later lines. As an example, Experienced 5 (Exp5) could not recall which claim was assumed as true in the wording "Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP)." Because the proof begins with the hypotheses of PPP (Zandieh, Roh, & Knapp, 2014), he inferred that "implies" meant to assume PPP and prove EFP. Exp5 initially interpreted that L1 assumed PPP was true. It was not until L6 when the proof applied EFP that the student decided EFP was the hypothesis and PPP the conclusion. He supported this by revising his understanding of L1 as assuming only the hypotheses of PPP rather than assuming the entire claim.

Some novice readers exhibited less productive examples of ongoing revision when they read L8 of the Pythagorean triples proof. They inferred that the contradiction denied the hypothesis in L1 rather than the hypothesis "d is a factor of both b and c" from L7. Once they concluded that L8 stated that (a, b, c) is not a primitive Pythagorean triple, they rightly expressed difficulty making sense of the argument when the object in question was not in the relevant category. Our moment-by-moment methodology uniquely provides access to this type of ongoing forming and reforming of models for what proofs claim to be true.

Identifying and Stating Warrants

The final notable pattern of ROMP activity we present in this report dealt with the ways students sought and stated warrants for inferences made in proofs. The interview protocol often invited students to explain why particular lines were justified, which for us meant to identify warrants. More experienced readers tended to be more adept at this practice and we observed key differences among the kinds of warrants sought and produced. Nov1's reasoning about L5 above exemplifies an *enacted warrant* in which he justified the inference by describing how particular manipulations could be made to show that d would be a factor of an expression. This constituted a mini-proof of the relevant warrant. Nov1's reasoning about L9 above is an example of *justification by example*, which does not constitute a valid warrant, but nevertheless provides some support for the claim. More experienced students were more often observed trying to *state warrants in general form*. For example, they articulated that L5 is justified because the sum [or difference] of any two multiples of d is also a multiple of d. Finally, Graduate 1 was able to *cite a relevant warrant* for L5, namely that any linear combination of multiples of d is also a multiple of d. Across our interviews, we observed a range of ROMP activities within which students with more experience exhibited greater tendency to seek warrants and where more adept at identifying particular inferences as instances of a general mathematical fact.

Discussion

This paper presents findings from our adaptation of the moment-by-moment reading assessment methodology to the reading of mathematical proof. We identified several novel ROMP activities that emerged in our interviews that justify the value of the methodology. The first phenomena distinguishes between the kinds of practices that students used to make sense of the proof texts and relates to the textual metafunction of mathematical texts. We anticipate that this finding that novice readers try to make sense of proofs using expectations from other mathematical texts could be fruitfully explored in the context of introduction to proof instruction. This pattern of ROMP sense making may help explain why Inglis and Alcock (2012) found that novice readers attended more closely to equations in proofs while experts attended to the surrounding text, which contains logical connectives. We hope that these other ROMP activity constructs can be further harnessed in later investigations to better understand how students make meaning of proofs they read and how that process develops over time. Ongoing work intends to find ways to adapt this methodology into an efficient assessment tool that can be more quickly administered and coded. This will contribute more insights about the process of reading to supplement the existing assessments of end reading comprehension and proof validation.

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Professors Intentions' and Student Learning in an Online Homework Assignment

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Homework accounts for the majority of undergraduate mathematics students' interaction with the content. However, we do not know much about what students learn from homework. This paper reports on a pilot study of why professors chose particular homework problems, what they hoped students would learn from them, and whether students' engagement with the problems reflected those outcomes. Results show students gained the desired familiarity with notation and procedures. The results also speak to how professors manage the content between what they discuss in class, homework problems, and intentional overlap between the two.

Key words: online homework, sequences, instructional triangle, calculus

Background and Theoretical Perspective

One way researchers have conceptualized mathematics instruction is as "interactions among teachers and students around content, in environments" (Cohen, Raudenbush, & Ball, 2003, p. 122). In this perspective we can think of instruction as a triangle that relates the teacher, the knowledge at stake (content), and the student (Figure 1). The student vertex includes the mathematical tasks students work on and the milieu in which they experience those tasks. A milieu is a "counterpart environment[s] that provides feedback on the actions of students" (Herbst & Chazan, 2012, p. 9). The interactions among the vertices are governed in part by the didactic contract (Brousseau, 1997), the set of implicitly-negotiated expectations between teachers and students. For example, students expect professors to provide opportunities to learn the knowledge at stake. Professors expect students to do the tasks (or other activities) that represent learning opportunities. At the undergraduate level, an important component of the didactic contract is the expectation that students will spend significant time out of class interacting with the knowledge at stake (Ellis et al., 2015).

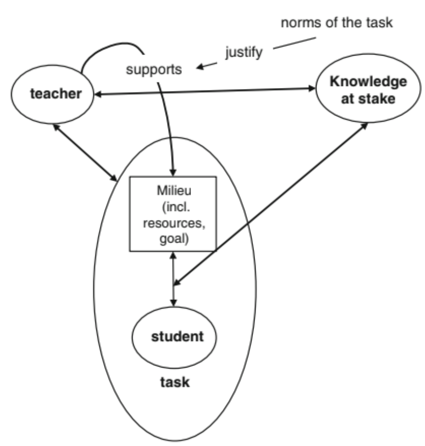

Figure 1. Instructional triangle (Ellis et al., 2015, p. 270; Herbst & Chazan, 2012, p. 10).

Homework represents the primary milieu for students' out-of-class learning. University calculus I students spend more time doing homework than they do in class (Ellis et al., 2015;

Krause & Putnam, 2016). As such, homework accounts for the majority of students' interaction with mathematics content and mathematics tasks (White & Mesa, 2014). White and Mesa (2014) found instructors view homework in general as a way for students to learn through repetition, understand algebraic manipulations, and apply mathematics to realistic situations. However, we do not know much about what students learn from homework.

LaRose (2010) found homework improved students' ability to do procedural integration problems. There is evidence that students frequently complete textbook exercises by focusing on superficial features and finding procedures to mimic (Lithner, 2003). We also know that in the case of online homework, students sometimes guess answers (Dorko, 2018; in preparation; Hauk & Segalla, 2005) or sometimes type entire problems into search engines (Krause & Putnam, 2016). However, there is also evidence that students engage in mathematical sensemaking when doing online homework (Dorko, 2018; in preparation; Krause & Putnam, 2016). Homework has the potential to be a powerful learning environment and research about the nature of students' reasoning while doing homework and what they learn from different sorts of homework tasks can help instructors design homework assignments that more effectively influence students' cognitive activity.

Toward that end, this paper reports on a pilot study of student learning from homework. I sought to answer the research questions (1) *why did two calculus II instructors choose the particular problems they did and* (2) *did nine calculus II students learn what instructors intended they learn from each of fourteen problems in an online homework assignment about sequences?* While limited in scope to one assignment, the results provide initial information about what students might reasonably learn from an online homework assignment. Additionally, themes in the professors' intentions for the problem talk back to the theory by providing insight into how the professors managed the knowledge at stake across multiple milieu.

Data Collection

The data presented here come from video recorded interviews with two calculus II professors, and video recordings and follow-up interviews with 9 calculus II students. The data were collected in the fall and spring semesters at a large public university in the U.S. Calculus II at this university is a coordinated course in which a course coordinator chooses a set of online homework problems for each section. Each professor assigns some or all of the problems the course coordinator chose. In the professor interviews, which occurred prior to the student interviews, each professor viewed the coordinator's 14 chosen problems for section 10.1, sequences. The professors had each taught the course numerous times and the problems were not new to them. For each question I asked, "would you assign this problem and why or why not?" If the professor would assign the problem I asked, "what would you hope students would learn from this problem?" I transcribed both interviews and listed what the professors hoped students would learn from each question. I then wrote questions for the student interviews based on this list, with the goal that students' verbal answers and written work would lend insight into whether the student had achieved the professors' goals for the problem. For example, in Question 1 (Figure 2), Professor B hoped students would "solidify their understanding of factorials", so I asked students "how familiar are you with factorials?". Professor A said of the sequence -1, 1, -1, 1, … with general term $\cos(n\pi)$ would

Excerpt 1. Professor A discussing the sequence -1, 1, -1, 1 with general term $\cos(n\pi)$

Professor A: [This one] is sort of a good idea because it shows that a sequence can be simpler than the way it's defined.

Hence I asked students "were you struck by the fact that the sequence -1, 1, -1, 1, … is fairly simple, but is defined by a trig function?" As another example, Professor B chose not to assign a question that asked about the convergence of $a_n = (3/8)^n$, but Professor A said he would assign this because

Excerpt 2. Professor A discussing a question that asked about the convergence of $a_n = (3/8)^n$

Professor A: I think this will naturally get them thinking more about this as a discrete set of numbers [instead of a continuous function].

Hence I asked students if they had a mental image of that sequence and if so, I asked them to describe or sketch their mental image. If students described or sketched a line, I asked "do you envision this sequence as a line or a set of points?"

Match each sequence with its general term. (Assume $n \geq 1$)

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (b) -1, 1, -1, 1, … (c) 1, -1, 1, -1, … (d) $\frac{1}{2}, \frac{2}{4}, \frac{6}{8}, \frac{24}{16}, \ldots$

$\cos(n\pi)$ $\frac{n!}{2^n}$ $\frac{n}{n+1}$ $\sin(n\pi)$ $(-1)^{n+1}$

Figure 2. Question 1 in the online sequences homework.

Each calculus II student met with me twice. In the first session, I video recorded them doing their online 10.1 homework. I did not interrupt students except to ask what they had typed on calculators. I photocopied students' written work from the session, their class notes, and any supplemental materials they viewed. In the second session, the student and I watched the video. I paused the video to ask students questions about what they did and why and to ask each question I had generated from the professors' lists of the knowledge at stake.

Data Analysis: Why Professors Chose the Problems that they Did

I employed the constant comparative method (Strauss & Corbin, 1994) to identify themes in why the professors chose the questions they did. The professors stated their reasons for choosing the problems in terms of what they hoped students would learn from them, so the data source for this analysis was the same list generated above. I read through the list looking for similarities in the motivations the professors expressed with the items. For example, professors mentioned including problems because a particular notation or operation would be important in future topics (e.g., the notation shown in Question 4, Figure 3). Other problems they included to elicit shifts in students' cognitive activity. For example, in Question 4, Professor B said

Excerpt 3. Professor B discussing question 4 (figure 3)

Professor B: I know from experience that many of them are going to say c_1 is 1/5, c_2 is 1/8, c_3 is 1/11… because this notation is, is very unfamiliar to them, this idea of a sum with a variable at the end. [I hope] they would get comfortable with the idea of this kind of notation for a partial sum, and so when in the next section we start doing this all the time, they've at least done it for themselves one time.

I continued searching for similarities until I believed I had exhausted them all, then wrote an initial set of categories and their criteria. Following this, I applied the criteria to code all the items again, which allowed me to refine the criteria and to ensure each item belonged to at least one category (that is, to ensure the categories adequately described all the data). The resultant themes are presented in the next section.

Calculate the first four terms of the given sequence, starting with n = 1.

$$c_n = \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \cdots + \frac{1}{3n+2}$$

c_1 = ______ c_2 = ______ c_3 = _______ c_4 = ______

Figure 3. Question 4 in the online sequences homework.

Results: Why Professors Chose the Problems that they Did

Table 1 shows the themes in why the professors selected the problems they did. These themes are not mutually exclusive, and any particular instance of student work could be coded as representing multiple themes.

Table 1. Themes in professors' reasons for assigning particular questions

Category	Examples
Students will engage with a skill/concept specific to the content of section 10.1.	Excerpts 1, 2, 3
Students will engage with a skill/concept that is important for a future topic.	Excerpt 3
Students will make a connection back to a prior skill/concept (either from the current course or a past course).	Both professors hoped students would recall the use of dominant terms to find the limit of the sequence $a_n = \frac{7+n-3n^2}{7n^2+3}$
Students will build number/operation sense (familiarity with numbers and operations.	The professors wanted students to gain familiarity with factorials, powers of -1, powers of 2, etc.
The professor would refer to the problem in class.	Excerpt 3
Students will experience a cognitive shift (think about something in a different way).	Excerpt 2, 3

The theme 'Students will engage with a skill/concept specific to the content of section 10.1' seems obvious, but this category helped distinguish statements professors made about students connecting back to a prior skill from statements about something new in section 10.1 (e.g., notation). I discuss these findings later in the paper.

Data Analysis: Did the Students Learn what the Professors Intended?

To analyze whether students learned what instructors intended from each question, I took each item from the previously-generated list of what professors hoped students would learn from each question and identified what would suffice as evidence that a student had met that outcome. Identifying what I would take as evidence was an iterative process in which I looked at the data from all students (all 9 students' written work and answers to the interview prompts) while I was trying to determine what would suffice as evidence. Because this was a pilot study and there is so little literature about student learning from homework, I did not know how to define "learning" for this context or what might count as evidence of it, so I was unable to establish a priori what evidence of learning might be. Looking at what students had done for each question helped me determine what it might mean for students to learn a particular part of the knowledge at stake. For example, in many of the questions (e.g., Question 4, Figure 3), the professors wanted students to gain familiarity with notation. Looking at student data for these questions helped me determine that if a student answered those questions correctly, they had made sense of the notation.

Results: Did the Students Learn what the Professors Intended?

On the whole, the nine students achieved the goals the professors stated regarding gaining familiarity with notation, operations (e.g., factorials), number sense, vocabulary, and procedures. However, the students seldom noticed nuances the professors hoped they would notice in particular problems. I present brief examples of each below. These students' responses were representative of the entire group.

Familiarity with notation, operations, number sense, vocabulary, and procedures

There were two problems in which professors hoped students would gain familiarity with notation. One (Question 2) was making sense of subscripts: students were given a formula for a_n and directed to generate terms for $b_n = a_{n+1}$, $c_n = a_{n+3}$, and $d_n = 2a_n - a_{n+1}$. Professor A said notation "tends to trip them up", so I inferred he wanted students to become familiar with subscripts. Professor B said "I want them to be very comfortable with what a_{n+1}, a_{n-1}, what that does in the sequence." All students computed the terms correctly (some taking multiple attempts), which I took as evidence that they had made sense of the subscripts. I also asked students to describe what the subscripts meant, and they made statements such as "you would just go to like the term after… on this one you had to go to the third term after". Question 4 (Figure 3) was the other problem the professors thought was important for notation. Professor B said "I know from experience that many of them are going to say c_1 is 1/5, c_2 is 1/8, c_3 is 1/11… because this notation is, is very unfamiliar to them, this idea of a sum with a variable at the end. [I hope] they would get comfortable with the idea of this kind of notation for a partial sum, and so when in the next section we start doing this all the time, they've at least done it for themselves one time". Five of the students computed the terms correctly on their first try, indicating they had made sense of the notation. The other four made the error Professor B predicted, then computed the terms correctly on their second try. I took these correct computations, and students' descriptions of how they thought about the problem, as evidence that they made sense of the notation. For example, one student said at first she thought the 1/(3n+2) was "the pattern, like in the previous question" in which she had been given a general term. She described that after seeing her initial answers were wrong, "I realize[d] it was adding the terms and not just like the individual [fractions]". Another student said "I learned what to look for, and the difference between… a sequence and a sum."

Like the analysis regarding whether the students made sense of the new notation, my criterion for items the professors stated about familiarity with operations and number sense was primarily whether or not the students answered the problems correctly. For example, the professors felt questions like number 1 (Figure 2) and number 3, computing terms of $9^n/n!$, would help students "recogniz[e] you know factorials, powers of 2, changes of sign… they're getting to practice that" and "they're getting to use powers, maybe a power they're not familiar with… so they'll see how those numbers come out". I took correct answers to problem 1 (Figure 2) as evidence students gained familiarity with factorials, powers of 2, and changes of sign because students either seemed to do this in their heads (writing nothing or typing nothing into a calculator) or took the general terms and wrote the first several terms of the sequences before matching the answers. That is, students appeared to engage in the computations with the general terms in question 1 and this engagement represented their gaining more familiarity with particular powers and operations. In question 3, computing terms of $c_n = 9^n/n!$, six students either did the factorial calculations in their heads and/or wrote the factorials as products (e.g., $(9^4)/(1*2*3*4)$). I took this as evidence that they gained familiarity with the factorial. Three students typed the $9^n/n!$ into their calculators. The calculators outputted simplified answers, which did not give students an opportunity to see powers of 9 or how the factorial affected the terms.

Five questions with various sequences were stated like Question 9 (Figure 4) in which students selected a multiple choice option that Professor A considered "checking vocabulary." Though students did not answer all these questions correctly, with one exception, they always checked 'converges' if they inputted a limit that was a real number and 'diverges' if they inputted ∞ or $-\infty$. Two students looked at the definition of converge in the textbook or their notes. I took this and the internal consistency of all students' answers to the two parts of the problem as evidence that they either knew what 'convergence' and 'divergence' meant before starting the assignment, or (in the case of the two students who looked up the definitions) they learned it while doing the assignment.

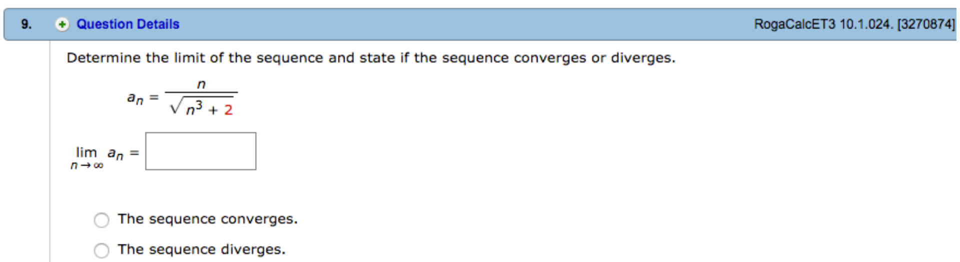

Figure 4. Question 9 in the online sequences homework.

In questions like question 1 (Figure 2), questions 2 and 3 (described above), and question 4 (Figure 3), the professors described they wanted students to become familiar with the procedure of generating terms of a sequence. All of the students answered these questions correctly, which served as partial evidence that they had gained familiarity with the procedures in each case. In summary, on the whole the nine students achieved the goals the professors stated regarding gaining familiarity with notation, operations (e.g., factorials), number sense, and vocabulary.

These results support the efficacy of an online homework program with multiple attempts per question for helping students make sense of new notations, gain familiarity with operations and powers of numbers, learn the meanings of new vocabulary, and practice procedures. An important caveat is that if the goal is for students to gain familiarity with operations (e.g., factorials), it may be best to encourage students to write computations by hand instead of relying on a calculator.

Nuances

While the homework problems supported students in learning or practicing notation, procedures, and operations, students largely missed the nuances the professors hoped students would notice in the problems. This may be because the professors' goals for the problems asked for something that was not a necessary conception for the students to have in order to get the correct answer, and not a connection that was directly asked of the students. For example, no student picked up on Professor A's desired take-away for $\cos(n\pi)$ and the sequence -1, 1, -1, 1, … (Excerpt 1). Calvin, one of the students, said

Excerpt 4. Calvin discussing $\cos(n\pi)$ *and the sequence -1, 1, -1, 1*

> *Interviewer*: Did it surprise you… so we have a sequence that's fairly simple, right? Because it's -1, 1, -1, 1 but it's defined with a trig function.
>
> *Calvin*: I mean not really. I mean back when I was first learning trig stuff, like the emphasis on the graph and how it was alternating… I didn't really think about it.

Similarly, students did not imagine $a_n = (3/8)^n$ as a discrete set of points, as Professor A intended (Excerpt 2). The students who described mental images of this sequence described or drew continuous functions.

In the next section, I discuss the results and make connections between the themes in why the professors chose the problems and whether students learned what the professors intended.

Discussion and Conclusion

These results support others' findings that online homework can improve students' fluency with procedures and notation (LaRose, 2010). However, students missed some of the nuances the professors hoped they would take away from the problems. There were two problems professors hoped would cause cognitive shifts for students, but only one problem was successful in doing so. Professors chose problems that helped students recall prior learning and connect it to the new content, problems for students to practice content particular to sequences and their computation, problems that would provide a foundation for future content, problems they wanted to talk about in class, and problems that they hoped would cause students to think about something differently. This list could be informative for new instructors in thinking about what to include in a homework assignment.

White and Mesa (2014) found variation in the cognitive orientation of tasks across milieu (homework, worksheets, exams) and instructors. The professors in this study selected problems that were largely procedural, expressing they wanted students to be exposed to these problems beforehand so they could discuss the details in class. For example, in a problem that directed students to use limit laws and theorems to find the limit of $a_n = 9^n/n!$, Professor B said he did the example $11^n/n!$ in class …

Excerpt 5. Professor B discussing $a_n = 9^n/n!$

> *Professor B*: ... so it, it went up for a bit longer before it started to come down… they either didn't know the limit or they thought the limit was infinity. And then you know I talked through factoring everything and realizing that after we get to this peak, things start to come down, and they start to come down kind of fast because we're multiplying by these numbers that are less than 1 all the time….We have to actually manipulate the factorial as a product now to, to see the answer.

Similarly, the professors wanted the students to have experienced computing partial sums so they were familiar with it for the lesson on series (Excerpt 3). In summary, the findings suggest the professors made intentional decisions about managing the knowledge at stake across milieu.

These results have many implications for future research. One avenue would be investigating what students learn from homework problems that are more conceptual in nature. In particular, online homework platforms have the advantage of allowing students multiple attempts and providing immediate feedback, and research should examine how we can leverage these systems to influence the cognitive bases of students' activity.

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“Zoom in Infinitely”: Scaling-continuous Covariational Reasoning by Calculus Students

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Abstract: Recently, Ely & Ellis (2018) described a new mode of covariational reasoning—scaling-continuous reasoning—and conjectured that it might support productive student thinking in calculus. We investigate that hypothesis by analyzing how calculus students employed scaling-continuous covariational reasoning when discussing differential calculus ideas. The interviewed students who took a course based on a “local straightness” approach to calculus used scaling-continuous reasoning in their description of the derivative at a point, particularly in their imagery of zooming in on a function at a point to reveal its slope. The interviewed students who took a course based on an “informal infinitesimals” approach to calculus used scaling-continuous reasoning in their account of how zooming in on a neighborhood reveals the coordination between a bit of x (dx) and the corresponding bit of y (dy), a relationship that gives a differential equation for that curve.

Keywords: covariation, differential, infinitesimal, local straightness, derivative

Ely and Ellis (2018) proposed the category of scaling-continuous variational/covariational reasoning and hypothesized ways it could productively support student reasoning in calculus. We build on this idea by investigating if and how scaling-continuous reasoning could support student understanding in single-variable differential calculus.

Theoretical Background

The idea of scaling-continuous reasoning is grounded in significant ongoing research on variational and covariational reasoning (e.g., Carlson, Jacobs, Coe, Larsen, and Hsu, 2002; Carlson, Persson, and Smith, 2003; Castillo-Garsow, 2012; 2013; Castillo-Garsow, Johnson, and Moore, 2013; Confrey & Smith, 1995; Saldanha & Thompson, 1998; Thompson, 1994; Thompson & Carlson, 2017; Thompson & Thompson, 1992). We briefly summarize several categories that are prominent in this research, as recently synthesized by Thompson and Carlson (2017). For a single quantity, *chunky-continuous variational reasoning* involves imagining that changes in a variable’s values occurs only in completed iterated chunks, but without a clear image of how the variable actually takes on the intermediate values within each chunk. For two quantities, *chunky-continuous covariational reasoning* describes chunky reasoning with two quantities simultaneously: one quantity is taken in chunks, with corresponding chunks in the other quantity, but with no clear image of variation co-occurring within the chunks. *Smooth-continuous variational reasoning* entails an image of a changing quantity that smoothly changes in time. The reasoning can imagine the variable’s magnitude increasing in bits, but simultaneously anticipates smooth variation within each bit (Thompson & Carlson, 2017). *Smooth-continuous covariational reasoning* involves smooth variation in both quantities at the same time, including the understanding that smooth change in one quantity, no matter how small, can correspond to simultaneous smooth change in the other quantity. According to Thompson & Carlson (2017), smooth-continuous variational and covariational reasoning requires reasoning in terms of something moving in time. They describe smooth-continuous covariation essentially in terms of two quantities parametrized by an underlying time variable: “The coordination of quantities’ values is like forming the pair $[x(t), y(t)]$, where “t” stands for a value of conceptual time” (2017, pp. 444-5). Smooth-continuous reasoning has been shown to be robust and productive in calculus (e.g., Castillo-Garsow, 2012; Castillo-Garsow, Johnson, and Moore, 2013).

Scaling-continuous variational reasoning entails the image that at any scale the continuum remains continuous and that a variable takes on all of its values in that continuum. The continuum can be zoomed in on arbitrarily or even infinitely, and at no scale will it be revealed as discrete or having holes. *Scaling-continuous covariational reasoning* involves imagining re-scaling or zooming in on an increment of one variable quantity and coordinating that with an associated re-scaled increment of another variable quantity. For instance, one can envision shrinking or expanding a window of x-values and at every scale is a corresponding re-scaled window of y-values determined by the correspondence between increments of x and y. Unlike smooth-continuous reasoning, this does not fundamentally rely on an image of motion or an underlying time parameter. Scaling-continuous reasoning itself entails the idea that it is possible to zoom arbitrarily to any (finite) scale, but it plausibly requires another mental act to generalize or encapsulate this to develop an image of zooming in *infinitely*, revealing infinitesimal increments. We also note that scaling-continuous reasoning does not by itself entail the ability to effectively calculate at any scale (just as smooth-continuous reasoning does not alone entail the ability to effectively calculate change in one quantity in terms of change in another).

Method

Each author taught a calculus class using different non-traditional approaches—local straightness (Samuels) and informal infinitesimals (Ely)—conducting various semi-structured interviews investigating the reasoning of students in the classes. For this study, we analyzed these interviews with an eye to how different types of covariational reasoning manifested.

Setting 1: A Calc I class with a local straightness approach

Author 2 (Samuels) taught a Calculus I class using local straightness as a cognitive root (Tall & McGowen & DeMarois, 2000) for the derivative and the integral. Local straightness is the property that zooming in at one point on the graph of a function of one variable reveals a (nearly) straight line when the function is differentiable at that point, and the slope of the line is the derivative at that point (Samuels, 2017). Student-centered guided discovery activities were at the core of the curricular design.

Students first developed the idea of the derivative at a point by engaging in activities using an applet with two windows. One window contains the graph of the function on a fixed scale. The second window graphs the function centered at a variable point on the graph on a variable scale. (The point and the scale can each be manipulated by the user with sliders; A box in the first window indicates which portion of the graph appears in the second.) After zooming in, students see a (mostly) straight line, and learn to associate the slope of that line with the slope or derivative at that point. (If the function is not differentiable at that point, a straight line never comes into view.) Questions and activities for the students included: describing what is visible during the zooming process, estimating slope at a point, and making a table of slope values. For a more detailed description of the approach, see (Samuels, 2017). Algebraic limits and their application to the slope difference quotient typically are presented as an entrée to the derivative (e.g. Stewart, 2012) and are seen as a necessary precursor to understanding the derivative (Zandieh, 2000); in this curriculum, they are reserved until the end of the course. The geometry of local straightness replaces the symbolic formalism of the limit definition as a way to conceive of the derivative. Further, in this approach, the slope object is not an encapsulation of a limit process, as it is when you move the second point along the graph toward a fixed point and secant lines must be understood to approach a tangent line. In that process, secant lines are first constructed as additional mathematical objects. Instead, the local slope is in some sense already there to be "found" for the student; once one zooms in close enough one can see the graph as being straight (enough) and thus having a slope. Here, no additional mathematical objects are constructed; rather, we take a different view of the existing graph.

Setting 2: A Calc I class with an informal infinitesimals approach

In Fall 2016, the first author (Ely) taught a Calculus I course that used an "informal infinitesimals" approach in a large lecture (110 students). His purpose was to build calculus ideas in such a way that the notation transparently referred to quantities, rather than serving as a shorthand for the result of a limit process. This is in keeping with the imagery Leibniz had in mind when developing the notation we still use for calculus: dx denotes an infinitesimal amount of x and $\int$ represents a sum of infinitely many infinitesimal bits. The class used Leibniz' heuristics for imagining infinitesimals, and his consistent rules for working with them. The purpose was to allow students to work directly with infinitesimal quantities using regular arithmetic and algebraic operations. For instance, dy/dx was a quotient of two infinitesimal quantities, not code language for $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Although the development of the hyperreal numbers in the 1960s offers a formal system sufficient for rigorously grounding Leibniz' approach (Robinson, 1961), the informal infinitesimals calculus class used Leibniz' notation and imagery with very limited reference to the formal hyperreal numbers. For a detailed summary of how infinitesimals can be rigorously developed in this manner, see the appendix of (Ely, 2017).

An infinitesimal is a number or quantity smaller than any real number but larger than 0. In lecture, the instructor used the image of an infinitesimal distance as being revealed by zooming in infinitely on the real number line. For instance, if you zoom infinitely on the point 100 on the real number line, using an infinitesimal scale factor of ε:1, you can see a little neighborhood or "monad" around 100 that contains an entire world of numbers that are all infinitely close to 100, including numbers such as 100 + ε and 100 - 3ε. Infinitesimals such as $100 - 4\varepsilon^2$, are still indistinguishable from 100 at this zoom factor of ε; these are revealed by zooming *again* infinitely at 100 by another scale factor of ε:1, and thus are considered second-order with respect to the infinitesimal ε. This image can be formalized in the hyperreal numbers (e.g. Keisler, 1986), although the informal infinitesimals class did not do so.

Focusing just on differential calculus, the class developed methods for deriving differential or "bits" equations from "amounts" equations. For example, $y = x^2$ was seen as an equation that gives an amount y in terms of an amount x; its bits equation dy = 2x·dx provides a bit of y (dy) in terms of a bit of x (dx), which relies on the value x near which the variation is occurring. Later, after working with bits equations, we divided both sides of a bits equation by an infinitesimal to find the quotient of two bits near a particular x: i.e., dy/dx = 2x. In the special case where y is a function of x in the original amounts function, this quotient will also be a function of x, which enables the defining of that amounts function's derivative function. Bits equations were also used extensively in the course as a basis for definite and indefinite integrals, a development that is beyond the scope of this article (but see Ely 2017 for more detail).

Data collection

Both authors conducted semi-structured clinical interviews with students about a variety of topics from a Calculus I class they were just completing, 7 students that used the informal infinitesimals approach and 25 that used the local straightness approach. Interviews were analyzed for student use of various types of covariational reasoning. For this paper, we focus on how scaling-continuous reasoning manifested and supported student understanding and explanation of several important ideas in differential calculus.

Results

Scaling-continuous student reasoning in local straightness calculus class

Multiple students discussed the derivative using scaling-continuous reasoning. This occurred both in their general conception and in solving specific problems. For brevity, here we relate three excerpts, each with illustrative verbal and graphical components.

The interviewer asked Young to describe his process of determining the derivative of a function at a particular point x. He said that "[while zooming in] the line straightens out in the zoom window." Subsequently, to explain this process, he drew the picture in Figure 2a.

He indicated his focus on a single point with a black dot on the graph. He indicated his zoom action by drawing, first, a box around this point, and second, that box magnified. (The paper was rotated during the discussion.) Young's work suggests he is using scaling-continuous covariational reasoning. He zooms, then draws a re-scaled window to show the imagined result of the zooming action. In the magnified image, the zoomed-in neighborhood on the graph is represented as continuous, unbroken, and (essentially) straight. This straightness allows him to coordinate the vertical and horizontal variation in order to find a slope of the graph in that neighborhood.

Figure 2. Tangent line sketches by: (a) Young (b) Carl (c) Sam

To explain the derivative at a point, Carl drew a graph with a tangent line at one point. He then elaborated, "To get this tangent line, we learned from the lab that it could be there, and there (draws 3 lines going from curvy to straight, in Figure 2b), you zoom in enough, and it becomes a straight line. It's got to become a straight line or you don't have a derivative." He focused on a unique point, and the nature of the function at multiple levels of zooming, a strong indication of scaling-continuous reasoning.

A third student, Sam, also used scaling-continuous covariational reasoning in his description of the derivative at a point and how it can be calculated. He goes further than the other two students in that he also distinguishes between zooming arbitrarily to get an approximate value and zooming infinitely to get an exact one:

J: What is a derivative?

Sam: Derivative is slope at a point. That's the bottom line. … If the graph is like this (draws image in Figure 2c), the derivative, as you zoom in, this is the tangent line. The derivative becomes more and more accurate.

J: So you also mentioned the tangent line. What does the tangent line have to do with the derivative?

Sam: The tangent line is the slope at a point. As the tangent line moves this way (gesturing to the right), it gets more and more steep. So that's the derivative.

J: When you find the derivative, when you give an answer, is it approximate or exact?

Sam: It's approximate.

J: Is there an exact answer?

Sam: If you zoom in infinitely. It's not perfectly accurate. The main concept of finding the derivative, I think, is seeing this curve as a collection of straight lines. But it's not really a collection of straight lines, it's a curve. And the straight lines are the tangent lines.

Sam's account of slope at a point uses scaling-continuous covariational reasoning in several ways. He indicates his focus on a single point with no secondary point with a single black dot on the graph. Like Young, he describes "zooming in" to find a derivative at that point, suggesting that each zoom entails a coordinated horizontal and vertical re-scaling. With each zoom, the derivative becomes more accurate, but it is still "approximate." This indicates he is picturing scaling revealing covariation at an arbitrary level. Then he explicitly adds that one can "zoom in infinitely" to get an exact answer. His description indicates that he is generalizing his image of arbitrary re-scaling: at the infinitesimal scale there is still smooth covariation, and the graph has become perfectly straight, enabling the determination of an exact slope. His "collection of straight lines" metaphor is a way to hold both finite and infinite scaling conceptions; it was, in fact, also used by Leibniz (Katz, 1998).

Scaling-continuous student reasoning in informal infinitesimals differential calculus

Several of the students who took the informal infinitesimals calculus class employed scaling-continuous covariational reasoning when interviewed in their reasoning with differential notation and differential equations. For sake of brevity, we describe this with an illustrative segment of one interview. In this segment, the interviewer has asked the student, Roan, to describe the relationship between the amounts equation $y = x^2$ and its corresponding bits equation $\mathrm{d}y = 2x{\cdot}\mathrm{d}x$. The interviewer asks what the terms in the bits equation mean. Roan describes how the dx refers to an infinitesimal difference between two x values, and the dy refers to an infinitesimal increment between the two corresponding y values. The interviewer then asks what the x is doing in the equation. After some discussion, Roan asks if he can illustrate his thinking with the dynamic graphing program Desmos on his computer. He graphs the function $y = x^2$ and then says that the bits equation $\mathrm{d}y = 2x{\cdot}\mathrm{d}x$ needs an x in it because for this function the dy's will be different sizes depending on the dx's. The interviewer then asks him to explain his thinking in terms of dx and dy increments.

Roan's computer has a touch screen which enables him to zoom in and out on the graph in the Desmos program by using two fingers. He zooms in on the graph at the origin, and points out that near 0 "the proportion to dy to dx is not much at all," gesturing a vertical increment (dy) that is small in comparison with the horizontal increment (dx). Roan then zooms back out and says:

Roan's words	Roan's gestures
Yeah, 'cause you can see, like, as you go across this distance,	gestures with two fingers significantly separated
y doesn't change as much as here, like if you go from here to here,	gestures with two fingers close together a dx increment in one place and then another same-sized dx increment further to the right
y goes up more in relation.	gestures with the corresponding vertical dy increments of two different sizes, the right one being significantly larger than the left one
Or from here to here,	zooms in
it goes up this much, so it's going up more and more in comparison.	gestures a fixed small horizontal increment from x = 0 to 0.2 and then again from x = 0.2 to 0.4, then a few more times, moving the increment to the right
So the change isn't affecting y as much and then you keep going over. Now when x changes,	zooms out, then drags the graph over and indicates a small dx increment in a different spot
y goes a lot.	gestures a vertical increment
Then when you keep going over,	drags graph over and indicates another same-sized small dx increment yet another spot further right
when you change your x, y changes a lot.	gestures a large vertical increment

Roan then zooms out further. The interviewer asks about how this relates to the x, and Roan says, while pointing at the indicated parts of the equation $\mathrm{d}y = 2x{\cdot}\mathrm{d}x$, "Because this [d$x$] stays the same, and this [x or maybe $2x$], is giving the proportion, where this [dx] is fixed…" He describes then how as you move to the right, x gets larger, and the dy increment gets larger even though the dx stays the same.

In this segment, Roan treats the increments dx and dy as small differences in the variable quantities x and y in the graph of $y = x^2$. He describes how the bits equation $\mathrm{d}y = 2x{\cdot}\mathrm{d}x$ shows the coordination of uniform-sized dx increments with varying-sized dy increments, and that this variation depends on where in the x direction the increments are being considered.

Roan's continual gesturing shows how scaling-continuous covariational reasoning supports his understanding of this coordination between dx and dy. In two minutes, Roan zooms in or out on the graph no fewer than twelve times. He zooms in on the graph usually when he is talking about a particular increment dx and its corresponding dy. This suggests that his image is that an "infinitesimal" (as he often calls it) difference or increment is obtained by zooming in near some point x. When it comes time to talk about how a dx-dy pair at one spot x_1 relates to another dx-dy pair at another spot x_2, he zooms back out again so that the overall shape of the graph is more apparent, gesturing how the dy's are different in size at these two locations. Scaling in is part of his image of how one sees a pair of infinitesimal increments in the two coordinated variables at a particular location. Scaling out is part of his image of how the coordination between the dx and dy itself varies from point to point on the larger graph.

In his image, there seems to be an operational coordination between increments of x and increments of y at every scale, which also presumes that scaling never reveals non-intervals in either quantity. Because this coordination is available even, according to Roan, at the infinitesimal scale, he can envision a distinct coordination of dy and dx "at every x."

Discussion

Neither calculus course was designed or taught with the idea of scaling-continuous variational/covariational reasoning in mind—indeed, at the time neither instructor had heard of the idea. Yet some of the students in the courses ended up displaying these modes of reasoning, and these modes seem to support these students' reasoning about some key ideas in differential calculus. In this section we discuss how scaling-continuous reasoning can be seen to support robust understandings of some key ideas in differential calculus that are aligned with the goals of the two classes.

Students in the local straightness calculus class frequently exhibited scaling-continuous covariational reasoning when discussing a derivative at a point. They anchored focus at a single point, which they indicated both verbally and with a graphical mark, and pictured zooming in as far as needed, with a technology tool or with mental or written images, to reveal a straight line segment. They then estimated the value of the slope and assigned it the meaning of the derivative of the original function at that point. In this last step, they turned to coordinating increments in both quantities at a single point, recognizing that the arbitrary zooming of scaling-continuous reasoning was necessary to make that meaningful.

Also, it is notable that this can serve as a foundation for the conception of the derivative as a function, as demonstrated by Sam (and by many students in class). He described taking the straight line at a point and moving it to the right and recording the derivative at every point. This indicates he had encapsulated his scaling-continuous construction of the tangent line, to recreate it at any point.

In the informal infinitesimals calculus class, scaling-continuous variational reasoning provides a crucial image that at each scale the values of a continuous variable form a continuous unbroken increment on which variation occurs. This idea can then be generalized to an image that each infinitesimal increment looks the same way, a generalization that Sam and Roan both appear to have made. A robust image of infinitesimal entails generalizing or encapsulating the process of scaling involved in applying scaling-continuous variational reasoning.

With this in mind, scaling-continuous covariational reasoning gives the student a way to imagine a coordination between each continuous increment of one variable and a continuous increment of another, at every scale. Roan tacitly assumes that coordination when gesturing and speaking about the relationship between bits, differences, and changes in x and y. While scaling-continuous covariational reasoning only includes this coordination for arbitrary scales, for the informal infinitesimals approach it is important for this coordination at some point to be generalized to the infinitesimal scale. The reason is that this provides a basis for the productive interpretation of a bits (differential) equation as an algebraic description of the relationship between an infinitesimal amount of change in, say, x and a corresponding infinitesimal amount of change in, say, y. Because these amounts are infinitesimal, this coordination can be envisioned at every value of x, and depending on that value of x. This is illustrated when Roan describes and gestures how the coordination he imagines between dy and dx is established at different points, and how this in turn varies from location to location.

In both classes, the encapsulation of scaling-continuous covariational reasoning at a single point is a crucial element as students form their conceptions of single variable differential calculus, even though it manifests differently.

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Mathematicians' Validity Assessments of Common Issues in Elementary Arguments

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This study explores how mathematicians view validity in the face of explicit validity issues within written mathematical arguments in the context of the Introduction to Proof (ITP) setting. An internet survey of 30 arguments was constructed leveraging common issues in validity at the ITP level, and widely distributed to research-active mathematicians in the United States. The results suggest that there is no consensus as to the effect of any single validity issue on the overall validity of an argument, lending credence to the notion that argument validity lacks a consistent set of criteria from one mathematician's point of view to the next.

Keywords: Proof, Validity, Mathematician Practice

The nature of a valid, mathematical proof is difficult to define. In fact, several studies have established that even mathematicians have a number of points of disagreement on what constitutes a valid proof (Inglis & Alcock, 2012; Inglis, Mejía-Ramos, Weber & Alcock, 2013; Weber, 2008). This finding corroborates mathematicians' own accounts that there may not be any fixed set of standards for determining what is or is not valid within the mathematical context (e.g., Rav, 2007). The study presented here extends this tract of research by exploring on a large scale how mathematicians judge the effect that specific flaws within an argument can have on the validity of an argument at the Introduction to Proof (ITP) level. The effort is to explore in depth what standards might currently exist and what perceived requirements might lead to disagreement amongst mathematicians. Specifically, this research aims to answer the following questions:

- To what extent do mathematics professors agree about whether basic deductive arguments (at the ITP level) are proofs?
- What characteristics of deductive arguments account for disagreement in mathematician's validity assessment?

Background

In response to the assertion that argument validity is an important criteria when exploring undergraduate mathematics major's understanding of proof (Selden & Selden 2003), researchers have focused on mathematicians' ideas concerning validity to clarify existing standards and determine the consistency and importance of validity within mathematics at large and in the undergraduate mathematics classrooms (Inglis & Alcock, 2012; Inglis, et al., 2013; Weber, 2008). Weber (2008) investigated both the contextual criteria and strategies research-active mathematicians used when validating both elementary and advanced arguments. Weber found that there were a number of extra-mathematical criteria that the eight mathematicians from his study used in considering the validity of the arguments, including who the author of the argument was. One of the most important criteria for many of the mathematicians when looking at elementary proofs was the question of what had been established to be true. This key characteristic hits at the heart of any validity judgement as the building of a specific set axioms, theorems and the like within any setting – or the lack thereof – may require further argumentation on the authors part when constructing an argument. While Inglis and Alcock's (2012) main focus concerned the differences between novice and expert approaches to validating tasks, their findings concerning the 12 mathematicians in the study support the notion that mathematicians do not exhibit a uniform consensus of what might count as valid. Inglis, et al. (2013) expanded upon this idea by exploring how these disagreements in validating might arise in terms of a mathematician's area of expertise within mathematics, as well as exploring mathematicians' assessments of their own validity judgments in terms of their perception of the how other mathematicians would validate a proof. In the end, this study of 109 mathematicians and the two prior studies point to the same overall conclusion that validity is, as yet, a poorly defined construct which is case and individual dependent.

While each of these studies has helped to clarify the relationship between mathematician, context, and expertise and the role the latter pair play in validity judgments, none of them have offered deeper insight into individual, specific criteria relating to argument creation that might affect the validity of an argument. Meaning, for example, it is unclear how mathematicians might react in the face of a warranting issue within an argument or to an argument that begins by assuming the conclusion and showing the antecedent as a direct result. Are mathematicians consistent in their judgments of some set of perceived validity issues, but less consistent in others?

Framing

The idea of proof is nuanced in the mathematics education literature ranging from the overtly mathematical in nature (e.g., Healy & Hoyles, 2000; Knuth, 2002; Mariotti, 2000) where logic and deduction are stressed at the expense of all else, to the cognitive or social perspectives each focusing on aspects of conviction, and communal acceptance (e.g., Balacheff, 1988; Harel & Sowder, 2007). For this study, I adopt Stylianides' (2007) definition:

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291; emphasis in original)

From this definition the understanding is gained that a proof is a mathematical argument which is defined by three distinct characteristics concerning statement, modes of argumentation, and representation. For this research, the scope of what is a proof is limited to statements and representation which are common in the ITP classroom (David & Zazkis, 2016), and modes of argumentation that consist specifically of direct proof[1].

Validity

Selden and Selden (2003) called proof validation, "the reading of, and reflection on proofs to determine their correctness" (p. 5). While correctness may imply some sort of universal standard, the validity standing of an argument is often subjective. I treat proof validation is the act of judgement or evaluation which leads the reader to identify whether a given argument appropriately proves a statement.

Finally, for linguistic clarity, I take *argument* to represent the body of all purported proofs regardless of their validity. Thus, to ascribe a series of logical (or illogical) statements as

[1] Direct proofs in this research included proof by cases.

an argument is to remove any notion of validity from the conversation. Arguments are valid-neutral. On the other hand, identifying an argument as a *proof* is to remove its valid-neutrality and assert that it is valid.

Common Validity Issues

To give context to the validity judgements that mathematicians made, this study leverages the idea of *issues in proof writing* which focuses on the prevalent validity issues amongst undergraduate mathematics major's own written arguments (Hazzan & Leron, 1996; Selden & Selden, 1987, 2003). The validity issues considered for this study fall into one of six categories as presented in Table 1. Each argument which was initially coded as invalid had the inclusion of a single example of one of the six issues where each argument was intended to present a single validity issue to the participants. Though often the validity issues were simple in nature, they were chosen or constructed because they represented issues that are considered common in undergraduate mathematics (Alcock & Weber, 2005; Hazzan & Leron, 1996; Selden & Selden, 1987, 2003; Weber & Alcock, 2005; Weber, 2001).

Table 1. Common Validity Issues

Issue (Abbr.)	Definition
Assuming the Conclusion (AC)	An argument assumes the consequent (conclusion) of the proposition it is claiming to prove and attempts to show that the antecedent is a direct consequence.
Circular Reasoning (CR)	An argument assumes the consequent (or antecedent) of the statement it is claiming to prove and comes to a trivial conclusion, namely the consequent (or antecedent) once again. Equally, within an argument a claim is made and used to argue to trivial ends, the claim itself. ($P \to Q \to \cdots \to P$)
Logical Gap (LG)	An argument omits a portion of reasoning; the argument has a hole. This could be thought of as a lack of explicit warranting where such would seem prudent.
Misuse of Notation (MN)	Within an argument, proper notation or variable naming conventions are not adhered to, or notation and variable naming conventions are used inconsistently.
Warranting (W)	Within an argument, an error in justification is made either explicitly or implicitly. This can take the form of an incorrect explicit warrant, or an incorrect implicit warrant which may emerge as an arithmetic or computational error.
Weakening the Theorem (WT)	An argument proves less than what is implied by the statement being proven or begins by assuming more than is permissible.

Methods

Qualtrics, an internet survey system, was used as the main resource for data collection in this study to obtain a large sampling of mathematicians. The survey itself consisted of 30 arguments to 22 different propositions considered germane in the ITP setting (David & Zazkis, 2016). For each argument, the participants were first asked, "Is the argument for the included proposition a valid proof?" and given the binary option of "Yes - valid," or "No - invalid." Participants were initially warned against grading the proofs as though they were student proofs, but to instead answer for themselves the question, "Does this argument actually prove the proposition in a way that I feel is appropriate, based upon what I believe is requisite for an argument to be valid?" In this way it was left to the participant to infer what they felt was requisite for an argument to be a valid proof.

If the proof was initially coded as invalid and the participant disagreed (i.e., they chose "valid" as their response) the participant was presented with the proposed validity issue and asked how the presence of said flaw affected their initial response, and then given the chance to change their minds about the validity of the argument[2]. If the participant did not change their mind, they were asked to share why they felt the flaw did not invalidate the argument. Additionally, for each argument that was initially coded as invalid, if the participant agreed that it was in fact an invalid argument, they were also presented with the flaw and asked if it was the reason they choose invalid. If it was not the reason, participants were asked to state why they thought the argument was invalid. For all arguments which were initially coded as valid, if the participant disagreed and chose invalid, they were asked to justify their views by stating why they thought the argument was invalid.

The arguments themselves were clustered into one of seven groupings based upon their initial validity coding and issue. Participants were then randomly presented with an argument from each cluster to ensure that they saw an argument whose flaw came from each area of the framework as well as being presented with an argument which was considered to initially be valid. No participant saw the same argument twice. In total, 1528 survey invitations were distributed via email to research-active mathematicians across the United States, of which 228 submitted responses to the survey. Of the 228 participants, 178 completed all 7 argument sets with which they were presented, all others completed no less than 2 argument sets.

All free responses were analyzed using thematic analysis (Braun & Clarke, 2006). The analysis began with open coding of the free responses for each of the 30 arguments independently and categorizing responses relative to each argument in terms of their appropriateness. All nonsensical free-responses led to a cycle of analysis of the quantitative data supplied by the author of said free-response to ensure the author was not supplying malicious data[3]. Malicious data was omitted from further analysis. Following open coding, themes were identified, categorized and condensed for each argument. No cross-argument analysis occurred as the questions for this study do not focus on how responses to one type of validity issue are correlated to responses to other validity issues.

Results

Figure 1 comprises the final validity judgements to all 30 arguments including those of which were initially coded as valid (i.e., arguments V1-V5). The chart represents the percentage of mathematicians that deemed each argument to be invalid calculated by taking the total number of "No - invalid" responses along with the number of mathematicians who changed from "Yes – valid" to invalid and then dividing by the total number of responses. For both the set of valid and invalid arguments, the number of responses for each argument was not uniformly distributed due in part to the random design of the survey and the inclusion of partially completed responses. Disagreements among mathematicians was found in every category, and while there are cases

[2] There were four exceptions where no follow-up was requested. Three of these four arguments contained a logical gap which initially it was unclear if the gap would affect the validity of the argument.
[3] Quantitative data was considered malicious if the entire survey had been completed in under 10 minutes, any one validity set – the initial validity question and all follow-up questions – was completed in less than one minute, or if all validity questions were homogeneous and all *other* follow-up free response questions were left blank.

where 100% of mathematicians agreed that something was invalid (AC1-AC3), no one category was free from disagreement.

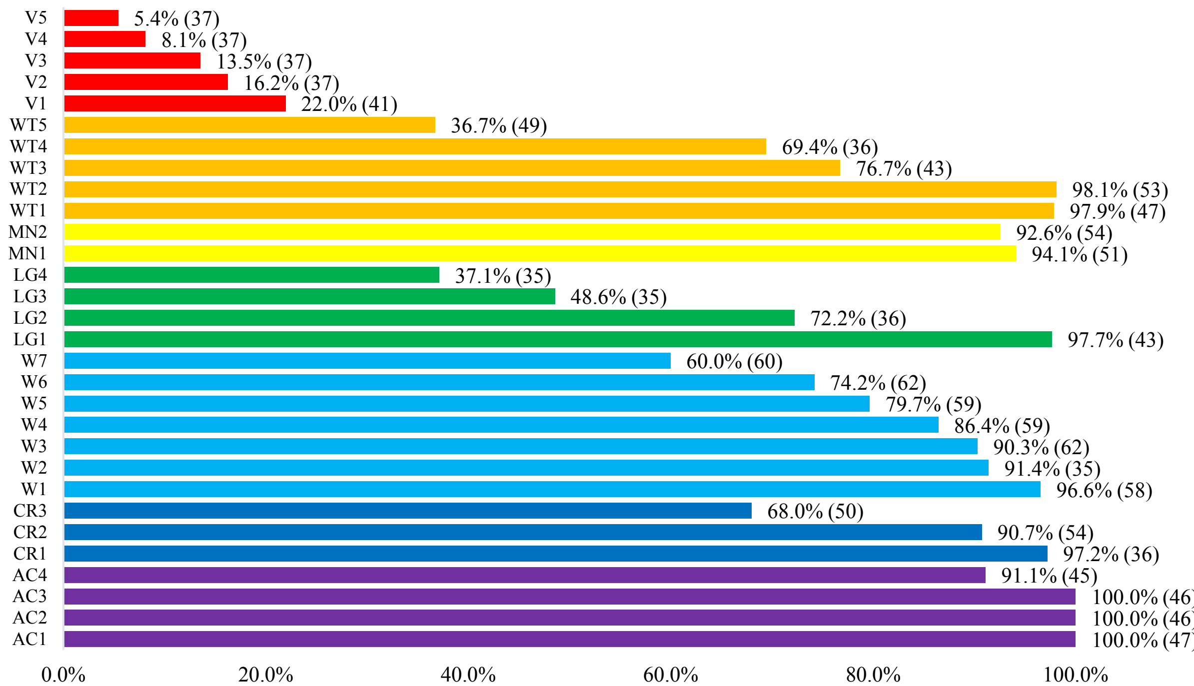

Figure 1. Percentage of mathematicians who thought the argument was invalid (number of responses). Each argument that was initially coded as invalid was given a name and number based on its included validity issue (i.e., WT – weakening the theorem, MN – misuse of notation, LG – logical gap, W – warranting, CR – circular reasoning, and AC – assuming the conclusion).

A Weakening the Theorem Example

One of the more interesting weakening the theorem results, argument WT5's (Figure 2) validity issue was that the argument did not account for the negative integers and zero when defining the parameter x as odd (i.e., "$x = 2a + 1$ for some $a \in \mathbb{N}$" instead of "for some $a \in \mathbb{Z}$"), thus arguing for something weaker than what was intended to be implied by the proposition. Mathematicians who felt this was not enough to invalidate the argument fell into two general groups, the first arguing that the proposition itself does not clearly define odd as to mean odd integers versus odd natural numbers. The second, and perhaps more pertinent group of mathematicians thought that, though the argument failed to account for the negative integers and zero, because the structure and logic of the argument was intact the weakening of the theorem that had occurred did not invalidate the argument. For instance, one mathematician said, "The heart of the argument is understanding that odd numbers are 1 mod 2 and that an odd number squared is 1 mod 2, which remains valid. The error is minor because of its consequence. If this was a proof involving absolute values and the negative numbers [were] not properly dealt with that would be much more damning." Thus, despite the inaccuracy these mathematicians felt the argument was valid.

Proposition: If x is odd, then x^2 is odd.

Argument: Suppose x is odd. Then $x = 2a + 1$ for some $a \in \mathbb{N}$. Thus we have

$$x^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1.$$

Since $2a^2 + 2a \in \mathbb{N}$, then $2(2a^2 + 2a) + 1$ is odd, and therefore so is x^2.

Figure 2. Argument WT5 – 36.7% of mathematicians thought the weakening that occurred was enough to invalidate the argument.

A Warranting Example

The validity issue of warranting lead to unclear results in term of agreement. Arguments W5 (Figure 3) and W7 (Figure 4) argued for the same proposition and had identical arguments save for an explicit warranting issue which occurred at the same point within each argument. The difference came about in the perceived reasonability of the warranting issue which lead 60% of mathematicians to conclude that W7 was invalid, while 79.7% of mathematicians thought the warranting issue in W5 was sufficient to invalidate the argument[4]. Many of the mathematicians who claimed that W7 was valid cited the "minor typo" that occurred did not underscore the soundness of the argument as a whole. On this fact many mathematicians made statements like, "Yes, it is the incorrect term for the property being used; however, the property actually used (multiplication on R is commutative) is certainly true, so the argument is still valid." This contrasts with the general sense that though commutativity was also correctly used in W5, there was a much stronger negative reaction in term of validity to the claim that "multiplication is an equivalence relation in $\mathbb{R}$." It should be noted that no one argued that either was a true statement.

Proposition: For $a, b \in \mathbb{R}$, if $a < b$ and $0 < a$ then $a^2 < b^2$.

Argument: Let $a, b \in \mathbb{R}$ and suppose that $0 < a < b$. Now, consider the following

$$\begin{aligned} 0 < a < b &\Rightarrow a \cdot a < b \cdot a \\ &\Rightarrow a^2 < b \cdot a \\ 0 < a < b &\Rightarrow a \cdot b < b \cdot b \\ &\Rightarrow a \cdot b < b^2 \end{aligned}$$

But since multiplication is an equivalence relation in $\mathbb{R}$, then $b \cdot a = a \cdot b$. Thus we have that $a^2 < b \cdot a = a \cdot b < b^2$, thus $a^2 < b^2$ as required.

Figure 3. Argument W5 – 79.7% of mathematicians thought the claim that "multiplication is an equivalence relation in $\mathbb{R}$" was enough to invalidate the argument.

Proposition: For $a, b \in \mathbb{R}$, if $a < b$ and $0 < a$ then $a^2 < b^2$.

Argument: Let $a, b \in \mathbb{R}$ and suppose that $0 < a < b$. Now, consider the following

$$\begin{aligned} 0 < a < b &\Rightarrow a \cdot a < b \cdot a \\ &\Rightarrow a^2 < b \cdot a \\ 0 < a < b &\Rightarrow a \cdot b < b \cdot b \\ &\Rightarrow a \cdot b < b^2 \end{aligned}$$

But since multiplication is associative in $\mathbb{R}$, then $b \cdot a = a \cdot b$. Thus we have that $a^2 < b \cdot a = a \cdot b < b^2$, thus $a^2 < b^2$ as required.

Figure 4. Argument W7 – 60% of mathematicians thought the claim that "multiplication is associative in $\mathbb{R}$" was enough to invalidate the argument.

An Assuming the Conclusion Example

Even in the case of arguments which assume the conclusion, thus having major structural issues there was some amount of disagreement. Argument AC4 (Figure 5) argues the converse of

[4] The difference between the two proportions is statistically significant with $p = .0196$, 95% CI [0.0195,0.3582] with continuity correction.

the proposition, and despite having this fact pointed out to them, four mathematicians held that the proof was valid making statements like, “I would say this argument is almost correct rather than invalid,” or “It could be modified quite quickly for the proof to be correct. The main idea is still there.” Thus, despite arguing the converse and even though these mathematicians agree the argument is not *correct* they felt it was valid.

Proposition: The sum $x+4$ is odd whenever x is also odd.

Argument: Assume that $x+4$ is odd, then there exists an integer n such that $x+4=2n+1$. Thus we have that $x=2n-4+1=2(n-2)+1$. Since $n-2\in\mathbb{Z}$, then x is odd.

Figure 5. Argument AC4 – 91.1% of mathematicians thought the argument for the converse was invalid in light of the proposition.

A Logical Gap Example

Finally, mathematician’s sense of the affect of logical gaps lead to an interesting result with argument LG3 (Figure 6). Here, the argument presented trivializes the proving process at many points with unsupported statements, twice using the phrase “which implies” in place of an actual argument. This lack of overt justification divided the mathematicians’ validity stance with 48.6% of mathematicians claiming the argument was invalid.

Definition: The symmetric difference of A and B is defined as $A \triangle B=(A-B)\cup(B-A)$.

Proposition: For any sets A, $A\triangle A=\varnothing$ and $A\triangle\varnothing=A$.

Argument: Let A be a set, then by the definition of symmetric difference $A\triangle A=(A-A)\cup(A-A)$. But $(A-A)\cup(A-A)=\varnothing$ which implies that $A\triangle A=\varnothing$, as required. Furthermore, also by the definition of symmetric difference $A\triangle\varnothing=(A-\varnothing)\cup(\varnothing-A)$. But here $(A-\varnothing)\cup(\varnothing-A)=A$ which implies that $A\triangle\varnothing=A$. Thus we have shown that for any set A, $A\triangle A=\varnothing$ and $A\triangle\varnothing=A$.

Figure 6. Argument LG3 – 48.6% of mathematicians thought the lack of overt justification was enough to invalidate the argument.

Conclusion

In response to the first research question, the data from this study reflects that even in direct, deductive proofs at the elementary level, there is a substantial disagreement amongst mathematicians over validity. This finding corroborates the findings from Inglis et. al. (2013) that mathematicians use different standards in judging an argument’s validity. The divergence was not unexpected in terms of more subjective proof aspects such as the allowable size of a logical gap. However, these divergent validity standards were apparent even when an argument had a major structural issue: assuming the conclusion.

The disagreement over validity has implications for instruction. Particularly, the fact that these inconsistencies may counter the dominant narrative that mathematics is universal. Furthermore, inconsistency across instructors could lead to cognitive dissidence in student’s proof writing and reading as they progress through a tract in undergraduate mathematics, and perhaps beyond.

Finally, taken together with past research, this data suggests that not only do mathematicians have different standards for what is and is not valid, but they might not have a good sense of what valid means generally as such a notion may not exist in a binary sense (Rav, 2007). This in turn leaves some questions about whether we as mathematics education researchers have a good feel for what validity is as well. If nothing else, future studies should be careful in making claims about validity in terms of absolutes, as there may be no absolute standard, at least not for elementary arguments.

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Is Statistics Just Math? The Developing Epistemic Views of Graduate Teaching Assistants

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Research has shown that teachers and instructors' views about the discipline they teach inform their instructional approaches. As a foundation for investigating this relationship in statistics, we explore how (or whether) beginning graduate students in statistics perceive statistics as distinct from mathematics. Using the lens of epistemology, we share findings from interviews with four, first-year graduate students who served as graduate teaching assistants (GTAs) in a statistics department. Using data collected from interviews across their first year, we constructed three models that explain how the GTAs conceived of the nature of statistics in relation to mathematics. Additionally, we identified two continua that reveal how participants came to understand the nature of doing statistics. We discuss how these models and continua form the basis of a unified statistical epistemology that has implications on their views for statistics education.

Keywords: Statistics Education, Graduate Teaching Assistants, Epistemology

Research in mathematics and science has revealed that instructors' pedagogical decisions are deeply influenced by their views about the nature of the discipline they teach (Abd-El-Khalick, Bell, & Lederman, 1998; Cross, 2009; Speer, 2008). As a young and evolving discipline, statistics proves to be an area ripe for investigation on this matter. Statisticians and statistics educators have identified multiple perspectives and routes for disciplinary engagement in statistics, even going so far to suggest statistics to be validly seen as a liberal art, an area of scientific inquiry, and a branch of mathematics (De Veaux & Velleman, 2008; Lindley, 2000). With so many different perspectives of statistics available, our research investigates how instructors develop and refine their own views of statistics. We view this line of inquiry as a precursor to understanding how instructors make certain pedagogical and curricular decisions.

Our research focuses on the views of Graduate Teaching Assistants (GTAs) in statistics. GTAs represent a unique subset of instructors, distributing their time between the roles of teacher and student. This report documents the views of four, first-year statistics GTAs as they respond to questions about the nature of statistics as a discipline and what constitutes a statistical problem. In particular, this paper focuses on how these GTAs distinguished statistics from mathematics as a unique discipline. We explore the perceived distinctions and commonalities the GTAs expressed between the two disciplines, concluding with thoughts on how these views might guide their teaching. We address the following question: How do new statistics GTAs discuss and conceptualize statistics in relation to mathematics?

Conceptual Framework

We approach our topic through the lens of epistemology to target GTAs' deeper philosophy about the purpose and structure of statistics as a discipline. *Epistemology* is focally concerned with an individual's views on the nature of knowledge and the means by which we know (Hofer & Pintrich, 1997). Research in recent decades supports the stance that epistemic views appear to be contextual and domain-specific (Buehl, Alexander, & Murphy, 2002; Op 't Eynde, De Corte, & Verschaffel, 2006). The case for domain-specific lenses on epistemology can be readily made when one considers aspects that are privileged in classrooms of different domains. In the mathematics classroom, Schoenfeld (1992) noted that it is typical for students to think there is only one right answer and that mathematics is a solitary activity. Contrast this perspective with norms in the Humanities, where students more readily recognize peer review and negotiation as a means to developing knowledge (Donald, 1990).

To frame our investigation of statistics GTAs' epistemic views, we first explored work on epistemology from a domain-general perspective to identify common dimensions of thinking that permeate multiple disciplines (e.g., Hofer & Pintrich, 1997). Second, noting that statistics has origins deeply rooted in both mathematics and science (Stigler, 1986), we examined epistemology as it has been framed in these two disciplines (Op 't Eynde et al., 2006; Russ, 2014; Schoenfeld, 1992; Tsai & Liu, 2005). From these models and discussions, we generated a four-dimension framework to capture the epistemic views of statistics GTAs (Table 1). Diamond and Stylianides' (2017) paper, which examined the personal epistemologies of statisticians, was also influential for designing our interview questions and interpreting our results. That study, however, adopted a domain-general framework (Hofer & Pintrich, 1997); thus, it did not contribute new ideas to the dimensions of our statistics-specific framework.

Table 1. Statistical Epistemology Framework

The Nature of Statistics	The Nature of Statistical Knowledge	The Nature of Knowing Statistics	**The Nature of Doing Statistics**
The distinction and purpose of statistics	The origin and openness of statistical knowledge.	The nature of proving and negotiating statistical knowledge	**The role of creativity and subjectivity in engaging in statistical problems**

In this paper, we focus primarily on participants' responses related to the first and fourth dimensions of this framework—the nature of statistics and the nature of doing statistics—which yielded the richest dialogue on the comparisons of statistics to mathematics. As Diamond and Stylianides (2017) recognize, there is an "inextricable link between statistics and mathematics…statistics teaching is frequently embedded in mathematics" (p. 336). Unfortunately, as the authors also recognize, such conceptions of statistics are incomplete. An expert framing of statistics recognizes the dynamic and flexible nature of the discipline based on its attentiveness to context (Cobb & Moore, 1997). Cobb and Moore expand on this by describing statistics as exploring and explaining variation in the world, with mathematical methods acting as part of the toolkit individuals use to accomplish this goal. We focus on disciplinary aspects where the GTAs drew meaningful connections and distinctions between statistics and mathematics, as well as points of struggle.

Methods

Setting

This study took place in a statistics department with 200 graduate students and 62 GTAs, housed in a large, public university in the U.S. In this department, new GTAs are assigned grading or recitation duties for the first two semesters, with the possibility of becoming a solo instructor the following summer or fall if they complete the department teaching workshop in the spring. During the 2017-2018 academic year, 12 new graduate students were awarded teaching assistantships, all of whom were invited to participate in an interview at the beginning of their first term, with continuing invitations for more interviews throughout the year. This study documents the views and experiences of four of these GTAs who a) had no previous teaching experience, b) participated in all six interviews for the full year (Fall 2017-Summer 2018), and c) were awarded solo teaching positions during the final semester of the study. Details about the participants are displayed in Table 2.

Table 2. Participants

Pseudonym	Gender	Nationality	Highest Degree
Kathy	Female	U.S.	B.S. Mathematics & B.S. Human Health
Li	Male	Chinese	B.S. Mathematics
Mindy	Female	U.S.	B.S. Mathematics
Sahil	Male	Indian	B.S. & M.S. Statistics

Data Collection

The GTA-participants were involved in a larger, yearlong study, with each completing six one-on-one interviews across a full year. The aim of the full study was to chart participants' epistemic views, pedagogical views, and influential experiences throughout the year, culminating in observations and discussions of their teaching during the following summer. The interviews and observations were conducted and facilitated by the first author, a U.S. white male who had previously been a GTA in the department. As such, the interviewer was familiar with the GTA responsibilities and program of training in place in the department.

This paper focuses on data collected from the first and third interview. The first interview was relatively informal, allowing the interviewer to get to know each of the participants (Corbin & Strauss, 2006) and explore their initial thoughts about the discipline of statistics (e.g., How would you define statistics? What does it mean to do statistics?). The length of the first interview was between 30-45 minutes for all participants and took place during the first week of classes during their first semester. The third interview included more in-depth exploration of each participants' epistemic views, reflecting themes included in the framework presented in Table 1. The third interview lasted between 70-90 minutes and took place midway through their second semester. Each interview proceeded in a semi-structured format, leaving time for the interviewer to probe certain ideas more if they were relevant. Additionally, each of these interviews connected participants' views about statistics to their views about statistics pedagogy, which we plan to document more fully in future research.

The epistemology questions used in the interviews were inspired from a number of sources, including questions used to assess the epistemologies of statisticians (Diamond & Stylianides, 2017), students in mathematics (Op 't Eynde et al., 2006), and students in science (Tsai & Liu, 2005). Items to address statistics GTAs' views about an introductory course curriculum, statistics teaching, and statistics learning were borrowed or adapted from items on existing surveys used to understand the pedagogical views of statistics GTAs (Justice, Garfield, & Zieffler, 2017). Additional items that were written specifically for this study were reviewed by three other researchers who have published work on statistics GTAs.

Methods of Analysis

The first interview provided an early glimpse of each GTA's developing statistical epistemology, revealing ideas and distinctions to explore further. The third interview allowed for more in-depth probing of each participant's epistemology. The first author transcribed the interviews and coded responses according to the dimension they fit. After creating a data matrix that included all relevant responses divided by dimension, further coding was conducted to identify response themes (e.g., the interdisciplinarity of statistics) that helped connect ideas across participants (Miles, Huberman, & Saldaña, 2014). Some of these themes were isolated to a specific question, while other themes were present across responses to several different questions.

This paper highlights several themes that emerged regarding GTAs' views about statistics. We discuss three models of thinking related to views of the Nature of Statistics and two continua of thinking regarding the Nature of Doing Statistics: statistics as flexible versus methodical and statistics as experienced-based versus knowledge-based. These models and continua represent how we saw the participants making sense of statistics, and specifically how they were relating statistics to mathematics. We briefly highlight each idea and conclude with a discussion of how their statistical epistemologies have implications for their instruction.

Findings

The Nature of Statistics

All four participants discussed statistics as being centrally concerned with data and agreed that statistics is closely related to mathematics in nature and structure. Participants also discussed statistics as being concerned with interdisciplinary applications. Primary differences were rooted in articulations of the purpose of statistics and whether the discipline was better understood as a form of mathematics, an extension of mathematics, or as its own distinct subject. We highlight each of these three models below.

Statistics as applied mathematics. In discussing the nature of statistics, Kathy found statistics to be inherently similar to mathematics in structure. She described both as having assumptions and utilizing fixed methods, making them both "hard and fast sciences." Kathy differentiated statistics from mathematics as being more concerned with applied questions, noting that statistical problems often necessitate extracting information from a paragraph. She said of statistics, "it's not just learning the equation, it's learning how to interpret the equation and what it means, and I think that's just as important as getting the right answer." When Kathy discussed her experiences in mathematics, it became clear that she had few experiences to work through applied problems. As a result, she largely associated statistics with more applied problems and mathematics with abstract problems.

From Sahil's perspective, mathematics exists fundamentally for its own sake and is not centrally concerned with modeling reality. Statistics, in contrast, exists for the purpose of application. According to Sahil, mathematical concepts, specifically integration and differentiation, exist in their own right. Statistical topics, like hypothesis testing, differ in that application was necessary to give it identity. Sahil was hesitant to call statistics a subset of mathematics, but viewed it more as an alternative use of mathematics, concerned with methods created for the purpose of understanding and modeling real-world data. The model of statistics Kathy and Sahil discussed (see the left most panel in Figure 1) represents the domains of statistics and mathematics as a spectrum between application and theory.

Statistics as extending mathematics into context. In many ways, Mindy shared a similar perspective of statistics being focused on application, but knew that mathematics still included applied elements. Mindy noted that mathematics is more observable than statistics and could be described better as an exact science involving certain formulas. To Mindy, statistics certainly involves real-world observation, but not in the same way that mathematics does. Mindy viewed statistics as more situational and assumption-based, explaining that every time you use a particular statistical test, you need to check that assumptions are met first (e.g., random, independent sample). It seemed to Mindy that using mathematics does not require assumptions: rather, mathematical methods we use are essentially always appropriate.

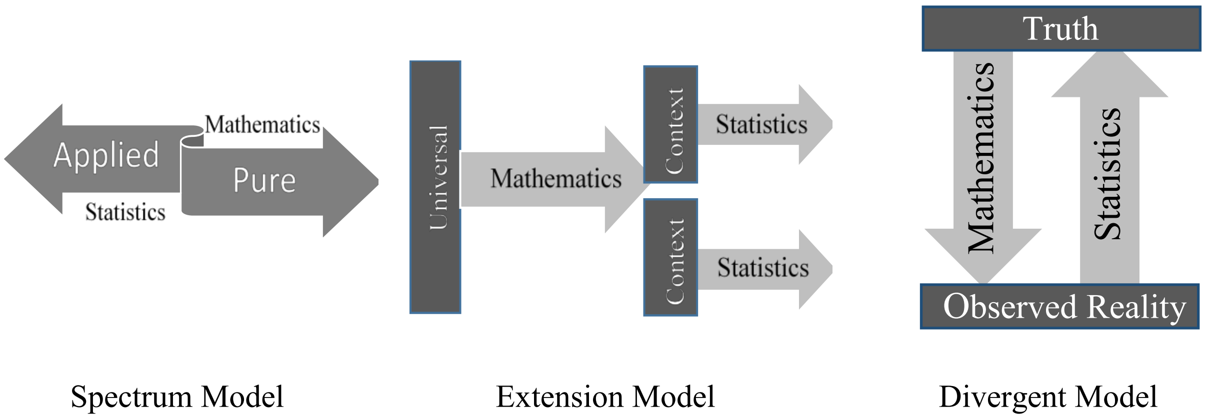

Figure 1. Models for the Relationship between Mathematics and Statistics

As we understand it, Mindy seemed to recognize that applying statistical methods and principles has to start from the situation in which the data were collected. In this way, statistical methods (e.g., a 2-sample t-test) are limited, whereas mathematical methods (e.g., addition, the associative property), are seemingly always appropriate. In the middle panel of Figure 1, we represent what we see as Mindy's model, showing statistics as an extension of mathematics into specific contexts, with mathematics extending application universally.

Statistics as diverging from mathematics in purpose. Li saw statistics and mathematics as sharing similar purposes in helping us understand and explain the world. For Li, however, the approaches to this goal are diametrically opposed for the two subjects. According to Li, statistics rests on the philosophy that we can never figure out the truth. Mathematics tries to prove truth under starting assumptions (e.g., Euclidean Geometry) and attempts to create a comprehensive and logical story. Statistics by nature cannot provide a full story, but simply a reasonable story. He described statistics as starting from the bottom (i.e., data through observable reality) and attempting to reach the top (i.e., the truth), while mathematics starts from an assumption-based top and logically proceeds to the bottom (right-hand panel of Figure 1).

The Nature of Doing Statistics

In the previous section, we unpacked three models that reveal the foundational paradigm through which the participants conceived of statistics as a discipline in relation to mathematics. In this section, we discuss how the participants' discussed the nature of doing statistics. We distinguish participants' responses using two continua: flexible versus methodical and experience-based versus knowledge-based. Within each of these continua, we illustrate how the participants discussed statistical problem-solving in relation to mathematical problem-solving. We view these two continua providing a richer picture of each participants' statistical epistemology with respect to mathematics that generally complement the disciplinary models they conveyed. We detail these two continua with examples below.

Continuum 1: Statistics as flexible versus methodical. At times during in the first interview, Kathy talked about doing statistics as similar to following the scientific method. She connected this to the type of work she did during a previous summer internship, discussing statistical analysis as essentially running experiments. Other times, Kathy used very mathematical language to describe the process of doing statistics, such as identifying “variables,” using “formulas, manipulation, and computing,” and finding the “right answer.” Both types of statistical work had in common a rather strict protocol—there is a correct way to do statistical and mathematical work. She described both as having “a process and a right answer,” suggesting uniformity in their approaches to problem solving. The primary difference between statistical and mathematical work was that statistical work included a broader spectrum of responsibilities and practices (e.g., applied work) while mathematics was primarily the procedures themselves.

Kathy did recognize some level of flexibility existing in statistical work, for example choosing a procedure or test to use. She remarked that two people using the same methods should have the same result, but that statisticians will often approach problems differently depending on their theoretical orientation (e.g., Frequentist or Bayesian). Sahil expressed an additional layer of flexibility in his responses by discussing openness in developing new methods and generating theory. He described the goal of such statisticians as trying to come up with the most “elegant” methods, not over- or under-fitting, but creating something simple yet robust. He went farther than Kathy in this respect by noting that statisticians utilize a mixture of pre-determined procedures and creative approaches, allowing researchers to add their own impression into their work. In this process of doing statistics and employing creativity, Sahil saw mathematics as the medium through which statisticians were playing with models and solutions. He did not view the two fields as remarkably different in the way theorists express creativity in their work. The difference is primarily in the fact that mathematicians are working more directly in the abstract while statisticians use mathematical tools to work more directly in the real world. With this perspective, Sahil was consistent with his model for statistics and mathematics being on different ends of the spectrum of applied versus pure.

As a volunteer data analyst for a school sports team, Mindy noted that statistical problems often have multiple approaches and valid solutions. Analytical approaches can then be flexible depending on the purpose they serve, such as choosing how to assess performance and improvement in sports. Underlying this flexibility is theory that, she supposed, must be objective. In comparison to mathematics, it appeared statistics leaves more room for creativity in deciding how to go about solving a problem, yet it is still dependent on a set of truths, which Mindy essentially equated with mathematics. This perspective aligned with Mindy’s model of statistics as being much like mathematics, but fitting context rather than universal stipulations.

Continuum 2: Statistics as experience-based versus knowledge-based. Having completed undergraduate degrees in both mathematics and human health, Kathy instinctively paired statistics and mathematics as categorically similar disciplines with “right answers,” while fields like human health were based on principles and individualized truths. She shared sleep as an example in health class for which students can share their own experiences and feelings as it related to healthy living, but joked that similar student-centered experiences do not make sense in statistics: “For [statistics], how do you feel about correlation? Where do you see correlation? [chuckling]…It’s just not as discussion based.” From Kathy’s perspective, both statistics and mathematics existed independent of students’ experiences and could not feasibly be approached in a similar manner as human health. Both mathematics and statistics existed within an objective framework of truths.

Li offered a contrasting viewpoint by stating that experience is an important part of statistical work. He described doing statistics well as a skill—like playing the piano or drawing calligraphy. Rather than simply applying knowledge, Li viewed doing statistics almost as an art where instinct essentially guides you in the same way that instinct guides a musician in the moment. Li did not distinguish the work of statisticians from that of mathematicians on this point; what he did distinguish was the type of experience that guided practitioners and researchers in each field. While mathematicians are inspired by reality in their pursuit to understand abstract ideas, statisticians are work from instinct and observation to better understand and model real-world phenomenon, just as a scientist would.

Summary

While Kathy and Sahil both shared a model of statistics and mathematics on a spectrum of applied to pure, they articulated differences in the nature of solving problems. The extent to which Kathy recognized flexibility in statistical work came in how statisticians might choose different methodical solution path (e.g., choosing a test); Sahil saw both disciplines informed by work that could be creative and open. Li and Mindy shared more alignment in their views, with Li sharing a more philosophical description of statistics in terms of pursuing truth and Mindy bringing a more practitioner perspective through discussing universal versus contextual applications. Li was much more detailed in discussing both statistical and mathematical work as experience-based, but in slightly different ways. Mindy's uncertainty in the theoretical components of statistics produced more hesitant responses on this front, but she also recognized flexibility and experience as core components of statistical work.

Interestingly, none of the participants described statistics as the pursuit to explain variability, which is regarded by many as the fundamental distinction of statistics (e.g., Cobb & Moore, 1997). That is not to say that anyone's answers were inherently incorrect. In fact, all participants shared ideas about statistics and its relation to mathematics that were sensible.

Conclusion and Implications

In future work, we plan to document how each participant's epistemic views connected to their vision for introductory statistics. In reference to the findings in this paper, we simply note how each GTA's statistical epistemology sets up different perspectives on the purpose of statistics education and the nature of problems they would likely envision students working on. For example, from Kathy's epistemology we hypothesize that students would complete procedural problems similar to those they would see in a mathematics class, but with a context attached. In contrast, Mindy and Li recognized statistical problem solving as being more flexible and experience-based, suggesting they might be more open to having students complete projects that interest them, or that their classes might more readily explore how different approaches or measures are judged for validity based on how well they meet the context and nature of the problem. Sahil envisioned a more theory-based statistics course that engaged students in the beauty and construction of statistical methods (i.e., statistical reasoning).

In order to prepare the next generation of undergraduate students for a world of data, their instructors must first understand what is truly unique about statistics and how an introductory statistics course differs from such courses as College Algebra and Calculus. The epistemic views of this study's participants reveal striking distinctions in how they understand the purpose and role of statistics. We believe this topic requires more research, including explicit connections between GTAs' epistemic views and instructional decisions.

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Discovering the Linearity in Directional Derivatives and Linear Approximation

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Linear functions of more than one variable exhibit the property that changes in the dependent variable are linear combinations of changes in the independent variables. Although multivariable calculus makes frequent use of this linearity condition, it is not known how students reason about linearity within this context. This report addresses this question by analyzing how three students incorporate linearity into their schemas for linear approximation and directional derivative. The students in this report showed a progression in their understanding from not using linearity within their reasoning to incorporating linearity into first their scheme for linear approximation and finally into their scheme for directional derivative. The results indicate that the context of linear approximation was useful for developing concepts of linearity and aiding their development of the concept of directional derivative.

Keywords: Multivariable Calculus, Linearity, Schema Theory, Directional Derivative

Introduction & Literature Review

There is a recent surge of interest in student learning in multivariable calculus, which is a crucial course for all STEM majors (PCAST, 2012). Although there is a rich body of research investigating students' understanding of rate of change (Johnson, 2013; Lobato & Siebert, 2002; Stump, 2001; Teuscher & Reys, 2010) in general and the concept of derivative (Zandieh, 2000; Park, 2011; Samuels, 2012; Orton, 1983) in particular, there is limited research on how students interpret these concepts in multivariate settings. Recent studies have indicated that students may struggle to generalize the concept of slope from two to three dimensions (McGee, Conner, & Rugg, 2011) and that by the end of a multivariable calculus class few students have developed a meaningful conceptualization of total derivative (Trigueros Gaisman, Martinez-Planell & McGee, 2018). Additionally, research in physics education shows that students having completed multivariable calculus struggle to appropriately apply partial differentiation in physics contexts (Thompson, Bucy & Mountcastle, 2006). However, there is evidence that the use of physical manipulatives may support students' conceptions of rate in multivariable calculus (Samuels & Fisher, 2018; McGee, Moore-Russo & Martinez-Planell, 2015).

Linear functions of one variable exhibit the important property that changes in the dependent variable are always proportional to changes in the independent variable. Calculus takes advantage of this fact when using a linear approximation to estimate nearby values of a function with the equation $\Delta y \approx f'(x) \cdot \Delta x$. Using the language of differentials this property can be summarized with the equation $dy = f'(x)\ dx$. However, in multivariable calculus the concept of linearity takes on the additional property that changes in the independent variables are additive when determining the change in the dependent variable. As a linear approximation for functions of two variables, we have that $\Delta z \approx f_x \cdot \Delta x + f_y \cdot \Delta y$; the corresponding differential property is $dz = f_x\ dx + f_y\ dy$. At a given point, these expressions become linear combinations. It is similar for the directional derivative: where $\mathbf{v} = (\Delta x, \Delta y)$, $D_{\mathbf{v}} f = f_x \cdot \Delta x / |\boldsymbol{v}| + f_y \cdot \Delta y / |\boldsymbol{v}|$. Although there is significant research on student understanding of linearity within the context of one variable functions (e.g. Ellis, 2007; Greenes, C., Chang, K. & Ben-Chaim, D., 2007; Moschkovich, J., Schoenfeld, A. &

Arcavi, A., 1993) and an emerging body of research on linearity within linear algebra (Wawro & Plaxco, 2013; Wawro, Rasmussen, Zandieh, Sweeney & Larson, 2012), there is an absence of research on how students experience linearity in multivariable calculus. This study adds to the literature by exploring student conceptions of linearity in this context. In particular, we seek to answer the question: what conceptions of the role of linear combinations do students form in the context of linear approximation and directional derivative for multivariable functions?

Theoretical Framework

This report aims to analyze student understanding through the lens of schema theory. Schema theory has a long history of development with many significant influences (e.g. Bartlett, 1932; Piaget, 1926; Anderson, 1984) whose models of cognition subtly differ from one another. For this reason there are multiple definitions of the word schema throughout the literature. For the purposes of this study we will define a *schema* as an internal framework used to guide encoding, organization and retrieval of information (Stein & Trabasso, 1982). In this way a schema characterizes the relations among its components (Anderson et al., 1978). From our perspective, schemas are *functional* in the sense that they are continuously undergoing change (Iran-Nejad & Winslerin, 2000), reshaping themselves as the individual undergoes new experiences and reflects upon past experiences. This reshaping can occur in three ways: *accretion*, in which new facts are assimilated into the existing knowledge structure, *tuning*, in which the knowledge structure is slightly modified without changing relationships, and *restructuring*, in which new knowledge structures are created. There are four types of tuning: refining accuracy, generalizing, exemplifying, and creating an archetype. There are two types of restructuring: patterned generation, in which an old schema is modified into a new schema, and schema induction, in which a recurrent relationship among schemas is retained as a new schema. The latter is the most difficult and rare form of learning (Rumelhart & Norman, 1978).

Methodology

The data for this report were obtained from semi-structured task-based interviews with three students working together as one group as they encountered the ideas of multivariable linear approximation and directional derivative for the first time. The students were enrolled in a multivariable calculus course incorporating physical manipulatives using the Raising Calculus to the Surface materials (Wangberg & Johnson, 2013). The interviews took place in two separate sessions. The first session consisted of open-ended questions and tasks designed to elicit their prior understanding of rates of change in single and multivariable calculus followed by a series of activities designed to explore the ideas of linear approximation and directional derivative. The second session revisited the linear approximation and directional derivative tasks to assess further changes in the way the students viewed these concepts. The sessions were video recorded and analyzed.

During the analysis the authors identified instances in which the students actively described or utilized schemas which incorporated aspects of rate of change or linearity. These instances were then analysed from the perspective of schema theory in order to identify the pattern of connections evoked by the students related to rate of change and linearity. These patterns were then analyzed over the duration of the interviews to determine significant changes within the students' schemas as a result of their explorations during the task.

Results

Students' Prior Knowledge

In response to the open-ended questions prior to the linear approximation task, each student exhibited a robust understanding of rates of change in single variable calculus. Their initial schemas included a description of derivatives as measurable rates of change in geometric and contextual situations arrived at through a limiting process. The students were then able to extend these ideas to a two-variable setting by adding an element of directionality to their mental framework. They were able to evoke this rate of change schema in order to measure partial derivatives in multiple settings: on a three-dimensional physical representation of a surface, in the applied context of a heated plate, and on a contour map.

At this point in the interview the students' primary use of directionality was to reduce a three dimensional problem to a problem of only two dimensions by looking at the traces of the surface on the coordinate planes. This is described below with the first evidence that the students were also considering directions other than those along the coordinate axes.

Interviewer: You mentioned earlier that the idea of derivative is connected to the idea of tangent line. Is there any sort of similar idea that holds in multivariable calculus?

Willy: When you're dealing with more dimensions, kind of like a plane, which plane the rate of change is happening, the *xz*-plane or the *yz*-plane.

Mo: You have to specify a direction.

Interviewer: You were nodding, was your description of plane similar to [Mo]'s description of a direction? (*Willy nods*) What sort plane are you thinking about?

Willy: There are infinite amounts of planes (*gestures vertical planes in many directions*). So you have to specify which direction the tangent line is in.

As we see in the above excerpt, the students were able to consider rates of change in many directions; however, prior to the linear approximation tasks they did not demonstrate an ability to measure or calculate rates of change along directions other than the coordinate directions. When asked if there is a relationship between the rates of change in different directions, Willy responded, "No… I don't think there is any relation between one slope and another." Similarly, when given two partial derivatives and asked whether a directional derivative would be positive or negative, James made a wavy hand gesture and stated "It would depend on the way the temperature's changing on the curve."

The Linear Approximation Task

In order to explore linear approximation in multivariable calculus, the students were given a physical surface representing a two-variable function with one point on the surface identified with a blue dot. The students were given the following task:

A. The surface represents the density of gold (in grams per cubic mile) beneath the ground. You own a small mine located at the blue dot. Estimate the density of gold at your mine and measure how the density of gold changes in the north and east directions.

B. You want to buy one of three mines which are for sale; their locations (relative to yours) are given below. Estimate the density of gold at each mine using only your previous measurements.
Mine A: 1.2 Miles North
Mine B: 1.2 Miles North and 0.8 Miles East

Figure 1: Linear Approximation Task

The students were able to apply their prior knowledge about partial derivatives to quickly answer Part A of the task finding that the height of the surface at the blue dot was 3.5, the rate in the north direction was 0.28 and the rate in the east direction was -1.1. When beginning Part B the students were quickly able to incorporate the fact that changes in the density will be proportional to changes in the north direction in order to approximate the density at Mine A.

James: Oh yeah, right, so we're at 3.5 right. So they're 1.2 north. And the rate of change is -1.1 per inch.
Willy: No that's the east direction, north is 0.28
James: Alright, [a rate of] 0.28, that means that gets slightly taller.
Willy: Or is it 0.28 [the rate of change] times 1.2 [the change in distance]?
Mo: That is exactly what it is. This is all we need to do. If it says north, we times it with dz/dy if it says east we times it with dz/dx.

As the students attempted to approximate the density at Mine B, they needed to consider changes in both the north and east directions simultaneously.

Willy: But what about, like, [Mines] B and C where it moves both north and east?
Mo: You can add them?
James: But look – you have to multiply the rate of change by the direction and add that to our mine. You get what I'm saying? Because it's 1.2 inches north, so you have to multiply that by the rate of change, and you have to add that to the mine, to see the height at that mine. Because it's moving .25 grams per mile to the fourth in that direction. So we multiply that by 1.2 and then add that to the 3.5 to see where their height, quote unquote, would be.
Mo: I see what you're saying, yes...
Willy: Yeah, that makes sense.

In the above excerpt we see that the additive property of linearity for changes in the dependent variable came naturally for the students. The students offered varied justifications for the linearity of their solutions when asked specifically why they believed it was appropriate to add the two components together. In the quotation below Willy argues that adding these changes together is similar to adding together vectors in three dimensional space.

Willy: So, say this is the original point (*indicating the blue dot on the surface*). Then when we went east it decreased a bit, and when we went north it increased a bit. So we are basically adding them. So, it's basically vectors. Like, you have $3i + 4j$ and you are basically adding them up, something like that… It's like, think of that parallelogram thing we learned. We're getting the resultant vector from the north and east. So, we are adding them up, basically.

As seen in the above excerpt, Willy has made connections between this activity and his prior knowledge of vector addition, which incorporates the key properties of linearity, addition and scalar multiplication. It is not clear whether he is recognizing that the partial derivatives can be represented as vectors on a tangent plane or whether he is just acknowledging that the additive behavior and directionality seen in vectors is similar to the approximation calculation.

Mo subsequently embraced the vector-style reasoning:

Mo: They're not vectors but they behave like vectors. (*He draws a rectangle with vectors as the edges*.) If you want to get to this point you have to do this plus this. So I guess it's like, they act like vectors, but they're not really vectors.

James constructed a justification from a different point of view.

James: When you multiply them out ... you're left with the change in z … you get the same unit as this one. And the same thing goes for here so you can add them all up.

He confirmed that in this context the units became the same for each term. While not a complete justification for linearity, it demonstrates its plausibility, as the inverse scenario would rule it out.

After successfully approximating the density of gold at Mine B the students quickly applied the same principles to approximate the density at Mine C. Following this task they were able to work together to create a generalized formula for linear approximations at any point in the domain. Furthermore, upon returning for the second session of the interviews the students immediately applied the same additive approximation scheme when given a similar task.

Directional Derivatives

Immediately after the development of their linearity schema for linear approximations, the students were asked if, given the partial derivatives of a two-variable function, they could evaluate the derivative in another specific direction. Their initial response is in the excerpt below:

Interviewer: Let's make this vector more precise. Let's make it $1i$ [plus] $2j$. Could we figure out what the rate of change is in that direction?
Willy: 2 over 1.
Mo: It's actually the magnitude.
Willy: Its 2 over 1.
James: No it's not the magnitude. The magnitude is the length of the…
Willy: Remember Pythagoras theorem,
James: It's the length of the steepest point.

In the above excerpt we see the students attempting to connect this problem to several prior experiences in mathematics, but significantly they have not connected this problem with the just completed linear approximation activity, and have not invoked any part of their linear combination schema.

During the second interview session the students were once again tasked with finding the directional rate of change. The function had partial derivatives $f_x = 0.41$ and $f_y = -0.19$, and they were given the direction of $3.5i + 1.25j$.

James: What if we add both rates, shouldn't it give you that rate?
Mo: Actually… yes.
James: Yeah, it should because it's going to give you the same points…
Interviewer: So, tell me what you're going to add?
James: Wait let me see if it makes sense first? [cross talk] To get from point A to point B, you just add them.

Interviewer: So where did the .22 come from?
James: I just added the rates.

Here we see the students begin to explore incorporating addition into their problem solution; however, they are adding the rates and not the changes in the values. Thus we see that from their linearity schema they have utilized addition but not scalar multiplication.

A short while later they recognize that they need to multiply the rate by the change in distance, but they still have not connected this process to the linear approximation schema developed earlier. This observation is finally made in the following excerpt.

Mo: Ok, I get it. So this rate is not for this distance, it is for anywhere. This is how much it is changing for a unit distance. So we need to multiply by this distance.
James: Wait, yo, it's what we did originally. It is.
Mo: Yeah, I think the rate is this (*writes 3.5 * 0.41 + -0.19 * 1.25*). 3.5 times what was the rate, 0.41, times the distance, plus, again the rate, -.19, time the distance 1.25, and you're going to divide it by the square root of it, to get this distance (*writes square root of* $3.5^2 + 1.25^2$). That's it. (*does a victory fist pump*)

When asked why they needed to divide in the above expression, the students responded:

Mo: I had the rate in this direction (*indicates the x-direction*), but I had to multiply it by the distance. But, since I don't want the rate times distance in this direction (*indicates the direction of the directional derivative*) I had to divide by the distance.
James: Yeah, to get the unit vector of unit rate.

Immediately following this excerpt the students extended their result and wrote a generalized formula for the directional derivative as a linear combination.

Discussion & Conclusion

Over the course of the two interviews, we saw a progression of the students' schema for linearity and its connectedness to linear approximation and directional derivatives. Initially the students did not display evidence of a connected linearity schema, arguing that there should be no relationship between rates of change in different directions. However, engaging with the approximation task prompted the students to introduce linear combinations into their linear approximation schema. Their construction of a correct procedure and answer for linear approximation represented a restructuring of their schema by schema induction. They were able to generalize their result, indicating an act of tuning. During discussion the next day on linear approximation they each comfortably utilized linearity in the same fashion, indicating that their schema had been strengthened.

They offered varied justifications for implementing linearity. Two students had a justification for linearity which was context-free (vector addition) and one was more context dependent (adding like units). The ability to justify the use of linearity in appropriate contexts is a significant development since prior research shows that students often apply linearity and its properties to mathematical scenarios where it is inappropriate (De Bock et al., 2007).

In spite of this schema development, the students were not able to evoke linearity once the students changed tasks to determine the value of a directional rate of change. This indicates that, at that time, there was no connection between their schemas for directional derivative and for either linearity or linear approximation. Instead they went about re-creating the linearity schema within the context of directional derivatives. Finally, after the linearity property was re-created by the students, James exclaimed “Wait, yo, it’s what we did originally!” This appears to be the moment that James recognized that he could use his linearity schema from the linear approximation task and adapt it in order to reason about directional derivatives. The other group members made the same realization and quickly incorporated linearity into their problem solution for directional derivatives. Their resulting schemas thus had connections between linearity, linear approximation and directional derivative. Given their inability to calculate the last two previously (in multivariable calculus), this indicates a significant restructuring of these schemas.

It is significant that the expression of linearity within the context of linear approximation did not immediately lead to the use of the schema when finding directional derivatives. This is reasonable, as the former deals with total change, whereas the latter involves a rate, and an additional division must occur. Indeed, in the construction of the directional derivative expression, this division is the final step the students took. It is notable that it was, in fact, the recognized connection between linear approximations and directional rates of change that allowed the students to complete their formulation of the directional derivative. Many major textbooks (e.g. Stewart, 2012) do not make the connection between these topics explicit. This development points to several areas of possible future research. Is this connection between linear approximation and directional differentiation commonly observed among students? How does the instructional sequence of linear approximation followed by directional derivative compare to other alternative instructional sequences? How does the choice to contextualize the linear approximation task impact a student’s development of a linearity schema? Finally, do the observations reported here generalize to large student populations?

This study has contributed to the body of research in multivariable calculus by observing how three students invoked their linearity schemas at varying levels of robustness while investigating linear approximation and directional derivative, by analyzing the connections they made, and has suggested new lines of inquiry as well.

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The Institutionalized Paradox: Our Teachers Are Not Trained To Teach

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In a culture where STEM preparation is rapidly becoming of utmost importance to the nation's economy and educators are challenged to increase diversity and equity amongst students, quality mathematics instruction at the collegiate level is critical. Yet the majority of undergraduate mathematics teachers are not formally trained in pedagogy. This is a systemic issue, an institutionalized paradox, which originates in the mathematicians' training grounds - mathematics PhD programs. This paper provides background on this issue and focuses on a survey of university mathematicians concerning their formal academic training and their outlooks and prioritization of pedagogical training. Attention is drawn to the disconnect between university mathematicians' beliefs about the important role of pedagogical education in mathematics program and their resistance to promoting its implementation as a basic institutional requirement. A call for action is suggested to remedy these institutionalized systemic paradoxes.

Keywords: Pedagogical Training, Pedagogy for Mathematicians, Undergraduate Instruction

Introduction

It is a curious phenomenon that those whose role it is to prepare the next generation of mathematics learners at the college level most often lack basic training in the fundamentals of teaching. Although this pedagogical vacuum is present in other subject matters, the lack is most critical in mathematics, where future success is based on a mastery of progressively complex predecessor functions and disciplines. Mathematics teachers rarely learn how to teach; they learn how to be mathematicians. Granted that to be a good mathematics teacher strong mathematical knowledge is required – as subject matter expertise is a primary critical component of teaching. However, in most cases subject matter knowledge is not sufficient to reliably result in effective and excellent teaching.

The central thesis of this paper is that a discrepancy exists between the way mathematicians think about the importance of pedagogical training for teachers of mathematics and the priority they place on actually implementing pedagogy into mathematics PhD programs. This discrepancy contributes to a situation in which the overwhelming majority of mathematics PhDs will become college level mathematics teachers (i.e. adjuncts, instructors, lecturers, and professors), while mathematics PhD programs across the United States, and often around the world, have little or no pedagogy development to assure that these programs will produce good mathematics teachers. These programs are designed to prepare mathematics researchers. Our future teachers are rigorously trained to "do" mathematics and are not trained to "teach" mathematics. Research shows that this is having an impact on student retention and attrition in undergraduate mathematics programs.

The aim of this paper is threefold: (a) to review research which discusses this systemic problem, (b) to assess contributing factors to this systemic problem by addressing the perspective of university mathematicians on the importance of pedagogical training, and (c) to suggest fundamental changes in the way we approach integrating course requirements for mathematics PhD programs.

The paper is organized as follows: (a) Section 2 is a literature review and discussion of studies related to this paper's central thesis; (b) Section 3 presents a survey on pedagogy and an analysis of survey results, which were conducted at an international conference of mathematicians and at a research seminar at an American university in spring of 2018; (c) Section 4 discusses the ramifications of the literature review and the survey results; (d) Section 5 draws conclusions and suggests a call to action for fundamental change in the curricula of mathematics PhD programs and proposes a study to assess the value of that change.

Literature Review

There is a rich amount of research showing that strong mathematics knowledge is not necessarily an indicator of strong mathematics teaching skills (Bass, 1997; Kennedy, 1991). Universities, have competing goals when they hire faculty: research and teaching. These two goals often conflict. Many universities have a value hierarchy and regard research as more pivotal and will hire faculty primarily for their research abilities, regardless of their pedagogical training and skills (Brand, 2000). Although universities generally do have a process for assessing teaching capabilities of its subject matter experts, it is most often a limited process consisting of a brief model lesson and an observation lesson each semester for beginning teachers and student evaluations (NRC, 2003). Whereas this system may screen out teachers with "poor skills" it almost never results in formal pedagogical training or deep professional development. The system is missing the fundamental step of providing formal pedagogical training prior to graduates becoming teachers.

Mathematics instructors in college mathematics vary widely, from tenured full time professors and full time lecturers with many years of teaching experience, to adjuncts either with PhDs or enrolled in PhD programs, and varying teaching experience especially in the beginner mathematics college courses (Haycock, Majors, & Steen, 2004). Implementing Shulman's directive that to understand a profession one looks at its nurseries, to understand the profession of college mathematics teachers one should look at the mathematics PhD programs (Shulman, 2005). PhD programs most often do not focus on preparation for college teaching, despite teaching being a fundamental component of an academic life (Adams, 2002). Many of these professors and instructors have had no formal pedagogical training. As Bass states, "academic mathematical scientists, who typically spend at least half of their professional lives teaching, receive virtually no professional preparation or development as educators, apart from the role models of their mentors" (Bass, 1997). Moreover, since teaching is often not the primary focus for many university teaching faculty, this results in minimal time to focus on building teaching skills and tends to rely on "learning on the job" to gain classroom skills despite the availability of resources (Boyer, 1990; Fairweather, 2005).

The systemic issue is that regardless of the intention and perspective of the teacher, lacking pedagogical skills often negatively impacts the students in the classroom (Gibbs & Coffey, 2004). Furthermore, it leads to disinterested students and can discourage students from continuing their pursuit of a mathematics degree (Seymour & Hewitt, 1997). Is it okay to have an entire system dependent on idiosyncratic teacher performance? Acknowledging that there are many mathematicians who profoundly care about teaching and who have developed excellent teaching skills on the job (Oleson & Hora, 2014), should there be a systematic approach to developing excellent teachers? Research shows that educators with teacher training are more successful educators than teachers without professional teacher training (Darling-Hammond, 2000). Moreover, teacher practices and skill are not innate but something that is learned

(Darling-Hammond, 2012). Formal pedagogical training is ubiquitously accepted as fundamental and required in the K-12 level of schooling, yet this consensus is not an established norm at the college level despite the prevalent need and public concern (TAC & NRC, 2001).

The field of mathematics education, which was established to study the fundamental issues of pedagogy in mathematics, was founded over a century ago by the renowned mathematician, Felix Klein (Bass, 2005; Eves 1969). Naturally it would seem that a positive symbiotic relationship between mathematicians and mathematics education would ensue. Yet, there is an unfortunate disconnect between the fields (Dörfler, 2003). Under the umbrella of mathematics education there is a plethora of rich research, knowledge, tools and resources that focus on pedagogy for postsecondary mathematics instruction, e.g. Transforming Postsecondary Education in Mathematics (TPSE Math) (Holm & Saxe, 2016). There exist communities of scholars and programs consisting of mathematicians and mathematics educators that focus on pedagogical related issues for undergraduate teaching in mathematics, e.g. programs such as SIGMAA on RUME, the Mathematical Association of America (MAA) Project NExT, the Preparing Future Faculty (PFF), and the International Commission on Mathematical Instruction (ICMI), to name just a few. Furthermore, there exist teams of scholars addressing mathematicians' knowledge of teaching (Loewenberg Ball, Thames, & Phelps, 2008) and active studies (e.g. see Miller, 2017) finding the best teaching methods at the collegiate level. Nevertheless, the majority of mathematicians are generally unaware of these resources (Nardi et al, 2005). Many PhD programs that are training future mathematics educators fail to acknowledge and integrate this fundamental body of knowledge. This failure can have tremendous impact on the quality of teaching and hence negatively impact the quality of mathematics learning at the collegiate level.

The principal problem is that this body of research does not enter the curriculum of mathematics PhD programs. It is just not part of the system. There are some PhD mathematics programs that have begun to require and offer pedagogy training in the form of mentoring, but even a rigorous mentoring program is not sufficient for ensuring student learning. In addition, most mentoring training programs fail to offer basic courses such as a methods class, or a multicultural mathematics education course which would better equip teachers in increasingly diverse populations of undergraduate classrooms. Many PhD mathematicians are not pedagogically trained at all, as is highlighted in the survey below.

Survey on Pedagogy for Mathematicians

The Survey

During his 90th birthday celebration mathematician Dr. Henry Pollack humorously told the crowd that when he teaches his mathematics education students mathematical modeling he tells them "I'll teach you math, and you'll teach me how to teach." This sentiment resonates with many mathematicians. To highlight this perspective, which resonated deeply with me while training to be a mathematician, I decided to conduct a survey of fellow mathematicians to ascertain what they thought about the importance of pedagogy for mathematicians. To date, I have conducted the survey with two groups: (a) at a recent international mathematics research conference consisting of a group of mathematicians actively engaged in advanced mathematics - faculty, postdocs, and graduate students in PhD mathematics programs who often have teaching requirements at the undergraduate level; and (b) at an American university mathematics seminar in advanced mathematics. For both groups, the survey was intended to elicit participants' thoughts on their training and their views on the value of pedagogical training for mathematicians.

In total, 64 participants completed the survey. The majority of the survey participants have teaching obligations at the undergraduate level in mathematics. The responders consisted of 32 faculty, 13 postdocs, 16 students, and 3 unidentified. The international conference had a total of 77 participants from 50 different universities worldwide. In total 57 responded. The survey at an American university was given after a research seminar talk to a small group of 7. The questions on the survey were chosen to be direct, short, and easy to answer in order to attract a high volume response rate. There were four questions:

1. How many pedagogical courses have you taken during the course of your mathematics education? (a) none, (b) 1 or 2, (c) 3 or more.
2. How important is it to have pedagogical training for mathematics PhD programs? (a) Not important, (b) Somewhat important, (c) Very important.
3. Should mathematics graduate programs offer courses in pedagogy? (a) No, (b) Yes, (c) Unsure.
4. If you answered yes to #3, should the courses be required? (a) No, (b) Yes.

Following these questions, the survey included a section for comments, and an option to describe the individual's position as faculty, postdoc, or student.

Survey Results

The questions and corresponding responses are indicated in Table 1. The numbers indicate the number of responses for each option per question; adjacent are the corresponding percentages with respect to the total number of responders for that particular question indicated as well.

When administering the survey at the international conference I requested that responders write the name of the country in which they took their pedagogical training (if they had any pedagogical training). The participants in the conference were from a diverse collection of countries. 21 of the 30 responders who had pedagogical training identified the country in which they took pedagogical courses, (see Table 2). It will be interesting for further research to determine whether there is any significant variance amongst geographical locations concerning the perceived importance of pedagogical training.

Table 1. Survey questions along with participants' responses.

Survey on Pedagogy for Mathematicians Results						
1. **How many pedagogical courses have you taken during the course of your mathematics education?**	**None**		**1-2**		**3+**	
	34	53%	25	39%	5	7.8%
2. **How important is it to have pedagogical training for mathematics PhD programs?**	**Not important**		**Somewhat Important**		**Very Important**	
	9	14.3%	33	52.4%	21	33.3%
3. **Should mathematics graduate programs offer courses in pedagogy? With three options to respond:**	**No**		**Yes**		**Unsure**	
	8	12.7%	37	58.7%	18	28.6%
4. **If you answered yes to #3, should the courses be required?**	**No**		**Yes**			
	16	44.4%	20	55.5%	--	--

Table 2. Countries where participants received teacher training.

Country where pedagogical courses where given			
Country	# of responders	1 or 2 courses	3 or more courses
Hungary	1	--	1
Israel	1	1	--
Japan	1	--	1
Korea	1	1	--
US	17	16	1

The results led to insightful findings highlighting the systemic issue, as follows:

1. The majority (53%) of respondents did not have any pedagogical training.
2. The overwhelming majority (86%) replied that pedagogical training for mathematics PhD programs is "somewhat" to "very important".
3. Less than 15% answered that it is "not important".
4. The majority (59%) did agree that graduate programs should offer courses in pedagogy.
5. Less than 13% answered that mathematics graduate programs should not offer pedagogical courses, and approximately 29% were unsure.
6. Of those who responded yes to question 3 (Should mathematics graduate programs offer courses in pedagogy?), approximately 55% replied that it should be required.
7. Very few (less than 8%) have taken 3 or more pedagogical courses.

Many were eager to complete the survey and expressed concern about the lack of focus on pedagogical training and attitudes about pedagogical training in mathematician circles. Of note, the majority of mathematicians in this survey were not trained formally in teaching, yet an overwhelming majority believe pedagogical training for mathematics PhD programs is important. 55% of those who responded yes to question 3 (59% of total participants) about whether mathematics PhD programs should offer pedagogical training courses said it should be required. This means that in total, only 31% of all responders believe that pedagogy training courses should be required. This underscores a common perspective amongst mathematicians, namely that many mathematicians do not think that formal pedagogy training is essential yet they still acknowledge that it is important. Most outstanding is that only 8% of all the responders had formal training of 3 courses or more.

After conducting the survey many responders reached out to discuss the issue of teacher training in mathematics PhD programs. Additional anecdotal insights provided by the survey participants conveyed that there was a sense that pedagogical training is not an issue that mathematicians think deeply about but is something that is vital. Few expressed concern about how ill-prepared they felt to fulfill their teaching obligations. Some were proud to praise the programs their universities had to guide their students in teaching. One faculty responder noted that his program for requiring students to take a 2-semester course on lecturing created the outcome that "our students result in the best presenters, regardless of the nature of their content".

The survey responses conveyed similar results with slight variation when observing data based on cross tabulation of the following subgroups (a) faculty, (b) postdocs, (c) students, (d) research seminar, and (e) international conference.

This survey highlights several key critical findings: (a) the lack of pedagogical training for mathematicians, (b) the overall belief that training is important, and (c) there exists a disconnect and reluctance of mathematicians regarding the fundamental importance of formal pedagogy training in PhD programs as a required part of the curriculum. The fact that only 31% think pedagogical courses should be required highlights the disconnect and lack of awareness of the vast body of knowledge in undergraduate mathematics education supporting the vital role of pedagogical training in the development of mathematics educators.

Discussion on Ramifications of Literature Review and Survey Results

There is substantial support, both in the literature and amongst the sample surveyed, that pedagogical training for college mathematics is important to produce good teachers, and more importantly, to produce good mathematics learners (students). This viewpoint runs counter to the notion that students' innate affinity for mathematics is the major determinant of successful mathematics learning and that students' lack of innate affinity for mathematics is the major determinant of failure to learn well (Rattan, Good, & Dweck, 2012).

There are two inherent paradoxes that emerge from the survey and literature review. (a) The majority of mathematics educators on the collegiate level are not trained in pedagogy; simply speaking, our teachers are not trained to teach. (b) The majority of mathematicians in the survey think pedagogy training is important, yet only a minority believes it should be required.

The failure to have pedagogically trained teachers contributes to poor outcomes of collegiate teaching in the STEM fields and blocks the emergence of mathematical talent across many demographics. This is of utmost concern given the global economic paradigm shift from agricultural-and industrial-based jobs to STEM-based careers, creating a need to prepare generations of students who are STEM-career ready. Yet we are not producing an adequate number of STEM degreed graduates to meet our national need (Hall et al, 2011). Moreover, those who are graduating are predominantly non-diverse—this is in part due to lack of interest in the field projected by ineffective teachers (Nardi, 2007). A significant cause of attrition is not students' ability but rather poor pedagogical practices by faculty (Seymour & Hewitt, 1997).

Understanding the needs of the student body and being equipped to teach collegiate students in mathematics are crucial for student success. The undergraduate curriculum for STEM-oriented majors requires proficiency in the "gateway" courses of calculus and linear algebra. Moreover, many students coming into college are missing basic mathematics skills and are placed in remedial mathematics courses such as college algebra or pre-college algebra (Bryk, & Treisman, 2010). These students require well-trained educators to succeed. Those students who don't pass entry-level courses either are blocked from furthering any STEM-based education or they drop out because their failure has caused them to believe that they cannot succeed (Bellafante, 2014).

To address this gap, critical care must be given to train the teachers who will be responsible to teach all students including (a) students that are insufficiently prepared and (b) an increasingly diverse student body. It is telling to note that students who are taking "beginner courses" are often taught by adjuncts, who are mostly PhD graduate students who are learning and researching upper level mathematics and have little to no pedagogical training (Harris et al, 2009). (As noted, if the teacher is a full-time professor, often they also don't have pedagogical training.) This systemic failure is having a damaging effect on our students and can be rectified.

Conclusions, Limitations, Future Directions, and A Call to Action

Conclusions

The results of this survey highlight that although most mathematicians have limited training in formal pedagogy, the majority believes that pedagogical training for mathematicians is important. Paradoxically, whilst they believe it is important, only a minority of mathematicians endorse that pedagogy is vital enough to be a basic requirement. The disconnect between the well-researched importance of education in mathematics instruction and the level of education training among mathematicians is represented in this survey and speaks to a surprising gap in current mathematics educational practice.

Lack of pedagogical instruction for university teachers is a systemic and detrimental problem. Absent any pedagogical requirements, and coupled with the dominant viewpoint that teaching is the second fiddle to the virtuoso performance of research, university teaching cannot be expected to be efficient – let alone excellent. The literature on the subject of teacher training overwhelmingly demonstrates that trained teachers produce better mathematics learners than untrained teachers, however brilliant these teachers may be in their field. Yet university systems impose no formal pedagogical requirements on their teachers.

Limitations and Future Directions

The survey was conducted on a small convenience sample of mathematicians in a particular area of pure mathematics. Thus, a larger sample size is needed to more broadly assess the attitudes and beliefs of mathematicians across a broader range of university settings and across a more random sample of mathematics specialties.

This study's survey provides preliminary evidence of a gap in mathematicians' knowledge about the importance of formal pedagogical training and readiness to promote educational change on an institutional level. Further research is needed to explore the nature of this discrepancy, its driving/contributing factors, and the impact of raising mathematicians' awareness and increasing knowledge for teaching. A pilot study in which beliefs and attitudes of a randomized group of mathematicians are assessed before and after exposure to pivotal articles in mathematics education research would shed light on whether a lack of crosstalk between the fields of pure mathematics and mathematics education is an important contributing factor to the gap highlighted in this paper. The study would also assess whether individual attitudes and beliefs about the importance of formal pedagogy in mathematics generally, and as a basic requirement in teaching undergraduate mathematics courses specifically, are changed pre- and post- exposure to the selected articles.

Call to Action

To solve this systemic problem requires a systemic solution. Both institutional requirements and public policy require change. The following possible solutions are suggested. Institutionally, mathematics PhD programs should offer and require pedagogical training for all their students, and universities should require all members of their teaching staff to be trained in pedagogy. In order to implement such change, public policy must be affected at a national level. Notwithstanding that change can happen systematically, it may require a grass roots effort (one school at a time) to adopt this policy. Further research is needed to better understand attitudes and beliefs among influential mathematicians toward and in resistance to imposing these educational requirements are crucial to guide future effective and targeted public policy change.

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A Mathematician's Instructional Change Endeavors: Pursuing Students' Mathematical Thinking

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To reform instruction by moving towards student-centered approaches, research has shown that faculty benefit from support and collaboration (Henderson, Beach, & Finkelstein, 2011; Speer & Wagner, 2009). In this study, we examined the ways in which a mathematician's instruction unfolded during his participation in a faculty collaboration geared towards reforming instruction and aligning it with inquiry oriented instruction (IOI) (Kuster, Johnson, Andrews-Larson, & Keene, 2017). Results indicate the participant's mathematics background and research interests influenced how he used student thinking in his instruction. More specifically, when mathematics content specifically aligned with the participant's research interest he often guided students to view differential equations as he did; whereas, when the content was not aligned with his research interest, he was more open to the using his students' thinking to drive the class forward. Implications and future research directions are discussed.

Keywords: Instructional Change, Faculty Collaboration, Student Mathematical Thinking

Over the last decade there have been numerous calls for reform in undergraduate mathematics education (e.g., President's Council of Advisors on Science and Technology [PCAST], 2012). These calls for reform draw on research that has shown the benefits of student-centered instruction (e.g., Freeman et al., 2014). To address these calls, change is needed in the instruction of undergraduate mathematics. For example, *A Common Vision* gave a general call that instruction should move away from traditional lecture as the sole instructional method in undergraduate mathematics (Mathematics Association of America [MAA], 2015).

Given these calls for instructional reform, faculty want to make changes to their instruction. However, research has shown that even when working with research-supported curricular materials, mathematics faculty are often unprepared to undertake the challenge of changing their instruction (Henderson et al., 2011; Wagner, Speer, & Rosa, 2007). Current endeavors are providing mathematics faculty with support needed to change their instruction.

There are also calls for departments and faculty members to collaborate specifically on the pedagogy (MAA, 2011). One research-based method of support is faculty collaborations geared towards collectively improving instruction (e.g., Nadelson, Shadle, & Hettinger, 2013). In particular, researchers are studying how mathematics faculty come to use research-based instructional strategies in their classrooms in the context of faculty collaboration. This study explored the experiences of a mathematician who participated in one such faculty collaboration that addresses the numerous calls for reform in undergraduate mathematics education and instruction. The study addressed the following overarching research question: *1) In what ways does one mathematician's experiences in an online faculty collaboration on inquiry oriented differential equations relate to his instructional practice?* And the following sub research questions: *a) How does his instructional practice unfold over his first implementation of inquiry oriented differential equations and in what ways does it align with inquiry oriented instruction? b) How does his participation unfold in the online faculty collaboration?*

Theoretical Backing and Literature Review

Our study and the instructional strategies we sought to disseminate to the mathematics community are rooted in Freudenthal's (1991) theory that mathematics is a human activity. This is manifested in the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999) on which inquiry oriented mathematics is based. In this section, we briefly describe this instruction and relevant research on instructional change.

Inquiry Oriented Mathematics

The faculty collaboration focused on inquiry oriented mathematics and instruction. Rasmussen and Kwon (2007) defined inquiry oriented (IO) environments as teaching where students are inquiring into the mathematics, while the teachers are inquiring into the students' mathematical thinking. In this study, we specifically focused on inquiry oriented differential equations (IODE) which has been shown effective for student understanding of differential equations (Kwon, Rasmussen, & Allen, 2005).

Inquiry oriented instruction. In inquiry oriented mathematics, it is clear that the role the teacher plays is important for advancing the mathematical agenda. Kuster et al. (2017) recently defined four focal components of inquiry oriented instruction (IOI): generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. The focal components of instruction are guiding principles of IOI. It is important to note that the four focal components very rarely occur independently; oftentimes, these components overlap and occur in the complexities of an IO classroom. Further, there are local practices of IOI. The local practices of IOI (see Table 1) are an elaboration on the four focal components of IOI. While the focal components are guiding principles of IOI, the local practices are specific actions that instructors do in an IO classroom.

Table 1. Inquiry oriented instructional local practices (Kuster et al., 2017).

Local Practice	Description
1	Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point.
2	Teachers elicit student reasoning and contributions.
3	Teachers actively inquire into student thinking.
4	Teachers are responsive to student contributions, using student contributions to inform the lesson.
5	Students are engaged in one another's thinking or reasoning.
6	Teachers guide and manage the development of the mathematical agenda.
7	Teachers introduce language and notation when appropriate and support formalizing of student ideas/contributions.

Overview of Faculty Instructional Change

Here we first describe barriers to instructional change and then what the research community knows about facilitating and sustaining instructional change.

Barriers to instructional change. One barrier to instructional change is faculty's knowledge for teaching with student-centered instructional strategies. Research has shown that some faculty lack the necessary skills to enact student-center instruction (Hayward, Kogan, & Laursen, 2015), sometimes because they lack specialized content knowledge relating to instruction and being prepared to respond to student questions productively (Wagner et al., 2007). Further, faculty have stated that student resistance, lack of student buy-in, and student attitudes of school are reasons why they do not use student-centered instruction (DeLong & Winter, 1998). The most often cited environmental reason by faculty to not use student-centered instruction is how much more time it takes than teacher-centered instruction (Henderson & Dancy, 2017). Likewise, faculty say they stray away from student-centered instruction because they have a certain amount of material that needs to be covered over the course of one semester (Hayward et al., 2015).

Facilitating and sustaining instructional change. Henderson et al. (2011) outlined four categories of instructional change strategies that are elaborated on in this section: disseminating curricula and pedagogy, developing reflective faculty, enacting policy, and developing a shared vision. Borrego and Henderson (2014) elaborated on these four categories of change by defining eight change strategies that fit within the framework. Our study considered two of these change categories: scholarly teaching and faculty learning communities. Scholarly teaching is when "individual faculty reflect critically on their teaching in an effort to improve" and faculty learning communities are when a group of faculty come together and "support each other in improving teaching" (Borrego & Henderson, 2014, p. 227). These two strategies can work together to improve undergraduate mathematics instruction.

Methods

This study focused on one participant from an IODE online faculty collaboration (OFC). This qualitative instrumental case study (Stake, 1995) was bounded by the participant's participation in the OFC and his classroom teaching. This work comes from the BLINDED project, which supported university mathematics faculty in shifting their practice towards an IO practice. BLINDED offered three supports: the IO materials (in this case IODE), a summer workshop, and the weekly OFC. Here we first highlight pertinent details on the OFC.

Online Faculty Collaborations

The IODE OFC met weekly during the semester they are teaching IODE, virtually via Google Hangouts to conduct lesson studies that were modified Japanese lesson studies (Demir, Czerniak, & Hart, 2013) led by a facilitator. The main goals of the OFC were to: 1) aid teachers in making sense of the instructional IODE materials, 2) thinking through the sequences of tasks, how students might approach the tasks, how to structure instruction around the tasks to support student learning, 3) assist teachers in developing and enhancing their instructional practice, and 4) develop a safe and supportive community.

Participant

The focus of this study is one participant from the IODE OFC, Dr. DM. The OFC consisted of the facilitator (Dr. GG), two graduate research assistants (GRA1 and GRA2), and five faculty teaching the materials for the first time (Drs. DM, AB, PR, CD, ST). The sampling of Dr. DM was purposeful in nature (Yin, 2013) and there were several reasons for that choice. First, he was and is passionate about his participation in BLINDED and to this day continues with IOI in his IODE classroom. Second, he became a facilitator for the project in future semesters following his participant experience. Furthermore, Dr. DM filmed every class of the semester, which was more than was expected of the other BLINDED participants, affording a plethora of possible data sources and a semester-long look at instruction.

Data Collection and Analysis

Data were collected from Dr. DM's classroom instruction, the OFC he participated in, and two interviews during his participation in the project.

Classroom data. Video data from Dr. DM's classroom were collected. Classroom video data were chosen to match the units covered in the OFC lesson studies (i.e., Unit 6 and Unit 9). In addition to those units, Unit 1-2 as an introductory unit and Unit 12 were analyzed. All units lasted a different amount of time. The IOI framework discussed above (Kuster et al., 2017) was designed to capture IOI in action. Consequently, we used the framework as an a priori analytical framework for coding Dr. DM's classroom instructional practice to answer research question 1a. In particular, we used the local practices (LP) of IOI. The IOI framework also contained "evidences," not shown above, of each LP; these evidences served as codes that were collapsed to each LP. LP1 was not coded for unique observable instances in the data. After the first round of coding, we went back again and revisited analysis logs and made adjustments to the coding as necessary. In this step, we looked for emergent themes from the data.

OFC data. Each OFC was screencast using software. All weeks of the OFC were analyzed except week 6 because the data was corrupted and week 8 because Dr. DM was unable to attend that week (in total 9 OFCs were analyzed). Weeks 1 and 2 were introductory weeks. Lesson study 1 took place over weeks 3-5 and lesson study 2 took place over weeks 6-10. Lastly, a debrief OFC occurred during week 11. All videos were transcribed. To analyze Dr. DM's participation in the OFC we coded the transcripts with a priori codes and frameworks: the role of the speaker (production design from Krummheuer, 2007), the role of the listener (reception design from Krummheuer, 2011), and conversation categories (Keene, Fortune, & Hall, under review). These frameworks were adapted to fit the context of this study and are discussed in the results. In a broad sense, we considered Dr. DM's active versus passive participation.

Interview data. The interview data served as a third data source to relate Dr. DM's experiences in the faculty collaboration to his instructional practice. Furthermore, this data offered Dr. DM's personal perspective on being part of a faculty collaboration. Entrance and exit semi-structured interviews were conducted. All interviews were audio recorded and transcribed. Transcripts of both interviews were open coded (Yin, 2013).

Results

Instructional Practice

Central to IOI is the facilitation of mathematics where students are actively inquiring into the mathematics while the teacher is actively inquiring into the students' mathematical thinking (Rasmussen & Kwon, 2007). Dr. DM's instruction focused predominantly on LP2, eliciting student ways of reasoning and contributions (see Table 2). Dr. DM less often actively inquired into why his students were making such contributions (LP3), used those contributions to push the agenda forward (LP4), and had students engage in one another's thinking (LP5; although this happened frequently in Unit 1-2). Note that frequencies were scaled and rounded to represent the same amount of class time as each unit lasted a different number of days.

Table 2. Frequencies of Dr. DM's Local Practices of IOI.

Practice	Unit 1-2	Unit 6	Unit 9	Unit 12
2	58	52	66	26
3	17	24	16	4
4	17	16	15	8
5	42	26	14	2
6	14	16	6	4
7	3	14	8	2

Table 2 is very telling of Dr. DM's instruction. He was very interested in generating student contributions. While some of the questions asked were ones from the IODE tasks themselves, he often would ask his own questions in his own way as a means to address something that he wanted to focus on or have his students think about. While students had opportunities to engage in others' contributions as they were written on the board, they less often had opportunities to engage in others' thinking, as Dr. DM did not follow up with questions to have students elaborate on their thinking as often. Essentially, after students made contributions, Dr. DM would more often move on. We cannot know for sure if Dr. DM was so in tune with the students in his class and the mathematics itself, that he did actually know why his students were thinking along certain lines. However, LP3 and LP4 are about making explicit to the rest of the class such thinking and thus Dr. DM's LP frequencies were reflective of the fact that he did not often make public his inquiring into student thinking.

Comparison of instructional units. Dr. DM's instruction did not necessarily change from the beginning of the semester to the end of the semester. As discussed across the totality of Dr. DM's instruction his most frequent LP was LP2, eliciting student ways of reasoning and contributions. However, when comparing the four units of analyzed instruction there were contrasts between the units. Namely, the way Dr. DM's instruction unfolded was tied to 1) how and when he used student thinking in his class and 2) his mathematical beliefs, rooted in his mathematical research arena.

First, Units 1-2 and 6 were when Dr. DM frequently (more often than any other unit when comparing across scaled time) engaged students in one another's thinking. In particular, these units were the units where his students' thinking was most at the forefront of the class and he oftentimes used that thinking to advance the mathematical agenda. When student thinking was made prevalent to the rest of the class, Dr. DM's instructional reflected that. For example, when introducing phase lines one student made a claim that the solution will never reach 8 (i.e., an equilibrium solution) and the following 8 minutes focused on that one claim. During that time students were responding directly to each other [LP5] or prompted to do so by Dr. DM [LP5]. Dr. DM asked clarifying questions [LP3] such as "and that assessment was based on what?"

Second, when the mathematics of the unit was associated with Dr. DM's mathematical research interests he would focus on getting students to get to "the way [he] view[s] the mathematics" rather than having his students' work or ideas at the center of the development of the mathematical agenda. Unit 9 dealt with the development of the phase plane which was a crucial tool in Dr. DM's research. The instructional portrait of that unit had the highest amount of eliciting student ways of reasoning and contributions [LP2] and in comparison, a very low frequency of LP3-5 (the other practices associated with student thinking). Many of the questions that Dr. DM asked were of his own accord and not generated from the whole class discussion. Because he knew the mathematics so intimately, he was most interested in getting students to see the mathematics the way he does, rather than letting the mathematics emerge from the students.

Dr. DM specifically discussed in his exit interview how he would want students to view mathematics as he does, in particular, the subset of differential equations closely related to his research field: phase planes.

> *Dr. DM [interview]*: And so, um I see DEs, like that's my goal is for students to be able to start to see that. And for that reason, I have to push that kind of phase plane agenda to start to be able to talk about that. ... By viewing myself as the curator of their discussion and just picking apart things and building towards my mathematical agenda allowed me to inject a lot of my personality back into the course and talk about things that I'm really passionate about. … And that agenda is largely because of the way I see DEs used in my research. Uh, I want students to have a taste of that.

Similarly, in class Dr. DM would point out his bias of use of the phase plane.

Dr. DM [class]: This is my home; phase planes are where I live. ... All of my research is based in the phase plane, in phase space. ... That is a sufficiently strong hint that says I will allow my bias to show and I will promise you many questions on the phase plane on the next celebration of knowledge [Dr. DM's tests]. I can't help it. I find it exciting.

Participation in OFC

Recall the goal of the OFC was to support cohorts of mathematicians as they came to learn about IOI and IODE. Table 3 highlights the participation frequencies based on role and conversation. For the purposes of space, we only discuss active and passive participation here rather than all the more specific roles adapted from Krummheuer (2007, 2011). Additionally, we adapted frameworks from our previous work (Keene et al., under review) but here only include four broad conversation categories rather than each individual conversation topic.

Rather than growth throughout the semester, Dr. DM immediately jumped into the active role in the OFC and that active role was consistent throughout the semester. Similar to his classroom instruction there was not a change but rather how his role looked depended on the content of each OFC. For example, if the week focused on doing mathematics, he rarely authored topics because he simply was partaking in the conversation, however, he was very active in those weeks as he has a real passion for mathematics. Additionally, when the OFC focused on sharing of his videos, he authored frequently those weeks and the conversation focused on pedagogy as he sought advice on, for example, how to speed up his class because he was running out of time at the end. Table 3 highlights Dr. DM's most active role related to pedagogical issues.

Table 3. Frequencies of Speaker / Listener Codes by Participation / Conversation Category.

Conversation Category	Speaker		Listener	
	Active	Passive	Active	Passive
Pedagogical Issues	137	16	82	55
Mathematical Issues	70	6	72	40
Student Issues	63	2	20	23
OFC Issues	97	24	91	156

Discussion and Conclusion

In this section, we discuss how Dr. DM's instruction related to his participation in the OFC. In our analysis we observed numerous relationships, but in this report, we specifically focus on how his mathematics background impacted his teaching and his participation in the OFC.

Dr. DM's mathematics background played a role in how his instruction panned out throughout the semester and how he participated in the OFC. In both cases his mathematical content knowledge (rooted in his background and research interests) was placed on top of his interest in enhancing his pedagogical practice. By that we mean, in his teaching, his view of mathematics sometimes was the view of mathematics that he was guiding his students towards. Likewise, in his participation in the OFC, his mathematical understanding was one of the driving factors for his interest in enhancing his pedagogical practice. Namely, he had a deep geometric understanding of differential equations and sought support on how he can get his students to that same level of awe and understanding. Dr. DM desired to reform his instruction but struggled to put aside his prescribed view of mathematics in lieu of his students' mathematics.

This conclusion supports previous work from Speer, Wagner, and colleagues (Speer & Wagner, 2009; Wagner et al., 2007). In their work, they considered the concept of analytic scaffolding necessary for mathematicians to facilitate whole class discussions in inquiry-driven classrooms. They considered analytic scaffolding to be how one supports the mathematical goals of discussion. They remarked, "Gage's [their participant] analytic scaffolding ... was met with only limited success, despite his strong understanding of the mathematical content, clear vision of the learning goals for the lesson, and commendable ability to elicit contributions from students" (Speer & Wagner, 2009, pp. 558–559). In this quote, numerous parallels can be made between Gage and Dr. DM. Firstly, both had strong understanding of the mathematical content. Second, both had a clear vision of the learning goals. Third, both were very able to elicit contributions from students. Recall that Dr. DM's most used IOI LP was LP2, eliciting student ways of reasoning and contributions.

However, there are important distinctions that shed light on this topic and provides discussion for faculty collaborations going forward. Most importantly, it brings to the forefront of discussion the subtle notion of a mathematician's mathematical content knowledge. In their work, Speer and Wagner noted that their participant had a strong understanding of the mathematical content but that did not help in terms of his analytic scaffolding (i.e., meaning facilitation of discussion). Similarly, Dr. DM also had a strong understanding of the mathematical content across all units. However, the difference lies in the fact that in some units he was able to provide analytic scaffolding, namely, he was able to use his students' ideas in the class (LP3: actively inquiring into student thinking, LP4: being responsive to student contributions, LP5: engaging students' in one another's thinking, LP6: guide the mathematical agenda). Yet, he was more likely to do that when the mathematical content wasn't his specific research interest. Consequently, we concur with Speer and Wagner and posit that one's mathematics background is not sufficient to successfully use student thinking in one's class. Additionally, however, the level to which one understands that content makes a difference in their instruction.

In the case of Dr. DM, his focus, for some of the content from the course, was to get his students to his view of the mathematics. This ultimately leads to a tension between his teaching agenda and inquiry. If in inquiry, student thoughts are central to the development of the mathematical agenda (Kuster et al., 2017), then imposing one's own view of mathematics does not align with an inquiry perspective. The reason this causes a tension is because being passionate about your research inherently is not a bad thing, nor trying to get your students to see the beauty of mathematics. However, in so doing, one privileges their understanding over that of their students. We know from extant literature that mathematicians often struggle to implement novel teaching (if it is new to them) and in particular struggle with how to respond to and deal with student contributions in a productive and successful way (Wagner et al., 2007). However, this was not an issue for Dr. DM as he was in an OFC supporting his instruction. He never noted that he was unsure what his students were going to do. Yet, he seldom actively inquired into his students thinking. This indicates he either knew what his students were thinking or simply did not probe into their thinking; we cannot know which one.

This area of research is ripe for future investigation. The instruction of undergraduate mathematics courses is a hot button item in undergraduate mathematics education research today. More importantly, the research community still needs to know more about how we can support endeavors to reform instruction, how can they be scaled up, and how do we measure success? In this qualitative instrumental case study, while not generalizable, we can conclude that the OFC supported Dr. DM's desire to reform his instruction. This work has highlighted how those faculty collaborations can be improved moving forward and most importantly highlights that instructional change is possible if the time and effort are put into it.

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A Possible Framework for Students' Proving in Introductory Topology

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Advanced mathematics courses require that students possess sophisticated proving techniques. Topology is one such course in which students' proving behaviors have not been extensively studied. In this paper, we propose that visual methods play an important role in undergraduates' discovery of the key idea of a proof, and we describe a potential framework for students' proving processes in a first course in undergraduate topology based on Carlson and Bloom's (2005) problem solving framework.

Keywords: topology, proof, visualization, representation, key idea

Background

Proof is of great importance in mathematics, but it is known to be a difficult concept for students (Dawkins, 2016; Harel & Sowder, 1998). Harel and Sowder (1998) define *proving* to be "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). Indeed, proving is a composite of two processes: "Ascertaining is the process an individual employs to remove her or his own doubts about the truth of an observation. Persuading is the process an individual employs to remove others' doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241).

When proving, a mathematician's primary goal is often the discovery of the *key idea* of the proof: "A key idea is an heuristic idea which one can map to a formal proof with appropriate sense of rigor. It links together the public and private domains, and in doing so gives a sense of *understanding* and *conviction*. Key ideas show *why* a particular claim is true" (Raman, 2003, p. 323). A heuristic idea is an informal idea, often represented by a picture, which gives the *individual* an understanding of why a conjecture is true, but which may not lead to a rigorous proof. Determination of the heuristic idea may be the primary goal of visualization: "The drawing of a diagram was not a goal in itself but a means to aid them in gaining more information for the problem situation. Mathematicians *anticipated* that a figure would provide them with specific information – the drawing of a diagram was not simply a vague step forward in the solution of the problem" (Stylianou, 2002, p. 310). However, the key idea is necessary for the construction of a formal proof, as the prover must convince not only herself, but she must provide an argument which will convince others as well.

A diagram constructed in search of a heuristic idea may be thought of as a type of example. Watson and Mason (2005) define an *example* as "anything from which a learner might generalize" (p. 3). Students often use specific and generic examples to help make sense of a definition or theorem. Such examples make up part of the student's *example space* for the given topic (Mason & Pimm, 1984; Watson & Mason, 2005). This example space serves as a starting point when encountering definitions to be used in other contexts. Examples, along with definitions, theorems, actions, and images associated with an idea, constitute the individual's *concept image* (Tall & Vinner, 1981).

Moore (1994) observed that students often use definitions to generate examples. These examples then help to develop their concept image, which informs the students' understanding of the original definition. The chosen examples transition from a *model of* the definition to a *model for* the more sophisticated knowledge necessary for proof construction (Cobb, Yackel, &

McClain, 2000). Moore identified the scheme "Images → Definitions → Usage" to describe a successful trajectory used by students in his study. The scheme "Images → Usage" often failed students in his study. When examples were used to guide students toward a deeper understanding of a definition, the *definition* became more useful during proof construction.

Building on existing frameworks for individuals' problem-solving processes, this study proposes a framework for students' proving processes. Through observations of the proving behaviors of Stacey, an undergraduate taking a first course in topology, we propose a framework for students' reasoning in proving. Our results show that students' approaches to proving and problem solving are similar to those of experts but exhibit some key differences.

Theoretical Perspective

We examined our data using the Multidimensional Problem-Solving Framework (MPS Framework; Carlson & Bloom, 2005). Research into problem solving has shown that mathematicians use visual and analytic methods in a cyclic process to help them solve problems (Carlson & Bloom, 2005; Stylianou, 2002; Zazkis, Dubinsky, & Dautermann, 1996). A precursor to the MPS Framework, the Visualization/Analysis Model (VA model) describes a process of alternation between visual and analytical strategies employed when solving problems (Stylianou, 2002; Zazkis, Dubinsky, & Dautermann, 1996). The MPS Framework elaborates on this idea, proposing a cycle of four phases through which expert mathematicians proceed when solving problems: Orienting, Planning, Executing, and Checking. The VA Model is encapsulated in the Orienting and Planning phases, during which the mathematician familiarizes herself with the problem, often by drawing a picture or creating a manipulative, and comes up with a strategy to solve the problem. A sub-cycle of *conjecture-imagine-evaluate* takes place during the Planning phase. The strategy is applied in the Executing Phase, and in the Checking phase, the mathematician looks back at her work and determines if she was successful in solving the problem or if she needs to try another approach.

Visualization plays a pivotal role in the Orienting and Planning phases. The construction of an appropriate diagram not only helps the problem solver to make sense of the problem scenario, but we argue that it may lead to the realization of the *key idea* (Raman, 2003) of the proof, allowing one to transition from the Orienting phase into the Planning phase. Our data suggest that students progress through the same four phases of the MPS Framework that were observed in expert mathematicians, but that students' ways of executing and checking are different from those of experts.

Methods

Four students (three undergraduate and one graduate) taking an introductory course in topology participated in at least one weekly, hour-long "Group Study Session" in which the students were asked to prove a true statement and to disprove a false one. The first author acted as the facilitator for all Group Study Sessions. One undergraduate student, Stacey (all names used in this study are pseudonyms), attended all sessions: the data presented here focus on Stacey's behaviors throughout the semester. The facilitator attended all class sessions (excluding exams); proof tasks were chosen based on material that had been covered recently in class. Group Study Sessions were video recorded. Students were encouraged to speak aloud as they worked and to work together with other students in the session. To maintain an authentic study atmosphere, students were permitted to use textbooks and notes as they wanted. As compensation, the facilitator offered extra office hours for participants to receive help with topology.

Using deductive thematic analysis (Braun and Clark, 2006), codes were applied to the data. Initial coding focused on identifying instances of students producing drawings, generating examples, and writing proofs. During this round of coding, it became evident that Stacey (as the only student to be present for all sessions) frequently used drawings to help her visualize definitions or to represent aspects of the problem scenario, and that these drawings seemed to influence her proving strategies. Based on the results of this first round of coding, a second round of coding identified instances of students constructing examples or drawings related to a definition, instances of students arriving at the key idea of a proof, and instances of students monitoring their work (either checking their own ideas or checking with the facilitator for logical consistency), as well as evidence of students' transitions through the phases of the MPS Framework. The patterns observed in these data resulted in the Topology Proving Framework proposed in this paper.

Data

The data presented here focus on Stacey's behaviors throughout the semester. Because this paper describes a framework for the construction of proofs of true statements, we describe the "prove" condition from two sessions; future work will focus on the "disprove" condition. Though Stacey produced drawings in Session 1 and Session 4 when prompted to do so by the facilitator, she did not spontaneously produce a drawing until Session 6. We present here two examples of Stacey's proving activities.

Session 6: Prove: ***A subset* A *of a topological space* $(X, \mathcal{T})$ *is said to be* <u>dense</u> *in* X *if* $\bar{A} = X$*. Prove that if for each open set* $O \in \mathcal{T}$ *we have* $A \cap O \neq \emptyset$*, then* A *is dense in* X*. (Note: the notation* $\bar{A}$ *indicates the topological closure of A in X.)***

For Session 6, Stacey was joined by Tom. The idea of a *dense subset* had not been discussed in class prior to this session, and Stacey had not previously encountered this idea. Tom had previously encountered this term in his introductory analysis course. After a brief reading of the problem, Stacey began by silently producing the drawing in Figure 1A.

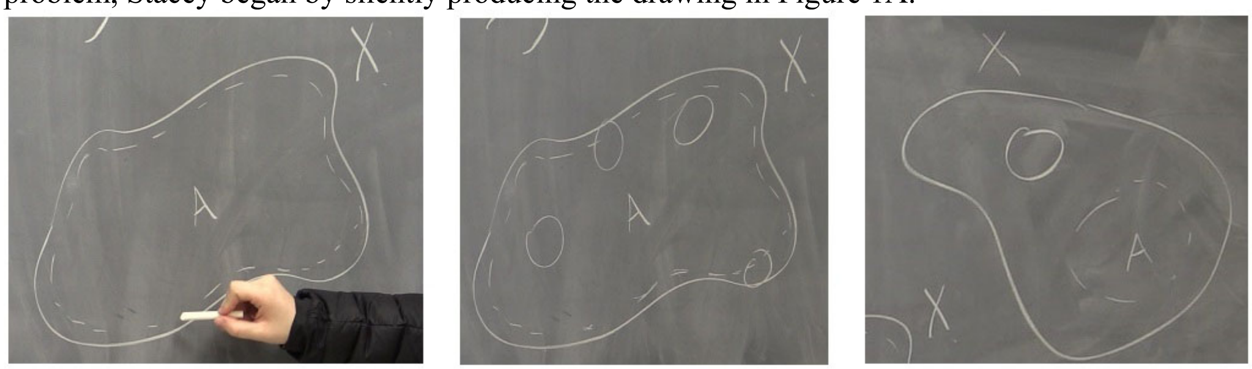

Figure 1A-1C: Stacey's drawings of a dense subset A of X. Figures 1A and 1B represent a dense subset; Figure 1C shows a subset A of X that is not dense.

After Stacey finished drawing, she explained:

> I can't really show it with a picture because I can't draw, like, a dashed line over a straight line, or like, a solid line, but we have X on the outside, and then we have the set A, which is represented by the dashed, which I wish I could get closer to this [pointing to the border of X], but I can't. So if we had the closure of A, then it would just be the same as that solid line [tracing the border of X with her hand]. So then if you take any open set

[drawing circles on her diagram, Figure 1B] anywhere, there has to be some kind of intersection with A. So if it wasn't, like if you take… if the intersection could be closed, er, could be, not closed, um, the empty set… [draws the diagram in Figure 1C] You've got X here… and A here, and you could have an open set here, and their intersection would be the empty set. [*code: recognizes key idea*] But then this closure wouldn't be equal to X. I get it conceptually I think, but I'm not sure how to prove it.

The preceding quote was coded as Stacey orienting herself to the problem. In the following excerpt, we see her transition into the Planning phase:

Stacey: We probably have to use the definition of closure in it… So we could say like… take x in… I don't know, either A or X, I'm not sure which one… and then a neighborhood of that point x…

Fac: Is there maybe a general strategy that you're thinking about? Or how are you thinking about approaching this problem?

Stacey: Um, I think contradiction, that's what's in my head right now.

Fac: If you had to outline your procedure – I know you don't have the whole thing fleshed out, but – how would your contradiction look? How would you set that up?

Tom: For the contradiction for this statement, it's gonna be "For each open set O, we have this *[points to* $A \cap O \neq \emptyset$*]*, but A is not dense in X. So the closure of A is not X." Right?

Following this exchange, several minutes were spent trying to determine whether the point *x* should be chosen from the set $\bar{A}$ or from *X*. Once it was agreed upon that *x* should be chosen such that it lies in *X* but not in $\bar{A}$, Stacey and Tom collaborated to write their proof. Stacey wrote "Let $x \in X$ and $x \notin \bar{A}$." Tom contributed, "So when you have this, when you have *x* is not in the closure of *A*, it means there is a neighborhood of *x* where it, intersect with *A*, will give the empty set," looking to the facilitator for confirmation of his reasoning. He followed this up by saying, "It doesn't seem right," but wrote this statement on the board, calling this neighborhood *N*. He then said this was the contradiction: "Now you have an open set that, intersect with *A*, gives you, uh, empty set." When the facilitator asked if *N* was an open set, Stacey concluded the proof by responding, "[The open set] is within the neighborhood… So there's *O*, subset of *N*, whose intersection with *A* is equal to the empty set." This resulted in a correct proof.

Session 8: Prove: Let $(X, \mathcal{T})$ be a topological space. A <u>separation</u> of X is a pair U, V of disjoint open subsets of X whose union is X. X is <u>connected</u> if no separation of X exists. If the sets C, D form a separation of X and if Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$.

This was Stacey's first encounter with the idea of a *separation* of a topological space. Stacey was the only participant in Session 8. She began by drawing the diagram in Figure 2A to orient herself to the problem.

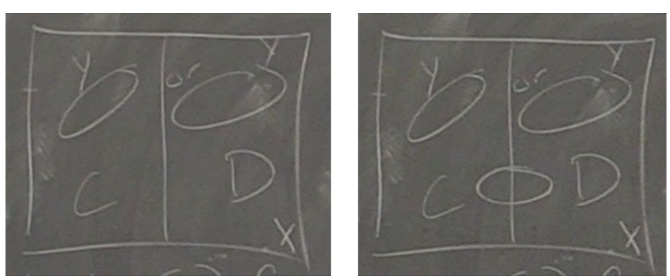

Figure 2A-2B: Stacey's drawings of a separation and a connected subspace Y.

She explained,

> If you have the X, the ambient space, and then you have the sets C and D, they form a separation, so that means that they're disjoint, so they don't have any of the same elements, and that their union is X, so that is satisfied for this. And then if Y is connected, which means it's not in these sets that are disjoint whose union is Y, it's just one cohesive set, then it has to be either in C or D, it can't be in both. Because if, if it was like that [draws the subset in Figure 2B], it would be disjoint. [*code: recognizes key idea*]

Stacey misspoke at the end of this explanation; throughout this session, she frequently said "disjoint" instead of "disconnected." In this last sentence, we observe Stacey's transition from Orienting to Planning.

She then began Executing her strategy, proceeding with her proof by way of contradiction. The facilitator provided guidance with logic and notation. Stacey frequently expressed correct ideas, such as the necessity to assume (for a contradiction) that some elements of Y lay in C and some elements lay in D. However, her initial notation read "Let $Y \subseteq C$ and $Y \subseteq D$." Because Stacey verbalized correct ideas, such as "If we do it like, by contradiction, and we say that there's intersection with both of them, and then we could show that Y can't be connected," we attribute errors like this to a lack of experience writing formal proofs, and specifically inexperience writing proofs in topology, rather than to a lack of understanding of the underlying ideas. When she changed her notation to a more appropriate statement, she checked with the facilitator to ensure that her new statement was accurate.

Stacey continued reasoning through the proof:

> Then you would say that… the points x and y are in disjoint spaces… From our assumption that C and D form a separation… So that would mean that… Y would have to be [disconnected] as well… Is there some kind of definition that says, like, a [disconnected] space that intersects all parts of another [disconnected] space is also [disconnected]? Is there something like that?

Stacey was talking about the fact that Y intersects both components C and D of X, which leads directly to the desired contradiction, as the sets $Y \cap C$ and $Y \cap D$ form a separation of Y, contradicting the connectedness of Y. As before, she appears to have the correct idea, but she lacks the experience to know exactly what she can do and how to formulate it correctly.

Discussions and Conclusions

The data presented here led to the creation of the Topology Proving Framework. It should be stressed that this is merely a *potential* framework; the small number of participants in this study makes it impossible to make generalizations with any reliability. This framework resembles the Multidimensional Problem-Solving Framework (Carlson & Bloom, 2005) in that it retains the idea of the four phases: Orienting, Planning, Executing, and Checking. Recall Stacey's behavior in Session 8: Stacey began investigating this conjecture by drawing a diagram to represent a separation, a clear sign of orientation to the problem. She then put forth the idea of proof by contradiction: *What if Y has intersection with both C and D?* "If we do it like, by contradiction, and we say that there is intersection with both of them, and then we could show that Y can't be connected." Here, Stacey has moved into the Planning phase and shows evidence of the sub-cycle of *conjecture-imagine-evaluate.*

Stacey's time in the Orienting phase often took a particular form. Beginning in Session 6 when she first began to produce drawings without prompting, her drawings frequently began as a visual representation of a key definition in the conjecture, occasionally becoming a representation of the entire problem scenario. As was reported by Stylianou (2002), this appears to have been a directed effort: the drawing seemed to stimulate Stacey's entry into the *conjecture-imagine-evaluate* cycle in the Planning phase, as it facilitated her ability to consider *What if?* questions. Furthermore, it is at this point that Stacey most frequently recognized the key idea (Raman, 2003) of the proof. For instance, in Session 6, Stacey drew two diagrams: one to represent a dense subset and one to represent a subset that is not dense. This seemed to motivate her to choose the strategy of proof by contradiction, and to recognize that if A is not a dense subset of X, then there must be some open subset O of X such that $A \cap O = \emptyset$. Ideas like this one do not always come fully-formed, as we saw in this example where Stacey seemed to have only a vague notion that contradiction *should* work. There was no guarantee that Stacey would necessarily know *how* to implement the key idea right away, as in this instance, in which Stacey wanted to begin her proof by choosing a point *x* which lay in either the set *A* or in the set *X*, but the determination of which set would be more productive seemed to require significant effort.

An interesting twist on the MPS Framework (Carlson & Bloom, 2005) as applied to Stacey's behavior arose when she entered the Executing and Checking phases. Carlson and Bloom's data show that experienced mathematicians proceed through the Planning, Executing, and Checking phases in a cyclic fashion until the mathematician is satisfied with her solution. Stacey, on the other hand, typically established a plan and then *alternated* between Executing and Checking activities. Furthermore, the experienced mathematicians Carlson and Bloom interviewed relied on their own *internal* resources to check their work. As a relative newcomer – not just to topology, but to proof writing in general – Stacey frequently checked with the *facilitator* to confirm notation, phrasing, and logical consistency, as seen in the following exchange from Session 5, in which part of the "Prove" condition asked Stacey to prove that the empty set and the set *X* are both closed in the topological space $(X, \mathcal{T})$:

Stacey: X is in $\mathcal{T}$, and then X is open, by definition.

Fac: Correct.

Stacey: And then the complement of X is the null set, and that's… closed.

Fac: Because…?

Stacey: Because… um… I mean, the null set is just like one, it's one element…

Fac: What reasoning did you apply to get there? X is in $\mathcal{T}$ …

Stacey: X is in T, so X is open. Well… So is the null set also in $\mathcal{T}$?

Fac: By definition, right?

Stacey: So that would also be open. So it's an open set… and the complement is the null set… And then the null set's also open, so then… it's a closed set?

Fac: Yeah.

Stacey: Same thing the other way around? So the null set can be open or closed, depending on the situation?

Fac: Well, not open or closed, but it's open and closed, simultaneously.

Stacey: But it's a different kind of open and closed than this, right? [points to the interval [0,1)]

Through her verbalizations, Stacey demonstrated the ability to self-monitor; external validation from the facilitator was not always necessary. However, this sort of external checking was common for Stacey, and it typically happened in conjunction with the execution of her proof construction, as part of an ongoing process of *Execute-Check-Execute-Check* which continued until the conclusion of her proof. We observed this kind of behavior with Tom as well in Session 6, as he unpacked what it meant for *x* to lie outside the closure of the set *A* while looking to the facilitator for confirmation of his reasoning.

The combination of these observations led to the development of the following Topology Proving Framework (TPF).

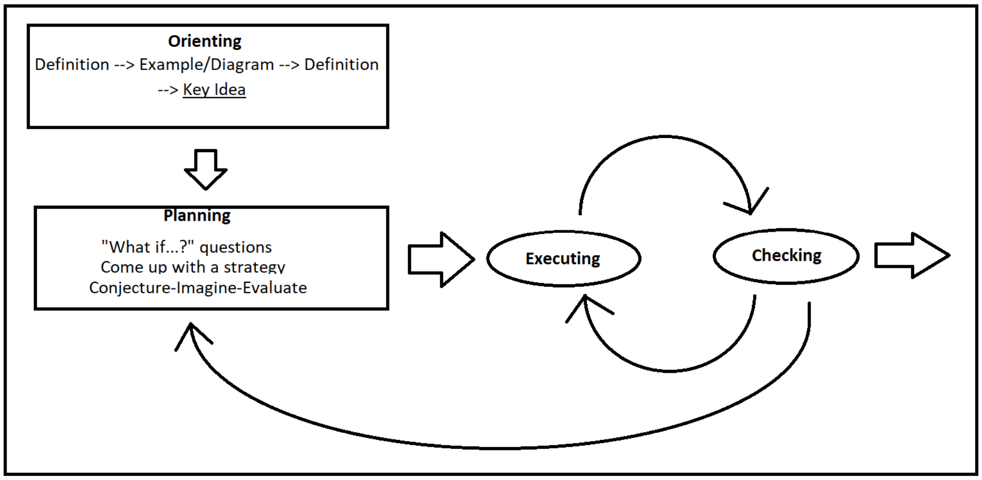

Figure 3:The Topology Proving Framework.

In keeping with the MPS Framework, the TPF begins with the student orienting herself to the problem. Most often, this took the form of the student converting a definition into a diagram or coming up with examples which gave a better understanding of the definition. This led to the realization of the key idea of the proof, which allowed the student to transition into the Planning phase.

With a visual representation of the key definition, the student was better equipped to ask *What if?* questions and to develop a plan, such as using proof by contradiction or direct proof. The recognition of the key idea gave the student a sort of "target," a sub-goal which, if proved, would result in the completion of the required proof. With a plan in mind, the student then began attempting to execute this plan. The execution was not always smooth and sometimes required some intense thought or trial-and-error. Throughout the execution of the plan, the student performed monitoring activities to ensure that she was still making progress toward her goal. These activities sometimes took the form of internal checks within herself, and other times they occurred as dialogue with the facilitator. Such external validation is not uncommon for students learning to prove or learning to prove in a specific content domain (Harel & Sowder, 1998). The alternation of execution steps and checking steps continued until a check resulted in the recognition of an error (which may reset the process back to the Planning phase) or in the student's perception that the proof was complete. The results of this study indicate that leveraging a key definition through visualization may be critical to success in identifying the key idea and producing a satisfactory proof in topology. Our future work will examine how this cycle is similar and different when tackling statements that require disproof.

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Responsiveness as a Disposition and Its Impact on Instruction

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There is evidence that instructors who are responsive to students' thinking tend to provide more positive learning experience for students. Additionally, effective instruction relies on an instructor's ability to respond to student thinking, which is especially relevant due to the increased attention on improving college mathematics instruction. In order to investigate instructor responsiveness to student thinking as a disposition (that guides action) and responsiveness to student thinking as an action (the enacted evidence of the underlying disposition), eight college Calculus instructors were interviewed three times over the course of one academic year. A thematic analysis of the task-based interviews indicated that instructors who exhibited a responsive disposition to their students' thinking enact this through eliciting student thinking, reflecting on student thinking, and responding to student thinking. Further, these instructors view themselves as decision-makers, and thus feel empowered to act on their responsive disposition.

Keywords: Instructor dispositions, decision-making, noticing

Effective instruction relies on an instructor's ability to attend and respond to students' mathematical understandings and strategies (Jacobs, Lamb, & Philipp, 2010). Additionally, there is evidence from the K-12 literature that an instructor's disposition towards student thinking also influences instructional decisions, including how they interact and respond to students (Sherin & Russ, 2014; Thornton, 2006; Schoenfeld, 2008). Specifically, it has been shown that teachers who are more responsive to their students' thinking are generally recognized as more effective teachers who provide more positive learning experiences (Thornton, 2006). This is especially relevant due to the increased attention on improving college mathematics instruction, and in particular, the focus on student-centered instruction. In order to further our understanding on how to most effectively implement such instruction, it is important for us to consider everything that contributes to college mathematics instructors' teaching practice. In this talk, we focus generally on responsiveness as a disposition, identify components of responsive instruction (enacting a responsive disposition), and compare this work to other existing frameworks examining similar qualities in teachers.

Research Related to Responding to Student Thinking

Although instructor dispositions towards teaching and students play an integral role in how instructional activities are chosen and enacted, it is not always clearly articulated what is encompassed by one's "disposition." Dispositions can refer to an instructor's beliefs, inclinations, values, attitudes, and ability, among other things (Splitter, 2010). For this talk, we draw upon Thornton's (2006) definition of "dispositions in action" that arose as a result of her work studying middle school teachers' dispositions:

> Dispositions are habits of mind including both cognitive and affective attributes that filter one's knowledge, skills, and beliefs and impact the action one takes in classroom or professional setting. They are manifested within relationships as meaning-making occurs with others and they are evidenced through interactions in the form of discourse (p. 62).

This definition highlights that dispositions are more than simply latent values or beliefs, and that these interact with knowledge and influence instructional practices. Based on other's previous work and our own experiences, we hypothesize that an instructor's disposition towards student thinking influences how they interact with students during class or office hours, how they elicit and respond to student thinking, how they prepare for a lesson, and how they approach grading and thinking about student work.

In her work on dispositions, Thornton (2006) developed a continuum of examples using classroom discourse analysis that describes teachers' orientations to student interactions ranging from a *responsive disposition* to a *technical disposition*. *Responsive dispositions* are those that are responsive to the needs and learning of students, including emotional, cultural, and development needs, and *technical dispositions* are those that involve going through the motions of teaching, but not engaging on a deeper level to probe, understand, or facilitate student learning. Thornton (2006) notes that with technical dispositions, instruction varies little from student to student and from situation to situation. This framing of teacher dispositions on a continuum lends itself to distinguishing between teachers who view themselves as in-the-moment decision-makers and those who do not. More specifically, one would expect teachers who view themselves as decision-makers to continually direct classroom interactions in order to align them with their goals for student learning as well as with their students' current thinking (exhibiting responsive dispositions). Conversely, teachers who carry out their role technically are expected to follow their prescribed lesson plans or pedagogical goals without deciding to adapt to the needs of the class or students (exhibiting technical dispositions).

One of the most developed models for considering teachers as decision-makers comes from Alan Schoenfeld who has worked to describe how knowledge, goals, and beliefs interact to shape instructional practices and decisions (Schoenfeld, 1998). His work provides evidence that an instructor's knowledge about the content, context, and pedagogy influences the types of things that they attend to during instruction and why they make certain decisions. Additionally, an instructor's goals (short or long term) influence how they decide to respond in the moment. For example, if a student asks a question in class, the instructor has to decide how they want to answer (with a mini-lecture, class discussion, etc.) and how long they want to spend answering the question and when (either now or later); these will vary depending on the instructor's immediate and long-terms goals for student learning. Further, an instructor's beliefs and dispositions influence which goals they prioritize. Schoenfeld (1998) notes that certain beliefs, knowledge, or goals can be strongly activated at a particular moment during instruction (either because of planning or an interaction) and that this can influence how the instructor decides to respond.

Schoenfeld (2008) has also noted that teaching is a system that involves coherence between teacher commitments and values. He highlights that even when teachers are flexible and responsive to student thinking in their classroom, attending to multiple or conflicting goals, it is possible to model their decisions with consistency. This illuminates the connection between an instructor's underlying beliefs and the instructional decisions they are making, further highlighting that responsive dispositions can be enacted through decision-making.

Another framework that unpacks how teachers act as decision-makers is that of professional noticing (Mason, 2002; Sherin, Jacobs, Philipp, 2011), which has been used as a way to connect an instructor's knowledge and practice with their disposition towards student thinking (Hand, 2012). This framework focuses specifically on how a teacher decides to respond to students' mathematical understandings, complementing Schoenfeld's framework that models all the decisions an instructor makes while teaching. Jacobs, Lamb, and Philipp (2010) describe noticing as: attending to, interpreting, and deciding how to respond to student strategies and understandings. An instructor's disposition to student thinking has been shown to impact the types of things that they attend to during instruction, impacting how and what they respond to (Sherin & Russ, 2014). We conjecture that in order for teachers to effectively attend, interpret, and respond to their students' understandings, they must have a responsive disposition that values student contributions and allows them to capitalize on their role as decision-maker.

The frameworks discussed above focus on in-the-moment decision-making, highlighting different processes and aspects that impact instructional decisions. Schoenfeld links knowledge, goals, and beliefs with decision-making, and Jacobs, Lamb, & Philipp link attending and interpreting with how an instructor interacts with specific students' understandings. However, neither framework attends explicitly to the underlying disposition that guides instructors' behavior - their responsiveness to student thinking - and how instructional decisions shed light on this underlying disposition. This study is guided by the following research question: How do college calculus instructors exhibit responsiveness to student thinking? In particular this work investigates instructor responsiveness by focusing on both responsiveness as a disposition (that guides action) and responsiveness as an action (the enacted evidence of the underlying disposition). This distinction will be discussed more thoroughly throughout the paper.

Research Design and Methodology

This study is part of a larger mixed-methods studying investigating the influences of college calculus instructors' dispositions towards student thinking. For this talk, we focus on the qualitative data collection and analysis component of this study.

To understand responsiveness as a disposition and how it impacts college mathematics instruction, we focus our study at one university and in one content area - calculus. We chose calculus because this is a course that impacts a vast array of students, with varying interests and educational goals, and is taught by a vast array of instructors with their own varying experience, interests, and educational goals (Bressoud, Mesa, & Rasmussen, 2015). For this study, eight Calculus 1 instructors from one highly selective institution were interviewed. Four participants were new graduate teaching assistants (GTAs) who were leading recitation sections, two were experienced GTAs who were instructors of record (with multiple semesters experience teaching Calculus 1), and two experienced teaching faculty. Of the teaching faculty, one was in her first year at this institution, but had several years of experience teaching as a graduate student at another highly selective institution. The other teaching faculty had received her PhD at this institution and had ten years of experience teaching Calculus 1 (and other courses) at this institution. These participants were selected because of their varying levels of experience instructing and interacting with students. Additionally, this variation of roles and responsibilities related to the instruction of calculus is likely to influence their perception of their role as decision-maker, and consequently provides greater insight into responsiveness as a disposition and how this is enacted in instruction.

We conduct this work from a situated cognition learning perspective which emphasizes the importance of context in the development of understanding and knowledge. From this perspective, it is essential to consider the multiple facets (i.e. content, level of instruction, teacher knowledge, teacher beliefs) that are tied to and interact to give rise to various knowledge impacting teaching practice (Putnam & Borko, 2000). Specifically, in trying to research and improve teacher practice, we must attend to teachers' dispositions as a part of this surrounding context. Although interviews were conducted outside of a teaching context, instructors were asked to consider their teaching practice in addition to examining student work, which is a common and authentic practice for most teachers.

A series of three interviews were conducted with each of the participants over the course of one academic year. The first interview was designed to learn about the participants' experiences teaching, career goals, and perspective on what it looks like to be a good instructor. The second interview was a task-based interview adapted from one used previously to exam college instructor mathematical knowledge for teaching where instructors were asked to work through calculus prompts, interpret student work to those prompts, and then discuss how they would respond to the students' thinking (Speer & Frank, 2013). The third interview was designed to facilitate a discussion revolving around various responsive instructional practices.

The interviews were audio-recorded and transcribed for analysis. The interview data were analyzed using thematic analysis (Braun & Clarke, 2006), by first highlighting all utterances related to a consideration of students or their thinking. These segments were then coded as either demonstrating *responsiveness in action* or *responsiveness as a disposition*. Segments coded as *responsiveness in action* included segments where instructors were responding specifically to students' work (e.g. "I would just go back over the definition with them"), and segments coded as *responsiveness as a disposition* were segments that demonstrated a general attending to students' needs, learning, and understanding (e.g. "I [try to] put myself into [the students'] positions, thinking about if I am first learning this concept."). We then used open-coding to determine themes, paying specific attention to how responsiveness as a disposition influenced responsiveness in action. After arriving at three general categories that described how instructors' were enacting responsive dispositions (or not) in their practice, we coded the interviews using these categories, developing subcategories as necessary.

Findings: Towards Understanding Instructor Responsiveness

The thematic analysis shed light on how an instructor elicits, reflects, and responds to student thinking and mathematical understandings (demonstrating responsive instruction) serves as a proxy for understanding their underlying responsive disposition.

Eliciting Student Thinking

The thematic analysis of the interviews highlighted a few underlying reasons why instructors might elicit student thinking, shedding light on their underlying disposition of responsiveness. Instructors that elicited student thinking either sought to draw out understandings they anticipated students would have (either correct or incorrect), or sought to gain insight into student thinking in order to gauge understanding. There were also instances where instructors did not elicit student thinking explicitly; these tended to be situations where either the instructor was able to interpret student thinking from the student's work or they sought to interpret the work without prompting for student thinking (e.g. "Well, I'd first have to figure out what they were getting at in answering this question."). The following excerpts demonstrate possible motives for eliciting student thinking.

Eliciting to draw out common student errors: "I have been spending time every week coming up with five challenging problems, and I think, 'What's all the stuff they mess up on the test?' And I can put them all into [these] problems … I said I'll work through all of these with you so they don't just blatantly do all the mistakes … They'll kind of know that they are not sure what they're doing, ... and so I have noticed that by me kind of drawing these to the forefront … [they] seem pretty good when [there are] similar … stumbling blocks on the later assessments."

Eliciting to gain insight into student thinking: "The first thing I would ask them is for them, now that they have the opportunity to take as much time as they want, try to explain to me what they were thinking."

Eliciting to guide a student through a problem: "I would probably just ask them like what's going on throughout time - like which car is moving faster. And then based on that, which one went farther during this time."

These excerpts shed light on instructors' underlying disposition of responsiveness to student thinking. Instructors that exhibit a more responsive disposition tend to demonstrate a variety of motivations for eliciting student thinking, drawing out student thinking in a variety of situations. This ties back to their role as decision-makers who capitalizes on opportunities to incorporate and respond to student thinking. The most common of the subcodes listed above was that of *eliciting to gain insight into student thinking*. This is likely due to the fact instructors were asked to respond to students' work several times throughout the interviews and they felt they needed more information about how the student was thinking in order to respond accordingly.

Reflecting on Student Thinking

Instructors who regularly reflected on their students and their students' understandings demonstrated a responsive disposition towards student thinking. These instructors tended to reference students or their thinking when discussing the motivation behind various instructional practices and decisions. The following excerpts come from one instructor's interview - note the variety of ways that this single instructor attends to students and reflects on student experiences and thinking. These excerpts together highlight a responsiveness (as a disposition) to student thinking, and provides insight into why they make certain instructional decisions enacting this disposition.

Reflecting on students' affect: "I have felt that my students have a lot of anxiety just because they are trying to prepare for this test … I am supposed to be very conscientious about how much information I share with my students, and I get that because we want the experience to be uniform. So if I am telling my students more than other instructors, then that is not fair … I personally don't care about fairness, but I understand that fairness is a consideration … And it's one way for me to alleviate my own anxiety, and my students' anxiety was just to tell them what I wanted them to know … I was still able to help them to focus on the things that I thought were important."

Reflecting on student difficulties with content: "Right now my students across the board - so students who I know came in with strong backgrounds and students who came in with maybe weaker backgrounds - are all having trouble with sigma notation and writing down Riemann sums."

Reflecting on student thinking when grading: "Definitely when I am grading I have more time and space to think, 'Oh you've written down this thing in this weird way,' let me try to figure out where it is coming from."

Reflecting on student thinking when planning: "I mean ideally when I plan a lesson I think about what my students will struggle with and what they will feel very natural [with], but I don't always do a good job of it. I don't always have the time and space to really think about what exactly is going to be the challenging part, and I also don't always do a good job of predicting what is going to be the challenging part."

Reflecting on specific student thinking and understanding: "I would want for them to draw me a picture, … because if they drew me just a single point, then I am worried that they are only thinking of this as single point instead of a single point in a continuous function. But if they are thinking of this as a single point in a continuous function, then I think that they have some understanding of what is going on with the limit."

Other instructors demonstrated a responsiveness to student thinking by reflecting on common student errors, by trying to anticipate student thinking (e.g. "I just try to put myself inside their head as best as I can"), or by reflecting on their own experience as a student (e.g. "We try to think ... through the first time I learned this, what was tough for me. And we write that on the board and go over it. And I think a lot of the times we get it sort of correct, and some of the times we don't."). The most common theme among instructors was a reflection on common student difficulties or, in response to interpreting student work, reflecting on what the student might be thinking. Instructors who demonstrated a more responsive disposition towards student thinking reflected on students regularly throughout the interview regardless of specifically being asked to consider student thinking, which seems to impact how they enact this disposition in the types of decisions they are making.

Responding to Student Thinking

Instructors' responses about how they would help students after interpreting their work in the interviews fell into a few categories that shed light on their underlying disposition of responsiveness towards student thinking. On one end of the spectrum, instructors responded to specific student work by selecting examples or explanations tailored to the student's understanding, enacting a responsive disposition towards student thinking. Further, these instructors tended to discuss additional ways in which they responded to student thinking (during planning, grading, writing assessment, in-the-moment instruction, etc.), highlighting their role as a decision-maker enacting this responsive disposition. On the other end of the spectrum, there were instructors who demonstrated a lack of responsiveness to student thinking, or a technical disposition (to borrow Thornton's (2006) term). These instructors typically responded directly to the mathematics prompt explaining how they would solve the problem instead of building off the student's demonstration of understanding.

The following interview excerpts show the spectrum of ways instructors demonstrated responsive dispositions (or lack thereof) to students and their thinking.

Responsive to specific student's thinking: "If [the student] drew another graph for me where it was decreasing and then increasing, then I would know they don't really understand what the sign of the derivative means. Then I would have to go back into this idea. I could … *<describes specifically what they would do>...* Whereas if they drew a correct graph, then I would know they were kind of grasping for where to go with this, and then we could talk specifically [about] if you realize that was a minimum, what should you have looked at next."

Responsive to student thinking on homework/exam problems: "If I think [a problem is] going to send them down a completely wrong road, I either might change the problem a little bit or give them a hint, say, 'Hey you notice this thing.' But I think it's important to at least be in the mindset when you're writing down homework problems or exam questions or any of that, you have to be in the mindset of what somebody who doesn't know the stuff very well would try."

Not responsive to student thinking: "I would just abandon [the student's] answer, and just start over with - I know this is the graph of f', what does this tell me about the slope? Or what does f' tell you about the original function?"

Most of the segments coded as *responding to student thinking* were in the instructors' responses to the prompt, "If this student were to come to your office hours, what sorts of things would you do to help them have a better understanding?" after examining the student's work. It is important to note that individual instructors typically demonstrated various types of responses to student thinking throughout the interviews. Thus it is important to consider how an instructor responds to specific student thinking along with the other aspects of responsive instruction that shed light on their underlying disposition of responsiveness, namely how they elicit student thinking and reflect on student thinking.

Discussion and Implications

This analysis has illuminated the distinction between responsiveness as a disposition and responsive instruction. Instructors who exhibit a responsive disposition to their students' thinking enact this through eliciting student thinking, reflecting on student thinking, and responding to student thinking. Responsive instruction is instruction that includes regular eliciting of, reflecting on, and responding to student thinking. We argue that instructors who exhibit responsive instruction have an underlying disposition of responsiveness. These instructors view themselves as decision-makers (Thornton, 2006), and feel empowered to act on their responsive disposition.

Instructors were even aware of this distinction between an underlying disposition of responsiveness and of what it looks like to be enacted through responsive instruction. This is demonstrated in the Findings section by the segment coded as *responsive to student thinking on homework/exam problems*. Here the instructor highlights the importance of being "in the mindset" of considering what students might do (demonstrating an underlying disposition of responsiveness) when he is writing exam or homework problems (enacting responsiveness through instructional decisions).

As mentioned in the review of the literature, previous work has focused on understanding instructors' decision-making and noticing has highlighted the great variety in ways that instructor can be aware of and respond to their students' needs (Jacob, Lamb, and Phillip, 2010; Schoenfeld, 2008). In this work, we have begun to unpack the underlying disposition of responsiveness that enables or constrains instructors' actions as decision-makers responding to their students. By better understanding responsiveness, we can learn how it can be developed and utilize it to impact instructors' practices as decision-makers.

Currently, much professional development surrounding student-centered instruction focuses on the teaching practices and the logistics of facilitating such learning. But since there is evidence that dispositions can be reshaped and developed (Hand, 2012), we should strive to not only improve instruction, but to foster responsive dispositions towards student thinking. As we gain a greater understanding of these underlying dispositions and how they impact responsive instruction (in how they elicit, reflect, and respond to student thinking), we can create professional development that more specifically targets this underlying factor that impacts instructor decisions and practice. Further, this area of research has the potential to drastically impact undergraduate instruction; since when we better understand how we can foster responsive dispositions and responsive instruction, we can better support students through student-thinking centered instruction.

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Prospective High School Teachers' Understanding and Application of the Connection Between Congruence and Transformation in Congruence Proofs

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Undergraduate mathematics instructors are called by recent standards to promote prospective teachers' learning of a transformation approach in geometry and its proofs. The novelty of this situation means it is unclear what is involved in prospective teachers' learning of geometry from a transformation perspective, particularly if they learned geometry from an approach based on the Elements; hence undergraduate instructors may need support in this area. To begin to approach this problem, we analyze the prospective teachers' use of the conceptual link between congruence and transformation in the context of congruence. We identify several key actions involved in using the definition of congruence in congruence proofs, and we look at ways in which several of these actions are independent of each other, hence pointing to concepts and actions that may need to be specifically addressed in instruction.

Keywords: geometry, transformations, secondary teacher education

Instructors of undergraduate teacher preparation programs face a transition in geometry instruction. In the past several decades, geometry has been taught primarily from a perspective based on Euclid's *Elements* (Sinclair, 2008); in recent years, geometry from a *transformation perspective* has come to the fore in secondary standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) and guidelines (NCTM, 2018).

These changes in geometry standards have implications both mathematically and pedagogically. For instance, consider the well-known triangle congruence criterion "Angle-Side-Angle (ASA)": *If* ΔABC *and* ΔDEF *are triangles such that* $\overline{AB} \cong \overline{ED}, \measuredangle BAC \cong \measuredangle EDF,$ *and* $\measuredangle ABC \cong \measuredangle DEF,$ *then* $\Delta ABC \cong \Delta DEF.$ In secondary and college geometry texts using an *Elements* approach, this criterion is often taken as a postulate: it is intended to be accepted as mathematical truth without proof (e.g., Education Development Center, 2009; Musser, Trimpe, & Maurer, 2008; Serra, 2008; Boyd, Cummins, Mallow, Carter, & Flores, 2005). These and other texts help students establish conviction in ASA through empirical exploration – a scheme for conviction, that taken by itself, can be unproductive when the objective is to construct a deductive proof (Harel & Sowder, 2007). In contrast, from a transformation approach, if a student is to show that two triangles ΔABC and ΔDEF in a plane are congruent, they must show that no matter the triangles' locations, there exists a sequence of translations, rotations, and rotations that map ΔABC to ΔDEF. (See Wu (2013) for a schematic for such a proof.) In the transformation approach, even if empirical exploration is beneficial, a teacher must also help students move toward deductive proof. In the *Elements* approach, a proof would be mathematically impossible.

It is critical for prospective and practicing teachers to understand not only the abstract notion that different axiom systems result in different proof approaches (Van Hiele-Geldof, 1957), but also that they may be teaching students from an axiomatic system different from the one they learned first. Consequently, teachers – including prospective teachers who are undergraduate students – may not be familiar with what can be proven, what cannot be proven, or how particular proofs operate. We address this problem from the perspective of developing knowledge for teaching prospective teachers, including understanding how prospective teachers learn. In this document, we report on a study guided by the question: *What concepts are entailed in prospective teachers' construction of congruence proofs?*

We focus this study on establishing congruence proofs because, as suggested by the example above, it is an area fundamental to the study of geometry at the secondary level where differences between *Elements* and transformation approaches are salient. We address our research question by analyzing data from prospective teachers for potential key developmental understandings (Simon, 2006) related to constructing congruence proofs.

Conceptual perspective

Transformation approaches to school geometry, though only recently sanctioned in standards documents such as that of the Common Core, are not new. Following Usiskin and Coxford (1972), we take a *transformation approach* to geometry as one that features:

- Postulation of preservation properties of transformations:
    - in particular, reflections, rotations, and translations are assumed without proof to preserve geometric properties such as length and angles; and
    - these transformations are defined as maps from the plane to the plane;
- Definition of congruence in terms of transformations: two subsets X and Y of the plane (e.g., two triangles) are said to be congruent if there exists a reflection, rotation, or translation, or sequence of these transformations[1], that maps X to Y;
- Definition of similarity in a corresponding way, via transformations.

The details of these features may differ across texts, for instance, different statements of postulates of transformations may be taken, but they have in common that the postulates are about transformations, rather than congruence criteria for particular objects such as triangles.

Hence, from a transformation perspective:

- **[T-to-C]** *To establish a proof of congruence of two objects in the plane, such as two triangles, one constructs a sequence of assertions that show that there exists a single one of or a sequence of reflections, rotations, or translations that maps one object to the other*,

where the assertions can be justified with reasoning and represented in ways that the community learning these concepts understands (Stylianides, 2007). Moreover,

- **[C-to-T]** *When two objects are congruent, the transformation perspective provides that there then exists a single one of or a sequence of reflections, rotations, or translations that maps the first object to the other.*

We emphasize and name the "T-to-C" (transformations are used to establish congruence) and "C-to-T" (congruence provides a sequence of transformations) statements for two reasons. First, they represent an unpacking of the two directions of the definition of congruence from a transformation approach, when the definition is taken as an if-and-only-if statement. Second, they are essential to the tasks used in the reported study.

We take an *Elements approach* to be one that features the postulation of at least one triangle congruence criterion (e.g., SSS, ASA, or SAS), and definition of congruence similarity in terms of individual geometric objects (e.g., congruence for triangles is defined separately from congruence of circles).

[1] Note that glide reflections can be expressed as compositions of reflections and translations.

As Jones and Tzekaki (2016) reviewed, there is "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" (p. 139). To our knowledge, there have been few studies on teachers' conceptions of congruence *proofs* from a transformation perspective. One exception is Hegg, Papadopoulos, Katz, and Fukawa-Connelly (2018), who examined how teachers managed their prior knowledge of congruence criteria when showing the congruence of two triangles. They found that teachers preferred to use triangle congruence criteria rather than transformations, but could, when asked, successfully complete proofs using transformations. However, their study did not examine the case of proving congruence of figures that are not triangles.

Hence, because of the novel nature of this study, we pursue an inductive analytic design, and we present related literature in the discussion section rather than in the introduction. This structure is "most suitable for the inductive process of qualitative research" and allows related literature to be "a basis for comparing and contrasting findings of the qualitative study" (Creswell & Creswell, 2017, p. 27).

Data and Method

Data

A post-hoc analysis was conducted of 20 prospective secondary teachers' responses to two congruence proof tasks, the Line Point Task and the Boomerang Task (below). The tasks were distributed as part of an in-class midterm examination in a mathematics course taught by one of the authors in Fall 2017.

- **Line Point Task.** Let ℓ, m be lines. Among all the points that are a unit distance from ℓ, choose one point P. Among all the points that are a unit distance from m, choose one point Q. Prove that no matter what points P and Q you chose, it is always true that $\ell \cup P \cong m \cup Q$.
- **Boomerang Task.** Let ΔABC and ΔDEF with congruences marked as shown. Let O be a point on the inside of ΔABC and P be a point on the inside of ΔDEF so that the angle measures $\alpha = \gamma$ and $\beta = \delta$ as shown. Given the all the above, prove that $\Delta AOB \cup \Delta ABC \cong \Delta DPE \cup \Delta DEF$ (Figure 1).

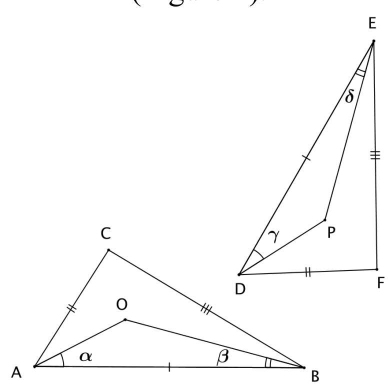

Figure 1: The Boomerang Task was distributed with this representation of $\Delta AOB \cup \Delta ABC$ *and* $\Delta DPE \cup \Delta DEF$

Analysis

The analysis focused on identifying potential *key developmental understandings* (KDU: Simon, 2006) used in constructing congruence proofs. A full conceptualization of KDU is beyond the scope of this brief report, but we emphasize that a KDU affords a learner a different way of thinking about mathematical relationships (Simon, 2006). For our analysis, this meant that to determine whether something may be a KDU, we must be able to identify how having or not having the KDU could make a difference in learners' capacity to construct congruence proofs. We proceeded by coming to consensus about the logic of each prospective teacher's response to the tasks, then generating potential descriptions of ways of thinking about congruence and proof that account for differences among responses. These descriptions became provisional codes. We consolidated or distinguished codes based on how and whether the use of the definition of congruence changed what was possible mathematically later in the argument.

Rationale for Task Design

The Line Point Task and Boomerang Task were part of a sequence of tasks intended for developing prospective teachers' understanding of using definition of congruence from a transformation perspective to prove the congruence (or non-congruence) of given figures, especially when the proof requires showing the extension of transformations from a proper subset of figures to entire figures. The prospective teachers' responses to these tasks suggest that there are KDUs underlying the doing of the tasks; responses to the tasks indicated different understandings of the role of the definition of congruence and the need for showing extensions of transformations. Moreover, in-class discussions indicate that understandings were more likely to develop as a result of reflection and multiple experiences than through direct instruction.

The second author selected and designed this sequence using variation theory; in brief, this theory holds that knowledge of a particular idea develops from tasks that keep constant the use of the idea while varying other aspects of tasks (Lo, 2012). The sequence included tasks co-designed by teachers, mathematics educators, and mathematicians to support this goal (Park City Mathematics Institute, 2016), beginning with prospective teachers' discovering that, from a transformation perspective, the statement that "two line segments of equal length are congruent" required proof. Building on the transformations used in a proof of this statement, prospective teachers then used extensions of these transformations for proofs involving triangles and other unions of line segments during class and for homework. Prospective teachers were then asked to prove that two rectangles of equal dimensions are congruent, which requires showing that a candidate sequence of transformations can extend from mapping parts of a figure to mapping entire figures as desired. Two of the authors designed the Line Point and Boomerang Tasks as variations of the rectangle task.

Results

Decomposition of using the definition of congruence in congruence proofs

Using the prospective teachers' responses, we first decomposed the definition of congruence into the concepts C-to-T and T-to-C, and then decomposed each of these concepts. In particular:

- Using C-to-T involves prospective teachers explicitly using known congruence between two figures, known theorems, or axioms to infer the existence of a sequence of rigid motions mapping one figure to a second figure.
- Using T-to-C involves two actions:
  - the teacher consistently states that in order to establish congruence one must establish a sequence of rigid motions to map one figure to the other *and*
  - the teacher establishes rigid motions or a sequence of rigid motions to map one figure to another to show congruence between the figures.

Using these criteria, we found that using C-to-T does not predict using T-to-C, or vice versa. With this independence of C-to-T and T-to-C in mind, we then analyzed how prospective teachers' responses invoked C-to-T and T-to-C. Our analysis resulted in two potential KDUs. Due to space limitations we only describe illustrative examples for the first result; we elaborate upon the results in the presentation.

Potential KDU 1: Understanding that applying the definition of congruence to prove congruence of two figures means establishing a sequence of rigid motions mapping one <u>entire</u> figure to the other <u>entire</u> figure.

Prospective teachers without this KDU may know that rigid motions are involved in congruence proof, but they may not understand that figures remain fundamentally un-altered with every motion. For instance, we found responses that established rigid motions and thus congruence between parts that compose a whole (such as between ℓ and m as well as P and Q, or ΔAOB and ΔDPE as well as ΔABC and ΔDEF) but that did not necessarily establish congruence of entire wholes ($\ell \cup P$ and $m \cup Q$, or $\Delta AOB \cup \Delta ABC$ and $\Delta DPE \cup \Delta DEF$).

To illustrate, in the Boomerang Task, some responses used the premise that $\overline{AB} \cong \overline{DE}$ to claim abstractly the existence of a transformation mapping $\overline{AB}$ to $\overline{DE}$, but then the responses concluded that $\Delta AOB \cup \Delta ABC \cong \Delta DPE \cup \Delta DEF$ because $\Delta AOB \cong \Delta DPE$ and $\Delta ABC \cong \Delta DEF$ – and not because the transformations could extend to the unions. (See Figure 2 for an example.)

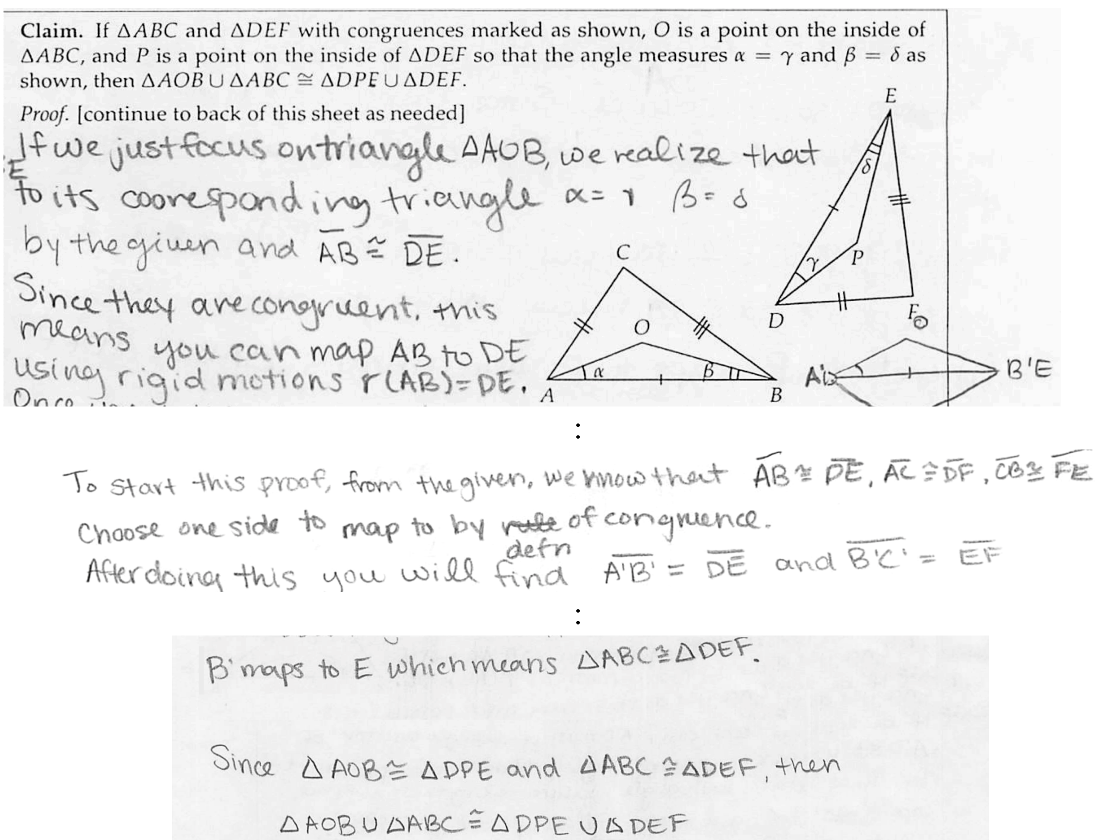

Figure 2: These show key steps of one teacher's work on the Boomerang Problem. In the first part the teacher used C-to-T. Just before the second part above the teacher concluded using these rigid motions that ΔAOB maps to ΔDPE. In the third part we see that the teacher did not use T-to-C to conclude congruence of the unions.

Additionally, some prospective teachers' responses described rigid motions that mapped some or all corresponding parts of the first figure to the second, but the rigid motions constructed did not extend to the entire figures – in this case, the responses exhibited different rigid motions for different components that could not extend. Other responses constructed rigid motions that did extend to the entire figure, but this extension was not recognized explicitly in the responses. Furthermore, some prospective teachers defined a transformation that did "double-duty", that is the teacher noted that two parts of the figures are congruent and therefore claimed the existence of a single transformation that mapped both pieces to their corresponding parts at the same time.

Potential KDU 2: Understanding that using a sequence of transformations to prove that two figures are congruent means justifying deductively that the image of one figure under the sequence of transformations is exactly the other figure.

To understand the necessity of proving that two figures need to be superimposed, one must conceive of the possibility that they may *not* be superimposed. Being able to conceive of this possibility allows for a learner to realize that there is more to show than identifying a candidate sequence of transformations.

Teachers without this KDU may declare the proof complete after defining the transformations or providing minimal justification. For instance, on the Line Point Task, some prospective teachers defined a sequence of rigid motions and claimed that $\ell \cup P$ had been mapped to $m \cup Q$ without further justification. Several other prospective teachers minimally attempted to justify superposition by stating that rigid motions preserve distance. We note that in this case, prospective teachers showed evidence of potential KDU 1 but not potential KDU 2.

Discussion/Conclusion

In this study, we analyzed prospective teachers' responses to tasks, designed using variation theory, for underlying understandings that support constructing congruence proofs. Based on this analysis, we proposed an empirically-based decomposition of and two potential KDUs for the use of the definition of congruence in congruence proofs. We now discuss our findings in relation to previous results in the literature. We highlight two such results; our findings corroborate one result and add nuance to the other.

First, as Edwards (2003) described, students at middle school, secondary, and undergraduate levels predominately hold a *motion view* of transformations. From this perspective, a transformation is conceptualized as the movement of a geometric object, which sits "on top" of the plane, from one location to the next. This contrasts with a *map view* (Hegg et al., 2018) in which objects are perceived to be subsets of the plane, and transformations to be maps of the plane. Multiple subsequent studies suggest that prospective middle school and secondary mathematics teachers may also hold a motion view (Portnoy et al., 2006; Hegg et al., 2018; Yanik, 2011), and that this view may make it difficult to construct proofs of congruence from a transformation perspective.

Our analysis corroborated the "motion-versus-map" findings of previous studies, instantiated as expressed conflation of pre-images and images. For instance, after applying a transformation to $\ell \cup P$ in the Line Point Task, some prospective teachers continued to refer to the image as $\ell \cup P$. We interpreted this notational usage as the consequence of a movement conception of transformation rather than a map conception. In contrast, when teachers used notation such as $r(\ell \cup P)$ or $(\ell \cup P)'$, we interpreted this as the consequence of a map conception. However, some teachers who used notation consistent with a movement conception nonetheless otherwise produced valid arguments for congruence, suggesting that this conception is not necessarily a barrier to understanding the structure of congruence proofs.

Second, as far as the ability to construct congruence proofs, Hegg et al. (2018) found that, after participating in a course which incorporated transformational geometry content, prospective teachers could successfully use transformations to establish congruence between two triangles. In our findings we also found this to be true; however, our data suggest that prospective teachers may not be as successful in establishing congruence for other objects, and that they encounter difficulties in applying the definition of congruence. The design of our study allowed us to examine prospective teachers' capabilities for writing congruence proofs beyond standard triangle congruence proofs. These tasks required not only finding sequences of transformations between familiar objects, but showing that a sequence could simultaneously map the objects in a *union* of these objects to another *union.* Furthermore, our data included working with lines and points—objects which, though familiar—are not often discussed in the context of congruence proofs.

We now make some points about the relation of our proposed KDUs to successful completion of congruence proofs from a transformation perspective. First, these potential KDUs are necessary but not sufficient for teachers to successfully complete congruence proofs. For instance, a teacher who has attained potential KDU 2 may know that further justification is necessary after defining a sequence of transformations but be unsure as to what justification to use. It also appears possible that a teacher may have one of the above KDUs but not the other, as with responses demonstrating KDU 1 in the Line Point Task but not KDU 2.

Additionally, we note that the conceptual link between transformations and congruence in the context of congruence proofs involves understanding C-to-T (the fact that the congruence of two figures implies that there exists a sequence of transformations carrying one figure to another) and T-to-C (the fact that the existence of a sequence of transformations carrying one figure to another implies that the two figures are congruent). A teacher who applies C-to-T in a mathematically valid way will use known congruences between two figures to infer existence of rigid motions mapping one figure to a second figure. A teacher who applies T-to-C in a mathematically valid way will both (a) consistently state that in order to establish congruence one must establish a sequence of rigid motions to map one figure to the other and (b) construct or declare rigid motions that carry one entire figure to another. A few additional ways of thinking related to the above concepts have also been noted. As the above actions are all teacher actions that appear to be prerequisites to the creation of mathematically valid and complete congruence proof construction, these are skills that instructors will likely need to address.

While the above actions may be conceptually related, they appear in this data set to be independently adopted by prospective teachers, with prospective teachers sometimes engaging in only one or two of the corresponding actions at a time. As a result, an instructor may need to keep in mind that successfully addressing only one or two of these concepts and actions may not be sufficient in helping prospective teachers create mathematically valid and complete congruence proofs.

Applications of this work may include the construction of lessons, assignments, and assessments that directly address each above potential KDUs and conceptual links. Such materials may help instructors as they attempt to help prospective teachers learn the subtle concepts listed above in addition to those involved in notation. Future work is needed to interrogate the accuracy of these KDUs.

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Investigating How Students from the Biological and Life Sciences Solve Similar Calculus Accumulation Tasks Set in Different Contexts

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Calculus teaching and learning is a topic of great interest in the mathematics education research community. Specifically, the definite integral and accumulation have received quality attention in the past few years (e.g. Jones, 2015; Sealey, 2014). However, even though approximately 30% of our introductory calculus students are planning on careers in the biological and life sciences, little research exists concerning how students from these fields reason about calculus. In this study, task-based interviews were conducted with 12 undergraduate students majoring in the biological and life sciences. Students were asked to complete two similar calculus accumulation tasks, one a traditional kinematics task and the other set in the context of plant growth. Data were analyzed via open coding. Results indicate students interpreted the given information in the tasks differently, they were more likely to view the rate of change curve as representing the total accumulated quantity in the plant growth tasks.

Keywords: Calculus, Definite integral, Biology

Introduction and Background Literature

The biological and life sciences are among the most popular fields for student enrolled in introductory calculus; approximately 30% of the students in traditional Calculus I courses intend for careers in the biological and life sciences (Bressoud, 2015). Unfortunately, the traditional Calculus I course "is designed to prepare students for the study of engineering or the mathematical or physical sciences" (Bressoud et al., 2013, p. 691) and research on student thinking in calculus has largely been done on students with a physics and engineering background. Therefore, there is a significant gap in our collective attention in undergraduate calculus - how students from the client disciplines of calculus other than engineering and physics reason about calculus within their own fields. Our goal with the current study is to answer the research question: How do students majoring in the biological and life sciences solve two calculus accumulation tasks which differ primarily by context?

Integration and accumulation serve an important role in differential equations, which are used extensively in modeling within the biological and life sciences. Additionally, there is a need for quantitative reasoning in the training of future biologists as the field of biology now depends on advanced mathematical and computer programming techniques (Bialeck & Botstein, 2004; Cohen, 2004; Gross 2004; Hastings & Palmer, 2003; NRC, 2003). Researchers have investigated student conceptions of the definite integral and have found that calculus students are good at using the standard antiderivative techniques taught in introductory calculus (Ferrini-Mundy & Graham, 1994; Grundmeier, Hansen, & Sousa, 2006; Mahir, 2009; Orton, 1983) and that focusing on the multiplicative structure of the Riemann sum is a productive way to conceive of the definite integral when compared to using only area under a curve (e.g. Jones, 2015; Sealey, 2014). Researchers have also noted that students struggle to make meaningful connections between rate of change and accumulation in definite integral tasks (Bajrachara & Thompson, 2014; Beichner, 1994; Thompson, 1994). In addition, research has shown that context can impact student solution strategies, both within calculus and mathematics more broadly (Arleback, Doerr, & O'Neil, 2013; Bajracharya & Thompson, 2014; Jones, 2015b; Herbert & Pierce, 2012; Moore & Carlson, 2012). To better serve students from the myriad client disciplines of calculus, we must understand how students solve calculus tasks set in contexts relevant to those fields and whether those contexts play a significant role in their mathematical reasoning.

In this study, our goal was to investigate how students from the biological and life sciences solved two similar calculus accumulation tasks that differed only in context - one set within a familiar kinematics context and one in plant growth. We hope to provide a foundation on how problem context and background knowledge play a significant role in calculus teaching and learning.

Theoretical Perspective

The perspective of learning that influenced this study is constructivism. In a constructivist theory of learning, learners are viewed as actively participating in the development and re-organization of the cognitive structures making up their understanding of the world (von Glasersfeld, 1982). Furthermore, we consider it necessary that to explore a given individual's understanding of mathematics, one must consider the "social and cultural influences of a tribe (group)" (Confrey & Kazak, 2006, p. 317). This perspective on learning maintains a focus on the individual learner while acknowledging that social and environmental factors necessarily play a pivotal role in that learning. For this study, such a perspective serves as the foundation for analyzing each individual's approaches to the calculus tasks while situating them within the influence of those individual's backgrounds (in this case, as undergraduate students majoring in the biological and life sciences) and the interview setting itself.

Methods

The current study was part of a larger project aimed at understanding how students from the biological and life sciences reason about calculus accumulation tasks. In the larger study, we utilized task-based interviews with twelve undergraduate students majoring in the biological and life sciences at a large public university in the Southeastern United States, which we refer to as Southern State University (SSU), in the spring of 2016. Data were open-coded via methods from constructivist grounded theory (Charmaz, 2000). The current study focuses on two of the five tasks from interview sessions and with it we seek to answer the question: How do students majoring in the biological and life sciences solve two calculus accumulation tasks which differ primarily by context?

The participants were selected from the population of all undergraduate students majoring in the biological and life sciences at SSU. SSU is a large, public university serving approximately 24,000 undergraduates. SSU is considered "very selective" with 46% of applications admitted per year (The College Board, 2017). Students majoring in the biological and life sciences at SSU at the time of this study were required to take at least two semesters of calculus. Participants were solicited by visiting second semester calculus courses specifically designed for students studying in the biological and life sciences as well as upper-level courses within the biological and life sciences. Twelve students were interviewed, half of which were freshman or sophomores while the other half were juniors or seniors. The students were predominantly female (8 of 12) and Caucasian (11 of 12).

During the task-based interviews students completed five calculus tasks concerning accumulation with each interview lasted approximately one hour. In each of the five tasks, the students were presented with a rate of change function of some quantity and asked questions about the accumulation of said quantity over various periods of time. In the current study, we are focusing on two of the tasks that were utilized to specifically address how students might solve two mathematically similar tasks that were set in two different contexts, one a standard calculus kinematics tasks concerning velocity and distance traveled and the other regarding plant growth and total number of plants. The two tasks are shown in Figure 1.

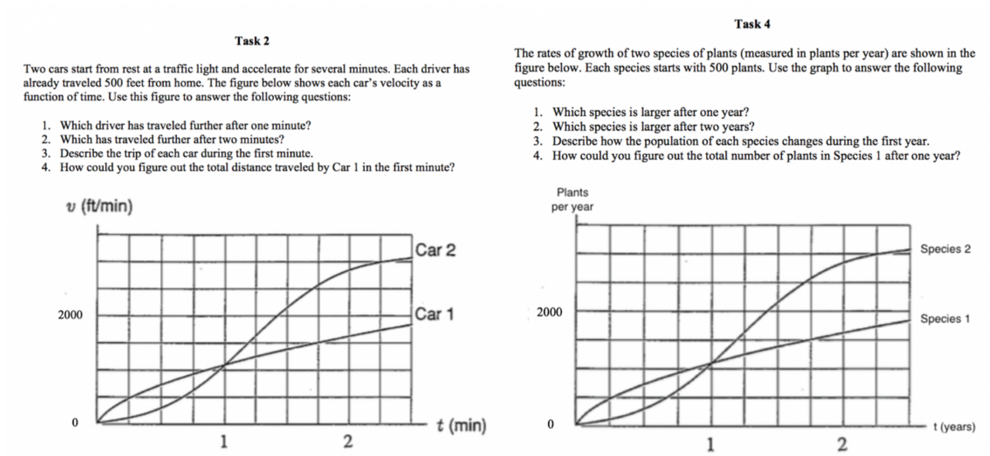

Figure 1. Interview tasks.

Data Analysis

The analysis procedures were developed out of a constructivist grounded theory approach (Charmaz, 2000) in which data were open-coded and categories of responses were allowed to naturally emerge. Constructivist grounded theory, like other forms of grounded theory (e.g., Glaser & Strauss, 1967; Strauss & Corbin, 1990), allows the researcher to explore the data without an assumed framework for results. Each interview task was first annotated independently using language as close to the students' language as possible. Annotating each task independently meant we did not begin with an assumption of uniformity in student approaches. However, we determined that due to the similarity in student approaches and the frequency with which students talked about both tasks at the same time during the interviews, that the annotations for each task should be merged into a single codebook. These annotations were then collected and grouped for similarity, becoming the initial codes in the codebook. The data was then coded and independently coded by external researchers to ensure validity and reliability.

Results

Student Interpretations of the Tasks

Table 1 shows how many of the students were given that code in that task, therefore there is a maximum of 12 for each cell in the table. The first two codes, *Graphed Function Represents Quantity* and *Graphed Function Represents Rate of Change*, tell us about how the students described and reasoned about the function they were given. The intended interpretation for each graph was as a rate of change, so the second code is considered more mathematically accurate.

Table 1. Frequencies of Codes Regarding How Students Interpreted the Tasks.

Code	Description	Example	Task 2	Task 4
Graphed Function Represents Quantity	Student reasons about the graphed function as representing quantity, either total number of plants or total distance traveled.	"And then after two years, species two is larger, that's pretty obvious…this is plants [vertical axis] and this is years [horizontal axis]."	3	9
Graphed Function Represents RoC	Student reasons about the graphed function as representing rate of change of quantity, either plant growth or velocity.	"Okay, so after one and some years, the growth rate was fifteen hundred plants per year."	12	5
Intersection of Curves Implies Quantities Equal	Student claims that when the curves intersect, it implies the quantities will be equal.	"Okay so I mean they're [number of plants] the same at, after one year. Um…"	7	9
Intersection of Curves Implies RoC Equal	Student claims that when the curves intersect, it implies the rates of change are equal.	"Even though they both end at the same rate at year one"	8	1

As we can see in Table 1, students interpreted these graphs differently. In solving *Task 2*, each of the 12 students reasoned about the function as representing rate of change while only 3 of 12 students reasoned about the function as distance traveled. However, in solving *Task 4*, 9 of 12 students interpreted the function as denoting total number of plants whereas 5 of 12 interpreted the function as representing the rate of plant growth. Here we see a clear distinction in students' interpretations where the function in *Task 2* was more likely to be interpreted as intended, as a graph of rate of change.

For *Task 4*, we see similar findings when considering the intersection point as compared to when we considered the graph more generally; more students (9 of 12) interpret the intersection of the curves as implying the number of plants in each species is equal compared to those who explicitly discussed the intersection as implying the rates of growth to be equal (1 of 12). However, for *Task 2* we see something surprising in that more students were coded as *Intersection of Curves Implies Quantities Equal* (7 of 12) as compared to those who reasoned about the graph as representing distance traveled (3 of 12). We still see a reasonably large number (8 of 12) of students who saw the intersection point as denoting that the two rates of change were equal. Therefore, at first glance it would seem students are interpreting the graph simultaneously as a rate and an accumulated quantity. This discrepancy and a potential explanation are discussed in more detail in the next section.

Student Reasoning Through the Tasks

Table 2 contains the frequencies of the codes relevant to how students reasoned with the given information to solve the tasks. The first code, *Greater Rate of Change Implies Greater Quantity*, gives us a little insight into our apparent contradiction regarding how students interpreted the curves in *Task 2* as compared to their intersection point.

Table 2. Frequencies of Codes Regarding How Students Reasoned in the Tasks

Code	Description	Example	Task 2	Task 4
Greater RoC Implies Greater Quantity	Student states that a greater rate of change value implies that the quantity will be larger.	"If you have a higher velocity your speed is faster, so feet per minute, so you're traveling more feet per minute, so he's traveled farther."	10	4
Kinematics Derivatives	Student recalls the calculus relationship between displacement, velocity, and acceleration.	"The derivative of distance traveled is velocity"	7	0
Kinematics Influence	Student explicitly reasons about the task utilizing their knowledge and/or language of kinematics	"It's telling us the, the rate of…so we have the, I guess 'velocity' of plant growth here"	0	3
RoC Multiplied by Time Equals Net Quantity	Student multiplies a rate of change value by time to calculate the change in quantity over a given time period.	"They're going a thousand feet per minute for one minute, so we guess they went a thousand feet"	5	1

Most (10 of 12) students reasoned that a greater rate of change implied a greater distance traveled. However, there are students who assumed that a greater velocity at any given point in the graph necessarily implies a greater distance traveled at that point. This means that a student could interpret the graph as giving a rate of change but the intersection point as implying the cars had traveled the same distance, shedding some light onto the apparent discrepancy. The results from *Task 4* are in line with our previous findings that they were largely interpreting the graph as giving the total number of plants instead of a growth rate since only 4 of 12 students were coded as *Greater Rate of Change Implies Greater Quantity* during their work on that task.

Students frequently discussed kinematics derivatives (e.g. velocity is the derivative of displacement or acceleration is the derivative of velocity) while solve *Task 2* (7 of 12) but never while solving *Task 4*, even though three students were coded as *Kinematics Influence*, meaning they discussed kinematics in some way while solving *Task 4*. The final code, *Rate of Change Multiplied by Time Equals Net Quantity*, shows us again the discrepancy in how students approached each task. While 5 of 12 students reasoned about multiplying a rate by a time to find a total quantity during *Task 2*, only one student was found to do so for *Task 2*.

Discussion & Implications

Students interpreted these tasks in diverse ways, which resulted in their solution strategies differing as well. In their work on *Task 2*, the kinematics task, students were more likely to interpret the graph correctly as a rate of change as compared to their work on *Task 4*, the plant growth task. However, this does not imply they successful in completing the task as many of those students also reasoned that if the two cars had the same velocity at one minute then they would therefore have the same distance traveled. Overall, these data showcase a rather insufficient understanding of rate of change and how it relates to accumulation, supporting the findings of previous researchers that students do not tend to have a robust conceptual understanding the connections between rate of change and the definite integral (Bajrachara & Thompson, 2014; Beichner, 1994; Thompson, 1994).

Previous research has shown students conflate rate of change and total amount (e.g. Arleback, Doerr, & O'Neil, 2013; Beichner, 1994; Monk, 1992). Beichner (1994) highlights that one of the "most common errors students make when working with these kinds of graphs are thinking that the graph is a literal picture of the situation" (p. 751). The kinematics task used in this study has been used in various forms in other studies. For example, Monk (1992) found that students tended to interpret the intersection of the velocity curves as implying that the cars will be in the same place or that one car will be passing another. Our results support and expand upon this finding as we saw that some students making this claim were actually reasoning about the rates of change and not just interpreting the graphs as if they were giving a distance or a position. For example, one student, Jake, stated that "because they've got the same velocity and they've got the same time so, say velocity is five and time is one, you divide that to get um, the distance and it's gonna be the same." Jake focused primarily on the equation distance equals rate times time, which he seemed to have been using extensively in a physics course.

When asked directly about the differences between Tasks 2 and 4, students noted some interesting distinctions. Mary felt that the plant context was "more concrete," Tom claimed that area under a curve was easier to conceptualize while solving the plant task but that he felt more comfortable with *Task 2* because he had seen tasks like that in the past. Andy indicated that *Task 2* was easier because "you can picture [velocity] in your mind, at least I can personally picture. It's harder to picture plant growth, or plant growth to population." Similarly, Jake also claimed

Task 2 was easier because of his experience with physics. When asked whether one task was easier than the other he said, "I think they're about the same. Maybe, I think maybe this one [*Task 2*] because I was more used to doing that." However, physics was not a universally positive influence on students' work on the tasks as Gina claimed *Task 4* was easier since *Task 2* "reminds me of physics and I failed physics." Clearly, students felt differently about the two tasks even though they were very similar mathematically. This finding is in line with previous research that has shown students have more familiarity with kinematics tasks than with contexts like work (Ibrahim & Rebello, 2012) and area and height (Herbert & Pierce, 2012).

Conclusions

Our goal with this study was to answer the question: How do students majoring in the biological and life sciences solve two calculus accumulation tasks which differ primarily by context? Students seemed to interpret the graphs differently, where they were more likely to reason about the rate of change represented in the kinematics task as compared to the plant growth task. This study lays a foundation for building a corpus of knowledge on the ways students reason about rate of change and accumulation within contexts meaningful to the biological and life sciences. Much more work needs to be done to explore these contexts more fully and we recommend a few directions for further inquiry.

First, teaching experiments and design research should be completed to develop good tasks and further our understanding of student reasoning within the context of population biology. Additionally, more work needs to be done to explore how students reason in other relevant contexts for calculus. We have so many fruitful areas to pull from as calculus educators and we do our students a great disservice to only utilize one or two contexts, like kinematics, in our calculus courses. In their work on problem solving, Carlson & Bloom (2005) include orienting oneself to the given task as part of their problem-solving framework and Moore & Carlson (2012) have noted that students will utilize their image of the problem context in the refinement of their solutions. This implies that students' background knowledge concerning the contexts we choose in our calculus courses plays a pivotal role in whether they are successful, before and while they apply their calculus knowledge. Finally, we recommend calculus instructors to start branching out and looking for myriad contexts they can use for changing quantities, specifically those that are most relevant to their students. Students' backgrounds are not utilized to their greatest potential when we do not present them with calculus tasks situated in their own fields of inquiry.

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Examining Questions as Written Feedback in Undergraduate Proof-Writing Mathematics Courses

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Abstract

The practice of providing written feedback on an undergraduate student's proof in the form of asking questions is striking in that professors do not know whether the student attempts to answer the questions. This phenomenon leads us to investigate the reasons why professors ask questions as written feedback. We analyze the written questions of four professors teaching abstract algebra and real analysis at a medium-sized, rural, comprehensive public university in the northeast. We find that these four professors most frequently ask questions that either seek further explanation from students or address a mathematical detail within their proof. In some cases, the professors answer the questions they ask as written feedback. Overall, the professors ask questions as written feedback to encourage students' thinking, thereby engaging students in the proof-writing process and improving the students' proof production skills.

Keywords: Feedback on Proof, Questioning, Proof Instruction, Written Feedback

Introduction

Professors who teach undergraduate courses often give students written feedback on their writing and assessments, but some feedback results in miscommunication between the professor and student (Price, Handley, Millar & Donovan, 2010). Many students find written feedback unhelpful or are uncertain of how to use to it (Amrhein & Nassaji, 2010; Price, et al., 2010; Vardi, 2009). In proof-writing mathematics classes, students misinterpreted the intention of written feedback (Byrne, Hanusch, Moore, Fukawa-Connelly, 2017). As professors report spending considerable time writing feedback on student proofs (Moore, 2016), it is desired that the feedback will maximize student understanding. The first step in evaluating the effectiveness of feedback is to research the current written feedback practices of collegiate mathematics professors, which responds to the call from Speer, Smith & Horvath (2010) for additional research on collegiate mathematics instruction.

We defined written feedback as the language and annotations that a professor leaves on a student's written work. Two types of written feedback are excluded from our analysis: check marks and numeric scores. Check marks are excluded because they convey little information, except that the professor read and accepted that component of the proof. Although feedback is often provided to justify numerical scores (Glover & Brown, 2006; Price et al., 2010), we disregard numerical scores. This decision is for privacy concerns and FERPA regulations, but also because inconsistencies were found between the feedback professors provided and the scores they assigned, as professors assigned significantly different scores to the same proof (Lew & Mejía-Ramos, in press; Moore, 2016; Miller, Infante, Weber, 2017).

This paper is part of a larger project examining all written feedback on proofs, and this paper examined the phenomenon of a professor asking a question as written feedback. This subset of the feedback had an inherent contradiction, as the question should be answered, yet none of the professors required the students to formally answer these questions. An example of this phenomenon is shown in Figure 1, where the student introduced a new variable on an abstract algebra homework assignment but failed to describe the properties of this variable. The professor chose to use the question “What is x?” to provide feedback.

Furthermore, professors occasionally answer their own questions in writing on their student’s paper. Sometimes the answer is provided as a statement, but in other cases, such as in Figure 1, the answer is provided in a more detailed follow up question.

In this article, we addressed the following research questions:

1. What types of questions do professors write on student proofs?
2. Why do professors choose to ask questions as a form of written feedback?
3. Why do professors choose to write answers for some of the questions they ask?

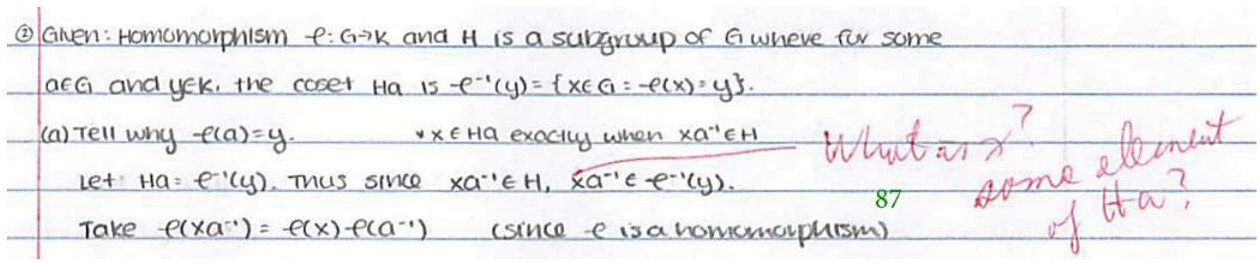

Figure 1 An instance of using a question as written feedback.

Literature Review

Theoretical perspective of feedback

Evans (2013) proposed a constructivist model which views feedback as an exchange between the instructor and student, termed the feedback landscape. Within this landscape, all instructors and students interpret the feedback and work through a personal buffer zone which is informed by the individual’s social and cognitive factors. Since these buffers are individualized, it is within these zones that the intended meaning of the feedback can be lost.

Theoretical perspective of questions in instruction

Rowe (1986) described classroom interactions as a game with two players: the teacher and the set of students. This game has four moves:

1. Structuring: giving directions, stating procedures, suggesting changes.
2. Soliciting: asking questions.
3. Responding: answering solicitations, expanding on a structuring move, reporting data, or continuing a line of reasoning.
4. Reacting: evaluating statements made by self or other player (Rowe, 1986, p. 46).

These moves can be initiated by any player and satisfaction is highest when each player utilizes all four moves. Rowe argues that when the teacher increases the wait time between moves 2 and 3 and between moves 3 and 4, then the students complete moves 1, 2 and 4 more frequently.

For this study, we viewed these four steps as a complete questioning sequence, no matter who completed each move. We focus on written assessments (homework, quizzes and exams) where the professor chooses the proof tasks, the student responds in writing, and then the professor reacts through the written feedback. As such, the professor always completes moves 1, 2 and 4, and the student always completes move 3. However, during move 4, the professor sometimes initiated a new questioning sequence by asking a question within their feedback. These sequences are usually incomplete, as moves 3 and 4 may not be completed.

Written feedback on writing

Several studies investigated written feedback on writing in undergraduate courses. University science students found written feedback on their work useful, especially feedback that helped them to understand where they had gone wrong (Brown and Glover, 2006). Walker (2009) expressed that written comments should be classified as feedback only if it is "usable" or can be implemented by students. Unfortunately, many studies concluded that a high proportion of comments are considered unusable to students and recognized the need for an improvement in the practice of commenting on written assignments (Amrhein & Nassaji, 2010; Mulliner & Tucker, 2017; Vardi, 2009; Walker, 2009). The present study took a step toward improving feedback in the context of proof writing in undergraduate mathematics courses, although we do not investigate the student perspective directly at this time.

Written feedback on proofs

While several studies examined written feedback on writing assignments, only a few focused explicitly on feedback on mathematical proofs. Professors valued several characteristics when evaluating student proofs, including logical validity, clarity of writing and demonstration of understanding (Moore, 2016). Linguistic and notational conventions are also valued by the professors, such as using $\emptyset$ to mean "the empty set" rather than just the word "empty" (Lew & Mejia-Ramos, in press; Moore, 2016). Scoring varied greatly between professors, with ranges up to 48% observed on the same proof (Miller, Infante & Weber, 2018; Moore, 2016). The professors assigned scores based on their perceptions of student thinking, the severity of the error, and whether the proof was written in a timed or untimed setting (Lew & Mejia-Ramos, in press; Miller, Infante & Weber, 2018, Moore, 2016). These three studies established that grading and providing feedback are complex practices with competing priorities and beliefs.

Only one study, Byrne, et al. (2018), investigated feedback from the student's perspective. In this study, the undergraduate students interpreted the written feedback on sample proofs, and then rewrote each proof incorporating the feedback. The students usually addressed the feedback in the rewrite, even when they could not express the rationale for the comment. Furthermore, the students attributed much of the feedback to linguistic conventions in mathematics, even when the feedback addressed the logical validity of the proof.

All four of these studies utilized clinical interviews, and none occurred in a classroom setting. The clinical setting removed genuine communication from the feedback process and restricted the opportunities to observe the buffer zones of the faculty and students. Additionally, the previous studies focused on proofs at the transition-to-proof level. This study, in contrast, used the feedback given by professors to their own students and more accurately reflected genuine instructional practice. The mathematical content of real analysis and abstract algebra added an additional layer of complexity that allowed us to see how feedback on the mathematical content interplayed with feedback on general proof techniques and proof writing.

Questioning in class

While there is significant research on the value of written feedback, no research focused on the specific phenomenon of providing feedback in the form of a question. On the other hand, many researchers found oral questioning to be valuable in the K-12 classroom, especially for probing student thinking and promoting higher-order thinking (Acar & Kilic, 2011; Almeida, 2010; Burns, 1985; Martino & Maher, 1999). We note that Speer et al. (2010) established that instructor questioning practices have not yet been researched within collegiate mathematics classrooms.

Methods

Subjects and data sources

The subjects in the study were two instructors of abstract algebra and two instructors of real analysis at a comprehensive undergraduate institution, with one section of each course offered in a fall semester and one section of each in a spring semester. Each professor had taught the course multiple times previously and held the rank of associate professor or professor. The professors maintained full control over their course during the study, including the textbook, the nature and frequency of assignments and assessments, and how they chose to give feedback to students. The graded papers were scanned before being returned to students. The papers were then redacted to remove all identifying information about the institution, the professor, the students, and to remove grade information. Table 1 shows the number of student participants, the number of homework assignments, quizzes, and exams that were collected in the course, and the total number of questions asked in writing by each professor.

Table 1 A summary of the participants and the items considered in the study.

Course	**Fall Algebra**	**Fall Analysis**	**Spring Algebra**	**Spring Analysis**
Professor	A	B	C	D
No. of students	5	10	15	8
No. of HW/Quiz/Exam	5/6/2	9/10/2	24/3/4	24/0/2
No. of questions asked	59	134	128	247

Analysis technique

After redacting each document and numbering the feedback, we assembled a spreadsheet containing the text of every question. We did not include question marks with no text, because a question mark conveyed significantly less information to the students than a question with text.

We utilized the constant comparative method (cf. Creswell, 2013), to sort the questions into clusters. Eventually we established five clusters: *drawing attention to detail, seeking further explanation, questioning assumptions, expressing confusion*, and *addressing proof structure*. We note that the descriptions of the clusters are not mutually exclusive, and we made a judgment call regarding which description seems most reasonable when more than one cluster applies.

After our initial coding, we interviewed all four professors asking them to describe why they chose to write questions as feedback in general. Then, we asked each professor to review a purposeful sample of 12 items of feedback, and to provide an explanation as to why they chose to use a question for the feedback. We used the interviews to triangulate our coding and found that the professors' descriptions aligned with our coding in all but five cases. In each case, we originally considered multiple clusters, but the professor emphasized a different cluster than the one we assigned. In such cases, we changed the cluster to match the professor's description.

Results

Types of questions

The analysis process resulted in five clusters for classifying written questions*: drawing attention to details, seeking further explanation, questioning assumptions, expressing confusion*, and *addressing proof structure*. We defined *drawing attention to detail* as questions that ask if the details provided by the student are sufficient. Within this category, there were two subcategories: mathematically focused and language focused. Mathematically focused questions address the student's use of notation or computational work, whereas language focused questions pertain to specific word choice and phrasing made by the students.

Since justification is a cornerstone of proof, many questions *sought additional explanations* from the students. A common example of this type of question was "why is that true?" or "how do you know…", and an indication as to which line needs the explanation. The cluster, *questioning assumptions*, concerned questions that point out false assumptions made by the student. This cluster differed from *drawing attention to detail* in that *questioning assumptions* pointed out false assumptions that were made or cases that were forgotten. Some *seeking explanation* questions appear to question assumptions, but in those instances the assumptions made are typically true and simply require further explanation.

The *expressing confusion* cluster contained questions where the professor indicated confusion about the student's writing. Common questions included "what does this mean?" and "huh?" The professors also used question marks to indicate confusion such as Professor B who claimed, "But a question mark by itself really means, 'This doesn't make any sense.'"

Finally, questions in the *addressing proof structure* cluster addressed the choices made by the student regarding the type of proof or the completeness of the proof. These questions focused on the framework of the proof, instead of being detail or explanation focused.

Each professor asked questions in each cluster, except professor A. Across the four classes, seeking explanation and mathematically focused questions were the most prevalent. These findings were unsurprising as we expected students to support their claims in proof-writing classes and to occasionally make errors using new notations and concepts.

Table 2 A summary of the types of questions asked as written feedback for each professor.

	Fall Algebra n=59		**Spr. Algebra** n=129		**Fall Analysis** n=134		**Spr. Analysis** n=247		**Overall** n=569	
Detail-Math	n= 15	25.4%	n=33	25.6%	n= 45	33.6%	n=100	40.5%	n=193	33.9%
Detail-Lang	n= 3	5.1%	n=12	9.3%	n=12	9.0%	n=5	2.0%	n=32	5.6%
Detail-Other	n=0	0%	n=0	0%	n=0	0%	n=2	0.8%	n=2	0.4%
Explanation	n= 29	49.2%	n=66	51.2%	n=37	27.6%	n=99	40.1%	n=231	40.6%
Assumptions	n=3	5.1%	n=3	2.3%	n=33	24.6%	n=21	8.5%	n=60	10.5%
Proof Struct.	n=0	0%	n=8	6.2%	n=1	0.7%	n=14	5.7%	n=23	4.0%
Confusion	n=9	15.3%	n=7	5.4%	n=6	4.5%	n=6	2.4%	n=28	4.9%

Although the sample is too small to support generalization, the data suggested that course subject may impact the types of questions asked by the professors. In both algebra classes, roughly half of the questions sought explanations from students, and drawing attention to the mathematical details accounted for another quarter of the questions. In contrast, the seeking explanation cluster was a smaller portion of the questions asked in the analysis courses and drawing attention to mathematical detailed was a larger portion of the questions asked.

Why professors ask questions as feedback

While all the professors asked their students questions, they have not examined their reasons for asking a question as opposed to another form of written feedback. Professor D explained, "I don't know if it's a thoughtful, considered decision. 'Let's see, should I ask a question, or should I write a declarative statement?' I don't know if I'm thinking about it that carefully." When asked about his decision to pose a question, Professor C repeatedly responded, "I think that's just a personal style."

Professors also asked questions to alert students that their work is unsatisfactory in a "non-insulting way." Professor C explained, "It's also the same as telling them that I think something is inadequate without saying it that way." Similarly, Professor A said she sometimes avoids explicitly telling her students they are wrong, preferring to pose a question. Therefore, questions are perceived by the professor as a less harsh method of critiquing students' work.

Collectively, the professors stated that they asked their students questions, so the students will reflect upon their work. Professor D claimed he asks his students questions to

> … guide them to ask the right question to kind of correct their mistake by just knowing which question they should be asking themselves. Because, I guess as I think about it, having been a math student myself for many years, sometimes if you know the right question to ask yourself, you're well on the way to answering the question correctly.

The professors used questions as a mechanism to guide students to improved self-reflection, with the aim of improving future proof-production.

Asking questions as written feedback gives professors the opportunity to stimulate students' thought process and lead students to correct solutions. Professor D attempted to "redirect [students'] thinking by asking them a question that maybe would get them on track." Professor C asked questions as written feedback because "it's something I think a student ought to think about." Therefore, professors asked questions as written feedback to encourage students' thinking with the intent of students arriving at the answer on their own.

Answering questions

The professors provided written answers to 6% of the questions to ensure their students learn from their errors. Professor C claimed, "Well, I think it's just a form of, of telling them, 'You've got something wrong, and here's the direction you should have gone.'" Professors A and B claimed that they answer their own question when they observed students repeatedly making the same error. Professor B concluded, "Maybe if I think there's something particularly tricky going on, or maybe if I think a particular student is persistently making the same kind of error, then I wanna make sure they understand what I'm trying to say." Professor A emphasized that she may answer questions "to help them think about what they might have done incorrectly." Thus, the professors believe they answered questions to ensure the student gets an answer.

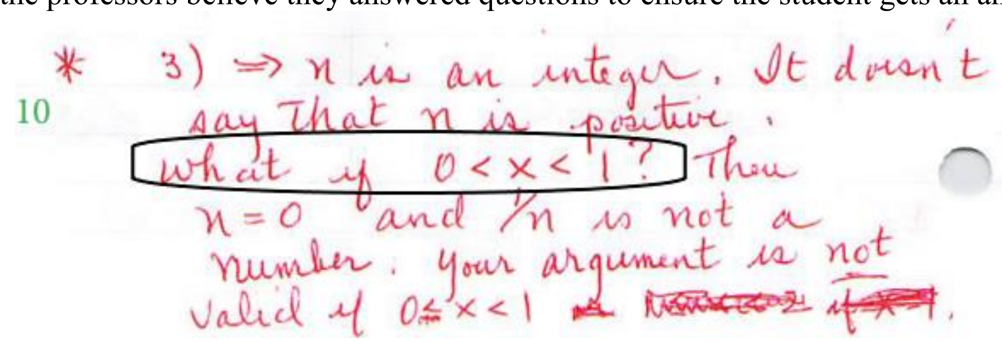

Figure 2 An example of asking and answering a question, attributed to stream of conciousness.

Some of the professors described asking and answering questions in a stream of consciousness. On a homework assignment in Professor D's class, a student used the variable *n* as a positive integer, even though the task needed to be proven for any integer. Professor D questioned the student's assumption by asking, "What if $0 < x < 1$?" as seen in Figure 2, and explained to the student what would happen if that were true, thereby answering the question for the student. Professor D examined his practice, and said

> So, I guess I could have asked him to think about a case where it's not true. But, see, actually, when I mark a statement like this, I just naturally, as I read this, I, myself, give a counterexample to show that the student's logic is not true. So I guess I just put it down on paper. Might have been just a gut reaction to write that. I don't know if it implies that I wouldn't trust the student to create their own example, to show that what they've written is not true. I guess it does. Because maybe if it was an easy one, I would just say, 'Figure it out.' But here, I thought I should maybe say a little bit more.

Here, Professor D gave multiple reasons for why he chose to answer the questions he posed to students. First, he explained that answering his own question is not always intentional, but a "gut reaction." Professor D also reasoned that answering the question is also appropriate, if it is probable that the student cannot come to the solution on their own. Finally, Professor D concluded that answering the question allowed the student to see the correct solution.

In general, professors answered the questions they pose to improve students' proof production skills. Specifically, professors answered their questions to draw attention to an error with the expectation of the student not making the same mistake again, and to emphasize course concepts. Finally, the act of asking and answering the question may simply be a stream of consciousness. Regardless of the reason for answering the question, professors asked questions as written feedback to enhance students' mathematical understanding and proof writing skills.

Discussion

Written feedback is a common instructional practice in upper-level mathematics courses to help students improve their proof writing (Moore, 2016). In this paper we investigated the practice of leaving written feedback in the form of a question. We found five clusters of questions: *drawing attention to details, seeking explanations, questioning assumptions, structuring the proof,* and finally, *expressing confusion*. Additionally, we presented two explanations for why the professors leave feedback in the form of a question. First, the professors asked questions to prompt the students to think, including training the students to ask questions themselves. Second, the professors claimed the questions mirror their thought process and personal grading style, including the desire to communicate corrections less harshly. Finally, we presented two explanations for why professors occasionally answer the questions they ask as feedback: to ensure students learned from their errors by having access to the answers to the questions, and because the professor asked and answered the question for themselves during the marking process.

The findings of this study are consistent with the research on oral questioning in the classroom as the professors asked written questions for the same reasons they asked questions in classrooms, specifically to probe students' thoughts and to encourage reorganization of students' thoughts (Ellis, 1993; Martino & Maher, 1999). Thus, professors asked questions as written feedback for student self-reflection and to promote higher-order thinking.

Questions as written feedback have limitations in their usefulness because the questioning sequence is incomplete. The students were not asked to revise and resubmit their proofs in any of the classes in this study, and as such, the responding and reacting move did not occur. The incompleteness may explain why the professors occasionally chose to answer their own questions; they desired to complete the moves of the questioning game.

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The Role of Multiplicative Objects in a Formula

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The goal of this article is to propose a way to think about the role of a multiplicative object in reasoning about formulas quantitatively and covariationally. Building off the works of others on the importance of constructing multiplicative objects when reasoning about graphical representations, I adapt their definitions to be able to include a meaningful way to discuss what it means to construct a multiplicative object with a formula. I then use the analysis of six sessions of a semester-long teaching experiment with a preservice secondary mathematics teacher to illustrate what it means not to construct and what it means to construct a multiplicative object with a formula.

Keywords: Cognition, Precalculus, Preservice Teacher Education

One of the upcoming avenues of research in the quantitative reasoning literature is studying the role the construction of a multiplicative object has in a meaning for a graph "as a continuum of states of covarying quantities" (Saldanha & Thompson, 1998) (e.g., Frank, 2016, 2017, in press; Stevens & Moore, 2017; Thompson, 2011; Thompson & Carlson, 2017). In this paper, I build on the research done with graphical representations by discussing the role constructing a multiplicative object has in a meaning for a symbolic representation (namely, a formula) that represents the varying measures of attributes identified in a situation. I propose a way to conceive of a multiplicative object with a formula. I then demonstrate the role of conceiving of a multiplicative object when constructing a formula to represent quantities in a situation. To do so, I will use the results of a four-month long individual teaching experiment designed to support a preservice secondary mathematics teacher's covariational reasoning and construction of formulas through dynamic geometric environments.

Background

What is a Multiplicative Object?

The notion of a multiplicative object first stemmed from "Piaget's notion of 'and' as a multiplicative operator—an operation that Piaget described as underlying operative classification and seriation in children's thinking" (Thompson & Carlson, 2017, p. 433) (e.g., Inhelder & Piaget, 1964; Piaget, 1970). Frank (2017) described Inhelder and Piaget's notion of a multiplicative relationship as schemas that invoke an image of simultaneity. The general idea is for an individual to construct a new attribute that simultaneously incorporates two other identified attributes. For example, Frank (2017) noted that a person can conceive of objects that are red, objects that are circular, and simultaneously, objects that are red circles. The final object is a uniting of the two other attributes, and thus, involves a multiplicative operator.

Saldanha and Thompson (1998) extended the idea of multiplicative objects by discussing it in terms of quantities (i.e., measureable attributes). For Saldanha and Thompson (1998), a multiplicative object involves constructing pairs of values. They described it as entailing a coupling of two quantities so that "one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p.

299). In 1990, Thompson defined a quantity's value as "the numerical result of a quantification process applied to it," which at that time to him meant that either "direct or indirect measurement" was taking place. He has since updated his definition of quantification (see Thompson, 2011), but in that update, he did not offer a new definition for a value. Thus, I offer an updated definition of values that is rooted in the understanding of quantities' magnitudes as Wildi magnitudes (see Thompson, 2011; Thompson, Carlson, Byerley, & Hatfield, 2014; Wildi, 1991). He argued that a quantity's magnitude (or amountness) is invariant of the unit used to measure it. I argue that the amount is the same regardless of the unit, but the value of a quantity necessarily depends on the unit chosen to measure it. Thus, a quantity's value refers to an obtained or anticipated measure of a magnitude using a defined unit magnitude for the quantity. The resulting measure is expressed numerically.

For the sake of clarity, when I refer to magnitudes, I refer to students' images of quantities or unmarked bars representing the students' conception of that quantity's amountness (e.g., the red and blue bars in Figure 1) without explicit attention towards units. When I refer to values, I refer to measurements (using either assumed or anticipated units) expressed numerically or symbolized within formulas. Thus, I update Saldanha and Thompson's (1998) definition of a multiplicative object by replacing "value" with "magnitude" in order to distinguish between reasoning about quantities vs. measurements. That is, a multiplicative object entails a uniting of objects so that one tracks a quantity's magnitude with the immediate, explicit, and persistent realization that, at every moment, the other quantity (quantities) also has (have) a magnitude(s).

Researchers have primarily discussed multiplicative objects in the context of graphing activities (Frank, 2016; (Frank, 2016; Stevens & Moore, 2017; Stevens, Paoletti, Moore, Liang, & Hardison, 2017). Frank (2016) discussed how to conceptualize a point in the Cartesian coordinate system as a multiplicative object. Figure 1 shows two quantities' magnitudes represented on a pair of axes. The plotted point on the graph is the result of the uniting of the two quantities. Thus, each point on the graph represents the magnitudes of two quantities simultaneously. The result can be expressed as values in a coordinate pair using (x, y). Students do not always interpret a point in a Cartesian plane as representing a multiplicative object (Frank, 2017; Stevens & Moore, 2017), and in this paper, I will demonstrate that the difficulty of representing multiplicative objects extends into reasoning with formulas as well.

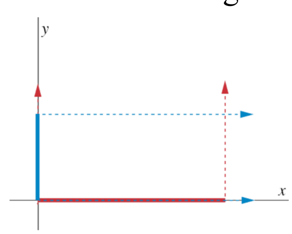

Figure 1. Frank's (2016) image of a projection of two quantities' magnitudes represented on axes and then projected to construct a single coordinate pair.

Why is a Multiplicative Object Important in Covariational Reasoning?

Based on the definition of a multiplicative object, there is an understanding that as two quantities' magnitudes covary in Figure 1, the resulting location of the point will change with it. How students reason about the covarying of the two quantities is split into six levels of covariational reasoning (Thompson & Carlson, 2017, p. 441). Covariational reasoning, in general, occurs when students conceive of situations as composed of quantities that vary in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), and researchers have deemed it important to understanding ideas about rate of change (Ellis, 2007; Johnson, 2015; Oehrtman, Carlson, &

Thompson, 2008; Thompson, 2011). A student cannot be classified in the top three levels of covariational reasoning if she has not constructed a multiplicative object. In Frank's (in press) study of interviews from three pre-calculus students, she noted how the two students who engaged in emergent shape thinking (i.e., constructing a graph as an emerging representation of a covariational relationship) attended to the quantities' values represented on the axes (i.e., the blue and red bars) as a way to help them conceptualize two attributes uniting. They represented this uniting by constructing a coordinate pair (x, y). She and others have concluded about the importance of supporting students in organizing images of varying quantities to construct meaningful representations. In the following section, I discuss how the idea of a multiplicative object is relevant to the construction of symbolic representations; namely, formulas.

To illustrate an example of the process of constructing a multiplicative object and then reasoning covariationally, I use the city task in Saldanha and Thompson (1998) in which students "engaged in a sequence of tasks centered around the activity of tracking and describing the behavior of the distances between a car and each of two cities as the car moves along the road" (p. 300) (Figure 2). In looking at the situation, the two quantities in the situation are highlighted using dotted line segments; namely, they are the respective distances the two cities are from the car. The quantities are represented as perpendicular magnitudes to the left of the image, isolated from the remainder of the situation. By identifying unit lengths for the quantities, it is possible to construct a Cartesian coordinate system by partitioning along the magnitudes and beyond. (In doing so, the student also has a unit magnitude identified with which to produce values.) The point P in this Cartesian coordinate system now represents the correspondence of the magnitudes of the distances between both cities. Based on Thompson and Carlson's (2017) levels of covariational reasoning, this correspondence is the first evidence that a multiplicative object has been constructed. To reason covariationally with this newly constructed multiplicative object, the student must anticipate changes in the magnitudes situated on the axes resulting in changing the correspondence point (i.e., the multiplicative object) as the car travels. If the point P is traced, a graph relating the two quantities emerges. For more details on the construction of the graph, see Moore and Thompson (2015).

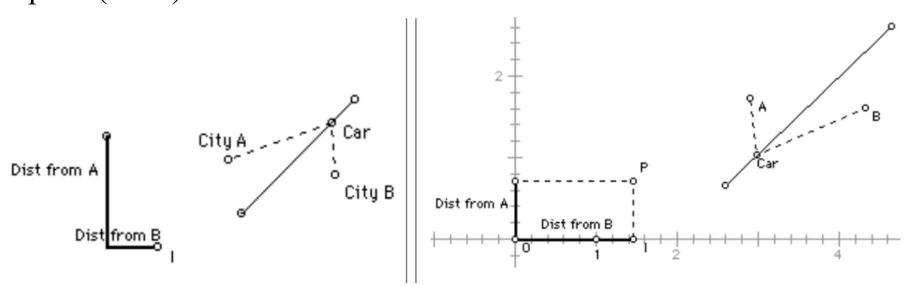

Figure 2. Saldanha and Thompson's (1998) image of the City Travels Problem.

How Does Constructing a Multiplicative Object Support Quantitative and Covariational Reasoning with Formulas?

The previous example motivates a need for students to unite cognitively two quantities' measures. For graphical representations, the purpose is clear; in order to construct a quantitative image of a graph, the student must construct a point P as a multiplicative object. What is unclear in the literature is how the role of a multiplicative object plays a role in either reasoning with formulas.

Consider a known formula: $A = ½bh$, a commonly presented formula for the area of a triangle. Students first use this formula in the 6^{th} grade "to find the area of right triangles,

triangles... by composing into rectangles or decomposing into triangles and other shapes" (National Governors Association Center for Best Practices, 2010). To do so, students identify a measure for a base, b, and its corresponding height, h, to calculate the measure for the area of the triangle, A. In this context, there is no intellectual need for a multiplicative object because there is no variation in the quantities. However, consider the case in which the measurement of the triangle's height varies. Then, simultaneously, the measurement of the triangle's area also varies. To use formulas to represent this covariation of quantities, a student must be able to unite the values of both the height and area of the triangle within the formula so that the united image of the quantities persists through the variation.

There are a few difficulties to consider when constructing a multiplicative object of a formula rather than a graphical representation. First, there is no single object within the representation that simultaneously represents the two quantities' measures as there is in a coordinate system. Rather, the uniting is an anticipation the student has that for any given instance of a triangle, there is a single pair of values (assuming the student has established units) to represent that instantiation. Secondly, and relatedly, there are no magnitude bars present in a formula; in a Cartesian graph, a student can identify magnitudes representing the values of quantities on the pairs of axes, and these magnitude bars change as the values for the quantities change. For a formula, however, glyphs (i.e., symbolic inscriptions) represent the values of the magnitudes of the quantities and these symbols do not alter as quantities in the situation vary. It is left to the student to have a meaning for those symbols that enables them to anticipate changing quantities' values in either their image of the situation, of corresponding magnitude bars, or a sequence of numbers that the individual can imagine running through (Oehrtman et al., 2008). Lastly, in the same way that a situation has quantities a student has to push to the background of their mind so that they can instead focus only the two quantities under consideration, in a formula, the student must isolate the symbol (or group of symbols) that represent the two quantities under consideration. Figure 3 illustrates this idea by using colors to bring attention to the two quantities (height and area) that the student attempts to reason about covariationally (Figure 3). One can imagine the colors shifting to different quantities represented both in the image and the formula as the student conceptualizes varying different pairs of quantities.

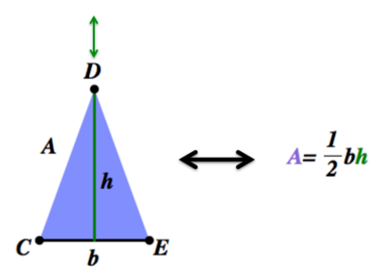

*Figure 3. Constructing a multiplicative object between the height and area of a triangle using the formula **A**= ½ b**h**.*

Methods

I explored how students construct and use multiplicative objects with formulas as part of a semester-long teaching experiment with three undergraduate students in a preservice secondary education mathematics program at a large public university in the southeastern U.S. The reason I chose preservice teachers is because of their vast mathematical experiences and their commitment to understanding secondary mathematics ideas through their undergraduate study. During the study, these students were enrolled in a course based on the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015) in which they learned about quantitative and covariational reasoning. Each student participated in 12-15 teaching sessions, totaling 18-19.5 hours of interview time per student. I video-recorded and screen-captured students' work on a tablet and made scans of student work. At least one observer was present at all but one interview. During and after each interview, we took notes of students' activities and planned future teaching sessions. Throughout the sessions, the importance of constructing and using multiplicative objects emerged, and thus, the analysis for this portion of the study focused on students' development of that idea by coding videos of the data. In this particular study, I focus on Lily's meanings for her formulas through the theoretical lens of her construction and use of a multiplicative object with two known formulas. I also attended to her levels of covariational reasoning based on Thompson and Carlson's (2017) framework. I limited the analysis to Lily's first six interviews because it was in these interview that she was working on problems with familiar formulas and first constructing multiplicative objects. I conducted a conceptual analysis (Steffe & Thompson, 2000) so that I could develop second order models of her thinking.

Task Design

In the first sessions of the teaching experiment, I updated a task based on the results of a previous study with preservice secondary teachers (Stevens, 2018). The task consisted of three parts, one given at each of the first three teaching sessions, each with the same starting prompt: "How would you describe the relationship between the height and area of isosceles triangles?" I particularly limited them to isosceles triangles in an attempt to limit the images students could have of what it would mean to vary the height of the triangle. In the first part, I gave the prompt without any other associated image. In the second part, I asked the question with a given static triangle. In the third part, I asked them to consider what would happen if the height of the triangle changed, providing them with a sketch created with dynamic geometry software in which they could drag one of the vertices of the triangle to change its height (as in the triangle in Figure 3 but without the green segment visible).

Lily started working on the Painter Problem in her fourth interview. This problem is similar to the growing rectangle problems other researchers have used (Ellis, 2011; Kobiela, Lehrer, & VandeWater, 2010, May; Matthews & Ellis, in press; Panorkou, 2017). In this problem, I ask the student to "relate the length that Kent [who is painting a wall in his home] has pulled the paint roller and the area that he has covered in paint" (Figure 4).

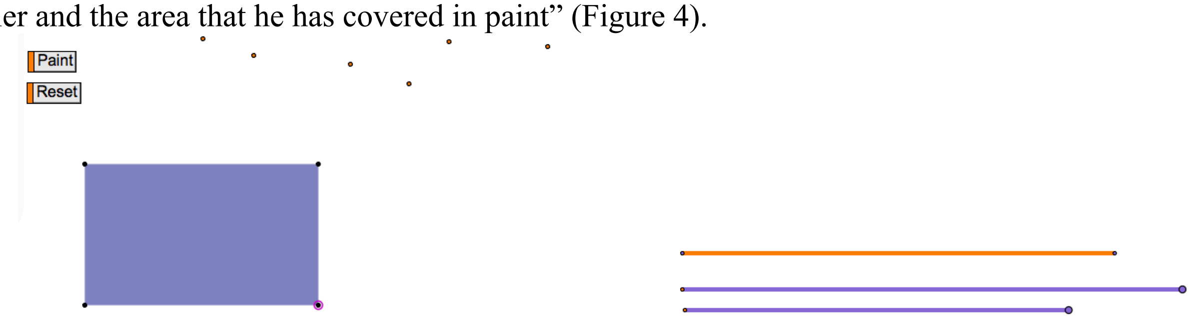

Figure 4. The Painter Problem.

Results

In the following section, I report on the results of the teaching sessions. The results are split into two parts characterize students' thinking as it relates to constructing and using multiplicative objects as it relates to formulas. First, I describe instances in which Lily did not construct a multiplicative object with her formula and then I describe Lily's first construction of a multiplicative object with her formula in the Painter Problem.

No Multiplicative Object Constructed in a Formula

One of the main aforementioned components of constructing a multiplicative object is to isolate two quantities. In the Triangle Problem, students are asked to consider the relationship between two quantities, the height and area of a triangle. Lily, when given this prompt, quickly identified the formula for the area of a triangle as the normative $A= \frac{1}{2} bh$. However, she struggled to continue to relate the height and area because of the presence of the b in the formula. She wanted to express the relationship symbolically with only A and h symbols present. The following transcripts show evidence of this reasoning. The parentheses beside the name indicate which interview number the statement occurred.

Lily (1): This is the area formula. So we know that our area is 1/2 -- area of any triangle is 1/2 base, height. But this is asking for the relationship between the height and the area, so the base is kind of like a -- I mean, I guess I'm trying to say that it's like not explicitly just between the height and the area, and the base is like in that [formula].

Lily (2)*:* I [*sees image of static triangle*] – [*pause*] That's my triangle. [*pause*] Area. [*pause*] But I want to relate just the area to just the height, so I need to get rid of that [*b in her formula*]. Not get rid of it, but write it in terms of area and the height, because I'm specifically trying to relate [*sighs*] area and the height. So I'm going to do -- I didn't want to write *A*. [*pause*] Obviously area equals area but it blows my mind.

As illustrated in the two transcript excerpts above, Lily wanted to use a formula to represent the relationship between the two quantities. However, she was dissatisfied with the presence of a third quantity, b, in her formula. In the second excerpt, she attempted to “get rid” of it by solving for b in her area formula and then re-substituting it into the formula. This resulted in her writing $A=A$. She was not satisfied with the outcome because then only one quantity, A, remained in her formula, rather than A and h.

Lily’s reasoning here is an example of the importance of understanding that the relationship between two quantities might be influenced be a third quantity (or more), and that the presence of a symbol representing that quantity in a formula does not exclude that formula from representing the relationship between the two quantities under consideration. Because her meaning for formulas entailed an understanding that a formula relates all the quantities present in a formula, Lily did not isolate two quantities to construct a multiplicative object with her formula in the way illustrated in Figure 3.

Coordinating Values Between Quantities as Evidence for Construction of a Multiplicative Object

When given the Painter Problem in her fourth interview with the bars, Lily tried to solve for h (the length of the paintbrush) in her formula $A=bh$ (A= area painted, b is length rolled) as she did in the Triangle Problem. However, she suddenly switched to consider how her formula could be used to describe the directional covariational relationship she identified in the situation (i.e., as the length swept out increases, the area painted increases). She pointed to the b and A in her formula [*underlined in Figure 5*], wrote down the calculations on the bottom right of Figure 5 and stated the following:

Lily (4): Yeah, the height stayed constant and we just changed the base [*motioning along the orange highlighted base in Figure 5*], and as it got bigger, the area got bigger [*pointing to the results of her area calculations, 5 and 10*]. Just because there's more space too, that he painted [*motioning along rectangle*]. Like if you stop here [*draws in dotted line*], [*focusing on the rectangular image*] the base would be smaller and there's not as much of an area. But if it gets bigger, there's more of an area. This got bigger [*motioning along the* *orange highlighted base*]. This Dimension. And the amount [*motioning along rectangle*] got bigger as well.

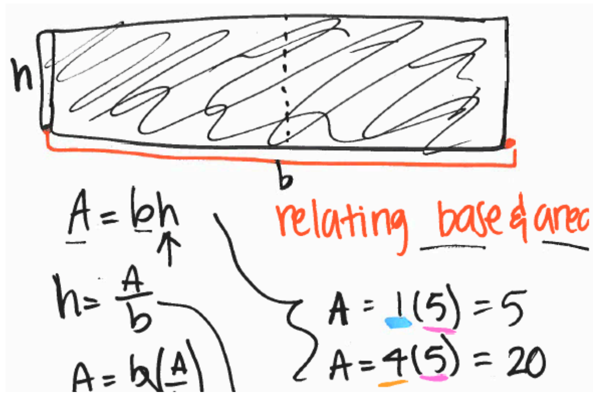

Figure 5. Lily's initial activity on the Painter Problem.

In order to isolate the two quantities in a formula, Lily connected how the quantities were changing or staying constant in the situation with her formula. That is, she noted that the height stayed constant in the situation, so h now also represented a constant (i.e., 5) to her. This idea enabled her to focus on what was changing, the base, and so she was able to consider different values for b. She then noted the results of her calculations as varying values for A, which she connected back to her situation by discussing "more space." Thus, her construction of a multiplicative object when she was able to coordinate different pairs of values for h and b with a connected image of how those different values corresponded to the quantities in the situation. It is important to note that Lily's activity here does not demonstrate that she envisioned changes in quantity's values, and so she can only be said to have a coordination of values rather than images of covariation. In fact, evidence for her using her constructed multiplicative object to reason in a way in which she could connect reasoning about amounts of change in her situation with values in her formula were not present until the end of her sixth interview. However, her coordination of values here is example of the first level to include the construction of a multiplicative object.

Conclusions and Discussion

Lily's activity over the course of six interviews demonstrated how her meaning for a formula developed as she was able to construct a multiplicative object within her formula that she could connect to her understanding of the dynamic situation. For her, writing the two calculations in Figure 5 was crucial to her conceptualizing the formula as able to represent pairs of values between two quantities using one formula that contained symbols for quantities that she was not trying to relate. I argue that providing Lily with a dynamic situation helped support her in accommodating her meanings for formulas in a way that enabled her to isolate quantities in her formula and construct a multiplicative object. Overall, I argue that in the same way that multiplicative objects are important for covariational reasoning within graphical representations, it is also important for symbolic representations, particularly formulas.

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Inquiry Does Not Guarantee Equity

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Our field has generally reached a consensus that active learning approaches improve student success; however, there is a need to explore the ways that particular instructional approaches impact various groups of students. Here we examined the relationship between gender and student learning outcomes in one particular context – abstract algebra, taught with an Inquiry-Oriented Instructional (IOI) approach. Using hierarchical linear modeling, we analyzed content assessment data from 522 students. While the performance of IOI and non-IOI students was similar, we detected a gender performance difference (men outperforming women) in the IOI classes that was not present in the non-IOI classes. In response to these findings, we present avenues for future research on the gendered experiences of students in such classes.

Keywords: undergraduate mathematics, gender, inquiry-oriented, assessment

Broadly speaking, 'active learning' approaches to instruction in undergraduate science, technology, engineering, and mathematics (STEM) classes have been tied to improved student success and learning, with Freemen et al.'s (2014) meta-analysis of 225 studies providing compelling evidence. Additionally, a number of more isolated studies have suggested that active learning may be more equitable for students from historically marginalized groups (e.g., Laursen, Hassi, Kogan, & Weston, 2014; Eddy and Hogan, 2014). For instance, Laursen et al.'s (2014) study found that students who took lecture-based mathematics classes exhibited substantial decreases in their mathematics self-efficacy with women disproportionately underestimating their ability. In contrast, the decrease in self-efficacy was less drastic for students in Inquiry Based Learning (IBL) classes and consistent across genders – perhaps helping to "level the playing field" (p. 415) for women and men.

However, the mechanisms linking active learning approaches to more *equitable* student outcomes are not well understood and the generalizability of these findings has been questioned (e.g., Hagman, 2017). In order to understand and replicate the positive results found for general student populations (e.g., Freeman et al., 2014) and the results for particular student groups (e.g., Laursen et al., 2014), it is important to identify the critical features of active learning that are empirically and theoretically linked to improved student outcomes. Indeed, Eddy and Hogan (2014) argue that any classroom intervention will impact different groups of students in different ways, and they extend Singer and colleagues' (2012) call for identification of critical features in order to explore the ways that particular approaches impact various student sub-populations.

In light of Laursen et al.'s findings, and in accordance with Eddy and Hogan's (2014) and Singer et al.'s (2012) call to better understand the ways in which particular instructional practices may impact particular groups of students, we examined the relationship between gender and student learning outcomes in one very specific context – abstract algebra, taught with an Inquiry-Oriented Instructional (IOI) approach supported through an ongoing and substantial professional development program. Our work draws on data collected in an NSF-funded project, Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES). The TIMES project, in an effort to support instructors learning to teach in an inquiry-oriented manner, provided participants with curricular support materials, summer workshops, and weekly online workgroups as they worked to implement a set of inquiry-oriented instructional materials. Here, we restrict our analysis to those instructors implementing the *Inquiry-Oriented Abstract Algebra (IOAA*) curriculum (Larsen, Johnson, & Weber, 2013). IOAA is a research-based, inquiry-oriented curriculum that actively engages students in developing fundamental concepts of group theory and is designed for use in upper-division, undergraduate abstract algebra courses.

We analyzed 522 completed Group Theory Content Assessments (Melhuish, 2015) to investigate performance differences between students whose instructors implemented the IOAA curriculum (with support from the TIMES project), and those who instructors did not. Specifically we address the following two research questions:

1) What is the relationship between inquiry-oriented instruction, as manifested by the TIMES program, and student performance on a content assessment?
2) Is this relationship consistent across genders?

Based on the work of Freeman et al. (2014) we would expect to find a performance advantage for the students in the IOI classes. Further, given the similarities between IOI and IBL, we expect to see Laursen et al.'s (2014) findings replicated in our study – i.e., we expected to see more differences between the performance of women and men in a comparison group than in the IOI population. Confirmation of these hypotheses would corroborate research supporting active learning in general and inquiry-approaches in particular, whereas contradictory findings might provide insights into the differential ways that particular instructional approaches impact various populations.

Literature Review

The intention of IOI is to reposition students as central to the process of constructing and reinventing important mathematical ideas. Informed by the instructional design heuristics of Realistic Mathematics Education, IOI curricular materials leverage students' informal and intuitive ways of reasoning as starting points from which to build more sophisticated and formal mathematical understandings (Freudenthal, 1973). Specifically, the IOAA curricular materials include instructional units on groups and subgroups, isomorphism, and quotient groups. Each unit includes both a reinvention phase and a deductive phase. During the reinvention phase, students work on a sequence of tasks designed to help them develop and formalize a concept. Initial tasks in the sequence evoke student strategies and ways of thinking that anticipate the formal concepts. Then follow-up activities, and teacher guidance, leverage these ideas to develop the formal concepts. The end product of the reinvention phase is a formal definition and a collection of conjectures. The students then prove theorems that are typical of those found in other introductory group theory courses (Larsen, Johnson, & Weber, 2013). The cycles of inquiry and formalization, supported by the task sequence and guided by the instructor, are usually carried out in collaborative small-groups and whole-class discussions.

Research carried out prior to the TIMES project suggests that IOI in general, and IOAA in particular, has the potential to improve student learning by supporting the development of more robust conceptual understandings (e.g., Larsen, Johnson, & Bartlo, 2013, Rasmussen et al., 2006) and by improving student retention (Kwon, Rasmussen, & Allen, 2005), as compared to students from more traditional courses. These findings from IOI courses align with the meta-analysis of

Freeman et al. (2014), which found that across undergraduate STEM courses "student achievement was higher under active learning" (p. 8411). They also align with the findings of a study on one form of active learning in undergraduate mathematics known as Inquiry Based Learning (IBL). Laursen et al.'s (2014) work found that "students in IBL math-track courses reported greater learning gains than their non-IBL peers on every measure" (p. 409). Further, Laursen et al. found that IBL may be more equitable for women, reporting that, even with equivalent success rates in subsequent math coursework, "in non-IBL courses, women reported gaining less mastery than did men, but these differences vanished in IBL courses" (p. 415).

Laursen et al.'s (2014) findings are particularly relevant for our work because of the similarities between IBL and IOI. Laursen et al. (2014) characterize IBL as follows:

> …students construct, analyze, and critique mathematical arguments. Their ideas and explanations define and drive progress through the curriculum. In class, students present and discuss solutions alone at the board or via structured small-group work… (p. 407)

As this description is fairly consistent with (though more general than) the conceptualization of IOI adopted in the TIMES project, we had reason to believe that IOI classrooms would similarly support a "leveling of the playing field" for women and men.

That being said, there may be aspects of IOI (but not necessarily of IBL) in which the opportunities for student experiences, shaped by their interactions with their peers and their instructor, to create a dynamic that may negatively impact students from historically marginalized groups. For instance, implicit bias (Hill, Corbett, & St Rose, 2010) and stereotype threat (Good, Rattan, Dweck, 2012) may impact the ways peers interact during small group work. Furthermore, whole class discussions are shaped by instructor choices. Such decisions have varying implications for how different students may experience the class.

When considering the gendered experiences of students in collaborative classroom settings, there is reason to believe that these setting offer both affordances and constraints for women. Some literature suggests that classrooms emphasizing collaborative work, problem solving, and communication may be supportive for women (Du & Kolmos, 2009; Springer, Stanne, & Donovan, 1999). Moreover, there is research suggesting high school girls acclimate better than boys to learning environments that emphasize work on open-ended problems and conceptual understanding (Boaler, 1997; 2002). However, other research suggests that instructional approaches requiring students to develop their own problem-solving strategies may favor boys and men (e.g., Fennema, Carpenter, Jacobs, Franke, & Levi, 1998). Hyde and Jaffee (1998) offered a possible sociological explanation of such findings: the use of standard algorithms aligns with traditionally-valued feminine traits like compliance and meekness, whereas the use of invented strategies aligns with traditionally-valued masculine traits like confidence and independence. Research on the nature of social interactions in collaborative decision-making and facilitated discussions also offer insights into the way students may experience mathematics classrooms in gendered ways. Studies in non-mathematical collaborative settings have found that, when groups are tasked with arriving at a decision, women in groups made up predominantly of men spoke less and were interrupted more than men (Karpowitz, Mendelberg, & Shaker, 2012). Additionally, research indicates that during facilitated whole-class discussions in math classrooms students often receive qualitatively and quantitatively different opportunities to participate in ways that follow patterns of gender, race, and class (Black, 2004; Walshaw & Anthony, 2008).

In summary, Laursen et al. (2014) findings suggest that active learning approaches similar to IBL may have the potential to both improve student learning, and improve gender disparities, in undergraduate mathematics. However, the research literature also indicates that active learning classrooms have the to potential to reorganize the nature of classroom inequities – perhaps in ways that further marginalize historically under-represented populations. Our study has the potential to either corroborate Laursen et al.'s (2014) finding that active approaches like IBL can help eliminate gender disparity, or to problematize these findings and push us to more clearly articulate the conditions under which active learning classrooms are more equitable for various groups of students.

Methods

To investigate how IOI relates to student performance, we quantitatively analyzed data from 522 student content assessments. Of those assessments, 147 were completed by students of the TIMES fellows; the remaining 375 were from students in the national comparison sample. Here we detail the TIMES program, the instrument, our samples, and our analysis.

As part of the TIMES project, 13 mathematics instructors participated as abstract algebra TIMES Fellows. These fellows were provided support for implementing the IOAA curricular materials, which are formatted as task sequences that include rationale, examples of student work, and implementation suggestions. Due to documented challenges associated with implementing IOI (e.g., Speer & Wagner, 2009; Wagner, Speer, & Rossa, 2007), and IOAA in particular (Johnson & Larsen, 2012; Johnson, 2013), the TIMES Fellows were provided both prior and ongoing support. Summer workshops, held just prior to the instructors' implementation of the IOAA materials, had two main goals: to help the instructors develop an understanding of the curricular materials, including an overview of the mathematical development of the concepts; and to develop a shared vision of IOI, focusing on the roles of the teacher, the students, and the tasks (See Kuster et al., 2017). Online workgroups, held throughout the term in which the IOAA materials were being implemented, were hour-long weekly meetings with two components: an open forum devoted to addressing issues and concerns for the Fellows as they arose (e.g., facilitating group work, particularly difficult class sessions) and lesson studies. During the two lesson studies, the workgroup would first discuss the mathematics of the lesson, followed by a discussion of student learning goals and implementation considerations. After instructors taught the unit, they would share video-recorded clips of their instruction for group reflection and discussion. Throughout the sessions, the workgroup attended to the critical components of IOI – generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation (Kuster et al., 2017).

The TIMES Fellows asked their students to complete the Group Theory Content Assessment (GTCA) (Melhuish, 2015). This assessment, developed to measure conceptual understanding of key concepts in group theory, spanned the topics of binary operations and their properties, group structures (including subgroups, quotient groups, and cyclic groups), element properties, and functions (homomorphisms and isomorphisms). The GTCA instrument was informed by textbook analysis and literature on student thinking and was designed to be applicable across a wide range of group theory courses. Instrument validation was achieved through a combination of expert review and multiple rounds of pilot testing (including clinical interviews) in which open-ended tasks were converted to a multiple-choice format based on student responses. (For specific details regarding the instrument development, see Melhuish, 2015.)

From the 13 TIMES Fellows, there were a total of 174 students, 147 of whom (84%) completed the GTCA. For our control, we have a national sample (Not-TIMES), with 375 students from 33 institutions. For Not-TIMES students we can presume (but not verify) that they did not experience IOI, as reports indicate that nationally the proportion of teachers using any form of non-lecture instructional approaches in abstract algebra is less than 10% (Keller, Johnson, Peterson, & Fukawa-Connelly, 2017).

Between the treatment (TIMES, n = 147) and control (Not-TIMES, n = 375) groups, we have a total of 522 participants: 275 students who identified as male, 240 as female, and 7 who otherwise identify or declined to identify their gender. The gender makeup was not significantly different (p = .229) between Not-TIMES and TIMES (48% and 42%, respectively, identified as women). We address our two research questions in stages. First, we investigate the relationship between IOI, then gender, and student performance on the GTCA via an exploratory univariate analysis. We calculated descriptive statistics and ran t-tests to look for evidence of performance differences on the GTCA between TIMES and NOT-TIMES students (with regard to Research Question 1) and to look for evidence of gender differences on the aggregate and within subgroups (with regard to Research Question 2).

The univariate analysis did provide evidence of significant differences when looking at the gender differences between the TIMES and Not-TIMES groups. Thus, in an attempt to control for compounding factors and to account for the nested structure of our data, we developed a Hierarchical Linear Model (HLM) to determine the robustness of the effects of IOI and interaction between IOI and gender. The appropriateness of a multi-level modeling approach for this data was determined by the sufficiency of the intraclass correlation (ICC) of the unconditional model (17%) and the results of the likelihood ratio test (χ^2= 38.368, p < .001) comparing the 1-level and 2-level null models.

As this was not a randomized treatment-control study, the inclusion of institutional nesting provides a means for accounting for differences between the TIMES institutions and the larger national sample. We conjectured that important institutional variables such as level of selectivity, 75th percentile mathematics SAT scores (referred to as "SAT" for the rest of the paper), and Carnegie classification may account for performance differences on the GTCA. To test this, we developed an HLM model these variables as effects. Of these variables, only SAT was statistically significant; results indicating that a student at an institution one standard deviation above average would be estimated to score roughly half an item (0.564) higher on the GTCA (*p=0.034*). As a result, we incorporated normalized SAT as part of our model. Finally, we leveraged Snijders and Bosker's (2012) guidelines to determine our effect sizes on a Cohen's *d* (1988) scale, where effect sizes were calculated via looking at the cumulative effect of a variable of interest and dividing by the standard deviation of the control group.

Results

In looking for performance differences between students of TIMES Fellows as compared with the control group (i.e. Research Question 1), we see that TIMES students slightly outperformed Not-TIMES students by about half an item (6.64 vs. 6.21), but this difference is not statistically significant (t = -1.520, df = 520, p = .129). To investigate Research Question 2, we compared the GTCA performance by gender of the students in the two instructional groups (Figure 1). We found no significant difference in the Not-TIMES group where, on average, men outperform women by about half an item (p = .098). In the TIMES group however, men outperformed women by nearly 2 items on average (p < .001).

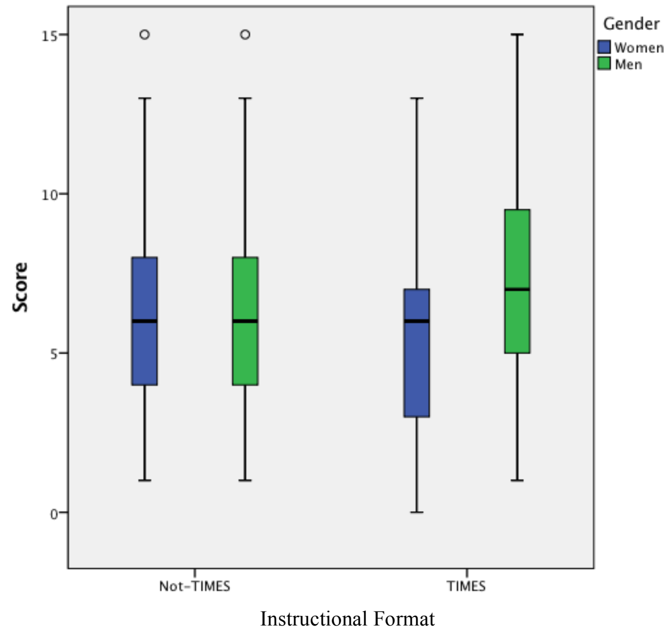

Figure 1. Gender Comparisons on GTCA Performance

Our initial univariate exploration provided evidence that there was no significant TIMES effect – i.e., TIMES students did not significantly outperform Not-TIMES students. However, this (non)effect of TIMES was *not* consistent across genders. While men in TIMES classes significantly outperformed the women in TIMES classes (and men in non-TIMES classes), this gender performance difference was not seen in the Not-TIMES classes.

Given the nested structure of the data, the univariate analysis does not rule out the possibility that these differences are better explained by differences in the instructor or by differences in the insitituion. Thus, we developed a series of HLMs to assess the robustness of the TIMES/Gender interaction effect. In revisting our first research question, this time controlling for instructor and SAT, we look at our simplified model. In this model, the estimated score for a TIMES student is 6.47 items while the estimated score for a not-TIMES student is 6.20 – a performance discrepancy between groups that is not statistically significant (p = .600). Thus, after accounting for potentially confounding variables, we again find no significant differences between the performance of TIMES and Not-TIMES students.

In revisting our second research question, again controlling for instructor and the inistutions' SAT, we look to the full model (see Table 1). This model verified that the interaction between gender and TIMES was robust and remained a significant factor (p = 0.014) even when nesting students within instructors, accounting for institutional differences in terms of SAT, and controlling for the global gender effect favoring men (p = 0.086). This model estimates that, for students at institution with mean SAT, a man in TIMES scores 7.23, a not-TIMES man secores 6.44, a not-TIMES woman scores 5.91, and a TIMES woman scores 5.86. So, while women are scoring roughly the same in TIMES and not-TIMES classes, men are scoring statistically significantly higher under the TIMES treatment.

Table 1
Model with TIMES and Gender Variables

	Coeff	SE	df	p
Intercept	5.913820	0.326461	67.321603	<.001
TIMES	-0.580915	0.600143	70.121264	0.336390
Level 1				
Man	0.526703	0.306206	458.653547	0.086089
TIMES*Man	1.372867	0.557205	453.700395	0.014115
Level 2				
SAT75	0.434603	0.225540	34.585700	0.062234
	Variance	SE		
Level 1 Residual	4757.087187	861.270931		
Level 2 Residual	1904.227364	1030.501765		

Discussion and Future Research

We found no difference in the performance of men and women in the national sample; however, under the TIMES treatment, a difference was present. Notably, this difference came from TIMES men outperforming Not-TIMES men, while the performance of women remained unchanged. While we see the detection of a gender performance difference within the IOI setting as an unfortunate finding, we are not arguing that the TIMES project, nor the implementation of IOI, is detrimental to women; in fact, both men and women under the TIMES treatment performed as well or better than students in the national comparison sample. However, the difference in learning outcomes between men and women among the TIMES population indicates that implementation of this curriculum is far from a guarantee of equitable instruction.

We suspect that there are important instructional differences between IOI and IBL that may impact different groups differently. This includes the routine use of student presentations in IBL classrooms (Hayward, Kogan, & Laursen, 2016), which are often distributed evenly across students and thus may remove barriers to equal participation; and the reliance on small-group work and whole-class discussions to develop the mathematical agenda in IOI, which may provide more opportunities for microaggressions and implicit bias to emerge. Indeed, preliminary analysis of 42 TIMES Fellows' instruction (across all content areas) suggests that, similar to the findings of Black (2004) and Walshaw and Anthony (2008), the TIMES instructors directed mathematically substantive questions at women at lower rates than men, and they re-voiced and elaborated contributions made by women at substantively lower rates than those made by men (Smith, Andrews-Larson, Reinholz, Stone-Johnstone, & Mullins, 2018).

We are hopeful that our future studies – investigating the gender performance difference we found in the IOI classes – will help us continue to refine our understandings of how features of student-centered instruction in undergraduate STEM can support robust student learning gains and equitable outcomes for all groups of students. It is our intention to use our findings to inform a critical examination of the effect of our interventions on the gendered experiences of our students and call on others in the field to do the same.

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Potential Intellectual Needs for Taylor and Power Series within Textbooks, and Ideas for Improving Them

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Unfortunately, students far too often have little or no intellectual need for learning the second semester calculus topic of Taylor and power series. In this study, we examine the "potential intellectual needs" (PINs) provided by commonly used textbooks. While the textbooks used different approaches, they both often lacked problems developing intellectual need, suggesting that instructors must incorporate intellectual need by themselves. To assist in this endeavor, we focus part of the paper on a discussion of including PINs for this content. We found that it may be difficult to incorporate genuine problems for first-year *students through an approach based on a "family of series" meaning for Taylor/power series, but that stronger problems could be incorporated through an approach based on an "extension of linear approximation" meaning.*

Keywords: calculus, intellectual need, Taylor series, power series

Harel (2013) observed from his experience that "most students… feel intellectually aimless in mathematics classes because we (teachers) fail to help them realize an *intellectual need* for what we intend to teach them" (p. 119). While this observation likely holds true for many topics, we have especially seen it manifested in the calculus topic of Taylor and power series,[1] despite this topic's importance in mathematics and science. We have often heard students express confusion and frustration as to the purpose of learning them. If our work as mathematics educators is focused toward the ultimate goal of improving student learning, this topic cries out for our attention. We propose extending the nascent research work on Taylor/power series by examining the potential *intellectual needs* (Harel, 2008a, 2008b, 2013) offered to students in commonly used textbooks (Hughes-Hallett et al., 2012; Stewart, 2015). Our study was guided by the research question: what potential intellectual needs for Taylor/power series are offered to students in commonly used textbooks? After presenting our results, we devote a portion of this paper to a discussion of redressing the stark absence of intellectual need we observed.

Brief Recap of the Limited Body of Related Research

Research on series has mostly been restricted to basic series, $\sum_{n=1}^{\infty} a_n$, which precede Taylor/power series. Two such studies related to our paper claim the importance for students to see sequences as functions from the naturals to the reals (McDonald, Mathews, & Strobel, 2000), and to also see partial sums as functions from the naturals to the reals (Martinez-Planell & Gonzales, 2012). As for power and Taylor series specifically, very little research has been done (see Speer & Kung, 2016). In some of the only studies on the topic, Martin (2013) and Martin & Oehrtman (2010) described distinct ways convergence can be imagined, such as looking at a specific *x* value, or looking at the function as a whole. Martin, Thomas, and Oehrtman (2016) then built on these results to develop a virtual manipulative in which multiple representations support connections between sequence convergence and Taylor series convergence. We note that this work is centered on Taylor series, rather than generic power series, which has connections to

[1] A power series is given by $\sum_{n=1}^{\infty} c_n(x-a)^n$. A Taylor series is a power series where $c_n = f^{(n)}(a)/n!$ for some analytic function $f(x)$. A Taylor polynomial is the first *N* terms of a Taylor series, $\sum_{n=1}^{N} f^{(n)}(a)/n!\,(x-a)^n$.

our discussion in this paper. However, we also note that a missing component to this research is the *intellectual need* that students might have for learning this topic in the first place.

(Potential) Intellectual Need

Intellectual need is a part of Harel's Duality, Necessity, and Repeated Reasoning framework (2008a; 2008b), and he has claimed that providing students with an *intellectual need* is paramount to learning (Harel, 2013). Harel (2013) explained that given a piece of knowledge possessed by an individual or community, intellectual need refers to the problematic situation that motivated the construction of the piece of knowledge. That is, the piece of knowledge *resolves* the problematic situation. Harel (2013) also differentiated between a major problem, referred to as a *global intellectual need*, and a smaller problem that becomes apparent along the way of resolving the global intellectual need, referred to as a *local intellectual need*.

In the context of education, Harel (2008c) claimed that an intellectual need must be a genuine problem or puzzlement to a student in order for them to construct the intended knowledge. When a sufficient intellectual need is provided to a student, she/he is driven to solve an intrinsic problem. Otherwise, a student may only be driven to satisfy a teacher's expectations or to improve her/his economic status by passing a required math classes for a certain degree.

Because we focus on examining textbooks' presentations of the Taylor/power series topic, and not on observing students directly, we operationalize *intellectual need* as motivations contained in a written curriculum that students might potentially adopt as their own intellectual need for the content. To distinguish against intellectual needs as actually experienced by students, we use the term *potential intellectual needs* (PINs) to refer to intellectual needs that curriculum might offer, including *global potential intellectual needs* (G-PINs) and *local potential intellectual needs* (L-PINs).

Intended Knowledge for Taylor/power Series

Because intellectual need depends on the intended knowledge, we find it important to preface our study with two different ways of knowing that instructors might use to scaffold their approach to Taylor and power series: *family of series* and *extension of linear approximation*.

Family of series is based on perceiving that certain basic series have similar structures and that they can be grouped together into a common generalized format. For example, the series $\sum_{n=1}^{\infty}(1/2)^n$, $\sum_{n=1}^{\infty}(-2/3)^n$, and $\sum_{n=1}^{\infty} 4^n$ can be thought of as individuals from a larger family with the same structure, and can be grouped under the common template $\sum_{n=1}^{\infty} x^n$. This general representation can now be explored without having to explore each individual series one at a time. In an instructional approach based on this way of knowing, power series would come first, with Taylor series being introduced later as an outgrowth or application of power series.

Extension of linear approximation is a fundamentally different way of knowing, that grows out of function approximation concepts rather than series concepts. In linear approximation, the line tangent to the graph of $f(x)$ at a point $x = a$ is seen to produce reasonable approximations to f near $x = a$, because they have the same value and first derivative (or rate of change) at $x = a$. However, tangent lines are limited in their approximation because the graph may curve away from the line. By seeking another simple function that can curve along with the graph, one can advance to finding a "tangent parabola" at $x = a$. This "tangent parabola" is required to have the same value of f at a, the same derivative as f at a, *and* the same curvature of f at a through equal second derivatives. The second-degree polynomial can now, typically, approximate better and over a greater interval than the tangent line. This process can be extended to "tangent cubics," and indeed to any order of polynomial, provided f is infinitely differentiable. In an instructional approach based on this way of knowing, Taylor polynomials come first as an extension of linear approximation, which are then extended to Taylor series and then general power series.

Methods

For our study, we first identified commonly-used textbooks. According to a large study done by the Mathematical Association of America (Bressoud, Mesa, & Rasmussen, 2015, as reported in Park, 2016), the most commonly used textbooks were Stewart (2015), Hughes-Hallett et al. (2012), and Thomas, Weir, and Hass (2014), which for convenience we refer to simply as Stewart, Hughes-Hallett and Thomas. Because Stewart and Thomas follow similar approaches, for brevity in this report we focused only on Stewart and Hughes-Hallett. We examined sections 11.8-11.11 in Stewart and sections 9.5-10.4 in Hughes-Hallett. Prior to the main analysis, we classified each book by whether it seemed to use an approach based on *family of series* or *extension of linear approximation,* according to the descriptions given earlier.

To begin the main analysis, we identified G-PINs and L-PINs of the textbooks as follows: A G-PIN was defined as the problem (if any) posed or contained within the very first part of each section that could be resolved by the intended knowledge of that section. For L-PINs, we then broke each section into smaller "intended knowledge units" (hereafter referred to as "units"), which were each smaller component of the intended knowledge of the section. For each unit, the L-PIN was the problem (if any) given within that unit that could be resolved by that specific piece of intended knowledge. We then analyzed the G-PINs and L-PINs by attending to (a) whether a problem was actually present, (b) whether that problem was explicitly stated, and (c) whether the textbook attempted to provide rationale as to its importance. If no problem was found within the unit, we still looked for whether the textbook attempted to motivate the content by describing, without posing a problem, that it was important for other uses. From this we created a ratings system of A, B, C, D, or E for each section's G-PIN and each unit's L-PINs, as follows: "A" was assigned to a section's or unit's PIN if there was an explicitly stated problem with rationale given for its importance. "B" was assigned to a section's or unit's PIN if there was either an explicitly stated problem without attention given to its importance, or if there was an implicitly contained problem whose importance was somehow demonstrated. "C" was assigned if the section or unit contained a problem, but the problem was both implicit and not motivated as important. "D" was assigned to a section or unit that had *no* problem (and technically no PIN), but that discussed where that intended knowledge might be useful, such as in advanced mathematics or science. "E" was assigned to a section or unit with *no* motivation at all, neither in terms of a problem (explicit or implicit) nor in terms of where it was useful in other areas.

The three authors then independently rated each section's G-PIN and each unit's L-PIN within both textbooks. Our independent ratings had reasonable consistency, in that at least two authors agreed on approximately 85% of the ratings. After clarifying discussions, our agreement was strengthened to near 100% agreement between at least two authors (only one unit did not). Following the final ratings, for any instance where one author did not have the same code as the other two, we defaulted to the code agreed on by the other two authors.

Results

G-PINs and L-PINs within Stewart

Stewart's approach generally seems based on the *family of series* way of knowing by introducing power series as a specific class of series (11.8), describing how power series can represent certain functions (11.9), exploring specific Taylor and Maclaurin series (11.10), explaining how Taylor polynomials are used in application (11.11).

Section 11.8. Not only is there no problem posed at the introduction of power series in Stewart, but there is no explanation to precede it at all. Stewart simply states the intended knowledge by defining a power series and giving its characteristics. This means that the (lack of a) G-PIN received an E rating from all three authors. For the majority of the units in this section, there are no problems, whether explicit or implicit. However, we identified two instances of B ratings, including the book explicitly illustrating the problem that the Ratio and Root Tests will always fail when x is an endpoint of the interval of convergence. Yet, importance for finding the interval of convergence in the first place is never discussed in the book.

Section 11.9. In this section, there are a few expository remarks, but nothing that provided any type of problem for the intended knowledge of the section. The book even admits, "You might wonder why we would ever want to express a known function as a sum of infinitely many terms" (p. 752). However, the introductory remarks do provide some importance for this intended knowledge by explaining that the topic can be useful in various applications, meaning that it was given a D rating. Most units in this section have no problems posed, meaning the (lack of) L-PINs receiving E ratings. However, there was one instance of a L-PIN with a D rating where the book stated that it can be useful to differentiate and integrate power series. There was also one A rating for the last unit's L-PIN where the book explicitly attends to the problem that integrating $1/(1+x^7)$ by hand is "incredibly difficult" (p. 756). The book then places importance on this problem by explaining that even computer algebra systems (CAS) return different forms of the answer that are all extremely complicated.

Section 11.10. In this section, Stewart does create a global problem by asking "Which functions have power series representations?" and "How can we find such representations?" While these problems are explicit, the book provides no discussion of importance for this problem, meaning the G-PIN was rated as a B. Of the nine units in this section, five have no problem contained in them. For example, the majority of the chapter explores how to find the Maclaurin or Taylor series of various "important functions" without suggesting any reasons why this would be useful. The section does contain a couple of units with B-rated L-PINs, because the book explicitly describes the issue of knowing whether a Taylor series actually converges to its function. However, nothing in the discussion of power and Taylor series in the book gives reason why knowing such a thing would be important.

Section 11.11. In this final section, the (lack of a) G-PIN received a D rating because Stewart does not provide a problem, but it does mention that Taylor polynomials are important within the mathematics and science communities. Despite the low G-PIN rating, this section does have better L-PINs than previous sections, including two As and one B. For instance, Stewart illustrates the problem of knowing the accuracy of an approximation of a function and the problem of knowing how large to take n in order to achieve a desired accuracy. Additionally, towards the end of the chapter, the textbook discusses the need for accuracy of approximations within the science topics of special relativity and optics.

G-PINs and L-PINs within Hughes-Hallett

Hughes-Hallett's first section (9.5) is based on *family of series* by introducing power series as a special class of series with special characteristics. However, from sections 10.1 to 10.4 the textbook's approach switches to being based on *extension of linear approximation*, by introducing Taylor polynomials as increasingly accurate approximations of functions (10.1), discussing Taylor series as an extension of Taylor polynomials (10.2), exploring how to find and use Taylor Series (10.3), and discussing the error in Taylor polynomial approximations (10.4).

Section 9.5. Hughes-Hallett introduces power series in much the same way as Stewart, without a problem that can be resolved with the intended knowledge of power series and interval of convergence. However, the text does at least open this section by explaining that power series can be used to approximate functions, "such as e^x, sin(x), cos(x), and ln(x)" (p. 521), meaning it received a D rating. Four of the seven units in this section were rating as E, for having no problem or discussion of importance. The book simply defines and outlines characteristics of power series. However, we identified two B-rated L-PINs, similar to Stewart, where the text demonstrates the problem of the Ratio and Root tests failing to show whether a power series converges or diverges at its endpoints.

Section 10.1. Hughes-Hallett does not provide a problem for the global intended knowledge for this section, but the book does explain that Taylor series are used to approximate functions. Thus, the G-PIN received a D rating. Of the section's four units, two lacked any motivation (E rating), while the L-PINs for the other two were rated as a B. For one B-rated L-PIN, the text creates a problem by explaining that ln(x) cannot be centered at 0 as it is undefined for $x = 0$.

Section 10.2. This section's (lack of) G-PIN received an E rating, since there was no motivation given at all. The text only says that a Taylor series "can be thought of as a Taylor polynomial that goes on forever" (p. 546). In this chapter, three of the four units did not have any motivation or L-PIN. The remaining L-PIN received a B letter grade for explicitly raising the problem that the Taylor series for ln(x) does not converge at certain locations.

Section 10.3. Here we identified the first A-rated G-PIN in our study. The textbook explains that it can be laborious and difficult to repeatedly take derivatives of certain functions in order to find the coefficients of their corresponding Taylor polynomials. This is likely a problem to which many students can relate, and the problem can be resolved through the different methods to find Taylor series illustrated in the chapter. This section also contained an A-rated L-PIN for presenting the problem of integrating e^{-x^2}, a B-rated L-PIN for posing the problem of identifying which of two functions has larger values, and a C-rated L-PIN for estimating the value of π (unfortunately without explicitly identifying the problem, nor its importance).

Section 10.4. In this final section, Hughes-Hallett explicitly brings up the problem of knowing how to bound an approximation's error "in order to use [that] approximation with confidence" (p. 560). This section yielded our only other A-rated G-PIN. Two units in this section had L-PINs with a B rating, by pointing out problems of bounding error and findings ways to approximate functions like cos(x). Unfortunately, discussion as to the importance of these problems was missing.

Summary

In summary, both textbooks unfortunately generally lacked PINs to offer students for the Taylor/power series sections. If the ratings are assigned the numbers A=4, B=3, C=2, D=1, and E=0, the G-PINs in Stewart had an average score of 1.25 and the L-PINs had an average of 1.36. In Hughes-Hallett, the G-PINs had an average score of 1.6 and the L-PINs an average of 1.33.

Where to Go from Here? Ideas on Providing Intellectual Need for Taylor/Power Series

The textbook analysis in this study showed a general lack of PINs, suggesting it may mostly be up to instructors to inject PINs into their own classrooms. As such, we consider it important to discuss hypotheses for providing PINs to our students, which we intend to examine in future studies. Here we consider what might be added, by way of explicitly posed important problems, to these sections to provide PINs. We focus here only on the larger-grained *global* needs that need to be developed. While one might infer from the results that both approaches lack PINs to offer, we believe that one approach can be more easily imbued with strong PINs than the other.

Intellectual Need for an Approach Based on Family of Series

Both textbooks initially introduced power series with no discussion or motivation, let alone a problem that could provide intellectual need. What problems could be posed at the beginning of the introductory section to provide it? Because the intended knowledge is the definition of a power series and intervals of convergence, the problem would have to be resolvable by that content. We identified one possible problem to be, "Can we determine convergence of a whole class of series through examining one generic series?" This problem seems related to Harel's (2013) *need for computation*, though the importance of this question to students would be highly dependent on the intellectual need they have for knowing whether basic series converge, or not, which is questionable in our experience. In the next section of Stewart (11.9), the intended knowledge is representing known functions as power series and learning how to manipulate them to determine additional power series for functions. The problem answerable by this intended knowledge may simply be, "Can we describe functions as power series?" However, since students are likely already comfortable with the existence of functions such as ln(x), cos(x), and e^x, there does not seem to be a genuine issue with needing to express these functions in other ways. Rather, the students, as Stewart explicitly puts it, would likely "wonder why we would ever want to express a known function as a sum of infinitely many terms" (p. 752).

These types of underlying problems for this content seem more the purview of advanced, proof-based mathematics focused on questions of existence, necessary conditions, or exhaustive cases. In fact, both textbooks state early on a theorem for all possible cases for radii of convergence. Stewart then goes on to provide theorems on the existence of derivatives and anti-derivatives, the existence of Taylor series, and the necessary conditions for a Taylor series to converge to the function. Because the vast majority of students in first-year calculus are planning on studying science and engineering, with only a very small number of pure mathematics majors (Bressoud et al, 2013), we believe these problems would likely *not* be seen as important by them, and would likely not produce intellectual need. Thus, while it is true that genuine problems can be found in an approach based on *family of series*, we personally find it difficult to see that they would become intellectual needs for many typical first-year calculus students.

Intellectual Need for an Approach Based on Extension of Linear Approximation

Past the first section on introducing power series, Hughes-Hallett generally follows an approach based on *extension of linear approximation*, so we focus only on sections 10.1-10.4 here. In section 10.1, the intended knowledge is the notion of improved approximation through linear approximation, quadratic approximation, and higher-order approximation. What underlying problem could be answerable by this intended knowledge? While Hughes-Hallett fails to actually provide the problem, it does state that the intention is to "see how to approximate a function by polynomials" (p. 538). However, this statement could be converted into a true problem by posing the following issue: Ask students to take out a calculator, or their phone, and input expressions like $\sqrt{1.03}$, $e^{0.15}$, or ln(1.1). The fundamental problem can be posed as, "How does your calculator determine the values of these (and essentially infinitely-many other) possible inputs?" This is also related to *need for computation* (Harel, 2013), but this need seems much more relevant to the experiences of first-year calculus students. The problem posed here is answerable by the content in section 10.1 of Hughes-Hallett. Note that this PIN is *not* dependent on the intellectual need students may or may not already possess for knowing whether basic series converge or diverge.

The intended knowledge in the second section (10.2) is identifying Taylor series and radii of convergence for the common functions cos(x), sin(x), and e^x. Again, the book fails to provide a problem, but we see a problem as being easily attached to this content, though using different functions to start. In conjunction with the previous problem, one could ask the students, "Can the process of approximation developed in the first section be used to program a calculator to return values for these functions for *any* input?" As students attempt to build Taylor polynomials for different inputs, some students may identify some inputs, such as ln(4), where the series diverge to infinity. This problem leads to the need to identify the domains over which these Taylor series can be used to program calculators.

In the last two sections (10.3 and 10.4), we note that the G-PINs were actually strong in Hughes-Hallett and that we would not see a need to change these. As a final point, though, we note that the first section (9.5) provides no G-PIN for general power series. As such, we would recommend *not* introducing power series first, but to place them *after* the complete development of Taylor polynomial approximation, Taylor series, and error analysis. In this way, power series may be introduced as a mathematical generalization of the already-motivated Taylor series.

Connections to other research. These suggestions for providing PINs for Taylor/power series are in line with Martin et al.'s (2016) virtual manipulative focused on helping students see connections between the *global* and *local* convergence. We believe the importance students may attach to such connections and representations are inherently connected to the problems we have recommended for developing G-PINs for Taylor and power series. That is, such manipulatives would be strengthened if students felt an intellectual need for them. On a different note, we believe that the stronger PINs that can be added to an approach based on *extension of linear approximation* can help develop the idea of partial sums as a function from the natural numbers to the reals (Martinez-Planell & Gonzales, 2012; McDonald et al., 2000). In an approach based on *family of series*, power series are stated as "fact" in the form of a long list of added terms, which seems to build the "series as a list" conception. However, *extension of linear approximation* would focus on creating higher-order Taylor polynomials, which can make it more plausible that students would attend to mappings between integers n and partial sums.

Conclusion

In conclusion, we find that, unfortunately, textbooks may often not provide PINs that instructors can use in their courses, and that instructors may need to do the work on their own of infusing PINs into their classrooms. We believe that while an approach based on the *family of series* way of knowing can be motivated by true problems, those problems may be more appropriate for advanced calculus concentrated on proof and theory. For first-year calculus, we instead posit that an approach based on the *extension of linear approximation* way of knowing can provide stronger PINs for the students in those classes. While this textbook analysis serves as an introductory study on the topic, we call for research to test these hypotheses regarding PINs for Taylor and power series. We are certainly open to the possibility of other types of PINs that could be used to motivate this topic, and we welcome discussion and debate on this point. As Harel (2013) suggests, we believe such attention to intellectual need is necessary if students are to learn this important topic well.

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Teachers' Reasoning with Frames of Reference in the US and Korea

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Our theory of what entails a conceptualized frame of reference is explained, along with items and rubrics designed to illuminate how teachers do or do not reason with frames of reference. We gave 551 teachers in the US and Korea frame of reference tasks, and coded the open responses with rubrics intended to rank responses by the extent to which they demonstrated conceptualized and coordinated frames of reference. Our results show that our theoretical framework is useful in analyzing teachers' reasoning with frames of reference, and that our items and rubrics function as useful tools in assessing teachers' meanings for quantities within a frame of reference.

Keywords: Frames of Reference, Mathematical Meanings, Secondary Teachers, Quantitative Reasoning

A frame of reference is an organizing tool most familiar in physics, yet it is also applicable to any mathematics task that involves quantities, or measurable attributes of objects (Thompson, 1993). Every time a person thinks about a quantity, its meaning is only fully understood within the frame of reference within which it was measured. To say a plane is flying at 35,000 feet only has meaning when we know height was measured in a frame where the reference point is sea level; to say a ball's free fall velocity changes by -9.8m/s/s only has meaning when we know that acceleration was measured within a directionality where the measurements are always away from the center of the Earth.

If professional development programs and education researchers wish to address issues with how teachers help their students with the mathematics they teach, we first need more nuanced information about the teachers' own understandings of the mathematics. Many current assessments that focus on mathematical knowledge for teaching (Hill, 2005) categorize teachers' MKT by whether or not they can give normatively correct answers to tasks. Project Aspire wished to take an alternate approach by analyzing teachers' responses by what those responses told us about the teacher's current meanings (Thompson, 2016), and to compare different meanings by how productive they might be for helping students to develop coherent meanings. We did so by writing items and rubrics and analyzing responses from over 500 teachers in the US and Korea. Our work can be connected to critiques of the deficit model (Bak, 2001), in that we are interested in identifying what teachers *do* understand, in whatever ways they do.

In this work, we draw on data from part of an assessment that was developed to analyze teachers' mathematical meanings for frame of reference. The research question for this analysis is: *In what ways do teachers reason about quantities within frames of reference on our tasks?*

Past Literature and Theoretical Perspective

When we first began to write about teacher responses to frame of reference items (Joshua, Musgrave, Hatfield, & Thompson, 2015), our search of math education and physics education literature revealed no cognitive definitions of frame of reference. By 'cognitive definition' we mean a definition of what mental actions a student must engage in in order to use a frame of reference productively to solve tasks. Instead, the definitions we found in both textbooks and academic articles referred to physical objects, such as "a set of rigidly welded rods" (Carroll &

Traschen, 2005), “a set of observers” (de Hosson, Kermen, & Parizot, 2010), or “a coordinate system and a clock” (Young, Freedman, & Ford, 2011) among others. Several studies looked at ways in which students struggled with frame of reference tasks (Bowden et al., 1992; Trowbridge & McDermott, 1980) or reported results of interventions meant to improve performance on frame of reference tasks (Monaghan & Clement, 1999; Shen & Confrey, 2010), and one identified common student misconceptions about frames of reference (Panse, Ramadas, & Kumar, 1994). None gave a clear cognitive definition of frame of reference, which we concluded was needed to have a productive conversation about student or teacher reasoning on frame of reference tasks.

When we speak about a person who has fully conceptualized a frame of reference, the frame of reference itself is not the primary object of consideration. Rather, the person is using one or more frames of reference as a systematic way to think about and organize the measures of quantities and their meanings, as well as the quantitative relationships between those quantities. This places our constructs of conceptualizing and coordinating frames squarely within the domain of quantitative reasoning. This clarification guided our eventual definition:

> An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *frame of reference* if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both. […] An individual is coordinating two frames of reference if she conceives each frame as a valid frame, stays aware of the need to coordinate quantities’ measures within them, and carries out the mental process of finding a relation between the frames while keeping all relative quantities and information in mind. (Joshua et al., 2015)

We wish to emphasize that we are certainly not claiming that people explicitly say to themselves “I have decided to commit to a ____.” For most people these commitments are made implicitly, and are only observable indirectly by looking at how individuals reason through tasks and inferring the presence or lack of commitments that explain their responses. Our theory therefore functions as an explanatory framework for how people think about quantities.

Methodology

From 2012 to 2015, the Project Aspire team created the 48-item assessment Mathematical Meanings for Teaching Secondary Mathematics (MMTsm). A major goal of Project Aspire was to provide information to professional development leaders. We tried to write descriptions of rubric levels that would capture certain ways of thinking, without requiring that the scorer be familiar with the nuances of those ways of thinking. The Project Aspire team and the BEAR team at UC Berkeley ran several rounds of inter-rater reliability (IRR) and used the results to refine the items and rubrics.

The second author translated each item into Korean. A Korean high school mathematics teacher who was a mathematics Ph.D. student translated the items back into English. The second author and the third author reviewed the back translations and the second author made adjustments to the Korean versions (Behling & Law, 2000; Harkness, Van de Vijver, Mohler, & fur Umfragen, 2003). We collected U.S. teacher data in 2014 and 2015 from multiple professional development settings and scored by the Project Aspire team, with some overlapping scores with which to run IRR. The Korean data was collected in the summer of 2015 and scored by English-speaking Korean teachers that tested sufficiently high on the rubrics after training. The second author then scored a subset of responses to run IRR.

Figure 1 shows the Willie Chases Robin task; this paper analyzes responses for Parts B and C. Willie Chases Robin "is a frame of reference context where an individual uses one clock to time two events that begin at different times…Thus, when an individual uses both times in the same expression and in the same unit, she must offset one time from the other to account for the differences in *elapsed* time. In addition, the item's references to times are in two different units—speed (distance relative to time) measured in miles per hour, and the difference in their elapsed time measured in minutes".

Robin Banks ran out of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.

Part A. Let u represent the number of hours since Robin passed the café. Write an expression that represents the number of hours since Willie left the café.

Part B. Here are two functions. They each represent distances between Willie and Robin.

$$f(x) = 65x - 50\left(x - \tfrac{1}{6}\right), x \geq 0.$$

$$g(x) = 65\left(x - \tfrac{1}{6}\right) - 50x, x \geq 1/6.$$

i) What does x represent in the definition of f?

ii) What does x represent in the definition of g?

Part C. Functions f and g both give a distance between Willie and Robin after x hours. But $f(1)=6.67$ and $g(1)=4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Figure 1. Willie Chases Robin MMTsm Item. ©2014 Arizona Board of Regents. Used with permission.

We then scored the teacher results with the rubrics in Figure 2 and Figure 3.

B4 Response:	The teacher said <u>both</u> of the following things: - x in $f(x)$ represents number of hours (or elapsed time) since Willie left café - x in $g(x)$ represents number of hours (or elapsed time) since Robin left café
B3 Response:	Matches B4 response except that x in $g(x)$ is since Robin left <u>bank</u>
B2a Response:	Matches B4 response except no reference points (café, bank) mentioned
B2b Response:	Matches B4 except teacher switched meanings for x in $f(x)$ and in $g(x)$
B1 Response:	Teacher gave same meanings for x in $f(x)$ as in $g(x)$
B0 Response:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, or is off-topic, but isn't blank or just the statement "I don't know".

Figure 2. Willie Chases Robin Part B MMTsm rubric. ©2014 Arizona Board of Regents. Used with permission.

Part B of the Willie and Robin item (see Figure 1) aims to see whether teachers would interpret the meaning of parts of function definitions by analyzing them quantitatively and with explicit reference to their domains. The highest level for this item, B4, is for responses where the teacher distinguished between both independent variables by the reference point of their magnitudes. The only way for two non-equivalent functions' definitions to represent the same quantity (distance between the men) is for the independent variable in each to have different meanings, which is why responses that said both x's have the same meaning were placed at the level B1. Levels B3, B2a and B2b were for responses that articulated the difference to some degree but did not specify the exact quantitative meaning of the x's. Figure 2 summarizes our rubric for Part B.

C2 Response:	Teacher said $f(1)$ and $g(1)$ represent distance between men at two different moments in time, or made same statement for x=1 in $f(x)$ and in $g(x)$.
C1 Response:	Teacher said x=1 has different meanings in both functions but a) did not elaborate on the meaning of x, b) described both x's as representing distances, or c) described $f(1)$ and $g(1)$ as representing time passed; or, described $f(1)$ and $g(1)$ as representing distances but not specifically distances between men.
C0 Response:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, or is off-topic, but isn't blank or just the statement "I don't know".

Figure 3. Willie Chases Robin Part C MMTsm rubric. ©2014 Arizona Board of Regents. Used with permission.

Part C of the Willie Chases Robin item (see Figure 1) is designed to see whether teachers could articulate why two very functions could represent the same quantity yet have different values for the same independent value. The answer, as in Part B, is that the meaning for x in each function is different. For example, if Robin passed the café at 4:00pm, then the distance between the two men at 5:00pm is given by either $f(1)$=6.67 and $g(1.16)$=6.67. Variables (and quantities) have no useful meaning without specified reference points from which we are measuring. Figure 3 summarizes our rubric for Part C.

Earlier we said that our theory therefore functions as an explanatory framework for how people think about quantities. By writing item-specific rubrics that described precisely what types of responses belong to each level, we sought to create rubrics that could categorize teachers' meanings for frame of reference without requiring the scorer to fully understand the theory of what constitutes a conceptualized frame of reference. Our item and rubrics can then be used to either assess the needs of a particular group of teachers for teaching, research, or professional development purposes, or to function as pre- and post- items to evaluate the efficacy of an instructional intervention.

Results & Discussion

In this section we discuss what individual responses can tell us about the teacher's meanings for quantities within a frame of reference, by studying several representative examples through the lens of our theoretical framework. Korean responses were translated into English and handwritten by the second author, and the country of origin of each sample response is not identified (gender pronouns were selected randomly).

Willie Chases Robin Part B Results

Part B elicited a wide range of responses, and so we built a rubric that looked at all three of the commitments necessary to fully conceptualize a frame of reference: unit, reference point, and directionality of comparison. Figure 4 displays three teacher responses to Part B.

Part B. Here are two functions. They each represent distances between Willie and Robin.	$f(x)=65x-50\left(x+\frac{1}{6}\right), x\geq 0.$	$g(x)=65\left(x-\frac{1}{6}\right)-50x, x\geq 1/6.$
i) What does *x* represent in the definition of *f* ?	i) What does *x* represent in the definition of *f* ?	i) What does *x* represent in the definition of *f* ?
x represent the number of hours since Willie left the cafe	the number of hours Robin passed the café	Related to Robin given he is 1/6 hr ahead of Willie
ii) What does *x* represent in the definition of *g*?	ii) What does *x* represent in the definition of *g*?	ii) What does *x* represent in the definition of *g*?
x represents the number of hours since Robin passed the cafe	the number of hours Katchim passed the café	Related to Willie given that he is 1/6 hr behind Robin

Figure 4. Teacher responses that were scored at a) Level B4, b) Level B2b and c) Level B0.

The response in Figure 4a was scored at the highest level of B4 because of three aspects we deemed important, all of which allow us to build a hypothetical model of how the teacher was reasoning while answering this item. The teacher clearly specified "number of hours" and so was identifying each *x* as representing a quantity; other responses merely referred to "time" which could apply equally to the passage of time or the time of day. The teacher also specified reference points and used the appropriate reference points (leaving the café for both men) to make sense of the function definitions. Without reference points for a quantity's measurement, there is no clear unambiguous relationship between a given measurement and the quantitative situation it represents. Finally, the teacher correctly identified that *f* gave the distance between the two men in terms of Willie's time since leaving the café, where *g* is in terms of Robin's time since leaving the café. In order to correctly identify each function's independent value, the teacher had to reason about how one would adjust each man's time in terms of the others to calculate his distance from the café, in terms of his speed times the number of hours he drove. Our model for how this teacher reasoned was that she conceptualized the quantity with an internal commitment to unit, reference point, and directionality of comparison.

The response in Figure 4b was scored at Level B2b because it is identical to a B4 response *except* that the teacher reversed the meanings of the *x* in the definition of *f* and the *x* in the definition of *g*. The definitions he gave do not allow for *f* and *g* to represent the distance between Willie and Robin. The teacher's response is consistent with using one directionality of comparison to define each measurement of time, but the opposite directionality of comparison to define each man's time in the other man's frame of reference. Our model for how this teacher reasoned about Part B was that he conceptualized the quantity with an internal commitment to unit and reference point; we hypothesize that instead of committing to a directionality of comparison the teacher took the +1/6 in (x+1/6) to indicate a later time and therefore a description of Willie's behavior. In this case, the increase is seen as an indicator of "largeness", instead of an indicator that *x*'s value needs to be augmented to represent the appropriate meaning within this frame.

The response in Figure 4c was scored at Level B0 because it did not fit any higher levels, and we can see why when we look at this response in terms of the commitments the teachers did and did not make. This teacher identified the difference in the x's by a general indication that each one has something to do with one person in the context and referred to the difference of 1/6 hours in starting time between the two men. We can see that the teacher is hinting at something relating to the difference in reference points for each man's measurement of time, but she does not know how to interpret that difference by defining two quantities with different reference points. Our model for how this teacher reasoned is that she did not define either x in terms of any quantities (precise or vague) at all, so she made no commitments to unit, reference point, or directionality of comparison in this response.

Willie Chases Robin Part C Results

Part C was particularly difficult for teachers from both countries, as shown in Table 1 in the next section. In deciding how to differentiate responses in a meaningful way, we decided that the most valuable information from Part C responses was in how the teachers did or did not commit to a reference point. Therefore, our rubric for Part C is built around assessing commitment to a reference point. Figure 5 displays three teacher responses to Part C.

Part C. Functions f and g both give a distance between Willie and Robin after x hours. But $f(1)=6.67$ and $g(1)=4.17$. Why are $f(1)$ and $g(1)$ not the same number?		
f(1)=distance between 2 men when officer has left cafe & traveled 1 hr g(1)= distance between 2 men since Robin has travel 1 hr beyond cafe-Cop is catching up	f(1) represent the distance Robin has gone in 1 hour. g(1) represent the distance the Officer has gone in 1 hour.	Because Willie started $\frac{1}{6}$ hrs at the café, the distance between two people should be represented by g(x). f(1) is fixed at café where Willie is.

Figure 5. Teacher responses that were scored at a) Level C2, b) Level C1 and c) Level C0.

The response in Figure 5a was scored at the highest level of C2 because this teacher described f(1) and g(1) as both representing the distance between the two men, but at different points in time because of the different meanings of x in each function. The prompt in Part C sets up a seeming contradiction and asks the teacher to reason why the contradiction does not, in fact, exist. To do so, this teacher had to think about the quantitative meaning of the independent value x in each function, and realize that different reference points for the inputs necessarily implied different meanings for the dependent values as well. Our model for how this teacher reasoned is that he conceptualized all four quantities x [in $f(x)$], x [in $g(x)$], $f(x)$ and $g(x)$ with commitments to reference points.

The response in Figure 5b was scored at Level C1 because this teacher described $f(1)$ and $g(1)$ as representing distances at different points in time, but not specifically distances between men. To reach this conclusion, she had to keep her commitment to the definitions of each x, but not make the same conclusions about the dependent values as the teacher in Figure 5a. Our model for how this teacher reasoned is that she conceptualized x [in $f(x)$] and x [in $g(x)$] with commitments to reference points, but not $f(x)$ or $g(x)$.

The response in Figure 5c was scored at Level C0 because it did not fit any higher levels, and we can see why when we look at how this teacher was not able to resolve the seeming contradiction posed to him. We do not have enough information to speculate about how he conceptualized the quantities represented by x in each function, but we can conclude that he did not conceptualize the quantities $f(x)$ or $g(x)$ with commitments to reference points.

Scores For All Teacher Responses

Table 1 shows a breakdown of responses to Part B and Part C, with 186 US and 365 Korean teachers responding. I/X responses consisted solely of "I don't know" or were blank. We do not make any quantitative analyses or conclusions in this paper, but we find the distributions of interest to those who want to see how large samples of teachers answer this question. Though the purpose of our paper is not a comparison between countries, we must note that when we only had US data, many objections were raised to Project Aspire papers and presentations by saying that our items were too difficult and therefore inappropriate to be given to secondary teachers. Our Korean data shows that this is not necessarily true.

Table 1. Responses to Willie Chases Robin. ©2014 Arizona Board of Regents. Used with permission.

	Part B Results	
	US	Korea
B4	23 (*12.4%*)	144 (*39.5%*)
B3	5 (*2.7%*)	8 (*2.2%*)
B2a	5 (*2.7%*)	73 (*20.0%*)
B2b	35 (*18.8%*)	25 (*6.8%*)
B1	29 (*15.6%*)	33 (*8.8%*)
B0	81 (*43.0%*)	74 (*20.0%*)
I/X	9 (*4.8%*)	10 (*2.7%*)

	Part C Results	
	US	Korea
C2	12 (*6.4%*)	56 (*15.3%*)
C1	24 (*12.9%*)	139 (*38.1%*)
C0	140 (*75.3%*)	160 (*43.8%*)
I/X	10 (*5.4%*)	10 (*2.7%*)

Conclusion

While other professional development projects continue to administer the MMTsm, the data discussed here shows that our theoretical framework is useful in analyzing teachers' reasoning with frames of reference. Our rubrics, correlated to different levels of productive meanings for quantities within a frame of reference, allow us to analyze teacher responses to our tasks and characterize to what extent each teacher reasoned about quantities within frames of reference on our tasks. The MMTsm was designed specifically to investigate mathematical meanings *for teaching*; we are interested in modeling the kinds of meanings that teachers might convey to students in their classroom. While a teacher with productive meanings for quantities within frames of reference is not guaranteed to help her students develop productive meanings, it is certainly true that a teacher with unproductive meanings will have difficulty in doing so.

One limitation of our data is that the teachers were pulled from voluntary participants in professional development programs (for the US) and voluntary participants taking exams mandatory for teachers finishing their fifth year of teaching (in Korea). Neither population is representative of their country as a whole. However, our non-random results do suggest that many teachers are probably not prepared to help their students reason through such tasks. It is important that mathematics and mathematics education professors are aware of teachers' weak meanings for frame of reference and address them during undergraduate instruction and professional development settings. We cannot begin to address a problem until we have identified it, and teacher reasoning with frames of reference is an important yet heretofore unidentified area in need of further study and intervention.

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Analyzing Students' Understanding of Isomorphism

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The purpose of this study is to analyze student understanding of isomorphism as it is taught in a university level mathematics course. We collected and studied student responses to course assignments covering the concept of isomorphism. The findings of this study support previous research that suggests student understanding of isomorphism is largely reliant on an imaged-based concept of symmetry. We found that student understanding is supported by an image-based radical constructivist approach and detail the techniques students use when first working with isomorphic mappings.

Keywords: Isomorphism, Abstract Algebra, Qualitative Methods

The concept of isomorphism is a core component of all Abstract Algebra courses. It holds the power to traverse mathematical operations and meaning between different groups and between different realms of mathematics. Mena-Lorca and Parraguez (2016) describe it as a "difficult" concept for undergraduate students (p. 378). This is partly because the concept of isomorphism builds from multiple other concepts in mathematics. To understand isomorphism, one must first have thorough knowledge of functions, one-to-one correspondence, groups, and homomorphism (Pinter, 1990). Furthermore, understanding isomorphism is significant because it requires a high level of abstract thinking that is not often reached in lower level mathematics courses (Larsen, 2013). In this way, understanding isomorphism transitions students from lower level mathematics concepts to more advanced concepts. Altogether, a strong comprehension of isomorphism can equip students to successfully study group theory and other theoretical mathematics topics. Hence, there is a substantial need for analysis of how students understand the concept of isomorphism. This research seeks to gain knowledge of student understanding of isomorphism as it is introduced in an upper-level university mathematics course that practices radical constructivism. The purpose of this study is to describe how students develop an understanding of isomorphism to improve the quality and effectiveness of undergraduate mathematics education.

Background Literature

Relatively little attention has been given to teaching methodologies that aim to minimize the void between confusion and understanding for undergraduates studying upper-level mathematics. Moreover, almost no research is dedicated to the study of how students understand elementary Group Theory topics such as isomorphism. It has been the opinion of current researchers that "the teaching of abstract algebra cannot be considered a successful endeavor" because students must work with unfamiliar, abstract concepts when they have previously relied on strict, procedural proof techniques (Mena-Lorca & Parraguez, 2016, p. 378). The most recent studies of this topic (Mena-Lorca & Parraguez, 2016; Larsen, 2009, 2013) seek to address how students' understanding of isomorphism stems from their pre-existing informal knowledge.

A case study teaching experiment of two students investigated how students could reinvent the ideas of groups and isomorphism using pre-existing knowledge (Larsen, 2009).

Larsen's guided reinvention approach used basic concepts such as the symmetries of an equilateral triangle to support student discovery. This study identified informal student strategies used to grasp the concepts at hand and suggested how these strategies could be evoked to support the reinvention process and learning of formal concepts (Larsen, 2009). In a similar study, Larsen (2013) formed a series of design experiments to support the reinvention approach to teaching group and isomorphism concepts. Most recently, a large-scale study published in 2018 captured a representative, nation-wide sample of student responses while working with the concepts of subgroups, cyclic groups, and isomorphism. This study expanded previously conducted, non-representative studies, establishing the expanse of different student conceptions and re-analyzing current theories (Weber, 2001; Weber & Alcock, 2004) on student understanding of isomorphism, suggesting that students take a slightly more semantic approach when working with isomorphism than once perceived. That is, in the study, students tended to explore groups structurally rather than within the formal definition when determining isomorphism (Melhuish, 2018).

Methods

Participants

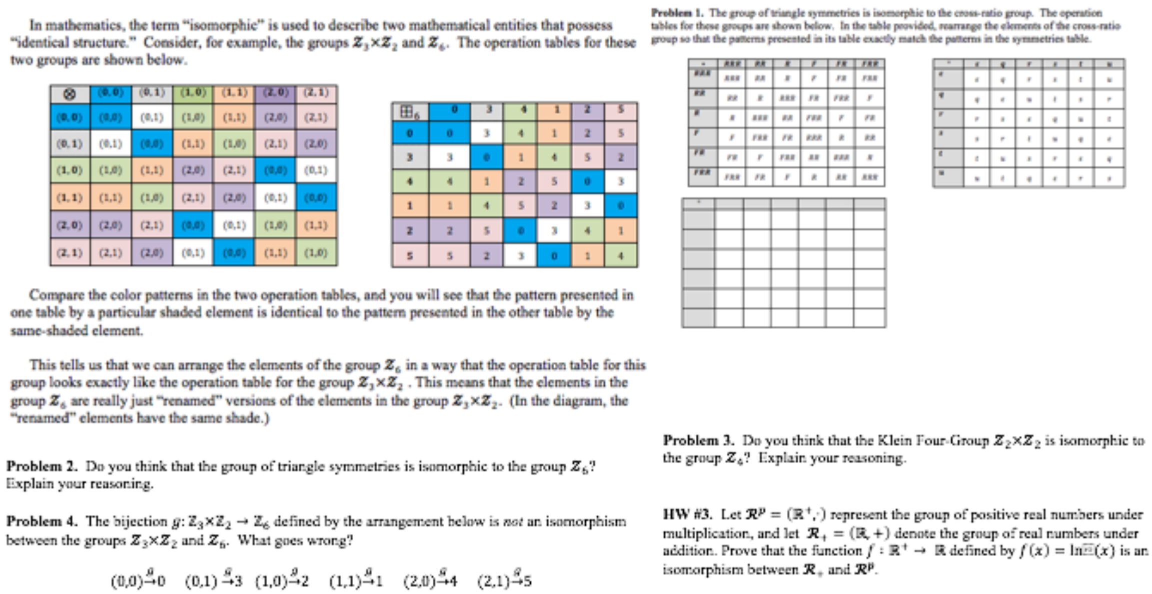
The participants of the study consisted of students majoring in mathematics or mathematics education enrolled in an Abstract Algebra I course at a southeast university. Data was collected from a total of 19 students in two classes over two semesters. Abstract Algebra I is considered the first upper-level mathematics course for the participants hence, these students had no previous course study in upper-level mathematics topics such as Analysis, Graph Theory, or Number Theory that may also cover types of isomorphic structures and relationships.

Task/Context

The instructor of the course utilizes a radical constructivist approach (Glaserfeld, 1995) as the learning through to develop a set of materials called, *Pathways to Abstract Algebra*. These materials view the classroom as a place for exploration of concepts through creating conjectures and making discoveries. The role of the instructor is to create learning situations in which this exploration can happen. In class, students work on investigations covering basic Group Theory topics in groups of two to five students. The instructor facilitates and monitors small group discussion, periodically leading full class discussion over questions and tasks in the investigation being completed. This study focuses on the isomorphism investigation. This investigation is designed to allow students to develop an intuition that motivates the properties of isomorphism.

This investigation begins with tasks that prompt students to use previously learned concepts and rudimentary skills such as matching to construct their own understanding of isomorphism. Initially, the students are encouraged to reason from the perspective of "labeling" groups as a way of motivating the function-based definition. Students are shown an example of two isomorphic groups, $Z_3 \times Z_2$ and Z_6, along with their corresponding, color-coded operation tables. In problem 1, students are asked to recreate similar corresponding tables for the group of triangle symmetries and the cross-ratio group. Through this exercise, students should form a visual relationship between the given isomorphic groups. After students gain a mental picture of isomorphism through this exercise, they work on questions that help winnow away false strategies that they may be using to determine if two groups are isomorphic to each other (i.e. exhaustively checking arrangements). Problem 2 asks students to determine if the triangle symmetries group and Z_6 are isomorphic and to explain why they come to their conclusion.

Problem 3 similarly asks students to determine if the groups $Z_2 \times Z_2$ and Z_4 are isomorphic and why. These questions give students the opportunity to recognize reoccurring properties of isomorphism that have not yet been revealed in the investigation.



In mathematics, the term "isomorphic" is used to describe two mathematical entities that possess "identical structure." Consider, for example, the groups $\boldsymbol{Z}_3 \times \boldsymbol{Z}_2$ and $\boldsymbol{Z}_6$. The operation tables for these two groups are shown below.

⊗	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)
(1,0)	(1,0)	(1,1)	(2,0)	(2,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(2,1)	(2,0)	(0,1)	(0,0)
(2,0)	(2,0)	(2,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2,1)	(2,1)	(2,0)	(0,1)	(0,0)	(1,1)	(1,0)

$\boxplus_6$	0	3	4	1	2	5
0	0	3	4	1	2	5
3	3	0	1	4	5	2
4	4	1	2	5	0	3
1	1	4	5	2	3	0
2	2	5	0	3	4	1
5	5	2	3	0	1	4

Compare the color patterns in the two operation tables, and you will see that the pattern presented in one table by a particular shaded element is identical to the pattern presented in the other table by the same-shaded element.

This tells us that we can arrange the elements of the group $\boldsymbol{Z}_6$ in a way that the operation table for this group looks exactly like the operation table for the group $\boldsymbol{Z}_3 \times \boldsymbol{Z}_2$. This means that the elements in the group $\boldsymbol{Z}_6$ are really just "renamed" versions of the elements in the group $\boldsymbol{Z}_3 \times \boldsymbol{Z}_2$. (In the diagram, the "renamed" elements have the same shade.)

Problem 2. Do you think that the group of triangle symmetries is isomorphic to the group $\boldsymbol{Z}_6$? Explain your reasoning.

Problem 4. The bijection $g: \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ defined by the arrangement below is *not* an isomorphism between the groups $\boldsymbol{Z}_3 \times \boldsymbol{Z}_2$ and $\boldsymbol{Z}_6$. What goes wrong?

$$(0,0) \xrightarrow{g} 0 \quad (0,1) \xrightarrow{g} 3 \quad (1,0) \xrightarrow{g} 2 \quad (1,1) \xrightarrow{g} 1 \quad (2,0) \xrightarrow{g} 4 \quad (2,1) \xrightarrow{g} 5$$

Problem 1. The group of triangle symmetries is isomorphic to the cross-ratio group. The operation tables for these groups are shown below. In the table provided, rearrange the elements of the cross-ratio group so that the patterns presented in its table exactly match the patterns in the symmetries table.

∘	RRR	RR	R	F	FR	FRR
RRR	RRR	RR	R	F	FR	FRR
RR	RR	R	RRR	FR	FRR	F
R	R	RRR	RR	FRR	F	FR
F	F	FRR	FR	RRR	R	RR
FR	FR	F	FRR	RR	RRR	R
FRR	FRR	FR	F	R	RR	RRR

∘	e	q	r	s	t	u
e	e	q	r	s	t	u
q	q	e	u	t	s	r
r	r	s	e	q	u	t
s	s	r	t	u	q	e
t	t	u	s	r	e	q
u	u	t	q	e	r	s

∘						

Problem 3. Do you think that the Klein Four-Group $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ is isomorphic to the group $\boldsymbol{Z}_4$? Explain your reasoning.

HW #3. Let $\mathcal{R}^p = (\mathbb{R}^+, \cdot)$ represent the group of positive real numbers under multiplication, and let $\mathcal{R}_+ = (\mathbb{R}, +)$ denote the group of real numbers under addition. Prove that the function $f: \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$ is an isomorphism between $\mathcal{R}_+$ and $\mathcal{R}^p$.

Figure 1. Problems 1, 2, 3, 4 and HW #3 of the Isomorphism investigation

Following this process, students are introduced to the formal definitions of operation preserving functions, homomorphism, and isomorphism. These definitions are presented through a mini-lecture with the goal to show students that the assignments they have been using to determine isomorphism, in fact, correspond to a function. Students then work on questions that serve to clarify their current understanding of the definition. Problem 4 defines an arrangement between $Z_3 \times Z_2$ and Z_6 that is not an isomorphism and asks students to determine "what goes wrong" within the given assignment. Next, the preservation of identity and inverses property of isomorphism is finally presented as a theorem. Homework problem 3 asks students to prove that a given function is an isomorphism between $\mathcal{R}_+$ and $\mathcal{R}^p$. While this problem does not cover new information, it is a significant indicator of what understanding the students gain from class learning and discussion. Given the structure and goals of these materials, we aim to answer the following research question: how do students develop an understanding of isomorphism?

Data Collection and Analysis

We collected all assignments that included the topic of isomorphism. This includes in-class assignments, homework assignments, quizzes, and exams. We also audio recorded class and small group discussions during the isomorphism investigation. However, for the purpose of this paper we will focus on their written and audio responses to the investigation and homework. All data was blinded and given pseudo-names for analysis.

The problems from which we analyzed data mirror the problems described in the methods section. While students' written work is shown, this is only to give readers a visual idea of student responses; the data analyzed is more extensive than the work displayed in this section and includes recorded discussion. We chose not to analyze data from 7 students because they had previously taken Abstract Algebra I, chose to not be recorded, or were absent during the isomorphism investigation, resulting in a total of 11 students' data.

To describe students' understandings of isomorphism, data was analyzed qualitatively, open-coding for the different aspects of isomorphism, to identify understandings and misunderstandings (Creswell, 2007). We double coded all student responses to each question detailed in the methods section and resolved disagreements through discussion. Specifically, we focused on the techniques through which students completed each problem (i.e. creating operation tables, checking for certain properties). We examined the codes and determined themes of student understanding of isomorphism (i.e., table reliant understanding). These themes are detailed in the results section.

Results

The goal for Problem 1 is for students to develop an intuition for the meaning of isomorphism between two groups and informally recognize properties of isomorphism, such as the preservation of identities and inverses. Altogether, students should focus on the structure of the triangle symmetries and cross-ratio groups rather than the label of each individual element in these groups. Upon analysis, data showed that 91% (n=10) of students successfully found an isomorphic mapping between the triangle symmetries and cross-ratios groups. The same 91% of students began problem 1 by identifying the identity elements of each group and mapping them to each other. An example is shown in Figure 2. Here the elements RRR and ε are first circled in each table and then the ε's are positioned in the bottom table to match the placement of the circled RRR's in the triangle symmetries table. Then the assignment $\varepsilon = RRR$ is made (while this notation is incorrect, the students have not yet been introduced to correct notation).

Of this 91% (n=10) of students who began by identifying the identity elements, five students moved on to map self-inverting elements to each other, three students moved on to mapping non-self-inverting elements, and two students were unable to make more progress and began randomly guessing full mappings. The ten students who were able to make progress and complete the isomorphic mapping tended to assign colors or shapes to the elements they mapped to each other, mimicking the example set forth by the instructor at the beginning of the investigation. Suzie's work in Figure 2 demonstrates this by her markings in the given table of triangle symmetries. Finally, about one third of the students recognized that there are multiple isomorphic arrangements between the group of triangle symmetries and the cross-ratio group.

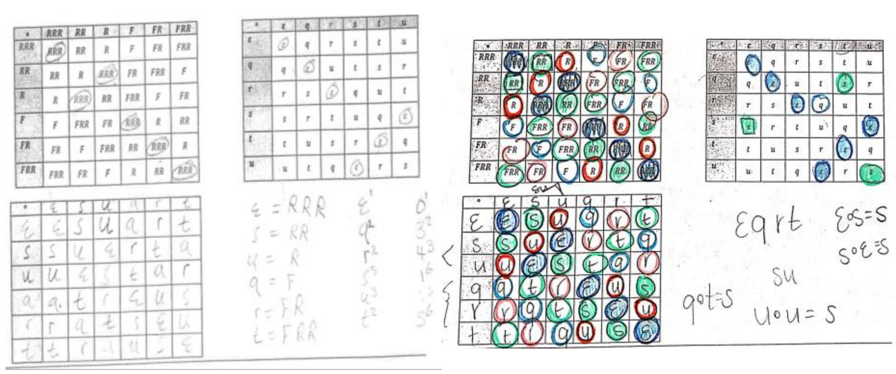


Figure 2. William's (left) identification of the identity elements and Suzie's (right) use of colors

Ultimately, almost all of the students started by mapping identity elements and then self-inverting or non-self-inverting elements, suggesting that students were able to informally recognize the preservation of identities and inverses within an isomorphic mapping. Moreover, two students recognized that there are multiple isomorphic mappings between the group of triangle symmetries and the cross-ratio group, suggesting they started to develop a greater intuition for the meaning of isomorphism.

Problem 2 is designed to help students create a distinction between their mental picture and the isomorphism properties they found in problem 1. This problem aims to refine students' mental image of isomorphism by helping them see what it is not. That is, problem 2 establishes that there is more to the concept of isomorphism than simple matching; isomorphism is, in fact, centered around the structure of the groups. We found that more than half of the students reasoned that the triangle symmetries group and $\mathcal{Z}_6$ group are not isomorphic because they do not have the same number of self-inverting elements. Of these students who recognized the different number of self-inverting elements, all but one did not draw or create their own table to come to this conclusion. For example, Samantha states in her answer, "No, the number of times the identity appears across the diagonal is not the same," meaning she found different numbers of occurrences of the identity in the diagonals of the operations tables for each group.

Contrastingly, the 45% (n=5) of the students did not recognize the different number of self-inverting elements and drew tables for each group, reasoning that they were not isomorphic because they could not find a configuration of tables as they did in problem 1. These students did not consider self-inverting elements as the other half did and instead relied on the structure of the tables that they drew. Chase created several configurations of tables for the $\mathcal{Z}_6$ group and ultimately concluded that the two groups are not isomorphic because there is no way to make them "look the same," saying, "we can't get this (a table for the $\mathcal{Z}_6$ group) to look like that (the given table of triangle symmetries)." Another justification two students used was that there were "unequal instances of unique elements" on the main diagonal of $\mathcal{Z}_6$ and "inconsistencies between rows and columns" when comparing the triangle symmetries and $\mathcal{Z}_6$ tables.

All the students (n=11) were successful in concluding that the given groups are not isomorphic. In this problem, we see an almost even split between the number of students who were able to reason from the perspective of isomorphism properties and the students who reverted to their techniques used in problem 1. This suggests that the students who continued to use tables did not yet understand the identity and inverse preserving properties of isomorphism.

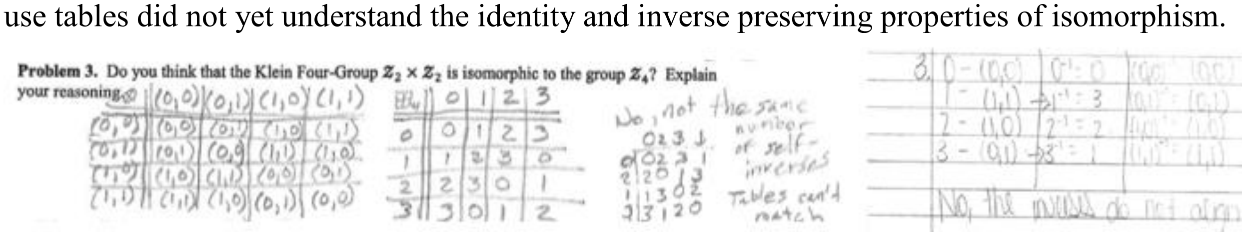


Figure 3. Jimmy's method (left) and Rachel's method (right) for completing problem 3

Problem 3 uses the $\mathcal{Z}_2 \times \mathcal{Z}_2$ and $\mathcal{Z}_6$ groups to reiterate the ideas presented in problem 2. The goal of this problem is to reinforce the students' conceptual understanding gained in problem 2. Upon analysis, data showed that 73% (n=8) of the students reasoned that the given groups were not isomorphic because each had a different number of self-inverting elements. Of these students who recognized the discrepancy in self-inverting elements, four used a table to come to this conclusion and four students did not use a table. In Figure 3, Jimmy draws two different configurations of the $\mathcal{Z}_4$ table and compares them to his written $\mathcal{Z}_2 \times \mathcal{Z}_2$ table before concluding that the groups cannot be isomorphic because they have a different number of "self-inverses." Rachel's work in Figure 3 is an example of the work of students who did not use a table to come to their conclusion but instead created a mapping and found the inverses of each element before concluding that “the inverses do not align.”

Alternatively, two of the students did not mention that the groups had a different number of self-inverting or non-self-inverting elements but instead came to the correct conclusion by drawing a table for each group and comparing. One student drew a table and found that the diagonal “contains unequal instants (instances) of unique elements,” but did not explicitly state that the two groups possess different numbers of non-self-inverting elements in their work or group discussion. In problem 3, almost three-fourths of the students recognized that the groups had different numbers of self-inverting elements. This suggests that some students were able to transition from table-reliant work to a greater understanding of isomorphism properties between problems 1 and 2.

The goal of problem 4 is to help students grasp the definition of operation preserving functions that has just been presented to them. Preferably, students will use the new definition of operation preserving functions to correctly answer problem 4. Every student was successful in finding a counterexample to show that the given mapping is not an isomorphism. We found that 36% (n=4) of the students did this by reverting to using written tables for the groups and comparing them. These students had more trouble completing the task than their peers who used the definition. One student described that it was hard to use the tables to find “what goes wrong” specifically because there are multiple “wrong” arrangements that make each table appear to not be isomorphic to its counterpart.

Conversely, 64% (n=7) of the students did not use tables to come to the correct conclusion. These students completed the task relatively quickly by finding counterexamples that did not preserve the operations of the groups. Samantha found a counterexample by checking if g preserved the operation of $\mathcal{Z}_6$ when operated on elements $(0,1)$ and $(1,0)$ from the $\mathcal{Z}_3 \times \mathcal{Z}_2$ group. Her work showed
“$g\big((0,1) \otimes (1,0)\big) \neq g(0,1) \boxplus_6 g(1,0)$ \\ $g(1,1) \neq 3 \boxplus_6 2$ \\ $1 \neq 5.$'

Altogether, the majority of the students successfully used the new definition of operation preservation to show that the given arrangement was not an isomorphism between the groups $\mathcal{Z}_3 \times \mathcal{Z}_2$ and $\mathcal{Z}_6$. Students who relied on the written tables encountered difficulties using this technique to solve the problem; their reluctance to use the new definition suggests that these students have developed a slightly weaker understanding of isomorphism than their peers.

Finally, homework problem 3 was used to determine if students gained an adequate understanding of isomorphism. Approximately three-fourths (n=8) of the students answered question 3 sufficiently, meaning they showed suitable work to prove that the given function was an isomorphism. Of these students, four explicitly cited the definition of isomorphism and four did not. The four students who did not clearly state the definition of isomorphism showed that the given function was bijective and operation preserving but did not conclude that these factors proved the function was an isomorphism. Since the majority of the students answered homework problem 3 correctly, it suggests a passable understanding of isomorphism. The fact that only half of these students used the definition of isomorphism explicitly in their work could suggests that half of the students do not understand or feel comfortable using the formal definition.

Discussion and Conclusion

Isomorphism is a significant component found in multiple realms of mathematics. Moreover, it is a core concept introduced in beginning Abstract Algebra courses. Previous research shows that, while significant, the concept of isomorphism is “seldom understood by students" (Mena-Lorca & Parraguez, 2016, p. 377), causing the teaching of isomorphism to be a difficult task for Abstract Algebra instructors. To pinpoint and address students' understandings and misunderstandings of isomorphism, we conducted an in-class study on student responses to an isomorphism investigation that utilizes a radical constructivist approach. The goal of this investigation is to allow students to use previously learned concepts and rudimentary skills to construct their own understanding of isomorphism.

Of the few studies that have been conducted on isomorphism, most analyze students' reconstruction (Mena-Lorca & Parraguez, 2016) and reinvention (Larsen, 2009, 2013) of theorems on isomorphism. This curriculum deviates from the guided reinvention approach by supporting student construction of the concept of isomorphism. We believe this study expands upon and supports current findings on students' understanding and provides new insight on student responses within the context of these new curricular materials.

In agreement with past studies, we found that students' have difficulties reasoning with the concept of isomorphism. This led us to conclude students are not prepared to learn the concept of isomorphism starting with the formal definition, but instead must initially gain an image-based understanding. Students who showed progress in their understanding tended to rely on either written operation tables or individual assignments when finding isomorphic mappings. Student responses to problem 1 of the investigation showed the most consensus and adherence to the instructor's goal for the problem when compared to student responses to other problems in the investigation. This suggests that problem 1 was the most successful at helping students construct an understanding of isomorphism. Students who relied solely on operation tables in their work throughout the investigation reasoned that for two groups to be isomorphic, their tables must look the same, suggesting that their understanding was purely image-based and supporting the theory that "the context of geometric symmetry can provide a rich and natural context for developing the concepts of group theory" (Larsen, 2009, p. 136). These students informally recognized the properties of isomorphism through conditions for their tables (i.e. the corresponding tables must have an equal number of instances of the identity elements in their diagonals informally requires that the groups have an equal amount of self-inverting elements) but found it difficult to recognize these properties outside of the tables. This suggests that while students' find the most progress in problem 1, the techniques learned in this problem have the danger of becoming "crutches" throughout the investigation. To attempt to resolve this problem future drafts of the materials could include a smoother transition in the investigation from the table-oriented problems to the formal definition of isomorphism. Contrastingly, students who were able to recognize and consistently use the properties of isomorphism in their work, showed a greater intuition when finding isomorphic mappings. In sum, we have detailed the techniques students use to approach varying challenges while learning the concept of isomorphism. The findings of this study support previous research that suggests students' understanding of isomorphism is largely reliant on an imaged-based concept of symmetry (in this study, operation tables). Moreover, we found that students who progressed from a strictly imaged-based reasoning to a property-based reasoning demonstrated greater understanding of isomorphism. Even with these findings additional research is needed on how students develop an understanding of isomorphism and the impact of different curricula on students' understanding.

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Different Epistemological Frames Give Rise to Different Interpretations of College Algebra Lectures, Yet Pragmatic Decisions About Grades Swamp Productive Beliefs

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In this study, we present a comparative case study of two students with different epistemological frames watching the same college algebra lectures. We show that students with different epistemological frames can evaluate the same lectures in different ways, including very different evaluations of the goals and important content. Moreover, we illustrate that even when students have seemingly productive epistemological frames might give way to pragmatic decisions about earning a good grade when presented with too much information too fast. We argue that students might have productive dispositions towards mathematics, but default to a procedural orientation, and, as a result, appear indistinguishable in a class, from those who only have a procedural view of mathematics. These results illustrate how a student's interpretation of a lecture is not inherently tied to the lecture, but rather depend on the student and her perspective on mathematics and factors in the control of the lecturer.

Keywords: College Algebra, Evaluation of lecture, Student thinking, Epistemological frames

Sitting in a lecture is one of the most common experiences that students have in a tertiary mathematics class from introductory classes through their proof-based work (e.g., Mesa, 2018; Johnson, Keller, & Fukawa-Connelly, 2018). At the same time, there is strong agreement among mathematics educators, and some mathematicians, that lecture is ineffective at helping students learn mathematics (e.g., Bressoud, 2011). While studies have investigated the results of student learning gains and attitudes following video watching of math lectures (c.f. CITES), few studies have explored the ways that students interpret and make sense of lecture. For example, Weinberg and Thomas (2018) asked 12 students to watch calculus lectures in video form and engage in reflective dialogue in real-time. They found that students attempted to self-monitor for understanding but were often doing so in ways misaligned with mathematical meaning. While Weinberg and Thomas identified some ways that students attend to particular moments within a lecture, these perspectives all require students to make identifications of queues within the lecture. More research is needed to determine what students value in mathematics lectures and why they value these. Student beliefs about mathematics and what it means to do mathematics hold promise for better understanding what they might take from a lecture. For example, significant evidence suggests that students believe that mathematics is about following rules (e.g. Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Schoenfeld, 1989), and, consequently, Schoenfeld (1989) argued that student's evaluation of teaching depended on whether the teachers clearly presented the rules and how to use them. Similarly, other researchers have argued that student's beliefs interfere with their learning (e.g., Alcock & Simpson, 2004; Bressoud, 2016; Dawkins & Weber, 2017; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016; Solomon, 2006). Including the claim from Weinberg and Thomas (2018) that students are not always able to monitor their own understanding. Subsequent research has repeatedly suggested that their behaviors are indicative of their beliefs (Muis, 2015). Yet, little of this work has focused on how students interpret mathematics lectures. One notable exception is the work of Krupnik, Fukawa-Connelly, and Weber (2017) who explored how two students' differing epistemological frames (e-frames) lead to different interpretations of the same real analysis lecture. We build on this work to explore the following questions:

1. What meanings of mathematics do students have?
2. How do those meanings shape how students interpret mathematics lectures?
3. What other heuristics do students use to evaluate mathematics lectures?

Like other recent studies, we explore student's evaluation of lecture via the use of video for methodological reasons.

Theoretical Framing

Following Krupnik et al. (2017), we adopt Goffman's (1997) concept of frames, further specifying that we focus on epistemological frames, based on Redish's (2003) description. For Goffman, a frame is a means for individuals to make sense of complex social spaces. As an example of a frame, we might consider the 'art museum frame' in which someone entering an art museum would expect to find art displayed on the walls of a building for perusal. At the same time, there are expectation of behavior for people entering that social situation that include things like; 'quiet discussion' and 'don't touch the art.' Violations of these heuristics are likely to lead to some sort of social sanction. Such a frame might be counter-productive at a children's museum, such as the Please Touch Museum in Philadelphia, which encourages people to touch and interact with the exhibits.

Physics educators, Redish (2004) included, refined the notion of frame for an academic setting. These epistemological frames (e-frames) guide people's expectations for pedagogical settings such as mathematics classes. They might then not develop the desired conceptual understanding. Krupnik, et al summarized e-frames as such:

> "These consist of an individual's responses to questions such as "what do I expect to learn?" and the related questions of "what counts as knowledge or an intellectual contribution in this environment?" and "by what standards will intellectual contributions be judged?" (Redish, 2003)(p. 174, 2018)

Krupnik et al. (2017) used this notion to explore how two students reacted to the same real analysis lectures. They described Alice as holding the position that one needs to define a concept in order to reason about it, and, consequently, that making claims and providing justification requires precise definitions. Relatedly, Alice believed that providing a definition was a mathematical contribution. For Alice, this meant that the idea of re-defining the rational numbers was a mathematical contribution because then she could make claims and provide justification for those claims about the rational numbers. They claimed that Brittany did not concern herself with a formal definition, and, instead believed that definitions were better when they were comprehensible and provided new insight into a concept. Because Brittany believed that she had a strong understanding of the rational numbers, she believed the re-presentation of the rationals to be relatively useless. While she might write proofs that comply with the norms of the class, she may not substantively change her conceptions, and, miss fundamental ideas in real analysis.

We might similarly reinterpret Weinberg, Wiesner, and Fukawa-Connelly's (2014) exploration of student sense-making in abstract algebra lectures. For example, Weinberg et al. (2014) showed an example where a professor drew a diagram off to the side of the main lecture notes. The stated goal of the lecture was to define the rational numbers as a set of equivalence classes. They claimed that a student, Jocelyn, used a communication-oriented frame to determine that the diagram on the side "was not the answers that he was looking for," while noting that the diagram was a diagram representing generic equivalence classes. Another interpretation is that the instructor might believe that making explicit connections between any particular example of equivalence classes and the abstract concept is a mathematical contribution and can help students build understanding of the abstract concept. In contrast, the student might believe that answering the asked question was the meaningful mathematical contribution. Our contribution is a further exploration of the relationship between student's conceptions of mathematics and their evaluation of mathematics instruction.

Methods

Participants

We solicited the participation of four students enrolled in a College Algebra class at a large, east-coast university although we only report on two here. The university requires four college-preparatory mathematics classes for admittance, meaning all of the students had passed, at the least, a precalculus class while in high school. We note that all four of the students were intending to major in some type of non-STEM education field. We do not know how this might shape their thinking about mathematics and mathematics teaching.

Data Collection

During the first interview, participants were shown two videos. The first was primarily procedural instruction of the mathematics topic while the second was a more conceptual viewpoint of the same topic. Following each video, participants were asked the same series of questions, which included the following:

1. What did you notice about the video?
2. What did you think was important to take away? Why?
3. What in the video did you notice, but not find valuable? Why?

To further probe thinking about the video content, participants were also asked about their prior knowledge of the content, as well as if they were confused by anything in the video content or presentation and if they thought the videos were similar or different in any way. Our purpose for asking these questions was to identify initial ideas about the e-frames of the participants. After the first interview, we listened to the audio of the interviews and developed initial hypotheses about the participant's e-frames which guided our selection of the video for the second interview. For the second interview we showed the students a video that contained a mixture of procedural and conceptual content and we asked the same three-question protocol as in the first. Finally, participants were asked a series of questions to elicit information about their e-frames, including:

- What does it mean to be good at math? Why?
- What do you hope to get from attending lectures in mathematics?
- What makes a good lecture? What makes a bad lecture?
- What do you think it means to understand a mathematical concept?
- What do you think makes a good mathematical explanation of a concept?

Data Analysis

The goals of our analysis were to develop a set of claims about the heuristics students evaluate instruction and ground those heuristics in their beliefs about what it means to know and do mathematics. As a result, after transcription, we followed Mason (2002) in our analysis of the student interviews and attempted to:

(i) give an account of the e-frames that each student holds,

(ii) give an account for the evaluations that each of the students gave to the respective mathematics lectures (videos).

We first coded each student's claims about what it means to know and do mathematics. While many of these were made in response to specific prompts about these ideas, students often made unprompted comments in their other responses. We identified such instances when they made explicit claims about 'math class' or 'doing math' that moved beyond the specific context being discussed. In our next round of coding, we summarized the student's comments about the different videos. We particularly attended to two types of claims, when the students gave a statement about the mathematical goals or contribution that the professor was intending to make, or, when the students made evaluative comments and comparative comments about the mathematics of the videos. We distinguished those that focus on mathematical content and those that focus on aspects of the presentation. Then, we categorized each comment as either supporting or contradicting an e-frame for each student, or, as needed developing a new hypothesis for an e-frame. We rejected any hypotheses when we did not find sufficient support for it (e.g., few supporting claims), or, we found significant inconsistent evidence. We present the data as contrasting case studies to illustrate how these students hold different e-frames and evaluated the videos in different ways but might all appear to have a procedural focus.

Data and Results

Lauren's Conception Of What It Means To Do Mathematics: Mathematics Includes Decontextualized Problems That Can be Solved Efficiently Through Memorized Equations

Lauren believes that someone who is good at mathematics, "can solve problems really quickly and everything like that but I've come to know that it really means like memorizing equations." That is, for Lauren, being proficient at mathematics means having equations memorized that she can then use to solve posed problems. She later claimed that "having those equations memorized" was a first step towards proficiency. For Lauren, the second step towards being good at mathematics requires, "knowing which equation goes with which type of problem." We interpreted this claim as meaning that being good at mathematics requires being able to select an appropriate procedure to accomplish a required task. She later specified that she felt proficient at mathematics because "I know which equations to use, like I know how to do it at this point." She repeatedly returns to the notion that "I prefer to see the equation," because she feels that following a procedure gives certitude and she would only attempt something new or different "when you're lost and don't know what to do." But, critically, "in the box thinking (procedural) is more important because math is very straight forward." When she does mathematics, she prefers to use, one, single procedure in the way that it was taught. She stated, "I feel like I always like to use the equation," which she contrasted with "outside the box" thinking. She reiterated a nearly identical claim repeatedly in the interviews, for example later claiming, "I always just think of it as, here's an equation, plug it in, and solve. I don't really think outside of the box I guess." We summarize her perspective on mathematics as believing that mathematics is best done via procedure, and, it requires both memorizing procedures and knowing which procedure to apply at a particular time.

Lauren's Evaluation Criteria: Mathematics Instruction Is About Presenting Procedures And Explaining When They Are Used.

Lauren's Heuristic 1.1: Good mathematics instruction involves clear presentations of procedures. When Lauren evaluates pedagogical presentations she uses a variety of heuristics, all of them tied to her goals for mathematical proficiency. The primary evaluative heuristic for a pedagogical presentation that Lauren uses is the clarity with which the instructor presents the steps in a procedure and when to use it. That is, she values a clearly presented procedure with examples of the process. When presenting an example of the procedure, this should also include an explanation of how each number was derived. She gave a positive evaluation of procedural videos, repeatedly noting that they are "really clear." For example, she claims "like if you were to just skip from negative twenty to positive twenty someone else might be confused by that and then how he just wrote it out and just explained how he got each product and then which lead to the answer for y like that was clear as well." In this quote, she specifically stated that the explanation of the derivation of a particular value was "clear" and she valued that he "explained how he got each product and then which lead to the answer," we interpreted all of this as her valuation of a detailed presentation of steps, including derivation of numbers.

Moreover, her only critiques of procedural videos came when she felt that the procedures or exemplification omitted details, for example, noting, "The only thing I got confused about… when he came up with the two for the vertex, I feel like he just pulled it out of thin air." The moment referenced by Lauren occurred when the lecturer in the video derived the vertex from equation y=(x-2)^2-5. He began by stating that (x-2)^2 was necessarily greater than or equal to zero. He continued by reasoning that since the vertex was a minimum it could only occur when (x-2)^2 was zero, and hence x must be two. However, his argument was constructed from conceptual mathematical reasoning rather than procedural steps that could be followed. We interpret Lauren's reaction that the value 2 was "pulled out of thin air" as part of an e-frame in which a procedureless justification was the same as no justification at all.

Lauren's Heuristic 1.2: Good mathematics instruction involves explanations of when to use procedures. Lauren also wanted to know when to use a procedure and repeatedly praised videos that made this explicit. For example, "it was a pretty good video. I guess it's like important that you would use this equation when you have a complex equation like that one where it's not so easy to find what x is and everything so it was a good video. It was really clear." Here, her evaluative focus is that the presenter specified that a particular process or equation can be used for a particular task. When evaluating a video focused on the different forms of a quadratic function she claimed, "I liked how he made the chart showing what each equation, what you can see and what you can't see, that was really nice. Because it's just good to know … which one you wanna use, or what to expect when using it." That is, she evaluated the presentation as good because it was explicit about when to use each form, again, giving rules for accomplishing a particular task. More, she specifically stated that a lecture should help a student understand, "why you use all the equations you use" where her use of 'why' means picking the right procedure to accomplish a task. The fact that these are Lauren's primary heuristics for evaluating a pedagogical presentation in mathematics is perfectly aligned with her beliefs about what it means to do mathematics; to know procedures and when to use them.

While Lauren repeatedly claimed to value conceptual explanations and used language that suggests this, such as, that she values "knowing what different equations mean regarding the shape of the function" we interpreted her claim as being able to link the graph of the function with the symbolic form, not that she can describe *why* the graph has that particular shape. More, she repeatedly demonstrated that she is content to have only memorized procedures, repeatedly making a claim like, "I memorized it" and "it just is what it is." That is, while she might use language that appears to value conceptual understanding, she appears to mean how and when to use a procedure.

Joseph's Conception Of What It Means To Do Mathematics: Mathematics Is An Exercise In Problem-solving.

Joseph considered mathematical skill to be the process of "just being able to figure out problems and stuff," where we interpret the term problems to represent a decontextualized mathematical task. He believes that mastery of mathematics includes, "Being able to be given a problem maybe that you haven't seen before, but that connects a few of the concepts you've learned and you can reason your way around the things that you've learned to figure out what you're supposed to do about that." When discussing the goal of mathematical knowledge, or the purpose of mathematical lecture, he often referred to math's future applicability during an assessment situation. On four separate occasions he cited tests as the times he would be actively using mathematics. He gave no other examples of times in which he might use math. Additionally, he claimed to identify mathematical understanding in himself when, "–I can see a problem and particularly ... I think the most understanding is when you see a problem and you know what you can do to it and how to do it." We interpret this to mean that he views mathematics as a set of problems to be identified and solved, as opposed to a set of concepts to be applied situationally.

Joseph's Evaluation Criteria 1: Mathematics Instruction Provides the Learner with Conceptual Information that Is Pragmatically Useful.

Joseph's heuristic 1.1: Good mathematics instruction includes generalizations of conceptual information. In addition to lessening the need for memorization, Joseph considered the generalizability of conceptual information to be more powerful than specific examples and evaluated instruction positively when it was included. He stated that he would prefer instruction that included a general problem over one with specific procedures because, "you can apply it to whatever example you're using." He described his preference this way:

> And it makes it easier to understand ... 'cause sometimes on past math tests, math tests I've taken in high school, sometimes there might have been a concept I didn't really understand, and then in the middle of the test because I understood multiple concepts ... or rather not concepts, more like a problem I didn't understand ... because I understood multiple concepts I could figure it out and figure out something that I had missed or that I hadn't studied, and then I'd be able to answer the question correctly because I understood what was at work behind the stuff I was supposed to be doing. Maybe if I had forgot an equation I could figure out a different equation made up of other ones that I learned.

We interpreted this to mean that Joseph values conceptual information for its general applicability for broad swaths of problem-solving situations. Joseph applied the application of general conceptual knowledge to a specific problem-solving situation during the interview. While solving a completing the square problem, he became confused by the video's final step at the same point where Lauren did, as described in her Heuristic 1.1. However, Joseph noticed that the process of completing the square had converted the parabola's equation into vertex form and was able to identify the vertex from this context rather than attempt to replicate the presenter's reasoning, in doing so, he was able to actively apply conceptual knowledge in order to mitigate procedural confusion.

Joseph's heuristic 1.2: Good mathematics instruction involves conceptual information because it reduces the need for memorization. In evaluating a conceptual video Joseph described some information as being of the type that "you wouldn't really need to know how to do as a student, but if you understand it, it makes other things a lot easier." We took this to mean that although Joseph values more conceptual instruction, he is pragmatic in his valuation. Although he expresses that conceptual information is in itself unnecessary because it is not included on class assessments, he values this information because it can be applied to problem solving in testing situations. He specifically contrasted the instructional content in the conceptual lecture with procedural information that would be "needed for the test." Joseph recognizes conceptual information as useful because "the more you actually understand the reasons behind what you're doing, the less that you have to memorize for the test." While Joseph recognized that procedural fluency is what is assessed in exams, he also valued conceptual information from the lecture because it reduced the demands for memorization. He continued by stating, "It'd also be less memorization, of just memorizing equations and signs and stuff; you don't understand why they're the way they are." Joseph acknowledged that although conceptual understanding of mathematical situations could be, "more valuable, it also takes more time and more effort to acquire." However, he justified this burden explaining, "There's more bang for your buck, I guess you could say." Joseph values instruction that includes conceptual content because it helps him to solve problems when he cannot remember memorized information.

Discussion

The purpose of this paper was to explore the relationship between what students believe to constitute mathematically valuable activity and their heuristics for evaluating mathematical pedagogical presentations. More, unlike Krupnik et al., (2017) work, these students were enrolled in the course for which they were evaluating the presentations, and, they could ostensibly derive benefit from the videos as they had not yet taken their final exam. In each case, the students' beliefs about what constitutes mathematical activity guided their evaluations of the different pedagogical presentations. We note some limitations, we only studied 2 students and a few presentations of very limited duration. It is possible that neither the students nor the videos had sufficient variation to capture enough meaningful differences. At the same time, the two students had very different beliefs about what counts as mathematics, and, as a result, gave very different evaluations of individual lectures and even different components within the lectures. While Lauren valued only the procedures, Joseph valued conceptual explanations for a number of reasons. Yet, when those conceptual explanations might prevent the student from learning the procedure, they would stop attending to the conceptual aspects. The students made a rational decision in that both students recognized that only procedural proficiency was required to be successful on mathematics exams. Thus, we note that while to an observer, it might appear that students only value the procedural aspects of a mathematics lecture this is not necessarily true. It might be a form of coping mechanism based on a rational decision-making process. More though, it means that even though students might have productive beliefs, these might not be visible to observers, instead it might appear that all students have a procedural focus. In none of the videos did the instructor attempt to explain what it means to do mathematics. As a result, there was nothing to challenge either of the students' beliefs, meaning students could only interpret the lectures through the beliefs that they already held. Perhaps by specifically teaching about meta-mathematical issues an instructor could change what students attend to and take from a mathematics lecture. Yet, as a final note, based on the very different desires of the students in terms of detail, it would be impossible to give a mathematics lecture that satisfies all students.

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The Transfer and Application of Definitions From Euclidean to Taxicab Geometry: Circle

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Research shows that definitions in mathematics are often not used correctly by students in mathematical proofs and problem-solving situations. By observing properties and making conjectures in non-Euclidean geometry, students can better develop their understanding of these concepts. In particular, Taxicab geometry is suggested to be introduced before other non-Euclidean geometries since it is a considerably simpler space. To further investigate this, APOS Theory is used as the framework in this analysis of responses to a real-life situation from students enrolled in a College Geometry course at a university. Through the perspective of APOS Theory, this report provides two representative illustrations of the conceptual understandings found among these students in relation to the definition of circle. By adapting and applying their knowledge of definitions from Euclidean geometry to Taxicab geometry, these students provided insight as to how Taxicab geometry concepts are assimilated into their existing understanding of concepts in geometry.

Key Words: Definitions, Geometry, Taxicab, Circle, APOS Theory

Introduction

Edwards and Ward (2008) found mathematics majors exist that do not understand the role of definitions in a mathematically acceptable way but have been deemed successful students in advanced mathematical courses. The authors explain that this should be addressed in undergraduate mathematics, and research is needed to determine pedagogical strategies that help facilitate student's understanding of the concept of definition. Emphasizing the importance of definitions in geometry, Güner and Gülten (2016) explain that geometry has three dimensions: definitions, images that represent these definitions, and their properties. In such context, since the properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition," (Fischbein, 1993, p. 141).

In college geometry courses, Euclidean geometry and its axiomatic system is thoroughly studied, but other axiomatic systems receive little consideration (Byrkit, 1971; Hollebrands, Conner, & Smith, 2010). This is despite the fact that research shows by exploring concepts in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). One example of a non-Euclidean geometry in which students can explore concepts is Taxicab geometry. This is the geometry that is the result of measuring distance as defined by the L_1 norm. Siegel, Borasi, and Fonzi (1998) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it easier for students to reason and thus abstract concepts. Consistent with this claim, Dreiling (2012) found that "through the exploration of these 'constructions' in Taxicab geometry… [students] gained a deeper understanding of constructions in Euclidean Geometry," (p. 478). For this report, we present results and discussion on the following research question: In what ways do students assimilate the definition of a *circle* in Taxicab geometry into their existing understanding of this concept?

Theoretical Framework

APOS Theory is a constructivist theory based on Jean Piaget's theory of reflective abstraction, or the process of constructing mental notions of mathematical knowledge and objects by an individual during cognitive development, (Dubinksy, 2002). In APOS Theory, there are four different stages of cognitive development: Action, Process, Object, and Schema (Arnon et al., 2014). In addition, there are mechanisms to move between these stages of cognitive development, such as interiorization and encapsulation. An Action in APOS Theory is being exhibited when a student is able to transform objects by external stimuli or perform steps to complete a transformation. As a student reflects on Actions, they are able to *interiorize* them, so they can imagine performing these Actions without actually doing so. In this case, we refer to interiorized actions as a Processes. A student can then *coordinate* processes with others within a schema in order to form connections between concepts. Once a student is able to think of a Process as a totality to which Actions or other Processes could be applied, we say that an Object is constructed through the *encapsulation* of the Process. Finally, the entire collection of Actions, Processes, Objects, and other Schemas that are connected to the original concept that form a coherent understanding is called a Schema (Dubinsky, 2002). We provide examples of evidence of the stages of cognitive development in APOS Theory for the concept of **Radius** within the context of this paper. When given two points and asked to find the center of a circle such that these two points are on the circle, if an individual does so by counting blocks or guessing and checking at radii lengths until they find an appropriate one, they are exhibiting an action conception of **Radius**. In the same scenario, finding the total distance between both points and dividing by two provides a possible radius for a circle whose center is equidistant from the two points. In this case, the individual is exhibiting a process conception of **Radius.** We note there are an infinite number of circles that can be constructed such that two given points are on the circle. If a student is aware of this and explains this implies there is more than one radius measure for which such a circle can be constructed, he or he is exhibiting an object conception of **Radius** since the student is performing an action of comparison on his or her **Radius** object.

A genetic decomposition is defined as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). For this study, a genetic decomposition was developed to identify development pathways students may follow to adapt their working understanding of the definition of a circle to incorporate concepts in Taxicab geometry. In other words, this report focuses on how students assimilate the concept of **Circle** in Taxicab geometry into their existing *circle schema.* The subconcepts of **Circle** as defined in this report are **Distance, Radius, Center,** and **Locus of points.** In order to construct a **Circle** process, a student must have a process conception of at least two of these subconcepts. Figure 1 shows a possible way a student can construct a **Circle** process by the coordination of his or her **Distance, Radius, Center,** and **Locus of points** processes.

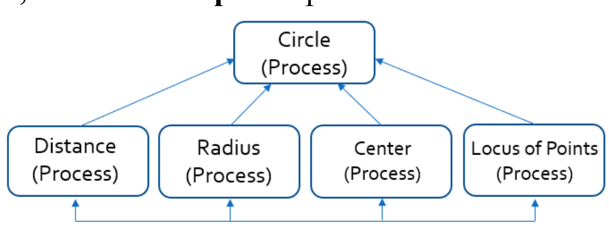

Figure 1. A construction of the Circle process.

A possible pathway a student may take in order to assimilate the concept of a circle in Taxicab geometry into their *circle schema* is shown in Figure 2. We show this assimilation using the subconcept of **Distance** but note that each of the subconcepts mentioned prior is expected to be assimilated into the *circle schema* in a similar manner.

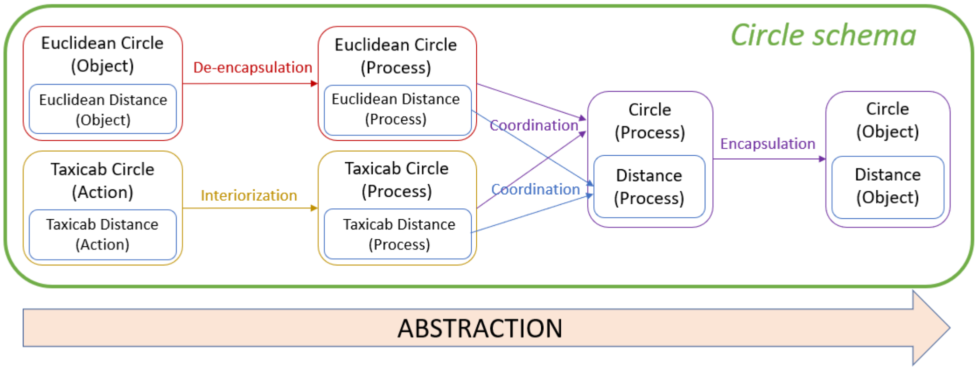

Figure 2. A possible way a student may assimilate Taxicab distance into his or her circle schema.

Methodology

This research study was conducted at a university in a College Geometry course during a Fall semester, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were both undergraduate and graduate students enrolled in the course. The textbook used in the course was College Geometry Using the Geometer's Sketchpad (Barbara E. Reynolds & William E. Fenton, 2011), written on the basis of APOS Theory. This study consisted of sessions of instruction on Taxicab geometry by one of the authors of this report, followed up with interviews conducted by the other author. The material of the course covered concepts and theorems in Euclidean geometry often seen in a College Geometry course and included Taxicab geometry for four 75-minute class sessions at the end of the semester. Written work from the semester and videos from the in-class group work and discussion during the Taxicab geometry sessions were collected and used as data in the study. After the semester but before final exams, semi-structured interviews were conducted with participants from the course. These interviews were conducted with 15 of the 18 students enrolled in the course who voluntarily signed up to participate in the interviews. All 18 students consented for their in-class group work and discussion to be recorded, as well as written work and exams throughout the course to be collected. Results from the analysis of student responses to a question on the final exam pertaining to concepts in Taxicab geometry are presented below within the context of APOS Theory. We focus our attention in this paper to responses from two of the 18 students enrolled in this course who were both secondary mathematics teachers and graduate students. We note prior to presenting results that students learned a continuous model of the Taxicab metric (or the L_1 norm). That is, distance between two points is measured continuously, not discretely. The problem on the final exam was stated as follows:

> Assume [a university's] campus and surrounding streets are designed explicitly in a grid pattern, i.e.- distance is measured by Taxi-distance. You are looking for an apartment near campus, but you want to make sure that from your apartment, the walking distance to [Building 1] (located at (-2, -2)) is the same as the walking distance to the [Building 2] (located at (4, 3)), since you have classes in both locations.
>
> a. Draw a graphical representation of where your apartment could be located, given that it needs to be equidistant from [Building 1] and [Building 2].
> b. What mathematical term would describe what you have drawn in your sketch?

The expectation for this problem (and an ideal solution) would be for students to recognize that there are an infinite number of places they could have an apartment so that its location is equidistant from the two buildings, with a midpoint having the shortest distance. Further, students should identify that the set of points equidistant (in Taxicab geometry) from both buildings is the equivalent of the Euclidean perpendicular bisector of the segment connecting the two buildings. The problem was open-ended without explicitly asking for students to identify a specific location for their apartment, but rather asked them to draw a graphical representation of the problem. For this reason, responses were expected to vary with regard to what mathematical term students associated with their drawing.

Results

As representative illustrations, we provide the APOS Theory based analysis of Kym's and Hannah's solutions to the exam problem as they correspond to this preliminary genetic decomposition.

Provided in Figure 3 is Kym's solution to this problem on the Final Exam. Kym was a graduate student and secondary mathematics teacher enrolled in the course who also participated in the interviews prior to this exam. Note Kym seemed to be operating with a discrete model of Taxicab geometry, as evidenced by her note "let 1 unit = 1 block."

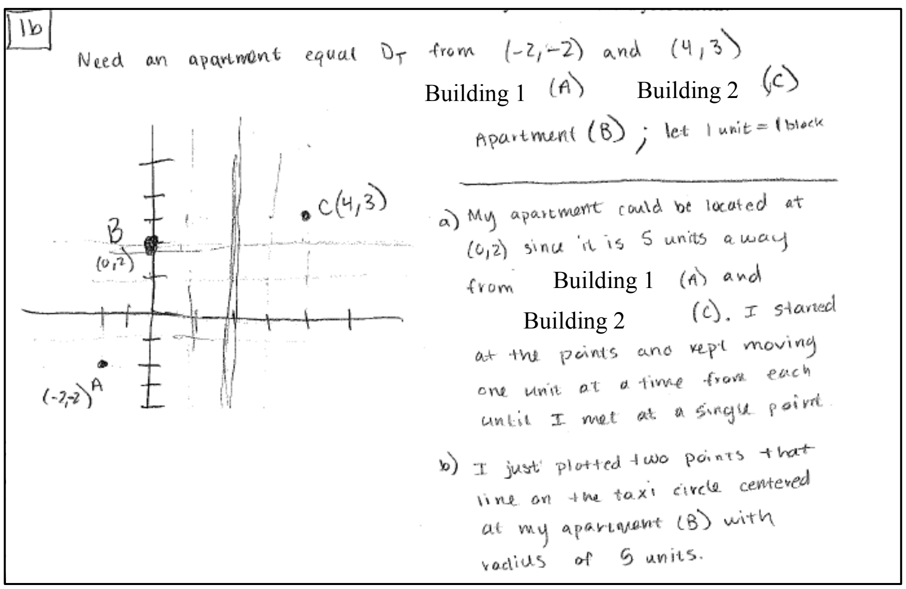

Figure 3. Part of Kym's solution to the given problem.

As we can see in the bottom right of Figure 3, Kym described her sketch as the Taxicab circle centered at her apartment with a radius of 5 units, mentioning prior that the two buildings would lie on this circle. Note that by saying she "plotted two points that [lie] on the taxi circle," she has in a way reversed the direction of the problem, since the problem was to find a point equidistant from these two points, not to plot two points equidistant from some fixed point. In any case, she demonstrated with this statement that she understood a Taxicab circle has this property of equidistance between the center and the points that lie on the circle. By saying in part (a) she "kept moving one unit at a time" to count out her distance between these points, it appeared that Kym was creating/constructing two radii of this Taxicab circle. Kym seemed to have at least a process conception of **Distance** and **Locus of points** since she could imagine a circle with the buildings lying on this circle. With the evidence provided in her solution, by stating in the past tense how she found this point/center (operating within the context of this problem) and counting blocks to define the radii of the circle, Kym was exhibiting an action conception of **Radius** and **Center.** Her solution point of $(0,2)$ was not actually the center of a circle with the buildings lying on the circle, since from her solution point to the buildings, one Taxicab distance is five and the other is six. However, we believe this inaccuracy was due to her operating with a discrete model of Taxicab distance.

Like Kym, the next solution presented was provided by a graduate student and secondary mathematics teacher who also participated in the interviews prior to the exam. The following was Hannah's solution where she provided an equation of this circle and exhibited an object conception of some of the subconcepts of **Circle** in a way not accounted for by the genetic decomposition.

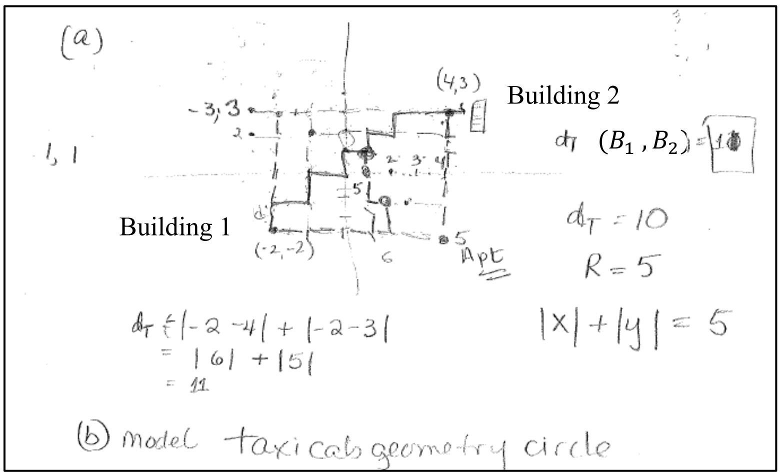

Figure 4. Hannah's solution to the given problem.

Seen in Figure 4 as her part (b), Hannah first described her sketch as a "model taxicab geometry circle." It appears as though Hannah first thought the distance between the buildings to be 10 units, as seen in the mid-right area of her work with what she wrote as "$d_T = 10$," although above this she corrected her initial calculation to be 11. This perhaps led to her labeling the radius to be 5 when she wrote "$R = 5$." A closer look at Hannah's drawing provides evidence she was attempting to find a center of a circle by constructing radii of length 5 or 6, with what looks like steps in her drawing. This is evidence that, like Kym, she most likely discretized the Taxicab metric which could be why she was counting radii of lengths 5 or 6. Hannah then tried to write the equation of the Taxicab circle she mentioned in part (b). By first finding a value for the radius and using this value to construct a possible circle, Hannah exhibited a process conception of **Radius.** Further, she attempted to plug this value into an equation of a Taxicab circle. Thus, Hannah had encapsulated her **Radius** Process into an Object since she was using it as an input into some function whose output was a Taxicab circle equation, performing an action on this Object.

Given the location she chose as her apartment of $(4, -2)$, indicated in the lower right of her graph as "Apt," the correct equation of this Taxicab circle would be $|x - 4| + |y + 2| = 5$. However, she wrote this equation as $|x| + |y| = 5$. It is possible with her equation she was attempting to indicate for the center of the circle and a point on the circle, "the change in x plus the change in y is equal to 5," but we do not have further evidence of this claim. Regardless, she was able to imagine that the solution would be the center of a circle, as indicated by her drawing and calculation of a possible radius for such a circle, which implies Hannah was exhibiting at least a Process conception of **Locus of points**. Although her equation of a circle is not indicative of a process conception of the algebraic representation of **Taxicab circle**, Hannah's geometric solution and approach to the problem provides evidence of at least a process conception of **Radius, Distance,** and **Locus of points** in Taxicab geometry and that she had coordinated some of these processes. Hannah also exhibited evidence of an object conception of **Distance** and **Radius**.

The mental structures necessary to write or derive the equation of a circle were not explicitly considered in the genetic decomposition. To write the equation of a circle, a student would need to identify an appropriate length for a radius and the center of a circle that is this distance from both buildings, specifying the metric used. It is possible that a student can write the equation for a circle but not understand or be able to explain how each part of this equation is a result of the definition of a circle and its subconcepts. In this case, he or she would most likely be memorizing a template for this equation. If the student has a process conception of several subconcepts, they may not have coordinated them with one another to make necessary connections to understand the equation's derivation. In this case, a student exhibits an object conception of all of the subconcepts of **Circle**, since he or she is using them as inputs into a mental function, but does not have a coherent understanding of the underlying structure of the *circle schema*. This understanding may be gained by the student *de-encapsulating* his or her object conceptions of each of **Distance, Radius, Center,** and **Locus of points** and coordinating them with one another to observe these relationships. An illustration of this is provided in Figure 5. In particular, the blue arrows indicate the de-encapsulation of all of these objects into processes. The red arrows in this figure indicate the possible coordination that could then occur among these processes.

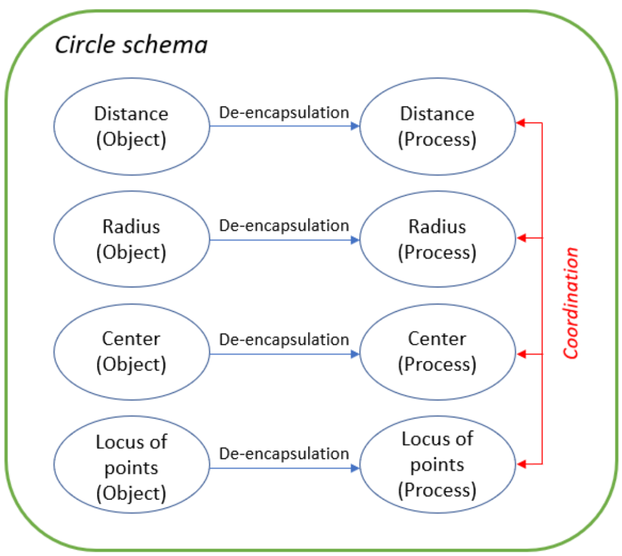

Figure 5. The de-encapsulation of objects to coordinate processes within the circle schema.

For this problem on the final exam, no students exhibited an object conception of all subconcepts of **Circle**. This may be a result of the manner in which the problem was stated since it did not necessarily require students to utilize an object conception. In the Discussion section, we present suggested questions that can help probe for this, as well as guide students in the construction of mental structures that are necessary to encapsulate these processes.

Discussion and Concluding Remarks

Fischbein (1993) explains that in geometrical reasoning, a major obstacle is the tendency to "neglect the definition under the pressure of figural constraints," (p. 155). By designing a problem where a student is essentially told a definition and has to derive the associated mathematic term, we hoped to overcome this obstacle in that it would minimize any misconceptions a student may have associated with a concept. Supporting this notion, although Taxicab circles look different than Euclidean circles, Kym and Hannah were able to use their geometrical reasoning skills to arrive at a solution to the given problem. They did so by applying their knowledge of definitions to correctly identify a mathematical term that satisfies the conditions of the problem. This exam problem also illuminated a misconception which became evident in other students' work in addition to Kym's and Hannah's: discretizing the Taxicab metric and not operating with it as a continuous measure. This led Kym and Hannah to somewhat disregard their understanding of the preciseness of the definition of a circle to identify locations which were almost equidistant from the buildings, but not exactly. This is consistent with Smith (2013) in that the author found it necessary to have conversations with students about how it was possible to draw line segments "through the grid" even though a car would not be able to drive through the blocks in a city. Referring back to Fischbein (1993) and this idea of a figural constraint creating pressure to neglect a definition, in these cases the figural constraint was the manner in which distance was defined. Thus, when introducing the Taxicab metric, educators should emphasize the continuity of the metric even when illustrating Taxicab concepts with situations that are discrete in real life.

In this paper, illustrations of various understandings of concepts in Taxicab geometry exhibited by two students in a college geometry course were provided. In the given problem, we hoped to help students develop a deeper understanding of these definitions and how to apply them. By using APOS Theory to analyze these students' solutions to a real-life situation, we were able to uncover some common misconceptions about Taxicab distance and circles. For example, multiple students believed it was not possible to travel in non-integer increments, i.e. – "split" units. This did lead to students attempting to optimize distance under a certain constraint, which was not intentional. These students could imagine a circle with a center that satisfied the problem but struggled to correctly identify a point that would actually be equidistant from the two buildings specified in the problem. We provide a suggestion to add as supplement to existing questions or to re-phrase the initial problem. By doing so, we hope to gather more details about how students understand the concept of **Circle**. This suggestion is as follows: Draw a graphical representation of how a Taxicab circle could be used to identify a location for your apartment, given that (i) You want to be exactly halfway between the buildings, and (ii) you do not want to be exactly halfway between the buildings. Is there more than one way to do each of these? What is this distance called in relation to the definition of a circle? Can you write the equation of either of the circles you have identified in (i) and (ii) using the definition of a circle?

Future research would investigate if these questions would help students to better assimilate the concept of a **Circle** in Taxicab geometry into their existing *circle schema.* There are other concepts that could emerge from the initial question posed such as **Midpoint** and **Perpendicular bisector**. Further research would investigate what questions could be asked for students to better develop their understanding of these concepts as well.

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Exploring Relationships Between Undergraduates' Plausible and Productive Reasoning and Their Success in Solving Mathematics Problems

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This study examines how the use of plausible and productive reasoning in mathematical problem solving (MPS) influences student performance on non-traditional problems. Data comes from ten individual, task-based interviews with College Algebra students. In general, students who demonstrated high use of plausible and productive reasoning had a higher percentage of correct answers on interview tasks than their peers. We propose reasons why a student may use plausible and productive reasoning and still arrive at an incorrect answer; we also consider how a student may use suboptimal reasoning and reach a correct answer.

Keywords: mathematical problem solving, plausible mathematical reasoning, College Algebra

Schoenfeld (1985) indicated that possession of relevant mathematical knowledge, facts, algorithmic procedures, and other domain knowledge were not sufficient for student success in mathematical problem solving (MPS); students often fail at MPS for other reasons. The purpose of this study is to explore the relationship between entry-level undergraduates' MPS practices and the correctness of their answers to mathematical problems. In particular, we focus on undergraduate students enrolled in a College Algebra course to explore the following research questions: a) To what extent is the amount of plausible and productive reasoning a student exhibits related to their success in accurately solving mathematics problems? b) What factors may contribute to perceived discrepancies between the amount of plausible and productive reasoning a student exhibits and their success in accurately solving mathematics problems?

Theoretical Perspective

The research literature contains several definitions for a mathematics *problem* (e.g., Schoenfeld, 1992; Wilson, Fernandez, & Hadaway, 1993). In our work, we adopt Lester's (2013) definition that "… a problem is a task for which an individual does not know (immediately) how to get an answer …" (p. 247). We distinguish a problem from a mathematical *exercise*, which we consider to be a routine scenario for applying mathematical knowledge and skills (Schoenfeld, 1983). By our definition, a particular mathematical task may be a problem for some students and not others (Schoenfeld, 1985). The problems we discuss in this paper are aimed at the audience of entry-level university students.

The process of MPS has also been described and defined by several researchers, and Campbell (2014) analyzed 25 research articles focused on MPS to characterize the process. He categorized the explicit or implicit definitions of MPS in the reviewed articles. Álvarez, Rhoads, and Campbell (in press) revised and refined Campbell's initial categorization and identified five key domains of MPS.

- **Sense-making:** Identifying key ideas and concepts to understand the underlying nature of the problem. Attending to the meaning of the problem posed.
- **Representing/connecting:** Reformulating the problem by using a representation not already used in the problem or connecting the problem to seemingly disjoint prior knowledge. Using multiple representations or connecting several areas of mathematics (e.g. geometric and algebraic concepts).

- **Reviewing:** Self-monitoring or assessing progress as problem solving occurs, or assessing the problem solution (e.g., checking for reasonableness) once the problem-solving process has concluded.
- **Justifying:** Communicating reasons for the methods and techniques used to arrive at a solution. Justifying solution method(s) or approach(es).
- **Challenge:** The problem must be challenging enough from the perspective of the problem solver to engage them in deep thinking or processes toward a goal, "without an immediate means of reaching the goal" (Wilson et al., 1993, p. 57).

We also draw on Lithner's (2000) characterization of undergraduate students' reasoning as they solve mathematical tasks. Lither argued that students' reasoning could be *plausible* or based on past *experiences*. In using plausible reasoning, students rely on "the mathematical properties of the components involved in the reasoning" (Lithner, 2000, p. 167). Formal proof is an example of plausible reasoning, although Lithner's definition also allows for less-rigorous reasoning, as long as it relies on mathematical principles to reach a conclusion. By contrast, reasoning based on experiences relies on the student's past experiences in mathematics class or elsewhere. In this type of reasoning, students draw conclusions based on what they have observed or experienced in the past, without connection to the underlying mathematical principles. For example, when given a quadratic expression as part of a problem, students may assume it can be factored if they have worked primarily with factorable quadratics in the past. Lithner illustrated how plausible reasoning was sparser than experienced reasoning, but emphasized that reasoning from past experiences can be a useful strategy in MPS when students also use plausible reasoning in the process.

In this paper, we describe undergraduate students' MPS in terms of both the MPS domain they employ (Álvarez et al., in press) and the type of reasoning that underlies the use of that domain (Lithner, 2000). For example, a student may make a choice in representing a problem (representing/connecting domain) based on sound mathematical principles, or they may use a representation based on their past experiences.

Research Methodology

Setting

The data for this study comes from the Mathematical Problem Solving Item Development Project, in which we aim to develop efficiently-scored survey items assessing undergraduate students' MPS in each of the domains described by Álvarez et al. (in press): sense-making, representing/connecting, reviewing, justifying, and challenge. As part of the project, we use an MPS survey consisting of five mathematics *problems* and a number of associated *items,* with each item linked to one MPS domain. The problems were designed to be open-ended and appropriately challenging for undergraduates, but do not require knowledge beyond secondary-school algebra. A sample problem is shown in Figure 1. (For additional survey information, see Álvarez et al., in press.)

> Fun Golf, a local mini-golf course, charges \$5 to play one round of mini-golf. At this price, Fun Golf sells 120 rounds per week on average. After studying the relevant information, the manager says for each \$1 increase in price, five fewer rounds will be sold each week. To maximize revenues, how much should Fun Golf charge for one round?

Figure 1. Sample problem from MPS survey.

The MPS survey was administered during the fall 2016 semester at a large, urban university in the southwest United States. The survey was administered in College Algebra and Calculus courses designed for undergraduates intending to major in a STEM degree. A pre-test version of the survey was completed by 492 College Algebra students during class time at the beginning of the semester.

Participants

Participants for this study were 10 students chosen from the pool of 492 College Algebra students who completed the MPS pre-test in fall 2016. Interview invitations were sent to various students in an attempt to interview a diverse group of students in terms of gender and their performance on the pre-test. However, due to a limited number of responses to invitations, participants mostly represented a convenience sample. Of the students interviewed, four were male and six were female. All except one were 18 years old. All except one were STEM majors. All had completed a previous mathematics course at a level beyond second-year school algebra, graduated high school in spring of 2016, and were now enrolled in their first year of university studies. Eight had their last mathematics course within the last year. Pseudonyms linked to participant identification numbers were assigned to the students interviewed.

During the fall 2016 semester, each of the 10 participants took part in an individual, one-hour interview with one of the researchers. An interview consisted of completing three problems, during which the student was asked to explain their work while solving each problem[1]. Of the three problems, one was new to the interview participant. The other two problems were selected from the MPS pre-test the student had already completed. After solving one of these older problems, the participant was given a chance to review their original work and explain any differences in approach. All interviews were video-recorded and later transcribed, and all physical work was collected for analysis.

Data Analysis

To analyze the data, we used thematic analysis (e.g., Braun & Clarke, 2006; Nowell, Norris, White, & Moules, 2017). Interviews were conducted to understand the MPS practices of entry-level undergraduates, and we were particularly interested in the five MPS domains specified in the theoretical perspective. As such, a preliminary coding framework for the interviews was designed to identify only usage of the MPS domains (e.g., Miles & Huberman, 1994). While coding, we discovered that identifying only instances of MPS domain usage was insufficient for describing the subtle differences in student work. The coding scheme was then adjusted using both inductive and deductive approaches to incorporate an array of subcategories within each MPS domain (e.g., Nowell et al., 2017). In particular, each specific instance of an MPS domain was simultaneously assigned two sub-categorizations intended to describe its *utility* and *origin,* respectively.

The utility of an instance of MPS was further coded as *productive, conditionally productive,* or *non-productive.* Productive use of an MPS domain involved using that domain in a way that brought the student closer to an acceptable answer or that helped them avoid an unacceptable answer. Non-productive use of a domain corresponded to the negation of productive use. Conditionally productive MPS corresponded to work that led to a correct answer in the interview, but may lead to incorrect answers on other, similar questions.

[1] Students also responded to corresponding MPS assessment items under development. Items are not discussed in this paper, but are described in Álvarez et al. (in press).

Along the other axis, we further granulated instances of MPS by examining the origin of the student's MPS reasoning process. We adapted Lithner's (2000) classification of reasoning styles as either *plausible* or based on past *experiences*—with the same dichotomy applied to MPS domain usage. We also made use of a third category, *indeterminate*, for cases when the origin of a student's reasoning could not be determined.

For example, Amy was able to make a rough sketch of three parabolas that helped her to make progress toward solving one of the problems, and this was coded as productive representing/connecting using plausible reasoning. By contrast, when solving a different problem, Amy guessed that the graph of a relationship would look like the graph of either a cubic function or a linear function. This inference did not help her to work towards an answer, and it was not clear on what reasoning her conclusion was based. This excerpt was coded as non-productive representing/connecting with indeterminate reasoning. As a final example, when working on a problem involving revenue, Liz claimed that as the sales price increased, the revenue would increase to a point and then "it'll start going down because people will stop buying." In the problem, it was mathematically the case that the revenue reached a maximum and then decreased, but her conclusion is not generalizable to other situations. In addition, Liz's reasoning was not based on mathematics but rather her past experiences. Hence, this situation was coded as conditionally productive sense-making using experiential reasoning.

Transcriptions of the interviews were coded independently by at least two researchers. At multiple points in the coding process, the researchers compared excerpts of coding to refine the coding scheme and resolve conflicts. Once all coding was complete, the results were analyzed and collated, again resolving any remaining conflicts.

To gauge student success on the MPS survey problems, it was necessary to establish a grading scheme for assessing their work generated during the interview. Although each student addressed three unique problems, the third problem was often not attended to with as much detail or rigor as the first two problems. So, we elected to score only the first two problems by assigning each question 50% of the student's overall score for the interview; then, any problem that was comprised of more than one sub-problem divided its 50% equally among those sub-problems. For example, a student who completed the problems Fun Golf (a one-part problem) and Air Travel (a three-part problem) during the interview could earn 50% credit for correctly answering Ken's Garden and an additional 16.6% credit for each of the three parts of Air Travel they answered correctly.

We then considered the correlation between instances of MPS coded in the interviews and the score on the interview problems. We were also interested in the factors that may contribute to perceived discrepancies between the amount of plausible and productive reasoning a student exhibits and their ability to accurately solve these problems. To explore this, we revisited the coded data and searched for possible explanations for discrepancies, using an iterative approach to refine these explanations (e.g., Yin, 2009).

Results

Observable Correlations

We noted strong, positive linear correlations between students' interview scores and two separate, but related, metrics: instances of productive MPS based on plausible reasoning ($r = .813$; shown as Plaus/Prod MPS # in Table 1), and the percent of MPS instances that were both plausible and based on productive reasoning ($r = .807$; shown as Plaus/Prod MPS % in Table 1).

Table 1. Percentage of Plaus/Prod MPS corresponding to "interview score".

Participant	Score %	Total MPS #	Plaus/Prod MPS #	Plaus/Prod MPS %
Jill	0%	9	1	11%
Amy	0%	11	2	18%
Zoe	0%	8	2	25%
Dan	25%	11	2	18%
Sara	33%	20	5	25%
Liz	33%	13	5	38%
Ian	50%	7	1	14%
Matt	50%	14	5	36%
Bob	75%	14	6	43%
Kim	100%	15	9	60%

Although the strong linear correlations exist, we recognize that interview scores were more categorical than continuous and inconsistent among students. For example, Kim was given two problems that were each single prompts requesting one answer. Dan completed two problems that encompassed five total sub-questions. We also recognize that each student demonstrated a different number of discrete instances of MPS during their interview (Total MPS # in Table 1). For students who demonstrated a low overall frequency of MPS, the corresponding percent of plausible and productive MPS is also undesirably categorical. Ian and Zoe's interviews were examples of this flaw, and removing them results in a large increase in both r-values (to .940 and .942, respectively).

Taking these limitations into account, we were interested in possible explanations for why the percent of plausible and productive MPS used by a student may not have provided a direct prediction for the percent score they made on the interview questions. We discuss possible reasons in the following sections.

Plausible and productive reasoning in concurrence with incorrect answers

We now discuss possible reasons why a student may demonstrate a nonzero amount of plausible and productive MPS practices but still earn an especially low score on the interview problems. In general, it is sufficient to note that a problem often requires more than one instance of "good" MPS to arrive at a correct answer.

For example, Amy was solving a problem about two runners in a race. In her solving process, she revisited the problem statement and identified an error in her work, which is an example of productive reviewing using plausible reasoning:

> *Amy*: Alright, so looking at it… It just says that Brett finishes the 100 in 16 so that means that the 80 he did not complete in 16 so automatically I need to change that [erases mislabeled diagram]. So at this point if I don't understand it, I'll just take a guess.

However, as shown in the excerpt, although Amy was able to refer back to the problem text to identify and avoid a mistake, she was then unable to use good sense-making to correctly orient herself in a more productive direction, and ultimately she decided to "take a guess" at an answer.

Another example can be found in the interview with Zoe, who worked to solve the problem shown in Figure 1 regarding revenue at a mini golf course. After reading the problem, she made sense of the given conditions:

Zoe: Okay, so \$5 for one round equals 120 rounds per week. And they're saying if they increase by \$1, which could be \$6 [per round], they will get 5 fewer rounds, which would be 115 rounds per week, and then they want to maximize their revenue, how much they bring in, so they have to charge a dollar decrease by \$1 for \$4 [per round] which would give them 125 rounds per week.

Zoe demonstrated productive sense-making using plausible reasoning, both when describing the effects of changing the price for a round of golf and when correctly attending to the meaning of the word revenue. However, she then went on to display non-productive sense-making by incorrectly interpreting how to maximize the revenue, arriving at an incorrect answer.

Limited plausible and productive reasoning in concurrence with correct answers

We now propose possible reasons why a student can achieve an interview score significantly higher than the percent of plausible and productive reasoning they exhibit. First, non-productive or conditionally productive MPS need not lead to incorrect answers; and second, students who have false starts are able to later correct themselves through a combination of appropriately plausible and productive reviewing and sense-making.

Kim's work exemplified the first point. Her use of representing/connecting illustrates how our classifications of domain use may contribute to a misleading characterization of an approach. Kim displayed three unique instances of representing/connecting in her work across two problems. Each instance was plausible, but two were non-productive. These non-productive instances were "trivial" in that they did not explicitly lead to either a correct or an incorrect answer. For example, while working a problem involving the area of a rectangular garden, Kim drew a simple diagram representing the garden, which did nothing more than extract the relevant dimensional information from the problem text. This qualifies as representing/connecting and uses plausible reasoning, yet is non-productive because the diagram itself does not play a meaningful role in Kim's approach to the problem. Had Kim used the diagram to robustly model the situation, it would have been productive. But by drawing the diagram, Kim lowered her percent of plausible and productive MPS but still answered the problem correctly.

Kim provided another example, this time of conditional productivity, while working the Fun Golf problem (see Figure 1). Kim claimed, "Yeah. Cause I thought it would just keep going up, but I realized maximize and minimum would mean quadratic." This is an example of experienced and conditionally-productive reviewing. Kim reasoned about the behavior of the revenue function using her experiential association of the word "maximum" with the vertex of quadratic functions. The revenue function in Fun Golf does happen to be quadratic, but certainly not every optimization problem involves second-degree polynomials. Thus, this particular instance of MPS is neither plausible nor explicitly productive by our definition. Still, it contributed to Kim's eventual success in the Fun Golf problem by helping her assess her progress, lowering percent plausible and productive MPS but contributing to a correct answer.

Finally, we consider a student who commits to an incorrect approach to a problem until recognizing a mistake and correcting herself with plausible and productive MPS. Sara worked on the Air Travel problem, as shown in Figure 2.

A commercial jet is flying from Boston to Los Angeles. The approximate distance in miles between Los Angeles and the jet can be found using the function $g(t) = -475t + 2650$, where t is the number of hours the jet has been flying. (i) Find a function, f, modeling the plane's distance from Los Angeles (in miles) in terms of

v, where v is the number of minutes the plane has been flying. (ii) How far has the plane flown after 12 minutes?

Figure 2. Air Travel problem.

When beginning part (i) of the Air Travel problem, Sara remarked, "So since v is the number of minutes, and then this one, t, is the number of hours, we'd have to do v times 60." This excerpt is an example of plausible representing/connecting, because Sara drew a connection between the units using the variables given in the problem text, but the MPS is non-productive, because the relationship she described is not correct. However, Sara soon made the following realization when using her non-productive MPS as a basis for her approach to part (ii):

Sara: f is equal to -475, 60 times 12 plus... [mumbling] ...is 720 minutes. Hmm. [using calculator] Mmkay, what I—sorry, I didn't write it down, what I was doing was trying to see-- I think you have to divide it by 60. Because you're dividing the minutes into the hours... And so, I just checked seeing what 12 divided by 60 was to see if it was a fifth and it is a fifth, so. It would-- this would be v over 60, I would think.

This excerpt exemplifies plausible and productive reviewing. Sara realized that 60 times 12 is 720 minutes, not 720 hours, as she had previously implied. She used this insight to evaluate an alternative—that minutes divided by 60 equals hours—and used a computation to judge that this relationship is more reasonable. In this way, Sara leveraged her initial non-productive MPS toward a correct answer by eliminating an incorrect possibility. A student who is often engaged in non-productive MPS may eventually arrive at a correct answer but with surprisingly low percent of plausible and productive MPS.

Discussion and Implications

Our results suggest that the amount of plausible and productive reasoning that undergraduate students use in solving mathematical problems may strongly correlate to their success on such problems. However, we also provide reasons why a student's plausible and productive reasoning would not need to be extremely high to answer problems correctly and why a student may use plausible and productive reasoning yet answer problems incorrectly. As shown in our data, reasoning based on past experiences is not necessarily detrimental to the solving process, and in fact, as Lithner (2000) found, reasoning based on past experiences may be helpful in MPS. In addition, non-productive solving paths do not necessarily lead to incorrect answers.

Nonetheless, as Lithner (2000) cautioned, students often generalize from the examples and exercises they see in mathematics class, sometimes inappropriately. This can lead to over-application of experiential reasoning that is not balanced by plausible reasoning. Solving more non-routine problems may offer an opportunity for students to see that experiential reasoning is not always useful. Undergraduate mathematics instructors may want to ask students to explain their reasoning as they solve such problems and be attuned to the types of reasoning being used.

Future research could explore whether the trends that we observed hold for a larger sample. A larger sample could also illustrate whether certain MPS domains are more often backed by plausible and productive reasoning (or experienced or non-productive reasoning).

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Factors Influencing Linear Algebra Instructors' Decision to Implement Inquiry-Oriented Instruction

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This study investigates factors that influence instructors' decisions to implement inquiry-oriented instruction. We analyzed entrance interviews with twelve Linear Algebra instructors, who participated in the Teaching Inquiry-Oriented Mathematics: Establishing Supports professional development project, to better understand the reasons why the instructors chose to shift from traditional lecturing to inquiry-oriented instructional approaches. We found three internal and three external factors that influenced the participating instructors' choice to teach the inquiry-oriented Linear Algebra course. Implications for future research are discussed.

Keywords: inquiry-oriented instruction, linear algebra

Student-centered instructional approaches have received significant attention over the last several years. Although lecture is still the predominant way of teaching undergraduate mathematics courses (Eagan, 2016; Johnson, Keller, & Fukawa-Connelly, 2017), researchers suggest that implementing active student-centered instructional approaches, such as Inquiry-Based Learning (IBL) or Inquiry-Oriented Instruction (IOI), may be more beneficial for students' achievement, affect, and persistence in undergraduate mathematics (e.g., Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, & Wenderoth, 2014; Laursen, Hassi, Kogan, & Weston, 2014). To support the claim that inquiry-based teaching promotes positive student learning outcomes, Freeman et al. (2014) conducted a meta-analysis of 225 studies that compared achievement outcomes of students in undergraduate STEM courses taught via either active learning or traditional lecture approaches. The meta-analysis concluded that using teaching approaches that gave students opportunities to actively participate, rather than passively listen, reduced student failure rates and raised students' scores on exams. Research has highlighted the effectiveness of IBL and IOI, so there is a need for training mathematics instructors to adopt such instructional approaches. Several projects have been designed to train and support instructors in student-centered teaching approaches, such as NExT Project, the Academy of Inquiry-Based Learning, and the TIMES (Teaching Inquiry-Oriented Mathematics: Establishing Support) project, which designed inquiry-based curriculum for undergraduate mathematics courses and provided professional development for instructors to implement student-centered instructional approaches. This study explores what influences instructors' decisions to pursue such professional development opportunities to learn to implement IOI.

We specifically focus on Linear Algebra instructors. Knowledge of Linear Algebra is vital in multiple areas of science. In many universities, the course of Linear Algebra is usually taken by students of diverse backgrounds and educational pursuits. Instructors' pedagogical approaches play a crucial role in influencing students' interest, motivation, and success in this course. Therefore, it is worthwhile to explore reasons why instructors choose to use a certain instructional approach to teach Linear Algebra. The purpose of the present study is to explore common factors that motivate instructors to use IOI to teach Linear Algebra. The following research question was addressed: What factors influence Linear Algebra instructors' decision to implement IOI?

Literature Review and Theoretical Perspective

Student-centered teaching approaches differ from what is considered traditional lecturing in mathematics courses. The aim of student-centered instructional approaches is to enhance students' problem-solving skills, giving them opportunities to generate ideas, ask their own questions, and develop strategies for answering them (Laursen, Hassi, Kogan, & Weston, 2014). In inquiry-oriented classrooms, students are actively engaged in producing their own mathematical ideas in solving problems, rather than repeating algorithms demonstrated by the teacher. Students present their solutions in front of the whole class or in small groups, while other students critically analyze their peer's solution and provide their feedback. IBL and IOI give students opportunities to "do mathematics like research mathematicians do mathematics" (Yoshinobu & Jones, 2012, p. 307). A growing body of research studies suggests student-centered teaching has positive effects on student learning in undergraduate mathematics (e.g., Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, & Wenderoth, 2014; Laursen et al., 2014). For instance, Kogan and Laursen (2014) analyzed data from 100 sections of 40 courses and found that students who had been engaged in an inquiry-oriented classroom were more likely to succeed in subsequent mathematics courses than students who had been taught via the traditional lecture approach. This highlights the benefits associated with incorporating student-centered instruction.

A historical predecessor of IBL is the Moore Method, named after mathematician R. L. Moore (Coppin, Mahavier, & May, 2009). The implementation of this method varies among instructors, but the core idea is that instead of using a certain textbook, the students are given a list of theorems, which they are expected to prove using given definitions. After the students prove a theorem, they present it in class, while their class peers evaluate the validity of the proof. There are several differences between the Moore Method and IOI, albeit they are both forms of student-centered instruction; an example of such would be that students' collaboration is prohibited in the Moore Method, whereas student collaboration is expected in IOI.

IOI is informed by Realistic Mathematics Education, an instructional design theory, which considers mathematics as a human activity (Freudhenthal, 1973). One of the most important heuristics of RME is that the instruction should provide students opportunities to reinvent key mathematical concepts with the guidance of the instructor (Stephan, Underwood-Gregg, & Yackel). In this guided reinvention process, mathematical concepts are not presented to students by the instructor, as in traditional lecture. In contrast, the instructor poses carefully designed mathematics tasks for the students to collaboratively work on. These tasks are designed to promote the emergence of the mathematical concepts, as students develop an intuitive understanding of the concepts. The instructor then formalizes the students' knowledge of the mathematical concepts. In short, guided reinvention in IOI involves the students reinventing mathematical concepts with the support of the instructor.

The instructor plays an essential role in IOI courses (Rasmussen & Kwon, 2007). Kuster, Johnson, Keene, and Andrews-Larson (2017) emphasized, "by inquiring into student thinking, teachers are able to support students in generating more sophisticated ways of reasoning" (p. 6). Cobb, Wood, and Yackel (1993) argued that the teacher plays an important role in developing students' conceptual knowledge and providing opportunities to share the acquired knowledge with peers through collective discussion. Along with asking students questions and facilitating discussions, instructors are responsible for establishing and sustaining classroom norms which allow students to share with their mathematical ideas (Stephan et al., 2014).

Several studies over the past decade have examined factors that influence instructors' decisions to move away from using traditional lecture to implement inquiry-based teaching approaches. Johnson, Keller, and Fukawa-Connelly (2017a) investigated what affordances and constraints on the use of non-lecture practices Abstract Algebra "lecturers" perceive. The authors administered a national survey to Abstract Algebra instructors, which gathered data on their typical teaching practices, beliefs about teaching and learning, and contextual affordances and constraints for using certain teaching practices. The data revealed a number of contradictions in the participants' responses. On one hand, several instructors suggested a lack of time, curricular resources, knowledge, and supports were reasons why they would not choose to use instructional methods other than lecturing. On the other hand, the same instructors claimed that they might have time for redesigning their instruction, they did not feel pressure from their departments to cover a certain amount of material, and there were funds available for teaching professional development opportunities. Despite this reluctance to adopt student-centered instructional practices, Johnson et al. found that 65% of lecturers from institutions that offer Bachelor's and Master's degrees and 48% of lecturers from PhD-granting institutions would consider switching to non-lecturing instructional approaches. In another study, Johnson, Keller, Peterson, and Fukawa-Connelly (2017b) explored Abstract Algebra teachers' beliefs, habits, and constraints at Bachelors-granting institutions, i.e. traditionally "teaching colleges." Johnson et al. (2017b) investigated the extent to which these Abstract Algebra instructors employed non-lecture approaches. They found that in these institutions, lecturing is the predominant way of teaching. The authors concluded that reformers still have a long way to go in helping instructors implement student-centered practices in mathematics. This motivated the present study to explore why instructors chose to switch to using a non-lecture approach.

We follow Henderson and Dancy's (2009) theoretical framework of aspects that influence instructional practices: experience with and attitudes toward teaching innovations, instructional goals, and perception of department support. We expand this theoretical framework by adding instructors' beliefs about students' difficulties in learning Linear Algebra. We also conduct more detailed analysis of external pressures that affect instructors' choice to use IOI. One of the goals of our research is to further explore Johnson et al.'s (2017a) findings regarding the influence of departmental pressure on the instructors' choice to use certain pedagogical practices. We also aim to discover other influential factors that were not previously found in the literature.

Methods

The following section describes the context of the study, the teaching experience of the participating instructors, and the methods we used for data collection and analysis.

Context of the Study

This study is part of a larger research program, the NSF-funded TIMES project, which is a professional development program designed to support undergraduate mathematics instructors of Linear Algebra, Differential Equations, and Abstract Algebra in learning how to implement IOI. The professional development program provided instructors with training in a three-day summer workshop, as well as support through the provision of curriculum materials and weekly online peer working groups. This study explores the factors that influenced Linear Algebra instructors to implement IOI through participating in the TIMES project.

Participants

Thirty-six undergraduate mathematics instructors participated in the TIMES project as fellows. This study considers a subset of twelve of those instructors, all of whom taught the Inquiry-Oriented Linear Algebra (IOLA) course. These instructors came from a variety of institutions across the United States. The participating instructors exhibited differences in their amount of experience in teaching Linear Algebra. Two (17%) of the instructors had taught Linear Algebra three or more times before, six (50%) of the instructors taught this course a couple of times, and four (33%) of the instructors of instructors had never taught the course prior to teaching the IOLA course. The instructors also exhibited differences in their previously used teaching practices (i.e. lecture, IBL, or a combination of both). Five (42%) of the instructors described their own teaching practice as mostly lecture, five (42%) of the instructors claimed they used mostly IBL methods, and two (16%) of the instructors claimed to use both methods.

Data Collection

Semi-structured interviews were conducted by project personnel with each of the instructors after they took part in summer workshops, in which they learned how to implement the IOLA curriculum. These interviews took place before the teachers began using IOI in the classroom. An interview protocol was written and administered in each interview to ensure the participants responded to the same questions. Some follow up questions were posed by the interviewer to elicit clarification or more detailed responses from participants. The questions prompted the instructors to describe their past teaching experiences, their reasons for wanting to implement IOI, and the nature of the support they received from their colleagues. The interviews were audio recorded and transcribed for retrospective analysis.

Data Analysis

The first author analyzed the interview transcripts using thematic analysis, coding common themes that emerged from the data (Roulston, 2010). Initial codes were produced based on the author's interpretation of the data. Similar codes were reorganized into categories during second cycle coding (Miles, Huberman, & Saldana, 2014). To ensure dependability of the qualitative analysis, both authors met to discuss the codes and their pertinence to answering the research question. The authors compared the different instructors' responses to find trends in the topics the participants discussed.

Results

As themes emerged from the data, we recognized the themes could be categorized as either internal factors related to instructors' interests, beliefs, and goals, or external factors related to departmental or student expectations. Our research findings are presented in the following two sections organized by the nature of factors that influence the instructors' choice to use IOI. The first section discusses the internal factors, and the second section describes the external factors that appeared to influence the instructors' choice to implement IOI. The discussion in each of the subsequent sections describes the nature of the different internal and external factors that were evident in the interview data.

Internal factors

In this section we describe internal factors that seemed to influence the instructors' decision to implement IOI. These internal factors include the instructors' interests in implementing IOI, beliefs about students' difficulty in learning concepts in Linear Algebra, and instructional goals.

Instructors' interests in IOI. The instructors' interests in IOI seemed to influence their choice to pursue inquiry-oriented instructional methods, so we explored the instructors' given reasons for their interests in IOI. Several instructors (42%) cited their past student experience as a reason to move away from traditional lecturing. Some of the representatives of this group had negative experiences as students of lecturers. One such instructor claimed, "Reflecting back on my own schooling, I fell asleep in Calculus and in most of my math classes because I only had experienced a lecture style." Other instructors experienced positive effects on their learning after participating in inquiry-based courses. One of these instructors asserted, "When I was an undergraduate student, I had IBL topology. I hated it when taking it, but it helped me greatly." These past student experiences were influential in motivating these instructors to develop interests in IOI.

Many instructors (25%) attributed their interests in IOI to their involvement in professional development events, such as conferences, professional seminars, and workshops. We also found 15% of the respondents mentioned they were satisfied by their lecture-based approach, but they were curious if there were other ways of teaching that could be more beneficial for students. Another 15% of the instructors were inspired by the successful inquiry-oriented practices of their colleagues and the desire of their department heads to incorporate innovative ways of teaching in mathematics courses. All of these reasons for instructors' interests in implementing IOI seemed to influence the instructors' decisions to change their instructional approaches.

Instructors' beliefs about students' difficulty in learning Linear Algebra. The instructors' perceptions of students' difficulty in learning Linear Algebra influenced their decision to use IOI in the classroom. When asked what they perceived as the most difficult aspect of the course for students, half of the instructors referenced the shift from doing basic computations to solving abstract problems. One of these instructors claimed the most difficult part of Linear Algebra is, "the abstract nature of the subject, especially for those who have just been through the calculus series." Additionally, 25% of the instructors cited formal proof writing as the most challenging part of the course, and 17% of the respondents argued that the greatest difficulty the students faced in Linear Algebra was understanding how "everything is interconnected." None of the instructors cited computation as a challenging aspect of the course.

These beliefs about what students struggle with in the Linear Algebra course contributed to the instructors' decision to implement IOI. Several instructors highlighted the usefulness of inquiry-oriented teaching in helping students gain deep understanding of abstract concepts. For instance, one instructor said, "I discovered that students have a quite difficult time when starting the concept of basis and span… so I started thinking this is where the IBL can be useful." The instructors perceived IOI as a method that could facilitate students' development of meaningful understanding of the abstract concepts in the course. Overall, instructors' beliefs regarding students' difficulty in learning Linear Algebra and their views regarding the potential benefits of IOI in helping students overcome these difficulties served as contributing factors that influenced the instructors' decision to implement IOI.

Instructional goals. The instructors' teaching goals seemed to influence the instructors' decision to implement IOI. The instructors seemed to believe implementing IOI would provide a way for them to achieve their instructional goals. Half of the instructors had instructional goals of helping their students be able to "build arguments," "explain their reasoning", "reflect on others ideas," and "provide critical feedback." One such instructor described how using IOI could help in pursuing these goals, claiming, "If [the students] have conversations with others early, later they can have conversations with themselves." This instructor seemed to believe IOI provided opportunities for students to build habits of communicating their mathematical reasoning, which would be useful for the students as they take future mathematics courses. This notion was echoed in the instructional goal held by 33% of the instructors, which was that of fostering greater mathematical maturity in their students and preparing them for other mathematics courses. These instructors viewed IOI as a way to achieve their instructional goals of giving students opportunities to communicate about mathematics and develop mathematical maturity. Therefore, the instructors' instructional goals were influential in their decision to implement IOI.

External factors

The following section addresses the external factors that seemed to influence the instructors' decision to employ inquiry-oriented teaching methods. These external factors include pressure from student evaluations, departmental support, and content coverage expectations.

Pressure from the effect of student evaluations on tenure status. The instructors' perceived pressure from the effect of student evaluations on their tenure status might have a negative influence on some instructors' decision to implement IOI. One instructor described waiting to try innovative teaching methods until he was tenured because he was cautioned "not to rock the boat with students until after [his] job is secured." Students' potential lack of appreciation for innovative teaching may be exhibited in poor student evaluations of instruction, which could have a negative impact on instructors' tenure process. Another instructor confessed that his department chair advised him not to try anything new until he had been tenured, since negative evaluations may adversely affect his pursuit of tenure. This fear of poor student evaluations might deter some instructors from choosing to implement IOI. However, the majority of the instructors (58%) participating in this professional development were untenured or were not on a tenure track, so this worry of negative student evaluations did not seem to deter them from choosing to implement IOI. Some of these instructors mentioned that they did not really worry about students' evaluations. Overall, some instructors felt pressure of potentially being negatively evaluated by students, but this did not deter them from choosing to implement IOI. However, fear of negative student evaluations can influence instructors to not choose to incorporate innovative teaching approaches.

Departmental support. Several instructors perceived supportive attitudes from their department chairs and colleagues regarding their intent to implement innovative teaching methods. One of the instructors specified the nature of this support from his department chair, claiming, for "anybody who goes in with a new idea, and whether it is about an instructional approach or instruction needs for the classroom, he is supportive in finding a way to make those things happen." The instructors claimed to be given full autonomy to implement whatever teaching methods they chose. The instructors asserted that, in general, most of their colleagues were very supportive of their decision to implement inquiry-oriented approach. This supportive departmental environment seemed to influence instructors' decisions to implement IOI, in that they did not feel any discouragement from colleagues that would inhibit them from doing so.

Content coverage expectations. The instructors generally felt no constraint to comply with covering specific topics other than those usually covered in Linear Algebra. One instructor specified that the only concepts he was required to cover were vector spaces, maps, eigenvalues, eigenvectors, and some proofs. Another instructor claimed, "I don't have to serve anybody else's wishes." Several instructors reported they were not required to cover a certain list of topics, assign certain homework assignments, or administer certain exams. One of the instructors mentioned that there was a textbook he was required to use, but he was encouraged to incorporate supplemental instructional material. The instructors generally did not feel much pressure from the department to cover certain content. This lack of curriculum constraints and freedom to use alternative curriculum materials seemed to serve as contributing factors in the instructors' decision to implement IOI.

Discussion

This study explored influential factors that seemed to affect instructors' decisions to implement IOI. We found three internal and three external factors that seemed to influence the participating instructors' choice to teach the IOLA course. The internal factors included instructors' interests in IOI, beliefs about students' difficulty in learning Linear Algebra, and instructional goals. With this finding, we propose expanding Henderson and Dancy's (2009) theoretical framework of aspects that characterize instructional practices by adding the instructors' beliefs about students' difficulties in learning mathematical concepts.

The instructors in this study viewed implementing IOI as a way to achieve their instructional goals. Future research can explore how professional development opportunities can leverage instructors' beliefs and goals to align them with the aims of the professional development program. Furthermore, these instructors had interests in the potential benefits of IOI, which influenced their decision to change their instructional approach to IOI. Therefore, dissemination efforts need to be made to increase mathematics instructors' awareness of the benefits of IOI and spark instructors' interests in using non-lecture teaching approaches.

The external factors that influenced instructors' decisions to incorporate IOI into their teaching include pressure from student evaluations, departmental support, and content coverage expectations. Typically, mathematics instructors' arguments against using non-lecture approaches like IOI reference departmental constraints and coverage concerns (Johnson et al., 2017a). Some instructors believe using primarily lecture-based instructional approaches helps them cover all the course content within certain time constraints, and they believe using non-lecture approaches would not allow them to do so. Departmental requirements and lack of support from colleagues can also deter instructors from implementing IOI. Contradictory to these typical excuses for not using innovative instructional methods, the instructors in this study generally did not receive discouragement from other faculty members for implementing IOI, nor did they reveal any pressure from requirements to cover a specific amount of content. The instructors generally felt a sense of support from their department chairs for choosing to use IOI in their Linear Algebra course. This finding could imply that instructors with supportive department chairs are more willing to try using IOI, or this could mean that arguments against implementing IOI concerning coverage constraints and departmental discouragement are ill-posed. Further research is needed to explore these hypotheses. There is also a need to investigate the source of mathematics instructors' perceived pressures to cover certain content and comply with supposed departmental expectations. Further research can explore how to help mathematics instructors, particularly lecturers, see the potential benefits of using IOI in their classrooms to allow for more widespread adoption of IOI in undergraduate mathematics courses.

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Variational Reasoning Used by Students While Discussing Differential Equations

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In this study we investigated how a small sample of students used variational reasoning while discussing ordinary differential equations. We found that students had flexibility in thinking of rate as an object, while simultaneously unpacking it in the same reasoning instance. We also saw that many elements of covariational reasoning and multivariational reasoning already discussed in the literature were used by the students. However, and importantly, new aspects of variational reasoning were identified in this study, including: (a) a type of variational reasoning not yet reported in the literature that we call "feedback variation" and (b) new types of objects, different from numeric-quantities, that the students covaried.

Keywords: Differential equations, covariation, multivariation, rate, feedback variation

Ordinary differential equations (DEs) are complex constructs that require reasoning about an interconnected set of relationships. A few researchers have provided deep insight into how students broadly understand DEs (e.g., Habre, 2000; Keene, 2007; Rasmussen, 2001). The literature on DEs reveals the importance of two mathematical concepts: function and rate of change. The importance of *function* is evident in results concerning student understanding of: solutions to DE's (e.g., Arslan, 2010; Dana-Picard & Kidron, 2008; Rasmussen, 2001; Raychaudhuri, 2014), understanding the quantities involved in DE's (Stephan & Rasmussen, 2002, Raychaudhuri, 2008), and existence and uniqueness theorems (Raychaudhuri, 2007). Donovan (2007) noted that when students were able to conceptualize first-order DEs as functions, they were afforded rich ways of reasoning about the solutions. Student notions of *rate* have emerged in the literature in a few different ways. For instance, Keene (2007) identified reasoning about time as a dynamic quantity in relation to other quantities as an important way of reasoning about solutions. Rasmussen & Blumenfeld (2007) outlined how students could use their notion of rate of change to construct solutions to systems of differential equations. In addition, Whitehead and Rasmussen (2003) identified *rate use*, where students used rate as a tool for determining solution functions.

These studies have provided important insight into how students reason about DEs and their solutions. However, there is a key component implicit to much of this work that has not been directly studied: how students use *variational reasoning* while thinking about DEs. Knowing how students use such reasoning could give insight to instructors both in terms of recognizing and eliciting student reasoning. In response to this gap in research, the study we present in this paper was guided by the question, "How do students use variational reasoning when interpreting and discussing DEs?"

Variational Reasoning

We use the generic term "variational reasoning" to mean reasoning about *any* situation involving a changing quantity. The term "quantity" refers to an object with an attribute that can be measured (Thompson & Carlson, 2017, p. 425). We use *co*-variation to mean how *two* quantities change in relation to each other, based on the extensive focus in the literature on two-quantity covariation (e.g., Carlson et al, 2002; Confrey & Smith, 1995; Johnson, 2015; Moore et al, 2013; Saldanha & Thompson, 1998). Thompson and Carlson (2017) have recently published a new covariational framework based on previous covariation research, consisting of six reasoning levels. We use this framework, except for the first level, in which a student does not engage in any coordination. Instead, we retain the first mental action, *recognize dependence*, from Carlson et al.'s (2002) original framework. Consequently, our first level is *recognize dependence*, in which a student perceives two quantities as being dependent in some way. The second level (now from the new framework), *precoordination*, involves imagining two quantities changing, but "asynchronously" (p. 441), meaning that the person envisions a change in one quantity first, then a change in the other. The third level, *gross coordination*, contains an image of two quantities changing together, but in a generic way, such as "this quantity increases while that quantity decreases" (p. 441). The fourth level, *coordination*, involves "coordinat[ing] the values of one variable (x) with values of another variable (y)" (p. 441). The fifth level, *chunky continuous reasoning*, involves imagining continuous change, but always by completed intervals, or "chunks," of a fixed size. In the sixth level, *smooth continuous covariation*, the person envisions the changes in the two quantities "as happening simultaneously" and with "both variables varying smoothly and continuously" (p. 441). Note that we use this framework as a "descriptor of a class of behaviors" (p. 441), rather than as a judgment of overall ability. That is, a student's usage of one level does not imply the inability to use a higher level.

Next, we use *multi*-variation for situations in which *more than two* quantities change in relation to each other (Jones, 2018). For our purposes, we use three types of multivariation. *Independent multivariation* involves two (or more) independent quantities influencing a third quantity, but where the independent quantities do not directly influence each other. *Dependent multivariation* involves three or more interdependent quantities where a change in one typically induces changes in *all* other quantities in the system simultaneously. *Nested multivariation* involves a chain of related dependencies, like the structure of function composition, $z = f(y(x))$. We use the term "multivariational reasoning" to mean the reasoning one does about the quantities involved in one of these types of multivariation.

Methods

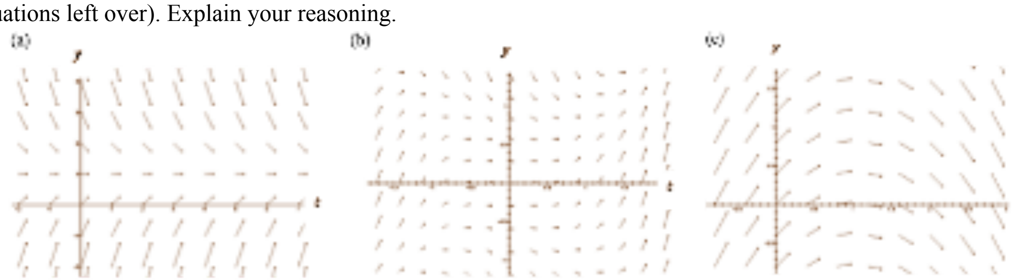
As a preliminary step to our study, we conducted a conceptual analysis for ourselves on how one might interpret basic DEs of the forms $y' = f(y)$, $y' = f(t)$, and $y' = f(t, y)$. This confirmed to us that there would likely be many aspects to variational reasoning in interpreting these equations. Encouraged, we conducted our study using pre-existing data that came from a series of five task-based, semi-structured interviews done with eight students enrolled in a traditional ordinary differential equation course (not taught by the researcher). This data was collected as part of an earlier investigation by one of the authors in an effort to explore the connection between ideas involving function and rate of change in relation to student understanding of differential equations. For the purposes of this study we identified four tasks within these interviews that had DEs matching the three basic forms, including two that had symbolically written DEs and two that had visual graphs associated with DEs. The four tasks are shown in Figure 1. Note that tasks I2T4 and I4T2 were adapted from the IODE curriculum (Rasmussen, Keene, Dunmyre & Fortune, 2017). For the purposes of this conference report, we chose a small sample of three of the eight students to analyze, based on those that were most talkative and that articulated their thinking.

We analyzed the interview data at two levels: holistically and per instance of student reasoning. The second author went line by line through the transcript to identify each instance of student variational reasoning, done by noting any time a student talked about two or more quantities at the same time. For each identified instance of variational reasoning, a timestamp and the associated utterance were recorded, as well as the type of variation (co-, multi-, or other) and the quantities involved. For covariational reasoning, the instance was coded according to the Thompson and Carlson (2017) framework, with the inclusion of the *recognize dependence* level. For multivariational reasoning, the instance was classified in terms of the type of multivariation. The conceptual analysis helped us be sensitive to certain ways students might using variational reasoning. Independently, the first author took a more holistic approach, identifying the general steps and lines of reasoning the student utilized to complete each task. Within each line of reasoning, the researcher identified the main types of variation, the objects being varied, and key mental actions associated with the approach. For each interview task, the two authors then met and discussed the findings until a consensus was reached.

Interview 1 Task 2 (I1T2): What does the following differential equation mean to you? $P' = 3P$

Interview 1 Task 3 (I1T3): Suppose the equation $P' = 2P + 2t$ can be used to model the fish population in the campus duck pond. How might it be used to determine the number of fish in the pond at a given time t?

Interview 2 Task 4 (I2T4): Below are three different tangent vector fields and six rate of change equations. Without using technology, identify which differential equation is the best match for each tangent vector field (thus you will have three rate of change equations left over). Explain your reasoning.



$$\frac{dy}{dt} = t-1, \quad \frac{dy}{dt} = 1-y^2, \quad \frac{dy}{dt} = y^2-t^2, \quad \frac{dy}{dt} = 1-y, \quad \frac{dy}{dt} = t^2-y^2, \quad \frac{dy}{dt} = 1-t$$

Interview 4 Task 2 (I4T2): Below you are provided with a graph of a rate of change equation rather than the equation itself (Note that *dy/dt* depends only on *y*). Figure out the long-term behavior of possible solution functions, illustrate your conclusions with a suitable graph or graphs, and state your conclusions about the long-term behavior of these solutions.

Figure 1: The four interview tasks we analyzed for this paper

Results

We organize our results by first describing how the students in this study reasoned about rate, and how they generally employed covariational and multivariational reasoning. We then discuss novel findings regarding the variational reasoning used by the students, including (a) the importance of dependence, (b) a new type of variational reasoning, and (c) new types of objects used in covariation.

Student Interpretations of Rate

Our students often referred to variables such as P' and *dy/dt* by name, but also as a slope, derivative, value or ratio, and sometimes represented them graphically with direction vectors. Our purposes here were to understand what quantities students were varying and the variational reasoning they used to do so, and as long as the students seemed to see some equivalence between these different interpretations of derivative, or “rate,” we did not analyze according to which interpretation was being used. That is, while the students expressed these various images for rate, we did not focus on the specific properties of the representation (e.g., slopes, variables, vectors, etc), but rather on how the students reasoned about changes in the values associated with those objects. Categorically speaking, the students used their different understandings of rate in two key ways: as a single quantity in its own right and as a quantity that could be unpacked to indicate how the values of two distinct quantities changed in relation to one another. These two ways of using rate were often associated with different relationships in the DE’s.

Students in our study often used rate as a single quantity/object with a numeric value when reasoning about the relationship explicitly defined in the DE. For instance, while discussing $P' = 3P$, Student 1 noted that the DE "is indicating that as x goes towards infinity the rate of change is increasing dramatically, it gets bigger as time goes on." Similarly, on the same task Student 2 noted "as P increases, or as t or whatever it is related to increases, the rate of change increases." Both of these students used the DE to determine how the value of one variable was related to the value of the other. While they were both sensitive to P' being a rate of change (of P), their reasoning did not rely on this intrinsic characteristic of the DE; their reasoning was similar to thinking of the relationship between y and x in the equation $y = 3x$ (where y and x share no other inherent relationship).

On the other hand, the students also often unpacked rate in different ways. For task I1T2, Student 1 used P' to indicate the general behavior of the solution function: "as x increases P is going to increase faster and faster as indicated by P'". While this statement does not accurately capture cases where P' is negative, it does represent the idea that P' can be used to compare how P is changing as x continually increases, namely that P increases "faster and faster." While completing the same task, Student 3 used rate as an indicator for both speed and direction in which P would change over time: "if this number [P'] is really big then in short amounts of time the population goes through a lot of growth. but ... if it was negative then the population would be decaying as time goes on." When completing I4T2, Student 3 noted "*dy/dt* is the derivative of y, its how y is changing at t." In addition he said soon after, "the thing is that since *dy/dt* is the derivative of $y(t)$ it's the slope of, it's basically y over t, it's the slope of y as a function of t." In both cases he unpacked *dy/dt* as an indication of how y was changing at a particular t value. However, the second statement is slightly more complex as he notes that this change depends on t ("the slope of y as a function of t").

While these two ways of interpreting rate are important independently, the students in our study often used them in combination while reasoning about the behavior of solutions. In fact, we feel that being able to *simultaneously* attend to a rate as single quantity *and* unpack it was a critical part of thinking about DEs for these students. Aside from instances when students engaged in the utilization of analytical solution methods, our students often made single statements that indicated both ways of interpreting rate. For instance Student 2 when working with task I1T3 said, "there is a positive correlation between the population of fish and the rate of change. So the more fish there are the faster the fish population will grow." Here, Student 2 reasoned about how P' was changing as P changed (both increase; "a positive correlation"), and simultaneously unpacked rate to determine how P was going to change ("the more fish there are the faster the fish population will grow"). It is important to note that this last determination was more detailed than simply direction or an amount of change in a single instance; the phrase "the faster the fish population will grow" indicates a comparison over various instances. Namely, Student 2 interpreted rate as a single quantity that changed in relation to P, and simultaneously unpacked rate to not only determine the fish population would increase (presumably with respect to changes in time), but that it would increase faster as time changed.

Students' Usage of Covariational and Multivariational Reasoning

General usage of covariational reasoning. Essentially every level of covariational reasoning was used by these three students, suggesting the need for fluency with covariation when reasoning with DEs. The only level not directly observed was *chunky* continuous reasoning, likely because the interview questions were not set up to prompt its usage. Importantly, despite three quantities (t, P, and P') being part of the DEs, the students often focused only on two of these quantities at a time. For example, while discussing $P' = 3P$, Student 2 stated, "As P increases, the slope of P increases." This coordination involved the two quantities directly present in the equation, without explicit attention to t, even though Student 2 had previously recognized its implicit presence. Even with equations with all quantities present, as in $P' = 2P + 2t$, the students often still focused only on two at a time. For example, Student 1 stated, "As t goes up, just, this number [i.e. $2t$] is going to increase, and since it is being added, P' is going to be greater." This coordination involved only t and P' in this statement. It was in a separate instance of reasoning that Student 1 coordinated P and P'.

The importance of recognizing dependence (and non-dependence). In our usage, the first mental action in covariational reasoning is to *recognize dependence* (see Carlson et al., 2002). This may seem quite trivial, as evidenced by the fact that *recognition* is not even explicitly a part of the new Thompson & Carlson (2017) framework. However, a significant part of our students' cognitive efforts in reasoning about DEs involved *recognizing* quantities that may or may not be dependent on each other. For example, when discussing $P' = 3P$, Student 1 stated, "Since P is a function of time, P' is also a function of time." Note that task I1T2 contains no mention of time, nor a variable t. Student 1 *recognized* that such a variable should be *implicitly* present. All three students made such recognitions, where the third variable was often envisioned as time, though Student 2 did acknowledge that it could be "x or t or whatever this P' is taken with respect to."

In the case of DEs, there also appears to be an important parallel mental action to recognizing dependence, wherein students recognize when quantities are *not dependent* on each other. For example, while discussing the equation $P' = 3P$, Student 2 stated that the variable t "is completely irrelevant in terms of the behavior" of the derivative P'. He clarified that a specific solution function $P(t)$ *is* dependent on the variable t, but that the rate of change, P', is not impacted by t. As another example, while Student 3 was working on task I2T4, matching equations to graphs, he explained, "The way this one [the first slope field] didn't change with t, this one [the third slope field] isn't really changing with y." This reasoning action greatly facilitated Students 2's and 3's identification the corresponding DEs on task I2T4.

Multivariational reasoning. While students often discussed only two quantities at a time, they at times engaged in *multivariational* reasoning. All three students invoked *dependent multivariation* by recognizing that t, P, and P' were interdependent quantities. For instance, Student 1 stated, "Since P is a function of time, P' is also a function of time." Sometimes students noted *non-dependence*, as described in the previous subsection, where they recognized that a change in one quantity might not correspond to changes in another quantity, suggesting *independent multivariation*. There was also an occasional use of *nested multivariation*, as seen in the excerpt from Student 2, given earlier, while discussing the equation $P' = 2P + 2t$. He explained that an increase in t first led to an increase in $2t$, which in turn led to an increase in P'. Thus, we can see the nested structure of $t \rightarrow 2t \rightarrow P'$. It appears, then, that multivariational reasoning may be an important aspect of interpreting DEs, in addition to two-quantity covariation.

A New Type of Variational Reasoning: Feedback Variation

In our study, we identified a type of variational reasoning not previously described in the literature. To exemplify, consider Student 3 discussing the equation $P'=3P$: "So, say as P increases, like if P is positive, the rate is positive, so then P would be increasing, and that would in turn increase the rate, then in turn increase P." Later, while discussing the equation $P' = 2P + 2t$, Student 3 also said, "As P changes, it's also affecting its own rate because of this equation." In typical covariation, it is imagined that changes in one variable (x) are related to changes in a *separate* variable (y). However, in this case, Student 3 was explaining how P is related to changes in *itself*. Student 2 made similar statements, by couching P in the real-world context of fish population: "As there is more fish, it supports more growth… If you have more fish, more fish make more fish." Like Student 3, we see Student 2 explaining how a quantity's value dictates how that same quantity will change. It is true that covariation between population and time is implicit, because population cannot change without elapsed time, but the student's focus is on the single quantity P, and how it influences changes in itself. In another task involving y and dy/dt, Student 2 stated explicitly that a DE "is representing what is the effect of [y] with respect to itself." We call this type of variation *feedback variation*, because of how it reminds us of a feedback loop in a microphone/speaker system. In the analogy, the output from the speaker continuously feeds back into the microphone and back out through the speaker, increasing the feedback volume. For $P'=3P$, one might imagine the speaker to be analogous to P and the microphone to be analogous to P'.

Further, we see a slight nuance to some of the articulations of feedback variation. Notice that in the first excerpt from Student 3, the flow of reasoning is that the quantity P has a value, *then* the rate is positive, *then* the quantity increases, *then* the rate increases, and so on. The language suggests imagining a sequence of discrete steps, similar to what Thompson and Carlson (2017) call *precoordination*. Thus we call this type of reasoning *precoordination of feedback variation*. Of course, the way Student 3 articulated his reasoning may simply be an artefact of attempting to communicate his thoughts to the interviewer. For example, in the second excerpt from Student 3 given above, he explained "As P changes, it's also affecting its own rate." This statement could indicate thinking not of discrete steps, but of a continuously evolving system in which P is always impacting its own rate of change. If a person envisions such a continuously evolving system, we call it *continuous feedback variation*.

New Types of Objects Used in Covariational Reasoning

At one point while discussing a DE involving y and dy/dt, Student 1 drew a graph of a solution function. When the interviewer asked if it was the solution function, Student 1 clarified that it was "one of the potential y of t functions, because there is an infinite [number of them], based on your initial condition." Student 1 recognized that different initial conditions would be associated with different specific solution functions, $y(t)$. Student 2 expressed a similar idea when he stated, "The *y-naught* allows you to put it to a specific situation… Then just literally sliding it [i.e. the graph] over to the point that you need." Here it appeared that Student 2 imagined a *continuously changing* solution graph that ranged over many possible initial conditions until it reached the desired initial condition. In other words, as the initial condition changed, the solution graph changed. Further, Student 3 talked more explicitly about how initial conditions might pair with different solutions. When discussing the task shown in Figure 2, he stated, "So, based on what initial conditions you have, wherever you start on the curve, you are gonna, like, if you start between -2 and 2, the curve will plateau off at 2. If you start below -2 it will plateau off at -2, and above, the curve will plateau off at 2." Despite the incorrect assertion for initial conditions below -2, the point is that he imagined changes in initial conditions leading to changes in the solution function.

These three students appeared to be *covarying* initial conditions and solution functions. Typical covariation usually deals with two numeric quantities, such as x and y (e.g., Carlson et al, 2002; Confrey & Smith, 1995; Johnson, 2015; Moore et al, 2013; Saldanha & Thompson, 2002). However, our students imagined covariation as happening between initial conditions and solution functions, which are different types of objects than discussed in the literature. Of course, depending on the definition of covariation, this may or may not even be considered "covariation," if covariation is only between *numeric-value*-type objects (see Thompson and Carlson 2017, p. 423). However, we suggest it may be appropriate to consider other objects to be *covarying* as we move into more abstract forms of mathematics. Our analysis suggests students have images of initial conditions and solution functions as changing (varying) together (co). Because these objects do not have "values" in the same way as numeric quantities do, some mental actions like coordination of values might not have matches for this context. However, Student 2 may even have employed *continuous covariational* reasoning by imagining a graph sweeping through initial conditions until the desired initial condition was reached.

Discussion

Our results show that variational reasoning is important for unpacking and understanding DEs. Our results further indicate that there may be unique aspects to variational reasoning for DEs. A key part of our students' mental work was recognizing what quantities are implicitly contained in a DE, what quantities are dependent on each other, and what quantities are *not* dependent on each other. This greatly expands the *recognize dependence* mental action in Carlson et al.'s (2002) original framework, and underscores its importance. Thus, we suggest that it should not be dropped from the new covariational framework (Thompson & Carlson, 2017), but be incorporated as an important skill students may need as they advance to more complicated mathematics. Additionally, we have identified a new type of variational reasoning outside of current covariation and multivariation frameworks (Carlson et al., 2002; Jones, 2018; Thompson & Carlson, 2017). DEs have a unique structure wherein a quantity is explicitly related to changes *in itself*. In other words, its current value indicates how it will change. This does not occur in covariation between, say, x and y, where x is free to vary as a, literally, *independent* variable. Finally, we saw that students appeared to employ covariational reasoning with new types of objects beyond what it typically described in the literature. In addition to covarying numeric quantities, the students covaried points (initial conditions) and functions (particular solutions). There are even likely different levels to covarying these types of objects. One could imagine a change in initial condition *then* a change in particular solution (*precoordination*), a generic imagine of the initial condition moving to the right as the solution function changes in some way (*gross coordination*), or one could imagine a "sweeping" initial condition with specific values that continuously passes through infinitely many specific particular solutions (*continuous covariation*).

Our work also illuminates the importance of reasoning with rate as both a single quantity and as a relationship between two varying quantities when making sense of DEs and their solutions. While the covariation literature discusses constructing rate by composing amounts of change in two related quantities, much of our students' mental work consisted of decomposing rate. That is, they reasoned with the DE as if it were a function to understand how the values of the various quantities changed, but then also unpacked the rate to make sense of how the intrinsic quantities behaved. They used both notions of rate to conceptualize the solution functions. They took a rate as a single, changing quantity, decomposed it into two quantities and used it to perceive the relationship between the two quantities so that they could construct a solution function. Further, they often simultaneously coordinated changes in the rate (as indicated by the DE itself) with changes in the two quantities from which it was composed. Our findings regarding students' frequent utilization of variational reasoning and the various ways of working with rate align with and add to the thematic nature of function and rate of change in the research literature on DEs. For instance, our findings bring together and strengthen Donovan's (2007) assertion regarding the importance of conceptualizing a DE as a function, Keene's (2008) work regarding student reasoning with rate of change, and Whitehead and Rasmussen's (2003) discussion of rate use. In this case, examining how the students reasoned about relationships between varying quantities allowed us to understand some of the ways these two concepts come together when reasoning about DEs.

This work suggests it is important for instructors to provide students with opportunities in which they engage in reasoning with DE's in two ways: as a relationship between bare variables, and as a relation between the value of a function and its corresponding rate of change at a particular instance. The latter may require focused and meaningful attention on the often implicit inclusion of the functions independent variable. Importantly, instructors must get students to consider both of these relationships simultaneously.

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More than Meets the I: Inquiry Approaches in Undergraduate Mathematics

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In the United States (US) and worldwide, undergraduate mathematics instructors are increasingly aware of the value of inquiry-based instruction. We describe the intellectual origins and development of two major strands of inquiry in US higher education in mathematics, offer an explanation for apparent differences in these strands, and argue that they be united under a common vision of Inquiry-Based Mathematics Education (IBME). Central to this common vision are four pillars of IBME: student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building. We conclude by calling for a four-pronged research agenda focused on learning trajectories, transferable skills, equity, and an educational systems approach.

Keywords: Inquiry-based learning, Inquiry-oriented instruction, Inquiry-based mathematics education, Active learning

In the United States (US), a growing chorus of voices is calling for post-secondary mathematics teaching to provide students with learning experiences that are rich and meaningful: centered on students' ideas, requiring their mental engagement in and out of class, and accountable to their prior understandings. These calls are grounded in evidence from education research that such research-based, student-centered teaching practices benefit student learning, attitudes, success and persistence in mathematics and related fields (see e.g., Freeman, et al., 2014; Kober, 2015). And, because success in mathematics courses is essential for many education and career paths, these experiences and outcomes also support students to pursue interests in many other fields. While research-based instructional practices are not yet the norm in North American classrooms, they are becoming more mainstream (Stains et al., 2018)—as, indeed, they must in order to have widespread benefit.

Such calls for reformed instruction are often motivated by national or regional concerns for economic competitiveness—for education that prepares STEM workers to fuel the innovation economy (e.g., President's Council of Advisors on Science & Technology (PCAST), 2012; Rocard, et al., 2007; West, 2012). As Artigue and Blomhøj (2013) note, such sociopolitical justifications merit critical consideration of the intellectual origins and pedagogical practices that are endorsed. Within the discipline of mathematics, leaders of professional societies (CBMS, 2016; MAA, 2017; Saxe & Braddy, 2015) have emphasized how students benefit in ways that in turn strengthen the discipline. These statements are both responses to and drivers of the growing visibility of active learning within mathematics.

As scholars who have studied active learning and teaching in postsecondary mathematics education, especially approaches known as inquiry-based learning (IBL) and inquiry-oriented instruction (IOI), we are encouraged to see this growing interest in educational practices we know to be effective for students. We have also observed growing concern for defining and differentiating particular strategies (e.g., Cook, Murphy & Fukawa-Connelly, 2016; Kuster, Johnson, Keene, & Andrews-Larson, 2017). Here we propose some key principles of mathematical inquiry in the undergraduate classroom, describe the history and development of two major strands of inquiry in US higher education, and offer an explanation for apparent differences in these strands Because the commonalities are more important than the differences, we argue for a common educational vision and research agenda for Inquiry-Based Mathematics Education, focusing on how inquiry experiences matter for students, instructors, mathematics departments, and the profession.

What is Inquiry in Mathematics?

We begin by situating inquiry within the broader landscape of active learning and teaching. Decades ago, Bonwell and Eison (1991) defined active teaching strategies as those that "involve students in doing things and thinking about what they are doing" (Bonwell & Eison, 1991, p. 19). Students may "do" and "think" by reading, writing, discussing, or solving problems, but they must take part in higher-order thinking tasks such as analysis, synthesis and evaluation. We add to this definition the explicit expectation that students talk to each other about what they are doing and thinking, as conversations are powerful in clarifying, solidifying, and elaborating learners' ideas. They also take advantage of the inherently social nature of classrooms and provide the instructor with the feedback needed to identify fruitful next steps toward her learning goals for students.

The instructor's role is to orchestrate this doing, thinking, and talking—to choose the important mathematical ideas and to develop tasks that enable students to meet and grapple with them. As Campbell and coauthors (2017) show in their multi-institution observation study, these cognitively responsive practices are often missing, even in classes that feature interactive and hands-on activities. Instructor skill and thought are required to make active learning truly active.

We consider inquiry a subset of active learning. As with active learning more generally, students in inquiry classrooms are engaged in doing mathematics, and the instructor is orchestrating and structuring student learning opportunities. Inquiry, however, has several additional distinguishing characteristics. First, inquiry curricula exhibit a longer-term trajectory that sequences daily tasks to build toward big ideas. These coherent task sequences scaffold students' mathematical work on challenging problems over weeks of instruction and may lead to proving a major theorem or (re)inventing a mathematical idea, definition, or procedure. To support such task sequences, instructors must deeply understand the mathematics so they can capitalize on students' mathematical ideas, thus recognizing and nurturing the seeds of student ideas that have the potential to grow and develop, without getting lost in the weeds. A good task sequence of course helps to provide the framework.

A second distinguishing characteristic of inquiry is the nature of students' mathematical work. In inquiry classrooms students reinvent or create mathematics that is new to them. They do so by engaging in mathematical practices similar to those of practicing mathematicians: conjecturing and proving, defining, creating and using algorithms, and modeling (Moschkovich, 2002; Rasmussen, Zandieh, King, & Teppo, 2005). Thus students develop not only deep mathematical understanding, but also a sense of ownership through creation and reinvention. Instructors, for their part, allow students intellectual space to be creative, while at the same time they seek ways to extend student ideas and connect these to formal or conventional mathematics. This requires adaptive and responsive facilitation skills, not just expertise in exposition and delivery of content.

A third distinguishing characteristic of inquiry is a consequence of the previous two: it offers students and instructors greater opportunity to develop a critical stance toward previous, perhaps unquestioned learning and teaching routines. A critical stance is "an attitude or disposition towards oneself, others and the object of inquiry that challenges and impels learners to reflect, understand and act in the milieu of potentiality" (Curzon-Hobson, 2003, p. 201). For example, inquiry provides occasions for students to reconsider their past experiences and think anew about what mathematics is, and about what it means to know math, to do math, and to teach math. For instructors, listening to and making sense of student thinking may challenge how they think about the process of learning something new—how ideas may develop, what it means to "cover" material, and how tentative ideas and errors contribute to the learning-teaching process. A necessary part of developing a critical stance is to have learning experiences that differ from past experiences, and the opportunity to reflect on those experiences. Inquiry classrooms can offer such experiences.

Inquiry learning in mathematics may seem distinct from how this term has long been used in science education (see Bybee, 2011, for a brief history and key references). Yet at the core, these approaches are the same in seeking to involve students in the behaviors and practices of expert scientists or mathematicians.

We focus on two main traditions of inquiry in U.S. post-secondary mathematics, known as inquiry-oriented (IO) instruction and inquiry-based learning (IBL). We argue that the similarities are more important than the (apparent) differences; to do so, we first trace their intellectual origins and practical reach in the United States.

IOI: Inquiry-oriented Instruction

Several different IO curricula cover a variety of content areas for post-secondary mathematics, including abstract algebra, differential equations, linear algebra, and mathematics for future elementary school teachers. A major intellectual source of inspiration and influence for this work (especially in differential equations and linear algebra) comes from the pioneering research of Paul Cobb, Erna Yackel and colleagues in elementary school classrooms (e.g., Cobb et al., 1991; Cobb & Yackel, 1996; Yackel & Cobb, 1996; Yackel, Cobb, & Wood, 1991). Their innovative, classroom-based work was grounded in both cognitive and social theories of learning. Their use of the term "inquiry" came from Richards (1991), who characterized inquiry classrooms as those where students learn to speak and act mathematically by discussing and solving new or unfamiliar problems. The classrooms Cobb and Yackel studied were characterized by students routinely explaining their own thinking, listening to and attempting to make sense of others' thinking, asking questions if they didn't understand someone's work, offering different solution strategies, and indicating their agreement or disagreement, with reasons. Such patterns of classroom talk represent *social norms* and could aptly apply as well to a science class or a history class (Yackel & Cobb, 1996).

Cobb and Yackel also identified classroom talk that was specific to mathematics. For example, when students routinely offer different solution strategies, a relevant mathematical issue is what constitutes a *different* solution. Is Angie's solution different from Juan's? If yes, how so and why? When someone explains their reasoning, what makes for a mathematically *acceptable* solution, or what constitutes an *elegant* solution? Difference, acceptability, and elegance are all criteria that fall under the realm of mathematics and are thus referred to as *sociomathematical norms* (Yackel & Cobb, 1996). While this work originated in second and third grade classrooms, the constructs of social and sociomathematical norms provide powerful and useful tools for researchers and practitioners in IO approaches at the university level (e.g., Rasmussen, Yackel, & King, 2003; Yackel, Rasmussen, & King, 2000).

Another cornerstone of IO curricula is their grounding in the instructional design theory of Realistic Mathematics Education (RME). Traditional curricula are typically designed based on expert understanding of the mathematics, but RME takes a bottom-up approach where curricula are designed based on how learners might reinvent important mathematical ideas and procedures

(Freudenthal, 1991; Gravemeijer, 1999). That is, rather than seeing mathematics as a collection of pre-established truths and procedures that learners must assimilate, RME offers a set of design heuristics where students can, with the support of their instructor, reinvent mathematics at successively higher levels. The classroom, design-based research approach is an ideal method for revealing and generating such routines and practices as well as the kinds of knowledge and dispositions that instructors need (Andrews-Larson, Wawro, & Zandieh, 2017; Johnson, 2013; Johnson & Larson, 2012; Kuster et al., 2017; Marrongelle & Rasmussen, 2008; Rasmussen, Zandieh, & Wawro, 2009; Wagner, Speer, & Rossa, 2007).

Visitors to IO classrooms would see students working in small groups on unfamiliar and challenging problems, students presenting and sharing their work, even if tentative, and whole-class discussions where students question and refine their classmates' reasoning. The students' intellectual work lies in creating and revising definitions, making and justifying conjectures and justifying them, developing their own representations, and creating their own algorithms and methods for solving problems.

IBL: Inquiry-based Learning

In contrast to the research-based history of IO instruction, IBL emerges from practical work by educators and the collegial community they formed. Key support for this community has come from the Educational Advancement Foundation (EAF). Former students of UT Austin topologist R. L. Moore, aided by the EAF, initially sought to commemorate and share Moore's distinctive teaching style, known as the "Moore method" (W. S. Mahavier 1999; W. T. Mahavier 1997; Parker 2005). Although student-centered pedagogies had appeared in the US and Europe well before the 1990s (Artigue & Blomhøj 2013), this Moore-derived movement developed largely independently of those concepts and practices, primarily through collegial exchange and a bootstrapping approach to professional development. Moore did not refer to his method as inquiry-based learning, but early leaders of the movement saw similarities between Moore's teaching and the general principles of inquiry-based teaching that were gaining momentum in higher education at the time (NSF 1996; Brint 2011); the term inquiry-based learning and the initialism IBL came into currency within this community at this time. As the movement grew in size and vitality, it broadened its conception of IBL teaching practices to what is known as the "big tent" (Hayward, Kogan & Laursen, 2016; also Ernst, Hodge & Yoshinobu, 2017; Haberler, Laursen, & Hayward, 2018; Haberler, forthcoming). Whereas IOI continues to develop through design-based research on different courses, the IBL community continues to grow as a lively place for practitioners to exchange ideas and deepen their practice—a network of people and events, such as workshops, conference sessions, and practitioner-authored publications.

Typically, IBL courses are based on a carefully scaffolded sequence of problems or proofs, set up so that as students work through these problems they jointly build up the big ideas of the course through discovering and explaining the mathematical arguments. Commonly, the problem sequences or 'scripts' are based in instructors' mathematical knowledge and classroom experience with how students may productively develop ideas. But they may not be grounded in instructional design principles from education research; they are shared colleague to colleague through informal networks or a course repository. While traditionally Moore method courses emphasized upper division topics such as real analysis and abstract algebra, today IBL approaches have been adapted to nearly all courses in the mathematics curriculum, and for general education, teacher education, and mathematics specialist audiences.

Visitors to IBL courses would see class work that is highly interactive, emphasizing student communication and critique of these ideas, whether through student presentations at the board or small group discussions. Whole-class discussion is used to aid collective sense-making, and instructor mini-lectures may provide closure and signposting. Instructors' classroom role is thus shifted from telling and demonstrating to guiding, managing, coaching and monitoring student inquiry. There is a long tradition of practical literature from reflective educators describing IBL teaching practices and curricula, but more recently, IBL practices have been characterized by a team of researchers who sought to understand student outcomes emerging from multiple IBL courses taught at four institutions (Laursen, 2013; Laursen, Hassi & Hough, 2016; Laursen, Hassi, Kogan, Hunter & Weston, 2011; Laursen, Hassi, Kogan & Weston, 2014). This research has in turn provided language and foundations for deeper practitioner inquiry. Thus, we do not describe IBL as "research-based" practice but rather as consistent with and supported by education research (Laursen et al., 2014).

Differences in the Research Bases for Inquiry Traditions

It is in the research studies of IBL and IOI where apparent differences arise between these traditions—largely due to different emphases in what are still small literatures. We will highlight differences in the types of study samples, study methods, and research questions of interest that give rise to these differences in the literature bases for IOI and IBL. Despite these differences, studies of student outcomes show broadly similar results, with greater benefits to students in inquiry classes than to their peers in non-inquiry classes across cognitive and non-cognitive domains. Some outcome measures show no difference; importantly, there is no evidence of harm done to students in inquiry classes, despite reduced content "coverage."

In addition to differences in focus and methods of the existing research studies, we note differences in the researchers' stance with respect to the teaching tradition. As mathematics-trained researchers, IO scholars were interested in developing and studying student reasoning about particular ideas, instructors' practices and the knowledge they find useful in IO teaching. Instructors taking part in these studies tended to be part of the extended research team, typical of design-based research. In contrast, Laursen and colleagues brought an external perspective to IBL. While the research team included people trained in mathematics as well as in other areas of natural and social science, they were not IBL instructors themselves, and began their work with a very practical orientation as evaluators, embedding themselves in the IBL community and attentive to its place in the broader national landscape of STEM higher education.

We describe these differences not to value one approach over another, but to point out some differences in the bodies of RUME scholarship emerging from these two inquiry traditions. These differences in the research questions, methods and perspectives may lead RUME researchers and practitioners to focus on the differences between IBL and IOI methods, rather than on their commonalities. But we argue that the commonalities are more significant for improving practice and for generating fruitful and impactful research.

The Four Pillars of Inquiry-Based Mathematics Education

Because these descriptions make clear that IBL and IOI mathematics share common foundational practices despite their different origins, we discuss them jointly under the term Inquiry-Based Mathematics Education, or IBME (Artigue & Blomhøj, 2013). In their study of student outcomes, Laursen and coauthors (2014) identified "twin pillars" (p. 413) that support student learning: deep engagement with meaningful mathematics and collaborative processing of mathematical ideas. Deep engagement occurs as students encounter and grapple with important ideas, in and out of class. And, as students discuss, elaborate and critique these ideas together, they deepen their understanding and build communication skills, collaborative skills, and appreciation for diverse paths to solutions. These pillars of learning emphasize what students do that leads to the good outcomes; they imply, but do not make explicit instructors' roles in selecting and staging meaningful tasks and orchestrating students' conversation about them. Rasmussen and Kwon (2007) characterized inquiry using two similar pillars and a third, instructor inquiry into student thinking. This pillar emphasizes the instructor's role to strengthen the student pillars by eliciting student ideas and making them public, building a classroom community where students can fruitfully engage with and refine those ideas together, and elaborating and extending student ideas—a role that requires that instructors value and attend to students' ideas.

We add to these three a fourth pillar, equitable instructional practice. The research base in undergraduate mathematics education does not reveal just how to accomplish this in inquiry-based college classrooms. Current studies show that inquiry classrooms can level the playing field for women (Laursen et al. 2014) and offer evidence and arguments for why this may occur (Hassi & Laursen, 2015; Tang, Savic, El Turkey, Karakok, Cilli-Turner, & Plaxco, 2017) but also show that this is not automatic (Andrews, Can & Angstadt, 2018; Brown, 2018; Ellis, 2018; Johnson et al., 2018). Research on high school classrooms offers useful lessons, however Boaler (2006) describes seven teaching practices that yielded higher and more equitable educational attainment and fostered students' respect and felt responsibility for each other. It is striking, yet no coincidence, that these practices overlap well with the first three pillars of inquiry. For example, asking students to justify their answers and share their reasoning is a form of instructor inquiry into students' mathematical thinking—but as Boaler's study showed, this also contributed to equity and respect, instilling a norm that students explain their own ideas and ask for others' explanations and help. However, equity-oriented practices such as assigning competence—publicly raising the status of a student's intellectual contribution—require instructor attention to interpersonal classroom dynamics as well as mathematics. Instructors must consider not just what students think but what they may feel and experience; they must notice whose thoughts are heard, acknowledged and valued and actively shape those experiences in ways that foster respect and responsibility.

To recap, four pillars of IBME support student learning. Two emphasize student behaviors and two emphasize instructor behaviors:

- Students engage deeply with meaningful mathematical tasks
- Students collaboratively process mathematical ideas
- Instructors inquire into student thinking
- Instructors foster equity in their design and facilitation choices.

Research Agendas for Inquiry-Based Mathematics Education

As core IBME principles, these four pillars are the foundations of effective IBME practice; they account for student learning and thus offer guidance to instructors seeking to develop their teaching practice. The four pillars also offer guidance to researchers interested in IBME about fruitful and important questions to pursue. We will make a case for four research agendas as important for researchers and practitioners to explore:

- *the learning trajectory agenda*: IBME classrooms offer ideal settings for surfacing student ideas and explicating learning trajectories. At the elementary and secondary school levels, research and development on learning trajectories holds great promise to make significant impact on learning and teaching (Daro, Mosher, & Corcoran, 2011; National Research Council, 2007). Comparable work at the post-secondary level, however, is relatively sparse, both in general and in particular to inquiry curricula.

- *the transferable skills agenda*: IBME classrooms emphasize collaboration, communication, teamwork, and other valued transferable skills. IBME classrooms are well suited to explicitly teach and assess transferable skills, so we call for researchers and practitioners to take up this agenda. Challenges for researchers include whether inquiry curricula do indeed generate such skills, and how to measure them, how to design curricula and identify teacher knowledge and practices that support students to develop transferable skills.
- *the equity agenda*: IBME classrooms offer opportunities and challenges for making mathematical inquiry fair and accessible to all and for understanding what practices and contexts best accomplish this goal. Attention to equity in IBME classrooms may mean designing studies that have the statistical power needed to unpack average gains or outcomes in more intersectional ways, or developing measures to probe particular phenomena classroom more deeply (e.g., Reinholz & Shah, 2018). There are opportunities to explore new theoretical perspectives (see Adiredja & Andrews-Larson, 2017) and build theory across multiple instantiations of IBME when examining topics such as teaching practice, classroom discourse and power, epistemological ownership, intersectionality and student identity.
- *the educational systems agenda*: IBME classrooms offer a distinctive space for considering how teaching and learning are affected by the broader disciplinary and institutional contexts where instruction occurs. Attention to systems may give rise to fruitful questions about whether and how instruction is changing within departments or in networked communities to align with recommended practices in the discipline (e.g., Apkarian, 2018). Fine-grained studies in multiple settings may reveal interesting variations in student experiences or outcomes that depend on classroom dynamics or instructors' facilitation skills or they may demonstrate ways to adapt IBME for different student audiences. Systems-focused studies must attend to variability, recognizing that one size does not fit all and accommodating that variability as a feature—not a bug—of the research design.

Conclusion

Investigation of these challenging, higher-order problems will benefit both research and practice. For research, these agendas will generate greater coherence of the body of knowledge across all IBME traditions and will focus scholars' attention on challenging educational problems of wide interest, with potential for significant impact. Practitioners will likewise benefit from greater commonality and coherence in the body of research-based advice for improving their practice. The shared agenda is reflected in the shared terminology and four pillars of inquiry-based mathematics education.

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Influences from Pathways College Algebra on Students' Initial Understanding and Reasoning about Calculus Limits

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The Pathways to College Algebra curriculum aims to build concepts that cohere with the big ideas in Calculus, and initial results suggest improved readiness for Calculus by students who have taken a Pathways class. However, less is known about how Pathways might influence students' initial understanding and reasoning about calculus concepts. Our study examines similarities and differences in how Pathways and non-Pathways students initially understand and reason about the calculus concept of the limit. Our findings suggest that Pathways students may engage a little more in quantitative reasoning and in higher covariational reasoning, and have more correct and consistent initial understandings. Further, the Pathways students were explicitly aware of how their Pathways class may have benefited their understanding of limits.

Keywords: Pathways College Algebra, Calculus, Limits, Understanding, Reasoning

A critical idea in mathematics education is *coherence* across curriculum (NCTM, 2006; NMAP, 2008; Newmann, Smith, Allensworth, & Bryk, 2001; Schmidt, Wang, & McKnight, 2005). Thompson (2008) argues that coherence should be viewed through *ideas* and *meanings* rather than topical structures and orderings. Such coherence seems to be lacking between calculus and its prerequisite classes, like College Algebra, which often focuses on calculations and procedures (see Blitzer, 2014; Sullivan, 2012). While knowing procedures can help students manually work out calculus problems, it is hard to see how these cohere with the *big ideas* in Calculus of limits, rates of change, and accumulation (see Kaput, 1979; Thompson, 1994).

To address this issue of coherence, a recent curriculum for College Algebra, *Pathways to College Algebra* (Carlson, 2016, hereafter referred to as "Pathways"), aims to build Algebra concepts through quantitative and covariational reasoning. The curriculum was developed specifically to cohere with big ideas in calculus and data has shown that students who used the Pathways curriculum tend to be better prepared to enter Calculus (Carlson, Oehrtman, & Engelke, 2010). However, little work has been done in documenting exactly *what* students who have used the Pathways curriculum do differently than their non-Pathways peers. This study examines one specific area, namely how a Pathways experience might influence students' understanding and reasoning about limits at the *beginning* stage of limit instruction. Our guiding research question is: What differences or similarities are there between calculus students who took non-Pathways algebra versus Pathways algebra, in terms of how they *initially* understand and reason about limits?

Brief Background on Pathways Curriculum

The Pathways curriculum (Carlson, 2016; Carlson, Oehrtman, & Moore, 2017) was developed to provide a coherent and meaningful course for students that would help them understand the foundational aspects of calculus. The Pathways curriculum was informed by research on learning functions (Carlson, 1995, 1998), the processes of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), mathematical discourse (Clark, Moore, & Carlson, 2008), and problem-solving (Carlson & Bloom, 2005). The curriculum contains modules based on research of student learning and conceptual analysis of the cognitive activities conjectured to be necessary to understand and apply the module's central ideas. Specific concepts that are targeted include rate of change; proportionality; functions: linear, exponential, logarithmic, polynomial, rational, and trigonometric; polar coordinates; vectors; and sequences and series. The curriculum also supports a problem solving approach to mathematics, where students are expected to engage in novel contexts and reasoning to construct mathematics.

Initial Understanding and Reasoning about Limits

In this section, we articulate our perspective on "initial understanding and reasoning about limits," based on the research literature. Of course, our perspective outlined here will not contain everything that may be involved in understanding or reasoning about limits, because of the fact that our study only deals with understanding at the *beginning* stages of learning limits.

We define "initial understanding" as a student's early *concept image* of limit at the initial stage (Tall & Vinner, 1981), which we would expect to be fairly narrow and incomplete. Also, because some misconceptions are nearly "unavoidable" (Davis & Vinner, 1986) and take considerable exposure to examples, counterexamples, and contexts to address (Cornu, 1991; Przenioslo, 2004; Swinyard, 2011), we are less interested in documenting misconceptions students have. Rather, we are interested in comparing students' initial understanding with standard *informal* definitions of limit. In our study, we examine cases of both the limit at a point, $\lim_{x \to a} f(x) = L$, and the limit at infinity, $\lim_{x \to \infty} f(x) = L$. While the students in our study had not yet discussed limits at infinity in their classes, we wanted to know how they might attempt to understand and reason about them with only the first day of limit instruction. Our informal definition of limit at a point is that the limit of $f(x)$ is L "if we can make the values of $f(x)$ arbitrarily close to L… by restricting x to be sufficiently close to a… but not equal to a" (Stewart, 2015, p. 83). Our informal definition of limit at infinity is that "the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large" (Stewart, 2015, p. 127).

We define "initial reasoning" through two aspects of reasoning that the literature has claimed are important for limits. First, Kaput (1979) has stated that "virtually all of basic calculus (the study of change) achieves its primary meaning through an absolutely essential collection of motion metaphors" (p. 289). As such, changing quantities are a part of early reasoning. However, standard curricula often focus heavily on algorithms for finding limits (e.g., Stewart, 2015; Thomas, Weir, & Hass, 2014). Nagle (2013) claims that this approach likely leads students to have "independent, unconnected conceptions" of limits that are based on quantities and computation (p. 3). Consequently, the way students use quantitative reasoning versus computational reasoning is one part of their "initial reasoning about limits."

Second, some researchers have noted strong relationships between covariational reasoning and understanding limits, due to a limit inherently dealing with two changing quantities (Carlson et al., 2001; Carlson et al., 2002; Nagle, Tracy, Adams, & Scutella, 2017). The informal, "as x approaches a, y approaches L," strongly suggests covariation between x and y. Carlson et al. (2002) even claimed that, "Students' difficulties in learning the limit concept have been linked to impoverished covariational reasoning abilities" (p. 356). Because of the importance of covariational reasoning, even at the early stage of learning limits, we consider how students use covariational reasoning as the other part of their "initial reasoning about limits."

Methods

Twelve Calculus 1 students at a large private university participated in the study. All students had taken College Algebra at the university during the previous year, with five having taken Pathways (P) and the other seven having taking non-Pathways Algebra courses (N-P). The students' Algebra grades and Calculus pre-test scores were similar across the two groups, though three of the N-P students had completed Calculus previously. Students were interviewed about limits the day of or the day after their initial lesson on limits in Calculus 1. Unfortunately, one P student had not attended his calculus class the day limits were introduced, so we excluded him from the study. Because we ended up with only four P students in the data, we are careful to state that the results of this study can only be suggestive, not conclusive. We label the P students as PA, PB, PC, and PD and the N-P students as N-PA, N-PB, N-PC, N-PD, N-PE, N-PF, and N-PG.

The interview contained four questions: (1) Explain the meaning of $\lim_{x\to a} f(x) = L$. (2) If you found the limit, $\lim_{x\to\infty} 4x^2/(x^2 - 5x + 6)$, what would you be finding? (3) Select the graph(s) [among six graphs given to the students] that correspond to each limit expression, (a) $\lim_{x\to\infty} f(x) = 1$, (b) $\lim_{x\to 0^-} f(x) = -\infty$, (c) $\lim_{x\to 3^-} f(x) = 0$. (4) The equation $v_{orbit} = \sqrt{GM/r}$ relates a satellite's required velocity for a stable orbit, v_{orbit}, with its distance from Earth, r (where M is the Earth's mass and G is a constant). What is $\lim_{r\to\infty} v_{orbit}$? After questions 2 and 4, students were also asked to identify any connections they saw from their college algebra class that might have helped them understand limits.

We analyzed the students' responses according to their reasoning and understanding as follows. We first analyzed how students used quantitative reasoning versus procedural reasoning in their responses. Quantitative reasoning was operationalized as using number sense and relationships between quantities to discuss the limits. Procedural reasoning was operationalized as using an algorithm or memorized set of steps to solve the problem, without explaining why the process worked, regardless of whether the student used the procedure correctly or not. However, if the student explained why the process worked, it was coded as quantitative reasoning, rather than procedural. We note that we applied this analysis only to the questions where students *could* potentially compute the limit, questions 2 and 4.

Second, we analyzed students' covariational reasoning behavior from all four questions by classifying individual responses according to the reasoning levels in Thompson & Carlson's (2017) framework: *no coordination*, *precoordination*, *gross coordination*, *coordination of values*, *chunky continuous covariation*, and *smooth continuous covariation*. We further grouped these levels into "high," "mid," and "low" categories. High covariational reasoning included *coordination of values, chunky continuous covariation,* and *smooth continuous covariation*, because these were less commonly exhibited types of reasoning. Mid covariational reasoning included only *gross coordination*, because it was the most commonly used reasoning level. However, during analysis two subcategories emerged within the mid level: (1) We attended to whether students were *specific* about the quantities involved in the covariation, or whether they used *imprecise* language to refer to the quantities (Leatham, Peterson, Merrill, Van Zoest, & Stockero, 2016). (2) Some students were close to the boundary between *gross coordination* and *coordination of values*, by explicitly attending to the limiting values that x and y were approaching. We labeled this as a new reasoning level, *gross with limiting values (GLV)*, as opposed to regular *gross coordination (GC)*, and we consider *GLV* to be on the higher end of the mid category. Finally, low covariational reasoning included *precoordination* and *no coordination*. Also, when students were consistently incorrect about the relationship between x and y (e.g., interpreting $x \rightarrow$infinity as $y\rightarrow$infinite), those instances of reasoning were also coded into the *low* category.

The last step of analysis was to infer student's initial understanding of limits by documenting their description of what a limit was by the end of questions 1, 2, and 3. We decided not to include question 4 here because the students generally struggled with it. We recorded whether the students' descriptions were mathematically correct according to our informal definitions. We also noted whether a student's descriptions were consistent across questions, including for *limit at a point* at the end of questions 1 and 3 and for *limit at infinity* at the end of questions 2 and 3.

Results

Procedural versus Quantitative Reasoning

We grouped the students into three categories based on their reasoning: reliance on quantitative reasoning, reliance on procedural reasoning, or reliance on combined reasoning. To illustrate an example of a student who relied on quantitative reasoning, consider N-PA's explanation for how he found the horizontal asymptote in question 2:

> *N-PA*: So as x gets increasingly large, only the most powerful exponents of x are actually going to have much of a difference. ... And so as you get bigger and bigger to 100 or 1,000, or 100,000, then these values here on the bottom become pretty much obsolete or irrelevant. At that point, you can just look at the highest exponent of x [circles $4x^2$over x^2]. In those really large number areas, we have two exponents that are equal to each other… so we know that in the end, it's approaching a positive value of 4 at some very, very far distance down the road.

Compare this example of quantitative reasoning in identifying the horizontal asymptote to an example of a student who mostly relied on procedural reasoning. When N-PE was initially asked question 2, he stated, "I don't have the slightest." He continued,

> *N-PE*: I know that I would search for asymptotes. That would be one of the first things that I would search for. … I would look for vertical asymptotes, which is where x is equal to 0 [points to denominator]. I would break that apart which would be -3 and -2, right? Yeah, -3 and -2 [writes $(x-3)(x-2)$ and points to numerator] and that doesn't break up into $x=-3$ or $x=-2$, so there would be vertical asymptotes when $x=3$ and $x=2$. That's where I would start.

This student, upon seeing a rational function, appealed directly to the procedure for finding vertical asymptotes, which is unproductive in this context. The point is that N-PE relied on trying to identify and use a procedure when encountering an unfamiliar question.

We identified a third category of combined reasoning, that we defined as students who used both quantitative and procedural reasoning during these questions. As an example, in Question 4, PD began by using quantitative reasoning to explain how the function $v_{orbit}=\sqrt{GM/r}$ behaves until he became stuck on an inability to remember a specific set of rules.

> *PD*: The way I'm thinking, as *r* is getting bigger, this fraction inside ... is getting smaller inside of the square root. And the denominator is going to continuously get larger ... then the fraction will get smaller. The fraction [pause]. I'm really not sure with this one. The problem is that I can't remember, because usually with square roots and things like that there are all of these rules. … I can't remember what would happen if you would take the square root of that. I mean the square root would give you a larger number or if it gets

even smaller. I can't remember… So I really don't know.

Three of the four P students and one of the seven N-P students relied on quantitative reasoning to answer questions 2 and 4. Three N-P students relied on procedural reasoning. One P student and three N-P students reasoned with combined reasoning. These results suggest a skew for the P students toward quantitative reasoning and a skew for the N-P students toward procedural reasoning. We also note that the three N-P students who relied on procedural reasoning were consistently unsuccessful in completing question 2 and 4.

Covariational Reasoning

In this subsection, we provide examples of high, mid, and low covariational reasoning, and explain the trends between the P and N-P groups in terms of their covariational reasoning.

High covariation. Of all instances of covariational reasoning among the four P students, 12% of their instances were coded in the high category, with three of the four students having instances in this category. Of all instances of reasoning among all seven N-P students, 3% were coded in the high category, with three of the seven students having instances in this category. To exemplify reasoning at these higher levels, consider PA, who displayed continuous covariational reasoning in his response to the expression $\lim_{x \to 3+} f(x) = 0$ in question 3. When asked to explain why he only looked at x-values coming from the right side, he said,

PA: The little positive symbol right there, by 3. It's asking for values that are just bigger than three… In graph number one [points to a graph], there is a hole and so at the value of 3 there is no output for $f(x)$. But if we were to get infinitely close to three, with values just bigger than three like 3.1, 3.01, 3.001. We're getting closer to the output value of zero.

Mid covariation. Students most commonly reasoned at this level of covariation. Of all reasoning instances for the four P students, 85% were in the mid category, and of the seven N-P students, 69% of all reasoning instances were in the mid category.

To illustrate the differences between *specific* versus *unspecific* reasoning, and *GLV* versus *GC* reasoning, consider the following examples. First, when N-PD was justifying his choice of graph for $\lim_{x \to 3+} f(x) = 0$ in question 3, he stated, "As *it* moves from the positive side, *it* looks like *it* will be 3" [emphasis added]. Note that N-PD used the ambiguous language "it." Further, this statement suggests basic *GC* because of the generic description of "increasing." Thus, we consider this reasoning instance to be *unspecific* and at the level *GC*.

By contrast, consider N-PD's response to the same question:

N-PD: As x approaches 0 from the negative side. We have $f(x)$ approaches negative infinity. So, this graph [point to a graph] is approaching zero from the negative side [motions horizontally across the left side of the x-axis]. And as it does the value of $f(x)$ plummets to negative infinity [motions vertically along the bottom half of the y-axis].

Unlike N-PD, PC always specified which quantity he was attending to, whether x or $f(x)$. Also, in addition to general statements about "increasing," PC was specific about the values the quantities x and $f(x)$ were approaching, zero and infinity. Thus, we consider this reasoning instance to be *specific* and to be at the level *GLV*.

Generally, the P students were specific in their reasoning more often than N-P students. P students were specific for 72% of all reasoning instances in the mid category, while N-P students were specific for 38% of all reasoning instances in the mid category. However, within just the mid category, the P and N-P students had similar percentages of reasoning instances at the *GC* level versus the *GLV* level, with about two-thirds of all mid category reasoning instances being at *GLV*. Table 1 summarizes the results for the mid category of covariational reasoning.

Table 1. Results for specific versus unspecific and GLV versus GC within the mid category

	Specific versus Unspecific		GLV versus GC	
	Specific	Unspecific	GLV	GC
P students	72%	28%	65%	35%
N-P students	38%	62%	67%	33%

Low covariation. Of all reasoning instances among the P students, 3% were in the low category, and among all instances for the N-P students, 28% were in the low category. A common example of this category was when students reasoned about a graph as an object, rather than as two variables coordinated together. For example, consider N-PF's reason why she believed that $\lim_{x \to 0-} f(x) = -\infty$ should be matched with a graph of the horizontal line $y = 1$ with a hole at $x = 0$:

> *N-PF*: So we have 0 to the negative, which means we're going to look at a point just smaller than 0… And you see that it's going to approach negative infinity. It is going to go on forever in the negatives [gestures horizontally along the x-axis towards negative infinity].

It appeared she was thinking of 0^- and $-\infty$ both in terms of x alone, without attention to y, meaning she was not coordinating two variables. Another typical reasoning among the low category instances, typified by N-PF for the same limit and graph, was, "This [graph] makes somewhat sense because the line goes on forever and it's at 1."

Another type of reasoning we categorized as low was when students attempted to covariate but mixed up x and y values. As an example, N-PG did so for the limit in question 2 when she said the limit would be approaching a vertical asymptote, imagining that y was approaching infinity, rather than x. She showed this same confusion in question 3 when she incorrectly claimed that the limit $\lim_{x \to \infty} f(x) = 1$ indicated a vertical asymptote at $x = 1$.

In summary, the results for the students' covariational reasoning suggest a similarity between the two groups in that both had a majority of reasoning instances in the mid category. However, we can see that P students' covariational reasoning was overall skewed somewhat higher than the N-P students, and that they were more specific in articulating that covariation.

Initial Understandings for Limit

Finally, no student had a perfect initial understanding for limits, as we expected. However, five of the students across the two groups had understandings that were "correct," according to our informal definitions, and whose descriptions of a limit remained consistent across the interview questions. As an example, by the end of question 1 PC had described a limit as, "We're not necessarily looking for the value of the function at a specific x point, but what $f(x)$ is approaching at that certain point, from both sides." This description remained consistent through question 3 as well. Three of the four P students were in this group, as well as two of the seven N-P students. Note that both of the N-P students in this group had completed calculus before, while none of the P students had. We find it impressive that these three P students had correct, consistent limit definitions after just one day of learning about them.

Three of the N-P students and one of the P students often referred to the limit as giving the slope of $f(x)$, possibly based on how their instructors introduced limits during class. The other two N-P students simply had many varying ideas throughout the interview or would mix up the input and output values when the limit approached infinity.

Connections to Algebra Experience

When asked for connections to their College Algebra course during the interview, students from both groups mentioned several topics they remembered. However, four of the seven N-P students only listed computational procedures, such as finding vertical asymptotes or finding the inverse of a function. The other three N-P students primarily cited graphing as a connection. The N-P students generally were explicit in stating that their College Algebra course was not helpful in learning about limits in calculus.

By contrast, three of the four P students discussed a change of thinking. Students said that their College Algebra courses helped them reason on their own or gave them an understanding of how concepts work. The remaining P student mentioned both this change in thinking and procedures like discontinuities and parent graph functions. All of the P students indicated that their College Algebra experience directly helped them in learning limits in calculus.

Discussion

In discussing the trends seen in the results, we again caution that our small sample is only suggestive, and cannot imply generalization to the larger Calculus student population. However, within our small sample, we certainly observed differences in trends for how the overall group of P students reasoned about and understood limits compared to their N-P counterparts. Of course, there was overlap in how the students reasoned about limits. For example, students from both groups were seen to reason quantitatively and procedurally. Students from both groups were seen to reason at lower and higher levels of covariational reasoning. Yet, taken in aggregate, P students were shifted more toward using quantitative reasoning than procedural reasoning, and were overall shifted somewhat toward higher levels of covariational reasoning. Their initial understandings for limits were also more on the correct/consistent side of the spectrum. This certainly does not mean that N-P students cannot engage in these types of reasoning nor hold those types of understandings for limit, as seen in our results. But it does suggest a small net effect for the students having taken Pathways College Algebra in using higher reasoning and having better developed personal meanings. In other words, the Pathways curriculum seems to *cohere* with initial limit instruction, which is an important aspect of sound curriculum (NCTM, 2006; NMAP, 2008; Newmann et al., 2001; Schmidt et al., 2005; Thompson, 2008). This coherence suggests a possible advantage for P students when encountering the difficult limit concept for the first time. While it may only be small, if it is combined with a net advantage for other concepts as well, such as the derivative and integral, it begins to build a picture as to why P students might be more successful in calculus (see Carlson, Oehrtman, & Engelke, 2010). In fact, the students themselves seemed aware of the ways in which their Pathways curriculum connected to the limit concept they were in the process of learning about.

We suggest building on this work by sampling a larger group of Pathways versus non-Pathways students to see if the trends observed in our small sample hold for that larger group. Our study suggests that there may be differences, and such future work would be needed to gain the desired generalizability to the larger student population.

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Detailing the Potentially Marginalizing Nature of Undergraduate Mathematics Classroom Events for Minoritized Students at Intersections of Racial and Gender Identities

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Undergraduate mathematics instruction contributes to marginalization among women and racially minoritized individuals' experiences. This report presents an analysis from a larger study that details variation in minoritized students' perceptions of potentially marginalizing events in undergraduate mathematics instruction. With past research on undergraduate mathematics experiences largely focused on students' post-hoc reflections and one or two race-gender intersections, this analysis extends prior work by attending to variation in students' in-the-moment perceptions of mathematics instruction across various race-gender intersections. Findings highlight how issues of underrepresentation, stereotypes, and instructor care contributed to interpretations of instruction-related events as potentially marginalizing. The report concludes with implications for teaching practices in undergraduate mathematics that academically support and socially affirm students from historically marginalized backgrounds.

Keywords: equity, gender, instruction, race, student experience

Undergraduate mathematics instruction contributes to marginalization among women and racially minoritized students underrepresented in STEM (science, technology, engineering, and mathematics; Bressoud, Mesa, & Rasmussen, 2015; Seymour & Hewitt, 1997). Prior research has also documented minoritized students' reflections about marginalizing experiences in undergraduate STEM, particularly at one or two intersections of race and gender identities (e.g., Borum & Walker, 2012; McGee & Martin, 2011). The present analysis extends past research by detailing variation in minoritized students' perceptions of potentially marginalizing events in instruction across various intersections of racial and gender identities. Such research is especially critical in entry-level undergraduate mathematics courses, such as pre-calculus and calculus, that operate as racialized-gendered gatekeepers into STEM majors (Chen, 2013; Ellis, Fosdick, & Rasmussen, 2016). By drawing on minoritized students' in-the-moment reflections on classroom events that they found potentially marginalizing, this study also advances past research that has largely focused on students' post-hoc reflections on their mathematics experiences.

Research Questions

This research addresses two questions to detail intersectional (namely, race-gender) variation in minoritized students' perceptions of undergraduate pre-calculus and calculus instruction:

1. What aspects of undergraduate pre-calculus and calculus classrooms, including instruction, leave women and racially minoritized students feeling marginalized?
2. Why do students from different intersections of racial and gender identities perceive these classroom aspects as marginalizing?

Findings can inform the design of more equitable undergraduate mathematics instruction that academically supports and socially affirms students from historically marginalized backgrounds.

Theoretical Perspective: Positioning Theory

The focus on undergraduate students' interpretations of pre-calculus and calculus instruction as marginalizing of their race-gender identities was informed by positioning theory (Davies & Harré, 1990; Holland, Lachiotte, Skinner, & Cain, 2001). Positioning theory considers how different actors develop expectations about themselves and each other, as well as highlights what norms are structuring participation in pedagogical contexts (Esmonde, 2009). The racialized-gendered nature of mathematics classrooms, as documented in extant research (e.g., Battey & Leyva, 2016; Borum & Walker, 2012; Rodd & Bartholomew, 2006), can position white women and racially minoritized students as being less welcome to participate or as feeling an increased pressure to demonstrate their ability through participation (Engle, Langer-Osuna, & McKinney de Royston, 2014; Suh, Theakston-Musselman, Herbel-Eisenmann, & Steele, 2013). Use of positioning theory in this study, therefore, guided inquiry into variation in how students from different intersectional backgrounds interpreted features of undergraduate pre-calculus and calculus instruction as positioning them in marginalizing ways.

Research Methodology

The central goal of this analysis was to capture intersectional variation in historically marginalized students' perceptions of the ways they found particular features of undergraduate mathematics instruction to be discouraging. From a critical race theory perspective (Solórzano & Yosso, 2002), the analysis foregrounded the voices of undergraduate white women and racially minoritized students to challenge exclusionary framings (e.g., color- and gender-blindness of ability) and enactments of undergraduate mathematics instruction. To do this, the study was designed so undergraduate students can take note, share, and reflect on details about potentially marginalizing events from their mathematics classroom experiences. The study methodology, as detailed below, created space for participants to further examine shared events and reflect on why they interpreted them to be potentially marginalizing for different race-gender identities.

Study Context and Participants

This study took place in a large, public research university in the northeastern U.S. with a diverse yet predominantly white student population. The analysis presented in this report is based on data collection that took place during fall 2017 and spring 2018. A total of 16 first-year undergraduate students enrolled in a section of pre-calculus or calculus were recruited, including 4 Black women, 3 Black men, 4 Latinx women, 2 Latinx men, and 3 white women.

Data Collection

Journaling. Student participants journaled about events in their pre-calculus and calculus courses that made them and others feel discouraged or uncomfortable. Participants were asked to begin journaling during pre-calculus and calculus classes to capture in-the-moment details about the events and their interpretations.. Journal entries included the date and time of occurrence, whether it happened in lecture or recitation, an event description, and a reflection of why they found the event to be problematic. Events submitted as journal entries included instructor-student interactions, instructors' general comments to the whole class, and peer interactions.

After compiling participants' journaled events, the research team organized them into categories (e.g., the instructor ignoring a student response, laughing at a student's contribution). These categories guided the development of an interview protocol centered around 4-5 stimulus events from categories that ranged from being less to more commonly occurring. For example, a more frequently occurring event was the instructor advising students to drop down a level in mathematics if they could not quickly complete steps to solving a problem. An example of a less frequently occurring event in the interview protocol was an instructor accusing a student of not owning a calculator that was provided by a university support program aimed to financially help underserved student populations at the university. Any details about racial and gender identities as well as emotionally-charged language from the submitted events were removed in the protocol, so participants had opportunities to experience stimulus events in different ways.

Interviews. The individual interviews with the 16 study participants were semi-structured, audiotaped, and lasted between 60-90 minutes. Participants were asked three sets of questions for each of the stimulus events. First, since events in the interview protocol may not have been submitted by the interviewed participant, we asked participants to describe what they saw happening in each event. Then, we asked if they found the event to be uncomfortable, why or why not, who they thought would feel uncomfortable, and if there is anyone who would not feel uncomfortable. Lastly, we asked participants if they saw the race or gender of the instructor or student(s) playing a role in their interpretations of each event. During the interview, interviewers probed about various student-generated themes that arose from their interpretations of the events.

Data Analysis

To address the first research question, the data analysis focused on aspects of undergraduate mathematics classrooms, including instruction, that participants described as positioning them or other students in marginalizing ways. We listened to the interviews multiple times and noted differences in participants' responses for each event, including whether or not they saw the event as potentially marginalizing, the extent to which race and/or gender played a role, and how they had or would have experienced the event as a student. After this initial pass through the interview data, we openly coded for features of undergraduate mathematics classrooms and instruction that influenced participants' perceptions of classroom events as potentially marginalizing. These codes were synthesized into three broad themes of features that made the events marginalizing: (i) underrepresentation, (ii) stereotypes, and (iii) instructor care.

To address the second research question, we examined similarities and differences in participant responses within each broad theme to document variation across as well as within intersectional subgroups. We used axial coding to identify such similarities and differences in participant perceptions across race-gender intersections of identity. For member checking purposes, we completed follow-up interviews with 10 of the 16 participants to ensure accuracy of the emergent themes. These member checks clarified participants' perspectives that were shared during the initial interviews and prompted participants to respond to themes from our analysis. Research team members (1 Black woman, 1 Latinx women, 2 Latinx men, 2 white men, and 3 white women) brought awareness of their respective positionality to the data analysis in efforts to minimize threats of both social proximity and distance to participants (Milner, 2007).

Findings

Below we elaborate on the three themes revealed across participants' perspectives about what can make events from undergraduate mathematics instruction potentially marginalizing: (i) underrepresentation, (ii) stereotypes, and (iii) instructor care. We infuse voices from participants across race-gender intersections to capture variation in students' perspectives within each theme.

Underrepresentation

Classrooms. Twelve of the 16 participants related the potentially marginalizing effects of instructional incidents to racial-gendered underrepresentation in undergraduate mathematics classrooms. Black and Latinx students, in particular, expressed how events would impact them emotionally if they were one of the only women or racially minoritized students in the class. Such emotional impact includes pressure to prove themselves (Beatriz, Quinton), self-doubt about participation (Jasmine), hypervisibility of race (Jasmine), and "feel[ing] uncomfortable" (Parker). In response to an event about an instructor suggesting students drop down a course level, Jasmine (Black woman) described the importance of having a "support system" of same-race peers who could counter the instructor's discouraging remarks. These same-race peers could also lessen the high stakes associated with the instructor's remark for racially underrepresented students like Jasmine, managing pressures of "feel[ing] like [they're] the representation of [their] entire ethnic group" in the classroom.

Quinton (Black man) similarly acknowledged how being the only Black student in an undergraduate mathematics class can limit opportunities to find affirmation from same-race peers about instructors' potentially racialized interactions. Responding to the event about an instructor laughing at and disregarding a student's question, Quinton described how a Black student in a predominantly white classroom experiencing this will not be able to check in with Black classmates about whether or not they also perceived the instructor's actions as racialized. Quinton reflected, "You're surrounded by white faces... a white professor… You're looking like you're the one who's the problem… There's no one to really say, 'No, you're [the professor's] wrong. You need to answer the question.'" Furthermore, Quinton interpreted the instructor's laughter and student disregard in the event as reflective of the instructor's possible perception that the student "didn't belong there [in the class]." He described how Black students, for example, are often viewed as getting into college through athletics rather than academic merit, leaving them with the burden of having to "prove [their] worth" and belongingness.

STEM fields. Participants also reflected on how racialized-gendered underrepresentation in STEM fields influenced their interpretations of instructional events as potentially marginalizing. Reflecting on an event where an instructor confused two students, Uzma (Black woman) conjectured that a woman would not feel as comfortable as a man because the "masculine presence in STEM majors" brings men to feel like they belong in the undergraduate mathematics classroom. Victoria (Latinx woman) perceived the instructor's whole-class comment about dropping down a course level as discouraging women from persisting in male-dominated STEM fields. She argued how women may interpret the comment as confirming gendered representation in STEM, bringing them to think "Maybe STEM isn't for me."

In addition, women participants used racialized-gendered STEM representation as a lens to interpret events as reflecting inequitable opportunities for classroom participation. Amy (white woman) described how instructors may perpetuate notions of STEM as a "predominantly masculine field" through "giving them [men] more time" to ask questions and receive support. To illustrate, Amy referred to gendered patterns in the quality of her mathematics instructor's responses to student questions that brought her to limit her classroom participation. Jasmine (Black woman) argued that racialized-gendered associations of STEM through representation shape instructors' differential responses to student contributions based on students' race and gender. For example, Jasmine referred to the lack of expressed gratitude for a woman or student of color correcting an instructor (a white or Asian man) as a "power move" because the instructor might perceive the correction as the student "encroaching on space that doesn't belong to [them].” She described these “very disheartening” classroom moments as contributing to the lack of representation and support for marginalized groups in STEM.

Summary. Participants, thus, varyingly interpreted the potentially marginalizing nature of events in relation to racialized-gendered underrepresentation in mathematics classrooms and across STEM fields. At the classroom level, racially minoritized students expressed how the absence of same-race classmates can bring them to interpret instructors’ actions and words with racialized implications about their academic potential and belongingness. Women participants raised how gendered representation in STEM can shape potentially gendered double standards of how instructors interact with students, such as allowing men to take up more space than women and deeming women’s contributions as less worthy of acknowledgment.

Stereotypes

Racial stereotypes. Fifteen of the 16 participants interpreted events being potentially marginalizing due to the activation of stereotypes in and beyond STEM. One set of stereotypes was related to racially minoritized students’ limited mathematics ability and lack of academic effort. Angelica (Latinx woman) interpreted the event of an instructor not reviewing an “easy problem” during class and claiming a student’s exam problem solution was “so wrong” as being more likely to happen between a white instructor and student of color. In particular, Angelica perceived this event as an implicit form of racial bias with an instructor positioning students of color as “trying to get more points because they don’t want to try,” thus “undermining their intelligence and the effort they put in on an exam.” Both Jasmine (Black woman) and Quinton (Black man), in responding to an event about a student with their hand raised being ignored, acknowledged how such deficit stereotypes about students of color can also frame racially minoritized instructors’ teaching practices. Jasmine, for instance, explained how “the culture of… ‘these are what we interpret as the smart kids’” in STEM can produce “implicit biases… even within minority teachers” that could bring women’s and racially minoritized students’ contributions to be deprioritized.

Participants also acknowledged how the racial stereotype that Black and Latinx people are criminals could play a role in the event when an instructor accused a student of not owning a university-provided calculator (Amy, Beatriz, Leonardo, Nadine, Parker, Sarah, Uzma, Victoria). Leonardo (Latinx man), for example, reflected on how the event would bring him to “feel like the teacher thinks [he is] a thief.” If the student in the event was a Black or Latinx student, Leonardo conjectured that the instructor’s remark may be bring classmates to “assume ‘Oh, well it isn't hers. She's black. Well, she must have stole it.’”

Gender stereotypes. Another set of stereotypes raised in participants’ reflections about how the classroom events could produce discomfort or discouragement was related to gender. Participants referred to the gendered stereotype that women are less mathematically able than men in explaining instructors’ potentially marginalizing actions through teaching (Delma, Sarah) and women’s pressure to challenge others’ underestimation of their ability (Anne). Sarah (white woman), for instance, described how this gendered perception of ability can explain the logic behind an instructor’s disregard of a women’s request to do a similar follow-up problem, “Just because this one girl has another question doesn’t mean I have to do it for the rest of the class.” Anne (white woman) interpreted the event of a student apologizing for asking a question that the instructor curtly refused to answer as potentially gendered, particularly because the student was likely a woman who felt she must apologize for asking something that was simple or obvious.

Summary. These student reflections capture how they perceived the operation of racial and gender stereotypes in framing what could be experienced as potentially marginalizing instances of classroom instruction. Racial and gendered stereotypes of academic ability were raised in explaining disparities of student acknowledgment and participation due to implicit biases among instructors, including those from minoritized backgrounds. Furthermore, as exemplified in Leonardo's reflection, the influence of an instructor is evident in how their stereotypical framings of classroom interactions can bring students to similarly position marginalized peers in deficit or negative ways.

Instructor Care

Getting to know students. Thirteen of the 16 participants, especially among women of color, interpreted events as being potentially marginalizing due to the level of care that instructors exhibited. For example, instructor comments were interpreted as them not caring to know their students personally. Nadine (Black woman), in reflecting on her submitted event where an instructor confused her with another women, shared how offended she felt when she learned that her instructor did not know her name mid-semester. As one of only two women in the classroom, Nadine described the instructor's confusion as "careless" which she took personally, especially since she had "taken the time to learn the professor's name and ... put effort into the class." Nadine states, "I always get really upset when that happens. It's an honest mistake, but the reaction after you're [the instructor] corrected shouldn't be like 'Yeah whatever.'... I'm a person with my own identity and my name is a part of that." Sarah (white woman) similarly discussed how such confusion of two students could reflect the instructor "group[ing] them off in their mind based on race or gender." She argued that this captures how "a teacher really does decide not to get to know their students" at an individual level.

Student support in understanding. Another interpretation of classroom events was instructors not caring to make sure students understand the material. Jasmine (Black woman), in response to an event with an instructor declining to review an "easy problem" and laughing at a student's request to earn more points, described how most mathematics instructors do not worry much about having rushed through the material and whether students understood what was presented in class. In particular, she commented on how instructors may not ask themselves, "Maybe I missed something? Maybe it was a rushed job? Maybe I didn't teach it at all?". Jasmine further acknowledged how, if she was the student in the event, the instructor's lack of care "discourages [her] from asking a question about [her] exam or just asking a question about a concept." Sarah (white woman) interpreted an event (namely, one with an instructor ignoring a student's question) as the instructor rationalizing that they can't "waste time" if only some students don't understand the material, thus communicating "a lack of care for explaining and helping other students." In Sarah's reflection, she described being brought to "feel a little unimportant" and, similar to Janiya's reflection about discouraged participation, may cause students in general to not ask questions because the instructor has "no interest in helping them." Beatriz (Latinx woman) commented on how instructors ignoring students' questions makes her feel as though she needs to "practice what [she] need[s] to practice and just look out for [herself]" since she "can't rely on the professor" to answer her questions.

Impact of classroom interactions. A final interpretation students had was that instructors may not be aware of the potential impact their behaviors and words on white women and racially minoritized students. For the event when an instructor asked a student if they owned the calculator that a university support program provided low-income students, Uzma (Black woman), perceived the instructor as having a "level of ignorance in how certain programs in the university work" that could bring the student to feel the instructor was not "sensitive to [their] situation." Sarah (white woman) believed that the instructor's actions for the calculator-related event could be an "innocent mistake." However, Sarah felt that the instructor should still be responsible in learning about the support program to avoid offending future students, "If a professor were to learn what it [the program] is, they would see why the mistake could be offensive." For the event where the teacher told students they should consider moving down a course level in mathematics, Victoria (Latinx woman) commented on how instructors might not realize some students, particularly from minoritized backgrounds, might interpret comments in discouraging ways, such as "If you can't do this, you might as well not be a doctor".

Summary. Participants perceived events as reflections of instructors' lack of care in building relationships with students, deepening students' understanding of content, and acknowledging students' social backgrounds and life circumstances beyond the classroom. These reflections highlight how such lack of care could be disrupted through instructors getting to know students more personally and providing more opportunities for student support in instruction.

Implications for Teaching Practice

Findings from this analysis raise implications for socially affirming teaching practices in undergraduate mathematics education across different intersections of students' racial and gender identities. The theme about underrepresentation captures the importance of teachers challenging racialized-gendered frames about students' ability to shape instruction in ways that establish equitable participation opportunities and affirm underrepresented students' sense of belongingness in STEM. In addition, the theme about instructor care raises considerations about the extent to which instructors design classroom learning opportunities to build relationships with their students and learn more about them as whole individuals. Participants reflected on how instructors learning more about their students, including their names and university program affiliations, could allow them to feel their individuality appreciated rather than being positioned as one of the only white women or racially minoritized students in the classroom. Such intentional considerations for the relational spaces of undergraduate mathematics classrooms is especially important in entry-level mathematics classes and larger institutions of higher education where high enrollment can present challenges in getting to know students personally.

Furthermore, the fast-paced instruction and lack of student support opportunities that characterized the theme of instructor care points to the significance of designing undergraduate instruction that prioritizes student understanding. Instructor acknowledgment of how students' questions and volunteered answers (regardless of correctness) advance the understanding of content can contribute to building supportive learning environments that challenge the construction of status or hierarchies of ability. With women and minortized students of color navigating deficit stereotypes of ability, such broadening of instructor support can minimize the discouragement that participants felt about asking questions and seeking help due to instructors' lack of care. Findings related to the role of stereotypes capture the importance of instructors being mindful of how whole-class messages can be interpreted in more or less discouraging ways among students from different social backgrounds and histories of educational experience. Findings from this study, thus, build on previous research by outlining how specific actions in undergraduate mathematics instruction might be marginalizing for students underrepresented in STEM, rather than attributing such experiences of marginalization to an ethos.

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Using a Computational Context to Investigate Student Reasoning About Whether "Order Matters" in Counting Problems

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Abstract. Students often struggle with issues of order – that is, with distinguishing between permutations and combinations – when solving counting problems. There is a need to explore potential interventions to help students conceptually understand whether "order matters" and to differentiate meaningfully between these operations. In this paper, I investigate students' understanding of the issue of order in the context of Python computer programming. I show that some of the program commands seemed to reinforce important conceptual understandings of permutations and combinations and issues of order. I suggest that this is one example of a way in which a computational setting may facilitate mathematical learning.

Key words: Combinatorics, Computational Thinking, Permutations, Combinations

Introduction and Motivation

Determining whether or not "order matters" in a counting problem is a perennial issue in combinatorics, and students often struggle with whether to use a formula involving permutations or combinations when they approach counting problems (if either formula is appropriate). In this paper, I report on a study in which students were given opportunities to engage in computational activity (in the form of elementary programming tasks) as they solved combinatorial problems. In this report, I elaborate episodes that demonstrate the ways in which computational activity may have served to advance students' mathematical thinking. Specifically, I focus on the particular case of students engaging in computer programming to reason about permutations, combinations, and differences between these two fundamental operations.

I seek to accomplish two goals in this paper. First, I want to highlight a potential pedagogical innovation that sheds light on our understanding of how students might reason about an important combinatorial idea in a meaningful way (namely, the difference between combinations and permutations). That is, I am interested in the combinatorial goal of identifying an activity, which involves computing, that might help students understand this important combinatorial distinction. Second, I want to provide an example of what computational thinking and activity might look like in a mathematical context. In this way I want to contribute to the conversation of how computing might be leveraged to help students to reason about mathematical concepts. I seek to answer the following research question: In what ways did programming commands help students to reason about whether or not "order matters" in a counting problem?

Literature Review and Mathematical Perspective

Literature on combinatorics. Permutations and combinations are two foundational combinatorial ideas, and they form the basis of much of the counting that students do. The key difference is that permutations count arrangements of objects – that is, differently ordered arrangements of elements of a set are counted as distinct outcomes. When counting combinations, on the other hand, differently ordered arrangements of elements are *not* counted as distinct outcomes. For example, suppose we have the set $S = \{1, 2, 3, 4\}$, and we wanted to count permutations (and combinations) of 3 of the elements of S. There are 24 such permutations

(Figure 1), and there are only 4 such combinations: 123, 124, 134, 234. An in-depth discussion of the formulas for permutations and combinations is beyond the scope of this paper.

123,	124,	132,	134,	142,	143
213,	214,	231,	234,	241,	243
312,	313,	321,	324,	341,	342
412,	413,	421,	423,	431,	432

Figure 1 – Permutations of three of the numbers 1, 2, 3, and 4

There is ample evidence that students struggle to learn and distinguish between these two ideas (e.g., Annin & Lai, 2010; CadwalladerOsker, Engelke, Annin & Henning, 2012). In particular, many researchers have cited that a common error and struggle for students is to determine when to use a combination formula or a permutation formula. Batanero, et al., (1997) cite "errors of order" as being one of the primary errors that students encounter, and Annin and Lai (2010) discuss difficulties that students have maneuvering issues of order in counting. Lockwood (2014) previously showed examples of students not being sure of how to differentiate between when "order matters" and when it does not. Lockwood reports that when solving a counting problem, an undergraduate student, Kristin, said "I'm doing the combination ones because I'm pretty sure order doesn't matter with combination" (Lockwood, 2014, p. 33). When asked why, Kristin said, "I'm not sure about that one (laughs). I just kind of go off my gut for it, on the ones that don't specifically say order matters or it doesn't matter" (p. 33). This response is perhaps indicative of students' approaches to the distinction between permutations and combinations – often they do not have well-understood ways to differentiate between the two. Some have reported on ways to try to address this. Lockwood (2013, 2014) contends that by focusing on the set of outcomes, students can reason about the nature of outcomes as a way to clarify what is being counted, thus helping to determine whether or not counting matters.

Literature on computational thinking and activity. Mathematics departments across the country increasingly emphasize the importance of computation. As evidence for this trend, consider a) departments that include "computational requirements" for their mathematics majors, b) the growth of the branch of computational mathematics, and c) the myriad applications of computational mathematics, ranging from work with big data to modeling real-world problems using sophisticated software. Science, Technology, Engineering, and Mathematics (STEM) education researchers have focused on computation in the last decade especially, and computer scientist Wing (2006, 2008) coined the term *computational thinking* as analytical thinking that "takes an approach to solving problems, designing systems, and understanding human behavior that draws on concepts fundamental to computing" (Wing, 2008, p. 3717). In addition, the Next Generation Sciences Standards (NGSS Lead States, 2013) includes "using mathematics and computational thinking" (p. 37) as one of eight key scientific practices. I currently adopt the following definition of computational thinking, adapted from Wing (2014): Computational thinking is the way of thinking that one uses to formulate a problem and/or express its solution(s) in such a way that a computer (human or machine) could effectively carry it out.

Weintrop, et al. (2016) developed a "taxonomy of practices focusing on the application of computational thinking to mathematics and science" (p. 128). I use this taxonomy of practices, especially the computational activities associated with Computational Problem Solving Practices, to characterize computational activity. These include preparing problems for computational solutions, programming, choosing effective computational tools, assessing different approaches/solutions to a problem, creating computational abstractions, and troubleshooting and debugging (p. 135). Practically, for the results described in this paper, the students engaged in basic programming tasks in Python, and this primarily included preparing problems for computational solutions, programming, and troubleshooting and debugging.

Theoretical Perspectives

Characterizing combinatorial thinking and activity. In considering students' combinatorial thinking, I use Lockwood's (2013) model, which frames students' combinatorial thinking in terms of three key components: *Formulas/Expressions*, *Counting Processes*, and *Sets of Outcomes*. *Formulas/Expressions* are mathematical expressions that yield some numerical value. A formula or expression is what a student may write as "the answer" to a counting problem. *Counting Processes* are the imagined or actual enumeration processes in which a student engages – that is, the steps or procedures that one completes when solving a counting problem. *Sets of Outcomes* are the sets of elements that are being counted. The cardinality of the set of outcomes typically determines the answer to the problem.

Reinforcing the relationship between counting processes and sets of outcomes. The relationship between counting processes and sets of outcomes is particularly important for students to develop. In terms of the model, one way to frame students' difficulties with counting is that students do not clearly connect their counting processes with the outcomes they are trying to enumerate (Lockwood, et al., 2015). Thus, a possible solution to improve students' combinatorial problem solving is "to reinforce the relationship between counting processes and sets of outcomes, and to help students integrate the set of outcomes as a fundamental aspect of their combinatorial thinking and activity" (Lockwood, 2014; p. 36). One way to establish and strengthen this relationship is through the systematic listing of outcomes.

Lockwood and Gibson (2016) showed that listing behavior (taken as partial and complete listing) was positively correlated with correctly answering combinatorial problems for novice counters. Lockwood and Gibson hypothesized potential reasons for this correlation, namely that in terms of the model, listing supports the relationship between counting processes and sets of outcomes. This prior work suggests that the activity of listing has the potential to strengthen the important relationship between counting processes and sets of outcomes, and thus serve as an avenue by which students can solve combinatorial problems more successfully.

Computational activities represent a natural extension of listing. Even though prior work has demonstrated that listing is a potentially valuable combinatorial practice (Lockwood & Gibson, 2016), solutions to combinatorial problems can be enormous (there are ten billion 10-digit PIN numbers, for example). It is often not feasible for students to generate complete lists of outcomes by hand. Partial listing also has limitations, as patterns do not always extend to all cases, and students often fail to detect subtle errors. Thus, there is a dilemma – we know that listing can be valuable, but listing by hand has clear drawbacks. This leads to a question of how we can move past limitations of by-hand listing in order to facilitate listing in more complex problems and contexts. Fortunately, there is a natural solution to this question: we can leverage technology and computational activities, allowing students to reap similar benefits of by-hand listing by designing algorithms and computer programs to enumerate lists. I hypothesize that such activity can potentially strengthen the relationship between counting processes and sets of outcomes, which can help students solve counting problems. As noted in the Literature Review, I adopt Weintrop, et al.'s (2016) taxonomy in defining computational activity. I particular focus on programing, trouble shooting, and debugging as the primary computational activities.

Methods

Participants and Data Collection. In this paper I report on data from a teaching experiment (Steffe & Thompson, 2000) that consisted of 15 hours of contact time with two students in 60-90 minutes sessions. The participants I discuss in this paper were two vector calculus students who were interviewed as a pair (pseudonyms Charlotte and Diana). Both were novice counters and had no programming experience in high school or in college, and they were chosen based on 30-minute selection interviews. They were paired together because they had relatively similar backgrounds and abilities, and they also had schedules that allowed them to meet together for 15 hours over the term. Charlotte was a sophomore and Diana was a freshman at the time of the interviews, and both students were majoring in chemistry with an interest in forensic science.

During the TEs, the students sat together and worked at a desktop computer in the programming environment PyCharm. I gave them paper handouts and also wrote the tasks and prompts in PyCharm, and the students used PyCharm to edit and run the Python code. To capture the interviews, I videotaped and audiotaped the interviews, and I also took a screen video recording of their work on the computer. This allowed me to view the students' on-paper work and their interactions, as well as what they programmed and how they used the computer.

Tasks. Over the course of the TE, I gave the students a variety of tasks in which they were asked to use the computer to determine the answers to counting problems. I created these tasks with the goal of targeting some fundamental combinatorial ideas, particularly focusing on the relationship between counting processes and sets of outcomes. The tasks overall followed a trajectory toward helping students reason about key combinatorial ideas including the multiplication principle, basic operations of permutations and combinations, and aspects of positional reasoning and encoding outcomes. For example, the tasks in Figures 2 and 3 represents typical tasks in the TE. Generally, I had them engage in programming directly by writing and running code, or I had them evaluate excerpts or outputs of code. I frequently asked for follow up questions or asked them to reflect on their thinking and activity. In this way, the interviews were interactive.

For the purposes of this paper, I focus especially on the tasks involving the development of permutations and combinations. In developing such tasks, I had considered some ways in which these ideas of permutations and combinations might be coded using Python. In particular, the task in Figure 2 shows how the symbol != helps to count permutations of 5 of the letters in the word PORTLAND. Note that != means "not equal to," and the if statements within the for loops indicate that the outcomes will not be printed if any of the characters are equal to previous characters. In this way, the inclusion of != in this code counts permutations in which repetition of characters is not allowed.

In a similar way, the task in Figure 3 shows how the symbol ">" might function in Python. By encoding the elements we want to count (books, in this case) as numbers, we can compare the elements using the greater than symbol. Thus, the "if $j > i$" condition will only consider arrangements for which a subsequent character is strictly greater than previous characters. Essentially, this would count something like 1, 2, 3, but it would not count 1, 3, 2 or 2, 1, 3, or any other arrangement of the numbers 1, 2, and 3. This is exactly what we want to count with combinations – subsets, but not arrangements, of some elements. The students were able to make sense of what the commands might mean and might do in terms of outcomes. As we will see in the results, the act of programming these ideas seems to have beceme meaningful and useful for them.

a) Can you write some code in order to create a list of all of the ways to arrange 3 of the letters in the word ROCKET?
b) Caleb had to answer the question: *How many arrangements are there of 5 of the letters in the word PORTLAND?* He wrote the following code to get the answer. Do you think he's correct? Why or why not? What numerical expression does his code suggest?

```
arrangements = 0
Portland = ['P','O','R','T','L','A', 'N', 'D']

for i in Portland:
    for j in Portland:
        if j != i:
         for k in Portland:
            if k != i and k != j:
                for l in Portland:
                    if l != i and l != j and l != k:
                        for m in Portland:
                            if m != i and m != j and m != k and m != l:
                                arrangements = arrangements+1
                                print(i,j,k,l,m)
print(arrangements)
```

Figure 2 – A task to elicit permutations

a) Suppose you have 8 books and you want to take a pair of them with you on vacation. How many ways are there to do this?
b) Consider the code below. Does this answer the same question as before? Why or why not?

```
arrangements = 0
Books = [1, 2, 3, 4, 5, 6, 7, 8]

for i in Books:
    for j in Books:
        if j > i:
            arrangements = arrangements+1
            print(i,j)
print(arrangements)
```

Figure 3 – A task to elicit combinations

Data Analysis. For the results shared in this paper, I reviewed transcripts, particularly episodes in which the students used, referred to, or reflected upon the "not equal to" or "greater than" symbols in their code. This allowed me to analyze the students' reasoning about these symbols, and I sought to understand and create a narrative (Auerbach & Silverstein, 2003) about their reasoning about and use of those symbols.

Results

In this section I describe the students' reasoning about the "not equal to" and "greater than" symbols as ways to express certain combinatorial constraints. In having to communicate with the computer via Python code, the students had to think about how the computer interpreted these different symbols and what the resulting output of the code would be. I will make the case that this experience helped the students make a meaningful distinction between these symbols and could clearly make a connection between these two ideas and what they did in terms of the outcomes. The students established meanings of these symbols as commands they gave the computer, and that this experience helped them to understand important aspects of counting.

The students first thought about the not equals to symbol (!=) in a problem in which they had to think about code that listed the number of ways to list arrangements of 5 people. In the excerpt below, we see the students initially interpreting and considering the != notation.

Int.: What do you this the code's doing.
Charlotte: Gosh a lot of code.

Diana: I think for sure that the statements have the exclamation point each time, that's making it so that these values will not repeat, which makes sense when you have five people because you can't just repeat a person.

Charlotte: Yeah, that makes sense. Yeah, kind of what she was saying, I think the code, yeah, just trying to figure out how many different arrangements each person can be in and then yeah, <u>each of these exclamation points, like Diana said, is to make sure John isn't sitting in two different seats at the same time</u>.

As we see in the underlined portion, the students were beginning to understand what the != symbol might be doing in terms of the context of the problem – namely, not allow for John to sit in two different seats at the same time.

Later in the teaching experiment the students were working on the Lollipop problem, which says, "How many ways are there to distribute 3 identical lollipops to 8 children?" The students had written code in which they used a greater than sign. Here they had established that they wanted to count sets of 3 numbers from the numbers 1 to 8, which would represent which children get lollipops. They noted that they did not want to count arrangements of these numbers because the lollipops are identical. In the excerpt below, we see Diana articulate the important fact that the use of "greater than" eliminates duplicates, in the sense of not allowing for both outcomes of 1, 2, 3 and 2, 3, 1 to be counted.

Charlotte: Because, yeah, then it eliminates the factor of duplicates.

Int.: Okay. And can you say again, how that 'greater than' sign eliminates the duplicates like you said?

Diana: So, like it says that k is not able to be less than j, it always has to be greater than. So, and in the example of the 1, 2, 3, it'll print 1, 2, 3, but then when it comes to printing 2, 3, 1, it won't be able to do it because k can't be 1 when these two are 2 and 3.

I suggest that, in these examples, the students were engaging in computational thinking. Diana's comments above suggest that she was considering what the computer would output, which suggests that she was thinking about what steps and procedures the computer was engaging in as it completed the program. In this way, the students seemed to be reasoning about the solution in such a way that they were considering what the computer must have done to carry it out.

Throughout the remainder of the interviews, the students continued to make this distinction and to use it in reasoning about problems. While I do not have space to detail each of these occurrences, I conclude these results with a wrap up discussion from the final session. We had explicitly asked the students some reflection questions about their coding and how they thought about certain aspects of their code. In the excerpt below, we see the students responding to a prompt that asked them to reflect on the difference between the > and != symbols.

Charlotte: Okay. So, the greater than symbol definitely plays an important role. In this problem with the alphabet, the greater than symbol played a role because you didn't wanna have A, E, I, O, and U not in alphabetical order. So, it helps arrange them in that order because A representing one, E representing 2, I representing 3, O for 4, and U for 5. You don't wanna have 3, 4, 5, 2, 1. So, that greater than symbol helps play a role for that. Do you wanna explain the not equal to?

Diana: Sure. So, the not equal to sign helps prevent the outcomes from being 1, 1, 1, 2, 2, 2.

> And in the case of the lollipop and the red balloon problem, you don't want one kid, which would be 1, 1, 1, getting all three lollipops. So, you use the not equal to statement.

In sum, while the students referred to particular problems in discussing the utility of each command, I contend that they were establishing ways of reasoning about these commands and issues of order in solving counting problems. They could clearly articulate the different commands and what they counted in terms of the sets of outcomes.

Discussion, Conclusion, and Implications

In this paper, I offered evidence of ways in which students reasoned about commands in Python in order to think about whether order should matter in solving counting problems. The students did eventually come to understand more general formulas for permutations and combinations, although they did not necessarily refer to them by those names. The point is that the students seemed to have established meaningful ways of thinking about generating outcomes through a program, and the symbols in the commands put certain constraints on what outcomes were being generated. In this way, the students were formulating a relationship between the counting process (the programs that involved nested for loops) and the outcomes that were being generated. By specifying that i != j or $j > i$, the students were imposing constraints that dictated the nature of the outcomes. I contend that the computing environment in particular leveraged this kind of activity and reasoning about these important combinatorial ideas.

There are obviously many different productive ways that students can reason about counting processes and outcomes. I am not claiming here that this is a superior way for students to reason, nor that it is the only way that they should reason about these ideas. But, the students seemed to demonstrate a solid and meaningful understanding of these ideas. Their understanding of what the greater than sign indicated in terms of duplicates stands in contrast to Lockwood's (2014) student who said she just went "off her gut." I certainly do not want to simply have mantras of "< means order doesn't matter" or "!= means order does matter", but I do not this was how the students were reasoning. Rather, it seems that by actually thinking carefully about what the program was doing in terms of those symbols, and thinking about both what those commands told the computer and how the computer implemented and carried them out, the students developed a better understanding of how the outcomes were being generated.

These findings provide an existence proof that meaningful mathematical ideas can be introduced and reinforced in computational settings. This suggests that there is more to study and learn related to the relationship between computational activity like programming in students' mathematical reasoning and activity.

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Conversations on Density of $\mathbb{Q}$ in $\mathbb{R}$

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We explore the notion of density of the set of rational numbers in the set of real numbers, as interpreted by undergraduate mathematics students. Participants' responses to a scripting task, in which characters argue about the existence of one or infinitely many rational numbers in a real number interval, comprise the data for our study. The framework of reducing abstraction is used in explaining the participants' mathematical behavior when coping with the task. The analysis reveals informal ideas related to density as well as unconventional understandings of density-related concepts of rational numbers and infinity.

Keywords: density, rational numbers, scripting tasks, reducing abstraction

The notion of density is one of the main characteristics of the rational numbers, which distinguishes these from natural numbers and integers. However, the notion of density has not yet received significant attention within the growing body of research in mathematics education at the tertiary level. While the notion of density is the main focus of this paper, we demonstrate how engaging students in a discussion on density brings to light some of their underlying ideas on the structure and nature of rational numbers. However, prior to presenting the details of our study, we supply an overview of the notion of density in mathematics education research, followed by a discussion on mathematical nuances related to the concept.

The Notion of Density in Mathematics Education Research

The investigation of learners' understanding of the notion of density in prior research was associated with the development of understanding of rational and irrational numbers. In this regard, Vamvakoussi & Vosniadou (2004) argued that the understanding of rational numbers requires a conceptual change, which is a lengthy and gradual process. They further assumed that the idea of discreetness, developed through experience with natural numbers, is a "fundamental presupposition which constrains students' understanding of the structure of the set of rational numbers" (p. 457).

In studies that focused on learners' ideas in relation to density, middle and high school students were often given a particular interval (such as "numbers between 0.21 and 0.22" or "numbers between 1/10 and 1/11"), and subsequently asked multiple variations of similar-idea questions – such as whether there exist any rational numbers in the interval, how many rational numbers exist in the interval, and so forth (e.g., Vamvakoussi & Vosniadou, 2004, 2007; Vamvakoussi, Vosniadou, & Van Dooren, 2013). The findings pointed to a natural number bias, in the sense that the discreetness of natural numbers, as well as the existence of a successor in natural numbers, were extrapolated to rational numbers. This resulted in frequent mistakes, reported both with common fraction and decimal fraction representations of rational numbers.

In several studies that explored teachers' understanding of irrational numbers, the issues related to density appeared as part of the tasks. For example, Sirotic and Zazkis (2007) focused on the density of both sets of rational and irrational numbers, and inquired into how prospective secondary teachers' "fit together" these two sets. In particular, they asked participants to determine whether it was possible to find a rational (or irrational) number between any two rational (or irrational) numbers. We note that in the density related items there was no specific attention to the option of a general interval of real numbers, that is where one endpoint may be rational and the other irrational. In the current paper we address this aspect and attend to the more general property – the density of the set of rational numbers in the set of real numbers. The following section elaborates on this issue.

On Density: Density of $\mathbb{Q}$ vs. Density of $\mathbb{Q}$ in $\mathbb{R}$

We observed that most of the studies that explicitly discuss the density of rational numbers attend exclusively to the set of rational numbers. However, the notion of density of the rational numbers is more general: not only that the set of rational numbers $\mathbb{Q}$ is dense, i.e., dense within itself, but it is also dense in the set of real numbers $\mathbb{R}$. Formally, we attend to the following definitions:

- Definition 1: Given a $X \subset \mathbb{R}$, we say that X is ***dense*** if for every $a, b \in X$ there is a $c \in X$ such that $a < c < b$.
- Definition 2: Given a subset $X \subset \mathbb{R}$, we say that X is ***dense in*** $\mathbb{R}$ if for every $a, b \in \mathbb{R}$ there is a $c \in X$ such that $a < c < b$.

Note that Definition 2 appears in formal mathematics texts (e.g., Bartle & Sherbert, 2011; Courant & John, 2012), while variations of Definition 1 are implied in the mathematics education research literature (e.g., Vamvakoussi & Vosniadou, 2010; Malara, 2001). That is, mathematics education research has primarily focused on the existence of rational numbers in a *rational* number interval, rather than in the interval of real numbers. However, the density of a set does not imply its density in $\mathbb{R}$. Consider for example the set $X = (0, 1) \cap \mathbb{Q}$, which is dense (meaning within itself), yet not dense in $\mathbb{R}$. Hence, the density of $\mathbb{Q}$ in $\mathbb{R}$ cannot be deduced from the density of $\mathbb{Q}$, and therefore requires a separate consideration. As such, our study attends to the notion of density of $\mathbb{Q}$ in $\mathbb{R}$, specifically as understood by undergraduate students.

Theoretical Framework: Reducing Abstraction

The framework of *reducing abstraction* was introduced by Hazzan (1999) when inquiring into students' struggles with concepts and ideas of Abstract Algebra. The basic premise of the framework is that when solving mathematical problems, students may operate on a lower level of abstraction than is intended by the task or the instructor. The framework is based on three different interpretations of abstraction discussed in the literature, described briefly below. It is important to note that these interpretations are neither mutually exclusive nor exhaustive.

a) The interpretation of abstraction level as the *quality of the relationship between the object of thought and the thinking person* is based on the idea that abstraction is not a property of an object, but rather on "a property of a person's relationship to an object" (Wilensky, 1991, p. 198). An illustration of this idea is provided by Noss and Hoyles (1996) who wrote "To a topologist, a four-dimensional manifold is as concrete as a potato" (p. 46).
b) The interpretation of abstraction level as *reflection of the process–object duality* is based on the process–object duality, suggested by several theories of concept development in mathematics education (e.g., Dubinsky, 1991; Sfard, 1991). Despite the differences in further elaborations, researchers agree that during learning stages of a mathematical concept, its conception as a process precedes – and as such is on a lower level of abstraction – than its conception as an object.
c) The interpretation of abstraction level as the *degree of complexity of the mathematical concept* is based on the assumption that a more complex object is more abstract. For instance, a particular example demonstrating a property is less abstract than a general

claim justifying a property; a particular element of a set, or a particular subset, is less abstract than the set itself; and so forth.

In addition to the initial work in Abstract Algebra (Hazzan, 1999), the framework was employed in different areas of mathematics, such as differential equations (Raychaudhuri, 2014) and a variety of topics in school mathematics (Hazzan & Zazkis, 2005). Hazzan (2003) provided a comprehensive report that illustrated the application of the reducing abstraction framework in a variety of situations and topics taken from undergraduate mathematics. In this paper we describe an application of the framework in analyzing students' ideas of density, and demonstrate the role of reducing abstraction in students' conceptions of real and rational numbers.

The Study

Participants and Setting

The participants of the study were 95 first-year undergraduate students enrolled in a Bachelor's degree in mathematics in a highly-ranked university in Brazil. At the time of data collection the students were enrolled in a "Foundations of Mathematics" course, which provided a foundation for subsequent Pre-Calculus, Calculus, and Real Analysis courses. It was assumed that the students were familiar (at least to some degree) with how rational numbers are defined, with different representations of rational numbers, and with the relation between different number sets (natural-, integer-, rational-, irrational-, and real numbers). During the course, special attention was given to the representation of numbers and intervals on the real number line. In the middle of the course, the students responded to a task that dealt with the notion of density, as described in the following section.

The Task and Research Questions

The task that was presented to the participants of the study belongs to the genre of scripting tasks. In such tasks, participants are typically given a beginning of a dialogue, referred to as a prompt, and are asked to extend the dialogue in a way they find mathematically and pedagogically fit. Scripting tasks were used in prior research in various mathematical contexts (e.g., Kontorovich & Zazkis, 2016; Marmur & Zazkis, 2018; Zazkis & Herbst, 2018), and their advantages were elaborated upon in detail (e.g., Zazkis, 2018). In particular, a significant feature of scripting tasks is that they provide script-writers the opportunity to consider or revisit the mathematical ideas related to the task, and offer researchers a lens on the script-writers' understanding of these particular mathematical concepts and relations.

The prompt for the particular task analyzed herein (see Figure 1) presents a disagreement on how many numbers can be found in a given interval of real numbers.

Pedro:	Hello, Maria! Did you manage to explore the applet[1]?
Maria:	Yes, it was quite nice. Here's my conclusion: Given two distinct numbers on the line, a and b, we can always find a rational number between a and b.
Pedro:	Wow, my conclusion was very similar to yours, but there is a difference. See: Given two distinct numbers on the line, a and b, there are infinitely many rational numbers between a and b.
Maria:	I don't think so, how did you come to that conclusion?
Pedro:	...

Figure 1: Prompt for the scripting task

[1] The applet (https://www.geogebra.org/m/nruYwQAd) provided an interactive and virtual environment to explore density. However, students' interaction with the applet is outside the scope of our analysis in this paper.

In addition to continuing the dialogue (Part-A of the task), the participants were asked to present a mathematical analysis reflecting their personal understanding of the issue (Part-B). This was in order to be able to distinguish between student-character statements that might represent a "student way of thinking", and statements that represent the script-writer's own ideas.

The task was designed to uncover the participants' informal ideas about the density of $\mathbb{Q}$ in $\mathbb{R}$, ideas on which the formal proof is built in a later course. Note that while the claims of Maria and Pedro are presented in a form of disagreement in the task, they are in fact equivalent as each claim implies the other.

Initially, the task was designed to address the following research question:

- What is revealed in the participants' claims in regard to their informal ideas about the density of $\mathbb{Q}$ in $\mathbb{R}$?

Through the examination of data, we added another research question, to which we attend herein:

- What is revealed in the participants' claims in regard to their understanding of infinity, as well as real and rational numbers?

Data Analysis

The data for this study are comprised of the scripted dialogues composed by the participants, together with their personal mathematical analyses of the issues at hand. As in prior research that used script-writing for data collection, we regarded the ideas expressed in the scripted dialogue, on which both characters agree, as ideas held by the student who composed the dialogue, unless explicitly stated otherwise in the mathematical analysis section.

In the first round of analysis we identified with which character (Maria or Pedro) the script-writers agreed. In the second round we focused on the arguments that were provided in support of one of the characters' views. While focusing on the existence of rational numbers in an interval, the participants revealed in the voices of their characters some unconventional understandings of rational numbers and ideas related to infinity, which are in discord with mathematical convention. Accordingly, in the third round of analysis we identified and analyzed these unconventional and at times idiosyncratic understandings by utilizing the framework of reducing abstraction (Hazzan, 1999). The findings from this round are presented below.

Findings

While the instruction of the task did not require the students to choose which statement they thought was correct, most of the participants explicitly agreed with one of the characters in the dialogue. In fact, out of the 95 participants, the majority (n=69) sided with Pedro. The other students either agreed with Maria (n=10), or with both (n=11), or did not voice any explicit agreement with either character (n=5). However, regardless of the chosen claim (Maria's or Pedro's), the students' arguments and justifications were at our focus of attention, as they provided a lens into their understanding of density and related concepts. In what follows we exemplify participant ideas related to density, though at times incomplete or erroneous, that illuminate their understanding of real and rational numbers.

Referring to a Ruler to Spread Rational Numbers on the Number Line

One method students employed in order to deal with the task was to first "spread" rational numbers all over the number line, typically represented with a ruler, and only subsequently place the points a and b in their accurate location, whilst already having rational numbers "ready-made" in between.

Pedro: Don’t you know that between two points on the number line we have several other points?

Maria: Yes, I know! But I don’t agree that there are infinite numbers.

Pedro: I'll explain with a ruler how I came to this conclusion and you're going to agree with me. When we get the school ruler we can see the cm because we have the traces, right? So we can also do with millimeters.

Maria: Yeah. But what does this have to do with what I said?

Pedro: Calm down, I'm getting there! After we have observed that between the cm exists the mm, and that to arrive at the value of 1 cm we need to count 10mm, then we can conclude that in order to arrive at the value of 1 mm, we will need to count another 10 of some value that we do not use normally and so on. As you can see, my points A and B are between 0 and 1, and when we partition that same measure we realize that there can be found infinite numbers between them. The more partitioned, the more numbers are found!

We regard this type of mathematical behavior as reducing the level of abstraction in the following three ways. First, we recognize this abstraction reduction as reflection of the process-object duality (Hazzan, 1999). That is, the students attend to the process of *creating* infinitely many rational numbers using “smaller and smaller” partitions, rather than to the *existence* of these numbers. We note that the above excerpt does not actually demonstrate the existence of infinitely many rational numbers between a and b, but only points towards a process that can continue indefinitely in order to produce them.

Secondly, we view the abstraction level in regard to the applicability, concreteness, and tangibility of the mathematical object. In this case, the rational numbers are related to a real-life application of measuring distances, and exist in a physical form as lines on the measuring ruler. Thirdly, we consider the abstraction reduction in relation to the logical complexity of the given statement. Meaning that instead of demonstrating the existence of rational numbers (whether one or infinitely many) for a *given* segment, the students herein swap the logical order by first creating rational numbers with a ruler, and only then positioning the segment on the number line. This mathematical behavior is in line with the logical difficulties observed by Dubinsky and Yiparaki (2000), where students confuse between AE and EA statements (i.e., $\forall x\ \exists y\ R(x,y)$ versus $\exists y\ \forall x\ R(x,y)$).

Particular Intervals and Sequences with Discernable Patterns

Many students chose to work with specific intervals in which rational numbers were searched for (that is, with particular choices for a and b), typically accompanied by a construction of a sequence with a clear pattern. The following excerpt illustrates this tendency:

Pedro: Now we can take as an example a number that is between 0 and 1, tell me all that comes to your mind.

Maria: Well, we can think of half of 1 = 1/2 = 0.5.

Pedro: Yes, we can, this number is certainly between 0 and 1 right there in the middle. But we can get a lot more numbers. Think of a few more.

Maria: Okay, how about these: 1/3, 1/4, 1/5.

Pedro: Perfect, those are certainly between zero and one. Did you notice that you can increase the denominator until you get tired?

Similarly, other sequences in various participants’ scripts followed an easy-to-guess pattern, such as the sequence 0.11, 0.101, 0.1001, 0.10001, …, given between 0.1 and 0.2. Most sequences approached one of the endpoints of the interval (in fractional or decimal representation), though some were placed somewhere “in between”, e.g., the sequence 0.1, 0.11, 0.111, 0.1111, …, in the interval $(0,1)$. This demonstrates an abstraction reduction towards the process (versus object), where students focus on the calculative aspect of producing particular sequences of rational numbers in an interval. Furthermore, we suggest that by producing simple-patterned sequences as illustrated above, the students were not attending to the arbitrary nature of the segment (a, b), and how rational and irrational numbers are situated in it.

Additionally, the level of abstraction is reduced here in relation to the degree of complexity of the concept of thought (Hazzan, 1999). Not only is there a preference towards particular numbers rather than arbitrary real a and b, but also a and b are always chosen as integers or rational numbers, thus reducing the complexity degree of the concept of an interval. Consequently, the level of abstraction being reduced is also manifested by students accepting particular examples as a valid justification (see Hazzan & Zazkis, 2005). In most cases we could find no evidence, neither in the scripts nor in the accompanying mathematical analysis, which demonstrated awareness that the particular examples were not generic, in the sense that the general case could not be concluded from the chosen examples. To the contrary, we witnessed cases in which the consideration of segments with irrational endpoints was explicitly rejected, demonstrating that working in a reduced level of abstraction was a conscious choice.

Pedro: I imagined A and B as integers…

Maria: But, does it work for my numbers? Is this a rational number that I find between A and B?

Pedro: I'm not sure. I think that for this rational number to be the midpoint it is necessary for A and B to be rational numbers. Imagine if the points were $\sqrt{2}$ and π. I think the midpoint would be irrational because $\sqrt{2}$ and π are irrational.

Maria: Ihh! It's already complicated. Let's stay with rational numbers for now?

Fractions are Small Numbers

As illustrated in the previous sections, we noticed that many students not only chose to work with specific intervals, but also situated the problem around the number zero. This led us to suspect that some students have a concept image (e.g., Tall & Vinner, 1981; Vinner, 1983) of fractions as “small numbers”, that is, what we refer to as positive proper fractions. The following representative excerpt supports this interpretation, exemplifying only positive proper fractions without attention to the interval in which rational numbers are being sought:

Pedro: Note that if I divide a unit into 2,3,4,5 parts and get one of them, ex. $\frac{1}{2}, \frac{1}{3}$, and so on ... I'm dividing this unit into smaller and smaller parts but I'll never get to zero. And as I can put any integer value, there will be infinite parts without reaching zero.

Maria: I had not thought of it this way, but that does not mean that my statement is wrong.

Pedro: Yes, I agree with you that we will always find a point between a and b. But my demonstration goes further and shows that we can find infinite points between a and b.

As Raychaudhuri (2014) elaborated, students can reduce the abstraction level of a problem or concept by ignoring or “freeing” the context in which it is situated. In the current case this is done by attending to rational numbers with no regard to the segment (a, b) in which they are to be found. Our interpretation of this tendency as illustrative of students’ abstraction reduction is further supported by Zazkis (2014), who regarded the students’ evoked example space (Watson & Mason, 2005) – which “accounts for what specific examples are actually used” (Zazkis, 2014, p. 34) – as indicative of how students reduce the abstraction level of a concept by attending to particular examples.

Personal Meaning of "infinite rational numbers"

Another phenomenon we observed in the data was students' preference towards their own personal meaning of mathematical concepts over conventional interpretations. Note that the Portuguese formulation of the task "infinitos números racionais entre a e b" literally translates to "infinite rational numbers between a and b", though for the purpose of this report was translated to "infinitely many rational numbers between a and b". However, some students interpreted the expression "infinite rational numbers" as a rational number that has an infinite decimal representation, rather than the intended meaning of an infinite amount of rational numbers. Once such a number was found (e.g., 0.666…), the subsequent conclusion was that Pedro's assertion was correct, and therefore there are "infinite rational numbers" between a and b.

Related to Hazzan (1999) interpretation (a) above, Raychaudhuri (2014) found that one way in which students reduce the level of abstraction is by referring to their own personal meaning of a concept. That is, they choose their own interpretation, which is based on their personal mathematical (and non-mathematical) experience, rather than search for and base their ideas on conventional mathematical meanings. By regarding 0.666… as "infinite numbers", the script-writer reduced the abstract nature of grasping a non-concrete infinite amount of numbers, and changed the meaning to a single and concrete number whose digits continue indefinitely.

Discussion and Conclusion

The current research was designed to gain deeper insight into undergraduate students' understanding of the density of rational numbers within the set of real numbers. The findings demonstrate the complexity of this notion for learners. In particular, the participants in this study demonstrated difficulties in justifying their chosen mathematical claims in an appropriate manner. This revealed unconventional yet somehow limited understandings of the relation between rational and irrational numbers, as well as the notion of infinity.

When analyzing the data, the framework of reducing abstraction proved to be a valuable tool in explaining the participants' mathematical behavior and their coping mechanisms with the task. Rather than attending to the general structure of a segment on the real number line, and how rational and irrational numbers interlay within it, it seems that the participants concentrated on specific examples, contexts, processes, and personal meanings, consequently reducing the intended abstraction level of the task. This also revealed certain mathematical conceptions and ideas held by the participants: a concept image of fractions as small (positive) numbers; a restricted view on the notion of infinity which is solely regarded as a process (e.g., Dubinsky, Weller, Mcdonald, & Brown, 2005); and a rational-number bias in the sense of: (a) a strong preference towards working with rational numbers whilst rejecting cases with irrational numbers, and (b) regarding particular examples with rational numbers as explanatory justifications of the general case which includes real numbers as well.

In conclusion, the contributions of our findings are twofold. First, our study expands on previous research in mathematics education, and explores not only learners' understanding of the density of $\mathbb{Q}$ (within itself), but also of the density of $\mathbb{Q}$ in $\mathbb{R}$. The findings suggest that by placing the discussion in the context of real numbers rather than rational numbers only, the level of mathematical complexity rises, which may also explain the resulting student behavior of reducing the level of abstraction. Secondly, when examining the scripts that are situated in this more general mathematical context, the findings demonstrate mathematical ideas that are held by learners not only in relation to rational numbers, but also in relation to irrational numbers and the notion of infinity. These insights into unconventional student understandings could in turn be utilized for the development of suitable teaching practices that address these student conceptions.

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Example Spaces for Functions: Variations on a Theme

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In this study we focus on example spaces for the concept of a function provided by prospective secondary school teachers in an undergraduate program. This is examined via responses to a scripting task – a task in which participants are presented with the beginning of a dialogue between a teacher and students, and are asked to write a script in which this dialogue is extended. The examples for functions fulfilling certain constraints provide a lens for examining the participants' concept images of a function and the associated range of permissible change. The analysis extends previous research findings by providing refinement of students' ideas related to functions and the concept of the function domain.

Keywords: function, script writing, example space

The Function Concept

The concept of a function is fundamental in mathematics, and it has been repeatedly regarded in the education literature as a central concept in the mathematics curriculum from school to undergraduate studies (e.g., Ayalon, Watson, & Lerman, 2017; Dreyfus & Eisenberg, 1983; Dubinsky & Wilson, 2013; Hitt, 1998; Paz & Leron, 2009). However, it has been demonstrated that undergraduate students often struggle with similar difficulties as those attributed to secondary school students. These include difficulties in recognizing what is or is not a function, especially in cases of "irregular" curves; difficulties in defining what a function is, and not alluding to the definition when working with functions; difficulties in linking and changing between different representations of functions; incorrect assumptions that all functions are continuous and smooth, or need to be expressed as a single formula, equation, or rule; overemphasis on graphic representation and reasoning (such as the vertical line test); and overreliance on procedural algebraic computations (compiled from Dreyfus & Eisenberg, 1983; Even, 1998; Hitt, 1998; Leinhardt, Zaslavsky, & Stein, 1990; Sánchez & Llinares, 2003; Steele, Hillen, & Smith, 2013; Thomas, 2003). As articulated by Huang & Kulm (2012), these types of mistakes are "serious and striking" (p. 427).

The current study is focused on function examples generated by a group of prospective secondary school teachers in an undergraduate program in response to an imagined mathematics-classroom situation. We analyze the generated examples, and demonstrate how the collective example space of the group provides insight into students' ideas and conceptions of a function.

Theoretical Underpinnings: Example Spaces and their Features

Watson and Mason (2005) introduced the notion of *example spaces*, which are collections of examples that are central in mathematical teaching and learning, in the sense that they "require the learner to see the general through the particular, to generalize, to experience the particular as exemplary to appreciate a technical term, theorem, proof, or proof structure, and so on" (p. 4). Example spaces include not only exemplifying mathematical objects, but also a range of related associations and construction methods (Goldenberg & Mason, 2008). Subsequently, Watson and Mason (2005) borrowed and extended terminology from Marton and Booth's (1997) Variation Theory to describe the structure of example spaces. They used the term *dimensions of possible variation* to address the generality of example spaces, meaning those example characteristics that may be varied without changing their exemplifying essence. Additionally, with the associated term *range of permissible change*, they referred to the defining "borders" of example spaces, meaning the extent to which each dimension may be varied. As explained by Goldenberg and Mason (2008), the latter term was introduced to address learners' "unnecessarily restricted sense of the scope of change available in any given dimension" (p. 187). Furthermore, Sinclair, Watson, Zazkis, and Mason (2011) described the following features of example spaces: *population*, meaning how scarce or dense available examples are within an example space; *connectedness*, that is whether different examples in a space are interconnected; *generality*, namely whether the example represents a class of related examples; and *generativity*, which regards "the possibility of generating new examples within the space using given examples and their associated construction tools" (p. 301).

Within the discussion on example spaces, special attention has been given to learners' capability of generating new examples in order to enlarge their example spaces and deepen their understanding of the related underlying mathematical structures. Accordingly, it has been argued that *learner generated examples* (LGEs) can be used as a valuable pedagogical tool to promote conceptual learning and understanding (Watson & Mason, 2005; Watson & Shipman, 2008). Zazkis and Leikin (2007) extended this argument, noting that LGEs are a valuable research tool, since the generated examples provide researchers with a lens into learners' cognitive structures.

The Study

The Participants, Course, and Scripting Task

The participants of the study were twenty prospective secondary school teachers who were studying in a teacher-education undergraduate program. At the time of data collection they were in their final term, enrolled in a course titled "Investigations in Mathematics". During the course the participants completed a series of scripting tasks, one of which is described below and serves as the data for our report.

The task that was presented to the participants of the study belongs to the genre of scripting tasks. In such tasks, participants are typically given a beginning of a dialogue between a teacher and students, referred to as a prompt, and are asked to extend the dialogue in a way they find mathematically fit. Scripting tasks were used in prior research in various mathematical contexts (e.g., Zazkis & Kontorovich, 2016; Zazkis & Herbst, 2018), and their advantages were elaborated upon in detail (e.g., Zazkis, 2018). In particular, a significant feature of scripting tasks is that they provide students the educational opportunity to consider or revisit the mathematical ideas related to the task, and offer researchers a lens on the script-writers' understanding of these particular mathematical concepts and relations.

The current study focuses on a particular prompt for a scripting task, presented in Figure 1. In addition to writing a script that extends the dialogue (Part-A), the students were asked to explain their choice of the presented instructional approach (Part-B). Furthermore, the participants were asked to note if their personal understanding of the mathematics involved in the task differed from what they chose to include in the scripted conversation with students (Part-C), providing us with a finer-tuned lens into their personal mathematical ideas. In the task the participants were presented with a table of values, and invited to explore an imaginary student question, whether there are functions other than $y = 3x$ that satisfy the same table of values.

From a mathematical perspective, the task was designed to address known misconceptions regarding the function concept that are attributed in the literature to undergraduate students. In particular, the task attends to the phenomenon of linear functions as "overpowering" prototypical examples (e.g., Dreyfus & Eisenberg, 1983) and the reported lack of understanding of the arbitrary nature of how a function may be defined (e.g., Even, 1990). From a pedagogical perspective, the design of the task draws on underlying principles of Variation Theory (Marton & Booth, 1997; Runesson, 2005), which regards variation as a pivotal role in the learning process, as it promotes and facilitates the learner's capability to discern and separate critical aspects of mathematical objects. Accordingly, effective task design should foreground variation against invariance of other aspects in the task (Watson & Mason, 2006). The current task sets the four points in the table of values as invariant "pillars", whilst promoting variation through the exploration of the range of permissible change in which functions satisfying the table of values exist. As claimed by Watson and Mason (2005), through the awareness of the dimension of possible variation, learners' example spaces may be enriched.

Teacher: Consider the following table of values. What function can this describe?
Alex: $y = 3x$
Teacher: And why do you say so?
Alex: Because you see numbers on the right are 3 times numbers on the left
Jamie: I agree with Alex, but is this the only way?
Teacher: …

x	y
1	3
2	6
3	9
4	12
5	
6	

Figure 1: A prompt for the Table of Values scripting task

The participants' responses to the "Table of Values" scripting task comprise the data corpus for this study. The scripts were analyzed with a focus on the particular examples of functions considered in the dialogues. The following research question guided the analysis: *What are the participants' example spaces for a function that satisfies the task? More specifically, what are the dimensions of possible variation and associated range of permissible change that are evident in the collective example space of the participating prospective teachers?*

Analysis and Results

The analysis is presented by the main themes that were identified in the scripts. Both authors independently categorized the different examples included in the scripts, and subsequently resolved any discrepancies by discussion and reconsideration of the identified themes. The structures of the exhibited example spaces were then examined in terms of their population, connectedness, generality, and generativity. We distinguished between examples used in Part-A, that could have been purposefully restricted in the scripts based on pedagogical and instructional considerations, and the examples mentioned in Part-B or Part-C, which pointed to participants' personal example spaces triggered by the task.

In designing the prompt, Jamie's question "is this the only way?" was intended to direct the script-writers to consider and explore alternative functions. Indeed, 11 out of 20 scripts included a variety of examples of other functions that satisfy the given table of values, which we categorized into five different dimensions of possible variation. Figure 2 indicates the frequency of occurrences of each cluster of examples pointing to a common dimension. Note that the overall number of occurrences (21) is higher than their associated number of scripts (11), as in most of these scripts multiple types of examples were considered. However, 9 out of the 20 participants did not produce any alternative functions, other than representational variations on the linear option $y = 3x$. Due to the limited scope of this paper, in the subsequent sections we focus only the first three dimensions of possible variation in the script-writers' example spaces.

Alternative options to $y = 3x$	11 scripts	
Single formula expressions	5	
Restricting the domain	5	Total:
Graphical representation	5	21
Piecewise functions	4	
Recursive relationship	2	
No production of functions other than $y = 3x$	9 scripts	
Different algebraic representations of $y = 3x$	4	
"Shield"	5	

Figure 2: Dimensions of possible variation in the generated examples

Single formula expressions. Five scripts included single formula expressions to describe functions other than $y = 3x$ that satisfy the given table of values. These included two possible options: the absolute value function $y = |3x|$ (three scripts) and a polynomial function (three scripts). We note that both these function types are continuous functions that are defined for all real numbers. Due to the mathematical challenge involved in generating a polynomial function that satisfies the given table of values, we focus our attention on this option, as illustrated in the excerpt from Logan's script:

Teacher: Well in all of these cases we have assumed something subtle. If we filled the table of values what would we get for the remaining y entries?

Alex: 15 and 18

Teacher: Does it have to be those values? What if I put 16 and 23?

Jamie: … Can you do that?

Teacher: Why not? The points could be modeling anything! There is nothing there that says it has to be a line.

Jamie: Can we find an equation for that though?

Teacher: Certainly, but I need to talk about degrees of freedom. In our table of values we could make up 6 values of y and therefore we have 6 degrees of freedom. Simple enough?

Jamie: Mhmm.

Teacher: So we need to find a polynomial with at least 6 degrees of freedom to describe it, that is a polynomial with at least 6 terms.

Alex: So a 5th order polynomial?

Teacher: Exactly Alex, we could find a polynomial of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ that fits the table of values.

Jamie: But how can we ever assume that any patterns we see in a table of values continues?

Teacher: An excellent question, short answer is we don't. When we make these equations we are assuming that the trend we observe will continue. When making this assumption we need to look for reasons to explain the trend and then ask if we expect those factors to stay the same. Maybe the data was showing the population of a species but at $x = 5$ more food is introduced or a predator is removed and the species can grow at a faster rate.

While general solutions are usually considered in mathematics as more valuable than specific ones, Zazkis and Leikin (2008) noted that often general examples point to an individual's inability to generate a specific one. In this case, the presented example of a polynomial function can be seen as a generality of Logan's personal example space, while it may also point to Logan's difficulty in producing an explicit formula for the polynomial.

While Logan noted the existence of a polynomial function, Corey provided the polynomial $y = x^4 - 10x^3 + 35x^2 - 47x + 24$ "out of the blue", and left it for the imaginary students in the script to verify that it is consistent with the entries in the table of values. In his commentary, Corey added that the polynomial was generated by a computer program, using matrices to solve systems of linear equations. He felt, however, that this material was inappropriate for secondary school students, and in Part-B he wrote: "The level of math needed to determine the final function is beyond what I consider high school level math. After being given the function the answer can be easily revealed, but it still is not easy." We note that Part-C of the task did not demonstrate any alternative higher-level mathematical explanations on how to find fitting polynomials.

Restricting the domain. Five scripts included an example of the function $y = 3x$ in which the domain was restricted to either integers or natural numbers, as demonstrated in the following excerpt from Jill's script:

Teacher: You plotted the points in the table of values, totally correct. Then you connected the dots using a straight line, what is the assumption here?

Alex: Assumption? ……

Teacher: The table of values only gives you the natural numbers, 1, 2, 3, and so on.

Alex: Oh, I guess I assumed that all the points in between follow the same pattern.

Jamie: Well, I guess so too. But now that the teacher mentioned it, maybe the points in between don't have to follow the same pattern?

Alex: I guess so… because they are not in the table of values anyways.

Teacher: That's right! So what other functions can you have?

[Alex and Jamie look at the graph and think.]

Alex: Can we just have those points in the table of values?

Jamie: Like this?

Alex: Yah. It looks a little wired. But it is still a function, right?

Jamie: Right, because it passes the vertical test. It is a function. How do we write the equations then?

[Alex and Jamie feel stuck here.]

Teacher: What is the difference between graph 1 and graph 2?

Jamie: Graph 1 has all the x values, and graph 2 only has natural numbers.

Teacher: Can you describe this difference in more mathematical terms?

Alex: They have different domains?

Teacher: Right, now, can you write the domains for both functions?

Alex: The first one is all real numbers.

Jamie: The second one is all natural numbers.

Teacher: Exactly, when you write the equations, you need to specify domains. By restricting the domains, you have different functions.

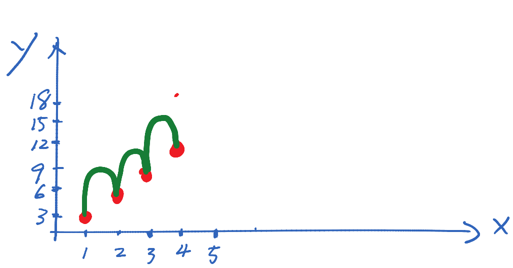
As opposed to the previous section, the function examples here are neither continuous nor defined for all real numbers, yet the domain consists of an infinite and unbounded set of numbers. Moreover, these examples demonstrate a recognized human tendency of "continuing the pattern" (e.g., Rivera, 2013), that is, assigning the same rule of multiplication by 3 to all integers. In this sense, the assignment of the same rule to a restricted domain demonstrates the arbitrary choice of the domain in the function concept, though not the arbitrary choice of correspondence between the domain and codomain. In terms of the features of example spaces, on the one hand we note the connectedness between the examples, highlighted through the different attributes of non-identical domains. On the other hand, we notice a "missed opportunity" for generativity, as these examples do not lead to additional generated examples in the scripts that allude to the various options for choosing the domain.

Graphical representations. While in the above excerpt from Jill's script, the teacher confronts students' tendency to connect the points, in other scripts "connecting the points" appears to be the convention that is either supported or invited by the teacher. Taylor exemplifies this tendency:

Teacher: Excellent question Jamie, what's your instinct, are there other ways?
Jamie: Well I don't know, I guess there could be, but how could we tell?
Teacher: Why don't we start by plotting these points. And by we I mean you.
[Students plot the points]
Teacher: Good, so how would it look if we used Alex's function?
Jamie: It would have a straight line through all the points.
Teacher: Yes, but how else can we connect these points?
Jamie: I suppose we could do a zig zag line.
Teacher: Sure, that would work. But we want this to be a function, so what rule do we need to follow?
Jamie: The vertical line test.
Teacher: Which is the easy way of remembering what?
Jamie: Each output can only have 1 input.
Teacher: Correct, so how can we connect these points then?
Jamie: Any way we want as long as we don't break the vertical line test.

In this excerpt, the teacher's question "how else can we connect these points?" leads students to explore alternative options to the straight line. All other scripts that used graphical representation as dimension of variation also alluded to the arbitrary choice of how to "fill the gap" in between the points, presented both via verbal explanations and graphical illustrations, including also non-continuous "step functions" (see Figure 3 taken from one of the scripts). All examples in this dimension explicitly or implicitly regarded the domain as the set of all real numbers. As in Taylor's script above, in the other scripts the determining factor for how to "connect" the points was the vertical line test, serving as the identifying criterion for a function.

While in this dimension, connecting the points extends the population feature of the example space, various ways of connecting the points "anyway we want" (in student words) indicate the generativity, as well as the generality, of the resulting example spaces. However, and in line with previous arguments, this generality may be accompanied by the participants' inability to produce specific algebraic representations for the graphically represented examples.



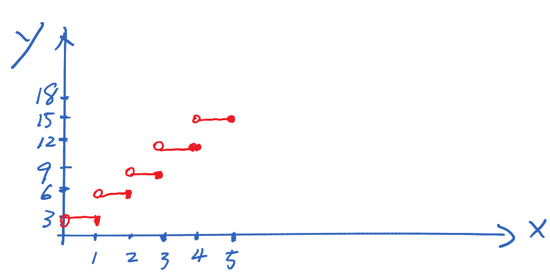


Figure 3: Graphically represented functions

Discussion

The scripts in response to the "Table of Values" task provided a lens into the participants' personal example spaces of functions. Whereas in almost half of the scripts the example space was limited (i.e., no production of functions other than $y = 3x$), the other scripts demonstrated example spaces that were well connected. Within these, the population feature of the participants' example spaces was not extensive; however, generality and generativity were featured in scripts that included multiple examples.

More specifically, the analysis led to two kinds of observations in regard to the participants' example space and concept image of a function (see Vinner, 1983). First, the participants' example spaces provided further support to features that have been previously discussed in the education literature. The students' examples clearly demonstrated the conception that a function should be represented by a single formula (e.g., Vinner & Dreyfus, 1989), typically describing a continuous function (e.g., Hitt, 1998) in which "other points follow the same pattern". Moreover, students' reliance on the "vertical line test" (see Wilson, 1994) was clearly present in the scripts as an identifying criterion for a function.

Secondly, the participants' example choices point to a specific identifying feature of undergraduate students' example spaces of functions, which was not elaborated upon in prior research: that the domain of a function is infinite and unbounded. Focusing on the domain, Bubp (2016) noted that in an attempt to prove mathematical statements, students often used "implicit, unwarranted assumption that the domain of the function f was $\mathbb{R}$" (p. 592) and that "a function cannot have a restricted domain" (p. 593). The current findings provide further refinement of this issue, by noting that even in the examples in which the domain was in fact restricted, it still included infinitely many points (integers or natural numbers). We note that no example of a finite domain or a function on a bounded interval was given by any of the students.

Viewing the findings in a broader context, we suggest that the analysis of scripting tasks not only can provide a theoretical contribution for research, but also a practical utility for undergraduate instruction. In the current case, we also used the analysis of the scripts to plan for follow-up activities that were based on the collective example space of the group, with the goal of extending the participants' personal example spaces, and in such extending their understanding of the concept of a function. These activities are elaborated in detail in Zazkis and Marmur (2018). However, as an illustrative example, one of these activities focused on generating an explicit formula for a non-linear polynomial function consistent with the given table of values. During this activity, we provided one of the examples from the students' scripts, a polynomial of degree 3, for classroom discussion. This led to recalling the Fundamental Theorem of Algebra, and to the subsequent realization that the example was not feasible, as there cannot be a cubic function that intersects a line in 4 different points. This discussion, which made an explicit connection between undergraduate and secondary school mathematics, highlighted the "borders" of the relevant example space, or in Watson and Mason's (2005) terms, the range of permissible change. To conclude, the seemingly simple task of considering a given pattern in a table of values – an exercise that often appears in middle school mathematics lessons – served to advance mathematical understanding of undergraduate students. This by utilizing the collective example space found in the scripts as a springboard for describing the structure of this space, examining what kind of functions belong to the space, determining its confining borders, and enriching the examples of functions that exist within it.

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What is a Differential? Ask Seven Mathematicians, Get Seven Different Answers

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The symbol "dx" is one example of a differential, which is a calculus symbol that is found in a variety of settings and expressions. We wanted to explore how expert mathematicians think about differentials in some of these settings and expressions, in order to see what levels of consistency might appear among their views. To that end, we created an interview protocol that contained differentials in the contexts of derivatives, definite and indefinite integrals, and separable differential equations, interviewed seven mathematicians, and analyzed their responses using a form of thematic analysis. Overall, we found no instances of total agreement among all subjects, but did find several common and recurring themes, including some that were unexpected and not found in our previous studies.

Keywords: Differentials, Calculus, Concept Image, Derivatives, Integrals

In this contributed report, we analyze how seven mathematicians view the roles, if any, that differentials play within various mathematical expressions and situations. When discussing the term "differential," we refer to a letter d followed by a second letter that is usually dependent on a particular context. Examples of these include dx, dt, and dA, and for this paper, we will use "dx" to reference a generic differential. These symbols are common in calculus, and can be found in many places, including Leibniz notation for derivatives, definite and indefinite integrals, the process of integration by substitution, and several types of differential equations.

We have found research in both mathematics and physics education literature that describes how students perceive the dx in a definite integral. For some students, this differential might have no meaning at all (Artigue, 1991; Hu & Rebello, 2013). If it does have a meaning, it might only serve to indicate the variable of integration (Artigue, 1991; Jones, 2015), or it could represent a small amount of a quantity (Artigue, 1991; Nguyen & Rebello, 2011; Von Korff & Rebello, 2012) or a small change in a quantity (Sealey & Thompson, 2016; Von Korff & Rebello, 2012). Outside of these particular student interpretations, a differential might function as a linear estimate (Henry, 2010; López-Gay, Martinez, & Martinez, 2015) or represent a formally-defined infinitesimal as found in nonstandard analysis (Keisler, 2012; Robinson, 1961).

Most of this particular literature discusses student interpretations of the definite integral, but only minimally addresses the interpretations of the instructors and expert mathematicians who teach these students. We have felt that there is an opportunity to broaden the above research by expanding the list of expressions containing differentials as well as exploring the interpretations of experienced mathematicians. Therefore, the main research question we address in this paper is "What concept image(s) (Tall & Vinner, 1981) do expert mathematicians hold of the differential throughout its various mathematical contexts?" Two other areas we wish to explore are analyzing each expert's interviews to see how consistent his or her responses are throughout the interview, and looking at each context in which a differential exists (e.g. indefinite integrals) and comparing each expert's views on the differentials in that context, to see what patterns or consistencies, if any, might emerge.

Preliminary work was conducted via two smaller-scale studies. An initial study involved four mathematicians who were asked about how they conceived the differentials in expressions involving integration, Leibniz derivative notation, integration by substitution, and ordinary differential equations. We concluded that, while some subjects gave common responses at times, there was no overarching formal concept definition for the differential (McCarty & Sealey, 2017). A second study included two mathematicians and one physicist who were interviewed about similar expressions and contexts, and found not only a similar lack of an overall formal concept definition for differentials, but also the suggestions of a split between mathematicians' views and physicists' views. (McCarty & Sealey, 2018). In this current paper, we focus only on mathematician interviews and leave physicist interviews for future research.

Theoretical Perspective

Discussing the notations $\frac{dy}{dx}$ and $\int f(x)\, dx$, Tall (1993) questions what relationship might exist between the two "dx" portions of those notations and notes:

> Giving a modern meaning to these terms that allows a consistent meaningful interpretation for all contexts in the calculus is possible but not universally recognized. On the other hand, failing to give a satisfactory coherent meaning leads to cognitive conflict which is usually resolved by keeping the various meanings of the differential in separate compartments. (Tall, 1993, p. 6)

Thus, one might use only one conceptualization for all differentials at all times, or one might possess and use different conceptualizations for differentials depending upon the context in which they are found (for example, viewing the dx in an indefinite integral as indicating the variable of integration and the dx in the derivative notation $\frac{dy}{dx}$ as a small amount of the quantity represented by the independent variable, x.)

Because multiple interpretations of differentials are possible, we believe that Tall and Vinner's (1981) *concept image* is an appropriate theoretical perspective for our research. Concept image is defined as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p.152), and if one has multiple interpretations of differentials, then the words "total" and "all" in that quote take on greater meaning. During our interviews, we attempted to gain as complete an understanding of our subjects' concept images as possible, with the following questions in mind: Within these possible multiple interpretations, would any subjects exhibit *potential conflict factors*, defined as aspects of their concept image that showed contradiction? If so, would they be aware of any of their contradictions, making them *cognitive conflict factors*? Would all subjects' responses be able to be distilled into a *personal concept definition* that fully defined how they viewed differentials, and if so, would multiple personal concept definitions be able to come together to form a possible *formal concept definition*?

Methods

For this study, seven mathematicians (pseudonyms André, Bryan, Christopher, Diane, Eugene, Francis, and Gustav) from the same large research university were given semi-structured interviews that used the interview protocol summarized in Table 1. Each subject was asked the same questions about the expressions and contexts given in the protocol, but follow-up questions were asked when needed to clarify subjects' initial responses. Including these additional questions and introductory questions that asked the subjects' background information, the average length of the interviews was approximately forty-five minutes. All interviews were videorecorded, with six interviews conducted in person, and a seventh conducted over Skype and recorded with Open Broadcasting Software.

Data analysis was done in the style of Braun and Clarke's (2006) thematic analysis. The videotaped interviews were transcribed and analyzed for data points, which we defined to be the specific instances in which differentials were discussed. These data points were assigned codes based on how we perceived the tenor of the subjects' views toward the differentials. The lists of codes from all seven interviews were analyzed, and similar codes found across multiple interviews were pulled together, to create an initial list of themes. The themes in this initial list were compared with one another to see which of them might be consolidated and streamlined into a smaller list of larger, overarching themes. Finally, the transcriptions were read one last time and compared with this final list of themes, to make sure that the themes described by this list encompassed all responses within the entire data set.

Table 1
A Summary of Our Interview Protocol

Description	The Specific Questions
Five Expressions Presented with no Context	• $\frac{dy}{dx}$, $\int_a^b f(x)\,dx$, $\int g(x)\,dx$, $\int_0^1 \int_2^3 f(x,y)\,dy\,dx$, and $dy = 2x\,dx$ • For each of these, subjects were asked how they conceptualized the differentials in the expressions, and whether they thought the differentials had (a) a graphical representation, and (b) a size.
Three Expressions Presented within a Context	• A "Law of Cooling" ODE: $\frac{d\tau}{dt} = -k\tau,\ \ \tau(0) = 20$ • A "Work" problem involving the integral $\int_0^{50} 700 - 3x\,dx$ • $du = \frac{1}{2\sqrt{t}}\,dt$, used in the evaluation of the integral $\int_1^4 \frac{\cos\sqrt{t}}{2\sqrt{t}}\,dt$
Three Additional Questions	• At the beginning of the interview, subjects were asked what the word "differential" meant to them. • After the word "Delta" was first mentioned by the subject, he or she was asked to clarify the differences, if any, between Δx and dx. • After their first use of a phrase like "infinitely/infinitesimally small," subjects were asked if they could clarify/quantify their phrase.

Data and Results

We found many themes during data analysis, some of which were expected from our prior research and our analysis of recent literature, some that were new to us, and some that were stronger than expected. We summarize the major themes below.

Algebra with Differentials versus "Algebra" with Differentials

The use of the quotation marks in this subtitle is to represent the idea that some experts were not willing to describe certain common manipulations of differentials by directly using words like "multiply," "divide," and/or "cancel." To give one example, when some subjects brought up "Chain Rule" notation, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, Bryan, Christopher, and Diane each had no problem with notating it this way, but stopped short at saying that what was happening was true division or cancelling of the dx. Christopher said that it was "as if" we cancelled the dx, Bryan said that "there's a little bit more going on than just cancelling," and Diane said that she wasn't sure if they cancel, and that books "come up with some funny, hand-wavy thing to explain what they're doing there."

Another example of "algebra" with differentials occurred during the discussion of the separable ODE $\frac{d\tau}{dt} = -k\tau$. Subjects who claimed either that the expression $\frac{d\tau}{dt}$ was not a ratio (Eugene, Francis) or that they weren't sure if it was a ratio (Diane) still ended up separating the expression when solving the ODE. This separation was rationalized by either claiming that this separation stood in for the integration $\int \frac{d\tau}{dt}\, dt = \int -kt\, dt$ (Eugene, Francis), or that we just "think of" dt as being a quantity and act like we're "multiplying" (Diane). It is perhaps worth noting that, even though some subjects refused to say personally that separation of variables entailed "multiplying by dt," none of the subjects would outright object if their students described their solution to a separation of variables problem this way. Five of the seven subjects said they would have no problem if their students used the words "multiply by dt," while the other two (Diane and Eugene) were not certain if they would allow their students to do this.

There were clearer statements of actual algebra made as well. Some subjects stated directly that one could manipulate differentials by multiplying or dividing, and there were statements that implied multiplication and division were acceptable, including André's and Christopher's separation of variables in the ODE without any qualms as to the legality of such multiplication. There were also contrary, clear statements that one could not multiply nor divide, and some of these "Yes, you can" and "No, you can't" statements were in direct opposition to one another. One example of this was Bryan and Christopher saying that the "$f(x)dx$" in $\int_a^b f(x)\, dx$ was an actual multiplication of $f(x)$ and dx and Diane saying that it was not a multiplication.

Subjects' Uneasiness with Differentials

Given the lack of consensus found in all of our studies and the lack of a clear formal concept definition for differentials, it was not surprising that some subjects admitted a level of uncertainty to some of their responses. This uncertainty manifested itself in various ways: some subjects claimed that they had no formal definition for some of our expressions, some claimed that they had an intuition about the expressions but could not put this intuition into words, and some gave a partial explanation while admitting that they knew there was "more" to the concept but that they could not put this "more" into words.

There were definite instances of cognitive conflict factors. To give one example, Francis noted and called attention to his conflicting statements when they occurred. After claiming the differentials in the earlier expressions $\frac{dy}{dx}$, $\int_a^b f(x)\, dx$, $\int g(x)\, dx$, and $\int_0^1 \int_2^3 f(x,y)\, dy\, dx$ had no size, he gave what we call the standard "linear approximation" explanation of $dy = 2x\, dx$, stating that these dy and dx were measurable quantities. He noted the inconsistency, saying "… now I'm being cognizant of what I think about this, and what I originally said, no. That these [pointing at the dy and dx] are not quantifiable. [Thinking] And I'd have to really think about rectifying this."

The dx is a Real Number or a Formal Infinitesimal

Our previous work as well as the recent literature shows that an interpretation of a dx as an unquantified, not formally-defined "small" amount is common and not unexpected; what was slightly unexpected in our research was the emergence of themes in which subjects specifically stated that the dx represented a real number or a formal infinitesimal. Francis mentioned one area in which textbooks commonly assert that dx and dy are real, the idea of linear approximation, usually represented as $\Delta y \approx dy = f'(x)dx$. André and Bryan described some dx as being on a smaller scale than every other entity in the problem, a description I liken to Courant and John's (1965) "physically infinitesimal." For example, Bryan defined his dx as "relatively small," and gave examples of dx possibly equaling 100,000 miles if one is discussing astronomical phenomena, but dx equaling one Ångström if one is discussing molecules. Either way, no matter at what scale one is measuring a specific problem, for these two subjects, the dx represents a real number. For the purposes of this report, we can define a positive, nonstandard analysis infinitesimal, ϵ, as $0 < \epsilon < r$ where r is any real number (Keisler, 2012). Gustav directly stated that one could view any dx as one of these formal infinitesimals, and while Eugene and Francis did not view differentials in this way, they acknowledged that others might, and that formal infinitesimals were a valid interpretation of differentials.

The dx is not Specifically Sized

This is a common theme, found both in the literature and in our previous work, though our current research has found more nuance to this theme than we reported previously (McCarty & Sealey, 2017). A differential might be described or implied to be "small" without a precise definition of what "small" means (as opposed to defining differentials as real numbers or formal infinitesimals, both concepts with precise definitions.) This occurred at the beginning of every interview, when the subjects were asked what the word "differential" meant to them, and all replies contained a reference to "smallness" that was not explicitly defined. Other versions of this were Diane describing differentials as "infinitely small" while claiming that "infinitely small" could not be defined, and Eugene claiming that the dx was a small entity that was the result of the limit of Δx going to zero.

We include in this theme comments that did not state directly but seemed to imply that the dx might be a real number or formal infinitesimal. Eugene described the dx in a definite integral as being a stage in the limit process. In this case, if Δx is going to zero step-by-step, and the dx represents one of those steps, must not dx be a real number? Other subjects made statements that might be interpreted as referencing nonstandard infinitesimals. André described the dx as being "what's left of Δx after it goes to zero," and Diane said that when the two points that define a secant line "are on top of each other", then we can think of the Δx as a dx. One might interpret both of these ideas in a nonstandard manner: in each case, the subject describes a process that goes through all real numbers and results in a distance of zero, yet the dx still exists. This might be possible if one views these dx as the epsilon described above: an entity that still exists yet is outside of the reals.

The dx Indicates a Variable or Process

It is also possible that a differential might not have a size because it indicates a variable or references a process. Differentials might only be used to call attention to a particular variable, as in the dx serving as an indicator of the variable of integration in an indefinite integral or the dy and dx indicating the "directions" of integration in the double integral. Differentials might also serve to indicate a process, with some subjects saying that the dx in a definite integral only represented that the limit of a Riemann sum was taken, and that a "u-substitution" made in the evaluation of an integral was a representation of the Chain Rule.

A small sample of the themes we found in the discussions of some of our questions can be found in Table 2. A quick look at this table and the number of themes found in it can determine our answers to the questions posed earlier in this paper: there is no formal concept image for the differential across all contexts, and only some areas of consistency within one expert or within one expression. Many experts stated that one's views on differentials also depends on the context in which the differentials were presented, and thus it is even possible to discuss inconsistencies at a level more fine than the level implied by this table.

Table 2
A Summary of Some of Our Results (Only the Expressions Presented without Context)

Expression	André	Bryan	Chris	Diane	Eugene	Francis	Gustav
$\boldsymbol{\frac{dy}{dx}}$	A, C, R	C, R, V	IR, P	P, S	V	II, P, V	I, V
$\boldsymbol{\int_a^b f(x)\,dx}$	II, P	A, R, P	P, S	P, S, V	C, S, V	P	C, I, V
$\boldsymbol{\int g(x)\,dx}$	V	N	A, S	V	V	N	V
$\boldsymbol{\int_0^1 \int_2^3 f(x,y)\,dy\,dx}$	C, II, P, V	C, P, R	S	P	P, S, V	C, P, S, V	V
$\boldsymbol{dy = 2x\,dx}$	P	C, U, R	A, IR, S	"A", U	P, U	R, U	A, R, S

The letters in the table correspond to the presence of the themes described above:
A: Algebra with Differentials, "A": "Algebra" with Differentials, C: Differential Interpretation Depends on Context, I: Differential is a Formal Infinitesimal, II: Differential is an Implied Infinitesimal, IR: Differential is an Implied Real Number, N: Differential Has No Meaning, P: Differential Represents a Process, R: Differential is a Real Number, S: Differential is "Small" (Not Specifically Sized), U: Subject Expressed Uneasiness about Differentials, V: Differential Indicates a Variable

Discussion

Given the number of themes we found in our research and the number of different opinions within each theme, it should not be surprising that we conclude there is no formal concept definition of the differential. To be more direct, we found no instances where all seven subjects agreed on the interpretation of any one differential in any one mathematical context. It appears that the second half of Tall's (1983) quote applies, and that the lack of one overarching meaning for dx means that our subjects' concept images of dx consist of many different meanings for the differential compartmentalized in separate "locations."

This leads to some implications for instruction and suggestions for future research. One might ask if it matters that individuals possess such disparate views of the differential. After all, these seven subjects are accomplished mathematicians and experienced instructors; the fact that each of them views differentials in their own way did not prevent them from earning their doctorates. However, one might counter that argument with the idea that many, if not most, notations in mathematics are not ambiguous at all. For example, we would submit that a study that asked subjects their interpretations of the notations "Σ", "$\sqrt[3]{\ }$", and "!" would show no ambiguity in subject responses. If many notations have only one clear, direct, single interpretation, one might argue that dx should have one clear, direct, single interpretation as well. Indeed, a few subjects in our study expressed personal discomfort when noting that sometimes differentials are taught in a "hand-wavy" way, without real support (Diane), or that instructors sometimes teach differentials less formally than they should (Francis). We suggest that the reason for this discomfort is the fact that there is no consensus on what a differential is. Perhaps further research could investigate how (or if) instructors having disparate views of the differential affects student learning.

Another teaching implication might come from the first past of Tall's (1983) quote: "Giving a modern meaning to these terms that allows a consistent meaningful interpretation for all contexts in the calculus is possible but not universally recognized. (p.6)" It is possible that the differential as a nonstandard analysis infinitesimal would be the most consistent approach. There are certainly textbooks that teach calculus this way (e.g. Henle & Kleinberg, 2003; Keisler, 2012), but, as Tall stated, such an approach is not universally recognized. Further research might explore the efficacy of such an approach. An idea for future research comes from the notion, mentioned above, that some subjects claimed that there were "Physics" and "Mathematics" approaches to differentials. This idea was touched upon in our pilot study (McCarty & Sealey, 2018) but not in this study. Further research might wish to explore how physicists view differentials and how consistent their views are with mathematicians', especially since many first-year physics majors take calculus classes that are taught by mathematicians.

We conclude this paper by quoting what Christopher said at the end of his interview, regarding the usefulness of differentials:

> Yeah, they're very useful, 'cause they have a lot of content. There's a lot of, sort of conceptual content in there, and if you shy away from them, you're robbing the students of sort of conceptual content where they can think about things – these things actually mean something, rather than being things that are so abstruse that they can only be handled with a course in advanced calculus. I think a lot of that – all that developed just from physical reasoning and – although the mathematics by itself is not rigorous, you can make it rigorous, and the reasoning is valid. So I don't see any reason to avoid talking about them

At this time, we are in no position to say with certainty that one view of differentials is superior to any other. If there were any conclusion we might make, it is that we are in agreement with our interview subjects who are not comfortable with textbooks or teaching methods that either ignore differentials entirely or give them short shrift. We agree with Christopher that differentials are useful and worthy of classroom discussion, and it is our hope that our research inspires and motivates further work that will help explore the utility of differentials.

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Abstract Algebra Students' Function-Related Understanding and Activity

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Functions play a fundamental role both in abstract algebra and earlier courses in the mathematics curriculum. Yet little attention has been paid to how students' understanding of function (informed by their prior experiences) supports or constrains their activity when dealing with functions in abstract algebra. In this study, we report on six abstract algebra students' understanding of function, their function-activity in abstract algebra tasks, and the degree to which their understanding of function from prior experiences is connected to their understanding in this new setting. We conclude with two cases contrasting the activity of two students with divergent levels of connection between their function understanding and the abstract algebra setting. In general, we found that function served an important role in students' activity and provides implications for instruction and future research.

Keywords: Abstract Algebra, Functions, Student Understanding

Functions are one of the core topics threaded throughout the mathematics curriculum. In abstract algebra, students encounter a number of important classes of functions including isomorphisms and homomorphisms. The treatment of functions in this setting is often more formal and abstract; however, students' extensive exposure to functions in prior courses likely plays a role as they grapple with new function contexts and definitions. The degree to which this occurs is particularly pertinent due to the extensive documentation of complexities involved with understanding functions at the secondary level (Oehrtman, Carlson, & Thompson, 2008). Understanding functions involves integrating function-properties (e.g., Slavit, 1997), flexibly understanding multiple representations (e.g., Schwarz, Dreyfus, & Bruckheimer, 1990), leveraging appropriate metaphors (e.g., Zandieh, Ellis, and Rasmussen, 2017), and moving beyond action conceptions to process and object conceptions (e.g., Breidenbach, Dubinsky, Hawks, and Nichols, 1992). In parallel, the abstract algebra literature illustrates that students often struggle with aspects of specific functions such as isomorphisms (e.g., Leron, Hazzan, & Zazkis, 1995), binary operation (Melhuish & Hicks, 2018), and homomorphisms (e.g., Rupnow, 2017). With these results in mind, we developed a survey and interview study to address:

1. What are students concept images of functions at the end of an abstract algebra course?
2. How do they see functions from prior courses as connected to functions in abstract algebra?
3. How does their understanding of functions play out in their abstract algebra activity?

Literature Review

The complexities involved in understanding function have been well documented. Students have been found to possess several alternate or incomplete conceptions of function that can persist even throughout the secondary and undergraduate level (Oehrtman et al., 2008). For example, students may interpret functions as necessarily having an explicit symbolic rule (e.g., Vinner & Dreyfus, 1989; Thompson, 1994). Students may also struggle with definitional properties such as delineating between the requirement for a well-defined function and that of a function being injective (Dubinsky & Wilson, 2013). Further, their conceptions of functions may reflect different degrees of sophistication such as in Breidenbach, Dubinsky, Hawks, and Nichols' (1992) documentation of students conceiving of functions as actions, process, or objects.

Understanding of function has been treated through different lenses including the aforementioned action, process, and object hierarchy. Slavit (1997) posited an alternate route of function understanding relying on important properties of functions and distinguishing between functions possessing or lacking properties. Another marker of understanding of function is proficiency with multiple representations of functions. Numerous researchers have documented students' preferences for a particular representation even when alternate representations would be supportive (e.g., Knuth, 2000), students' lack of flexibility moving across representations (e.g., Akkoç, & Tall, 2002), and even students seeing alternate representations as unique functions (Elia, Panaoura, Eracleous, & Gagatsis, 2007). As is the case with representations, students may also leverage multiple function metaphors while reasoning about functions. Such metaphors may reflect the input-output machine (Tall, McGowen, & DeMarois, 2000) or directionality between sets (e.g., Lakoff & Núñez, 2000). Zandieh et al. (2017) identified five clusters of metaphorical expressions with which students engaged in linear algebra: input/output, traveling, morphing, mapping, and machine. Properties, representations, and metaphors provide additional components to be situated in a students' larger concept image of function.

The concept of function then plays a vital role in more advanced courses such as abstract algebra. While little research has treated function explicitly at this level, existing literature in abstract algebra suggests that students struggle to develop rich conceptions of abstract algebra concepts that rely on functions (Dubinsky, Dautermann, Leron & Zazkis, 1994; Hazzan, 1999). Students in abstract algebra tend to struggle with particular kinds of functions such as isomorphisms and binary operation. For example, Leron et al. (1995) found that students struggled with constructing specific isomorphisms and formulating definitions about isomorphisms. Rupnow (2017) shared cases where students struggled with homomorphism when they did not have metaphor flexibility. Melhuish and Hicks (2018) documented that students may bring some of the same function representational limitations to the context of binary operations. In sum, the results from prior research suggest that explicitly studying student conceptions of function may provide insight into their abstract algebra activity.

Theoretical Orientation and Analytic Framework

In this paper, we rely on two key constructs to make sense of students' understanding: Tall and Vinner's (1981) *concept image* and Zandieh et al's (2016) *unified notion of function.* A student's understanding of function involves not only the words used to specify the concept (personal concept definition), but also all of the surrounding cognitive structures (concept image). These various components may or may not be coherent and they may or may not align with mathematics' communities accepted definition for a given concept. In terms of functions, a number of components have been associated with concept images including metaphors (e.g., Zandieh, et al., 2016), representations (e.g., Hitt, 1998), properties (e.g., Tall & Vinner, 1981), and evoked examples (reflecting a students' *personal example space*, Sinclair, Watson, & Mason, 2011). Due to space limitations, we share the specific categories from our analytic framework in Table 1.

Our primary research goal was to address each of these components of abstract algebra students' concept image of functions at the completion of an introductory course. Further, we use Zandieh et al's *unified notion of function* to address the degree to which students understood "various constructs [of functions] as examples of the same phenomenon" (p. 24). That is, did students see functions presented in abstract algebra as instances of their larger function concept? We address this question through analysis of both students' self-reported understanding and their activity as they engaged with relevant tasks. We conjectured the connectedness of their function understanding would play out through explicit questions about functions in abstract algebra, explicit reference to functions in their abstract algebra activity, and components of their function concept image implicitly playing out in their activity.

Methods

Data Collection

Surveys were given to four undergraduate-level modern algebra classes at two public universities. The survey was composed of one part concerning functions in general and another part concerning homomorphisms and kernels in group theory. In the first part of the survey, students were prompted to provide formal and informal definitions of function, examples of functions, and representations for functions. In the second part, students provided formal and informal definitions of group homomorphism, and kernel. They also were given a series of tasks where they needed to leverage the definition of homomorphism or kernel to address prompts in particular contexts (such as determining if a given map is a homomorphism.) In addition to the surveys, we conducted six semi-structured follow-up interviews (three at each university) with the goal of obtaining a more robust interpretation of the participants' survey responses. The interviews included additional tasks that the students were asked to complete including addressing homomorphisms in the context of Cayley Tables and function diagrams, and producing formal proofs of standard homomorphism and isomorphism prompts. Two such prompts include determining if the function diagram in Figure 1 could potentially be of a homomorphism, and identifying the kernel for the homomorphism in the following map from $\mathbb{Z}$ to $\{i,-i,1,1\}$:

$$\Phi(n)=i^{n}.$$

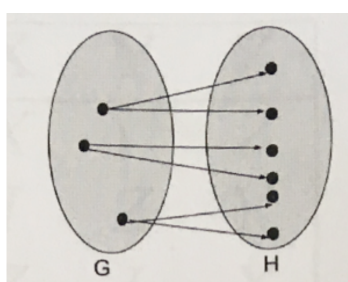

Figure 1. A Diagram Representing a Non-Function

At the end of the interview, the participants were prompted to reflect on functions in their abstract algebra class by identify if and what functions were in the subject. They were then asked to reflect on whether "functions in modern algebra the same as functions from high school?"

Analysis

To analyze the transcripts of the interviews, each of the four authors independently open coded (Strauss & Corbin, 1990) the transcripts looking across all prompts. Through this process, a coding framework was developed to target specific aspects of student thinking that were deemed pertinent to the research questions. In particular, this framework included the properties that students attended to, the metaphors (adapted from Zandieh et al. 2017) and representations (adapted from Melhuish 2015) and Mesa 2004) utilized, the students' evoked example space for functions and non-functions, and the similarities and differences that the students noted between functions in abstract algebra and functions in lower level courses. The four authors then independently coded the transcripts (for all items related specifically to functions as a general construct) using the framework. From these coded transcripts, profiles for of the six cases were compiled leveraging the five targeted categories.

We then returned to the transcripts to further unpack the activity on the second set of prompts: prompts where students engaged in representations and proofs related to homomorphism and isomorphism. These transcript portions were analyzed with the intent of exploring whether a student's conceptions of functions aligned with their abstract algebra activity (implicitly or explicitly), and the degree to which their function conceptions appeared to support or constrain their abstract algebra activity.

Results

In Table 1, we share the variety of ways functions were addressed by our participants in terms of their definitional properties, metaphors, representations, and evoked examples. These components stem from analysis of the general function prompts.

	Properties	Metaphors	Representations	Examples	Functions in AA
Student A	WD	T, Mp	S, G, V, E	F, AA	Different Domains, New Properties
Student B	WD, ED	Mp	S, G, V	F, AA	Different Reps, Expansion
Student C	WD, ED	IO, T, Mp	S, G, D	F, AA	Same, Restructured
Student D	WD, 1-1	IO, Mh, Mc, Mp	S, G	F	Different Reps, More Complex
Student E	NA	T, Mp, Mc	S, V	F	Expansion
Student F	WD	T, Mp	S, G, V	F	Expansion

Properties: WD: well-defined, ED: everywhere-defined, RB: relation-based, EB: equation/rule-based;
Metaphors (adapted from Zandieh et al., 2016): T: traveling, IO: input/output, Mp: mapping, Mc: Machine, Mh: Morphing
Representations (adapted from Mesa, 2004; Melhuish, 2015): S: Symbolic Rule, G: Graphical, V:Verbal, E:element-wise defined, D:Diagram
Examples: F: familiar secondary level algebra functions, AA: abstract algebra context

Table 1. Evoked Components of Abstract Algebra Students' Concept Image of Function

Across our six cases, we note significant differences in the students' evoked concept images. In terms of definitional properties, five of the six students articulated some understanding of well-definedness. However, in one case, the student had not delineated well-defined from a map being one-to-one. In a second case, the student relied on a function as a rule with no additional required properties. In terms of metaphors, mapping was leveraged by all of our students. Various individuals leveraged it to greater or lesser success depending on a number of other factors. In terms of representations, all students had a dominant image with explicit symbolic rules. This is not surprising in light of both the literature and the common usage of such representations (as in Melhuish's (2015) curriculum analysis.) Student D, E, and F particularly leaned on symbolic rules. In terms of examples, we saw a similar trend where Student D, E, and F shared examples of typical (explicit, symbolic rule) functions from earlier settings such as $f(x)=x^2$. In contrast, Student A, B, and C all provided examples that were particular to abstract algebra such as functions whose domain was dihedral group.

We then analyzed how students were seeing functions in abstract algebra as the same or different from high school. In general, the students reported that functions in abstract algebra expanded ideas from functions including new qualities such as properties or representation types. However, we note that even though students made these statements, half of the students did not provide examples in an abstract algebra context specifically even with explicit prompting (after having engaged with abstract algebra function prompts) leading us to question the depth of this declared unified conception. To further instantiate the trends in our data, we share two contrasting cases: Student D and Student C.

Case 1: Student D

Throughout the interview, Student D used typical functions from the secondary algebra and calculus settings when prompted to provide examples of functions and struggled to provide specific examples of functions in an abstract algebra context. Moreover, when asked if she viewed functions in abstract algebra as the same as functions in previous courses, she stated that they are "completely different" and explained:

> That's what threw me off from the very beginning, was the functions weren't the same. It was a totally different way of thinking. I mean, you're not thinking in terms of ... I'm thinking in groups.

She explained that she had previously relied on graphical representations of functions to aid her understanding and that the lack of graphs to represent functions in abstract algebra presented challenges for her understanding. Student D suggested that functions in abstract algebra were too large to draw pictures, as they could involve sets such as the set of integers. Overall, throughout the interview Student D's responses did not suggest that she connected her understanding of functions in previous courses to this context.

In the second half of the interview regarding the concepts of homomorphisms and kernels, Student D's disconnect between functions in prior settings and functions in abstract algebra is made particularly clear. When presented with Figure 1 and asked if the function diagram could represent a homomorphism, Student D responded that "I would say [...] is a homomorphism because all of the elements in G get mapped to a particular H value," Thus, Student D is attending to the need for every element in the group G to be mapped to some element in H.

However, when explicitly asked if the diagram represented a function, Student D correctly identified that this diagram fails to meet the requirements of a function: "two x values with different y values in it wouldn't be able to be defined as a function." Meanwhile this does not perturb her previous classification of this diagram as a possible homomorphism. We interpret this as further evidence of Student D's disassociation of the concepts of function and homomorphism.

This disassociation continues to play out in the portion of the interview regarding kernels. While Student D's definition of a kernel of a group homomorphism, "the set of elements in group G that mapped to the group H, to the identity element", is largely correct; she continued:, "One group, one set of elements is going to map to another set of elements, but, in a sense, the reversal map from that final group to the initial group is what the kernel is, so it maps." Thus, we see that Student D does not necessarily see the kernel of a group homomorphism as a pre-image, but rather the image of an inverse function.

Her response to the Kernel Task provides further evidence that she may be working with an action conception of the homomorphism. When asked for the kernel she explained, "... I wrote that the kernel was zero[1], because you would get one, which was your identity element in H." Student D's kernel candidates focused on identifying a single element of the set of integers which maps to the identity in H. She was testing individual values in the function, but not considering the full preimage of the identity. Such focus on individual pairs of input/outputs likely reflects an action conception of this mapping. If a student lacks a process understanding of function, they may be limited to proceduralized ways of dealing with inverse and preimage (Oehrtman, et al., 2008). When explicitly asked if kernels can have multiple elements, Student D agreed. When further probed about this particular map, she identified one more element, but remained focused on individual inputs and outputs.

Student D presents a case of a student who did not appear to have robust connections between her prior function knowledge and the abstract algebra setting. Further, we may reasonably conjecture that her limited conceptions of function implicitly constrained her ability to work with the kernel concept, a concept that necessitates ability to deal robustly with preimage.

Case 2: Student C

In contrast to Student D, Student C flexibly leveraged function metaphors, attended to important properties of functions, and provided an array of examples and representations of functions. Notably, Student C provided examples of functions situated in the abstract algebra context throughout the interview pairing standard secondary algebra examples (e.g., $f(x) = x^2$) with abstract algebra examples ($h(a,b)=b^2$) when prompted to share examples of functions. This integration was further evidenced when Student C was asked explicitly to address functions in abstract algebra listing out typical functions in this setting including isomorphisms, and homomorphisms. When asked if functions seen in their abstract algebra class are the same as functions that they've seen in other classes, Student C explained:

> But I mean, when we go through the isomorphisms and the homomorphisms, we're really going back to those simple kind of equations that we did in the beginning of algebra. Or in linear algebra kind of thing. It's not necessarily like we're coming up with whole new ideas. It's just restructuring them.

---

[1] Note: This student also suggested another singleton candidate for the kernel as she worked to make sense of the identity in H. Unpacking this portion of her response is beyond the scope of this paper.

[2] Note: The student is treating the input element as an ordered pair, but only included the single set of parentheses in their notation.

In contrast to several of the other participants in our interviews, Student C treated functions in abstract algebra as naturally connected to functions from other courses.

We also saw this play out in Student C's engagement with abstract algebra specific tasks. She responded to the diagram in Figure 1 by immediately evaluating if the diagram was of a function.

> The second one is kind of what I was talking about earlier with function that everything has to be taken to exactly one spot. I feel like reverse it would've been fine. Like it was taking *H* to *G*. I'm trying to think of a function that would do this and really there's not one because it's not a function.

She concluded the map could not be a homomorphism because it is not even a function. This consideration to a homomorphism being a function evidenced her connected knowledge.

A second case where we witnessed Student C's connected function knowledge was addressing kernels. Student C explained the kernel as, "…the group of elements, like if you have a homomorphism, let's call it Φ from *G* to *H*. It's the group of elements in *G* that get mapped to the [identity] element of H." She leveraged mapping metaphors for function and was easily able to approach the preimage of a function without constructing a map from the codomain to the domain. For example, when identifying the kernel from the Kernel Task, Student C explained:

> So I said that the kernel of phi was actually the integers times 4. So with that, it was because we didn't have some element ... I just called it the first group *G*. Well, $\mathbb{Z}$. So some element of that takes *i* to that power and gives us out 1, which would be the identity element for *H*. Because is gonna i^1give us *i*. i^2 is gonna give us - *i*, so on. So in order for *i* to be taken to a certain power and give us 1, it needs to be a multiple of 4. And that's to do with ^2i is negative 1. So negative 1 times negative 1. And as long as that's a multiple of 4, we're good to go.

These instances were representative of the way that Student C engaged in tasks. Her function understanding appeared to play a supportive role in her abstract algebra activity.

Discussion

This report addresses six undergraduate abstract algebra students' understanding of functions and provides two cases to illustrate how these understandings play out in their abstract algebra activity. As this work is exploratory, we are not attempting to make generality claims. Rather these varied cases provide images of the complex ways that function knowledge plays out in abstract algebra. The students in this study ranged in terms of their evoked concept image of functions. While all students' concept images contained explicit symbolic rules, several students saw the rule as essential for functions. These same students tended to evoke examples of typical function families such as polynomials without connections to abstract algebra. In contrast, the other students had more varied representations and evoked examples including those explicitly connected to abstract algebra. As seen with Student D, this unified understanding may play a supportive role in abstract algebra activity.

This work has several implications. First, we see that even students in an advanced mathematics course towards the end of their undergraduate tenure can struggle to grasp the complex and nuanced concept of function. From a research standpoint, we may want to explicitly consider the role of function understanding in student activity in advanced mathematics. From a teaching perspective, instructors may want to take stock of students' function understanding even in advanced courses. Second, instructors may want to attend to the ways their students' function understanding plays out in courses and consider how one might more proactively connect their prior function experiences with the new types of functions found in abstract algebra.

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Abstract Algebra Instructors' Noticing of Students' Mathematical Thinking

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Examining teaching practices in advanced mathematics is a relatively new field of scholarship despite a long history in K-12 settings. One important research area in this setting is documenting teacher noticing of students' mathematical thinking. In this report, we extend this line of work to explore how undergraduate mathematics instructors attend to, interpret, and respond to student thinking (Jacob, Lamb, & Philipp, 2010) in abstract algebra. We surveyed 25 abstract algebra instructors with a range of experience. Overall, we found that our participants focused on student thinking to a greater degree than the elementary teachers in earlier studies. Further, their interpretations spanned two distinct foci: understanding of concepts and the formal representation system. Their proposed responses then reflected a wide span of teaching actions. This exploratory analysis unveiled a number of previously undocumented characteristics of instructor noticing at the undergraduate level which can serve to inform future research on teaching practices.

Keywords: Abstract Algebra, Teaching, Noticing

The mathematics education community has moved towards models of teaching where instruction is tied to students' mathematical thinking (Jacobs & Spangler, 2018). Jacobs and Spangler identified teacher noticing as one of two core instructional practices needed for this type of instruction. While this construct has been studied and unpacked from a multitude of lenses at the K-12 setting (e.g., Jacobs, Lamb, & Philipp, 2010; Sherin & van Es, 2005; Star & Strickland, 2008), very little is known of instructor noticing at the undergraduate level. A lack of research in this setting is unsurprising in light of Rasmussen and Wawro's (2018) recent look at research at the post-calculus level where teaching is just beginning to be studied.

In this report, we share results from an exploratory study unpacking a particular aspect of instructor practice: noticing of students' mathematical thinking. We adapt this lens from the work of Jacobs, et al. (2010) who have decomposed teachers' in-the-moment noticing into three related acts: (a) *attending to children's strategies*, (b) *interpreting children's understandings*, and (c) *deciding how to respond on the basis of children's understandings.* Our study is situated in the context of abstract algebra, a standard upper level undergraduate course. We leverage pieces of student work that reflect documented ways students reason about the core concepts of identity, subgroups, and cyclic groups. Through surveying a variety of instructors, we introduce analysis of how mathematics instructors are attending to, interpreting, and responding to the student responses. We pay particular attention to how these responses diverge from the responses documented in the K-12 literature in order to contribute to our knowledge of teaching practices at the advanced undergraduate level.

Background

In this section, we provide background both on noticing research at the K-12 level and the larger research base on teacher practices at the advanced undergraduate level.

Noticing at the K-12 Level

Noticing student thinking is a "core practice of high-quality mathematics instruction because it is foundational for teachers' in-the-moment decision making" (Jacobs & Spangler 2017, p. 192). Which aspects of student thinking teachers give their attention to and how they interpret what they see or hear, influences their instructional decisions (Jacobs et al., 2010; Schoenfeld, 2011). Researchers have documented that teachers and prospective teachers notice a multiple of things when engaging with videos of classrooms (e.g., Sherin & van Es, 2006; Star & Strickland, 2008). Jacobs and colleagues (2010) developed a framework to distill one aspect of this noticing: noticing students' *mathematical thinking*. This framework can be leveraged to explore teacher noticing in the context of written artifacts or short video clips of students engaged in mathematical tasks. As teachers engage in describing, interpreting, and deciding how to respond to artifacts of student work, they demonstrate their skill in noticing mathematical thinking.

A number of researchers have built off of this work from the elementary level to study varying populations of teachers including Simpson and Haltiwanger's (2017) recent work at the secondary level. As noted by Nickerson, Lamb, and LaRochelle (2017), expanding beyond the elementary level brings additional challenges including the availability of artifacts, the availability of well-articulated frameworks around student thinking, and the availability of expert responses. Such work may also require adaptations to the original framework in light of the new contexts (see Simpson & Haltiwanger's additional distinctions.)

The summative results from these studies reflect that (a) professional noticing of student thinking is an essential skill for teachers and (b) it is a skill that can be developed through appropriate support (Fernandez, Llinares, & Valls, 2013; Jacobs, Lamb, & Philipp, 2010; Miller, 2011). Examining noticing at the advanced undergraduate level likely requires both consideration to the elementary literature base and consideration of what aspects of noticing may be informed by the particulars of the advanced mathematics context.

Teaching at the Advanced Undergraduate Level

Few studies at the advanced undergraduate level have focused "directly on *teaching practice*—what teachers do and think daily, in class and out, as they perform their teaching work" (Speer, Smith, & Horvath, 2010, p. 99). A few exceptions have begun to unpack some of the relevant practices including the nature of lectures (Weber, 2004), question types in lecture (Paoletti, Krupnik, Papadopoulos, Olsen, Fukawa-Connelly & Weber, 2018), and grading student proofs (Moore, 2016). Little work has explored the nature of teaching practices directly related to engaging with students and their thinking. The studies that have begun unpacking this work are situated largely in the implementation of inquiry oriented-instruction (IO), a pedagogy that relies heavily on instructor use of student ideas as a component of lessons aimed to move from informal to formal understanding of ideas (Rasmussen & Wawro, 2018).

Instructors using both the differential equations IO materials and abstract algebra IO materials have been documented to struggle to support productive discussions and leverage student reasoning without strong pedagogical content knowledge (Speer & Wagner, 2009; Johnson & Larsen, 2012). While mathematician instructors likely have very strong mathematics content knowledge, their knowledge of student reasoning and connections to pedagogy may not be as fully formed. Pedagogical content knowledge provides the lens through which instructors can interpret and respond to student thinking. In this way, teacher noticing is a specific practice or skill, related to pedagogical content knowledge, that becomes critical for instructors striving to adjust their lessons based on student thinking, as is often the case in IO classrooms.

Johnson and Larsen (2012) and Johnson (2013) provide perhaps the most nuanced look of addressing and leveraging (or failing to leverage) student reasoning in advanced undergraduate settings through their look at IO curriculum implementation in the abstract algebra classroom. In particular, Johnson and Larsen highlight the role of *generative listening*. This listening occurs when an instructor is able to interpret students' reasoning and adjust the lesson trajectory accordingly. Johnson and Larsen noted that their case study instructor often lacked knowledge of the specifics of student reasoning such as seeing operating on symmetry elements left-to-right, and thus failed to appropriately respond. In Johnson's follow-up work, she provides contrasting images of abstract algebra instructors' productive mathematical activity that was needed to interpret and analyze student ideas, as well as make connections between these ideas and the larger mathematical goals of the lessons. These studies provide cases that establish the important role of noticing student reasoning in order to promote student-centered instruction. They also illustrate that the knowledge and skills involved in supporting students in abstract algebra is non-trivial.

Theoretical and Analytic Orientation

Our work is orientated towards teaching practices, the work teachers do in their daily lives as instructors (Speer, et al., 2010). In particular, we focus on their noticing of student thinking, and ultimately the nature of the responses connected to this noticing. We make the assumption that "teacher noticing is worthy of study because teachers can be responsive only to what has been noticed" (Jacobs & Spangler 2017, p. 192). We leverage the framework introduced by Jacobs, Lamb, and Philipp (2010) that unpacks noticing as three interrelated practices: (a) *attending to children's strategies*, (b) *interpreting children's understandings*, and (c) *deciding how to respond on the basis of children's understandings.* Each practice can range from noticing that is disconnected from students' thinking to noticing meaningfully and richly coupled with student thinking.

Beyond the scope of the original framework, we incorporate other theoretical distinctions to produce a more detailed image of instructors' interpretations and responses to students. First, at this level, interpreting can have both a semantic orientation, focused on concept understanding, and logio-structural (formal) orientation, focused on aspects of proof and the formal structure emphasized in advanced mathematics (c.f., Weber, 2004). We also parsed the nature of responding to not just how coupled the response was with student thinking, but also the nature of the response itself--what did these instructors say they would do next with this student? Responding is a practice that has substantial theoretical breakdowns at the K-12 level (e.g., Boaler & Humphreys, 2005; Herbel-Eisenmann, Drake, & Cirillo, 2009; Milewski & Strickland, 2016) focused on the nature of teacher questions and actions. To analyze our instructor responding moves, we leveraged various literature to identify key ways of responding in terms of question types (Sahin & Kulm, 2008), and other responding moves (Milewski & Strickland, 2016). We expand upon our categories in the next sections.

Methods

For this study, we surveyed 25 Abstract Algebra instructors representing a range of experience and institution types. Table # reflects the demographic information of the participants.

Table 1. Background on Participants.

Research Focus		Experience Teaching Algebra		Position		Institution Type (highest mathematics degree)	
Abstract Algebra	*n*=11	<5 times	*n*=8	Assist. Prof.	*n*=2	Ph.D.	*n*=10
Math Education	*n*=5	5 - 9 times	*n*=10	Assoc. Prof.	*n*=9	M.S.	*n*=8
Other Math Pure	*n*=9	> 9 times	*n*=7	Full Prof.	*n*=11	B.S.	*n*=6
				Other	*n*=3	NA	*n*=1

The Survey

The survey was directly adapted from Jacobs, Lamb, and Philipp (2010) and Jacobs, Lamb, Philipp, and Schappelle (2011). The instructors were given five pieces of student work. For each piece of student work, instructors were asked:

- Please describe in detail what you think this student did in response to this prompt.
- Please explain what you learned about this student's understanding.
- Pretend that you are the instructor of this student. Describe some ways you might respond to this student, and explain why you chose those responses.

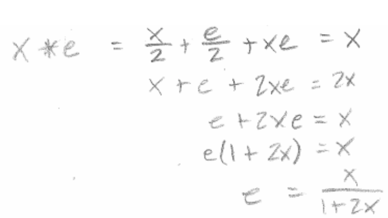
The student work stemmed from a large-scale project collecting data about student understanding in group theory (Melhuish, 2015). Table 2 contains three pieces of student work that are the focus of this report.

Table 2. Sample Student Work Provided to Participants

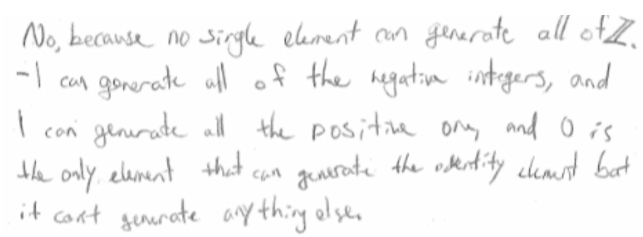
(1) Given L the set of all positive rational numbers, consider the binary * defined:

x*y=x/2+y/2+xy

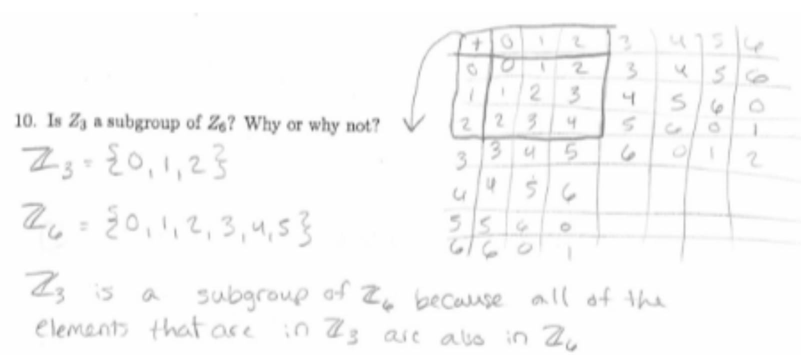
Determine if this operation has an identity. If so, identify the identity.



(2) Is *Z*, the group of integers under addition, a cyclic group?



(3)



Each response was selected due to its connections to established ways of thinking about group theory topics from the literature. Response one stems from a task identified by Novotná and Hoch (2008) as reflecting structure sense for operation where students may or may not recognize that an identity element must serve as an identify for all elements in the set. The second piece of student work reflects incomplete coordination of the binary operation with subgroups where students may rely on a subgroup test without attending to the differing operations between $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_6$ (e.g., Melhuish, 2018; Dubinsky, Dautermann, Leron, & Zazkis, 1994) The third piece of student work reflects a common conception of cyclic groups where elements only generate via repeated operation and thus do not take on negative powers (Melhuish, 2018; Lajoie & Mura, 2000).

Analysis

To analyze the data, we incorporated a two-fold approach. First, we analyzed the data using the original scheme developed by Jacobs, Lamb, & Philipp (2010) addressing whether instructors attended to student thinking, the robustness of their evidence of interpretation, and the degree to which their responding actions were connected to the student's thinking. Second, we took a more grounded approach to account for the fact that the advanced tertiary level may lead to substantially different characteristics of instructor noticing. The initial passes were done by the authors independently. In collaboration, we then arrived at a coding scheme where all responses were classified. A subset of the scheme can be found in Table 3. All instructor responses were then coded in tandem with discussion serving to settle disagreement in codes.

Table 3. Background on Participants.

Noticing Practice	**Categories** (*Codes*)
Attending	**Connected to Student Thinking** (*Y:Yes*, *N:No*)
Interpreting	**Evidence Level** (*N: No Evidence Provided*, *L: Limited evidence provided*, *R: Robust Evidence Provided*); **Aligned with Literature Interpretations** (*Y:Yes*, *N:No*); **Formal Representation System** (*D: Definition*, *A: Implicit Assumptions*, *P: Proof*, *Q: Quantifiers*)
Responding	**Connected to Interpretation of Student Thinking** (*Y:Yes*, *N:No*); **Nature of Response** (*E:Praise*, *T:Telling*, *G:Guiding*, *P:Probing*, *C:Command*)

Results

Attending & Interpreting

The abstract algebra instructors provided quite different profiles in terms of noticing. Compared to the documented literature on elementary teachers, these instructors were much more likely to attend to students' thinking. A typical interpretation looks as follows:

> The student understands which binary operation is in question. The student has done some elementary algebra correctly, from which the question could be answered. But I would infer from the response stopping at this point, that the

student thinks that there is an identity and that it has been found (referencing task 3).

In fact, across our three focal tasks, we documented zero instances of not paying attention to students' thinking and only 7% of responses provided largely evaluative statements. For example, one instructor made comments such as, "[I]mpressive written response, a good 'abstract algebra' presentation..." Such a response illustrates a focus that was more evaluative with language about the quality of the response, and less focus on the student thinking offered. While the instructors did largely attend to student thinking, there was range of evidence provided as (see Table 4).

Table 4. Percentage of Interpretations with Particular Characteristics

Task	Level of Evidence Provided	Aligned with Literature	Focused on Aspects of Formality
Task 1 (Identity)	R: 12% L:40% N:48%	Y:48%	Y: 60%
Task 2 (Subgroup)	R: 38% L:48% N:14%	Y:65%	Y: 54%
Task 3 (Cyclic Group)	R: 14% L:52% N:38%	Y:73%	Y: 60%

One feature that differed across our participants was attention to aspects of the formal representation system which split the interpretations. Non-formal interpretations included statements like "confused the notions of subset and subgroup." For formal interpretations, the role of definition was particularly prominent (52%[1]) (e.g., "the student does not appeal to the literal definition"), followed by hidden assumptions (14%) (e.g., "The student assumed the distributive law holds."), quantifiers (14%) "e.g., "Student has a weak grasp of the words 'for all'.", and issues of proof (12%) (e.g., "[I]t does [not] formally 'prove' that an identity element exists").

Deciding How to Respond

Deciding how to respond to a student is an important aspect of noticing, but is also its own area of research in K-12 scholarship on teaching with little corresponding research at the tertiary level. In terms of the original noticing framework, we note that almost all of the mathematician responding choices were connected to their interpretations of student thinking (97%). This attention was a dramatic shift from what was documented with elementary teachers where a sizeable portion did not attend to student thinking (Jacobs, et al., 2010).

Table 5. Percentage of Response Types

Type	Example Response	% [1] (*n*=72)
Command	"Please review the definition of subgroup which has three parts (closure, identity, inverses). Then come to my office hours."	8%
Praise	"First is praise the amount of good work that happened."	8%
Probe	"Tell me your reasoning for what you did? What does this answer	17%

[1] Interpretations could include multiple components. Percentages are not intended to sum to 100%

	mean?"	
Tell	"I remind you that identity cannot depend on x"	35%
Guide	"I think this student is ready for direct questions about their solution. For example: Is it a problem that x=1 and x=2 result in different values for e?"	65%

1. Percentages sum to greater than 100% because some responses included multiple response types

Even though responses were connected to student thinking, the nature of the responses varied. Guiding questions, questions intended at move students' mathematics to the correct mathematics were by far the most prevalent for these instructors. However, there was also substantial telling responses along with commanding, praising, and probing. See Table 5 for examples of each type of response and the respective percentages.

Discussion

The abstract algebra instructors provided a contrasting profile to what has been documented about teachers at the K-12 level. First, with a few exceptions, the abstract algebra instructors attended to the students' mathematical thinking. Second, their responses nearly always aligned with their description and interpretation of this thinking. As such, differentiating based on these categories was not a meaningful way to distinguish the nature of the instructors' noticing. However, when going beyond connections to student work, the instructor responses ranged across our participants. In terms of interpretation of student work, we found that attention to formality was particularly helpful to distinguish amongst responses. Close to half of the instructors focused exclusively on elements of concept understanding (without attention to formality) while the other half of participants focused on aspects of formality. Definition was the most common formal aspect attended to. This is not particularly surprising in light of how closely formal definitions are tied to concepts at this level. Further, the student work responded to prompts that may engage them with definitions, but not require formal proofs. In terms of deciding how to respond, the most significant difference across our participants was the nature of the teacher moves. Guiding questions seemed particularly prevalent among our sample. This exploratory analysis of our data can support a more nuanced look at these responses. For example, to what degree are the guiding questions intended to funnel students towards correct answers versus lead towards open exploration? Due to our sample size, we are hesitant to make generalizability claims. Follow-up research may look explicitly at the role that specific tasks and experience play in this noticing through larger samples or qualitative interviews.

Implications of our study are mostly research-based. This exploratory study provided evidence that mathematicians' noticing (in terms of interpreting and responding) ranged greatly in even a small sample. Noticing at the tertiary level likely includes parallel constructs of noticing conceptual understanding and noticing formal representation aspects. Further, the nature of the participants responses illustrated components of the practice of responding to an individual student. The nature of these responses was significantly different than question types that were recently documented during lectures (Paoletti et al., 2018), and as such may serve as a starting ground to examine the instructor responses outside of the traditional lecture.

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Examining Graduate Student Instructors' Decision Making in Coordinated Courses

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In an effort to improve teaching and learning in undergraduate mathematics courses and help graduate students learn how to teach, many departments across the United States have begun coordinating courses. Although coordination may provide structure and remove some variability in the classroom, there are still many decisions made in the classroom that cannot be coordinated. The purpose of this study was to examine the "uncoordinated" decisions that graduate student instructors made when enacting examples in the classroom. To examine this phenomenon, I studied the cognitive demand of the examples that graduate student instructors chose to enact and the roles that they took on while enacting high cognitive demand examples. As a result, I found that less than 27% of the examples that I observed were enacted at a high level of cognitive demand and that there were three roles (modeling, facilitating, and monitoring) that instructors took on while enacting examples.

Keywords: graduate student instructors, coordinated courses, cognitive demand, examples, decision making

The purpose of this study is to examine graduate student instructors' (GSIs) decision making in coordinated courses. In the department where I conducted my study, precalculus courses are primarily taught by GSIs and are highly coordinated. This coordination involves common lesson guides, student worksheets, WeBWorK homework assignments, and exams. These courses are coordinated primarily by a GSI who serves as the Associate Convener, but there is also a Faculty Convener. Although the high level of course coordination means that GSIs do not have to make many of the decisions regarding course structure and assessment, the lesson guides provided to GSIs allowed them flexibility regarding what examples they chose to do and how they chose to present them. So, for this reason, I chose to examine the examples that GSIs enacted in their classrooms by looking at both the cognitive demand and the roles (modeling, facilitating, or monitoring) that the GSI took on while enacting the example.

Background

The cognitive demand of mathematical tasks is something that has been widely studied in the literature (Boston & Smith, 2009; K. J. Jackson, Shahan, Gibbons, & Cobb, 2012; Kisa & Stein, 2015; Smith & Stein, 1998; Stein, Grover, & Henningsen, 1996). Studies have found that high cognitive demand tasks provide students with more opportunities to learn (Floden, 2002; K. Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Smith & Stein, 1998; Stein, Remillard, & Smith, 2007). Researchers have also found that high cognitive demand tasks are difficult for instructors to enact (Henningsen & Stein, 1997; Rogers & Steele, 2016). But what would it mean to have a high cognitive demand mathematical example? Examples are different from mathematical tasks that are primarily worked on by students. Examples may involve input from students or opportunities for students to work independently or in groups on parts of the example, but usually the teacher plays a leading role in working out or explaining the mathematics. Although studies have shown that students do not learn as much from observing a worked out example as they do from actively engaging in the problem solving process (Richey & Nokes-Malach, 2013), the examples that teachers use still play an important role in the learning process (Chick, 2007; Muir, 2007; Rowland, 2008; Zaslavsky & Zodik, 2007). In particular, Ball and her colleagues (TeachingWorks, 2017) identified "explaining and modeling content, practices, and strategies" as a high-leverage teaching practice.

Methods

The GSIs that I observed (Dan, Emma, Greg, Juno, Kelly, and Selrach) were all experienced graduate students who were teaching precalculus. These GSIs were experienced in two ways. First, they were in at least their third year of graduate studies, had earned their M.S. in Mathematics, and were working towards their Ph.D. Second, they were all teaching their respective course for at least the third time. It is also important to note that many, but not all, of the GSIs had went through a one-year course on *Teaching Mathematics at the Post-Secondary Level*. This 3-credit course was taught by a faculty member in the department who was the Director of First-Year Mathematics. All second-year GSIs were required to take this course in addition to their normal 9-credit course load, but were also given a course release during the fall semester to compensate for the extra time. Alex and Dan were in the first cohort of GSIs who took this course during Year 1. Greg was not required to take this course, but chose to with the first cohort. Emma, Juno, and Kelly were in the second cohort of GSIs who took this course during Year 2. Selrach did not take this course, as it was not offered when he started the program and he did not opt in to take it later. The goal of this course was to support GSIs as they became evidence-based practioners of mathematics education. So, the course aimed to help make GSIs aware of mathematics education research, issues, and terminology so they could apply what they were learning in their own classrooms and become reflective teachers.

For this study, I conducted semi-structured pre-observation interviews, classroom observations, and semi-structured post-observation interviews. I also collected copies of the lesson guides that were provided to the GSIs, the individual lesson plans that the GSIs prepared, and the student worksheets. During the pre-observation interviews, I asked questions about the previous and next class and focused on what examples they planned to use and why. During the classroom observations, I collected video data and took field notes. After each observation, I watched the video and selected one or two examples to discuss with the GSI during the post-observation interview and tagged interesting moments to use for video-stimulated recall.

Each enacted example was first coded using a modified version of Smith and Stein's (1998) framework for the cognitive demand of examples. A full description of this modified framework can be found in Miller (2018), but included four categories for the cognitive demand of examples: *memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics*. Next, I open coded the high cognitive demand examples to examine the roles the GSIs took on while enacting (note that I did not code low cognitive demand examples). Three roles emerged out of this open coding (modeling, facilitating, and monitoring), which I have defined below in Table 1. I then went back and recoded each high cognitive demand example using the final coding scheme for GSI roles.

For this study, I observed each GSI three times throughout the semester. In the first semester, I observed Alex, Greg, and Kelly and asked them to choose three dates (spread out from September-December) that worked best for them. During the second semester, I observed Dan, Emma, Greg, and Selrach and chose specific lessons that I wanted to observe. The lessons that I chose for the second semester were more procedural, because I thought they would provide me with an opportunity to see whether GSIs chose to present examples as *procedures <u>without</u> connections* or *procedures <u>with</u> connections*. Also, I only observed one day of instruction in the first semester, regardless of whether or not the lesson was spread out over two days. However, if a lesson was spread out over two days in the second semester, I observed both days.

Table 1. Definitions of the three types of roles (modeling, facilitating, and monitoring)

Term	Definition
Modeling	An instructor is modeling content, practices, and strategies if they are working through an example independently and expecting students to follow along by taking notes.
Facilitating	An instructor is facilitating a whole class discussion if they work through an example together with input from their students.
Monitoring	An instructor is monitoring if they are requiring students to work through an example independently or in small groups.

Results

Of the 93 examples that I observed, I coded 25 of them as high cognitive demand examples. When enacting high cognitive demand examples, GSIs used a variety of approaches. Although some GSIs took on primarily one role when enacting high cognitive demand examples, others transitioned back and forth between different roles. Figure 1displays the aggregate role profiles for the high cognitive demand examples that I observed each GSI enact. These role profiles were constructed by summing the total time each instructor spent in each role across all of the high cognitive demand examples that I observed and provide a glimpse of which roles each instructor tended to take on. In this paper, I will focus on three role profiles: modeling, modeling and facilitating, and facilitating and monitoring. Although there were several GSIs who enacted examples using these different role profiles, I will focus on specific examples enacted by Emma, Greg, and Kelly in order to illustrate the different ways in which these GSIs chose to enact high cognitive demand examples in their classrooms.

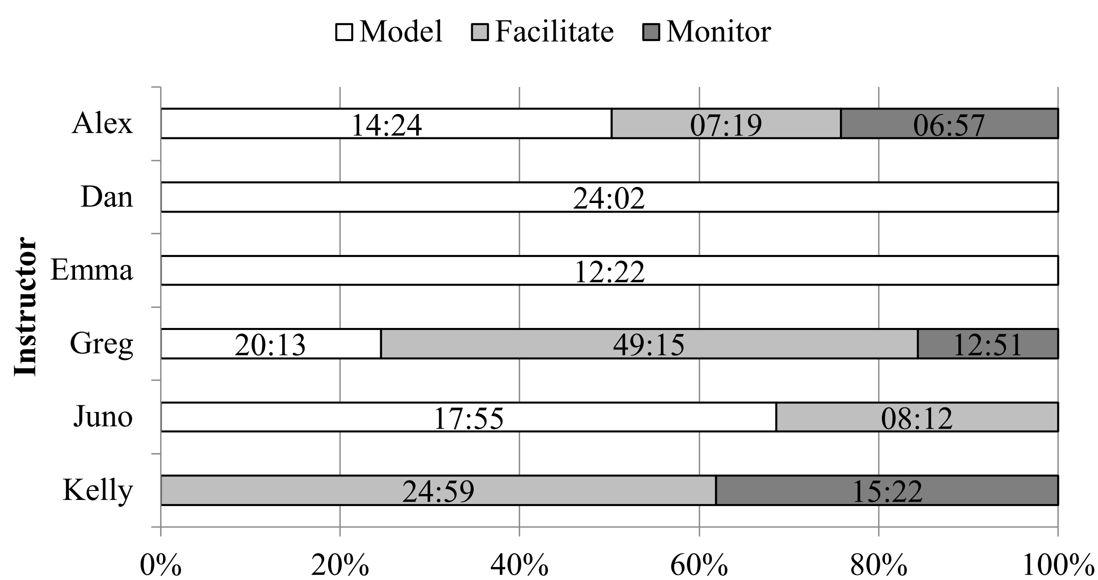

Figure 1. Aggregate role profiles for each GSI

Model: Emma

Many GSIs chose to take on different roles when enacting examples, but some chose to just model examples for their students. Although students do not have an opportunity to struggle with the mathematics in this type of setting, they do have an opportunity to have high cognitive demand processes modeled for them. In order to maintain the cognitive demand while modeling, GSIs focused on making their cognitive processes explicit and attending to student understanding. The example that I observed Emma enact at a high level of cognitive demand was situated at the end of a chapter on function transformations. Emma chose the example because it was a question on the chapter quiz that many of the students had struggled with. In particular, she wanted to reemphasize the connection between order of operations and order of transformations and explain how to check their work using an alternative method. The example gave the graph of a piecewise linear function and asked students to sketch a graph of $3P(t+1)-2$ for $0 \leq t \leq 9$ on a provided grid.

Since so many of her students had struggled with this problem on the quiz, Emma chose to model it for her students at the beginning of the next class. Emma worked through the example by first identifying the order of transformations. She emphasized that it did not matter if they did horizontal transformations before or after vertical transformations, but that they did need to attend to the order of the vertical transformations. To help her students understand why the vertical stretch had to occur before the vertical shift, she explained how function transformations are related to the order of operations. Next, Emma explained that they could transform the endpoints and corners of the graph and then connect these points with straight lines. Emma also noted that one of the transformed endpoints fell outside the domain $0 \leq t \leq 9$ and explained how to find the new endpoint. Since so many of her students had struggled with determining the correct order of transformations, Emma also presented an alternative method for graphing the transformed function that did not rely on memorizing information related to order of transformations. Instead, she explained how students could use the equation $3P(t+1)-2$, the original graph, and integer values in $[0,9]$ to graph the transformed function.

I coded this as a *procedures with connections* example because of the following reasons. First, Emma focused students' attention on the use of procedures for the purpose of developing deeper understanding of mathematical concepts and ideas. To help her students remember the order of vertical transformations, she focused on the underlying mathematical concept of order of operations. Also, to help her students find exact output values, she focused on the underlying concept of slope and how to interpret it in a way that is helpful for calculating non-integer values. In her example, Emma presented two different pathways that students could follow to solve the problem (using order of transformations to move points or using an input-output table). In explaining each pathway, Emma focused on the underlying conceptual ideas (order of operations and evaluating function compositions), instead of the narrow algorithms. The example involved graphical, algebraic, and tabular representations and Emma often made connections between each of them. Finally, the number of student questions and the prevalence of student struggle on the problem when it was presented on the quiz are evidence that the example required some degree of cognitive effort for students to follow.

Model and Facilitate: Greg

The high cognitive demand example where Greg switched back and forth between modeling and facilitating was situated in the second day of an extended lesson on finding all solutions to trigonometric equations. After spending the first day exploring the structure of the infinite families of solutions and working through simpler problems that did not involve shifts and stretches, Greg introduced more complicated sinusoidal functions. First, Greg did two examples that only involved vertical transformations. For his final example, Greg chose to find all solutions to $\sin(3\theta - 1) = 1/4$. Greg chose this function for several reasons. First, he wanted his students to learn how to find all solutions when the period is not equal to 2π. Second, he wanted to give an example with both a horizontal shift and a period change because he knew that problems of this type would come up on the online homework as well as the exam. Finally, he did not want to use a standard unit circle angle and instead force students to use arcsine.

Greg started by first modeling content, practices, and strategies for students. To make the equation more clear and appear less complicated, Greg decided to define the variable $X = 3\theta - 1$. Greg chose to do this because he wanted to remove the part of the equation that looked unfamiliar and highlight that first they needed to isolate the input of sine. Next, Greg switched to facilitating a whole class discussion. First, he asked how they could proceed from $\sin(X) = 1/4$ to solve for X. A student suggested that they could use arcsine, so Greg wrote $X = \sin^{-1}(1/4)$ and explained that this gave the first solution. When Greg asked where the second solution came from they were able to come up with $X = \pi - \sin^{-1}(1/4)$ with some assistance from Greg. From here, Greg switched back to modeling. He explained that since they had started with θs, they needed to end with θs and substitute out the Xs. Doing this resulted in the following two equations: $3\theta - 1 = \sin^{-1}(1/4)$ and $3\theta - 1 = \pi - \sin^{-1}(1/4)$. Before solving for θ, Greg paused to explain that this problem "was a little bit more involved than the other [examples] because we generate our initial solutions and then we have to keep working to…find the initial solutions just in terms of θ." From here, Greg worked through the algebra to solve for θ, which resulted in $\theta = 1/3(\sin^{-1}(1/4) + 1)$ and $\theta = 1/3(\pi - \sin^{-1}(1/4) + 1)$.

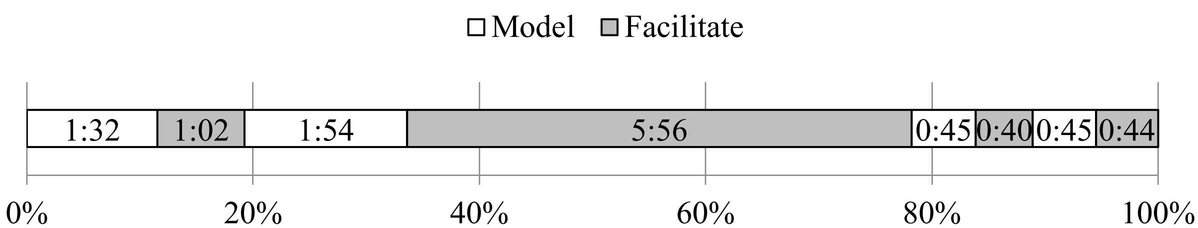

Figure 2. Role profile for Greg's example

At this point, Greg switched back to facilitating by pausing and asking for student questions. Students asked, "Why divide by 3? Where did the $1/3$ come from?" and Greg explained the algebraic step the student was stuck on. Next a student asked, "Will we still involve adding the period times k at the end?" Greg explained that was the next step and reiterated that the work they had done so far was all to get the initial solutions. Greg then moved on to talk about all possible solutions and reminded the class that they should be of the form $(\text{initial}) + (\text{period})k$. To start this conversation, he asked, "What is the period of [$\sin(3\theta - 1)$]?" After working collaboratively, the students were eventually able to identify that the period was $2\pi/3$ and then wrote up the final solutions. Throughout this conversation, Greg switched frequently back and forth between modeling and facilitating. At the end, Greg took the time to summarize the whole process and the general procedure that they had followed.

I coded this example as *procedures with connections* for the following reasons. Although parts of this example strayed into lower cognitive demand tasks, the majority of the problem was focused on the broad general procedure of using the initial solutions and the periodicity of sinusoidal functions to find all solutions. Greg consistently focused students' attention on the underlying structure of solutions to trigonometric equations: $(\text{initial}) + (\text{period})k$. There was a lot of algebra involved in getting the initial solutions and students struggled to find the period, but Greg always brought the focus back to this underlying concept. Although the example was computational, Greg emphasized the connections between the general form of solutions to trig equations and the specific families of solutions that they had found. Also, the number of questions asked by students is one form of evidence to support the claim that this example required some degree of cognitive effort for students to follow.

Facilitate and Monitor: Kelly

The high cognitive demand example that Kelly presented by both facilitating a whole-class discussion and monitoring students as they worked individually or in small groups was situated at the beginning of the lesson introducing exponentials. To start class, Kelly asked her students to work on a problem that asked students to compute the account balances in an account that earned simple interest and an account that earned compound interest. During this time, she asked a group to write the balances in both accounts after one year on the board. After a few minutes, Kelly brought the class back together to see if everyone agreed with what the students had written on the board. She then asked a student to volunteer the balances after two and three years and wrote those on the board. Kelly then asked, “Which one would you chose?” A choral of students said responded with the same answer and Kelly explained why that was correct.

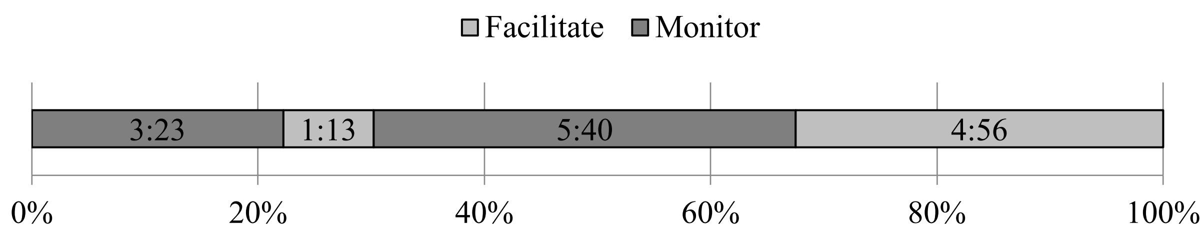

Figure 3. Role profile for Kelly’s example

At this point, Kelly gave her students a similar problem to work on: “Suppose you are investing $\$500$ at an annual rate of 4.5%. Create a table that shows the balance after 0, 1, 2, and 3 years. What is the balance after t years?” As students began working individually and in small groups on this problem, Kelly monitored their progress by walking around the room and interacting with different student groups. After almost six minutes of work time, Kelly brought the whole class back together for a discussion of the general formula. First, Kelly asked students what values they found for the table and verified that everyone had gotten the same answers. Then Kelly asked, “So how are we getting these numbers?” One student explained that they were using the formula $a(1+r)^t$ and Kelly acknowledged that this was correct, but she wanted them to figure out why that formula made sense.

To help start the discussion, Kelly asked, “How did we get from $\$500$ to $\$522.50$?” Another student responded with, “Times 500 by 0.045.” Kelly agreed that this would work, but asked if anyone knew an easier way of doing that. A new student piped up and said, “Times 500 by 1.045.” Kelly responded by explaining how we could factor out a 500 from both terms in $500 * 0.045 + 500$ and get $500(0.045+1)$. Next Kelly asked how they had found that $\$546.01$ was the balance after two years. A student responded with, “522.5 times 1.045,” which Kelly agreed with. Kelly asked, “What’s another way of writing 522.5?” After working together, the students were eventually able to refer back to the equation $522.5 = 500(1.045)$. Kelly then explained that to get 546.01, we needed to multiply that again by 1.045 to end up with

$500(1.045)(1.045) = 546.01$. After writing this all on the board, Kelly asked her students if they saw a pattern and if they could guess what the formula for t years would be. A student responded with $500(1.045)^t$. Kelly then encouraged her class to plug in $t = 3$ and verify that the value agreed with what they found in their table. Kelly asked for any final questions, with no response, and then asked, "So what kind of formula is that?" A student responded with exponential and Kelly explained that this is what the new chapter was all about.

I coded this as a *procedures with connections* example for the following reasoning. First, Kelly expected her students to be familiar with exponentials and know how to work with them computationally, but she really focused the example on the underlying concept of multiplicative growth. Students were not provided with any specific pathways to follow and Kelly encouraged them to solve the problem in different ways in order to check their work. Kelly also used tabular and algebraic representations of the problem. Finally, not every student was able to come up with a formula during their small group time, so we know that it required some degree of cognitive effort for students to complete.

Conclusion

In this study, I examined the decisions that GSIs made while teaching in highly coordinated courses. Using my modified framework for the cognitive demand of examples, I analyzed 93 examples that were enacted and found that 25 of them were enacted at a high level of cognitive demand. In these examples, I found that there were three roles that GSIs took on during the enactment: modeling, facilitating, and monitoring. Although some GSIs chose to just model examples for their students (e.g., Dan and Emma), others chose to switch between different roles. Juno also modeled examples for her students, but often asked for student involvement and switched to facilitating. On the other hand, Alex and Greg switched back and forth between all three roles, while Kelly chose to never model and instead just facilitated a whole class discussion or monitored her students as they worked on parts of the example independently or in small groups.

One limitation of this study is that the data I collected focused on the GSI and did not incorporate the student perspective. Therefore, I had to assess the cognitive demand of each example based upon the questions that students asked and the mathematical content of each example. Although I tried to define the four different levels of cognitive demand so that a classroom observer could categorize examples, it was still difficult at times to determine whether or not an example required cognitive efforts for students to follow or understand. Another limitation of this study was that is difficult to determine when an GSI is switching between modeling and facilitating. In particular, facilitating still requires contributions from the teacher, so it can be difficult to determine exactly when an GSI stopped modeling and started facilitating a whole-class discussion. Therefore, the role profiles should be interpreted as having a margin of error any time an GSI switched between modeling and facilitating.

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Learning Through Play: Using Catan in an Inquiry-Oriented Probability Classroom

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Research has documented the power of play to affect learning at all ages. This research shares the kinds of mathematical student thinking elicited by incorporating the board game Catan into an inquiry-based classroom. The class was composed of 25 students, not majoring in STEM fields, who were enrolled in a freshman seminar course intended to provide an opportunity to engage in research-like inquiry. Students had played Catan in class and to engaged inquiry-oriented instruction sessions focusing on the relationships between mathematics and Catan. Student work is provided for arguments related to the value of each resource and selecting locations for initial settlements, with connections between this work and topics traditionally taught in probability classes

Keywords: probability, games, inquiry-oriented instruction

The phrase "learning through play" often evokes images of elementary age students, however play can be an important part of the learning process for learners of all ages (Rieber 1996). In mathematics games have often been used to address relatively simple probability concepts such as rolling dice or choosing cards from a deck (Goering, 2008; Hoffman & Snapp, 2012/2013). The board game, Catan (Mayfair Games, 1995), provides unique opportunities for learners to engage with both these probability basics as well was several other mathematical ideas including more advanced ideas related to probability and expected value. Here I share evidence of how Catan provided a relevant context in which students used concepts of probability and expected value to address two important questions related to strategic game play: (1) determining which resource is most valuable and (2) deciding where to place one's initial settlements.

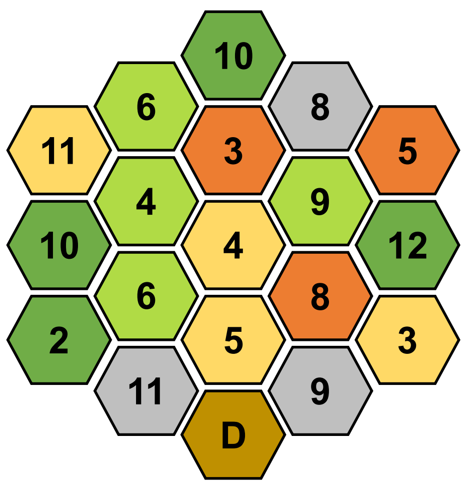
A Brief Introduction to Catan

Catan is a property building and resource trading based Euro-style board game for 3-4 players. Since its introduction in 1995, Catan has gained world-wide recognition, winning several awards, being translated into over 30 languages, and revolutionizing the board game industry (Law, 2010). At the start of every game the board is constructed from resource tiles, numbers, and ports. This makes Catan a perfect setting for abstract thinking about patterns and generalities rather than strategies that optimize play on one unchanging board. The benefits of this dynamic nature has already been noted by both mathematicians (Austin & Miller, 2015) and computer scientists (Szita, Chaslot & Spronck, 2010).

The board is assembled by arranging 19 hexagon tiles into a larger hexagonal formation as shown in Figure 1. These tiles dictate what resource will be produced at a location: brick/clay (3 tiles), lumber/wood (4 tiles), wool/sheep (4 tiles), grain/wheat (4 tiles), ore/stone (3 tiles), or nothing/desert (1 tile). Next each tile except the desert receives a number chip, this will dictate when the resource is produced. There is one ship for 2, one for 12, and two for every number from 3 to 11 excluding 7. Whenever the number on a tile is rolled by any player, the players with settlements or cities adjacent to that tile will collect the given resources. Finally, eight ports are distributed around the board, but they do not affect the mathematics discussed in this work. Settlements will be placed at the intersection of the corners of the hexagon tiles. Roads will be built along the edges of the tiles. Only one settlement or road may occupy a given space and there must be at least two edges between any two settlements on the board.

The game begins with each player placing two settlements in reverse draft order. Choosing initial settlements is a very important part of the game because it affects what resources you have immediate access to and which resources you may have the opportunity to build to; all new settlements must be connected to one of the player's initial settlements by at least two roads.

On a player's turn they roll two standard dice and all players who have a settlement on a tile with that number collect the corresponding resources. If a seven is rolled the player whose turn it is has the opportunity to move the robber, which is another detail which does not affect the mathematics in this research. Next the player whose turn it is has the opportunity to build additional road or settlements, upgrade a settlement to a city, or purchase a development card, which gives various bonuses. Each of these things contributes to a player earning victory points; the first player with 10 victory points wins. For complete rules please see, https://www.catan.com/en/download/?SoC_rv_Rules_091907.pdf.



Resource	Tile
Brick	
Lumber	
Wool	
Grain	
Ore	

Figure 1: This is one sample Catan board, but there are many more possibilities.

Setting and Methods

Data for this study were collected from a freshman learning community focusing on the relationship between mathematics and Catan; it consisted of the same cohort of students enrolled in both a common pre-calculus course section and a freshman seminar section which provided the majority of the focus on Catan. Students in the learning community were all first semester freshman at a large college in the southeastern US. The class included 25 students from a variety of majors, none of whom had officially declared majors in the university's science and math focused college. 24 students consented to participate in the study.

The freshman seminar course was intended to orient students to college life and to provide them with an opportunity to engage in some accessible research-like activities which do not have an easily found answer or explicit method for solving. The course met three days a week for 50 minutes, with roughly 1/3 of time dedicated to orientation content, 1/3 of time dedicated to playing Catan, and 1/3 of time dedicated to inquiry-oriented instruction (Rasmussen & Kwon, 2007) to explore the mathematics of Catan. The mathematical components of the course were presented to students as practical questions in the context of the game, which they were then asked to answer using mathematical reasoning. Explorations were often proceeded or followed by a game played on a board strategically chosen to highlight the concept in question. Methods of exploration included individual explorations, group work, debates, and whole-class student-led solutions. Previous analyses have shown that the course was effective in engaging students in mathematical reasoning (Molitoris Miller & Hillen, 2018). The goal of this report is to provide more detailed analysis of the kinds of mathematical reasoning the students used. These results answer the research question: What kinds of mathematical thinking are elicited by the board game Catan, in a student-driven inquiry-based classroom?

The data for this analysis came from a final exam item, which is very closely related to the student-centered inquiry-based theme of the course. It stated, “Describe five ways you can use mathematics to improve your chances of winning in Catan. For each mathematical application, describe it in detail and provide an example of how it works.” Students were given this prompt one week before the exam to think about it in advance but were not permitted to bring any prepared materials into the exam with them and two sample boards were provided. Although different students took more or less vocal roles in classroom or group discussions, the final exams better measure what each individual student eventually learned. Inductive coding was used to code the 120 responses from the 24 consenting students and group them into categories according to the topic they addressed. The results in this paper focus on the two largest overarching themes present in the student’s final exam responses, the value of each resource, and settlement location.

Determining Resource Value

The first main theme that demonstrates the kinds of student thinking elicited by the game focuses on assigning value to the resources. During the game, players may opt to trade resources with one another; thus, the value of each type of resource comes into question to determine if a certain trade is advantageous or at least fair.

First students began contemplating the usefulness of any resource in general as they relate to building in the game. In one approach, students highlighted that brick, lumber, wool, and ore are each used in two out of the four building processes, but grain is the only one used in three of the four building possibilities, thus they claimed that grain is the most valuable resource. Their argument rested on the idea that not having any access to grain would greatly limit ones’ ability to progress in the game.

Other students used a more weighted average, where the student determined how many of each card was needed to build one of everything. This strategy found brick, lumber, and wool were equally valuable because you would need only two of each of those resources. Grain and ore were also equally valuable, requiring two of each. Students then looked to the number of tiles of each type to determine that ore was more valuable because there are only three tiles which produce ore opposed to four tiles which produce grain. This strategy suggests that while you need access to both grain and ore, you would have a stronger advantage if you have slightly better access to ore.

Both of these solution methods were completely student-generated in class. This variety of arguably equally valid methods or measuring value provide opportunities to discuss the complexity of measuring more abstract judgement-based attributes such as value. This discussion could be taken further to include strategies like tracking how many of each resource is used in an average game, or a combination of general usefulness and rarity on a particular board.

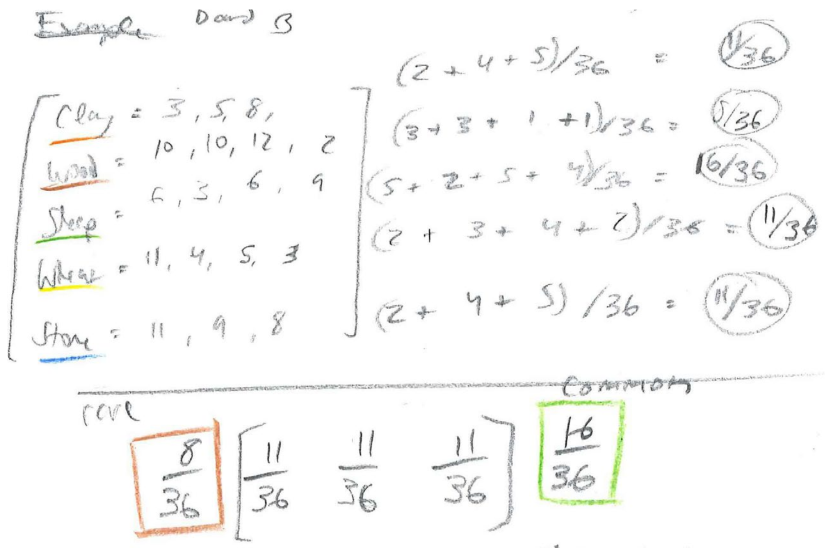
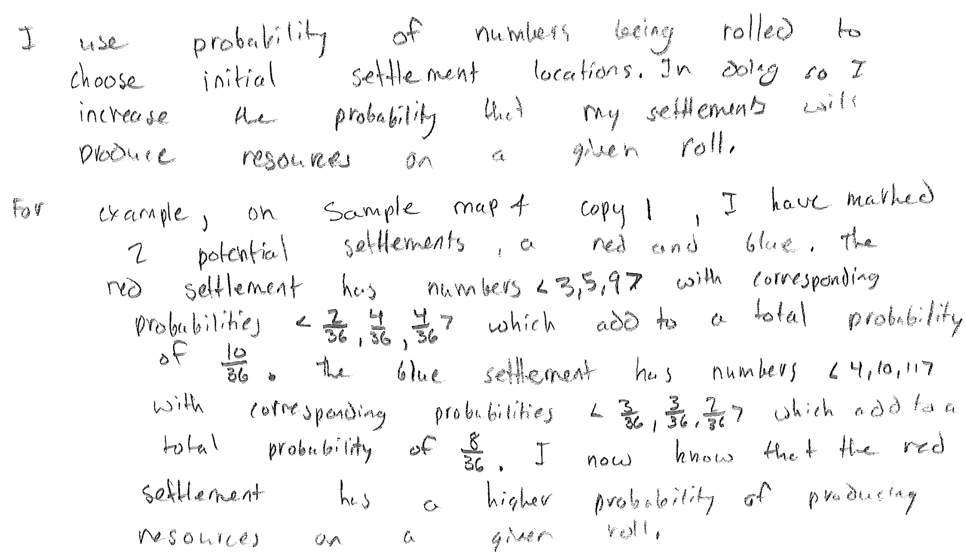


Figure 2: This student used probability to determine which resources will be rare and common

Raising the connection to the game board lead students to consider not only the resources' usefulness in general, but also the usefulness of each resource in terms of supply and demand on a given board. This question was encouraged by asking students to play on a certain board with one particularly rare resource. Students used the expected number of cards of each type produced on any given roll to determine "how many of each resource you would get if you rolled all 36 possible rolls each exactly once." Student work corresponding to this strategy can be found in Figure 2. This work involved considering the probability of rolling each number two through twelve with a pair of standard six-sided dice, as well as when adding probabilities is appropriate or not, and how to handle duplicate numbers, such as two grains tiles with 5's on them.


I use probability of numbers being rolled to choose initial settlement locations. In doing so I increase the probability that my settlements will produce resources on a given roll.

For example, on Sample map 4 copy 1, I have marked 2 potential settlements, a red and blue. The red settlement has numbers <3,5,9> with corresponding probabilities $<\frac{2}{36}, \frac{4}{36}, \frac{4}{36}>$ which add to a total probability of $\frac{10}{36}$. The blue settlement has numbers <4,10,11> with corresponding probabilities $<\frac{3}{36}, \frac{3}{36}, \frac{2}{36}>$ which add to a total probability of $\frac{8}{36}$. I now know that the red settlement has a higher probability of producing resources on a given roll.

Figure 3: This student compared two potential settlement locations based on expected number of cards produced.

Settlement Placement

Knowing which resources are most important in general or on a specific board is only one part of what informed student's mathematical justification of their decision process when choosing where to place initial settlements. The most basic intuition is that being on more resource producing tiles is better than being on fewer; however, when presented with this proposal, students were able to create examples where it could be statistically advantageous to be on a single very productive tile over a location with three low production tiles. After considering these ideas students began to evaluate each location based on the expected number of cards each location would produce, as shown in Figure 3. Students also considered resource rarity to determine which tiles were most important to settle near. This lead to an informal exploration of conditional probability and possible applications of reasoning aligned with Bayes formula.

Other Catan Applications

The student work provided above demonstrates the kinds of mathematical thinking that can be elicited by the use of a board game such as Catan in a course focused on exploring probability and expected value. Other topics explored included, the probability a certain number would be rolled before it is your turn to build again, the largest number of cards you could have in your hand with no more than three-of-a-kind and not be able to build anything, and various combinatorial considerations related to how to acquire the required ten victory points and win the game. These questions are not unlike others seen in probability classes but they are uniquely motivated because of the relevance to the game and game play. Recall that this work was completed with freshman non-math majors who were co-enrolled in a pre-calculus course. Employing similar techniques in a higher level probability or discrete math course intended for mathematics majors would likely lead to the same conclusions more quickly and provide further opportunities for extensions.

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Peer Tutors Attending to Student Mathematical Thinking

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Attending to and leveraging student thinking is known to be an effective teaching practice, but little research has been done to investigate the ways in which mathematics tutors attend to student thinking. This study will use the construct of decentering and Ader & Carlson's (2018) framework for analyzing teacher-student interactions to describe the ways in which tutors attend to student thinking in the moment. We will also provide examples of how written reflections and stimulated recall interviews can contribute to a tutor's ability to attend to student thinking.

Keywords: decentering, mathematics tutoring, observable behaviors, student thinking

Mathematics tutoring has been linked to improved grades (Byerley, Campbell & Rickard, 2018; Rickard & Mills, 2018) and increases in confidence and positive attitudes towards mathematics (Duranczyk, Goff, & Opitz, 2006). It is very common for universities in the United States to offer peer tutoring for entry-level mathematics courses such as Calculus (Bressoud, Mesa, & Rasmussen, 2015). While there is evidence that tutoring is effective, there have been few studies investigating exactly what happens in a peer tutoring session at the university level. Tutors working one-on-one with a student have the opportunity to make sense of and build upon students' thinking, however, it is unknown if tutors take advantage of this opportunity.

In the context of teaching, carefully listening to student thinking and using it to inform instructional decisions is considered an effective practice (NCTM, 2000). However, studies have shown that teachers are often not naturally able to implement the professional practice of attending to and building upon student thinking (Wallach & Even, 2005). Mathematics professors even have difficulty with making sense of their students' thinking despite their content expertise (Johnson & Larsen, 2012; Speer & Wagner, 2009). Thus, we cannot expect that it will be typical for tutors to make sense of and use student thinking in the moment without training in these skills. It has been found that dialogue patterns between tutors and students tends to focus on the steps of the solution, with the tutor choosing the path and asking the student to contribute to calculations (James & Burks, 2018; Van Lehn, Siler, Murray, Yamauchi, & Baggett, 2003).

In this paper, we address the research question: What is the nature of decentering among college math tutors who have no training in decentering? We present examples from tutoring interactions in which undergraduate mathematics tutors attend to student thinking at varying levels. We also give evidence that written reflections and one-on-one stimulated recall interviews can help tutors begin to consider mathematical thinking from the student's point of view.

Literature Review

Thompson (2000) describes an *unreflective interaction* as one in which the teacher is not attempting to set aside his/her own understanding to discern the mental actions that are driving the student's behaviors. In this instance he would say that the teacher is operating from a first-order model, projecting his/her own cognition onto the student. In contrast, a teacher who is *interacting reflectively* is actively trying to build a second-order model of their student's thinking by asking questions. The term *decentering*, first used by Piaget (1955), is used by Teuscher, Moore, and Carlson (2016) to refer to the teacher's action of trying to understand a student's perspective *in the moment*. Teachers who are motivated to decenter believe that the student has some viable set of meanings that contribute to his or her actions, even though the student may have an idiosyncratic understanding (Teuscher, et al., 2016). Thus the teacher actively asks questions and strives to make sense of the student's thinking without assuming that the student's thinking aligns with their own. Decentering has been used to analyze teacher decision making in the moment with Ader & Carlson (2018) and Teuscher et al. (2016) providing examples from authentic classroom interactions.

One way teachers have been trained to focus on student thinking is through discussions of video clips of their own teaching with their colleagues in which they craft questions that the teacher could have asked to probe student thinking. Participation in video clubs of this type has been shown to have an impact on teacher practice (Sherin & Van Es, 2009).

Much of the literature related to decentering has focused on classroom instruction (Ader & Carlson, 2018; Teuscher, Moore, & Carlson, 2016) and not one-on-one tutoring interactions. There are many ways in which tutoring differs from teaching. Undergraduate peer tutors are not content experts and have often not received training in pedagogy, but they have a skill set that is different than instructors. They have been successful students themselves, they know how the mathematics applied their subsequent courses, they can communicate informally with students, and they may be better able to sympathize with the concerns of students (McDonald & Mills, 2018). Since the context of teaching and tutoring are so different, it is necessary to examine what decentering could look like in the context of peer tutoring.

While the classroom is certainly an important part of student learning, much of the learning takes place when students are working through the homework, and interactions with tutors could be just as formative for students. Also, while university professors may be resistant to changing their classroom pedagogy, hourly paid undergraduate tutors can be trained to implement any approach that we deem beneficial to students. Thus, the impact of training tutors on student learning can be more immediate than attempting to change departmental attitudes towards progressive teaching methods.

Theoretical Framework

Decentering occurs in the mind of the teacher or tutor, and thus it is impossible to determine precisely whether or not a teacher or tutor is decentering. However, in order for a tutor to decenter and take on the perspective of another person, the tutor must do more than simply project their own cognition about the mathematics onto the student. Ader & Carlson (2018) focus their analysis on *observable behaviors* of the instructor as an indicator of decentering. We will briefly describe their four levels of interaction and associated observable behaviors.

Interactions in the first two levels of Ader and Carlson's (2018) model describe unreflective interactions when the teacher operates from a first-order model. In these interactions, the teacher does not attempt to understand the student's thinking, but assumes that the student's thinking is identical to his or her own. In a Level 1 interaction, the teacher does not pose questions aimed at understanding student thinking. Teachers engaged in a Level 2 interaction ask questions to reveal student thinking, but do not attempt to understand student thinking and rather guide the student to the teacher's own way of thinking. Levels 3 and 4 are reflective interactions in which the tutor asks questions to understand the student's thinking, taking on his or her perspective and building a second-order model to inform his or her instructional decisions. Level 3 interactions are characterized by the teacher "asking questions to reveal student thinking and then following up on student responses to perturb students in a way that extends their current ways of thinking," and attempts to move the student towards his/her way of thinking. In a Level 4 interaction, the teacher is focused on using and developing student's idiosyncratic ways of thinking by posing questions or giving explanations that are attentive to students' thinking.

Methods

The data were collected from a drop-in mathematics tutoring center in a Midwestern research institution. Tutors in this study are undergraduates who are trained to spend 5-10 minutes with each student and move around the room. The students that the tutors are working with are enrolled in a wide variety of classes from college algebra through differential equations. The tutoring center employs 40 tutors who work from 6-12 hours per week.

As part of their training, tutors were required to record a tutoring session, and they had a week to transcribe the session and respond to written reflection questions addressing the student participation level, student mathematical thinking, questioning, and the tutor's listening skills. They then scheduled a 10-15 minute interview with the researcher in which they watched the interaction together and the interviewer went through the session line by line asking questions such as "What do you think the student meant by that comment?" and "Why did you choose to ask that question?" and "What was your goal in giving that example?"

The written reflection and interview served two purposes. First, they gave the tutor a chance to elaborate on his or her in-the-moment decision making. This is helpful for training purposes because it allows the supervisor to better understand the tutor's methods. For this study, it has helped us to triangulate the data to build a better case for our classification of tutor moves. Second, it gives the tutor the opportunity to think critically about what they believe the student might have been thinking, and what the tutor could have done differently.

Several interactions were analyzed and heuristic cases which exemplify varying levels of tutor decentering were selected for presentation. During analysis, we examined the LiveScribe recording of the interaction and the transcript to look for observable behaviors of the tutors that indicated their level of decentering. We then read the written reflections and listened to the audiotaped interviews for further evidence of the tutors' attention to student thinking. We should note that in the same manner as Nardi, Jaworski, & Hegedus (2005), we are classifying episodes rather than tutors. Individual tutors can display varying levels of attending to student thinking even in the same 10-minute tutoring session, thus it is not feasible to label tutors according to their tendency to focus on student thinking.

Results

In the observation data that we collected, we have many examples of tutors leading the student through procedures without asking for the student to express his or her thinking. This is consistent with results in physics tutoring (VanLehn, et al., 2003) and undergraduate mathematics tutoring (James & Burks, 2018). Because we want to examine decentering in the context of tutoring, we focus on instances in which students were asked to explain their thinking.

Here we will present three naturalistic tutoring sessions illustrating varying levels of decentering in a drop-in tutoring environment. We will also give an example of how reflecting on a tutoring session can lead a tutor to think more deeply about student thinking and formulate questions that he or she could have asked.

Episode 1: Tutor Decentering In-The-Moment

In this episode, Abby was working with a student to compute the derivative of $f(x) = x^5(3-x)^6$. Abby had already asked the student if he knew how to take the derivative but the student did not suggest using the product rule, so Abby proceeded to walk him through it.

Abby: So the product rule is the first times the derivative of the second plus the second times the derivative of the first, so we take the first *(writes down* x^5*)* and we take the derivative of the second, so what's that derivative?
Student: ummm… six times three minus *x* to the fifth
Abby: Awesome. But soo… three minus *x* to the fifth. That is not just *x* to the fifth, right?
Student: Right.
Abby: So it has a function inside of it
Student: mhm
Abby: So that means you'll have to do that chain rule
Student: Oh okay, yeah
Abby: Okay so what's the derivative of three minus *x*.
Student: One
Abby: Ok, so… Why would you say one?
Student: Because three turns into zero and *x* the one turns into zero, so it's *x* to the zero, which is one.
Abby: Okay, so not quite, whenever you just have the equation $y = x$. What's that derivative?
Student: uhhh… one.
Abby: Right, so…. *y* prime is equal to one, but if you have – *x*, what would that turn into?
Student: Negative one
Abby: Right, so…
Student: Oh okay… It would be negative one.

When the student gave the unexpected answer that the derivative of $3-x$ is 1, Abby asked the student to explain his thinking further. She then followed up on the student's response to give examples that perturbed the student's thinking in a way that extended his current way of thinking. In her interview, Abby elaborated on her interpretation of the student's thinking.

Interviewer: And then the student is saying that the derivative of $3 - x$ is 1.
Abby: Mmm-hmm.
Interviewer: And then how did you... what were you thinking about that?
Abby: I'm like, there are so many ways that they could have gotten to 1, so I wanted to know how they got to 1.
Interviewer: So you really ask an open ended question there: "Explain why you said that?"
Abby: And they are like "because 3 turns into 0" that's right, and "*x* to the 1 turns into zero" so I assume that they mean *x* to the first, so they are trying to do the [power] rule, and they are like "that goes down, and so it's *x* to the zero and that's just 1." And so that's technically right, but…
Interviewer: They are missing the sign.
Abby: Yeah.

Interviewer: And so you ask a series of questions, or actually kind of give another example to lead them to that, so… Why did you choose to do that instead of just saying "Oh, you missed a negative?"
Abby: Because a lot of people had been not knowing, like they are like $3 - x$, I just don't, like so many people had been missing that negative because I had done this problem like three times before already. And so, I'm like, "well, if y is x, then…" So, it makes a lasting impression.

Abby asked the student to explain his thinking because she was genuinely curious how he "got to 1." When she listened to his thinking, she interpreted his explanation as a correct application of the power rule and deduced that he is just missing the negative sign in the derivative of $-x$. We can see that in this instance, Abby has the natural inclination that leading a student through a series of examples to perturb his thinking will make more of a "lasting impression" on him than just telling him that he missed the negative sign.

We label this interaction as a Level 3 in terms of Abby's decentering actions because she asked a question to reveal the student's thinking and then followed up with two simpler examples designed to perturb the student in a way that extended his current way of thinking.

Episode 2: No Evidence of Tutor Decentering In-The-Moment

Bernard was working with a student to solve a quadratic equation. We can see that the student explained his first step, but Bernard lead the student in a different direction.

Bernard: Okay so the problem is $3x^2 - 2x - 5 = 0$. So what we tried before was…
Student: Multiplying -5 and 3 to get -15 and getting the factors of that to get it.
Bernard: Right, so that's kind of along the right track. So we want to work on the rational roots theorem, which is we take the factors of the last term which in this case is…
Student: 1 and 5
Bernard: 1 and 5. So I'll put that on top here. So 5 and 1, and we'll put that over the factors of the first term which is...
Student: 3 and 1.
Bernard: 3 and 1 right. And so it could also be plus or minus any one of these values so what we could have is +1, +5, +5/3 and +1/3 does that make sense?

Although Bernard prompted the student to outline his strategy, he did not ask the student to elaborate on his thinking, but evaluated the student's response in light of his own way of thinking. He then posed questions that focused on procedures requiring little thinking on the part of the student. It is unclear what Bernard understood about the student's strategy and what he meant by "you are on the right track." In the interview, Bernard was asked to explain what he thought the student was thinking.

Bernard: So, he, I think like in the very beginning he tried multiplying the negative five and the three together and finding the factors of that.
Interviewer: So, what do you think he meant by that?
Bernard: I think, like, so I mean, that's kind of, sort of... meh... At least he was realizing that he needed something from the last term and something from the first term.
Interviewer: Right.

Bernard: I can't remember if there's a way... like another theorem or something where you do that, but I don't remember what...
Interviewer: Yeah. So, you're feeling like, he knew he needed something from that negative five and something from the three, but he was multiplying instead of dividing?
Bernard: Yeah. Yeah.

As experienced mathematics instructors, we realize that the student could be attempting to transform $3x^2 - 2x - 5$ into $3x^2 + ax + bx - 5$, where *a* and *b* are factors of -15 so that he could factor by grouping. Since Bernard did not probe the student's thinking, it is unclear whether or not the student was attempting to use this strategy. It seems that Bernard was not aware of this method and assumed that the student was incorrectly applying the rational roots theorem, although he acknowledged in the interview that the student may have been using a theorem that he didn't remember. This interaction is categorized as Level 1 because when the student gave a response that did not match with Bernard's own method, Bernard did not ask the student questions to understand the mental actions driving the student's behaviors.

Towards the end of his interview, Bernard said, "Yeah. It made me realize that I do a lot of the talking, and it's not as interactive as I'd like it to be. I still ask questions that make sure they understand what's going on, but maybe having them reproduce what I'm writing or having them write down what I'm saying and see if that maybe clicks with them." This gives further evidence that for this problem Bernard was focused on helping the student to adopt his own way of thinking rather than understanding and building upon the student's thinking.

Episode 3: Tutor Develops Awareness of Decentering

The student was attempting to find the intersection of the lines $r_1(t) =< 0,1,1 > +t < 1,1,2 >$ and $r_2(s) =< 2,0,3 > +s < 1,4,4 >$. The student had constructed six equations: $x = t$, $y = 1+t$, $z = 1+2t$, $x = 2+s$, $y = 4s$, $z = 3+4s$. Emma led the student to set the *x*, *y*, and *z* equations equal to one another. Then, the following interaction occurred:

Student: So does it want it in terms of *t* then or *s* or what?
Emma: So first let's, um, set them equal to each other and find what our *t* and s values are.
[... tutor and student set up the equations and solve them together]
Student: So subtract an *s*, and it's 3*s* = 3? And *s* = 1?
Emma: Exactly.
Student: So, and then I just use that to find *t*, and get *t* = 3?
Emma: Uh huh. So using that first equation
Student: So is that the point then? That intersects?
Emma: Um.
Student: Or is that, or it's in three dimensions, so…
Emma: So, uh, we, you found your *s* and *t* values, so, if we are looking at where they intersect and where they equal each other, we are looking at a point on both $r_1(t)$ and $r_2(s)$, so that means that if we plug in *t* for $r_1(t)$ or *s* for $r_2(s)$, we can pick which one, uh, we should be getting the same answer. So you just pick one of the variables and plug it back in.

This in-the-moment interaction is classified as Level 2. During the tutoring session, Emma did not ask the student to elaborate on his thinking, but continued to ask questions to lead him through the procedure of the problem. She showed interest in the student's thinking to the extent that it revealed the student's misconception, and then she attempted to guide the student to her own way of doing the problem.

> *Emma:* A lot of what he's been doing before is just solving for the variable, and the variable has been the answer, and so maybe here he was thinking, "the variable, is that the answer? But then those are different numbers, but that doesn't really make sense for an intersection... " I think he was just kind of confused on that.
> *[...]*
> *Interviewer:* So talk about what the student might be thinking there.
> *Emma:* So he's found the s and t variables, and I think he's thinking it's kind of like an x and a y, you know, two dimensional, but then he realizes that we're in three dimensions because we have x, y, and z, and so he's trying to figure out how to turn $s = 1$ and $t = 3$ into an answer.

Emma verified that she thought that the student initially thought that the s and t values were the solution, and then the student expressed cognitive dissonance when he realized that he needed a three dimensional answer. Whether she had this view of the student's thinking in the moment is hard to say, but in the written reflection, Emma said, "I could have asked what t and s are used for and why we're trying to find them." This question would be useful for developing a second order model of the student's thinking, and if she had asked it in the moment, the interaction may have been classified as Level 3. Thus, we can see that the process of going through the reflection afforded Emma the opportunity to construct a question that would further reveal the student's mental actions.

Discussion

The main contribution of this paper is the application of decentering to the context of undergraduate peer tutoring. We have given examples of tutoring interactions that display varying levels of tutor decentering in the moment and we have also shown an example of how the process of writing a reflection and re-watching the recording of their interaction can aid tutors in formulating questions to draw out student thinking. Our analysis of decentering was based on both the observable behaviors and the tutor's reflections. Many of the tutoring interactions that were analyzed for this study were Level 1, but we have presented examples of un-trained tutors decentering in the moment to varying degrees.

A limitation to this study is that the tutors self-selected the session that they wanted to transcribe, so the sessions may not be reflective of their typical tutoring. The stimulated recall interviews were not always consistent, and so some of the tutors may have been prompted more than others and the interviewer may have asked the student to comment on a moment that the student may not have spontaneously commented on. Another limitation includes our inability to determine if the questions constructed by the tutors in their reflections will enable them to decenter more frequently. The interactions that we recorded will be used to refine our reflection questions and design tutor training programs that incorporate real tutoring interactions. We can also design studies to investigate whether a focus on decentering in tutor training can improve tutors' ability to understand and leverage student thinking.

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Two Birds, One Instruction Type: The Relation Between Students' Affective Learning Gains and Content Assessment Scores

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The learning of mathematics is a complex phenomenon that is influenced by both cognitive and affective factors. Little is known about the relationship between students' affective and cognitive outcomes, as much work focuses on one or the other, but not the intersection of the two. Therefore, this study examines the relationship between students' affective learning gains as reported on the SALG survey and their content assessment scores for differential equations courses. The goal was to determine if there was a relationship and then investigate if this relationship held for male and female students, as well as those in inquiry-oriented classes. Mixed linear models were used to examine this relationship, while simultaneously taking into account the nesting of students within instructors. Results showed there are some significant relationships between affective learning gains and content assessment scores, but these relationships are not consistent across sub-groups by gender nor instruction type.

Keywords: Student Affective Learning Gains, Content Assessment, Differential Equations

In order to fully understand the complexity of mathematics learning, cognitive and affective factors must be explored. Cognitive outcomes relate to students' mental actions and the application of those actions. Some researchers define these outcomes in terms of "remembering, understanding, applying, analyzing, evaluating, and creating" (Burn & Mesa, 2015, p. 47). With this stance, researchers look at student performance, achievement, grades, and even in a broader sense, knowledge and skills (Dochy et al., 2003; Freeman et al., 2014; Kuster et al., 2017; Laursen et al., 2014; Lazonder & Harmsen, 2016). Only considering student performance can be misleading, as it does not fully capture students' abilities nor how students are participating in class. For example, research has shown that there is a difference in male and female performance in mathematics, with males having higher performance, however, Fennema and Sherman (1977) contribute this difference to low participation from females, as mathematics is typically stereotyped as a masculine subject. By only considering student performance, researchers would miss the intersection of cognitive and affective outcomes that can help explain the differences between males and females in mathematics.

In contrast to cognitive outcomes, affective outcomes consider students' internal factors. These are typically defined in terms of "beliefs, attitudes, and perceptions" (Burn & Mesa, 2015, p. 97). Although much research has been conducted on students' beliefs and attitudes toward mathematics, researchers are challenged with ways to infer beliefs and attitudes from student behavior (Leder & Forgasz, 2002). Because of the synonymous nature of words such as attitude, perception, value, and belief, it is difficult to define a belief (Leder & Forgasz, 2002), making this investigation even more difficult. Nevertheless, affective outcomes can include such things as confidence, enjoyment, persistence, interest, and even an approach to learning (Laursen et al., 2014; Sonnert & Sadler, 2015). These factors are more difficult to identify and investigate as they are personal and not easily observable. While, research has shown that there is a difference between male and female affective outcomes specific to mathematics (Chouinard & Roy, 2008), there is more work to be done to better understand these outcomes. Instead of obtaining a snapshot of what is happening in classes by focusing exclusively on cognitive outcomes, researchers are able to paint a clearer picture by matching cognitive outcomes with affective outcomes. Therefore, the purpose of this paper is to investigate student's affective learning gains (ALG) in relation to their content assessment (CA) scores.

Literature Review

As previously mentioned, it is often difficult for researchers to identify beliefs, as there is no one definition that captures their essence. Philipp (2007) states that "affect is comprised of *emotions, attitudes,* and *beliefs*" while beliefs are also "more cognitive than emptions and attitudes" (p. 2.59). Leder and Forgasz (2002) also deduce that beliefs and attitudes are "intrinsically related and that beliefs and attitudes have cognitive, affective, and behavioral components" (p. 96). Due to this overlap in cognitive and affective components concerning beliefs, it is important to note that this can create room for biases, as well as the use of multiple methods and frameworks, and can result in contradictory findings.

Throughout psychological literature, there is a focus on affective issues in relation to mathematics; demonstrating an expansive range of beliefs, which are measured in a variety of ways (Leder & Forgasz, 2002). Even though the methods and definitions vary, many studies show there is a difference in males' and females' motivation when it comes to mathematics (e.g., Chouinard & Roy, 2008). When males and females have similar levels of achievement, females demonstrate lower competence beliefs and more anxiety (Eccles et al., 1985; Kloosterman, 1990; Seegers & Boekaerts, 1996; Stipek & Gralinski, 1991). Males contribute their success to ability and failure to bad luck or lack of help (Hackett & Betz, 1992; Randhawa et al., 1993), which is in stark contrast to females who perceive their success comes from being determined, receiving help from others, or being provided with simple tasks (Stipek & Gralinski, 1991). This highlights the differences in males' and females' affective outcomes in mathematics.

Over time, there is a substantial transformation in student attitudes toward studying mathematics (Eccles et al., 1985; Fredricks & Eccles, 2002; Jacobs et al., 2002; Ma & Cartwright, 2003). Many high school students are more pessimistic when it comes to their ability to succeed in mathematics and they also place a lower value on their feelings toward mathematics (Chouinard & Roy, 2008). In their work, Chouinard and Roy (2008) examined high school students' attitudes towards mathematics. They specifically looked at whether their attitudes change over time, if changes are related to grade level, and if there are gender differences. Overall, results showed there was a regular decline in mathematics motivation throughout high school, especially between 9th and 11th grade, where the gradual drop represented a decrease between and within grade levels. These results confirm a steady decrease in students' attitudes towards the utility of mathematics for male and female students, and additionally indicate a more significant decrease for males than females.

Although Chouinard and Roy (2008) found a greater decrease in males' attitudes toward mathematics than females', other studies indicate a decline in positive attitudes toward mathematics has more of an effect on females than on males (e.g., Eccles et al., 1985; Fennema & Sherman, 1977). Additionally, researchers have found there are differences in confidence and anxiety between females and males, with females having lower levels than males (Leyva, 2017; Lubienski & Ganley, 2017). Not only are there differences between genders in terms of affective outcomes, research has shown that there are also differences in cognitive outcomes, based on the type of instruction provided. For example, Laursen et al. (2014) found that students in inquiry-based learning (IBL) classes had higher cognitive and affective outcomes than students not in IBL classes. Results from their study also showed that that students in IBL classes reported higher cognitive gains than those in non-IBL classes. Based on the self-report, IBL students had a better understanding and could think more deeply about the mathematics than their non-IBL peers. Although this research base has greatly contributed to the body of knowledge on student attitudes, beliefs, cognitive and affective behavior, future research is necessary to expand these findings. Therefore, we accept this charge and explore the relationship between students' cognitive and affective outcomes in inquiry-oriented (IO) classes, a more specific branch of IBL (Kuster et al., 2017). Specifically, we investigate the following questions:

1. What is the relationship between students' ALG and their CA scores?
2. Is this relationship the same for males and females?
3. Is this relationship the same for students in IO and non-IO classes?

Methods

This quantitative study uses a relational design to investigate the relationship between students' ALG and their corresponding CA scores, using data from related projects designed to support instructors interested in implementing IO instructional materials. The affective data used for this study stems from the Student Assessment of their Learning Gains (SALG) survey, developed by Laursen, Hassi, Kogan, Hunter, & Weston (2011) to help faculty gather insights about their instructional practices. The CA data comes from Hall, Keene, and Fortune's (2016) work on creating a common written assessment to better understand student learning in differential equations (DE) courses. To explore the relationship between students' reported SALG survey scores and their CA scores, we constructed a linear mixed model. Finally, we investigated whether the relationships identified for all students held when the groups were disaggregated by gender and IO instruction.

Participants

A total of 23 instructors were involved in this study. Of those 23 instructors, 16 were instructors who engaged in professional development focused on using IO materials. These 16 instructors were then asked to identify another instructor who was not participating in the IO project; these non-IO instructors were then recruited to participate in this study. Six out of seven comparison instructors came from the same institution, while the last was from a different university of comparable size located in the same city. The comparison instructors taught DE either in the same semester or within one year that their mapped IO instructor did. The instructors involved in the IO project define our IO sample; the non-IO instructors define our non-IO sample.

Students from these 23 instructors were then recruited to participate, resulting in 448 undergraduate students enrolled in DE courses across the nation. Of those 448 students, 296 (66.1%) of those students were taught by IO instructors, while 152 (33.9%) of those students were taught by non-IO instructors. In addition, from the students who reported gender, 225 (66%) students identified as male and 101 (29.6%) identified as female. Out of those students, 151 (33.7%) were males in IO classes, 68 (15.2%) were females in IO classes, 74 (16.5%) were males not in IO classes, and 33 (7.4%) were females not in IO classes.

Instrument and Data Collection

Affective data stems from the SALG-M survey developed by Laursen et al. (2011). The SALG-M survey is a modified version of the SALG survey, more specific to mathematics instruction. Laursen et al. (2011) modified the original survey to more effectively measure students' learning gains in mathematics classrooms. The SALG-M survey is broken four sections that measuring students' experiences during the course and two sections that measure their learning gains. The term learning gains encompasses cognitive, affective, and social gains as one holistic measure. Therefore, after conducting a factor analysis, some instrument items did not load to those three factors, and were not included in the latest revision, resulting in the modified SALG survey (Laursen et al., 2011). For the purposes of this study, we will only be looking at the 13 learning gains questions from the modified SALG survey that focus on students' cognitive, affective, and social gains. These items were rated on 6-point Likert scale indicated by 1 no gain, 2 little gain, 3 moderate gain, 4 good gain, 5 great gain, and 6 not applicable. The CA scores derive from a DE common assessment developed by Hall et al. (2016) to help support IO instructors implement IODE curriculum[1]. This common assessment consisted of 15 multiple choice items designed to evaluate students' conceptual understanding of DE. These questions focused on the following concepts: (1) solving first order differential equations analytically, graphically, and numerically (2) linear systems of differential equations, and (3) second order differential equations (Hall et al., 2016). This test was given to both IO and non-IO instructors to use at the end of semester.

Data Analysis

To begin, we compared students' mean CA scores by gender and instructional treatment using t-tests. Then, we examined the relationship between students' CA scores and their reported scores on the 13 ALG items from the modified SALG survey. Students' scores on the ALG items were centered, and scores of 6 (not applicable) were removed from analysis. This was done using a mixed linear model, which accounts for the effect of classroom instruction factors, and allows for the prediction of students' CA scores based on their Likert scores on the ALG items. Mixed linear models were also used to assess these relationships by gender and instructional treatment.

Results

Initial descriptive statistics indicate DE students' mean CA score to be 53.59 (SD=16.66). The mean CA score for males is 52.67 (SD=16.83) and the mean for females is 53.51 (SD=15.56). Despite the mean for females being slightly greater than that of males, there is no significant difference in the CA scores of males and females ($t(324)=-.43$, $p=.67$). In a comparison of IO (M=54.94, SD=17.11) and non-IO (M=50.97, SD=15.48) students, IO students were found to score significantly better than their non-IO peers ($t(446)=2.40$, $p=.02$).

Results of mixed linear models indicate that ability items 1, 2, 3, 6, 7, 8, 9, and 13 are significantly related to students' CA scores (Table 1). For example, ability item 1 asks students if they feel confident that they can do math. Students who strongly disagree with this statement (Likert score 1) are predicted to have a CA score of 47.07 ($t(74.61)=17.24$, $p<.001$). Also, for each one point increase in their Likert score on ability item 1, students are predicted to have a 2.51 point increase in CA score ($t(322.81)=3.18$, $p=0.002$). Accordingly, students who strongly agree that they feel confident that they can do math, are predicted to have a CA score of 57.11. Similar interpretations hold for ability items 2, 3, 6, 7, 8, 9, and 13. Ability items 4, 5, 10, 11, and 12, however, were not found to be significantly related to CA scores, as indicated by the results of the mixed linear models (Table 2). Therefore, increasing Likert scores on these items are not related to changes in CA scores.

[1] The CA asked students to report their gender identity as male, female, other, or prefer not to answer; accordingly, we use the language of male and female throughout this study to match their reported gender identity.

Table 1. Significant Results of Mixed Linear Models, by Ability Items

	t	Slope	Predicted Score if Student Reports 1 (Intercept)	5
Ability 1: I feel confident I can do math	3.18(322.81)**	2.51	47.07	57.11
Ability 2: Comfort working with complex ideas	2.70(318.34)**	2.30	47.85	57.05
Ability 3: Development of a positive attitude about learning math	2.86(314.17)**	2.06	48.37	56.16
Ability 6: Appreciation of mathematical thinking	2.88(319.88)**	2.35	47.50	58.90
Ability 7: Comfort in communicating about math	3.42(311.50)**	2.75	46.54	57.54
Ability 8: Confident you will remember what you learned in class	2.94(305.66)**	2.34	48.29	57.65
Ability 9: Persistence in solving problems	2.94(323.51)**	2.40	47.18	56.78
Ability 13: Ability to stretch your own math capacity	2.06(317.58)*	1.77	48.86	55.94

Note: Slopes indicate the predicted increase in CA score per one point increase in Likert score on the corresponding ability item.
*p<.05, **p<.01

Table 2. Non-significant Results of Mixed Models, by Ability Items

	t	Slope	Predicted Score if Student Reports 1 (Intercept)	5
Ability 4: Ability to work on your own	1.94(324.00)	0	49.63	49.63
Ability 5: Ability to organize your work and time	1.59(320.40)	0	50.97	50.97
Ability 10: Willingness to seek help from others	.02(317.41)	0	53.96	53.96
Ability 11: Ability to work well with others	-.10(321.55)	0	54.34	54.34
Ability 12: Appreciation of different perspectives	.38(317.19)	0	53.11	53.11

Note: As the relationships in this table between ability items and CA scores are not significant, students are predicted to have the same CA score regardless of changes in their Likert ability scores.

When separated by gender, increases in female students' Likert scores on all of the ability items with significant relationships for both genders continued to predict increases in CA score (Table 3). However, for males, the only significant differences in CA score based on ability items were for items 6 and 7. Thus, female students who strongly disagree with ability item 1 are predicted to have a CA score of 40.08, but for every one point increase in their Likert response to ability item 1, female students are predicted to have a 5.08 point increase in their CA score. This is not true for males, for whom a strong disagreement on ability item 1 corresponds to the prediction of a CA score of 49.67, but increasing Likert scores on ability item 1 are not predicted for male students. Thus, for males, increasing levels of confidence (ability 1), comfort with complex ideas (ability 2), positive attitudes toward math (ability 3), confidence in remembering ideas from class (ability 8), persistence (ability 9), and stretching one's mathematical activity (ability 13) are not related to higher CA scores; for females, these increases are related to higher CA scores.

Table 3. Results of Mixed Linear Models, by Ability Items and Gender

	Male *t*(df)	Male Intercept	Male Slope	Female *t*(df)	Female Intercept	Female Slope
Ability 1	1.60(219.91)	49.67	0	4.22(98.50)***	40.08	5.08
Ability 2	1.48(217.32)	49.84	0	3.47(98.42)**	41.67	4.59
Ability 3	1.46(214.22)	50.36	0	3.91(96.94)***	42.22	4.31
Ability 6	2.05(217.41)*	48.50	2.07	2.56(98.88)*	44.66	3.31
Ability 7	2.63(214.10)**	47.31	2.55	3.38(93.33)**	41.01	4.73
Ability 8	1.60(213.47)	50.40	0	4.36(95.21)***	39.43	5.98
Ability 9	1.97(218.04)	48.38	0	3.09(96.87)**	41.88	4.24

Ability 13	.90(216.80)	51.38	0	3.66(95.56)***	40.25	4.73

*p<.05, **p<.01, ***p<.001

Note: Intercepts indicate the predicted CA scores for students who report a 1 (strongly disagree) on the corresponding ability item. Slopes indicate the increase in CA score per one point increase in Likert score on the corresponding ability item.

Another result is that for males, all of the predicted intercepts are higher than those for females (e.g., ability item 1, 49.67 > 40.08). However, as the scores for males students are not predicted to increase significantly based on increasing Likert scores for ability item 1, it is predicted that male students with a Likert score of 5 on ability item 1 will also be 49.67. In contrast, the CA scores for female students are predicted to increase by 5.08 points per one point increase in their Likert score on ability item 1, giving female students who strongly agree with ability item 1 a predicted CA score of 60.4. Thus, when comparing the predicted CA scores of students who strongly disagreed with ability item 1, males outscores females; however, when comparing the predicted CA scores of students who strongly agree with ability item 1, females outscores males by more than 10 points. Similar trends are predicted on all of the ability items in Table 3 (1, 2, 3, 6, 7, 8, 9, and 13).

Disaggregating the data by instructional method results in similar trends (Table 4). Students in IO classes are predicted to have increased CA scores based on increasing Likert scores on all of the ability items with significant relationships for both types of instruction. Students in non-IO classes are only predicted to have increased CA scores related to increasing Likert scores for ability item 7. For example, students in IO classes who strongly disagree with ability item 1 are predicted to earn a CA score of 47.66, but for each point increase in their Likert score, they are predicted to have an increase in their CA score of 3.15 points. Students in non-IO classes are predicted to have a lower CA score regardless of their Likert score on ability item 1. Similar interpretations hold for ability items 3, 6, 7, 8, 9, and 13. Non-IO students who strongly disagree with ability item 2 are predicted to score higher than IO students who strongly disagree, but with a one point Likert increase, IO students are predicted to outscore non-IO students. Even on ability item 7, for which the relationship between Likert score and CA score was significant for non-IO students, non-IO students are predicted to have lower CA scores regardless of their Likert score, in comparison to their IO peers.

Table 4. Results of Mixed Linear Models, by Ability Items and Instructional Treatment

	IO			Non-IO		
	t(df)	Intercept	Slope	t(df)	Intercept	Slope
Ability 1	3.23(190.49)**	47.66	3.15	1.41(103.31)	44.01	0
Ability 2	3.29(167.65)**	47.20	3.38	.40(104.50)	48.18	0
Ability 3	3.05(158.78)**	48.61	2.76	1.13(104.67)	46.10	0
Ability 6	3.25(158.20)**	47.23	3.21	.85(102.61)	46.47	0
Ability 7	3.11(143.80)**	47.73	3.02	2.23(103.03)*	41.85	2.84
Ability 8	3.26(111.23)**	48.07	3.20	.94(103.10)	47.05	0
Ability 9	2.83(201.15)**	47.70	2.90	1.23(103.78)	45.29	0
Ability 13	2.47(178.69)*	48.77	2.54	.34(103.88)	48.18	0

*p<.05, **p<.01, ***p<.001

Discussion and Conclusions

When all students' CA scores and ability items were considered together, regardless of gender or instructional treatment, ability items 4, 5, 10, 11, and 12 were not significantly related to students' CA scores. Interestingly, these items are not related specifically to mathematical learning, but rather, to other metacognitive skills such as studying, time management, and group work. The other ability items each had a specific connection to mathematics, ideas learned in class or complex ideas, and problem solving, which are more directly tied to mathematics. Thus, while improving metacognitive skills is important, the improvement of such skills was not related to students' CA scores while increases in affective items directly related to mathematics, mathematical ideas, and problem solving were positively related to students' CA scores.

Considering gender, females and males did not score significantly differently on the CA. However, when the data was disaggregated to show differences in the relationships between students' affective items and their CA scores, it was demonstrated that affective increases for female students are predictive of higher CA scores, whereas they generally are not for male students. Thus, the inclusion of instructional practices that support affective gains such as those identified by the ability items support females in increasing their math achievement, as gauged by the CA. This supports previous research indicating that females tend to report lower affective levels, but have similar achievement to males (Eccles et al., 1985; Kloosterman, 1990; Seegers & Boekaerts, 1996; Stipek & Gralinski, 1991), and adds to previous literature by indicating that fostering female students' affective gains may foster higher achievement. Interestingly, while previous literature indicates that males tend to claim success comes from ability and confidence (Hackett & Betz, 1992; Randhawa et al., 1993) and that females believe their success comes from being determined and working with others (Stipek & Galinski, 1991), the results of this study show reported increases in confidence and determination are related to higher achievement more so for females, and increasing the ability to work well with others was not related to higher achievement for students.

Also, females have been previously shown to be more concerned with abilities, confidence, comfort, and persistence (Eccles et al., 1985; Fredricks & Eccles, 2002; Jacobs et al., 2002; Ma & Cartwright, 2003). The results of this study compliment these existing findings; perhaps female students' concern with these affective outcomes stems from their understanding that they tend to perform better academically when their confidence and persistence are supported.

Finally, we found that students in IO classes had statistically significantly higher CA scores than those in non-IO classes. Moreover, increases in IO students' ability item scores (1, 2, 3, 6, 7, 8, 9, and 13) were related to increases in their CA scores; for non-IO students, this was only true for ability item 7. Thus, as with females, increases in affective outcomes are related to increases in CA scores for IO students. This suggests that in an IO classroom, fostering students' affective growth is linked to higher achievement. This supports Laursen et al.'s (2014) findings, which suggest that IBL, or in this case IO, is beneficial for increasing students' affective and cognitive gains.

Taken together, these findings suggest that in IO classrooms, increases in affective and cognitive levels are related, whereas in non-IO classrooms, they generally are not. The implication is that IO instruction simultaneously addresses both the affective and cognitive needs of students, thereby metaphorically killing two birds (affective and cognitive issues) with one instructional stone. Conversely, in non-IO classrooms, cognitive and affective gains were disconnected. This study did not seek to make claims of causality, but rather, to offer one insight into the relationships between the increases, both cognitive and affective, that students make in mathematics classrooms. More research should address students' affective outcomes in relation to cognitive outcomes, particularly studying if a predictive relationship exists; this will provide a more holistic view of the field of student learning in mathematics.

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Mathematicians' Metaphors for Describing Mathematical Practice

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In the literature on metaphor, researchers have pointed out the importance of metaphor as a tool for sense-making and have demonstrated the impact of metaphor use on cognition. In mathematics in particular, metaphor has been shown to be a valuable tool for making sense of and reasoning with mathematics. To our knowledge, there has been no research on the metaphors that professors use when communicating the nature of mathematical practice to students in advanced mathematics lectures. In this paper, we present a particular metaphor, Learning Mathematics is a Journey, that we found in a corpus of 11 advanced mathematics lectures. We describe this metaphor we found and offer some speculative analysis regarding the implications of this metaphor.

Keywords: Metaphors, Mathematical Practice, Advanced Mathematics

A primary goal of contemporary mathematics instruction is to engage students in authentic mathematical activity. (e.g., Ball & Bass, 2000; Lampert, 1990; Rasmussen et al., 2005; Schoenfeld, 1992; Sfard, 1998). To achieve this objective, mathematics educators must grapple with the fundamental question of what it is like to do mathematics. The broad purpose of this paper is to shed light on this issue by exploring how mathematicians describe mathematical activity in their own words. In particular, we analyze the metaphors that mathematicians use when teaching advanced courses for university mathematics students.

We focus on the metaphors that mathematicians use in their lectures for two reasons. First, we know that mathematicians say an important goal of mathematics lectures is to help students understand what doing mathematics is like (e.g., Krantz, 2015; Pritchard, 2010; Rodd, 2003) and previous studies have illustrated some of the ways that professors have described doing mathematics to their students (e.g., Artemeva & Fox, 2011; Fukawa-Connelly, 2012). If we want to study how mathematicians describe their craft in a naturalistic setting, mathematics lectures are a suitable place to look. As we will document in this paper, a particularly common way that mathematicians convey what it is like to do mathematics is by describing mathematical activity metaphorically.

Second, students' perceptions of what mathematics is like are shaped significantly by their experiences in their mathematics classes. Psychological research has demonstrated that the metaphors used to describe a topic exert a powerful influence on how individuals think about that topic (e.g., Thibodeau & Boroditsky, 2011, 2013). Consequently, if we want to study students' perceptions of mathematics, it is important to analyze both the messages that mathematicians convey in their lectures and the ways that students interpret those messages. The analysis in this paper contributes to the first goal. By analyzing the metaphors that mathematicians use in their lectures, we will have a greater understanding of how mathematicians inform students about what it is like to do mathematics.

Theoretical Perspective: Metaphors Structuring Thought

Numerous cognitive scientists and linguists have noted that conceptual metaphors are ubiquitous in the ways humans use natural language, with many scholars claiming that the metaphors that we use structures our thoughts. (e.g., Lakoff & Johnson, 2003; Nuñez, Edwards,

& Matos, 1999; Nuñez, 1998; Reddy, 1979; Thibodeau & Boroditsky, 2011, 2013). In this paper, we follow Nuñez (1998) in defining *conceptual metaphors* (hereafter referred to simply as *metaphors*) as "cross domain 'mappings' that project the inferential structure of a *source domain* onto a *target domain"* (p. 87). As a well-known example, consider Lakoff and Johnson's (2003) claim that we metaphorically view Argument as War. In this example, "war" is the source domain of the metaphor, whereas "argument" is the target domain. This can be seen by many examples that commonly occur in our speech, such as "Your claims are *indefensible,*" or "He *attacked every weak point* in my argument" (Lakoff & Johnson, 2003; p. 4). Similarly, Lakoff and Johnson contended that we also metaphorically view Argument as a Journey, such as when we say "We have *set out* to prove that bats are birds," or "We have *arrived at* a disturbing conclusion" (Lakoff & Johnson, 2003; p. 90).

Lakoff and Johnson use these illustrations to highlight three points that will be relevant to this paper. First, metaphor usage is common when we discuss an abstract concept such as argumentation. Second, we can use different metaphors to discuss the same concept and these different metaphors highlight different facets of this concept. For instance, the Argument as War metaphor highlights the combative nature of argumentation, in which arguing is an adversarial activity with an attacker, a defender, a victor, a loser, and so on. The Argument as Journey highlights the sequential and rhetorical aspect of argument in which the individual presenting an argument is trying to lead her audience to a desired conclusion. The third point is the metaphors that we use to describe an abstract concept like argumentation structure our thought about this concept and significantly influence our reasoning about this concept. This third point is the most contentious point (c.f., Glucksberg & McClone, 1999) and we elaborate on this point below.

A central claim advanced by George Lakoff and other linguists is that our use of metaphors is not merely a rhetorical flourish on the part of the speaker. When we use a metaphor, we use our knowledge of the source domain in question to make novel inferences about the target domain. We further give primacy to the aspects of the target domain that are highlighted by the metaphor and less attention to the aspects of the target domain that are ignored by the metaphor.

Empirical support for this position is provided by a series of psychological studies conducted by Thibodeau and Boroditsky (2011, 2013). Thibodeau and Boroditsky noted that we frequently use metaphors when we speak of crime; we sometimes speak of Crime as a Beast in which criminals *prey* on victims and police *track* criminals, *hunt* them down, and *catch* them. We also sometimes describe Crime as a Virus where crime is an *epidemic* that can *plague* a city or *infect* a community. In a series of randomized controlled experiments, Thibodeau and Boroditsky (2011, 2013) compared the responses of participants exposed to different metaphors for crime when the participants were asked to propose measures to reduce crime. Participants who saw the metaphor that crime was a beast uniformly proposed developing better measures to capture and punish criminals. Participants who saw the metaphor that crime was a virus focused more on understanding the social causes of crime and educating the community on how to prevent crime. Thibodeau and Boroditsky concluded that metaphorical usage has "real consequences for how people reason about complex social problems like crime" (p. 1).

The key point to draw from this for the purposes of this paper is that we should not suppose the metaphors that mathematics professors use in their lectures are inert. They are not merely fancy ways of talking, but can say a lot about how mathematicians view their discipline. Further, Thibodeau and Boroditsky's (2011, 2013) studies suggest that these metaphors may influence how students subsequently engage in advanced mathematics.

Existing Literature on Mathematical Metaphors

Research on metaphor usage in mathematics can be divided into two broad areas of studies. First, some scholars have examined how mathematicians and students use metaphors to understand *mathematical concepts*. For instance, Lakoff and Nuñez have explored the use of metaphors in mathematical language in an attempt to understand mathematicians' cognitive underpinnings behind advanced and abstract mathematical ideas (e.g., Lakoff & Nuñez, 2000; Nuñez, Edwards, & Matos, 1999). As an example, mathematicians commonly use the preposition "in" to denote set membership. Lakoff and Nuñez (2000) argued that this use of language suggests that mathematicians metaphorically view sets as containers that are filled with objects. Sfard (1994) and Sinclair and Tabaghi's (2010) interview studies with mathematicians provide empirical support for Lakoff and Nuñez's (2000) theoretical claims. Mathematicians use metaphors as a powerful tool for doing, understanding, and communicating mathematical ideas.

Other researchers have examined how students understand various mathematical topics and concepts through metaphors (e.g. Oehrtman, 2009; Presmeg, 1992; Zandieh, Ellis, & Rasmussen, 2017). For example in interviews with ten undergraduate linear algebra students, Zandieh, Ellis, and Rasmussen (2017) found metaphors to be critical for understanding the varied ways students think about the function concept across high school and linear algebra courses. One student, for example, described one-to-one functions as functions in which, "for every output, there is one input *to get there*" (Zandieh et al., 2017; p. 35). Zandieh et al. identified the language *to get there* as being indicative of a travel metaphor for functions. Zandieh et al.'s (2017) work illustrates that metaphor usage can be used as a lens for studying students' cognition (e.g., the metaphors they use highlight a conception that they possess or are applying) and reveals that metaphors can provide an explanatory account for how students can develop a rich understanding of a concept (e.g., Zandieh and colleagues illustrate how blending metaphors enabled students to unify different conceptions of linear algebra concepts).

A second group of studies has explored metaphors as a lens to understand individuals' beliefs *about* mathematics (e.g. Latterell & Wilson, 2016; Noyes, 2006; Schinck, Neale, & Pugalee, 2008). To date, these studies have focused on students' and primary and secondary teachers' beliefs about mathematics. For instance, Latterell and Wilson (2016) asked prospective teachers to supply metaphors for mathematics. The authors then used these metaphors to understand preservice teachers' attitudes about mathematics. In one example, a student provided the metaphor, "Mathematics is like a tornado in Kansas" (Latterell & Wilson, 2016; p. 287). Latterell and Wilson (2016) suggested this reveals a view of mathematics as something that could cause risk, injury, or harm. Our current paper complements these studies by using metaphor as a lens to investigate mathematicians' beliefs about mathematical activity and how these beliefs may be communicated to their students.

Methods

Participants

In this study, we analyze the metaphors used by mathematicians when giving a lecture in an advanced mathematics course (i.e., a proof-oriented mathematics course for third or fourth year university mathematics students). We recruited participants by sending e-mails to every lecturer at three doctoral-granting institutions in the eastern United States who was teaching an advanced mathematics course. We asked to observe and audio record one of their lectures. Lecturers were not told the purpose of the study. Eleven lecturers agreed.

The Lectures

Each lecture was approximately 80 minutes in length. All professors gave "chalk talk" lectures (Artemeva & Fox, 2011) in which they presented formal mathematics (specifically definitions, theorems, proofs, and examples) on the blackboard. Each class had between seven and 30 students enrolled, with a mean of approximately 18 students. A member of the research team attended and audiotaped each lecture while transcribing everything the lecturer wrote on the blackboard. Each audio recording was transcribed. This transcription was the primary corpus of data used in our analysis.

Analysis.

We analyzed the data following Chi's (1997) scheme for quantifying qualitative analysis of verbal data, which Chi described as a practical guide for making sense of "messy" verbal data. In our first stage of analyses, the first two authors independently read the transcripts flagging for each instance in which a lecturer used a metaphor[1]. Any disagreements were resolved by conversation with all three authors of the paper. 1077 metaphors were identified across the 11 lectures. Of these 1077 metaphors, we found 216 pertained to the activity of doing mathematics.

The second stage of the analysis was thematic; we used an open coding scheme to generate common metaphorical archetypes with a common source domain and a mathematical activity as a target domain. Six metaphorical archetypes having a mathematical activity as a target domain were identified. In the third stage of analysis, we developed clear criteria for what types of utterances counted as an instance of each metaphorical archetype. In the fourth stage, the first two coders independently went through each metaphor that we had previously identified and evaluated if the metaphor belonged to any of the six metaphorical archetypes. Again, any disagreements were resolved through discussion with all three authors of this paper.

The third stage of the analysis was similar to the first. For each of the six metaphorical archetypes, we used thematic analysis to identify particular mappings between a component the source domain and a component of the target domain. For instance, with learning mathematics as a journey, there was often a particular mapping between progress on a journey and formal mathematics covered in a particular class. (e.g., a mathematics professor may say she wants to reach a certain theorem by the end of class, but "we'll see how far I get today"). We would identify criteria by which an utterance could be coded as an instance of this mapping. Then we would go through each metaphor in the metaphorical archetype and evaluate whether each individual metaphor was an instance of that mapping, going back to the original transcript for contextual details if necessary. Again, disagreements were resolved through discussion. The result of following Chi's (1997) methodology is that we can provide an in-depth analysis of the most interesting metaphors that individual professors used but also identify trends across our data set and describe how common these trends were.

As an important theoretical point, the coding scheme that we used is clearly highly interpretive. Following work of Reddy (1979), Lakoff and Johnson (2003), Lakoff and Nuñez (2000), and others, the meanings that we ascribed to the metaphorical utterances exist in the minds of the researchers; our research claim is that other mathematically knowledgeable

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[1] Coding for metaphor usage *in general* was theoretically difficult because metaphors pervade mathematical vocabulary (e.g., Lakoff & Nuñes, 2000). (For instance, is every instance of a professor using the word "in" to denote set membership a use of a metaphor?) However, in this paper, we only discuss metaphors for *mathematical activities* which did not introduce these theoretical nuances. Hence, for the For the sake of brevity, we do not discuss how we resolved disagreements that did not have mathematical activities as a target domain in this paper.

individuals would agree that our interpretations fit well with the data. We cannot be certain if the lecturers themselves intended to convey the meanings that we ascribed to their metaphorical utterances or if students would interpret the metaphorical utterances as we did. We elaborate on this point toward the end of the paper.

Results

Our analysis of these 216 uses of metaphors identified four metaphors used by the eleven mathematicians in the study: Learning Mathematics is a Journey, Doing Mathematics is Work, Mathematics is Discovery, and Mathematics is a Story. Table 2 shows the number of instances of each metaphor in each lecture and the total numbers of instances of each metaphor in the corpus of lectures. As Table 1 reveals, each metaphor was used by at least seven of the 11 lectures we analyzed. For space reasons, we will only provide commentary on the Learning Mathematics is a Journey metaphor here.

Table 1. Counts of Each Metaphor for Mathematical Practice by Lecture

Metaphors	Instances in Each Lecture											Total Instances
	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	
Journey	1	1	2	2	0	3	2	6	14	0	8	39
Work	10	5	2	4	3	10	7	4	12	1	1	59
Discovery	7	1	3	22	5	5	0	0	10	25	13	91
Story	5	0	2	0	1	0	1	8	7	0	3	27

Learning Mathematics is a Journey

Table 2. Metaphor Map for the Learning Mathematics is a Journey

Source Domain: Journey	Target Domain: Mathematics	Number of Instances**	Example
Progression along the journey	Progression in one's learning of mathematical content	16 (6)	L8 [abstract algebra]: "I'd like to do at least a little bit on group theory, *we may or may not get to that*."
Times/Locations along a Journey	Particular mathematical content learned at a particular time in one's mathematical career	16 (3)	L1 [set theory]: "I'd like to get to today, or very soon, is that this notion of cardinal arithmetic *will allow you to get away from the very explicit arguments that we've been doing the last few weeks*."
Journeys can have required landmarks or checkpoints to be crossed	Important Mathematical Ideas and Concepts that should be learned in mathematics	5 (5)	L2 [real analysis]: "So I think that you *can't possibly have gotten very far in math here without having seen Euler's number e*, which is usually defined this way."
Other		6 (4)	L3 [number theory]: "The proof of this will wait a little bit. Okay? We won't

** *Number of instances across the lectures (Number of lectures in which the instances occurred).*

On a long journey, travelers depart from an initial location and set out with a particular destination in mind. The traveler's journey may span multiple days in which they plan to traverse a certain distance and reach a certain point at the end of a day. In the lectures, mathematicians spoke of particular mathematical topics or results in terms destinations that they hoped to reach. The mathematical topics that were covered were analogous to the ground that could be covered on a particular leg of a journey.

The metaphor of Learning Mathematics is a Journey was often invoked at the start of the lecture; six lecturers initiated their lectures by using this metaphor to describe the planned itinerary for the day and the metaphorical location that they hoped to reach by the end of the lecture. For example, L8 used three metaphors of this type in the first six minutes of her lecture: "I'd like to do at least a little bit on group theory, *we may or may not get to that*", "at least *if we get through this* chapter 6, it'll be a nice ending for you *if we don't get further*", and "so *we'll see how far I get* today." The common theme in these quotations is that L8 wanted to cover certain topics (a little bit of group theory, the end of chapter 6) that he metaphorically described as locations that he would like the class to reach by the end of the lecture.

The discussion of L8 above used the journey metaphor in a local sense in describing what ground would be covered in a particular lecture. However, the lecturers sometimes used journey in terms of students' mathematical development, either in terms of the entire semester or even beyond that (see L2's quote in Table 2). In a journey, there may be particular landmarks that a traveler wishes to see or has seen in the past. The lecturers would describe important mathematical results as being these landmarks. For instance, in the last fifteen minutes of the lecture, L9 talked about the distance from and progress made toward a mathematical accomplishment in terms of distance from and progress toward a physical destination. After discussing how the real numbers are an extension field of the rational numbers, L9 said "here, *we are on the verge* of synthesizing or generalizing that approach" where the "verge" is "the edge or border of something" (Cambridge English Dictionary). The class was approaching an accomplishment of defining an extension field given an arbitrary field. In L9's language indicating they were *on the verge* suggests to us that the class was approaching a desired mathematical destination.

Meanwhile, we see that the class still had some ground to cover before arriving at this destination. L9 said, "now that's promising in that this sets this up as a direct parallel to this, but it doesn't yet, on its own, guarantee that *we have gone far enough* to find a root for P(x), okay? [...] Now, *we won't reach that pinnacle* today unfortunately." In this quote, L9 explained that the class has not yet *gone far enough*, or made sufficient progress, in their journey to reach the desired destination meaning that they have not covered the necessary content required to complete this generalization. As such, we see L9 describing the class's current or local progress on the journey in relation to the broader journey of learning mathematics. In particular, L9 also described the destination as a *pinnacle*, suggesting that this destination is not simply the next stop on the journey, but rather a local maximum in the domain being covered and a critical landmark that the students want to reach. Next, L9 paused to allow students to ask questions and continued "the *farther we go in this*, the more focused I can be in my anticipations of it." Here, L9's language suggests that as the class continues to make progress toward their anticipated mathematical destination, he will be able to better anticipate questions and guide the class on their journey.

Discussion

The main results of this study are that mathematics lecturers regularly invoke metaphors when they describe the activity of doing mathematics. Further, there was overlap in the metaphors that were used by different mathematicians. We first offer speculative thoughts about the implications of the metaphors we found, Learning Mathematics is a Journey, Doing Mathematics is Work, Mathematics is Discovery, and Presenting Mathematics is a Story. We then conclude the paper by delineating the limitations of our study and suggest directions for future research.

In undergraduate mathematics education, mathematicians frequently express an obligation to cover a certain amount of mathematical content. In contrast, most mathematics educators believe that content coverage by itself is useless if students do not understand the material that is covered (e.g,. Fukawa-Connelly et al., 2017). In the Learning Mathematics as a Journey metaphor, the lecturers frequently spoke of the ground they needed to cover and the landmarks they intended to reach. However, there were only one instance in which a lecturer (L3) mentioned losing students on the journey. One possibility is that lecturers' use of the Learning as a Journey metaphor provides a lens into their obligations as a teacher (covering ground and reaching destinations) and what are peripheral considerations (the number of students who are able to actually complete the journey). This also suggests how a change in the metaphor might lead some mathematicians to reconsider what they value. After all, a Sherpa who successfully scales Mount Everest would not be regarded as successful if the majority of his party perished along the way. Adding the notion of "survival rate" to the Learning Mathematics as a Journey metaphor could add the aspect of student learning to the metaphors that mathematicians used.

In terms of limitations, we emphasize that we only looked at mathematicians' metaphor usage in the context of lectures in United States classrooms. In light of recent inquiries into how language and culture shape mathematical curricula (Shinno et al., 2018), it would be worthwhile to investigate whether mathematicians from other cultures than our own used different metaphors to describe mathematical practice. The theoretical work of Lakoff and Johnson (2003) and the empirical work of Thibodeau and Boroditsky (2011, 2013) demonstrate how the metaphors that are used to frame a concept influence how people reason about that concept. It would be interesting to investigate how, if at all, the particular metaphors that mathematicians use influence students' mathematical reasoning and epistemologies. Questions regarding the impact of metaphors on student thinking and reasoning may have the potential to generate very interesting theoretical and empirical research.

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Calculus TAs' Reflections on Their Teaching of the Derivative Using Video Recall

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This paper addresses the characteristics of first time Calculus I TAs' teaching practice of the derivative and reflection on their teaching using video-stimulated recall. Our analysis using three views of function – correspondence, variation, and covariation – shows that regardless of representations that TAs adopted, their discussion mainly addressed correspondence, and most TAs used correspondence to justify the difference quotient (DQ) in the limit definition as a function, and transition from the derivative at a point and the derivative as a function. TAs also emphasized different uses of letters as an input of the derivative in such transition from students' point of view although they as mathematicians did not see the difference. Some TAs addressed the variational or covariational view in class and/or during reflections but in a limited way by simply acknowledging that quantities "change" without describing how they change.

Keywords: Video recall, Teaching, Reflections on Teaching, Teaching Assistants, Calculus

Graduate teaching assistants (TAs) for Calculus courses play a crucial role in educating Science, Technology, Engineering and Mathematics (STEM) students through significant interactions with students during their classes and office hours (Ellis, 2014). Unfortunately, the ongoing national effort to improve STEM education has found that many students leave STEM after their first year in college and report poor-quality teaching as their reason for leaving (Connolly et al., 2016) and many students still struggle with crucial ideas in Calculus necessary to understand various STEM phenomena (Park, 2013; Thompson & Carlson, 2017). Currently there are ongoing efforts to support TAs as novice teachers of STEM students (e.g., MAA, 2017), but what we as a field know about TAs' teaching practice is still limited. Based on this observation, this paper investigates TAs' teaching of one of the crucial Calculus concepts, the derivative by analyzing their video-recorded lessons and their reflections on their teaching by analyzing the interviews with them, in which they watched the videos and explained their instructional choices in class. The following research questions guided our study:

1. How did TAs discuss the derivative at a point and of a function in class?
2. How did TAs reflect on their class discussions on the derivative?

To answer these questions, we adopted three ways to conceptualize function – correspondence, variation, and covariation (Confrey & Smith, 1994; Thompson & Carlson, 2017) in our analysis of dominant approaches in TAs' classes and consistency between their teaching and reflection.

Theoretical Background

This paper builds upon two bodies of literature: video recall as a tool for teacher learning, and quantitative reasoning addressing mathematical concepts related to functions.

Teacher Learning Through Video Recall

Videos have been widely used as an effective tool in teacher education and professional development to enhance teachers' ability to notice or reflect on the recorded lessons or students' work. Researchers have argued that videos foster ways teachers think about teaching and learning. For example, van Es and Sherin (2006) studied the impact of videos with two groups of teachers who learned to notice different mathematical aspects of students' thinking depending on how the authors designed the use of videos during professional development. Rosaen et al.

(2008) showed that when videos were used for recall, pre-service teachers made more specific observation of their own teaching, focused more on teaching itself than classroom management, and commented more on students than themselves than when they were based on their memory.

Some researchers used videos of teacher's recorded lessons to help them reflect on and improve their teaching practice. Speer and Wagner (2009) used selected videotaped lessons where an instructor had trouble orchestrating class discussions to examine what occurred from the instructors' view point, and with the videos the teacher revisited what occurred at certain moments of teaching and critiqued his methods. Meade and McMeniman (1992) adopted the stimulated recall with videos to make teachers implicit beliefs and assumptions explicit, and Muir (2010) argued that videoed taped lessons are a powerful medium for teacher's deliberate reflection, and eventually led to change in teaching practice that was more effective for students.

This study also uses video footage to stimulate TAs recall and reflection of their teaching practice. Video stimulated recall is often defined as a research method, where "the subject is shown video records of his or her work on a task…immediately after the recoding" (Busse & Ferri, 2003, p. 257). However, due to the research design of this study, we adopted this method without the immediacy. Since we were interested in particular mathematical aspects of the lesson – how the TAs addressed the quantitative reasoning behind the discussion about the derivative – we, researchers, watched videos and selected the clips for crucial moments, before watching them with the TAs instead of immediately showing the whole videotaped lesson to TAs.

Quantitative Reasoning in Calculus

Calculus mainly deals with how quantities vary or covary in terms of rates of change or accumulated rates of change. To understand the teaching and learning of concepts in calculus, researchers have proposed multiple ways to investigate quantitative reasoning for functions which is pervasive in calculus concepts. The first view is *Correspondence* which conceptualizes a function as a process of building "a rule that allows one to determine a unique *y*-value from any given *x*-value," and thus "a correspondence between *x* and *y*" (Confrey & Smith, 1994, p.137), and another way of viewing function is through *Variation* and *Covariation* which focus on varying quantities involved in functions and their relations (Thompson & Carlson, 2017). Researchers who have advocated for variational/covariational reasoning emphasize the importance of understanding the context where functions were used, the quantities involved in functions, and how their coordination changes simultaneously (Confrey & Smith, 1994; Thompson, 1994). Thompson and Carlson (2017) further detailed levels for variational and covariational reasoning starting from no (co)variation towards smooth continuous (co)variation. We adopted and modified their levels by adding new categories to analyze our data specific to the derivative. Due to the limited space, we will only discuss the categories relevant to our data.

Regarding the derivative, one could consider two functions. First, while defining the derivative a point $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, one can consider the DQ, $\frac{f(a+h)-f(a)}{h}$ as a function of h with:

(a) correspondence where an h value corresponds to a specific value of the DQ,

(b) variation where DQ is changing either smoothly and continuously (smooth continuous variation), or simply approaches (gross variation), or

(c) covariational where DQ and h simultaneously covary smoothly and continuously (smooth continuous covariation), or simply approaching together (gross covariation).

Second, once the derivative at a point is defined, one can apply the definition on an interval where the derivative exists. One can conceptualize this process with:

(a) correspondence by considering each x value corresponding to the value of the derivative at that point,

(b) variation where the derivative changes over an interval either smoothly and continuously (smooth continuous variation), simply increases or decreases (gross variation), and

(c) covariation where the derivative and the independent variable simultaneously covary either smoothly and continuously (smooth continuous covariation) or simply increases or decreases as the independent variable increases (gross covariation).

Using these three approaches, we will investigate how the first year Calculus TAs addressed the quantitative reasoning in their teaching practice and their reflections on their teaching.

Research Design

This study is part of a larger study that includes a semester-long content-specific professional development (PD) for first time TAs for Calculus I. The PD consisted of five 75-min sessions during the semester starting from a week before the TAs started teaching the derivative focusing on varying and covarying quantities involved in the derivative. The current study focuses on TAs' teaching of the derivative and reflections on it. Five TAs, who taught Calculus I recitations in Fall, 2016 and Spring, 2017, participated in this study. Three TAs – Amy, Kay, and Dan – had no previous classroom teaching experience, and two TAs – Lia and Edi – had taught as an instructor before they entered graduate school. At the institution where the study was conducted, Calculus was offered as a large section for 80-180 students and taught by faculty instructors three times a week, and small recitations consisting of 30-35 students were taught by TAs twice a week. The study design consisted of three phases: video-recording of class, the first and second interviews for their reflections on teaching. In Fall 2016, we video-recorded TAs' recitation sections five times when they taught the derivative. The current study focuses on the first two lessons where they defined and used the derivative at a point and the derivative of a function in various contexts. Once the recording was done, we watched the videos and identified video clips for critical moments based on our framework. Then, we invited the TAs to individual one-hour interviews, where we showed each the selected video clips, and asked three questions:

1. What was the main idea that you want to discuss with your students here?
2. What do you think about your wording or representations? Do you have any thing that you want to modify? If so, why?
3. In this episode, do you see anything varying? What is or are varying mathematically?

The first interview occurred in the first week of Spring, 2017 before the PD started. The second interview occurred when they finished teaching the derivative towards the end of Spring, 2017 and the PD ended. The interviews were video recorded and transcribed.

Results

TAs' Approach to the Derivative at a Point with Symbols

While discussing the derivative at a point through the limit process on the DQ with symbols, all TAs addressed only the correspondence view in class except Kay, who additionally mentioned gross covariation (Table 1). During reflections, two TAs consistently addressed correspondence whereas the other two addressed variation or covariation. One TA, Kay addressed correspondence and mentioned gross covariation in class and during reflections.

All TAs started the derivative unit by defining and computing the derivative at a point with symbols and used the correspondence as the main approach in teaching and reflections. They conceptualized the DQ as a function, i.e., the corresponding value of the function DQ of the independent variable (e.g., h approaching 0). One of the TAs, Edi, explicitly used the word "function" for the DQ in class while emphasizing the correspondence by saying, "the limit as x goes to 1 of that function and just plug in 1 if the function exists there and negative 2 is the answer" for $\lim_{x \to 1} \frac{\frac{2}{x}-2}{x-1} = \lim_{x \to 1} \frac{-2}{x} = -2$. Edi and another TA, Lia, whose reflection also addressed the correspondence, used the word "function" again for the DQ and emphasized its existence at the point where the limit is computed. Edi even chose the correspondence as his main view on the limit process over the variational view; to the interviewer's question "what is or are varying?" in his algebraic computation of the derivative at a point, he responded, "it is a tangent line or a secant becoming a tangent line, but I guess I kind of see it more as evaluating the limit." It should be noted that Lia also mentioned variational view in her reflection as a response to the same question, but it was a simple mention about the limit symbol (e.g., $\lim_{h \to 0}$) by saying "h is varying here" without any connection to the DQ.

Another TA, Kay, used the function composition notation to compute the DQ for a given function, which also emphasized the correspondence view (Figure 2):

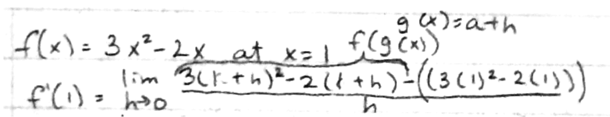

Figure 2. Kay's use of function composition for the DQ.

> A very difficult step, often when we're doing this limit definition of the derivative, is this plugging in step right here (gestures to $3(a+h)^2$-$2(a+h)$). It can be very difficult because you have to plug in a+h anywhere you see an x, and that can be confusing. But if you think of it as a composition, it might be a little easier. So I'm gonna let $g(x)$ be a+h (writes $g(x)$=a+h) and $f(x)$ is your function to begin with, then the first part here (puts a curly bracket around $3(a+h)^2$-$2(a+h)$ and writes $f(g(x))$)…wherever I see an x in my function f, I'm gonna plug in all of the $g(x)$.

It should be noted that in Figure 2, the input $g(x)$, which Kay emphasized, shed light on Kay's discussion of the quantity that varies in the DQ. With the limit symbol, $\lim_{h \to 0}$ attached to the DQ in computing $f'(1)$, h is an independent varying quantity in $(a + h)$. Therefore, a natural way to set up a function for $(a + h)$ would be $g(h)$ rather than $g(x)$. While watching this video, she did not comment on her use of x, but emphasized the substitution process to simplify computation.

The reflections of three TAs – Amy, Kay, and Dan – addressed variation or covariation. Both Amy's and Kay's reflections included that h and the DQ are varying simultaneously, but described this as a simple gross behavior of "changing" or "getting closer to." Dan, took a different approach from the first reflection to the second. In the first reflection, he addressed the gross variation on the location by simply mentioning "h" as "a variable…go[ing] to 0" in $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. In the second reflection, he identified both the DQ and the location as covarying at the gross level using graphical terms and connected it back to the algebraic expression:

> This (pointing to $\frac{f(x)-f(a)}{x-a}$ on the screen) is actually the slope of the secant line. So this one, corresponding, is changing. But, here (pointing to $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$) we are interested in the limit…Because x is approaching a…I expressed that in terms of x. And finally, when we carry out this calculation, x goes to a. So we can give out an exact number. We know where the limit is. So that's how we get the slope of the tangent line.

TAs' Approach to the Derivative at a Point with Graphs

Two TAs, Amy and Lia addressed the limit process with two types of graphs: non-linear and piece-wise linear. On both graphs, they drew a few secant lines approaching the tangent line at a point. Amy drew secant lines without marking x values and only described their behavior approaching the tangent line with gross variation. In comparison, Lia plotted both the x values and the secant lines and described how they covary with gross covariation.

TAs' reflections were consistent; Lia' reflection addressed the gross covariational view of the slopes of the secant lines and the corresponding location of x without explicitly mentioning how both quantities vary or covary. Amy's first reflection was also consistent with her teaching addressing the gross variation of the slopes of secant lines without the location on the non-linear function graph, but the gross covariation of the piece-wise linear function ("I can get as close as I want [bring hands closer to each other horizontally, and it [the slope of secant line] will always be positive"]). However, a piece-wise linear function only provides a limited context to discuss variational or covariational reasoning since the slopes of the secant lines are constant. During the second reflection, Amy explicitly addressed the missing component – the location, and the covarying relationship between the location and the change at the gross level: "the values of x are changing and the slopes of the secant lines are therefore changing."

TAs' Approach to the Derivative of a Function with Symbols

All TAs' discussions on the derivative as a function in class using symbols addressed correspondence. Their reflections on those discussions also addressed the correspondence. In the discussion of the correspondence between the input x value and the derivative value, TAs focused on the input, and some TAs also emphasized different uses of letters as an input.

In the discussions of the derivative function, all TAs used most of their class period computing the derivative of a function given as an equation applying the limit on the simplified DQ. The derivative process became only explicit when TAs substituted a number in for x to compute the derivative at a number. Three TAs – Dan, Amy, and Lia – introduced the derivative as a function using the correspondence between an input and the output value of the derivative. For example, Dan used a feeding mechanism analogy:

> [DF-1] What we have done *f* prime *a* is equal 2*a* *(writes* $f'(a) = 2a$*)*. So we can think of, like *the function* as a lazy dog…You feed the dog something and the dog will come up with something, so you feed in an *a*, and you get a result 2*a*. But if we changed the variable, here is a fixed number *(points to a)*, *x* equals to *a (writes x=a)*, but let's just say in general, cause *x* always equals to *a*, right? So if we feed in *x*, you will get *f* prime *x (writes* $f'(x) = 2x$*)*, right, so therefore f prime x is the function. So, we have done f prime negative 1 *(writes* $f'(-1)=$*)* and we'll get *a* equals -1, *(points to a in 2a)*, so it's negative 2 *(writes -2)*. So any number we're feeding, any real number will get an actual number (points to "$f'(x) = 2x$").

Here his analogy of feeding a lazy dog highlights the corresponding relation between an input and the corresponding output. Specifically, he interpreted the function as the correspondence between an input changed from *a* as "a fixed number," then "*x*" in general, and then a number *a* = -1, and its corresponding derivative 2*a*, 2*x*, and -2, respectively.

Three TAs – Dan, Amy, and Kay – emphasized the role of the input of the derivative function again during their reflections using a correspondence perspective. They specifically explained why *"a"* or "x_0" are different from "*x*" from the student point of view while acknowledging that a and x are the same mathematically. For example, in his second reflection about the lesson above [DF-1], Dan said:

> [DF-2] Here, from our point of view, it's [sic] just change x into a or change a to x. You can get from one to another. But um, from the learner's point, they may not see that easy, oh right, you just change the letter x uh into a, so you claim they're the same thing. Like, it's not that easy because again, usually a is a constant. x is a variable of a function that's not to be touched.
> [DF-3] I think here if I actually graph this, we can say okay, we can actually graph this, so this looks like it's a function, right? So, then I should probably say, okay, because a is, you can put in any number here. So why not just make it to be a variable? Instead of some fixed number.

Here, Dan said that mathematically "a" and "x" are not different, but explicitly differentiated "a" as constant or fixed from "x" as "not to be touched," ([DF-2]) and justified the transition from a fixed number "a" to the variable "x" by that any number can be substituted in "a" ([DF-3]). His use of "x" as something "not to be touched" is consistent of his use of "a=-1" instead of "x=-1" in his class ([DF-1]). In reflection, he explained that a direct transition "a" in $f'(a)$ "x" in $f'(x)$ would be hard for students because "they are not taught that we can substitute a constant by a variable."

It should be noted that two TAs identified varying/covarying quantities in the limit process on DQ rather than the change of letters or numbers as an input of the derivative function. Specifically, while reflecting on class discussions on the computation of the derivative at a point, Dan and Kay explicitly chose h in $\lim_{h \to 0}$, and then the corresponding DQ as changing whereas stating that they did not view changing the input for the derivative function from a letter x or a to a number as varying.

TAs' Approach to the Derivative of a Function with Graph

All TAs used graphs to discuss the derivative of a function while graphing it or finding its range in class using a correspondence perspective, and their reflections also addressed the correspondence view. However, but three TAs simply mentioned gross covariation between x and the derivative (e.g., when x changes, the slope changes) in their second reflection when they reflected on their teaching involving graphs. TAs' class discussions were limited from the variation or covariation point of view; graphing and describing the behavior of the derivative of a function was mainly based on the correspondence and the sign of the derivative of a function without addressing how the associated quantities vary or covary. For example, two TAs, Edi and Kay, drew the derivative of a function given as a graph but based on the correspondence rather than on variation or covariation. Specifically, Edi graphed the derivative of a piece-wise linear function, which provided only limited context for variational and covariational aspects of it. He mainly read the slope for each interval to graph the derivative function without discussing any variation. Kay drew the graph the derivative of a non-linear function, but only considered the sign of the slope of the function on the intervals partitioned by the critical values and graphed the derivative as if it were piece-wise linear (Figure 2):

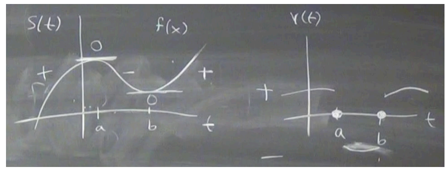

Figure 2. Kay's graph of the derivative of a function

Kay: You're gonna find out…the value of the slope, the exact value, the magnitude, but sometimes it's actually really more important just to know the sign of the slope. So, over here (points to most left), what is the sign of my slope?
Students: Positive
Kay: It's positive, right? So I'm just gonna write a little positive here. *(puts + next to curve)* Where is my slope zero?
Student: At the top
Kay: At this little crest? Right? And right here right? *(gestures to first and second critical points of curve, draws horizontal line, and writes 0 near).*

As Kay pointed out the point of discussion was to know the sign of the slope of the distance function on each interval instead of how the slope varied as a quantity. Moreover, how the original function behaved was not discussed in the class. Instead, the graph of the derivative was drawn completely by reading the sign of the slope of the original function from the graph without associating it with discussing how the original graph behaved.

Discussion and Conclusion

Our analysis of TAs teaching practice of the derivative showed that the correspondence was their main approach to discussing the derivative at a point through the limit process on the DQ, and the derivative as a function. Correspondence view was prominent in class, even when the TAs used graphs, which are often used to visualize the behavior of a function that is often overlooked with algebraic representations. Most TAs either chose limited contexts involving a linear original function whose rate of change is constant, and even when they chose non-linear functions as original functions, they graphed the derivative mainly focused on the sign of the slope on intervals partitioned by critical values without considering how the quantities involved in the function or the derivative vary or covary between those values. During the reflection, TAs often addressed a gross variational or covariational view in addition to correspondence. TAs, especially the ones who had not taught before, showed progress towards covariational reasoning by identifying missing quantities from their class discussions, and describing their relation in terms of how the quantities involved covary.

The results of this study provide valuable information for content-specific PD for TAs. TAs' adoption of correspondence was dominant even when they were using graphical representations which are often used to emphasize the behavior of changing quantities and their relationships. Their approach could be extended to include other approaches by letting them think about how different types of context, problems, and representations could promote discussion of varying and covarying quantities. Also, PD material should challenge TAs' own content knowledge for them to revisit the missing components and relations that could be in their teaching of calculus, based on which TAs could build up the mathematical knowledge for teaching to promote quantitative reasoning in students that is crucial in STEM fields.

Acknowledgments

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How Peer Mentors Support Students in Learning to Write Mathematical Proofs

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We study how the mathematical beliefs and knowledge of peer mentors in a summer mathematics program influenced their efforts to help high school students learn to write proofs in number theory. Using Schoenfeld's framework for understanding decision making, we analyze interviews of three undergraduate student mentors for evidence of how their views of the role of proof, norms for proof writing, and mathematical knowledge for teaching informed their pedagogical decisions. We find that each mentor developed a distinctive approach to providing feedback on student work consistent with their own values, and present evidence that the success of each approach depended on the mentor's resources for interpreting student work.

Key words: Mathematical Proof, Mathematical Knowledge for Teaching, Number Theory

Research in undergraduate mathematics education has documented some common challenges associated with teaching students to identify and produce correct mathematical proofs. These challenges are often rooted in issues of what students consider to be a valid proof: students are often found to possess ritualistic or empirical proof schemes and derive conviction from the form of an argument rather than its analytic content (Harel & Sowder, 1998). When reading an argument and deciding whether it is valid, students frequently focus on surface features such as whether the presentation of the argument engenders a feeling of "making sense," or the extent to which the proof represents reasoning symbolically or verbally (Bleiler, Thompson, & Krajčevski, 2014; Selden & Selden, 2003). The process of constructing proofs also presents challenges: for example, in addition to knowing relevant mathematical facts, students must also develop strategic knowledge of when proof techniques or theorems are likely to be useful (Weber, 2001).

Recent research also reveals complications inherent in the notion of "correct" mathematical proof. A study of mathematicians' validations of an elementary analysis proof has suggested that there is no single standard for validity shared by all members of this community (Inglis, Mejia-Ramos, Weber, & Alcock, 2013). Moreover, mathematicians disagree on whether nonstandard uses of language in mathematical proofs constitute breaches of convention, and their judgments of potential breaches may be influenced by the context in which proofs appear (Lew & Mejia-Ramos, 2017). Those tasked with teaching students how to write proofs thus face a doubly difficult task: they must help students develop the interpretive frameworks and strategic knowledge necessary to read and produce proofs, while also facilitating their enculturation into a community whose norms are not well-defined and may vary depending on context.

In this study, we investigate the beliefs and knowledge that guide undergraduate student mentors' efforts to teach high school students in a summer number theory course to write proofs that conform to these mentors' perceptions of standard conventions for mathematical writing. We aim to contribute to the growing body of research on the mathematical and pedagogical resources entailed in teaching students to read and write proofs.

Background and Theoretical Framework

We assume the theoretical stance, suggested by Schoenfeld (2010), that people's teaching decisions can be understood on the basis of their goals, their beliefs and orientations, and their knowledge and resources. In the context of our study of peer mentors' decision making about how to help students learn to write proofs, we construe "goals" to include not only the broadly shared goal of helping students become effective mathematical writers, but also subgoals associated with teaching students to write proofs that can achieve specific purposes envisioned by the mentors, such as convincing other students of the validity of a claim or helping others understand why the claim is true. Prior research on proof has enumerated different roles that proof can play in mathematical learning and practice; these include strengthening the audience's certainty of the truth of a mathematical statement, explaining why a statement is true in terms of its connections to other known facts, developing a formal axiomatic system of concepts and theorems, and transmitting mathematical knowledge to other practitioners (De Villiers, 1990; Hanna, 2000). We hypothesize that mentors' goals in helping students learn to write proofs may align with some of the purposes suggested by this framework.

Within "orientations" we include mentors' beliefs about attributes that "good" or "correct" proofs should have, along with beliefs about how they can most effectively develop in students an appreciation for these attributes and habits that will help them write proofs that meet these standards consistently. Lai, Weber, and Mejia-Ramos (2012) found that mathematicians believe that pedagogical proofs, which serve primarily to explain and communicate mathematical results to an audience of students, should contain introductory and concluding sentences, should format major ideas so as to emphasize their importance, and remove redundant or extraneous information in order to minimize confusion. We hypothesize that since proofs that students produce in a number theory course often have the same purposes, mentors' beliefs about desirable attributes of proofs may align with these preferences. In addition, mentors may have linguistic norms for proof that align more or less with those of the mathematical discipline, such as those that forbid allowing a symbol to represent two different objects or that discourage stating entire definitions that are external to a proof (Selden & Selden, 2014).

We conceptualize "resources" as knowledge that mentors deploy in the work of teaching. Numerous studies have illustrated distinctions between the mathematical knowledge used in teaching and the mathematical knowledge that people use in everyday life and in non-teaching careers (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008). Mathematical knowledge for teaching (MKT) includes both subject matter knowledge, including specialized knowledge that helps teachers interpret mathematical thinking, vet problem-solving strategies and select representations of concepts, and pedagogical content knowledge, which entails understanding students' ways of thinking about mathematics and ways in which concepts can be presented in classroom settings (Hill, Ball, & Schilling, 2008). In the context of a proof-based number theory course, mentors' MKT might include knowledge that helps them interpret student proofs with unexpected features and approaches, and strategies for helping students identify a productive problem-solving approach without directly advising them on how to approach a problem.

Guided by this framework, we address the following research questions:

1. How do peer mentors' goals, orientations, and resources influence their pedagogical approaches in helping students learn to write number theory proofs?
2. How are these pedagogical approaches reflected in their evaluation and marking of hypothetical and actual student proofs?

Method of Study

Our study took place at a six-week summer mathematics program for high school students in the United States in 2018. The program included 63 students, of whom 35 were first-time participants taking a course in number theory. While the program is highly selective, admitting less than 20% of applicants, our initial interviews of study participants suggested that most students had not had extensive prior experience with mathematical proof beyond what some had encountered in U.S. regional and national mathematics contests.

On a typical day of the program, first-year students attended a lecture in the morning, took other classes in the afternoon, and participated in a four-hour homework session during the evening. During these homework sessions, students worked in "study groups" of three or four to prove theorems that would be covered in subsequent lectures. At the start of the program, each study group was assigned a peer mentor who supervised the evening sessions; the small groups and their peer mentor assignments remained stable throughout the program. In addition to attending lecture in the mornings and working on developing proofs of theorems in the evenings, students attended afternoon problem sessions, led by peer mentors, in which they discussed solutions to homework problems they had completed and foreshadowed upcoming content.

We chose to conduct a case study (Yin, 2013) of peer mentors' approaches in helping students learn to write proofs because unlike the program faculty, who interacted with students primarily through lecture-based classes, the mentors had considerable opportunity to influence students' views on proof and proof-writing through daily problem sessions, extended interactions during homework sessions, and their marking of each day's completed homework. In addition, because peer mentors were typically undergraduate students in STEM disciplines, they were themselves in the process of learning to write and critique technical texts such as mathematical proofs; thus studying the work of mentors provided a unique opportunity to investigate the role of mathematical knowledge in teaching higher mathematics in a setting in which this knowledge was under active development.

To investigate how peer mentors supported first-year students in learning to write number theory proofs, we conducted interviews of four mentors (Table 1) during the second week of camp and during the final week of camp. During the initial interview, we asked questions about mentors' beliefs about the purposes of proof in the context of the number theory course, what it means for a proof to be correct or incorrect, and how participants supported their students in learning to identify correct proofs.

Table 1: *Participating Mentors and Demographic Information*

Mentor	**Age and Ethnicity**
Linda	17, Asian/Pacific Islander
David	19, White
Nina	18, Asian/Pacific Islander
Nathan	18, African American

Following are a few of the questions we asked in the initial interviews:

1. In the number theory course, proofs are given for most of the facts discussed in class. Why do you think the class does this?
2. What are some characteristics of good mathematical proofs?

3. When a student submits a proof, how do you decide whether the proof is correct?
4. How do you support your students in learning to construct correct proofs, and distinguish correct proofs from incorrect ones?

In the final interviews, we repeated some questions from the initial interview to track possible shifts in mentors' beliefs about proof and about teaching students to write proofs. We also asked each participant to read and mark a hypothetical student's proof of Euclid's lemma (that if *a*, *b*, and *c* are integers such that *a* divides *bc* and *a* is relatively prime to *b*, then *a* divides *c*). We also asked participants to explain their actual markings of several of their own students' proofs; this allowed us to gain insight about how participants' beliefs and MKT informed their approaches to the everyday work of mentoring. In these final interviews, we used a tablet to display scans of the hypothetical student proof and actual student proofs so that we could record mentors' markings on proofs as well as their spoken comments.

We transcribed audio from the initial and final interviews for each of the four mentors in our study. We analyzed transcripts using thematic analysis (Braun & Clarke, 2006) to identify themes in participants' beliefs about purposes of proof and reasons for learning to write proofs, beliefs about features that influence the quality or validity of a proof, and mathematical and pedagogical knowledge that was relevant to the work of helping students learn to write proofs.

Results and Analysis

In this section we discuss our analysis of our interviews of three of the four mentors; we selected these three cases because each revealed themes not readily visible in the other cases. For each case we include some extracts from our interviews with the peer mentor that shed light on their goals and orientations with respect to the teaching of mathematical proof; we also include some observations about their practices in marking proofs, as evidenced by their responses to the hypothetical student proof task and their discussion of their actual students' marked proofs.

Proof as Disciplinary Activity: The Case of Nina

When asked in her initial interview why the number theory course focuses on developing proofs of mathematical theorems rather than simply presenting facts and computational strategies, Nina discussed the role of proof in explaining how and why mathematical ideas work:

> I think it does this because the class is really focused on not so much accumulating facts and information, which you are doing as you go through the problem sets, but also understanding why each one of them works the way they do. We start with intuitive - sort of, quote, "simple" statements such as n times zero equals zero, and they're statements that we often take for granted. So when you dive into the axioms behind those and how they really work, you understand math from one different perspective, and then also a deeper perspective. You have a more solid understanding of it.

This and other responses from Nina suggested that one of her goals was to help students learn how to write proofs that would shed insight on conceptual underpinnings of and connections among mathematical ideas. The theme of proof as an avenue for deepening mathematical knowledge recurred in many of Nina's answers during the initial interview.

When asked about characteristics of "good" proofs, Nina highlighted the importance of developing an argument that is rigorous; when asked to clarify what "rigorous" meant, she described a rigorous proof as one that "explain[s] every step thoroughly and carefully," and that peers can understand without difficulty. She suggested that a proof should have "eloquence," observing that some of her students often used colloquial language or wrote in incomplete sentences. Finally, she noted the importance of validity, which she characterized as not assuming the conclusion, taking incorrect logical steps, or performing steps that could not be justified in terms of facts already proven. She also noted that a proof should minimize unnecessary steps. We view these norms for mathematical proofs as orientations that Nina might apply to the work of marking proofs; we note that some of these norms (such as omitting unneeded steps) are consistent with those described by Selden and Selden (2014) and Lew and Mejia-Ramos (2017).

Nina's initial interview also offered insight into her orientations regarding her role in helping students improve at proof-writing. She described a practice, shared by most mentors in the camp, of assigning "redos" and "rewrites" for proofs deemed to be inadequate. Nina characterized the distinction between a "redo" and a "rewrite" as based on the depth of errors in a proof; while a proof that contains a major error might warrant a "redo," a proof that contains a valid argument but has some writing errors (such as failing to introduce variables) might only receive a "rewrite." When asked about the pedagogical purpose of assigning redos and rewrites, Nina said:

> I think the point of a rewrite is to show them how to make their proof better, and show that you're missing a few steps here, and that *if you practice rewriting this, you'll write better proofs in the future*. I try to stress to my campers that it's not a bad thing to get a rewrite or a redo, it's not like you failed an assignment or you did poorly, got a bad grade. It's just that here's an opportunity for you to fix this proof, *and then next time you'll write an even better proof from there on*. [emphasis ours]

Thus for Nina, assigning redos and rewrites served as an opportunity to reinforce normative proof-writing practices for students.

During the final interview, we asked Nina to review two of her students' proofs that multiplying each element of a complete residue system modulo n by a unit produces another complete residue system. Both students had written proofs using similar approaches, but Nina had marked one proof correct while assigning the other a redo. When asked about this discrepancy, Nina explained that while the first student's writing suggested a sound understanding of the approach, the second student's writing did not:

> I think again, it's little things - you missed a word here, that shows that perhaps you're kind of writing things based off what you remember from presentation, but you're not fully there. It's really hard to define. It's subtle distinctions here. Two students can write almost the same amount of text, and one can just show that they understand better than the other did, just by the words they've selected and the way they have presented their proof.

This excerpt suggests one type of knowledge that Nina used in marking proofs; while a reader not concerned with individual students' understanding might have marked both proofs correct, Nina used her knowledge of content and students to discern the depth of a student's understanding of an argument. When a student's writing suggested a lack of understanding in tension with Nina's goals for students' proving activity, she asked the student to revisit it.

Proof as Persuasion: The Case of David

David's responses to questions about the purpose of proof in the number theory course focused on the roles of proof in verifying and communicating mathematical results. In both of his interviews, he showed commitment to the notion of a proof as a persuasive essay, and suggested that skills students developed in the number theory course could prove valuable elsewhere:

> I think by starting from the bare minimum, like the axioms, and building up on that, you're learning to justify everything you say, and that's a skill you need everywhere in life. If you're

> writing an essay, a persuasive essay, everything you say has to follow from some basic assumption, and you have to justify everything; otherwise a reader's not necessarily going to be persuaded. I think that's true in basically any field.

David's answer suggested that one of his goals was to help students learn to write proofs that could persuade peers of the truth of a mathematical claim.

In discussing norms for "good" mathematical proofs, David stated that a successful proof should be something that a layperson could understand, given a sufficient understanding of the problem under discussion. He noted the importance of using complete sentences and including explanation for each step of a proof. When asked how he approached the task of validating a proof, David suggested the notion of a skeptical reader who might identify holes in an argument:

> Everything could be technically correct, but if there isn't explanation behind each step, then it doesn't have value to somebody else. … If they aren't able to convince someone of something entirely, then I don't think it's correct. If I can, as a reader, think "oh, what about this case?" and they haven't addressed that, then I don't think it's a correct proof, because they have to be able to irrefutably convince you of something.

This suggests that some of David's beliefs about proof quality may have oriented him toward focusing on the flow of students' reasoning in proofs and their consistency in justifying steps and addressing all cases of a problem. However, we observed instances in which David's curricular knowledge of the number theory course may have imposed some limitations on this. In their proofs of the theorem that if b is nonzero and $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$, David's students used the fact that if $\gcd(a, b) = d$, then $\gcd(a/d, b/d) = 1$; they also used the fact that if $\gcd(a, b) = d$, then $\gcd(ak, bk) = dk$. The first of these facts can be proven using a relatively straightforward argument; the other is typically proven using Bézout's identity, for which this theorem is often a building block. Because David did not recognize that these facts would be systematized later in the curriculum (or was willing to accept these claims based on intuitive reasoning), he did not object to their use. When we asked David about these proofs, he initially judged the proofs to be correct, but revised this judgment when presented with the fact that the students' subsidiary claims about greatest common divisors had not yet been formalized.

Proof as Opportunity for Assessment: The Case of Nathan

Nathan framed many of his observations about the power and importance of mathematical proof in contrast with his experiences with traditional schooling. When asked why the summer program provides students with such extensive experience with mathematical proof, Nathan discussed the development of students' intellectual agency:

> I think having them write up their ideas and work on their own to prove theorems is very important, in the sense of not having a higher power or a teacher … do it for them. … If you're working with others who are trying to understand the same way you are and you guys are bouncing ideas off of each other - even if it's wrong at first, even if your proof is wrong, you work with each other to try to reach this conclusion of why this works, and how it's true. I think that's how people did it before there was anybody to really help them understand something - they'd bounce each other's ideas off of one another and come to a conclusion that was right and made sense to them.

In discussing his orientations regarding proof quality, Nathan consistently focused on two attributes of "good" proofs: rigor (which he characterized as attending to all of the details in a proof's reasoning) and clarity. His discussion of his teaching approach during both the initial and final interviews suggested an iterative approach to vetting students' proofs: when faced with a claim in a student proof whose justification was unclear or lacked rigor, he would engage in a one-on-one conversation with the author and ask questions to assess their understanding of the reasoning. If the student demonstrated sufficient understanding of the reasoning in the proof, Nathan would make a minor suggestion as to how the student might better convey this reasoning rather than assigning a more extensive rewrite. Thus Nathan used his knowledge of content and teaching to identify ways to honor students' agency in presenting and explaining their own reasoning – a goal he clearly valued – while maintaining standards for mathematical rigor.

Discussion

The cases of Nina, David, and Nathan illustrate ways in which peer mentors' goals and orientations might guide their norms for the proofs that students create as well as their approaches in helping students learn to create proofs consistent with these norms. They also suggest ways in which various facets of mentors' MKT might afford or constrain opportunities to make progress toward their self-defined teaching goals in ways consistent with their beliefs about mathematical proof and about teaching and learning.

The study took place in a setting in which students and mentors have co-constructed a distinctive set of norms for proof validity and quality that may not be consistent with those of the professional mathematicians who direct the program (Patterson & Cui, 2017). In particular, we hypothesize based on results from this and our previous study that peer mentors in this setting may have higher standards than most of the mathematical community for the amount of detail students must provide when justifying claims; for example, demands that students cite axioms for the integers, such as the commutative and distributive properties, fall off much later than they do in most number theory courses. In this study, however, we see that demands on students' justifications may originate from different pedagogical beliefs and intentions. While Nina's standards and practices seemed focused on maintaining the integrity of disciplinary norms for proof writing, David's emerged from a view of proof as argumentation, a practice that he viewed as transferable across disciplines. While Nathan had similarly stringent standards, he appeared to offer students greater flexibility in how they met these standards, and seemed interested in maintaining them in order to maximize students' opportunities to develop and demonstrate understanding. All three mentors responded in similar ways to an interview task that asked them to mark a hypothetical student proof, suggesting that they had similar norms for proof quality and comparable consistency in enforcing these norms; however, their motivations for enforcing these norms appear to be more diverse than we had originally hypothesized.

Limitations of Study and Next Steps

In this study we analyzed peer mentors' beliefs and practices for teaching students to write proofs in number theory. We do not yet understand how students interpret their mentors' feedback about the proofs they write, how these interpretations inform the iterative development of students' own beliefs about proof validity, or with how much fidelity students adopt the beliefs and norms of their mentors. We also hesitate to make broad inferences about mentors' MKT based on their responses to interview prompts, since some questions involved proofs that they had marked three to four weeks prior to the final interviews. Furthermore, the peer mentors work in an environment in which time for marking papers is scarce; failures to identify errors in proofs may be due to time constraints rather than gaps in mathematical knowledge.

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Distance Measurement and Reinventing the General Metric Function

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Real analysis is an important course for both undergraduate and graduate students. Researching the ways students reason about challenging and abstract concepts can inform and improve instruction in real analysis. In this report, I examine two undergraduate students' reinvention of a general metric function. To facilitate this reinvention, I conducted a 15-hour teaching experiment with undergraduate mathematics students that had completed the introductory sequence in real analysis. In this experiment, the students generalized their initial understandings of distance measurements on $\mathbb{R}$ to construct increasingly abstract measures of distance in various metric spaces, including sequence and function spaces. Their generalizing activity culminated in construction of a general metric function through reflected abstraction of operations relevant to distance measurement carried out in previous metric spaces. I explore the students' generalizing activity, as well as the abstractions that supported their generalizing[1].

Key words: generalization, real analysis, metric spaces, formal mathematics

Introduction and Review of the Literature

Success in real analysis can have substantial implications for undergraduate mathematics students, especially those pursuing graduate degrees. Along with abstract algebra, the majority of mathematics majors must take real analysis in some form as part of a core curriculum. Further, real analysis holds implications for mathematics graduate students as well, as it can be a major component of qualifying exams.

Despite its importance, real analysis is anecdotally difficult for both undergraduate and graduate students. In spite of this, we know very little of its teaching and learning, particularly in advanced settings of real analysis. While real analysis is the setting for various research agendas (e.g. proof, classroom instruction, student affect, understanding of definitions, etc.; c.f Alcock & Weber, 2005; Lew, Fukawa-Connelly, Mejía-Ramos & Weber, 2016; Weber, 2009), we know relatively little about how students understand real analysis topics outside of introductory contexts. While there have been a number of studies exploring how students understand formal limits (e.g., Adiredja, 2013; Cornu, 1991; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Gass, 1992; Roh, 2008; Swinyard, 2011; Swinyard & Larsen, 2012; Tall, 1992; Tall & Vinner, 1981; Williams, 1991), students' understandings of other key concepts in real analysis has generally not been explicitly studied.

Three exceptions to this are works by Wasserman and Weber (2018), Strand (2017), and Reed (2017). Strand used the Intermediate Value Theorem in the context of approximating an irrational root to draw out students' understanding of completeness on $\mathbb{R}$. Reed (2017) detailed a case study wherein a student reversed the roles of ϵ and N in point-wise convergence of functions as a result of a similar reversal in his understanding of real number convergence. Finally, Wasserman and Weber (2018) explored ways to use issues of classroom pedagogy in motivating underlying structure in introductory real analysis taught specifically to preservice teachers. While each of these studies explore different facets of student thinking in various real analysis contexts,

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there is still much we don't know about how students understand core and unifying concepts in real analysis.

In this report, I extend the literature by exploring how students understand more abstract concepts in real analysis, specifically in the context of metric spaces. A metric space, (X, ρ), consists of a set, X, paired with a measure of distance (i.e. a metric function), ρ. A function ρ is a metric if it satisfies the following four conditions for all $x, y, z \in X$: 1) $\rho : X \times X \to [0, \infty)$, 2) $\rho(x, y) = \rho(y, x)$, 3) $\rho(x, y) = 0$ iff $x = y$, and 4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. A productive understanding of metric spaces attends to the regularity of their topological structure across spaces in which such a pairing exists. For instance, sequences obey the same convergence structure, in that a sequence $\{x_n\}$ in a metric space converges to an element $x \in X$ if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\rho(x_n, x) < \epsilon$. Thus, a sequence of continuous functions under the supremum metric converges in the same way that a sequence of real numbers does under the absolute value metric.

I extend our knowledge of student thinking and learning in real analysis by exploring what advanced understandings students can construct through generalization of their understandings developed in introductory real analysis. Specifically, I report on the results of a 15-hour teaching experiment (Steffe & Thompson, 2000) that involved students' reinvention (Freudenthal, 1991) of a general metric space in real analysis.

Theoretical Perspectives

To reinvent the general metric function, the students engaged in generalizing activity that facilitated reflected abstraction (Piaget, 1975, 1980, 2001; Glasersfeld, 1995) of operations (referring to mental actions) involved in measuring distance in physical space. To describe both students' generalizing activity and the cognitive processes that support their learning while generalizing, I draw both from Ellis, Lockwood, Tillema, and Moore's (2017) Relating-Forming-Extending (R-F-E) generalizing framework and Piaget's (1975, 1980, 2001) notion of reflected abstraction. These two theoretical constructs have a synergistic relationship, in that Ellis et al. (2017) offer a nuanced analysis of the ways that students engage in the activity of generalizing, while Piaget's (1975, 1980, 2001) notion of reflected abstraction provides a language describing the underlying cognitive processes behind the students' learning through generalization.

Ellis et al. (2017) identify *relating* and *extending* as two broad categories of actions that students can take while generalizing. Relating occurs when " students [establish] relations of similarity across problems or contexts" (Ellis, et al., 2017, p. 680). This is a form of inter-contextual generalizing, in which students create relationships between two or more mathematical situations that they initially perceive as distinct. *Extending*, perhaps the more recognizable generalizing action, involves the application of established patterns, regularities, and relationships to new cases (Ellis, et al., 2017, p. 680).

Piaget's construct of reflective abstraction (specifically reflected abstraction; c.f. Piaget, 1975, 1980, 2001; Glasersfeld, 1995) complements the attention to the activities in which students engage as they generalize. Through reflective abstraction, we can make inferences about the cognitive mechanisms driving the students' generalizations, as well as the ways in which their knowledge is transformed through generalization. Situated as a mechanism of accommodation that facilitates equilibration (Glasersfeld, 1995), *reflective abstraction* is primarily characterized by two inseparable features: 1) a *réfléchissement* ". . . in the sense of the projection of something borrowed from a preceding level onto a higher one" (Piaget, 1975, p. 41), and 2) a *réflexion* ". . . in the sense of a (more or less conscious) cognitive reconstruction or reorganization of what has been transferred" (Paiget, 1975, p. 41). In this way, reflective abstraction captures the ways thinkers regulate their activity by first borrowing operations (mental actions) from one level of mental complexity (say, some Nth level of projected activity) and then reorganizing the operations on a new projected level (i.e. the $N + 1$st level). This reorganization produces new mental constructions enriched by the projected operations. Piaget acknowledged when this reconstruction occurs through explicit reflection on a thinker's activity by calling such conscious reflections *reflected* abstraction (Piaget, 2001; Glasersfeld, 1995).

These constructs frame the generalizing actions of the students in my study as they constructed the general metric function by first *relating* across their previous metrics, and then *extending* meaningful structures they identified through the process of relating. This involved explicit reflection on prior operations they had enacted in specific metric contexts that related to the properties of a metric, and so their extending activity occurred through reflected abstraction. I will examine their specific abstractions in the Results Section.

Methods

The data presented here was taken from a dissertation project involving two separate teaching experiments (Steffe & Thompson, 2000) with mathematics majors. Both teaching experiments entailed 8, 90-minute sessions in which students reinvented (Freudenthal, 1991) the general definition of a metric function. In this report, I focus on the final session of one teaching experiment involving a pair of students, Christina and Jerry. Both Christina and Jerry were mathematics majors (Jerry also studied physics while Christina was pursuing teaching credentials) that had completed the introductory real analysis sequence at their university. This two-term sequence covered topological results on the real line under the absolute value metric, as well as a rigorous treatment of basic calculus results including differentiation, integration, and point-wise and uniform convergence of function sequences. Importantly, the students had no previous exposure to metric spaces, or any form of measuring distance other than with the absolute value metric or the Euclidean measure of distance in real space. Thus, their activity with distances in more abstract contexts (e.g. the taxicab and supremum metrics in real space, sequence spaces, and function spaces) was truly novel for them. Figure 1 gives an overview of the specific spaces the students discovered during the latter sessions in the teaching experiment, as well as the major topics of discussion in each space.

$$\overset{\textit{Convergent Sequences}\qquad\textit{Distance Measurement}}{\underbrace{Sequences\ of\ real\ numbers}_{l_1,\, l_2\ and\ l_\infty}} \rightarrow \overset{\textit{Completeness}}{\underbrace{C([a,b])}_{L_1,\, L_2\ and\ L_\infty}} \rightarrow \overset{\textit{Openness}}{(X,\rho)}$$

Figure 1: Overall progression from sequence spaces to the general metric.

Throughout the teaching experiment, the students were given the goal-oriented prompt of characterizing sequential convergence in each new space they explored. This activity necessarily involved the construction of a distance measurement as well. The researcher then guided the resulting student activity primarily by facilitating moments of perturbation, as is consistent with teaching experiment methodology (Steffe & Thompson, 2000) and the RME heuristic of guided reinvention (Freudenthal, 1991).

The specific session that I am reporting on was the last session of the teaching experiment, wherein the students engaged in relating (Ellis, et al., 2017) by reflecting on commonalities across structures they perceived in the distances they had constructed throughout the previous sessions. This reflection on prior activity culminated in the students formally writing out the properties of a general distance function.

Video records were made of each session, and the video records were analyzed using the data analysis software MAXQDA. Specifically, each record was reviewed for moments of generalization, as well as moments of mathematical activity or discourse from which inferences could be made about the students' schemes and accommodations made to their schemes in the process of equilibration. Such instances were then coded according to the R-F-E framework (Ellis, et al., 2017), and also analyzed according to Piaget's constructs. The episodes were then further analyzed through Thompson's (2008) method of conceptual analysis (specifically Thompson's second use of conceptual analysis), consistent with Steffe and Thompson's (2000) concept of model building.

Results

For the purposes of this report, I will give an overview of Christina and Jerry's generalizing activity, and provide a representative sample of episodes that demonstrate their generalizing and abstraction. Recall that a function, ρ, is a metric if it satisfies the following four conditions: 1) $\rho : X \times X \to [0, \infty)$, 2) $\rho(x, y) = \rho(y, x)$, 3) $\rho(x, y) = 0$ iff $x = y$, and 4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. I will focus on Christina and Jerry's construction of properties 1-3, as property 4 (the triangle inequality) was generalized through a qualitatively different goal-oriented activity than those from which they generalized properties 1-3. Because of space restrictions, I will only demonstrate the progression of their abstraction of the third metric property, however I will briefly offer a description of the abstracted operations that contributed to the other properties as well.

The prompt and some initial generality

The session began with the students discussing their perspectives on a list of past distances they had constructed. The list, which I provided for them on the board, included the ℓ_1, ℓ_2, and ℓ_∞ distances (metrics) on $\mathbb{R}^n$ and sequences spaces, as well as the L_1, L_2, and L_∞ distances (metrics) on continuous functions defined on a closed interval. After some initial reflections, I reminded them of an earlier time in the experiment where they had expressed the desire to characterize their distances through a general distance function, A. Asking them to reflect on this activity, Jerry commented that " . . . each of these the process is the same, so let's call this thing a general distance A, and then write everything in terms of that." In this case, Jerry's use of the term 'process' conveyed that they each had the purpose of measuring distance on various mathematical objects, as later evidenced by his reflection that ". . . we found that there are other quantities [other distances] that satisfies the rules that we need [i.e. rules of distance measurement] but that may not necessarily look the same." These statements convey the relating that Jerry had engaged in, making explicit perceived relationships between different measures of distance in that they all behaved similarly, with the differences being the objects they acted on.

Jerry's statements were supported by Christina, who posited the "square root Pythagorean theorem [Euclidean distance] is like how we traditionally think of distance between two objects . . . and then the other ones kind of just go into things that we don't normally think about." The students' comments reflect that each new distance they constructed adhered to some collection of "rules" that distances should follow. Their formal defining of the metric function then came about through thematization at the general level (as it occurs in reflected abstraction[2]) of those rules that they had brought out earlier in the experiment. In this way, the students' formal statements of the properties of A were generalizations that occurred through reflected abstraction of the operations carried out in accordance with the "rules" of distance measurement as understood by the students. I will now give a representative sample of such operations that contributed to the students' defining of the third metric property, $\rho(x, y) = 0$ iff $x = y$.

The meaning of zero distance

Jerry and Christina's understandings of 0 distance were largely motivated by their explorations of sequential convergence in the various spaces they examined throughout the teaching experiment. Integral to the characterization of convergence is the tendency of the sequence approximations to the limit point to "tend towards 0". Early on, Jerry and Christina realized that convergence occurring in the way they intended necessitated a meaningful 0 distance measurement. In particular, if the students constructed initial distance measurements that resulted 0 distance measurements for non-similar mathematical objects, they then altered the form of their measurement to achieve a meaning consistent with what we know to be the general metric.

As an example of this activity, I reference the students' initial construction of the taxicab metric from an early session in the teaching experiment. The students initially used the formula $L = |v_x - w_x + v_y - w_y|$. To facilitate perturbation, I asked the students to measure the L-distance between the vectors $\vec{v} = [2, 1]$ and $\vec{w} = [1, 2]$. Upon calculating a 0 distance, I had the following conversation with the students:

Jerry: It seems weird. I don't like that.

Interviewer: And why?

Jerry: Because if we drew a picture, right? [draws two different vectors, $\vec{v}$ and $\vec{w}$, and a difference vector connecting them] We've got a — this is $[1, 2]$, so here's this vector and the other one is going $[2, 1]$. This vector, the distance seems like that should be a number greater than 0, but here we show that it is 0.

Interviewer: And why do you feel like the number should be something greater than 0?

Christina: Because when we're saying that the — in our lines — this is only looking at the convergence [their statement of $L_n \to 0$]. And we're characterizing that by distance of 0. However, if two vectors are converging upon each other, then they're becoming the same vector essentially, and those two things [the vectors Jerry drew] aren't the same vector but their distance is 0.

This discussion facilitated their altering of L to the standard ℓ_1 (taxicab) metric. This refinement of L in this instance constitutes generalizing through *extending*[3]. Their generalizing activity in this instance demonstrates the "rule" that they imposed on their new L distance, primarily that distance measurements of nonsimilar objects should be nonzero. To this point, Jerry later commented that "It seems like a good thing for our distance to be able to do, 'cause we want to use it to differentiate between vectors. If we can't, then sort of what's the point, I guess?" This conveys that Jerry was conceiving of using distance functions as a means of taking two

[2]Piaget took thematization to mean "to know [something] consciously and in an easily verbalized form" (Piaget, 2001, p.31)

[3]See *operating* in Ellis et al. (2017).

vectors and obtaining information about the differences between them based on the information given by the distance measurement. I next give another example from their sessions exploring functions spaces, where this operation of differentiating non-similar vectors was projected to a higher level of organization, and then conclude with the thematization of these operations at the general level during the last session of the experiment.

When exploring measures of distance between continuous functions defined on a closed interval, the first measure of distance the students initially constructed was similar to the L_1 measure of distance in the form of $d = \int_a^b f(x) - g(x)\, dx$. Facilitating a similar perturbation to that of the taxicab metric, I asked the students to measure the distance between the functions x and $x + sin(x)$ on the interval $[-\pi, \pi]$. Responding to this, the students made the calculations in Figure 2 and had a discussion, in which Jerry made the following comments:

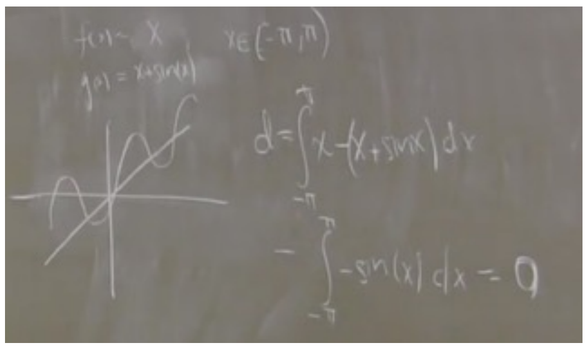

Figure 2: Calculating the 0 distance of x and $x + \sin(x)$

Jerry: . . . we know, visually, that our functions will look like this [draws the graph in Figure 2], and so those definitely aren't the same. . . . Like if I had a sequence of functions that converges to a function, I want to show that they end up becoming the same thing eventually. So it doesn't make sense for these [x and $x + sin(x)$] to have 0 distance but look different. . . .

As before, the students imposed the meaning of 0 distance on their new d function to adhere to some abstracted notion of distance measurement that gave specific meaning to measurements of 0 distance. While sequential convergence was a motivator for this meaning behind 0 distance, ultimately the students generalized by imposing meanings on these function distances generated from an abstracted construct of distance and distance measurement.

This meaning in the general setting was first voiced by Christina. During the final session, after revisiting the above L_1 example above, Christina said that ". . . to say that a distance is 0 means — that like two dissimilar things has distance 0 means that they're on each other essentially, but that means that they are the same thing." Formalizing their understandings, the students then wrote the two conditions $A(u, v)>0$ if $u \neq v$ and $A(u, v) = 0$ if $u = v$, which simplifies to the third condition of a metric. I infer that this formal statement was a written thematization of the operations that they had projected throughout the teaching experiment related to the activity of differentiating objects through interpreting the result of distance measurement. In terms of their generalizing activity, they engaged first in relating and then in extending (specifically through *removing particulars*[4]), as the specific contexts of the metric spaces were abandoned to reflect the general structure of the distances they wished to convey. Further, as

[4]See the subcategories of extending in Ellis, et al.'s, R-F-E framework (2017)

evidenced by their verbal and written thematization, this generalizing activity was reinforced by reflected abstraction of the operations involved in extracting meaning from a measure of 0 distance between two objects.

The other metric properties

This progression of generalizing and abstracting similarly occurred with each of the metric properties. Metric properties 1) and 2) emerged through attending to operations involved in distance measurement. In particular, the symmetry of the metric function emerged through abstracting operations involved in comparing measurements of reorderings of the objects being measured. Jerry described this property through the analogy of "... the distance between me to the wall is the same as the distance from the wall to me." Further, the first property of a metric emerged through Jerry and Christina both attending to distance as a measurement of "how far apart things are," and that conventional measures of distance primarily convey meaning through nonnegative measurements. This highlights that, for Jerry and Christina, carrying out operations involved in distance measurement (in the sense of conceiving distance measurements physically) was a productive and integral aspect of their generalizing activity to the level of a metric function. Consciously abstracting these operations (as occurs in reflected abstraction) then resulted in their production of the first three general metric properties.

Discussion and Concluding Remarks

This report demonstrates a productive way that students might learn and understand the metric function and its properties, and how the generalizing action of relating can facilitate reflected abstraction. Specifically, as the metric function is a measure of distance, conceiving of the activity of measuring distance, in this case physically measuring distance, can reveal certain operations inherent to distance measurement that can be abstracted to reveal the structural properties of the metric function. In the case of Jerry and Christina, they understood distance measurement as adhering to a certain set of "rules" that they imposed on their various constructed distances throughout the teaching experiment. These "rules" provided them with certain operations (such as the comparing of the similarity between objects in reference to their measurement value or comparing the measurements of the same object pair up to ordering) that they abstracted into increasingly general mathematical settings.

Through engaging in *relating* of their list of specific distances, (i.e. reflecting on the operations through which they constructed the collection of specific distance measurements) the students were able to reflect on and regulate their prior abstractions. In this way, the students engaged in reflected abstraction by "reflecting on reflection"(Glasersfeld, 1995, p. 105) and then *extended* the resulting metric properties to the general level by *removing particulars*. Thus, their generalizing activity was complemented by reflected abstraction of the specific operations that comprised the first three metric properties. Further, the generality of the metric structure was reinforced by the regularity of its occurrence across metric spaces (i.e. various spaces such as $\mathbb{R}^2$ and L_p).

Continuations of this research will investigate the impact of students attending to distance measurement in various abstract spaces prior to introduction of the general metric in a classroom environment. Future research will also explore the role that generalization plays in other constructs vital to real analysis, such as the measure function. This work builds a foundation of the ways that undergraduate students can reason about real analysis when transitioning to a graduate setting.

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Analyzing the Nature of University Students' Difficulties with Algebra in Calculus: Students' Voices during Problem Solving

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The aim of this research was to investigate the nature of difficulties with algebra in calculus problems from the perspective of students. We employed Skemp's (1979) theory to analyze two calculus students' difficulties with algebra in an interview setting. Our findings indicate that although these students were aware of their challenges with algebra, they struggled to resolve those issues in the context of calculus. Likewise, both seem to struggle in different ways with algebra outside the context of calculus. Implications for teaching based on our current research will be provided.

Keywords: algebra, calculus, path, director system, schema

Theoretical Background

Although, research on students' difficulties with school algebra has been prolific (e.g. Ashlock, 2010; Booth, Barbieri, Eyer, & Pare-Blagoev, 2014; Kieran, 1992; Hoch & Dreyfus, 2004; Stacey, Chick, & Kendal 2004), and students' difficulties with Calculus (e.g. Bressoud, Mesa, & Rasmussen, 2015, Tallman, et. al, 2015) has been conducted, research on students' school algebra shortcomings in calculus courses are scarce. Reeder (2017) addresses the fact that while students may be successful with algebra in high school, they often leave high school with a shallow, inflexible understandings. While understanding why and how this gap in student mathematical knowledge and skills exists is a complex endeavor, the fact that it does exist, is commonly known. Universities are keenly aware of the mathematical challenges of students entering the university. In light of this, many interventions have been developed to help students fill the necessary gaps in the mathematics knowledge and skills needed to be successful in university level mathematics courses. Unfortunately, according to McGowen (2017), one of the most common interventions, remedial mathematics courses, is not providing the needed support students need to be successful in university mathematics courses.

Recent research by the authors sought to understand the nature of student challenges with algebra in calculus settings. In a study by Stewart, Reeder, Raymond, & Troup (2018), participants in a Calculus I course were asked to solve a set of calculus questions and corresponding algebra questions that paralleled the algebra needed in the calculus questions. The findings of this study revealed, many students struggled with the algebra, inside and outside of the calculus context. Our research shows that many students, when confronted with algebra in a calculus context, tried to avoid the algebra required to solve the problem while others attempted the algebra and lost their way resulting in their inability to complete the problem correctly.

To examine the nature of the algebra difficulties in calculus context, for this study, we will employ Skemp's (1979) model of intelligence presented in his book, *Intelligence, Learning, and Action*. In remembering his work, most readers will recall his *relational understanding* and *instrumental understanding* (Skemp, 1976). Later, Skemp (1979) devoted an entire chapter on understanding (Chapter 10) as he developed his idea of a *schema*.

In conveying his theoretical ideas and connecting to his audience, Skemp (1979) made use of many everyday examples. In our theoretical stance we will draw on a coherent segment of his model and utilize some of his examples applicable to this study, in order to analyze calculus students' mathematical thinking and actions. Skemp's model claimed that most human activities are for survival and therefore goal orientated. In order to explain how humans organize their actions, he used the metaphor of a *director system,* which is central to his model. He defined a director system "that which directs the way in which the energy of the operator system is applied to the operand so as to take it to the required state and keep it there. .. for the rudder it is a valve mechanism" (p. 41-42). By an operand he meant, "that which is changed from one state to another and kept there…e.g. a ship's rudder, which is brought to the desired position and kept there" (p. 41). He defined "operators, as that which actually does the work of changing the state of the operand (..the position of the rudder) from its initial state to the state chosen by the …helmsman.)" (p. 41). In Skemp's view:

> Using swimming as an example, a non-swimmer is outside his prohabital if he is in deep water, not because of lack of muscular strength but because he cannot make the right movement. He is within the capacity of his operators but outside the domain of his (relevant) director system. A good swimmer caught in an off-shore current is also outside his prohabitat but for a different reason. He can make the right movements, but cannot swim powerfully enough to reach the shore, or he cannot keep it up for long enough. So he is within the domain of his director system, but outside the capacity of his operators. Both are non-viable because they are outside their prohabitats; but for different reasons. (p. 62).

Skemp defined the prohabitat as "… that region which is within both the domain of the director system and the capacity of the operators" (p. 62). Within Skemp's model he defined the idea of *knowing that*, as possessing an appropriate *schema*. In his views a "schema is a highly abstract concept" (p. 167). He defined "a *path* as a sequence of states and a *plan* consists of (i) a path from a present state to a goal state; (ii) a way of applying the energies available to the operators in such a way as to take the operand along the path" (p. 168). He further described "the connection between *knowing how* and *being able* to is the connection between having arrived at a plan, and putting it into action" (p. 184). In his view, "prerequisite for the production of these plans is understanding: the realization of present state and goal state within an appropriate existing schema" (p. 170). Some researchers have employed Skemp's model and drawn from his work. For example, Olive and Steff (2002, p. 106) used Skemp's work to build "a theoretical model of children's constructive activity in the context of learning about fractions." Berger and Stewart (2018) employed his idea of *schema*, to describe students' proofs in an introductory topology course.

The purpose of this study is to investigate student thinking as they encounter algebraic problems within a calculus context in order to shed light on the origin of these difficulties. More specifically, this study sought to answer the following questions: How did students react/respond to algebra processes in the context of calculus problems? What were their plans and what paths did they take to reach their goal state?

Method

The study employed a qualitative case study methodology for data collection and analysis using Skemp's (1979) model as a framework for making sense of the data. Students enrolled in Calculus I at a large university in the South Midwest United States were invited to participate in an one-on-one interview wherein they would be asked to complete a few problems and discuss their strategies and challenges with those problems (see Figure 1). Students were recruited from multiple Calculus I classes early in the semester. If interested in participating, they were asked to provide their name and email address. A member of the research team contacted each student and arranged for a time for a two-hour interview. Ultimately, four students participated in these interviews. Each participant student was given three common Calculus I tasks and were asked to choose two to complete. Based on the participants choices of calculus tasks, they were then given an additional two algebra tasks which mimicked the algebra skills needed in the calculus tasks. As they solved each of these four tasks, they were asked to think-aloud and describe what they were doing or thinking about. After students completed the four tasks, a semi-structured interview was conducted to further investigate students' thoughts, perceptions, confusions, and frustrations. The questions posed in the interview sought to elicit more of the participants' perceptions and thinking. Probing questions were asked by the interviewers when warranted by participants' responses. Once all data were collected, it was de-identified and think aloud interviews were transcribed verbatim. The data were analyzed using a variety of themes drawn from Skemp's model as described in this paper.

1. Evaluate $\lim_{x\to 3}\frac{x^2-9}{\sqrt{x}-3}$ 2. Use the limit definition of the derivative to find $f'(x)$ for $f(x)=(x-4)^2-x$ 3. Find the critical points of $f(x)=\frac{-(x^2+1)}{3x^2-6}$. 4. Simplify $\frac{x^2-9}{\sqrt{x}-9}$ by rationalizing the denominator. 5. Find $f(3+x)$ if $f(x)=\frac{(x-4)^2-x+4}{x-3}$ simplify your results by eliminating parentheses and collecting like terms. 6. Use the distributive property to solve $\frac{-2x(3x^2-6)-6x(-(x^2+1))}{(3x^2-6)^2}=0$.	1. What are your thoughts about the problems you just completed? 2. Which problems were particularly challenging? 3. How do you think you did on the problems? 4. Would you like to ask any questions about these problems? 5. What, if anything, do you find challenging about MATH 1823? 6. Do you feel you were well prepared for MATH 1823? Why or why not? 7. How confident do you feel in your ability to complete calculus problems? 8. How confident do you feel in your ability to complete algebraic problems?

Figure 1. Tasks and questions for student interviews.

Results

Based on our prior research (Stewart & Reeder, 2017; Stewart, et al, 2018), we have established two common types of calculus problems with corresponding algebra occurrences in those problems. These are presented as problems wherein the calculus proceeds the algebra (Type 1) (see figure 2 #1) and wherein the algebra proceeds the calculus (Type 2) (see figure 2 #2). Analysis of both these common types of problems presented in Calculus I classes, reveals that in Type 1 calculus problems, many students can take the first derivative, but are not able to carry out the many steps of algebra to complete the problem (Stewart & Reeder, 2017). Likewise, analysis of Type 2 calculus problems, reveals that many students either try to avoid the algebra in the first steps altogether, or have difficulty with the algebra that often involves rationalizing the denominator, factoring, which results in incorrect answers (Stewart, et al, 2018).

Calculus: Remember: • Use the product rule • Use the chain rule	**1. Find the absolute maximum and absolute minimum values of $f(x) = x\sqrt{x - x^2}$** Differentiating yields: $= x\dfrac{1-2x}{2\sqrt{x-x^2}} + \sqrt{x-x^2}$
Algebra: Simplify the derivative equation by adding the two algebraic expressions we find via common denominators.	Combine into a single fraction $= \dfrac{(x-2x^2)+(2x-2x^2)}{2\sqrt{x-x^2}} = \dfrac{3x-4x^3}{2\sqrt{x-x^2}}$
Calculus: To find the critical numbers, set $f'(x) = 0$ Algebra: Factor to find roots of the quadratic polynomial.	Set $f'(x) = 0$ $\dfrac{3x-4x^3}{2\sqrt{x-x^2}} = 0$ $\Rightarrow 3x - 4x^2 = 0 \Rightarrow x(3-4x) \Rightarrow x = 0$ or $x = \frac{4}{3}$
Calculus: Understand that to find the absolute max and min, we have to test the critical points and the boundaries.	Test boundaries and critical points
Algebra: Actually test the boundaries and critical points by evaluating the function at the appropriate values.	$f(0) = f(1) = 0$ absolute minimum $f\left(\dfrac{3}{4}\right) = \dfrac{3}{4}\sqrt{\dfrac{3}{4} - \left(\dfrac{3}{4}\right)^2} = \dfrac{3}{4}\sqrt{\dfrac{3}{16}} = \dfrac{3\sqrt{3}}{16}$
Calculus: Recognize the denominator prevents us from evaluating the limit directly. Algebra: Recognize the common factors in numerator and denominator, rationalize, factor, and cancel these common factors.	**2. Evaluate $\lim\limits_{x\to 3} \frac{x^2-9}{\sqrt{x-3}}$** Rationalize the denominator: $\lim\limits_{x\to 3} \dfrac{x^2-9}{\sqrt{x-3}} * \dfrac{\sqrt{x-3}}{\sqrt{x-3}} = \lim\limits_{x\to 3} \dfrac{(x^2-9)\sqrt{x-3}}{x-3}$ Factor the numerator: $\lim\limits_{x\to 3} \dfrac{(x+3)(x-3)\sqrt{x-3}}{x-3}$ Cancel the common factor: $\lim\limits_{x\to 3} \dfrac{(x+3)\cancel{(x-3)}\sqrt{x-3}}{\cancel{x-3}} =$ $\lim\limits_{x\to 3}(x+3)\sqrt{x-3}$
Calculus: Recognize that since the function is continuous, the limit can be evaluated directly. Algebra: Evaluate the limit directly by setting $x = 3$ and computing the result.	Evaluate at $x = 3$ $= (3+3)\sqrt{3-3} = 6(0) = 0$

Figure 2. Type 2 Calculus Problem (#1): Taking the first derivative (calculus) followed with many steps of algebra. Type 2 Calculus Problem (#2): Many algebra steps followed by the final step of taking the limit (calculus).

While participants in this study were given both Type 1 and Type 2 problems, for the purpose of this paper, we will only focus on one Type 2 calculus problem that all participants selected to solve. We will share findings resultant from the interview questions with two participants. The problem that all participants selected to solve was a limit problem requiring students to begin by rationalizing the denominator in order to get started (see Figure 2). The results will be presented as two cases.

Student 1 Case. Thinking aloud, Student 1 wrestled with solving this problem. He shared:

> … finding the limit as it pushes 3 but I can't put 3 in right now because that will put zero in the denominator and that doesn't work so I have to do something to this to make it work. So, I'll multiply both the top and the bottom by the square root of x minus 3... yeah... that way I can get rid of the square root in the problem? Yeah. And then I'll be able to work with the x minus 3 in the bottom. So, then to just get rid of the bottom part of the fraction... yeah... I can... yeah multiply the top and bottom by x minus 3 because that's the same as multiplying by 1... I think... yeah it is... it is. And then, I still have the square root of x minus 3 on the top, and there's nothing on the bottom, there's 1... or... no that doesn't work... because then I just have a different factor on the bottom... I'll just have that squared.

When challenges with algebra were encountered, he began to re-think his process:

> "I'll just start over. I don't think that first stuff was right anyway. There's probably something I could do with the conjugate but I don't remember, I don't know if that applies here. I don't think it does. Maybe it does. Um... I can... I can factor the top that's what I can do. Actually no, I'll go back I'll do the same first step again... that works... and so I'll do... multiply both the top and bottom by the square root of x minus 3, so that gets rid of the square root on the bottom and it's just x minus 3 and then I can factor... yeah factor the x^2 minus 9 into... because it's the whatever the difference of squares or something... it just works out... and that way I can cancel out the x minus 3 on the bottom now and I can take the limit with what I just have... that I can insert 3 into. So now I just have the x plus 3 times the square root of x minus 3. And I just plug in 3 because this is... this is real everywhere I'm pretty sure...yeah... yeah... no I can't. Can I? I don't think I can. Because I still have a problem with the square root on the top now. Maybe. Hmmm... no I can, I can take the square root of 0 that's fine. That's just 0. So... it's 3 plus 3, the square root of 3 minus 3 which is 0, so that's 6 times 0 which is just 0. So that's limit. Yeah."

This student made a plan and took a path that was not helpful, in trying to solve the limit problem. He then revised his plan and took a different path and was able to find the correct answer. In analyzing his work, we noted the connection between Skemps's *knowing how* and *being able to*. Although, he had a plan, due to the lack of algebra knowledge available, he was not able to reach the goal state on that chosen path in his first attempts at solving the problem.

When this student was asked about how they felt about the problems he had just solved, he immediately noted his challenges with algebra while working on calculus. "I don't know I think I just struggle with problems like this because it's hard for me to see what to do. Which I don't really know why because it's just... like... algebra. I don't know." This again can be made sense of in terms of an inability to determine a *path* when confronted with algebra in the context of calculus. When asked about which problems he felt more confident with, the calculus or the algebra, he indicated the calculus. This is interesting given the fact that the algebra problems paralleled the algebra needed with in the calculus problems demonstrating again Skemp's *knowing how* but not *being able to*. Clearly the student knows how to complete the algebra and has done so successfully many times but is not able to for these problems.

Student 2 Case. Student 2 initially approached the limit problem by trying to evaluate the limit without completing any algebra. After a few minutes, however, he began to try to simplify the problem. He shares his thoughts as he attempts to solve the problem and notes that he cannot recall how to complete this problem because it has been a few weeks since limit problems were the focus of study in class:

> …for evaluating the limit of x goes to 3, function being x squared minus 9 over root x minus 3. So, to solve this one... um... I am going to... let's see... I guess I could divide by the highest power of x which in the denominator is root x squared... yeah... hang on... do I need to do that? Well, … for the limit to exist the left-hand limit has to equal the right hand limit so... um... as x is approaching 3 from the negative side, from the left um... our denominator is getting closer and closer to 0. But that's going to be slightly less, so it's um... approaching 0 from the... the left-hand side though. But I don't think... that is only going to tell me if there's asymptotes in the graph, if I recall. Um... I mean it's been a while since I've done limits. And if we divided by the... I'm just going to go ahead and divide everything by the highest power of x but... yeah no... I don't want to do that. Um... Yeah, I would have to... honestly, I really don't remember... and I would have to... I would need to jog my memory... Which I think, I mean I've done them... if I were to jog my memory I think I would... I don't think I would have too much of an issue but... everything from the beginning of the semester I have really put on the backburner and I need to bring it back.

Avoiding algebra is one of the cases we see often in calculus questions (Type 2). This student made a plan to "divide by the highest power of x". He then questions that plan and abandoned it. Then he recalled some limit laws, and at this stage he is not thinking about performing any algebra, rather thinking more formally. Failing that, he decides to go back to his original plan and "divide everything by the highest power of x". However, his lack of algebra again lets him down. He does not say, how should I do that, instead he says: "I really don't remember", hence, he is not able to action his plan. Unfortunately, this student was not able to reach his goal state of solving this problem.

When this student was asked about how he general felt about the problems he indicated that he had some difficulty with the limit problem given it had been a few weeks since he had worked on them in class. "I think it's just that I haven't …done these honestly, not .. it hasn't been that long … a couple of weeks? And the thing is … I know how to do them, but I do not know it well enough…" In this case, utilizing Skemp's swimming example, we can see that the student knows how to swim but while swimming in new waters, or having not gone swimming for some time, he is unable to swim well.

Discussion and Implications

Calculus courses are widely considered a gateway to disciplines in Science, Technology, Engineering, and Mathematics (STEM), and as such, have garnered particular attention. Negative experiences encountered in gatekeeper or introductory math and science courses are significant contributors to more than half the attrition of declared STEM majors (Crisp, Nora, & Taggart, 2009; Mervis, 2010). In this way, calculus course often act as a significant obstacle or one that discourages students from pursuing STEM majors (Bressoud, Mesa, & Rasmussen, 2015).

Prior research (Stewart, et al, 2018) revealed that students' algebraic challenges included problems working across the balance point in equations, cancelling, operating with radicals, distributing, and incomplete algebra. That incomplete algebra was one of the most common errors was also both interesting and puzzling. Despite knowing what types of mistakes students were making it was difficult to ascertain *why* they made the mistakes. This study sought to better understand why calculus students make mistakes with algebra.

Utilizing Skemp's (1979) model to make sense of students' work helped to frame a better understanding of why calculus students are challenged with algebra. Skemp's swimming example is particularly useful. Successful students are also those strong swimmers who are able to swim within their boundary. They can swim regardless of the water, whether it is deep and unfamiliar, or shallow and calm. These students can work successfully with algebra within or outside a calculus context. They know what to do and are able to do it. Likewise, successful students are able to recognize when they are not being successful and choose a different path. According to Skemp (1979), "the greatest adaptability of behavior is made possible by the position of an appropriate schema, from which a great variety of paths can be derived." (p. 169). Unfortunately, the majority of students are not strong swimmers. Despite having completed several years of high school algebra and being placed in a university calculus course, many students seemingly *know how to* but are *not able to* successfully deal with the necessary algebra needed for most calculus problems. The students in this study would often begin a path but it would not lead them to their goal state.

In dealing with limit problems specifically, most instructors agree that first year calculus students struggle with conceptual and procedural aspects of limits. However, the nature of these struggles are not known. We believe that theorizing the situation will give insight in understanding the extent of students' difficulties and interventions for instruction. We also believe that more research in understanding students' difficulties with algebra in calculus is needed.

We agree with Tall (2017, p. 61) who suggests that mathematicians, curriculum designers, teachers, and learners need "to become explicitly aware of the underlying supportive and problematic aspects of long-term learning". Reeder (2017) suggests college instructors face the challenge of working with students everyday who can seemingly make sense of complex mathematical concepts but are unable to solve problems related to those concepts due to their difficulties with algebraic procedures. While resolving the algebra deficiencies that students bring with them will be challenging, "it cannot be simply ignored and remain as an everyday accepted or out of our hands part of teaching university level mathematics courses" (Reeder, 2017, p. 15).

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Constraints for Changing Instructional Approach? WE CAN DO IT!

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Although different instructional models for teaching mathematics have arisen over the past decades, lecturing continues to be the preferred approach of abstract algebra instructors. We identified facilitating and constraining factors of instructional change by analyzing thirteen instructor interviews. Factors were further classified as internal or external; as related to factors of community, sources, curriculum, procedures, empowerment and feelings; and as institutional, networking or change management. Additionally, different levels of resistance or support were identified for each factor. Some results of our analyses include finding that supportive faculty chairs and colleagues strongly facilitate attempts at instructional change while departments open to change serve as a moderately supportive external factor towards instructional change. Student resistance constitutes the most frequent constraining factor that instructors face.

Keywords: Abstract Algebra, Instructional Approach, Change Strategy, Resistance, and Support.

Despite active learning emerging as an alternative pedagogical model, lecturing remains the predominant instructional model in science, technology, engineering, and mathematics (STEM) education (Freeman et al., 2014). According to Terenzini and Pascarella (1994), such preference toward lecturing may be due to the myth in undergraduate education that traditional methods of instruction provide effective means to teach undergraduate students. Lecturing focuses professors' actions and role towards covering content instead of student learning (Barr and Tagg, 1995). Active learning by contrast "engages students to do meaningful learning activities and think about what they are doing" (Prince, 2004, p. 223). While there are a variety of interpretations of *active learning*, a core element is student engagement in learning (Prince, 2004; Roehl, Reddy, & Shannon, 2013). Rasmussen and Wawro (2017) describe Inquiry-oriented instruction (IOI), a particular form of active learning, as the kind of instruction in which students are engaged in doing mathematics and collaborating with peers. In inquiry-oriented instruction, instructors listen to students' ideas and use student thinking to advance the mathematical agenda. Rasmussen and Wawro report that this kind of instruction is beneficial for learners because it improves student success and promotes deeper learning and drawing connections between mathematics and real-world contexts.

Research outcomes have showed that active learning produces better student outcomes (Freeman et al., 2014; Rasmussen, & Wawro, 2017; Smith, Vinson, Smith, Lewin, & Stetzer, 2014). Freeman et al. (2014) analyzed scores of equivalent examinations, other assessments, and failure rates and found higher student performance for those in active learning rather than lecture classes. Authors like Terenzini and Pascarella (1994) and Prince (2004) both hold that active learning produces important gains in students' academic skills (such as thinking and writing) because non-lecture approaches provide more opportunities for students to explore and develop ideas for themselves.

Johnson, Keller, and Fukawa-Connelly (2017) identified that most abstract algebra (AA) instructors self-identify as lecturers and exhibit strong differences in pedagogical practices from non-lecture instructors. Thus, despite the benefits associated with non-lecturing models for instruction, lecturing continues to be the preferred instructional model in the teaching of AA. Additionally, despite the existence of different ways of changing the practice such as Action

Research[1] or professional development programs, change in undergraduate mathematics teaching has been minimal and strategies for change have been marginally incorporated (Henderson, Beach, & Finkelstein, 2011; Henderson & Dancy, 2007; Kezar, 2013). An important issue that arises then is identifying what factors may aid AA instructors in changing their teaching practice and what factors may constrain instructional change.

Our research question is: What are the constraining or facilitating factors that AA instructors face in their attempts at implementing new instructional approaches? The purpose of this study was to identify facilitating and constraining factors AA instructors face when they change their instructional practice toward inquiry-oriented instruction. These elements will provide benchmarks that can be considered in instructional change initiatives by the RUME community. Doing so will also provide useful considerations for individual instructors and researchers interested in facilitating local changes to instructional approaches.

Literature Review

Past literature has highlighted the importance of identifying and considering institutional and structural barriers and rewards (Henderson et al., 2011). Johnson, Keller, and Fukawa-Connelly (2017) found instructors' beliefs and institutional context were influential in understanding pedagogical decision-making and instructional change.

Facilitating Factors

By promoting change through developing reflective teachers, Henderson et al. (2011) found that change agents facilitate and encourage teachers towards change while defining change outcomes as well; "the change agent role is to use specialized knowledge to develop new environmental features that require or encourage new behaviors or attitudes that will lead to changes in instruction" (p. 962). Furthermore, reflection and peer support from their learning community helped faculty make improvements in their teaching. Henderson et al. (2011) found that successful change strategies consist of "coordinated and focused efforts lasting over an extended period of time" (p. 972), performance evaluations and feedback, and an intentional focus on changing faculty conceptions. Concerning the first aspect, holding workshops or short development programs are successful change strategies when the intention is focused on specific changes like the incorporation of new technology. With respect to the second aspect, Henderson et al. (2011) identified that one facilitator is providing feedback on teachers' practices. One of the forms of such feedback is through Action Research, whereby faculty take an active part in the study of their own classes. With respect to the third aspect, an intentional focus on changing faculty conceptions aligns with the idea that "meaningful educational change requires changes in beliefs" (p. 973).

Henderson et al. (2011) identified that developing reflective teachers as a strategy of change required certain support structures, such as dedicated faculty focused on instructional change to centers of teaching excellence. The most common facilitating factors were individual consultants and working groups.

Obstacles

Many reported obstacles come from K-12 education (Henderson et al., 2011; Johnson et al., 2017). Henderson et al. (2011) found that disciplinary affiliation, loose coupling, and reward

[1] Action Research (AR) is one strong tendency to change educational practice through research done by the same practitioner.

structures are features that affect the effectiveness of change strategies in undergraduate education. Johnson, Keller, and Fukawa-Connelly (2017) found that instructors report many constraints like time, content pressure, lack of curricular sources, knowledge, and departmental affordances. However, these authors report that instructors receive support from their departments for redesigning their course and for considering professional development opportunities. This highlights that departmental affordances seem to be a perceived resistant factor.

Following Henderson and Dancy (2007), situational factors are obstacles that prevent the use of alternative ways of teaching. In the teaching of physics, Henderson and Dancy (2007) identified the following obstacles to introducing research-based instruction in physics classrooms:

- Students' attitudes toward school: This refers to a lack of students' responsibility and their poor study skills.
- Expectations of content coverage: Teachers will not invest in research-based instruction if they must cover a lot of material. This constraint is also identified by Johnson, Keller, and Fukawa-Connelly (2017) in AA instructors.
- Lack of instructor time: Research responsibilities and teaching loads occupy instructors' time and thus lead instructors to avoid learning new instructional techniques.
- Departmental norms: It is difficult to implement a new instructional method if no other faculty are implementing it and there are no local role models who can help introduce this new instructional method.
- Student resistance: Some students do not like to interact with each other, and sometimes they are not prepared to think by themselves.
- Class size and room layout: It is hard to develop cooperative learning and formative assessments with large numbers of students.
- Time structure: One semester courses are not as conducive as year-long courses for identifying individual differences in learning needs.

In addition to these findings from undergraduate physics education, some constraints have been identified specifically with respect to implementing Action Research as an approach for changing instructional practice. In action research, the teacher is considered the best candidate for researching and changing his/her practice because he/she is the person who directly faces the problems studied. Gibbs et al. (2017) identified time management as one hindrance because research is regarded as a time-consuming activity, perhaps even more so when one is doing it oneself on top of teaching. Another barrier is resistance to change (Gibbs et al., 2017; Males, Otten, & Herbel-Eisenmann, 2010). In particular, Bianchini, Maxwell, and Dovey (2014) explicitly cite a variety of initiatives in Australia that proposed continuous reflection by academic staff. Such projects failed due to how ingrained the established system (which prioritizes commercial aims for higher education) was. According to these authors, universities had to sacrifice quality due to increasing political and economic pressures.

Even without the research component of action research, Henderson, Beach, and Finkelstein (2011) identify barriers in developing reflective teachers as a way to instigate instructional change; challenges arose when reform efforts did not align with institutional structures and pre-existing faculty beliefs. Other obstacles to individual change included the lack of recognition and rewards for improved instruction, lack of support, and lack of time. Lack of time was paralleled in Johnson, Keller, and Fukawa-Connelly's (2017) study of AA instructors who argued that their main reasons for lecturing was a lack of time to redesign the class in addition to covering content.

As can be noted above, there is more literature on constraining elements than supportive ones. In our research, we attempted to identify supportive factors along with constraining factors to instructional change.

Framework

Following Henderson, et al. (2011), instructional change can be understood as “alterations in classroom practices” (p. 953) done by the instructor. Thus, we take *supportive factors* of instructional change to mean any factors that stimulate, provide for, promote or facilitate becoming different a classroom practice. Similarly, *constraining factors* are factors that limit or delay the process of doing a different practice. Additionally, Gibbs et al. (2017) argue that barriers for staff development can be constructed internally as well as externally. We assume that constraining and supportive factors can be classified as internal or external. *Internal* factors are within the control of the instructor who attempts instructional change. *External* factors are outside of the instructor’s control.

We also consider the categorization that Hampton and Cruz (2017) propose regarding different factors that influence instructional change in undergraduate STEM education. We considered the following categories: *Change management* related to “the design and management of the change process itself” (Hampton & Cruz, 2017, para, 10); *institutional support* related to “the formal institutional support to the change initiative (Hampton & Cruz, 2017, para, 11); and *networking* regarding the relations with other members from the community, specifically relations with mathematics education community members. These authors also describe the category of *empowerment* as part of another category labeled as *faculty motivation*, the latter of which is defined as “factors related to the faculty´s willingness to adopt RBIS [Research-based instructional strategies] in their classes.” (para, 13) We adopt the category of empowerment, which refers the evocation of autonomy or change in students’ participation in their learning process.

Methods

We analyzed 13 pre-existing interviews from AA instructors participating in the TIMES Project (Teaching Inquiry-oriented Mathematics: Establishing Supports). This study is focused on the following subset of interview questions:

1. Is your department chair supportive of efforts to try new instructional approaches? In what ways is s/he (un) supportive?
2. Are others in your department supportive of efforts to try new instructional approaches? In what ways are these colleagues supportive/unsupportive?
    - Have you had any experiences in which your colleagues were resistant to efforts you or others have made to teach in innovative ways? If so, can you give me an example of such an experience?
3. Have you experienced any student resistance to attempts you’ve made to teach in innovative ways? Can you give me an example?

We open coded transcripts for three interviews together, highlighting sections that we considered relevant without considering supportive or constraining categorizations. In the following step, each researcher coded five interviews individually, using the same criteria but with the addition of the same codes for expressions that paralleled those from the first coded interview when appropriate. With these 13 interviews coded, we reviewed the resulting codes from each other's analysis, arriving at a total of 276 codes. We organized all codes recording interviewees' ID, the associated code and the portion highlighted from the transcripts corresponding to each code. In order to cut down on the number of codes, we then included all the codes in a sheet (S2) to identify which of them referred to the same principal idea. We organized codes referring to the same idea under new composite codes. This effort reduced the 276 codes to 19 codes. Then we created a final sheet (S3) which had three columns: the 19 codes, the different statements from the transcript that correspond to that code and the interviewer ID of the interviewee who expressed the statement.

We categorized the 19 factors as internal or external and included the frequency in which each code appears in the interviews. We also categorized the factors as *institutional, networking,* or *change management* depending on the nature of the factor. Because all instructors aimed to introduce inquiry-oriented instruction into their classes, we did not consider the category of faculty motivation. We also categorized the factors under the categories of *community, sources, curriculum*, *procedures, empowerment,* and *feelings* depending on the elements involved in the factor. For example, if the factor was associated with someone from the instructor's institutional context, then the factor was categorized in the community category.

Findings

The codes and the transcripts from S2 and S3 ground our analysis of the constraining and supportive factors we identified in the implementation of the new instructional approach. Frequencies of 23, 14 and 23 (Table 1) show that the majority of instructors' expressions refer to support from the chair, colleagues, and department respectively. In these cases, the chair and the department played an important role in affording instructors the opportunity to take a risk and try inquiry-oriented instruction in their classes. Additionally, the frequent constraining factor was student resistance, which appeared 25 times across the interviews (Table 1). In answering the third question, instructor J noted how some students "are like just very uh poignant about how I'm never taking an IBL course again" and instructor D mentioned student resistance because "my description for what I was going to do for the course, it didn't align with the course description in the catalog". These examples reveal student resistance lies in their lack of approaching toward depth situations that they must face in IBL as well as a break between the instructor's practice and students' expectations.

In table 1, we present 12 supportive factors and five constraining factors identified in S2. We also include the classification of each factor depending on their nature, the elements involved, and the corresponding classification according to the instructor's control:

Table 1. Supportive and constraining factors

	Frequency	Factor	Kind	Elements	Nature
Supportive factors	23	supportive faculty chair	External	Community	Institutional
	14	Supportive colleagues	External	Community	Institutional
	23	Department applies and encourages a new instructional approach	External	Curriculum	Institutional
	20	Previous elements that lead to change	Internal-External	Feelings-Procedures	Institutional
	9	Department open to change	External	Community	Institutional

	4	Department focuses on teaching	External	Curriculum	Institutional
	8	Colleagues interested in instructor's new instructional approach	External	Community	Institutional
	8	Funding support	External	Resources	Institutional
	6	Benefits introducing the new instructional approach	External	Empowerm ent	Institutional
	8	positive teachers' feelings	Internal	Feelings	Change management
	6	Connections with mathematics educators researchers	External	Community	Networking
	5	Relations with mathematics educators	External	Community	Networking
	11	not resistant colleagues	External	Community	Institutional
	2	No student's resistance	External	Community	Institutional
	15	Actions along the new instructional approach	Internal	Procedures	Change management
	14	No interference	External	Community	Institutional
Constraining factors	4	Colleagues' resistance	External	Community	Institutional
	7	Negative teachers' feelings	Internal	Feelings	Change management
	12	Communication Problems	External	Procedures	Change management
	18	Situations introducing the new instructional approach	External	Curriculum	Change management
	25	Students resistance	External	Community	Institutional

In the factor *Previous elements that lead to change,* some instructors mentioned their classes were previously lecture-oriented, and for various reasons, they realized a need for change. For instance, instructor I referred to the lack of student participation in the learning process:

Instructor I: I spent hours and hours and hours making up these detailed handouts and I was only talking to myself. I was having a little math party of one up at the front of the room and, [sic.] and nobody else was invited to that party. They were just watching.

Concerns over poor student learning outcomes and prior experiences with introducing some change (positive or negative), and existing relationships between department chairs and mathematics educators all positively impacted current attempts at instructional change.

Communication problems appeared as a frequent constraining factor. Some instructors also expressed problems due to misunderstanding the educational intentions or miscommunicating the methods of an inquiry approach. Due to instructors' autonomy in their instructional approach as well as for making decisions about researching, some instructors do not attempt instructional change because they do not perceive such research as valuable. The following excerpts are evidence of this finding:

Instructor E: We're not a, pretty much research is not part of anything. You don't have to do it if you don't want to.

Instructor F: No, I mean, its, its um, and there's not forced 'you have to do it this way'. This is, this gentleman is not going to be, you know, doing a bunch of inquiry- in his classroom. It's not gonna happen. But as long as we don't make him, you know- As long as he gets to do things how he wants to do it, everybody's happy.

Conclusions

We identified supportive and constraining factors AA instructors faced when they sought to change inquiry-oriented instruction. The 13 IOI implementing instructors in this study met more supportive than constraining factors (12 supportive factors vs. 5 constraining factors). This differs from past literature focusing predominantly on constraining factors. We found that supportive faculty chairs, colleagues, and a department that enact and encourage a new instructional approach were three frequent external-institutional supportive factors that support the implementation of instructional change.

Henderson et al. (2011) establish that institutional support is not enough for instructional change; individual faculty must also be willing to engage in some kind of development. Our findings parallel those arguments. Specifically, we found two factors were pivotal towards achieving change: (1) a department that encourages instructors to introduce change and aids them in that process and (2) instructors' willingness to moving outside their comfort zones. The latter may come about when instructors realize that their lecture-based instruction has not produced the desired effect, resulting in a change of beliefs as well as a search for more effective forms of instruction.

Additionally, to implement a new instructional approach, instructors must recognize the need to clarify the intentions behind their instructional decisions and to change instructional methods in the first place. That will decrease students' resistance, which was the most frequent constraint identified in this study.

Our study found that the nature of most constraining factors is change management. It may be then that these constraints can be prevented if the instructor is aware of them from the beginning and can plan accordingly. Although colleague resistance is an external factor outside of an instructor's control, it is not a strong constraining factor and thus may not influence instructors' instructional change efforts substantially.

As a parting note, this study provides hope to those instructors contemplating change by showing that change is possible and different constraints regarding time and expectations may not be as severe as prior research may make it out to be. For researchers interested in promoting ways for instructional change, this study provides different constraints factors that can be handle previously as well as supportive factors that can be used for strengthening such proposals.

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Instructors' and Students' Images of Isomorphism and Homomorphism

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This study uses thematic analysis to examine the conceptual metaphors used by two abstract algebra teachers to describe the concepts of isomorphism and homomorphism, both in interviews outside instruction and during class. These metaphors are compared to the metaphors used by their students to describe these concepts. While the two instructors utilized similar metaphors for isomorphism, they did not share metaphors for homomorphism. Further, when looking from interviews to instruction, there was again more alignment with isomorphism than with homomorphism, with metaphors used to discuss homomorphism during the interviews being less present during instruction than those used to discuss isomorphism. The students in these two classes appeared to incorporate the instructors' metaphors to varying degrees.

Keywords: Isomorphism; Homomorphism; Abstract Algebra; Conceptual Metaphors

Experts have identified isomorphism and homomorphism as two of the most central topics to abstract algebra (Melhuish, 2015). Although some research has been done on how students approach isomorphism (e.g. Larsen, Johnson, & Bartlo, 2013), research explicitly on students' understanding of homomorphism, or on instructors' understanding of and instruction on isomorphism or homomorphism, has been scarce. Thus, the purpose of this study is to examine teachers' and students' understanding of isomorphism and homomorphism through their use of conceptual metaphors and to examine how teachers' and students' metaphor usage is similar and different. Specifically I sought to answer the following research questions: (1) What conceptual metaphors do the teachers use to describe isomorphisms and homomorphisms and what relationship exists between these metaphors and the mathematical content in instruction? (2) What is the relationship between the mathematical content in instruction and conceptual metaphors the students use to describe isomorphisms and homomorphisms?

Related Literature and Theoretical Perspective

An isomorphism between groups is defined as follows: "The map $\phi\colon G \to H$ is called an isomorphism and G and H are said to be isomorphic or of the same isomorphism type, written $G \cong H$, if ϕ is a homomorphism, and ϕ is a bijection" (Dummit & Foote, 2004, p. 40). A homomorphism between groups is defined as follows: "Let $(G,\star)$ and $(H,\boxdot)$ be groups. A map $\phi\colon G \to H$ such that $\phi(x \star y) = \phi(x) \boxdot \phi(y)$ for all $x, y \in G$ is called a homomorphism" (Dummit & Foote, 2004, p. 39). An isomorphism can be thought of as a function that preserves the structure of a group in another group of the same cardinality; a homomorphism also preserves the structure, but can be formed between groups of different cardinalities. Two groups may or may not be isomorphic, but there is always at least one homomorphism between groups: the trivial homomorphism, by which every element of G is mapped to the identity in H. Quotient groups link isomorphism and homomorphism through a theorem known by many names, including the Fundamental Homomorphism Theorem (FHT): "If $\phi\colon G \to H$ is a homomorphism of groups, then $\ker(\phi) \trianglelefteq G$ and $G/\ker(\phi) \cong \phi(G)$" (Dummit & Foote, 2004, p. 97).

A theoretical lens for analyzing mappings in general is the conceptual metaphor construct (e.g. Lakoff & Núñez, 2000). "Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another" (Lakoff & Núñez, 2000, p. 6). Conceptual metaphors have been used to examine students' reasoning about many topics including linear transformations and functions more broadly (Zandieh, Ellis, & Rasmussen, 2016). Zandieh and colleagues examined the properties and metaphorical expressions students used within five metaphorical clusters: Input/Output, Traveling, Morphing, Mapping, and Machine. While these clusters informed background knowledge, every effort was made to ascertain whether or not these clusters were appropriate for the specific concepts of isomorphism and homomorphism,. As isomorphisms and homomorphisms are particular types of functions, these metaphors offer a starting place for this investigation. However, in addition to the functional aspect of these concepts, there are also structural properties (e.g., groups can be *isomorphic*). Thus, considering the literature on how students reason about function is necessary but not sufficient.

Previous studies have examined isomorphism in problem-solving, proof, and teaching contexts. Early studies mostly provided students with two Cayley tables or stated two groups and asked if they were isomorphic or how they could tell they were isomorphic. Dubinsky, Dautermann, Leron, and Zazkis (1994) found that when students considered isomorphisms between groups, they considered the cardinality of each group, but not whether the homomorphism property was satisfied. Leron, Hazzan, and Zazkis (1995) noted students' tendency to check the cardinality of a group as well as a general utilization of "sameness" as a stand-in for isomorphism, terming this "naïve isomorphism." In related studies, Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to prove theorems related to isomorphism and to prove or disprove specific groups were isomorphic. Later studies on isomorphism focused on developing local instructional theories to inform teaching isomorphism. In 2009, Larsen recorded a teaching experiment in which participants were expected to generate a definition of isomorphism. Later, Larsen et al. (2013) noted that the homomorphism property was more challenging for students to unpack than the bijection property. Additionally, Larsen (2013) noted, "students' use of the homomorphism property is usually largely or completely implicit" (p. 722).

Recently, Hausberger (2017) addressed students' understanding of both isomorphism and homomorphism through a textbook analysis and teaching experiment in which he observed the failure of textbooks to define "structure" in the context of "structure-preserving" isomorphisms and homomorphisms. Thus although some work on students' understanding of isomorphism has been addressed, such as the focus on sameness in naïve isomorphism, limited attention has been paid to students' use of language or images while considering homomorphism and teachers' conceptions of isomorphism and homomorphism have been ignored.

Methods

Participants included two faculty members and two students from each teacher's junior-level abstract algebra class. Both teachers had taught the course at least once before. Instructor A was tenure-track faculty, and Instructor B was a full-time instructor. The students' backgrounds varied; all had mathematics as at least one major and had previously taken an introduction to proof course, but some were double majors and other previous coursework varied. Teachers were recruited at the beginning of the semester from that semester's abstract algebra teachers. Students were recruited based on their responses to a survey as part of a wider project.

Data for this paper are drawn from classroom video and a round of interviews with students and teachers. The classroom video data was collected from days when isomorphism or homomorphism-related topics were discussed in class. Participants engaged in semi-structured interviews (Fylan, 2005) lasting roughly one hour each. The relevant interview questions focused on definitions, descriptions, and explanations for a 10-year-old of the concepts of isomorphism and homomorphism. Interviews with teachers occurred as they began teaching isomorphism. (Both taught isomorphism before homomorphism.) Interviews with students occurred after their class learned about the FHT and took an exam on group isomorphisms and homomorphisms. All interviews were audio and video recorded and any written work was collected.

The interviews were transcribed and coded using thematic analysis (Braun & Clarke, 2006). This included multiple iterations of coding (Anfara, Brown, & Mangione, 2002); first, transcripts were open-coded for vivid, active words that could indicate conceptual metaphors; next, statements were viewed holistically for mathematical approaches being conveyed by statements; finally, codes were generated and refined by repeating the previous stages. These codes were influenced by Hausberger's (2017) ideas of structuralism and Zandieh et al.'s (2016) work with functions; specifically the Input/Output, Morphing, and Traveling codes are similar to the latter's definitions. The codes generated from this process are given and defined in Table 1. The classroom video was selectively transcribed; segments when isomorphism and homomorphism were originally defined and when the FHT was introduced were completely transcribed. However, technical proofs or computations and difficult to hear segments were excluded. The classroom transcripts thus generated were coded like the interview data.

Table 1. Codes, descriptions, and examples.

Code	Description	Common Examples
Embedded	Structure inside a structure	"living inside"
Input/Output	Function machine language where entry leads to new result	"spit out," "pop out"
Matching	Elements or structures aligned	"match," "line up," "correspond"
Morphing	Elements or structures altered from original format	"collapse," "condense," "transform"
Relabeling	Names of elements rearranged	"relabeling," "renaming"
Sameness	Structures equivalent in some way	"same exact thing," "equivalent structures"
Sight	Visual imagery used	"reflected," "image"
Structuralism	Structure-based language of the formal definition	"operation-preserving," "structure-preserving"
Traveling	Element or structure moves from location to location	"from G to H," "go to," "send to," "hit"

Results and Discussion

Metaphors in Instructor Interviews

Class A. Instructor A used a variety of language to address isomorphism, including structuralism ("preserves the operation") and traveling metaphors (e.g. "a function from one group to another group"). However, most of her discussion of isomorphism centered on two metaphors: renaming and sameness. She seemed to view renaming as more indicative of isomorphism (the function) and sameness as indicative of groups being isomorphic:

> So if I was trying to explain isomorphic…I would say two things are the same, just with different names. If I was trying to find…[an] isomorphism, I'd say it was…how I decided to rename the things in one group as the things in another group.

Instructor A initially described homomorphism using structuralism, saying it was “a mapping that preserves operation.” Later descriptions used mostly sameness, traveling, and morphing language, often in conjunction with each other as she structured her thoughts around the FHT:

> So this is my domain and let’s say there's a bunch of elements in here….My homomorphism clumps them into like regions or sets. So this is kind of all working inside my domain, and then I have my function that goes over to my range, and now this set is sent to a single element over here.…The operation between these sets is the same as operation between those elements.

Class B. Instructor B used a variety of metaphors to discuss isomorphism, including matching, relabeling, and sameness, in addition to structuralism. When discussing isomorphism as a function, he used language like a “relabeling of elements,” a “correspondence that matches like things with like things,” and a “mapping between two algebraic structures that preserves the structure.” Common language for isomorphic groups included talking about “equivalent structures” or “there’s really no difference between these structures,” where “structures” meant algebraic structures like groups or rings. His preferred view of isomorphism was as a relabeling:

> From an algebraic point of view, there’s really no difference between these structures, and so…if you just took these elements and attached these other labels instead of the labels you originally had, you get the same exact structure. So that’s the idea I try to get across more than… a bijective function that…preserves such and such operation. So I think it’s really the relabeling is the most natural way to think of it.

When discussing homomorphism, he initially used structuralism and traveling language (a “map from one structure to another structure”). However, he later used more sight and sameness language to contrast with isomorphism: we “kind of don’t really initially see how the…structure within the…domain group is reflected in the…codomain whereas with isomorphism we…see that right away. Right we just see that it’s…equivalence of structures.” When pressed, he gave a more vivid picture of homomorphism that included morphing, sameness, and traveling language:

> I guess you could sort of view it as threads condensing into a single…element in the codomain and…then those would become equivalence classes modulo the kernel of…the map etc. etc. If we look at the…7 elements that get mapped to a particular element, then what we really have is this, this equivalence class modulo the kernel, and…if we mod out by the kernel then we can take any one of those things as a…representative.

Metaphors in Instruction

Class A. Instructor A used inquiry-oriented materials based on the local instructional theories developed by Larsen and colleagues (e.g. Larsen, 2013; Larsen et al., 2013) to have students reinvent the definition of isomorphism in class. This is significant because it meant her students talked about isomorphism before a definition was given. Pre-definition, most public language describing attempts to map between a mystery table of six elements and D_6 (dihedral group of six elements) was matching metaphors. For example, consider the following exchange:

> *Student:* I think it’s harder to find what each element corresponds with the letter because they’re self identities, but the ones that are not self-identities are D and G so it’s easier to see which ones…are the only two elements that are not self-identities.
>
> *Instructor A:* Right, so this is the game you’re playing, you’re trying to correspond these letters with D_6 elements?

However, when a definition was given, the language Instructor A used largely matched what she had said in her interview, while also incorporating the matching language the class had used:

> These correspondences we have been working with are potential *isomorphisms* that allow us to "rename" elements in G with elements in H and then verify the operation to show that G and H are essentially the same.

She introduced the homomorphism definition before teaching the FHT. Pre-theorem, most homomorphism-related discussion was based on the formal definition or traveling metaphors (e.g. "What's something that definitely gets sent to the identity?"). However, to describe the FHT, Instructor A utilized the imagery from her interview:

> We can use the homomorphism to…construct bands where all of these little elements get sent to the same thing so they're grouped together. And what they're grouped together into are their elements in the quotient group.…So all the little dots that get sent to x will form a coset in our quotient group….And the partitioning we would have under the quotient group is the same that we'd have under the homomorphism.

Although she used less vivid language to describe the theorem in class, her explanation did include elements of sameness (final sentence) and morphing (grouping) as in her interview.

Class B. Instructor B used similar language in class to describe isomorphism as he had in his interview. He mainly utilized traveling (e.g. "identity kind of has to go to identity") and matching (e.g. "what would have to go with what") language when discussing how to approach specific mappings with students. He also referred to isomorphism as "the rule that's doing the relabeling," utilized structuralism in a manner similar to his interview, and extensively referred to isomorphism as "essentially the same" or when "two groups have exactly the same structure."

Instructor B's metaphors for homomorphism in class differed from his interview metaphors to a large extent. In class, he frequently spoke of homomorphism as a function using traveling metaphors (e.g. "So homomorphism is essentially a map, and again this could be from structure to structure in general. In our case it's from one group to another group…"). He also used structuralism on a number of occasions (e.g. "…and so it is a map that preserves whatever operation we have, in this case the group operation, but is not necessarily a bijection."). Although he drew on morphing language to describe homomorphism in his interview, his description in class drew more on an embedding metaphor when first discussing the FHT:

> The way to think about this then is if you've got a surjective homomorphism, then the range H essentially is already living inside of G somehow. All the information about H is already here, and in fact we can recover H purely in terms of G by taking the factor group of G mod the kernel. So we get an isomorphic group where we don't even have to refer to H at all. It's just purely in terms of G.

Student Metaphors

Class A. The majority of the language used by both students from Class A focused exclusively on the formal definitions. For example, Student 1A described an isomorphism as:

> …basically a function that maps one group to another group such that the function is one-to-one and onto and such that the function of the combination of two values in the first group is equal to the function of the first value combined with the function of the second value.

However, he moved beyond the definition to sameness when asked how to describe an isomorphism to a 10-year-old: "If two groups of numbers or anything are the same." However, the idea of "sameness" seemed to confuse him as well. When he was trying to describe a homomorphism for a 10-year-old, he noted, "…when you explain that it's two groups don't have to be the same then it gets really confusing on what is a homomorphism and what isn't a homomorphism." In trying to distinguish between isomorphism and homomorphism, he seemed unsure how to take bijection away from isomorphism and still have a coherent mental picture.

In addition to the formal definition, Student 2A used matching and sameness metaphors for isomorphism. For instance, he used two circles of ten colored marbles in a matching metaphor:

> …then you number them also 1 through 10 but instead you…rotate it so you don't have 1's matching up with the 1's and… so the 1 in the red matches up with the 3 in the blue, and then…you figure out if you have 1 plus 3, that'll get you to marble 4. Well marble 4 matches to marble 6 or whatever, so something like that.

His example about work being independent of path emphasized sameness: "The idea is…regardless of how you go, it's the same ending spot, so what you're doing is actually the same operation; this just looks different."

He expressed ideas like "isomorphism is a fancy case of homomorphism" multiple times and did not make much effort to distinguish between isomorphism and homomorphism. When pressed on homomorphism, he returned to the marble example, noting this time you could have less marbles "and…now you're allowed to overlap." He maintained the matching metaphor across isomorphism and homomorphism, but did not retain the sameness metaphor.

Class B. Student 1B's isomorphism language aligned with Instructor B's to a large extent as he coordinated sameness, relabeling, and structuralism language: "I guess an isomorphism would be a function, which is bijective and it's structure-preserving…I mean… basically, you can just relabel the Cayley table, but that's formalized as f of ab equals f of a times f of b."

Student 1B's language for homomorphism drew on metaphors and the FHT like Instructor B:

> A homomorphism is just a function that preserves the structure…not necessarily all of the structures; it might just preserve one structure. Like the integers map to Z mod 2 or something, that could preserve the structure of like the evens and the odds, but it destroys a lot of the other properties of the integers….[Preserving the structure] would be that definition: that f of a product b equals f of a product f of b, but…it's intuitive for me to go back and think about the Cayley tables because they're just saying that wherever the product of these two things gets mapped to gets mapped to wherever the product of wherever these two other things gets mapped to, so…that's the structure right there that's being preserved: things still will be nice and well-defined and play nicely…..

Notice he used traveling language as he described the integers mapping to Z mod 2, much like Instructor B's use of traveling language. He utilized structuralism through preserving the structure. However, he seemed to use the word "structure" in two senses: the homomorphism definition and an imposition of order. This latter sense is similar to Instructor B's embedding description of the FHT given in class, in which the emphasis was on the structure of the domain.

Student 2B defined isomorphism as, "an operation through which you would transform an element of one group to the corresponding element in an identical group," which utilized morphing, matching and sameness metaphors. He also gave a vivid sight metaphor coordinated with sameness language when asked what he would say to a 10-year-old:

> …isomorphism is, is closer to the mirror….Like you get the same thing back….But you look in, just like a regular mirror straight on, it's pretty much the exact same thing back, but it's not you. It's just an image of you that retains all the characteristics.

Student 2B's language for homomorphism was in many ways similar to his language for isomorphism. His initial description coordinated morphing and sameness metaphors: "an operation through which you would transform an element in one group to a group with similar characteristics that is of lesser or equal size." When later prompted about how he would describe homomorphism to a 10-year-old, he again shared vivid metaphors. He expanded on the sight-based mirror imagery from isomorphism to compare and contrast with homomorphism: "…sometimes you have mirrors that make you look smaller like at the corners of hallways and hospitals. Sometimes it's a little bit smaller. That's like a homomorphism." He also gave a morphing metaphor: "Look at your dad and then look at yourself. Imagine…what part of your dad went to you sort of as a homomorphism….he took a part of himself and sort of condensed it to create you.…" Although he used a condensing image like Instructor B's interview response, his condensing image did not possess the clear FHT structure of Instructor B's response.

Discussion

Returning to research question 1, the teachers were largely consistent in their metaphor usage in the interview setting and in class. Both teachers focused on sameness (more for isomorphic structures) and renaming/relabeling (more for the isomorphism function). Both also relied on the FHT and morphing, sameness, and traveling metaphors to provide meaning for homomorphism beyond the formal definition. However, they structured their understanding around the FHT differently: Instructor A focused on morphing within the domain and then traveling to produce sameness between the groups whereas Instructor B morphed while traveling to produce sameness (interview) or viewed the relevant sameness as being embedded in the domain (in class).

Addressing research question 2, there was some alignment between metaphors used in class and metaphors used by students. All four students utilized sameness language for isomorphism like had been used in class, though Student 2A also used a lot of matching language and Student 2B incorporated morphing language for isomorphism. However, their images of homomorphism varied widely. Students 1A and 2A did not use sameness to describe homomorphism. Student 1A seemed to try separating isomorphism and homomorphism by removing sameness to reach homomorphism, but did not know where that left him. Student 2A used matching metaphors for both isomorphism and homomorphism but only applied sameness to isomorphism. Neither student from Class A used an FHT-based picture like their teacher had used, though it is possible that Student 2A's matching language was based on pre-FHT discussion around homomorphism. Students 1B and 2B had more distinct images for homomorphism and were closer to aligning with their teacher. Student 2B used condensing language to describe homomorphism, though he did not give evidence of attention to structure within the group being condensed. Student 1B was more aligned with his instructor's embedding view from class, based on his attention to some type of organization being highlighted and shared between the domain and codomain.

Conclusion

Isomorphism and homomorphism are concepts central to the study of mathematical structures, specifically within abstract algebra and in math more broadly. Thus deepening our understanding of how teachers and students think about these concepts and what conceptions are communicated from teachers to students is critical. In this study, the naïve isomorphism view of sameness (Leron et al., 1995) was broadly shared whereas the images of the FHT used by instructors were not broadly shared, and the images used by students varied widely. These varied metaphors revealed varied conceptions (e.g. elements traveling to elements, shared structure inside groups, transforming from group to group) that may be more or less useful when solving problems. Thus future work includes investigating what isomorphism and homomorphism problems students with these metaphors can solve, especially because most descriptions given by the students aligned (to some extent) with the definition. Furthermore, other teachers and algebraists may or may not share these teachers' FHT-based images of homomorphism, so ascertaining other expert views of homomorphism is essential for future study.

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Following Students in the Transition to Proof: Examining A Case Where Reasoning and Performance Conflict

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The transition to proof is difficult for students – what developments do students show while learning how to prove? I present a short-term longitudinal, qualitative analysis of 11 undergraduates taking a transition to proof course, of the developments seen in their proving. Within this, I follow one student whose proof reasoning grew but whose performance on the proof construction tasks declined. Investigating this single-subject case serves as an example of the complicated interplay between development and performance. It serves as a reminder for how attending to performance does not account for students' thinking and vice versa.

Keywords: Proving, Longitudinal, Transition to Proof, Problem Solving, Case Study

Learning how to prove can be notably difficult for undergraduate students (Moore, 1994). The transition to proof is a shift in the "game" of mathematics, from answering exercises that are largely procedural (Schoenfeld, 1992) to justifying and writing arguments. Through research, we know what students struggle with (Selden & Selden, 1987) and their strategies (Karunakaran, 2014) with proving at a snapshot in time, but we know less about development, as in what the learning process of proving looks like over time. Understanding this learning process is important to help students directly but also for designing courses that support undergraduates' transition to proof. The research question that guides this work is: How do undergraduate students' proof reasoning develop over the duration of a transition to proof class? But to examine this further, I present a short-term longitudinal case of a student whose growth in proving in terms of problem solving is not captured by performance. First, I present findings across all the participants. Then, I explore in detail this one participant.

Conceptual Framework

There are different perspectives from which to approach research in proof (Stylianides, Stylianides, & Weber, 2017). One common perspective is to consider proof as a form of problem solving (e.g., Savic, 2012). But even within this, there are many cognitive skills that make up what we consider to be proving. Selden & Selden (2007) discussed two major sources of difficulty for students when writing proofs, namely the formal-rhetorical aspects of writing and producing a proof but also the problem-centered aspects of proving, of finding and noticing relationships among concepts that are crucial to proving the claim.

In adopting the perspective of proving to be a form of problem solving, I take students' proving to be problem solving but in the context of proof, i.e., the work of constructing a proof for a given statement. However, a true problem for an individual is one in which a person faces impasses, experiencing the feeling of being stuck (Savic, 2012; Schoenfeld, 1992). For this reason, I focus on students' intentions when stuck. This conceptual framework is aligned with the notion of studying proving as a process rather focusing on its product (Karunakaran, 2014).

Method

Participants were N=11 undergraduate students taking a transition to proof course at a large Midwestern university. This work is part of a larger study about the cognitive and emotional aspects involved in the transition to proof. This course was designed to ease the transition from calculus-based courses to upper-level math courses that involve proofs. The primary population was students majoring and minoring in mathematics.

A series of four interviews were conducted with each of the participants across one semester. Within each interview, participants were asked to complete two proof construction tasks. Participants were given the statement to be proven and given 15 minutes to construct a proof. Each proof construction task was administered as a non-intervening think-aloud, in that students were asked to verbalize their thinking, but the researcher did not ask questions while they were working in order to minimize interrupting their problem solving process (Schoenfeld, 1985). Instead a debrief was conducted with the participant immediately after each task, during which they were asked questions about their thought process, places where they perceived they were stuck, and other points of interest. Interviews were audio- and video-recorded.

All proof constructions tasks were about basic number theory content: properties of integers and real numbers, even and odd, and divisibility. In order to measure development, tasks were designed to hold content area constant and minimize any special domain knowledge, as much as is possible; care should be taken when making content-free claims about proving (Dawkins & Karunakaran, 2016).

Participants were also asked to draw an "emotion graph" for each proof task, where the X axis represented time and the y-axis represented the intensity of emotion felt. Interviews were transcribed, and analysis was done using grounded theory (Glaser & Strauss, 1967) techniques to identify and group students' intentions when stuck into developments across participants. A coding rubric was developed for assessing performance (correct, partially correct, or incorrect) on the proof construction tasks.

Results

Developments Across Participants

Before diving into the single case, let us look across the participants to gain a broad view of the developments they showed. The developments most strongly grounded in evidence are shown in Table 1, defined as seen in at least three participants. I provide a brief description of each development, providing singular examples for illustrative purposes.

Table 1. Select developments in proving, by participant

	Change in how one chooses a proof technique	Harness awareness of how solution attempt is going	Check examples in conjunction with other strategies	Explore and monitor
Amy				X
Charlie	X		X	
Dustin	X			
Granger		X	X	X
Gabriella	X	X		
Joel		X		
Jordan	X			

Leonhard	X			
Stephanie	X			
Shelby	X			
Timothy	X	X	X	X

Harnessing awareness of how solution is going. Students showed a growing metacognitive awareness of how their attempt was going, usually when they felt they were on the wrong track. Being aware of how one's solution is going is normal and expected; what is key is examining what they did in response to this awareness and how that guided them to better solutions.

For example, Timothy became stuck multiple times over the course of proving: "If x, y are consecutive numbers, then xy is even." In reaction, he reasoning out loud about the mathematical relationships: "If [x,y are] not consecutive, they wouldn't have this relationship...what does this tell me?" He said he would continue to try it this way but did not know if it would work, saying he "can't think of any other way" right now. He explained during the debrief: "I finished it out because I just wanted to get something down but I didn't really like that one." After getting stuck twice more, he completed his proof but was unconvinced; he did not feel good about it. In the debrief, he explained that "I didn't really like that one [proof by contrapositive]. And then I went back because I really wanted to do something with this directly. I liked that better." This sense served him well, as indeed his first attempt was not correct and his next attempt was.

Exploring and monitoring. Working without already knowing how a solution would go was another development seen across participants. Students were used to tasks in past courses that lent themselves to clear methods, but during the transition to proof course, students became stronger at careful, intentional "winging it" – working and exploring without knowing what would happen and noticing when a key piece of information for constructing the proof arose.

As an example of this, Amy considered herself a planner from the start, always thinking ahead, as said in the second interview: "I plan everything super far in advance...I just feel like for everything, I just look ahead. Even when I'm doing math problems. I just like, in my brain, I think about what I'm gonna do before I start doing it." Amy noted this was how she did math, planning out her mathematical actions in advance and thinking ahead in the problem.

But Amy became comfortable with working on her feet as interviews progressed. In Interview 4 – Task 2, she became stuck and said out loud that she did not have a plan while working but that she would figure something out. During the debrief, she talked about what she had been thinking then: "I feel like this whole portion, I was kind of stuck, but I was just like, 'Just check through the algebra until you can get to something.' I was like, 'I don't see this going anywhere, but I'm sure it will. Just keep going.'" Indeed, this proved fruitful, as she noticed something contradictory in her exploration and argued why this was an impossible situation thus turning her work into a proof by contradiction. She worked without a specific goal in mind and when a potential avenue appeared, she pursued it. She became comfortable exploring when unsure what to do. The key however was that she noticed an insight when it arrived.

Checking examples in conjunction with other strategies. Students would check examples, as a strategy located within a temporal string of strategies. On Interview 4-Task 2, Timothy became stuck because he was not sure how to negate the statement to be proven: "If x, y are positive real numbers and x ≠ y, then $\frac{x}{y} + \frac{y}{x} > 2$." He was stuck because he was not sure how to negate the statement. He then reasoned out loud what his issue was and possible decisions he could take, trying to write the contrapositive and contradiction of the statement. This was akin to parallel processing in terms of assessing which of many solution paths was a good idea. He then checked some examples: “I’m just thinking of examples in my head now so like going at it straight [direct proof], so let’s say we chose 1 and 2, so ½ + 2 is greater than 2.” He then stopped and switched to contradiction. He had done the following: reasoned out loud about his problem, imagined multiple paths, and then checked examples in order to determine which path to pursue. An example is not a proof, but it can provide an idea for a proof, and students used examples in this nuanced way.

Changes in how students chose proof techniques. The most prevalent development that occurred across the participants were changes in how they chose which proof techniques to pursue when approaching constructing a proof. Proof technique refers to tools such as direct proof, proof by contradiction, proof by contrapositive, and cases. To see how this unfolded across a semester, we look at Leonhard, as an interesting case in that despite showing sophisticated growth in his rationales for his choices, his performance on the tasks in fact declined. For this reason, we examine Leonhard’s case in more detail.

Case: Following Leonhard’s Process for Choosing Proof Techniques

In the beginning, Leonhard’s baseline practice was to choose proof techniques based on what he knew and was familiar with. In Interview 1 - Task 1, Leonhard chose to use proof by contradiction, despite being a little stuck because he was not sure how to negate the conclusion (see Figure 1).

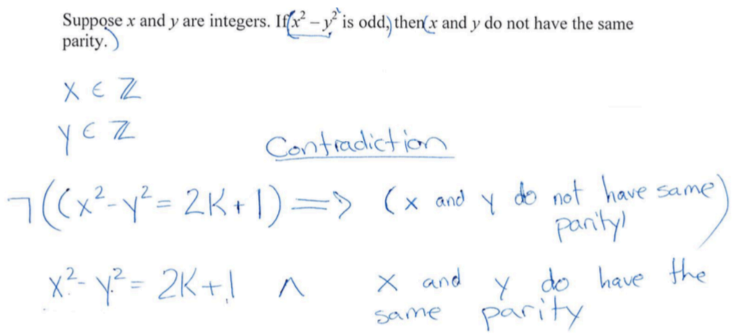

Figure 1. The beginning of Leonhard’s work on Interview 1 – Task 1

He reported that his rationale for that choice was that “A lot of time in class whenever we’re proving an implication, we use contradiction I guess so that’s why it’s my first thought.” He used contradiction because that is what they used in class and he was used to it.

Then in Interview 2 - Task 1, he wanted to do direct proof but became stuck because he was unsure whether what he wanted to do would work. He applied the definitions to x and y and then was stuck again over what method to use, direct proof vs. proof by contradiction. He became stuck again in choosing whether to do direct or contradiction. Ultimately, he chose contradiction and his reason was: “I decided to do contradiction because I know how to do it.” Leonhard chose what method to use based off what he felt he could do at that point in time, his own sense of fluency with methods.

As time progressed, there was clear growth in his reasoning for his choices. Interview 3 – Task 1 is an example where Leonhard cycled through a few options for proof techniques, as seen in his written work (see Figure 2). He used proof by contradiction but then became stuck when writing the negation, because his negation of the conclusion did not make sense to him. “One number is odd and all three numbers are odd” did not seem possible. He switched to proof by contrapositive but realized he had the same issue with how to negate the conclusion as before. He then switched again to direct proof.

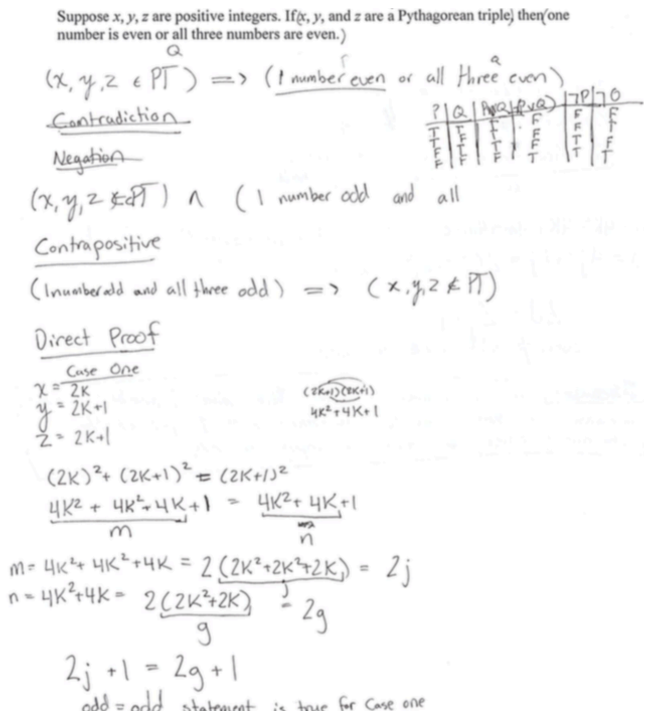

Figure 2. Leonhard's work on Interview 3 – Task 1

His rationale for why contradiction in the first place was as follows: "My mind goes straight there [to contradiction]. I like it the most because...at some point you usually run into something that just comes out sounding weird. So then you have to be right I guess." Contradiction was his favorite, so he tended to use it whenever he could. He liked it because of its unique nature in producing something nonsensical. He later added, "I don't know what possessed me to write this [contrapositive]," because he ran into the same issue. So Leonhard knew he liked certain methods over others and had some rationale - in how proof by contradiction results in a nonsensical claim and that he should have known better than to use contrapositive in this situation. His rationale was still general, however, in that contradiction was a technique he liked and that his fondness for it drove his usage of it.

Interestingly, he mused out loud about how his underlying idea may have been to check which proof techniques did not work well here and see what is leftover: "I guess this was a good way of crossing out the things that you can't do so you can find the things that you can do."

By the fourth interview, Leonhard displayed more precision in terms of detail in his rationales for his choice of proof technique. In Interview 4 - Task 2, he was stuck in the beginning and his subsequent actions were to identify the assumption and conclusion, test a couple examples for x and y, and then try proof by contrapositive (see Figure 3).

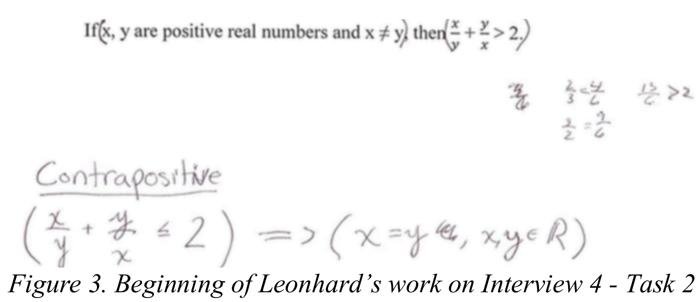

Figure 3. Beginning of Leonhard's work on Interview 4 - Task 2

His rationale for contrapositive was, “You can’t really do much with x not equal to y. But you can do a whole lot with x = y,” and “The contradiction wouldn’t give me anything to work with.” He wanted to start with x = y because he saw how an equality was more useful than having objects not equal to each other in proving, and neither direct proof nor proof by contradiction provided an equality. He decided what proof technique to use based on specifics of the statement to be proven. In addition, his rationale also explicitly explained why another proof technique (contradiction) would be less useful here. He had a rationale for why his chosen proof technique was a helpful approach and why other techniques would be less helpful. In the end, his proof was incorrect, as reaching a true statement ($2 \leq 2$) is not the same as showing the conclusion, but his rationale for why use contrapositive was coherent.

Making sense of Leonhard’s growth. Over the course of these interviews, the rationales Leonhard gave for why he chose the proof techniques that he did became more nuanced. He moved from choosing certain methods for (1) little to no reason to (2) having some rationale, with a general sense of one technique being better than others to (3) based on the statement itself. Leonhard showed clear growth, but if we look at his performance, Leonhard’s work was oftentimes incorrect. Across the interviews, he got one task correct (Interview 2-Task 1) and two tasks partially correct (Interview 1-Task 1 and Interview 2-Task 2). Moreover, his work on the last two interviews (all four tasks) was all incorrect, due to making substantial errors and/or missing crucial pieces of the proof. Interestingly, Leonhard’s perception was that his work was correct on three of these four tasks; he showed great confidence, as can be seen in his emotion graphs for these tasks in Figure 4.

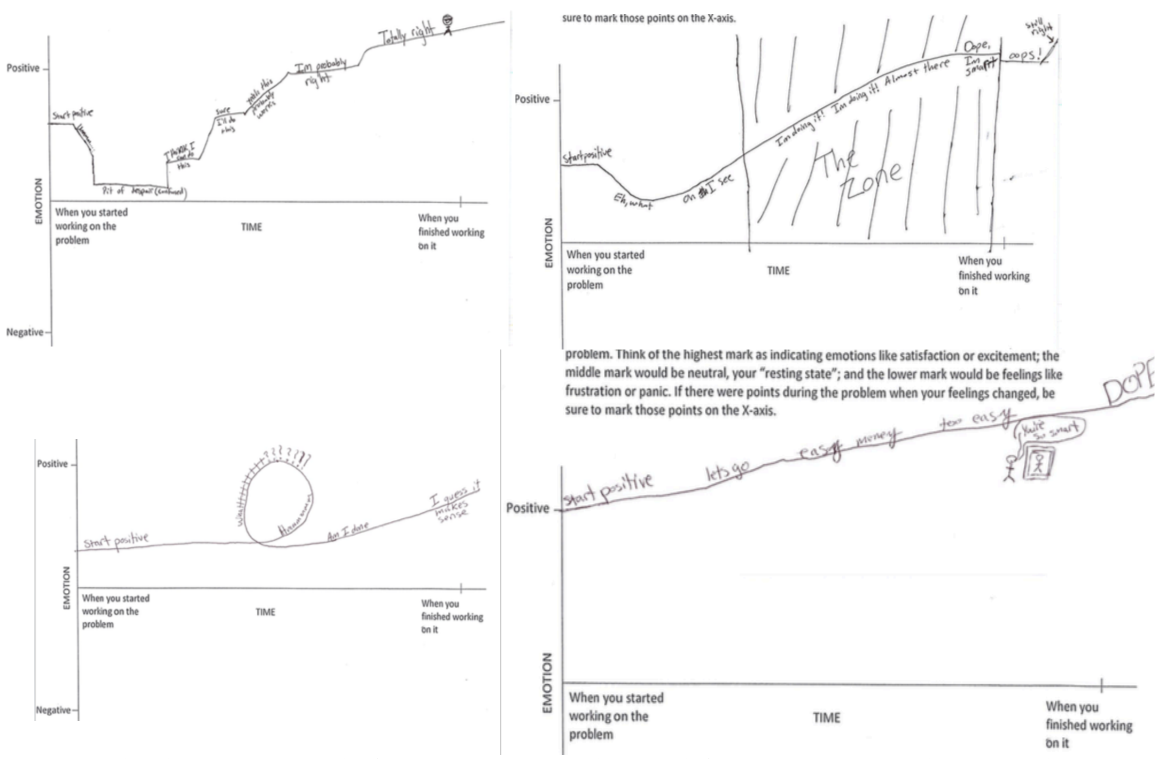

Figure 4. Leonhard’s emotion graphs for Interview 3 – Task 1 (top left), Interview 3 – Task 2 (top right), Interview 4 – Task 1 (bottom left), Interview 4 – Task 2 (bottom right). His graphs indicated high positive emotions about his work on Interview 3 and Interview 4 – Task 2 but his solutions were incorrect.

Conclusions

Over the interviews, Leonhard's rationales in deciding which proof techniques to pursue became more sophisticated, while his performance declined. Even though his success on tasks stagnated, Leonhard did show progress in terms of affect as well, in having confidence in his work. He had a positive orientation towards his work, but it is also worrisome when a student does not notice major flaws in their work. Leonhard is an example then of where a student's confidence is high and their reasoning and rationale for their decisions is high, but these do not necessarily lead to correct work. There is a difference then between reasoning and execution: Leonhard reasoned well but his execution was flawed. Can we say Leonhard is a better prover than before?

Another interpretation of this profile is that progress in terms of process does not always manifest itself in terms of performance, as measured by objective correctness. Judging a student based on solely their written work does not necessarily capture the thinking and reasoning behind their choices that was valid, which alone is valuable growth in proving.

This work – both the developments across all 11 participants and the misalignment of reasoning vs. performance in the case of Leonhard – highlights the need for a robust proving process framework. Such a framework would support the characterization of and assessment of students' proving as a process over short and potentially longer timescales.

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A Gendered Comparison of Abstract Algebra Instructors' Inquiry-Oriented Instruction

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Inquiry-Oriented Instruction (IOI) holds promise for providing equitable learning opportunities for men and women. We consider two abstract algebra instructors whose women exhibited different learning outcomes. We explore how this disparity in the women's achievement might be related to differences in these instructors' implementation of IOI, specifically regarding how they elicited and evaluated student contributions. We used Reinholz and Shah's (2018) EQUIP observation tool to investigate differences between students' participation opportunities in each class. We found a significant difference in the instructors' number of interactions with men and women during class discussion. We also found significant differences in the instructors' methods for soliciting and responding to student contributions in class discussion. A discussion of these differences in instructional practices, as well as implications for future work, is provided.

Keywords: equity analytics, gender, inquiry-oriented instruction, abstract algebra

The mathematics classroom is a gendered space, as socially constructed differences in achievement and participation exist between genders (Leyva, 2017). These differences might contribute to the undisputed underrepresentation of women in STEM. Lubienski and Ganley (2017) called for researchers to examine how women's choices to leave or persist in mathematics-intensive fields may be constrained by inequitable educational opportunities. Some researchers focus their efforts on identifying teaching practices that give women more access to learning opportunities and yield equitable learning outcomes. Laursen, Hassi, Kogan, and Weston (2014) advocated student-centered teaching approaches, such as inquiry-based learning, suggesting they "level the playing field" (p. 412) for men and women in mathematics. However, there is some dispute on the generalizability of equitable effects of student-centered instruction.

To explore the relationship between Inquiry-Oriented Instruction (IOI) and equitable student learning outcomes, Johnson et al. (under review) compared men's and women's learning outcomes in abstract algebra measured by their performance on the Group Theory Content Assessment (GTCA; Melhuish, 2015). Their participants included students whose instructors had participated in an IOI professional development project and students in the national sample. They found men outperformed women in the IOI sample. However, there was no achievement gap between men and women in the national sample. In recent analysis, a discrepancy was found in gender performance of students of two of the participating instructors from that sample, hereafter referred to as Dr. C and Dr. K. Both men and women in Dr. C's class outperformed the national sample, but men in Dr. K's class outperformed and women underperformed the national sample. These differences in learning outcomes may be attributed to differences in these instructors' practices. Our study seeks to highlight differences between these instructors' implementation of IOI to hypothesize potential instructional practices that may lead to equitable learning outcomes.

Literature Review and Theoretical Perspective

IOI is a student-centered pedagogical approach, which provides opportunities for students to inquire into mathematics and for instructors to inquire into students' mathematical thinking (Rasmusssen & Kwon, 2007). In IOI, students engage in meaningful tasks that allow them to develop informal intuitive understanding of concepts, from which they can develop more formal mathematical reasoning (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). The four principles of IOI are: "Generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation" (Kuster, Johnson, Keene, & Andrews-Larson, 2017, p. 2). These instructional principles emphasize inquiry into student thinking and formalization of mathematical reasoning. In addition to inquiring into student thinking, facilitating meaningful discourse, and guiding students' progress in the course, IO instructors need to sustain social norms in the classroom that are conducive to students' reinvention of mathematics (Stephan, Underwood-Gregg, & Yackel, 2014). These may include norms of students collaborating, explaining their reasoning, and participating in class discussion. Instructors also need to provide equitable opportunities for participation and position students as competent learners (Reinholz & Shah, 2018).

Equity research focuses on accounting for effects of past marginalization and mitigating systematic differences in ways students experience educational opportunities (Adiredja & Andrews-Larson, 2017; Gutiérrez, 2002; Reinholz & Shah, 2018). In the context of mathematics education, equity has been blurred with equality, in that educators might intend to provide equal access to curricular materials or learning supports (Gutiérrez, 2002). However, to account for differences in student backgrounds, student identities, and social biases, certain students need different learning opportunities to achieve fairness in the classroom. Reinholz and Shah (2018) proposed focusing on equality of participation opportunities as a necessary stepping stone for achieving equity in the classroom. We follow Gutiérrez (2013) and Adiredja and Andrews-Larson (2017) in taking a sociopolitical perspective, recognizing the interplay of knowledge, identity, and power in students' experiences and interactions in social educational contexts. Students associate mathematical success with power (Leyva, 2017), and as they construct their mathematical identities, perceptions of their own mathematical competence might be constrained by opportunities to participate in the classroom. This study considers participatory equity (Reinholz & Shah, 2018), discerning whether there exists a fair distribution of opportunities for participation given in two abstract algebra classrooms. Therefore, we explore the following research questions: What differences exist between two inquiry-oriented abstract algebra instructors' provision of opportunities for men and women to participate in class discussion? What differences exist in these instructors' teaching practices?

Methods

This explanatory case study explores the differences in practices of two abstract algebra instructors whose women students exhibited different achievement outcomes. Both instructors participated in an IOI professional development project in which they received training in implementing IOI in abstract algebra, curriculum materials, and support via online working groups with other instructors. Dr. C is a white man, Dr. K is a white woman, and both instructors teach at large public universities in the western United States. They both had over thirty students in their classes; Dr. C had twice as many men as women in his class, and Dr. K had about the same number of men and women in her class. We explore differences in Dr. C's and Dr. K's use of student contributions to gain insight on gendered experiences in their classes. We investigate which students participated in class discussions and how their contributions were elicited and responded to. We focus on gender to see how men and women's contributions were positioned during class discussion. Students' genders were inferred by the observing researchers rather than self-identified by the students, as this data was unavailable.

Instrument

The Equity QUantified In Participation (EQUIP) was developed by Reinholz and Shah (2018) as an observation tool to evaluate students' participation and instructors' practices of providing opportunities for students to participate in class discussion. The EQUIP rubric has seven dimensions, but we only used three of those dimensions, including solicitation method, teacher solicitation, and teacher evaluation (Figure 1), because they relate to the four principles of IOI (Kuster et al., 2017). Solicitation method refers to the type of strategy the instructor uses to initiate student participation. Teacher solicitation type refers to the type of question or statement the instructor uses to solicit student participation. Teacher evaluation describes how the instructor responds to students' contributions. We adapted the EQUIP tool by adding more specific codes for solicitation method and teacher evaluation.

Dimension	Solicitation Method	Teacher Solicitation	Teacher Evaluation
Levels	1. Not Called on 2. Random Selection 3. Called on Method 4. Called on Group 5. Called on Individual 6. Called on Volunteer	1. N/A 2. Other 3. What 4. How 5. Why	1. N/A 2. Revoice 3. Evaluation 4. Elaborate 5. Follow-Up Question or Task

Figure 1. Dimensions and levels from EQUIP (Reinholz & Shah, 2018) and coding assignments

Data Collection and Analysis

Instructor's classes were video-recorded by project personnel. The first author watched videos of five classes for both Dr. C and Dr. K. She transcribed only the student-teacher talk sequences that occurred during full class discussion. She then used the EQUIP observation tool to record the frequencies of the participatory and instructional practices observed in each student-teacher talk sequence that occurred during class discussion. Student demographics (e.g., gender) were noted for analysis of which students participated in the discussion. The codes were then discussed with the second author to resolve any uncertainties and were transformed to categorical values (Figure 1). A code was assigned to each instructor and gender to allow for comparison within the statistical analysis. We used three omnibus chi-square tests to compare the instructors for each of the three EQUIP dimensions. We then used three omnibus chi-square tests to compare each instructor by gender for each of the three EQUIP dimensions. Post hoc tests were conducted to examine statistically significant differences between instructors and gender. We then calculated the *equity ratio* (Reinholz & Shah, 2018), the "ratio of actual participation to expected participation" (p. 161) for men and women in each EQUIP dimension. Expected participation is the percentage of participation one would expect based on the demographics of the class. For example, if men comprise 40% of the class, they would be expected to participate 40% of the time. If men actually participated 50% of the time, the equity ratio would be 50/40 = 1.25. An equity ratio above 1 demonstrates an over-representation, and an equity ratio below 1 demonstrates an underrepresentation of participation from that group of students. Entrance interviews were also conducted with Dr. K and Dr. C by project personnel at the beginning of the semester they taught IO abstract algebra. We analyzed these interviews for data triangulation.

Results

Equity Analytics Results by Instructor

Dr. C's and Dr. K's instructional practices regarding student participation were compared using the three EQUIP dimensions. There was a total of 175 student-teacher talk sequences for Dr. C's class, and only 66 for Dr. K's class. The frequencies and percentages of each teacher's use of a certain solicitation method and type of evaluation are recorded in Table 1. The difference in instructors' *teacher solicitations* was not statistically significant. The difference in instructors' *solicitation method* was statistically significant $\chi2(4, N = 241) = 21.300$, $p < .001$, Cramer's V = .297. When looking at specific solicitation methods, neither instructor used random selection (e.g., drawing names from a hat). Dr. C's students were not called on more often than Dr. K's students were not called on. Dr. C called on individuals more often than Dr. K did. Dr. K called on volunteers and groups to participate more often than Dr. C did. The difference in *teacher evaluations* was statistically significant $\chi2(4, N = 241) = 28.687$, $p < .01$. Dr. C asked follow-up questions in response to student contributions more often than Dr. K did. They both primarily elaborated on student contributions, but Dr. K elaborated on student contributions much more often than Dr. C did. Both instructors responded by evaluating or revoicing student contributions about the same amount. Dr. C's responses were coded as N/A more often than Dr. K's were. This could be due to students responding to other students' contributions before Dr. C could respond.

Table 1. Code frequency and percentages for solicitation method and teacher evaluation by instructor.

Dimension	Instructor	Not Called on	Called on Method	Called on Group	Called on Individual	Called on Volunteer	Total
Solicitation Method	Dr. C	78 (44.6%)	0 (0%)	4 (2.3%)	41 (23.4%)	52 (29.7 %)	175
	Dr. K	24 (36.4%)	1 (1.5%)	9 (13.6%)*	6 (9.1%)	26 (39.4%)	66
	Instructor	Follow-Up	Elaborate	Evaluation	Revoice	N/A	Total
Teacher Evaluation	Dr. C	41 (23.4%)	54 (30.9%)	17 (9.7%)	14 (8%)	49 (28%)	175
	Dr. K	10 (15.2%)	42 (63.6%)*	6 (9.1%)	6 (9.1%)	2 (3%)*	66

*Represents significant differences in standardized residuals based on post hoc testing

Equity Analytics Results by Instructor and Student Gender

The difference between instructors' *number of interactions with men and women* was statistically significant, $\chi2(1, N = 241) = 5.540$, $p = .019$, Cramer's V=.152. Men in Dr. C's class participated in 104 of the 175 student-teacher talk sequences, while women participated 71 times. Dr. C's class had about twice as many men as women, but women participated proportionally more than men did, considering the equity ratios for the total amount of participation (see Table 2). Women in Dr. C's class had an equity ratio over 1, and men had an equity ratio under 1, implying women were over-represented and men were under-represented in the total amount of participation. Men in Dr. K's class participated in 50 of the 66 student-teacher talk sequences, while women participated only 16 times. Dr. K's class had an equal number of men and women, but women participated less than men did. Women in Dr. K's class had an equity ratio under 1, and men had an equity ratio over 1 for total participation, implying women were under-represented and men were over-represented in their total participation.

Gender differences in both Dr. C's and Dr. K's *teacher solicitation* and *teacher evaluation* were not statistically significant. The observed difference between Dr. C's *solicitation method* for men and women was statistically significant, $\chi2(4, N = 175) = 14.023$, $p = .007$, Cramer's V = .241. However, the observed difference between Dr. K's *solicitation method* for men and women was not statistically significant $\chi2 (4, N=66) = 3.662$, $p =.454$. The frequencies and percentages of each instructor's use of a certain method for soliciting participation from men and women are presented in Table 2. Slightly more men than women participated in Dr. C's class without being called on. However, in Dr. K's class, men participated without being called on much more often than women did. Dr. C rarely called on groups to participate, but when he did, only women shared their group's contribution. In Dr. K's class, men shared their group's contribution more often than women did. In both classes, however, men volunteered to answer more often than women did. Women in Dr. C's class were over-represented in participating in all levels of solicitation method except when Dr. C called on a volunteer. Women in Dr. K's class were under-represented in participating in all levels of solicitation method except when Dr. K called on individuals.

Table 2. Code frequency, percentages, and equity ratio for solicitation method by instructor and student gender

Instructor	Gender	Not Called on	Called on Method	Called on Group	Called on Individual	Called on Volunteer	Total
Dr. C	Men	43 (24.6%)	0 (0%)	0 (0%)	20 (11.4%)	41 (23.4%)	104 (59.4%)
		0.82		0	0.73	1.18	.88
	Women	35 (20%)	0 (0%)	4 (2.3%)	21 (12%)	11 (6.3%)*	71 (40.6%)
		1.36		3.03	1.55	0.64	1.23
Dr. K	Men	20 (30.3%)	1 (1.5%)	6 (9.1%)	3 (4.5%)	20 (30.3%)	50 (75.8%)
		1.67	2	1.33	1	1.54	1.52
	Women	4 (6.1%)	0 (0%)	3 (4.5%)	3 (4.5%)	6 (9.1%)	16 (24.2%)
		0.33	0	0.67	1	0.46	0.48

*Represents significant differences in standardized residuals based on post hoc testing

Descriptions of Dr. K's and Dr. C's Classes

To contextualize these numbers, we now turn to narratives from both classes. This section describes and compares teaching episodes by Dr. K and Dr. C, in which students developed formal definitions of isomorphic and isomorphism. In the classes prior to these episodes, students determined whether a given mystery Cayley table represented a group that was the same as D_6 (the group of symmetries of a triangle). This task aimed to give students an intuitive understanding of isomorphic groups and isomorphisms (see Larsen, 2013).

Dr. K's class. Dr. K began class by writing an informal definition of isomorphic groups on the board. She gave examples of isomorphic and non-isomorphic groups, and asked students to define isomorphic and isomorphism. Students worked individually, shared their ideas with their groups, and decided upon one idea as a group to contribute in whole class discussion. When Dr. K called on each group (numbered 1 through 8), a student voluntarily shared their contribution. A woman in group 1 claimed an isomorphism maps an element of one group to an element of the same order, and Dr. K elaborated on this. A man in group 2 commented that the isomorphism needs to be one-to-one and onto. Since few students discussed those terms in their groups, Dr. K gave a short lecture to explain the definitions and draw pictures of surjective, injective, and bijective set maps. Then, when Dr. K asked for other traits of isomorphisms or isomorphic groups, she called on a woman to share group 4's contribution. The woman claimed all mappings of corresponding elements had to be the same. Dr. K then asked for other traits, and a man from group 7 asserted the group tables (likely referring to Cayley tables) had to look the same. Dr. K wrote these contributions on the board and called on group 8 to share; a man from that group said isomorphic groups have the same order, but they wondered if they have the same operation. Dr. K elaborated on these contributions, and posed a new task to the class to decide whether isomorphic groups need to have the same operation. After working in groups, Dr. K gathered back the class. She asked if isomorphic groups can have different operations, and a student mumbled "yeah." Dr. K remined students of the example of the rotations of a square and $\mathbb{Z}_4$ having different operations but still being isomorphic groups. She then wrote the beginning of the formal definitions of isomorphic and isomorphism on the board, and she referred to group 4 and 7's informal ideas of homomorphisms. Students were asked to formalize those ideas using function notation, which they worked on for the rest of class.

Dr. C's class. Dr. C reminded the students of their previous task, in which they saw the group in the mystery table was isomorphic to D_6. He told students to write their own definition of isomorphic groups. Students (pseudonymed W# for women and M# for men) worked individually, and wrote definitions on whiteboards (see Figure 2), which were displayed at the front of the room. Dr. C had students talk with their groups about the different definitions. Dr. C initiated whole class discussion to address definition 2a; W1 said she had a similar definition. When Dr. C asked why, W1 explained her reasoning. W2 then stated a concern about the equal sign in the definition. Dr. C asked what was wrong with the equal sign, and W2 explained her reasoning. W3 said she did the same thing (definition 2d), but wrote "corresponds" instead of "equals." W2 added, "which implies there's a map." Dr. C discussed these ideas, and recalled the correspondence between elements of D_6 and the mystery table. Discussing definition 2e, students claimed $a * b$ is not necessarily in G, and they need to "split up the phi." Dr. C discussed these ideas, and acknowledged the idea of correspondence in the definition. Considering definition 2b, M1 said, "There exists a homomorphism G to H, and there also exists a homomorphism H to G." Dr. C mentioned that if M1's definition was true, then something about the definition 2b had to be wrong. Dr. C claimed definition 2c required a bijection, 2b required a homomorphism, and 2f required the existence of both. He then asked students to think of counterexamples to definitions 2b and 2c. After students worked in their groups, Dr. C called on W4 to share her counterexample. W4 described the trivial homomorphism that maps every element of G to the identity of H. When Dr. C asked W4, "Why is it not isomorphic?" W4 said this homomorphism could exist between groups with different numbers of elements. Dr. C asked for other comments, and M3 elaborated on W4's example. Dr. C then asked if groups with the same number of elements were always isomorphic; W5 explained her counterexample. Dr. C then led students to prove or give a counterexample to M1's idea, and M4 clarified that the homomorphism from H to G in M1's definition should be the "inverse that maps H to G." When Dr. C asked why, M4 explained his reasoning. Dr. C then explained this inverse homomorphism is a bijection, and he finalized the definitions of isomorphic groups and isomorphism.

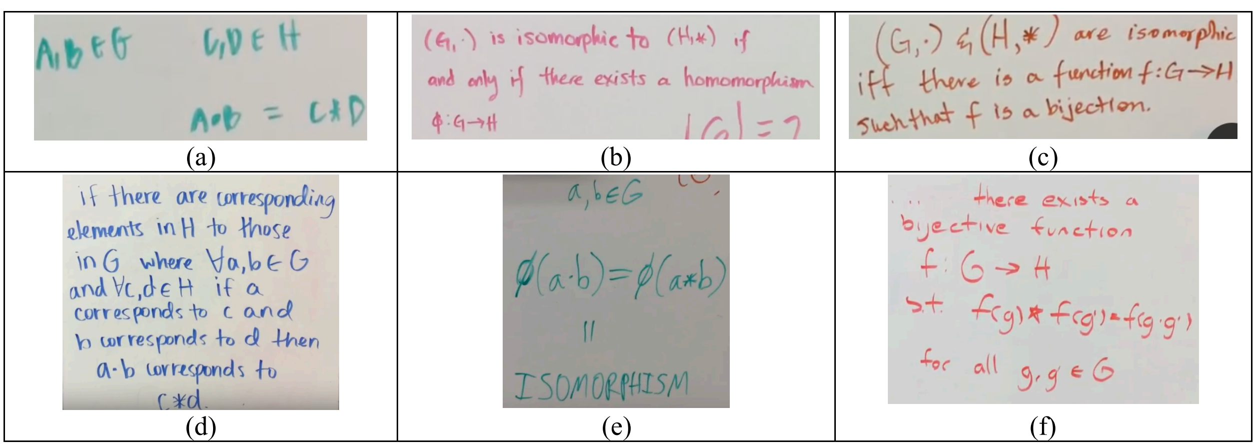

Figure 2. Student written definitions of isomorphic groups from Dr. C's lesson

Comparison of Dr. C's and Dr. K's practices. Here Dr. K and Dr. C exhibited differences in how they elicited student reasoning and contributions. Dr. K elicited characteristics students noticed about isomorphisms and isomorphic groups, but did not seem to elicit their reasoning. Dr. C, however, elicited students' reasoning by asking follow-up questions. Dr. K and Dr. C also exhibited differences in how they responded to student contributions and then used those to inform the lesson. Dr. K primarily elaborated on students' contributions; she accepted one-to-one and onto as characteristics of an isomorphism and then lectured on those definitions, without asking the students why or how they developed those ideas. Dr. C also elaborated on students' responses, but sometimes did not respond, allowing opportunities for other students to respond. Instead of accepting the definition of isomorphism in 2f, Dr. C led students to find counterexamples to see why it is necessary to have a bijective homomorphism between two isomorphic groups. Dr. K and Dr. C also exhibited differences in engaging students in each other's reasoning. Both instructors engaged students in each other's reasoning by assigning follow-up tasks in response to their contributions. However, during class discussion, Dr. K's students primarily talked to her and not each other, whereas Dr. C's students often spoke to each other about their reasoning without being called on. Although Dr. K seemed to do most of the talking in her lesson, she claimed in her entrance interview that she was trying to get better at not dominating class discussions. She said she wanted students to talk to each other instead of talking to her, but this did not seem to be enacted in her class discussion. In Dr. C's entrance interview, he claimed his overarching goal is to get everybody involved somehow. He also explained his class rule that his students are not allowed to judge each other. This might contribute to students' evident ease in participating in class discussion.

Discussion

Provision of equitable participation opportunities for students seems to be related to equitable learning outcomes. Since women in Dr. C's and Dr. K's classes had different achievement outcomes on the GTCA (Melhuish, 2015), we explored the differences in provision of opportunities for men and women to participate in class discussion. Women in Dr. C's class participated proportionally more than men did, while women in Dr. K's class participated less than men did, considering the equity ratios of participation. We found women in both classes were under-represented when Dr. C or Dr. K called on a volunteer. This finding aligns with Leyva's (2017) claim that mathematics is a masculine space, and men are more confident in volunteering to participate. The cause for the disparity in women's participation in Dr. K's class and women's limited volunteerism in Dr. C's class is unknown, yet prior research makes it reasonable to believe that women's lack of mathematical confidence may be a contributing factor (Ellis, Fosdick, & Rasmussen, 2016; Lubienski & Ganley, 2017). Future research can explore how equitable participation in class discussion relates to equitable achievement outcomes.

We also found differences in Dr. C's and Dr. K's instructional approaches during class discussion. We found significant differences in their methods for soliciting student participation and in their responses to student contributions. Our qualitative descriptions of Dr. K's and Dr. C's classes also highlighted some differences in their teaching practices, which we hypothesize might contribute to the differences in their students' achievement. We found Dr. K primarily elaborated on student responses without asking follow-up questions, whereas Dr. C asked follow-up questions to students' responses to inquire into their reasoning. Also, students in Dr. K's class primarily talked to her during class discussion instead of to other students, while Dr. C's students talked to each other, possibly because of his rule of no judgment. We hypothesize women might benefit from instructors inquiring into their reasoning, as this might position them as competent learners of mathematics. Women might also benefit from participating in class discussion in an atmosphere of no judgment, for this might enhance their mathematical confidence. Future research can explore which teaching practices give equitable learning opportunities for women.

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Characterizing Conceptual and Procedural Knowledge of the Characteristic Equation

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Research on student understanding of eigentheory in linear algebra has expanded recently, yet few studies address student understanding of the Characteristic Equation (CE). In this study, we explore students' conceptual and procedural knowledge of deriving and using the CE. Consulting Star's (2005) characterization of deep and superficial conceptual and procedural knowledge, we developed the Conceptual and Procedural Knowledge framework for classifying the quality of students' conceptual and procedural knowledge of both deriving and using the CE along a continuum. Most of our students exhibited deeper conceptual and procedural knowledge of using the CE than of deriving the CE. Furthermore, most students demonstrated deeper procedural knowledge than conceptual knowledge of deriving the CE. Examples of student work are provided, and implications for instruction and future research are discussed.

Keywords: procedural knowledge, conceptual knowledge, linear algebra, eigentheory

Considering recent demands for enhanced student understanding of concepts in science, technology, engineering, and mathematics fields, education researchers are tasked with exploring how students make sense of mathematical concepts in interdisciplinary settings. Our study focuses on quantum physics students' understanding of eigentheory in linear algebra. Eigentheory encompasses topics related to eigenvalues, eigenvectors, and eigenspaces. Students encounter eigentheory in a variety of contexts and courses, such as linear algebra, differential equations, numerical analysis, and quantum physics. Thus, it is essential for researchers to examine student understanding of eigentheory due to its interdisciplinary nature. This situates our work that investigates how students reason about and symbolize eigentheory in linear algebra and in quantum physics (*Project LinAl-P*, NSF-DUE 1452889).

A central tool often used to calculate the eigenvalues of an $n \times n$ matrix A is the *characteristic equation* (CE) of A, defined as $det(A - \lambda I) = 0$, for an $n \times n$ identity matrix I and scalar λ. The determinant of $A - \lambda I$ gives the characteristic polynomial of A, and the roots of this polynomial are the eigenvalues of A. In addition to symbolically representing a procedure, the CE is conceptually related to the Invertible Matrix Theorem (IMT, Lay, 2003). The CE can be derived from the eigenequation $Ax = \lambda x$, by subtracting λx from both sides ($Ax - \lambda x = 0$), introducing the identity matrix to get the homogeneous equation $(A - \lambda I)x = 0$, and making connections to the IMT. For example, one can recognize that for the equation $(A - \lambda I)x = 0$ to yield more than just the trivial solution for x (as eigenvectors cannot be the zero vector), the matrix $A - \lambda I$ must not be invertible, which implies that the determinant of $A - \lambda I$ must be zero.

Instructors want their students to conceptually understand these connections between the CE, the IMT, and related eigentheory concepts, yet some researchers posit that students struggle to do so (e.g., Bouhjar, Andrews-Larson, Haider, & Zandieh, 2018). Bouhjar et al. claimed:

> There is a disconnect between students' understanding of standard procedures for finding eigenvalues and the formal definition of an eigenvector and eigenvalue, and… students

> are more able to execute the standard procedure than draw on conceptual understandings aligned with the formal definition. (p. 213)

This disconnect seemed apparent in our own interview data with quantum physics students regarding their understanding of eigentheory. Although most of the students participating in our study successfully determined the eigenvalues of a given 2×2 matrix during an interview task, several students volunteered, sometimes unprompted, that they did not know why the CE is used or is true. When discussing why the determinant of $A - \lambda I$ must be zero to solve for the eigenvalues λ of A, some of the students explained, "That's just what I was taught," and the CE is true "because of something in linear algebra that says it needs to be this way." Another student explicitly expressed this focus on the procedure: "I remember learning why [using the CE] is the thing that I do. But… if I ever encounter a problem where I need eigenvalues, like, this is the first thing that comes to mind and not like where that comes from." This emphasis on the procedural use of the CE in our interview data led us to explore students' conceptual and procedural knowledge of the CE. We address the following research question: How do quantum physics students reason with and about the CE?

Literature Review

Various research studies (e.g., Boujar et al., 2018; Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Plaxco, Zandieh, & Wawro, 2018; Salgado & Trigueros, 2015; Thomas & Stewart, 2011) have explored student understanding of eigenvalues, eigenvectors, and related concepts, yet we have not found any that specifically focus on characterizing students' understanding and use of the CE. Thomas and Stewart (2011) focused on how students interpreted $Ax = \lambda x$, finding many students were comfortable with the procedural algebraic manipulations of matrices and vectors, as in Tall's (2004) symbolic world, but the students did not hold embodied ideas regarding eigenvalues and eigenvectors. They asserted that students' fluency in symbolic manipulations should be paired with understanding what the symbols represent. In particular, they pointed out the importance of understanding the resulting product on both sides of the equation $Ax = \lambda x$ is the same vector and understanding why the identity matrix is used in transitioning from $Ax = \lambda x$ to $(A - \lambda I)x = 0$, which many students in their study struggled to articulate.

Other studies demonstrated students' rich understanding of connections between concepts related to eigenvalues and eigenvectors (e.g., Larson, Rasmussen, & Zandieh, 2008; Salgado & Trigueros, 2015; Wawro, 2015). Wawro (2015) exemplified a student who made connections between statements in the IMT by reasoning about solutions to matrix equations, span, linear independence, null space, and the eigenvalue zero. Larson, Rasmussen, and Zandieh (2008) highlighted one student's ability to make connections between linearly dependent column vectors and the zero determinant of a matrix by reasoning about determinant graphically as the area of a parallelogram formed by two column vectors. More directly related to student understanding of the CE, Salgado and Trigueros (2015) described the reasoning of a group of three students who derived the CE without prior instruction by making connections to statements in the IMT, demonstrating conceptual understanding needed to reinvent the CE on their own.

Most relevant to our current study, Bouhjar et al. (2018) characterized students' responses to an open-ended written question that asked if 2 was an eigenvalue of a given 2x2 matrix. The authors claimed students who reasoned about the determinant used a more procedural approach, and students who reasoned about the matrix $A - \lambda I$ without computing the determinant used a more conceptual approach, as characterized by Hiebert and Lefevre's (1986) definitions of conceptual and procedural knowledge. However, the authors described their difficulty in classifying written work as demonstrating conceptual or procedural knowledge of the CE:

> It was often unclear from the responses of students who used the standard procedure [seeing if $\det(A - 2I) = 0$] whether they understood links among the equation used in defining eigenvectors, the solution set of $(A - \lambda I)x = 0$, and the equivalencies in the invertible matrix theorem that lead to use of the determinant as a tool for determining when the solution is non-trivial. (p. 212)

Furthermore, since many students simply used the CE to find the eigenvalues of A directly with no other explanation, the authors were unable to explore those students' conceptual understanding of derivation of the CE. Bouhjar et al. claimed more work is needed to distinguish whether a student using the CE to find eigenvalues just uses the procedure by rote or actually has deep conceptual understanding of why the CE works. Our analysis of students' interview responses about the derivation and use of the CE contributes toward this need by characterizing, along a continuum, students' conceptual and procedural knowledge in this context.

Theoretical Background

Conceptual Knowledge (CK) and Procedural Knowledge (PK) are qualitative constructs commonly used by mathematics education researchers to classify students' mathematical knowledge. Hiebert and Lefevre (1986) defined CK as "knowledge that is rich in relationships… a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3-4). They defined PK as "familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols" (p. 7), which consist of "rules or procedures for solving mathematical problems" (p. 7). Star (2005) argued that these definitions conflate students' *type* of knowledge with *quality* of knowledge, as if PK could never be as rich in connections as CK. Star further argued that holding CK is not necessarily better than PK; rather, both types of knowledge are essential for consummate understanding of mathematics. Thus, Star (2005) proposed classifying knowledge according to both quality (either deep or superficial) and type (either procedural or conceptual). He defined *deep PK* as "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment" (p. 408). A student demonstrates deep PK when (s)he can provide a "cogent explanation of how the steps are interrelated to achieve a goal" (Baroody, Feil, & Johnson, 2007, p. 119). *Superficial PK* is knowledge of procedures that is not richly connected (Star, 2005). Star characterized *deep CK* as richly connected knowledge of concepts, and *superficial CK* as knowledge of concepts that is not richly connected.

Classifying students' knowledge quality as deep or superficial can seem quite extreme, given that not all students exhibit strictly deep or superficial CK and PK. Therefore, we propose including a moderate knowledge quality as a classification for students who demonstrate deeper knowledge than students exhibiting superficial knowledge, yet less deep knowledge than those exhibiting deep CK or PK. We offer a framework for characterizing aspects of students' Superficial, Moderate, and Deep CK and PK of the CE, as described in the Methods section.

Methods

The data for this study consist of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 17 students enrolled in a quantum mechanics course. The interviews occurred during the first week of the course. Nine of the students were from a junior-level course at a large public research university in the northwest United States (school A), and the other eight were in a senior-level course at a medium public research university in the northeast United States (school C). All students are pseudonymed with "A#" or "C#." Interview questions aimed to elicit student understanding of several linear algebra concepts which they would use in the quantum mechanics course.

For this paper, we focus on students' attempts to recall, derive, and/or use the CE within their response to one particular interview question. Students were first asked, "Consider a 2×2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. How do you think about $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?" After follow-ups inquiring if they had a geometric or graphical way to think about the equation and how they thought about the equal sign in this context, students were asked how they thought about the equation if $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and to determine values of x and y that would make the equation true. Finally, students were asked to find the eigenvalues and eigenvectors of A, if they had not already done so. Note the interview question was designed so the terms "eigenvector" and "eigenvalue" were not used until the end; however, many students immediately recognized the first matrix equation as an eigenequation and often brought up eigentheory ideas on their own.

To begin our analysis, we wrote detailed descriptions for each student of their work on the aforementioned interview task, focusing on their reasoning about the CE. These descriptions contained evidence from the transcripts and images of students' written work. Using these descriptions, and consulting Star's (2005) definitions of deep and superficial CK and PK, we began to develop the Conceptual and Procedural Knowledge (CPK) framework for the CE (see Figure 1). The CPK framework describes characteristics of student work demonstrating both CK and PK across two dimensions: deriving the CE and using the CE. Through discussing the knowledge qualities demonstrated by the students in the context of the eigenvalue task, we developed lists of characteristics of student work demonstrating Superficial, Moderate, and Deep PK and CK for both deriving and using the CE. These lists were revised and organized into the CPK framework, which was used to code each student's response to the eigenvalue task.

	N/A	*Superficial*	*Moderate*	*Deep*
Procedural Knowledge of Deriving the CE	Does not attempt to write the CE	Incorrectly writes the CE (e.g., $A - \lambda I = 0$) and does not attempt to explain the symbolic derivation of the CE	Attempts to write the CE and make connections between $Ax = \lambda x$, $(A - \lambda I)x = 0$, and $\lvert A - \lambda I \rvert = 0$, but uses symbols incorrectly	Accurately manipulates symbols among $Ax = \lambda x$, $(A - \lambda I)x = 0$, and $\lvert A - \lambda I \rvert = 0$ to derive the CE, and writes the CE correctly
Conceptual Knowledge of Deriving the CE	Does not attempt to explain how the CE is derived	States they do not know where the CE comes from or gives irrelevant explanation of how the CE is derived	Gives explanation of how the CE is derived from $(A - \lambda I)x = 0$ that is relevant to the IMT, yet incorrect	Accurately explains how the CE is derived from $(A - \lambda I)x = 0$, while referencing connections to the IMT
Procedural Knowledge of Using the CE	Does not use the CE procedure to find eigenvalues	Correctly uses the CE procedure to find eigenvalues, without exhibiting fluency in algebraic manipulations or rigor in the calculations	Correctly uses the CE procedure to find eigenvalues, while exhibiting either fluency in algebraic manipulations or rigor in the calculations	Correctly uses the CE procedure to find eigenvalues, while exhibiting both fluency in algebraic manipulations and rigor in the calculations
Conceptual Knowledge of Using the CE	Does not recognize eigenvalues are the results of using the CE and does not use or discuss the resulting eigenvalues in the context of other eigentheory concepts	Recognizes eigenvalues are the results of using the CE but does not use or discuss the resulting eigenvalues in the context of other eigentheory concepts	Recognizes eigenvalues are the results of using the CE and makes only one connection between the eigenvalues resulting from the CE and other eigentheory concepts; OR makes two or more connections with at least one being incorrect	Recognizes eigenvalues are the results of using the CE and correctly makes two or more connections between the eigenvalues resulting from the CE and other eigentheory concepts.

Figure 1. Conceptual and Procedural Knowledge (CPK) framework for the CE

We now briefly explain each of the four rows of the framework. PK of deriving the CE entails symbolically moving from the eigenequation $Ax = \lambda x$ to the homogeneous equation $(A - \lambda I)x = 0$, and introducing the determinant to get $|A - \lambda I| = 0$. CK of deriving the CE involves making connections to the IMT to explain why the determinant of $A - \lambda I$ must be zero. PK of using the CE involves knowing the CE is an appropriate procedure to use to find eigenvalues and demonstrating fluency (i.e., ease of carrying out calculations) and rigor (i.e., making sure to write "$= 0$" at each step) in employing the CE. CK of using the CE entails understanding that the solutions of the CE are eigenvalues and making connections to other aspects of eigentheory (e.g., recognizing that the found eigenvalue 2 is the same 2 as in the original equation, plugging the found eigenvalues into $Ax = \lambda x$ or $(A - \lambda I)x = 0$ to attempt to find the eigenvectors, explaining what the found eigenvalues mean geometrically). For this last row, it is important to note that only the correctness of the connection was judged (e.g., plugging the eigenvalue into a correct equation like $Ax = \lambda x$ or $(A - \lambda I)x = 0$), not their knowledge of finding eigenvectors, or even what the equations in eigentheory mean. In the Results section, we explain how this framework helped us gain further insight into students' CK and PK of the CE.

Results

Responses of 3 of the 17 participating students were coded as "N/A" in all four categories, and one was coded as "N/A" in all but one category; thus, we focus our Results section on analyzing the remaining 13 students. Our four-part theoretical framework allowed us to unpack different aspects of students' understanding of the CE. The number of students exhibiting N/A, Superficial, Moderate, and Deep knowledge in each of the four categories is provided in Table 1. We share three prominent results from our analyses in the remainder of this section.

Table 1. Number of students exhibiting N/A, Superficial, Moderate, and Deep PK and CK of the CE

	N/A	Superficial	Moderate	Deep
PK of Deriving the CE	0	3	8	2
CK of Deriving the CE	0	10	2	1
PK of Using the CE	1	2	5	5
CK of Using the CE	1	1	3	8

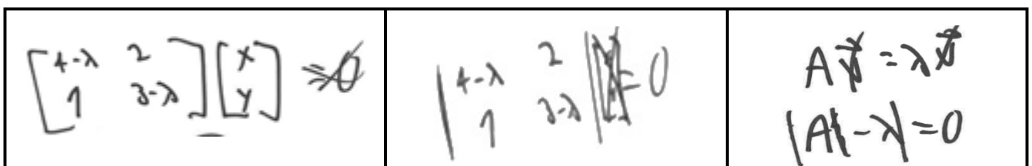
First, our CPK Framework for the CE illuminated that three students (A8, A11, and C5) showed relatively high sophistication in using and deriving the CE. In particular, two students demonstrated Deep knowledge in three of the four areas and moderate knowledge in a fourth, and another student showed deep knowledge in two areas and moderate knowledge in the other two categories. Taking A8 as a particular example, he first manipulated $Ax = \lambda x$ cleanly into $(A - \lambda I)x = 0$ (see Figure 2a), demonstrating Deep PK of deriving the CE. He then correctly stated there is a nonzero solution for x when $A - \lambda I$ is singular, connecting the CE to the IMT and exhibiting Deep CK of deriving the CE. When using the CE, A8 correctly calculated the eigenvalues with no apparent difficulty. However, his notation was somewhat improper, manipulating the polynomial in the CE by itself rather than as an equation (see Figure 2b). For this reason, we coded this as showing Moderate PK (partially due to this omission being associated with other common student errors in factoring). After finding the eigenvalues, A8 showed Deep CK by making connections both to finding eigenvectors using $(A - \lambda I)x = 0$ and to the previous part of the problem, where the eigenvalue 2 was given in an eigenequation.

Second, our data showed that the students in our study were better at using the CE than deriving the CE, both procedurally and conceptually. Students' CK of using the CE seemed

(a) (b)

Figure 2. (a) A8's symbolic derivation of the CE; (b) A8's use of the CE

stronger than their CK of deriving the CE, seen as 10 out of 13 students exhibited Superficial CK of deriving the CE, but only 1 out of 13 students exhibited Superficial CK of using the CE. Also, only 1 of 13 students exhibited Deep CK of deriving the CE, but 8 out of 13 exhibited Deep CK of using the CE. C3 exemplified this trend of exhibiting deeper CK of using the CE than of deriving the CE, as he demonstrated Superficial CK of deriving the CE and Deep CK of using the CE. In particular, when asked to find the eigenvalues and eigenvectors of A, C3 first wrote an appropriate homogeneous equation (Figure 3a), crossed out the "equals zero," and said it was the determinant of that which equaled zero (Figure 3b). He then explained he could cross out the $\begin{bmatrix} x \\ y \end{bmatrix}$ "because you're dividing it out," claiming the vectors in the eigenequation $Av = \lambda v$ cancel (see Figure 3c). Once C3 found 2 and 5 as the eigenvalues of A, he mentioned "you could have given me 5," in reference to the original $Ax = 2x$ equation, and he used $Ax = 5x$ to find other eigenvectors. Even though C3 did not seem to figure out a conceptual derivation of the CE, he recognized the CE solutions as eigenvalues and made connections between those and the eigenequation to find eigenvectors. This exemplar illustrates our result that our students connected the CE with eigentheory concepts, but they did not seem to know why the CE is true.



(a) (b) (c)

Figure 3. C3's written work for his derivation of the CE

Furthermore, students in our study seemed to have stronger PK of using than of deriving the CE (see Table 1). Most students wrote the CE correctly or made small mistakes writing it, but did not accurately make connections between equations like $Ax = \lambda x$, $(A - \lambda I)x = 0$ and $|A - \lambda I| = 0$. However, most students had little difficulty in using the CE to find eigenvalues. C3 exemplified this trend because he demonstrated Deep PK of using the CE and Moderate PK of deriving the CE. C3's symbolic manipulations (see Figure 3) revealed he could not accurately derive the CE from the eigenequation. Nevertheless, he fluently and rigorously used the CE.

Lastly, in comparing students' PK to their CK within both dimensions, contrasting trends emerged. In deriving the CE, all students demonstrated PK that was as deep or deeper than their CK. In some ways, this is not surprising as many students (10 of the 13) did not make any connection to the IMT in their explanation of deriving the CE, but most (10 of the 13) wrote the CE correctly and/or made connections to $Ax = \lambda x$ or $(A - \lambda I)x = 0$. By contrast, looking at using the CE, a majority of students (11 of the 13) demonstrated CK that was as deep or deeper than their PK. Looking back at A8 as a particular example, recall that he fluently found eigenvalues and connected them back to both the homogeneous equation and the equation given in the initial problem statement (demonstrating Deep CK for using the CE), but did not rigorously write "$= 0$" after each step in the calculations (demonstrating moderate PK for using the CE). We recognize this trend between PK and CK in using the CE is largely a result of the choices we made on characterizing "deep knowledge" within each dimension. In particular, we note that categorizing students who do not rigorously write "$= 0$" as having Moderate PK in using the CE (such as A8), and categorizing students who correctly connected the found eigenvalues to other eigentheory elements as having Deep CK in using the CE, regardless of their abilities to find eigenvectors or explain what eigenvalues mean, are subjective decisions. However, we feel our analysis highlights that many students *do* know how to find the values of λ that make $|A - \lambda I| = 0$ true, despite work that appears non-rigorous, and understand how this process produces the eigenvalues, which are essential to all other aspects of eigentheory.

Conclusion

Using the CPK framework to code students' interview responses allowed us to distinguish students' type and quality of knowledge of both using and deriving the CE. We captured student understanding of deriving the CE, which was not accessible in the written data in Bouhjar et al.'s (2018) study; hence, our work addresses their call to determine whether students employing the CE to find eigenvalues only know how to use the procedure or also understand how it works. We found the students in our study were better at using the CE than deriving it. Most students experienced little to no difficulty in using the CE to find eigenvalues and making connections to other eigentheory concepts, but they seemed to struggle with knowing how it is derived conceptually. To address this lack of Deep CK, instructors could intentionally enhance students' understanding of the IMT and help them make connections to it while deriving the CE. Instructors could also emphasize precision in symbolically deriving the CE to help students learn how to accurately manipulate symbols associated with eigentheory concepts.

This study offers a theoretical contribution regarding the addition of the classifications N/A and Moderate to delineating the quality of PK and CK; these allow for finer nuance in classifying the quality of students' CK and PK. In the CPK framework, we also offer the distinction of student understanding of deriving and using the CE to provide more insight into how students think about these different aspects. The CPK framework can be generalized for investigating student understanding of topics in linear algebra and other areas of mathematics, but the characteristics of student work listed in each cell of the framework may change, depending on the mathematical content and the nature of the tasks students perform. This framework seems most useful for analyzing student interview data, since interviewers can prompt students to both perform procedures and explain their thinking about concepts. To use the CPK framework with written data, the written tasks should elicit evidence of students' reasoning about both the derivation and use of the mathematical topic. Further research can explore how bidirectional relations form between CK and PK, as proposed by Rittle-Johnson and Schneider (2015), focusing on the how students' CK of the CE supports their PK of the CE, and vice versa.

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Examining Prospective Secondary Teachers' Curriculum Use and Implications for Professional Preparation

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In this paper, we share findings around four prospective secondary mathematics teachers' attention to varying curricula while planning an algebra lesson. We specifically address how their attention interacted with their interpretations of and responses to the curriculum materials via idea sequences, and further we study how the format of the curriculum materials plays a role in influencing these interactions. We discuss the result that sequences of interpretations and responses are always initiated by attention for PSTs, which itself is influenced by curriculum elements and format. We end with a discussion of implications around the need for curriculum use practices in teacher education and professional development.

Keywords: Curriculum, Graduate Teaching Assistant, Professional Development, Prospective Secondary Teachers, Teacher Training

Research shows that historically, written curriculum materials play a large part in a teacher's lesson planning. However, the literature suggests that teachers are not prepared to learn to use curriculum materials in adaptive and flexible ways (Drake, Land, & Tyminski, 2014). Since 80% of practicing teachers use some form of curriculum materials in their instruction (Banilower, et al., 2013), teacher educators need to support prospective teachers in learning to use materials. While research indicates that curriculum materials have the most direct influence on what teachers actually plan for and enact in their classroom (Brown & Edelson, 2003) and influence what students have opportunities to learn (Matsumura et al., 2006), we still know little about *how* curriculum materials exert this influence (Stein, Remillard, & Smith, 2007). This paper focuses on how prospective secondary teachers (PSTs) interact with curriculum materials by examining their *curricular noticing,* or what they attend to in curriculum materials, how they interpret what they attend to, and how they respond to the curriculum materials. Specifically, we describe how curricular interpretations and responses relate to curricular attention.

Theoretical Background

Throughout the past few decades, efforts have been made to develop research-based descriptions or models for how teachers use curriculum materials (Lloyd, Cai, & Tarr, 2017). And despite varying descriptions of this use (e.g. Brown & Edelson, 2003; Brown, 2009; Choppin, 2009, Geuedet & Trouche, 2009; Pepin, Geuedet, & Trouche, 2013; Remillard & Bryans, 2004; Sherin & Drake, 2009), each of these descriptions share the perspective that curriculum use involves some kind of interaction between teachers and materials. For instance, this interaction is described as participatory by Remillard (2005), where influence is bi-directional, meaning that the teacher influences the material and the material influences the teacher. Similarly, Geuedet and Trouche (2009) suggest that teachers engage with materials in "documentational genesis." They establish that documentational genesis involves two processes: instrumentation, the process by which curriculum materials influence what and how teachers use resources in the design and enactment of instruction, and instrumentalization, the process by which curriculum materials are influenced by the teacher.

Examining Curriculum Use via Curricular Noticing

We draw on the theory that teachers' interactions with resources are participatory (Remillard, 2005) and use the Curricular Noticing Framework (Dietiker, Males, Amador, & Ernest, 2018) to describe this interaction. This framework was informed by the work in the professional noticing of children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010). Curricular noticing refers to "the set of skills that constitutes the curricular work of mathematics teaching" (Dietiker et al., 2018, p. 524) and is comprised of three interrelated skills: *Curricular Attending, Curricular Interpreting*, and *Curricular Responding*. Curricular attending involves looking at information in curriculum materials to draw upon for the teaching and learning of mathematics, curricular interpreting involves making sense of what is attended to, and curricular responding involves making curricular decisions based on interpretations made. We visualize curricular noticing with Figure 1.

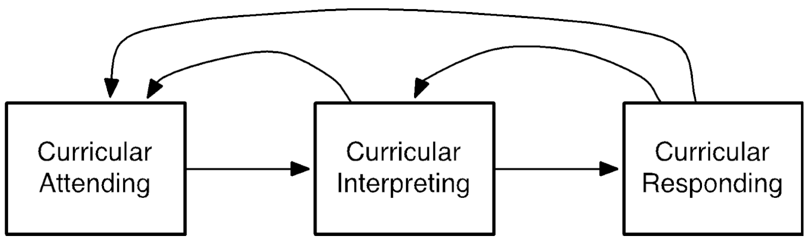

Figure 1. The curricular noticing framework (Dietiker et al., 2018).

While these definitions seem to imply a sequence, Dietiker et al. (2018) propose that this process may not unfold in a strictly linear fashion. For example, while a response relies on a curricular interpretation of something attended to, an interpretation may trigger a teachers' attention, or a particular response may result in attention.

Purpose and Research Questions

The main purpose of this paper is to describe how PSTs interact with curriculum materials, with a focus on the relationship between interpretations and responses and what teachers attend to, and how the curriculum materials influence attention. Specifically, we address the following research questions:

1. How do PSTs' curricular interpretations and curricular responses interact with their attention to the curriculum materials?
2. How do curriculum elements and format of each set of curriculum materials influence PSTs' attention?

By curriculum elements, we mean distinguishable parts of the curriculum materials such as sentences, phrases, representations, and images. We intentionally selected 'element' rather than 'feature' since features often include multiple sentences or paragraphs in curriculum materials. This allows us to describe the skills of curricular noticing piece-by-piece rather than by broad sections. The format that we refer to means the way elements are organized and how the curriculum appears. This includes not only color and location of student and teacher materials, but also 'embeddedness' of teacher supports (Beyer, Delgado, Davis, & Krajcik, 2009). A resource with embedded supports integrates teacher support within the directions and content for enacting activities found in the student version of the resource. Curriculum materials of this category often present teacher materials and student materials on separate pages. On the other hand, we see non-embedded supports in resources that have teacher support close to, but separate, from portions intended for students. Often this occurs on the same page.

Methodology

Participants and Data Collection

Our participants were four secondary mathematics PSTs who had not yet taken any mathematics teaching methods courses, but had completed much of the mathematics required for their degree. We engaged each participant in two semi-structured think-aloud Staged Planning Interviews, a popular style of interview to gain insight into teachers' use of curriculum materials (Males, et al., 2016; McDuffie, 2015; Reinke & Hoe, 2011). In one interview, teachers were asked to plan a hypothetical lesson using as a resource College Preparatory Mathematics (CPM) Algebra Core Connections (Dietiker, et al., 2014), and in another interview teachers were asked to plan using as a resource Pearson Education, Inc. (PEI) Algebra I Common Core (Charles, et al., 2015). We alternated which resource a teacher planned with first, meaning two PSTs planned a hypothetical lesson using CPM as a resource first, while the other two PSTs planned using PEI as a resource first. The two interviews for each participant were conducted at least a week apart.

Data Analysis

Documents and videos from the staged planning interviews were uploaded to a shared drive, and the glasses recordings and images of the curriculum pages were imported into Tobii Pro Labs (Tobii Technology, Inc., n.d.). Lastly, the glasses recordings and transcripts were imported into a qualitative analysis software program. In order to address attention, we used Tobii Pro Labs to map the gaze data recorded by the glasses to each of the curriculum pages. This data was in turn used to create timelines which illustrate when PSTs were attending to student and teacher materials (i.e., looking anywhere on the student or teacher portions of pages) and when they were not attending to the curriculum materials at all (i.e., looking at their written lesson plan, the interviewer, or other places in the room).

To address the interactions of interpretations and responses with attention, we coded the PSTs' transcripts. We assigned an *Interpret* code when a PST engaged in sense-making and a *Respond* code when PSTs made a curricular decision related to what to include (or not include) in their lesson plans. Once coding was complete, we studied PSTs' thought processes via idea units. We define an idea unit as a period of time within the transcript during which a PST focused on one big idea. Within these idea units, we identified idea sequences by recording the sequence of curricular attention, interpretations, and responses. For example, when Fay discusses her thoughts around the problems following the introductory problem in the PEI lesson, we generated the idea sequence in Figure 2.

Figure 2. An example of an idea sequence.

Results & Discussion

Figure 3 illustrates each PST's attention to the curriculum materials for CPM and PEI across the planning sessions. The black portions indicate times when the PST was not attending to the curriculum materials (e.g., looking at their lesson plan or other things in the room) whereas blue and yellow indicate attention to the student and teacher materials, respectively.

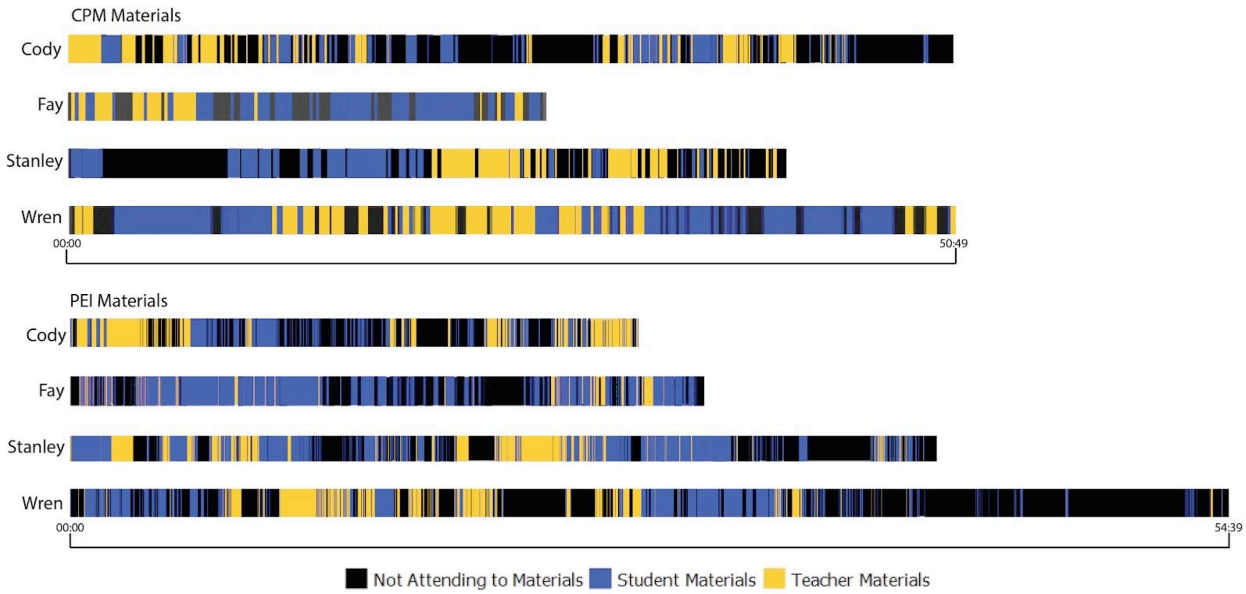

Figure 3. Attention across the planning session by curriculum and PST.

The timelines show that PSTs were shifting frequently between attending to student and teacher materials, with 40-85% of their attention time for both sets of materials devoted to student materials. When planning with both sets of materials, three PSTs spent more time attending to student rather than teacher materials. Cody was the opposite, spending more time attending to the teacher materials in both planning sessions. Looking across the curriculum materials, the timelines illustrate that PSTs shifted between teacher and student more frequently for PEI and that they attended for shorter amounts of time before switching compared to CPM.

While attending (blue and yellow in the figure), PSTs were simultaneously interpreting and responding to the curriculum materials. For instance, for three of the four PSTs, we see heavy concentrations of attention in the beginning of the CPM planning periods. Our idea sequences indicate PSTs were attempting to make sense of the unfamiliar format and content of the materials, often looking back at preceding portions of the text and spending considerable amounts of time interpreting. For example, during this time, PSTs were interpreting the reason for what seems to be provided answers in the student portion of the materials, such as Grant who states "I'm assuming that this…they ask me to write an equation at the top that represents the table below. But then they give me the equation?" Over the course of two and a half minutes, he comes to the realization that the bolded answers are not included in the materials given to the student. The unfamiliar content also seemed to necessitate more attention and interpretation. For example, Cody, who spent 22 more minutes planning his CPM lesson than his PEI lesson, struggled to make sense of the lesson content, specifically what was meant by a tile pattern. At the beginning of his planning sessions, he spent more time searching for information from the teacher materials (yellow in his timeline) and working out his ideas on his scratch paper (black in his timeline) as seen in Figure 4.

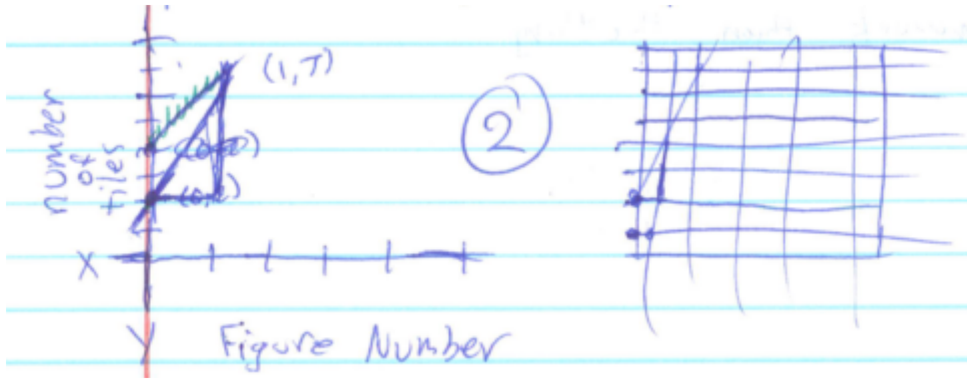

Figure 4. *An excerpt from Cody's scratch paper.*

Cody first thought that tiles meant a grid of some sort. Then he drew what appears to the left in the figure followed by what appears to the right as he said "So they want to look at tiles… something like that…I see they're trying to bring in some physical type of thing… but to me a normal grid just kind of makes more sense so I'd probably just keep going with the *x-y* axis."

Towards the end of the CPM planning periods, PSTs went back to portions they had initially attended to, attending again and then interpreting the intended trajectory or concept before responding based on the alignment of the perceived structure with their own beliefs on how a lesson on slope should be carried out.

In contrast, during PEI planning, we see heavy concentrations of attention throughout the entire planning period for each PST. Examining the idea sequences, we see that many more responses are made, along with interpretations, in the beginning half of these periods as compared to PSTs planning with CPM where responses were made towards the end of the planning periods. The most common interpretations involved PSTs making sense of the introductory slope problem and deciding quickly to adapt or supplement this because it was not "real-world" enough or approached in the way they would like, such as Stanley who says

> …But that's not how I would actually solve that problem in the real world. Because really you just want to take 1 over 0.25, equals 4. 4 over 1 equals 4. 7 over 1.75 equals 4. Use those comparisons. I know these are mathematically equivalent, but this is just a little more roundabout and confusing.

Our idea sequences indicated that PSTs began to work with new ideas by attending, meaning each of our idea sequences began with an Attend code. We also saw that, particularly for CPM, that attending to one curriculum element often led to attention (or repeated attention) to other elements. For example, after reading briefly through the CPM teacher materials around problem 2-12, when attending to the student materials Cody interprets problem 2-12 saying it "seems kind of obvious." He then initially responds by deciding not to use the problem in his plan. However, he goes back to the teacher materials and attends to the suggestions for problem 2-13 and notices that the problems are linked, with 2-12 providing valuable experience, so he decides to use both problems.

Our analysis indicated that idea sequences were different across materials. The average duration of the sequences was longer when PSTs were planning with CPM. In addition, when planning with CPM, in the first half of their planning period, PSTs had many more idea sequences that only involved Attend and Interpret codes (21 out of 53 idea sequences across all PSTs), while with PEI there were many more Respond codes in the beginning of the planning periods (32 out of 49 idea sequences across all PSTs). This means that PSTs made planning decisions much more quickly in their planning period for PEI than they did for CPM.

Implications for Teacher Education and Curriculum

This study provides insight into how different PSTs approach the same curriculum materials and produce a plan to enact in the classroom. Understanding the process of how PSTs plan using curriculum materials has implications for teacher education programs and curriculum development. First, this study suggests that format largely influences attention. All four PSTs in this study tended to switch back and forth more quickly between student and teacher materials in PEI, the non-embedded curriculum materials, and further made quicker decisions when planning with a resource of this format. These occurrences may result in many teacher suggestions being missed, or at the very least misunderstood. When attention is so short in duration to particular curriculum elements, it may be difficult for teachers, particularly early career teachers who have less experience with curriculum materials, to interpret and respond while planning for instruction. This points to the need for attention optimization in curriculum development.

Further, we see from this study that PSTs require opportunities to learn to use curriculum materials. Since this study showed that PSTs interacted differently with varying curricula, we advocate in the same way as Drake, Land, and Tyminski (2014): PSTs need opportunities to learn to use curriculum materials by interacting with different types. Teacher education also needs to guide PSTs in learning how to read curriculum materials. As this study exemplified, it is a skill to recognize and know what elements of curriculum materials are intended for teachers, and which are intended for students.

Extending study to undergraduate mathematics. Like PSTs, graduate teaching assistants (GTAs) who are teaching undergraduate mathematics courses are also early users of curriculum materials as teachers. More importantly though, GTAs may have even less opportunities to interact with materials before using them with students. Broadly speaking, GTAs have very little teacher education, and yet interact with curriculum while planning lessons on a weekly basis. Optimizing college curriculum materials for attention and engaging GTAs in learning how to enact curriculum is crucial for the success of the program.

Teaching preparation of GTAs first became a point of interest as a result of Speer, Gutman, and Murphy (2005). Speer and her colleagues pointed to K-12 professional development as a source from which to draw upon for GTA professional development, and also listed many directions of research to pursue from there. However, what little research we currently have on the topic largely focuses on case studies or development of GTA training programs. Even fewer studies point out the (often unmet) needs of the GTA population. We see interactions with curriculum materials as one such unmet need.

What little professional development GTAs are provided in graduate school is often the first training they will receive (Deshler et al., 2015), and more often than not, this training is provided simultaneously with the required teaching of courses (Ellis, 2016). For example, mathematics GTAs at the author's home university receive a three-day orientation the week before classes start. Examining curriculum interaction specifically, GTAs receive the opportunity of guided interaction with their department-provided curriculum for just an hour and a half duration out of their three-day orientation. This results in the majority of curriculum interaction occurring during the planning and enacting periods of teaching, a time when familiarity and teaching practices surrounding curriculum should already be developed. Further, this length of orientation time is only provided for first-time instructors of precalculus courses, with a curriculum that is set by the department and expected to be followed. GTAs which go on to teach other courses in subsequent years, then, have little to no training on what to look for in differing curriculum materials or how to interact with them in beneficial ways. We acknowledge that this curriculum interaction component is not necessarily representative of GTA orientations across the U.S., and so this further emphasizes the need for its integration into GTA training overall as teacher educators work to improve the number and quality of opportunities provided to GTAs in learning to use curricula.

Not only does this study point to the benefits of engaging GTAs in curriculum interaction in professional development, but also to the need to optimize college curriculum materials for attention. Since we saw that PSTs' noticing was driven by attention via the idea sequences, we understand that teachers, especially those with little to no teacher training, will more likely engage with curriculum materials that are designed to capture attention. Without this, PSTs, GTAs, and practicing teachers alike are not supported in their teaching practices and are likely to miss important components of curriculum materials, resulting in unintended effects on student learning.

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The Choice to Use Inquiry-Oriented Instruction: The INQUIRE instrument and differences across upper and lower division undergraduate courses

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In this study of mathematics teaching, we explore how to measure inquiry-oriented practices of college mathematics instructors. We offer a conceptualization of inquiry-oriented instruction organized by the instructional triangle (Cohen, Raudenbush, & Ball, 2003) and introduce an instrument developed to explore the extent to which elements of inquiry-oriented instruction are present in the teaching of university mathematics courses. This scale has been developed to explore what practices instructors currently use and eventually investigate the relationship between beliefs and practice. We show how we have operationalized inquiry-based instruction as self-report items and report preliminary findings that indicate our scales are performing well. We show that some inquiry-oriented practices are significantly more present in upper-division courses than lower-division courses. This suggests that at least some components of inquiry-oriented instruction are not reducible to individual differences (whether the instructor is an inquiry-based instructor), but also dependent in the context of instruction.

Keywords: Inquiry-oriented instruction, obligations, beliefs

College mathematics departments are faced with a growing need to develop innovative instructional practices to address the needs of increasingly diverse student bodies and declining numbers of mathematics majors (Holton, 2001; U.S. Department of Education, 2006). Even when instructors give a high-quality lecture, students often do not grasp the main ideas the instructor intends to convey (Goodstein & Neugbauer, 1995; Leron & Dubinsky, 1995; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016). Researchers have found that students learn better from active, student-centered instruction in college mathematics (Kwon, Rasmussen, & Allen, 2005; Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006) and that meaningful participation in inquiry-based instruction has been linked to higher achievement and persistence for women and students of color (e.g., Boaler, 1997; Laursen, Hassi, Kogan & Weston, 2014).

Many varied interpretations of inquiry exist. For researchers seeking to understand inquiry-oriented instruction or instructors trying to implement it, we need a shared understanding of its components and to what extent those components exist among instructors that presently are trying to make their classes more active. This study seeks to investigate what inquiry-oriented practices instructors implement, as part of a project that seeks to understand also why they do it.

Literature and Framing

Studying Inquiry-Oriented Practices

Inquiry-oriented learning involves following the methods and practices of mathematicians (Yoshinobu & Jones, 2012) or getting students to engage in "authentic mathematical activity" (Johnson, Caughman, Fredericks, & Gibson, 2013). Many studies have focused on the design and implementation of inquiry-based curriculum, such as with the linear algebra magic carpet ride task (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). In that study, Wawro et al. (2012) designed a task that allowed students to discover the concepts of span, linear independence, and linear independence and invent their own definitions of those concepts. Other studies have documented a handful of challenges that instructors face during implementation and how those challenges can be overcome. For example, Yoshinobu & Jones (2012) wrote about the coverage issue: that many instructors resist using inquiry-oriented practices because they fear they will not have time to cover the content their institution expects. No studies known to the authors of this paper attempt to document what various inquiry-oriented practices are being taken up by college instructors on a large scale. Some challenges of such a study are to identify the components of inquiry-based instruction that might be documented and to design a feasible method to study the practices of a large enough sample. Before subscribing effort to rate instructors using observational methods, it would be useful to do descriptive work in which one can study the feasibility of any such rating. A survey using self-reports is an economical way to discover and identify those components.

The Instructional Triangle

The instructional triangle (Cohen, Raudenbush, & Ball, 2003; Ball & Forzani, 2007) is a useful frame to organize the study of instruction and, in particular, to organize our identification of possible indicators of inquiry-based instruction. The instructional triangle is composed of the interactions between the teacher, students, the content, and the environment surrounding all three. The triangle suggests the situatedness of these interactions in environments in four possible dimensions of instructional actions (see Table 1).

The first dimension addresses how the students are given opportunities to engage with the mathematical content, for example, the extent to which they engage in authentic mathematical work. Education researchers have stressed the importance of students' school experiences aligning with the disciplinary practices of scholars across any subject (Bruner, 1960; Dewey, 1902; Schwab, 1978). The work of mathematicians involves contributing ideas, struggling with definitions, experimenting with examples, proposing conjectures, propositions, and theorems, and providing proofs and arguments for those claims. Lakatos (1976) wrote in *Proofs and Refutations* about an imaginary classroom dialogue surrounding the problem of finding a relation between the number of vertices, edges, and faces of polyhedra. As the class progresses, they consider examples and counterexamples, and construct various conjectures and proofs. The process is anything but linear and conjectures are continually revised as students encounter new evidence and arguments posed by each other. This exploratory discovery has been cited by mathematics educators to explain what it means to think mathematically (e.g., Schoenfeld, 1992). That text shows by example that doing mathematics not only involves solving problems, but also formulating hypotheses from observations and *problem-posing* (Silver, 1994, 1997).

The second dimension deals with how the instructor relates to the students vis-a-vis the knowledge to be learned, including, for example, how much they involve students in the development of new knowledge. Gonzalez (2013), a practitioner of inquiry-based learning, described his role as becoming more of a "'guide on the side' than a 'sage on the stage'" (p. 35). When a student is stuck, the instructor does not give the solution away, but helps by posing a question, getting other students to help, or finding a smaller problem or special case that can help them make progress on the larger problem (Yoshinobu & Jones, 2012). While strict lecture with the instructor dictating the lesson in the front of the classroom may be thought of as providing the least opportunities for inquiry, there are ways to make lecture more responsive. Burn, Mesa, and White (2015) used the term *interactive lecture* to refer to presenting material in an engaging way that included questions and answers.

The third dimension considers whether and how students have interactions with their peers as they engage in mathematical work. In inquiry-oriented classrooms, students are often asked to present their solutions to classmates and receive feedback on their reasoning (Gonzalez, 2013; Hayward, Kogan, & Laursen, 2016; Laursen & Hassi, 2010; Yoshinobu et al., 2011) either at the front of the classroom or in small groups (Yoshinobu & Jones, 2012). In small group discussions, students often work on problems together, while during presentations, one student leads the class in finding a proof or solution and other students can comment or ask questions. Practitioners like Renz (1999) report that when students interact, they theoretically gain motivation from their peers to check their own work carefully and present their ideas clearly.

Table 1. *Conceptualization of inquiry-oriented instructional practices*

Triangle Relationship	**Constructs**	**Description**
Student-Content	Open problems	Posing problems that either have multiple solutions or multiple nontrivial ways of arriving at a solution
	Constructing	Posing tasks that ask students to make conjectures and construct arguments
	Critiquing	Asking students to critique the reasoning of themselves and others
	Definition-formulating	Inventing or reinventing mathematical definitions
Teacher-Student	Interactive lecture	Incorporating interaction to whole class lecture: Requesting feedback from students, asking questions of students, and having students engage with the mathematics during lecture
	Hinting without telling	Guiding a student to work productively without directly telling the student a correct way to proceed
Student-Student	Group work	Creating an environment where students work together on mathematical tasks or problems
	Student Presentations	Having a student or students present completed or in-progress work to the class
Teacher-Content	Class preparation	Planning lessons to intentionally contain opportunities to engage in inquiry-oriented learning around the content being taught

And finally, the fourth dimension addresses how the instructor engages with the content, for example, through exploring the content on their own or designing resources meant to engage students in discovery. Instructors engage with the mathematical content to design and choose the inquiry-oriented problems or activities for their students (Gonzalez, 2013). An instructor's mathematical content knowledge is a prerequisite to this work (Wagner, Speer, & Rossa, 2007). An icon of the inquiry-based instruction movement has been the mathematician R. L. Moore, famous and infamous for his method of teaching students by engaging them in problems (Parker, 2005). Moore engaged with the content by understanding the mathematics and his students well enough to assign problems that were challenging enough to instill perseverance and pride, but not so challenging that students would grow discouraged and give up (Mahavier, 1999). We have attempted to review the literature to include inquiry-oriented practices within each of these categories, though we do not include a comprehensive literature review due to space limitations.

Instead of taking the simplistic view that some instructors implement inquiry-based learning and some do not, our multidimensional conceptualization of inquiry-oriented instruction allows researchers to anticipate various elements that might be present in some classes and not in other classes. We avoid rushing to a synthetic statement that inquiry-oriented instruction is one single thing and instead seek to examine whether we can identify some of its components and use them to characterize variability in the practices that present themselves as inquiry-oriented. Organizing the numerous components of practice around the instructional triangle allows us to measure the extent to which various characteristics of inquiry-oriented instruction are present.

With this framing in mind, we ask the following research question: How can we measure the inquiry-oriented practices of college mathematics instructors? In this paper, we begin exploring preliminary trends in the data collected in response to a survey that operationalized those four dimensions.

Methods

Instrument Design

The inquiry-oriented instruction review (hereafter, INQUIRE) instrument contains 62 items split into the constructs described in the framing. Each item reflects a literature-based inquiry-oriented practice which the participant can respond to on a Likert-type scale from 1-Never to 6-Multiple times per class. See Table 2 for an example item from each construct and Appendix A for additional examples. Cognitive interviews for the INQUIRE items were conducted with five mathematics doctoral students, four mathematics education doctoral students, and one mathematics department faculty member, all from two midwestern Research I universities. All had at least three years of teaching experience at the college level.

Sample

For recruitment, we used a comprehensive list of Research I mathematics departments in the U.S. We emailed the call for participants to each mathematics department and requested that they forward it to their instructors that had a minimum one-year teaching experience. Though many participants have completed the INQUIRE instrument (*N=247)* here we report the characteristics of participants that have completed both lower-division and upper division sections of the survey (*N=69*). This narrows the sample because many instructors, especially graduate students, have not taught upper-division courses.

Our sub-sample[1] consisted mostly of graduate student instructors (*N=20, 29.9%)* and non-tenure-track faculty (*N=14, 20.9%)*. The remaining instructors were postdoctoral fellows (*N=12, 17.9%)*, tenure-track faculty *(N=11, 16.4%)*, or tenured faculty (*N=9, 13.4%)*. The mean experience teaching was 9.14 years (SD=7.14). There were 34 males (50.8%), 31 females (46.3%), and 2 chose not to specify. There were 21 (31.3%) instructors who claimed to use inquiry-oriented or inquiry-based instruction, 19 (28.4%) who claimed to not use it, and 27 (40.3%) that either had not heard of it or were unsure.

[1] Based on *N=67* of the *N=69* participants, due to two participants not completing the background survey.

Table 2. *Examples of items in the INQUIRE instrument for each construct*

Triangle Relationship	**Constructs**	**Example Item**
Student-Content	Open problems	How often do you task students with problems where there are multiple solutions?
	Constructing	How often do you ask students to generalize a claim?
	Critiquing	How often do you provide students with arguments for them to critique?
	Definition-formulating	How often do you ask students to revise a definition?
Teacher-Student	Interactive lecture	While teaching the whole class, how often, after demonstrating how to solve a problem, do you ask students to try a similar problem?
	Hinting without telling	If a student asks you to look at his or her work, how often do you respond without evaluating whether or not it was correct?
Student-Student	Group work	How often do you have students work together in groups?
	Student Presentations	How often do you have students present work to the class?
Teacher-Content	Class preparation	How often do you design a sequence of problems so that students will discover something?

Results

The reliability fit statistics for item grouping in the INQUIRE instrument are satisfactory. A common cutoff for Cronbach's alpha is to consider values over 0.7 as acceptable and those below 0.5 as unacceptable (Kline, 2005), and inter-item correlations (IICs) should range between .15 and .50 (Clark & Watson, 1995). All item groupings had at least acceptable alpha scores, showing good internal consistency as shown in Table 3. Four IICs were too high (lower-division presentations and upper-division presentations, critiquing and group work), indicating that the items associated with those questions may be too similar. We can remedy this issue for creating scores later by removing some extra items.

The descriptive statistics from the INQUIRE instrument are shown in Table 4. We conducted a paired sample two-tailed t-test for each construct, results also shown in Table 4. For many of the categories, instructors report engaging students in significantly more inquiry-oriented practices in upper-division courses than lower-division courses. The only practices that were not practiced more in upper-division courses were interactive lecture, group work, and class preparation. For instructors newly attempting to implement inquiry-oriented instruction, these areas might seem more feasible or accessible.

Directions for Future Research

This study offers a method to study the inquiry-oriented practices on a broad scale. As more instructors attempt to implement more innovative practices, it could be useful for informing mathematics departments, inquiry-based-learning centers, or other stakeholders to understand what practices are currently used nationally and what factors predict their use. For our study, our first steps with the INQUIRE instrument will be to conduct a factor analysis to refine the items in our scale. The INQUIRE instrument is one of five instruments completed by all participants. We then will use methods from classical test theory and structural equation modeling to understand the relationships between beliefs, professional obligations (Herbst & Chazan, 2012), and the inquiry-oriented practices of college mathematics instructors. Early analysis has shown indications that beliefs do predict practices, but for some practices, professional obligations improve the model. We intend to continue investigating what inquiry-oriented practices can be better explained with a social lens in addition to an individual lens.

Table 3. *Reliability Statistics*

Relationship	**Constructs**	**Lower-Division**		**Upper-Division**	
		IIC	**α**	**IIC**	**α**
Student-Content	Solving open problems	.44	.79	.33	.71
	Constructing	.45	.85	.31	.76
	Critiquing	.39	.81	.56	.90
	Definition-formulating	.43	.79	.33	.71
Teacher-Student	Interactive Lecture	.25	.73	.36	.82
	Hinting without telling	.45	.77	.48	.82
Student-Student	Group work	.41	.87	.70	.95
	Presentations	.54	.89	.59	.93
Teacher-Content	Class Preparation	.29	.79	.35	.83

Table 4. Mean, standard errors, and comparison test results for the INQUIRE instrument *(N=69)*

Relationship	**Constructs**	**Lower-Division Courses**	**Upper-division Courses**	**Difference**
Student-Content	Solving open problems	3.21(0.13)	4.56(.14)	-1.36(.15)***
	Constructing	2.52(.13)	4.04(.16)	-1.52(.11)***
	Critiquing	2.42(.12)	3.93(.16)	-1.51(.13)***
	Definition-formulating	2.43(0.13)	3.78(.16)	-1.34(.12)***
Teacher-Student	Interactive Lecture	4.44(.09)	4.28(0.10)	.16(.75)
	Hinting without telling	3.39(0.08)	4.16(.98)	-0.76(.10)***
Student-Student	Group work	3.32(.16)	3.47(.17)	-.14(.13)
	Presentations	1.83(.12)	2.22(1.17)	-.39(.12)**
Teacher-Content	Class Preparation	3.83(.10)	3.90(.12)	-.07(.12)

**p<.05, **p<0.01, ***p<0.001*

Appendix A: Sample INQUIRE items

Student-Content Interaction

1. How often do you ask students to propose a definition?
2. How often do you ask students to construct mathematical arguments (e.g., justifying a solution or claim)?
3. How often do you give students problems that can be solved more than one way?
4. How often do you give students a sequence of tasks to solve that will lead them to discover something?
5. How often do you ask a student to find an error in a finished proof or solution?

Teacher-Student Interaction

6. While teaching the whole class, how often do you make an effort to elicit questions from students (e.g., by having them fill out exit slips, use clickers, giving them time to think of questions they might have, etc.)?
7. While teaching the whole class, how often do you pause your presentation to ask students to work on a problem or problems?
8. While you are solving a problem or constructing a proof with the whole class, how often do you ask students for suggestions of what to do next?
9. If a student is stuck on a problem and asks for help during class, how often do you give them a hint on how to proceed?
10. If a student is stuck on a problem and asks for help during class, how often do you help them by reminding them of an approach or strategy they've already learned?

Student-Student Interaction

11. How often do you have students give feedback to student-presenters?
12. How often do you ask a student to study and present a new topic to the class?
13. How often do you have students discuss a problem with each other?
14. If a student asks a question, how often do you redirect the question to other students?
15. How often do you encourage students to question each other's reasoning?

Teacher-Content Interaction

16. How often do you prepare worksheets for students to work on during class?
17. How often do you search in textbooks (including the one you're teaching from, if you are) or other resources to find material that will help students learn the course content?
18. How often do you design or search for problems or activities that aim to guide students to discover something you want them to learn?
19. How often do you design your lesson to include experiences you have had learning mathematics?
20. How often do you design your lesson to include experiences you have had doing mathematics?

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Erratum

We found a survey implementation error that caused a nonrandom portion of our sample to skip the remainder of the survey. If participants selected the responses “1-Never” to the question, “How often do you ask students to revise a definition?” they were skipped past the remainder of the survey, including the lower-division student-student, teacher-content questions, and the upper-division questions. Thus the statistical significance we reported in Table 4 of increased inquiry in upper-division courses may be due to the systematic missingness from the participants that took those items.

Categorizing Professors' Feedback on Student Proofs in Abstract Algebra and Real Analysis

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Abstract: Mathematics faculty spend considerable time scoring and providing feedback on student-generated proofs. Yet there is very little research on the feedback professors provide on proofs during the grading process. In this paper, we discuss a coding scheme developed for categorizing the feedback professors write on student-generated proofs in abstract algebra and real analysis. We then explore the types of annotations that professors make on student proof attempts. The results show that professors generously use annotations (like checkmarks) as informal grading tools or to signify things they have read when grading, most feedback focuses on a particular part of the proof that is no more than a few lines, and the majority of feedback does not convey why the feedback was given.

Keywords: Proof Evaluation, Feedback On Proofs, Proof Instruction, Abstract Algebra, Real Analysis.

Introduction and Literature Review

Proof writing is difficult for undergraduate students (cf., Stylianides, Stylianides, & Weber, 2018) and one of the primary proficiencies mathematicians want students to learn in advanced mathematics classes (cf., Weber, 2004, Lew, et al, 2018). Mathematics faculty report spending considerable time marking and commenting on student proof-attempts (Moore, 2016), believing that the significant amounts of time spent augmenting grades with detailed comments and corrections will improve student proof-writing. Although prior studies have focused on how professors grade their proofs (cf., Moore, 2016; Miller, et al, 2018), the content and efficacy of written feedback remains largely unexplored. This study explores the types of annotations professors make on student proof attempts, and, offers implications for student learning.

Three studies have explored how undergraduate mathematics faculty evaluate student proof-productions. Moore (2016) identified features that professors value in well-written proofs, including logical correctness, clarity, fluency, and demonstration of understanding. Two studies have found that there are disagreements among mathematics professors about how to assign points to incorrect proofs (Moore, 2016; Miller, Infante & Weber, 2018). For example, Moore (2016) found that disagreements about how severely individual errors should be penalized led to wide score variations among graders who identified the same errors. Miller, Infante & Weber (2018) found that faculty might agree that a vital portion of the proof was missing, but showed little consistency regarding how many points should be deducted. Both studies suggested that the scoring inconsistency – among graders who essentially agree on which portions of proof are in error – can be connected back to a professor's perception of student cognition. Moore (2016) explained that professors "sometimes differ in their evaluation of a student's proof because they differ in their perceptions of what the student was thinking, and consequently they arrive at different judgments on the seriousness of errors" (p. 269). Miller, et al. claimed that "a sizeable minority of the participants would give the benefit of the doubt to a student who they perceived to be strong while the majority of participants would be suspicious of a high quality proof written by a student of low perceived ability" (p. 8). These findings suggest that proof scoring relies on both a professor's perceptions about what constitutes a correct proof, as well as assumptions about the abilities of the student.

There is virtually no research on proof grading as an instructional practice. However, in moving from an analysis of scoring to an analysis of the annotations that professors make on proofs there are a number of concerns that arise. For example, some annotations might be "ticks" to indicate that a particular section has been read (e.g., mainly for the professor's organization) while others might be meant to communicate ideas to the student. We might explore the form and content of these annotations to better understand how and why professors annotate proofs, what they intend to convey to students, and, as a means to investigate what students might learn by reading the annotations professors leave on their proofs. The goal of this study is relatively modest, namely, to explore the form and content of the annotations professors make on student proof attempts. We then draw some inferences about what students might learn from the feedback, but these inferences are meant to be hypotheses that form the basis for further study.

Framing--The Nature of Feedback

Evans (2013) proposed a constructivist model in which an exchange between a professor and a student exists on a Feedback Landscape. When the professor comments on a proof, these comments are created in a buffer zone of mediating social and cognitive factors. The professor's perception of the student's understanding is one such factor. The student receiving the feedback parses the information through a similar buffer zone. It is within these buffer zones that the meaning and utility of feedback can be misconstrued or lost entirely. Glover and Brown (2006) argued the students often cannot derive actable meaning from feedback. In proof-based mathematics, Byrne, Hanusch, Moore, and Fukawa-Connelly (2018) found that students reading professor comments on proofs in a transitions-to-proof course could not reliably describe normatively correct logic for the professor's requested changes, suggesting that even when they could make the changes, that they would not derive transferable learning.

The present study developed and used a coding scheme for professors' annotations as a means to make claims, independent of professor intent or student interpretation, about the form and content of these annotations. We used the coding scheme to address the following questions:

- What types of annotations do professors commonly write on student work?
- What meaning might be conveyed to students by common annotations?
- How do the annotation patterns change over a semester-long course?

Participants & Coding Methods

The study was conducted at a medium-sized, rural, 4-year, public university in the Northeast. The participants consisted of four professors who were identified as teaching a single semester, proof-intensive abstract algebra (n=2) or real analysis (n=2) course. Both courses have an introduction to proof course as a prerequisite, and different semesters of the course should cover the same material.

Each professor volunteered to participate in the collection of students' homework, quiz, and test papers throughout the fall semester of 2017 and the spring semester of 2018. The number of homework assignments collected for fall algebra (n=4) and fall analysis (n=11) was smaller than the number of homework assignments collected for spring algebra (n=24) and spring analysis (n=26). This discrepancy has implications for using raw counts to draw inferences. To protect the students' identities, each student was assigned a numerical identifier that was kept consistent across a semester. The scores of each homework, quiz, and test item, as well as the overall scores on the students' papers, were redacted prior to coding. Each piece of written feedback was numbered so that the coders would be able to uniformly identify what constituted a separate piece of feedback. For each student paper, all the professor's annotations were analyzed by two coders working independently. The separate codes would then undergo a tie-breaking process wherein a third coder would reconcile any discrepancies.

We began the creation of our coding system using Vardi's (2009) coding system that analyzes three aspects of each item of instructor feedback, namely, characteristic, manner and scope. The characteristic category is our evaluation of the content of the professor's annotation; for example, it might be about proof structure, mathematical notation, or validity. While our focus is on the annotation, we also review the student's work in the evaluation. Vardi's characteristic codes proved inadequate for proof writing, so we used a thematic analysis through several iterations to develop new codes. In our system, each characteristic code is hierarchical, with a general group code and a detail subcode. We present a summary table of our characteristic codes in Table 1, and Table 2 shows an excerpt of our coding manual for three detail subcodes.

Table 1. General Characteristic Codes with Definition and Detail Characteristic List

Characteristic - General	Characteristic – Detail
General Academic Feedback (GAF) - Feedback that is non-specific to either proof or subject.	Mechanics (Mech)
	Completion (Comp) Sources (So)
	Fundamental Math Skills (FMS)
General Proof Feedback (GPF) - Feedback relating to the clarity and logical construction of proofs	Mathematical Language & Notation (MLN)
	Proof Framework (PF)
	Proof Presentation (PP)
	Referencing (Ref) Validity (V)
Content Specific Feedback (CSF) - Feedback on the nature and usage of content specific to the class being analyzed.	Subject Matter Notation (SMN)
	Def/Thm Content Statement (CS)
	Def/Thm Choice (DTC)
	Def/Thm Operationalization (Op)
Other Feedback (OF) - Feedback which fits none of the categories above.	Omission (Om)
	Overall (All)
	Other/Unclear

Table 2. Excerpt of coding manual

General Characteristic	Detail Characteristic	Detail Definition
General Academic Feedback (GAF)	Fundamental Math Skills (FMS)	Feedback which addresses skills and symbols from prerequisite, non-proof, mathematics courses, including algebraic manipulation and trigonometric facts.
General Proof Feedback (GPF)	Mathematical Language Notation (MLN)	Feedback on math language and notation, including idioms and phrases, set theory notation, functions and notation of non-specific symbolic logic.
Content Specific Feedback (CSF)	Subject Matter Notation (SMN)	Feedback on notation specific to the subject or course, or that repurposes previous notation in a subject specific way.

The manner codes describe the methods by which the professor expresses feedback, and the scope codes describe the breadth of application of an item of feedback within the proof. Many of Vardi's manner and scope codes were transferrable to the context of proof writing despite originally being designed for traditional academic writing. We show our manner codes in Table 3 below.

Table 3. Manner Characteristic Codes with Definition

Manner Codes	**Description**
Direct Edit (DE)	Feedback where the instructor directly edits the student's work. Something must be crossed out or inserted.
Explanation (Exp)	Feedback that explains why a change is required, that explains a mathematical concept, or explains the marker's reasoning or thinking
Prescription (Pre)	Feedback that prescribes a change to be made by the student. The change needs to be described, but not done for the student.
Question (Q)	Feedback in the form of a question.
Question Mark (?)	A question mark without other text, possibly accompanied by an underline, circle or other indication.
Comment (Com)	Feedback that makes an observation about the proof production, but does not indicate a specific correction.
Indication (Ind)	Feedback that indicates an aspect of the proof, but provides little other information, such as underlining or circling.
Evaluation (Eval)	Feedback that provides an evaluation of the student's work, such as "good" or "weak." X's over student work will fall into this category.
Personal comment (PC)	Feedback that addresses issues outside of the work, e.g., "I hope you're feeling better."
Checkmarks (Chk)	Feedback in the form of indicative marks. Often used for scoring purposes.
Other/Unclear	Any feedback that does not fit into the above categories

For this report, the scope codes we report on are *local* and *global*. *Global* feedback was directed to the proof as a whole, whereas *local* feedback appeared to be directed at a piece of text that was part of a sentence, an entire sentence, or a few lines. The critical distinguishing feature of *local* feedback was that the text could not be considered a proof of any proposition on its own (e.g., could not show that an operation was closed on a particular set), even if that proposition might be a subproof in context of the proof task. We note that our coding system includes an intermediate, *regional*, code, which we do not analyze here.

We made the decision to allow multiple codes within the characteristic and manner dimensions for a particular piece of feedback. We found this critical because in the context of abstract algebra or real analysis, there are both content-specific and "generic" proof proficiencies required to produce a proof.

At the end of the coding process, we analyzed the collected codes in several different ways. Initially, we computed the frequencies and relative frequencies of the four main code types (general characteristic, detail characteristic, manner, and scope) on each assignment (homework, quiz, test). We used these frequencies for a longitudinal analysis. We also explored coding patterns on different types of assignments such as homework, quizzes, and tests.

Data and Results

Examples of Coding

We first illustrate our coding scheme with 2 annotations that the professor made, then explain some patterns in coding we observed. For example, consider the professor's note shown in Figure 1 which makes an observation about the student's proof.

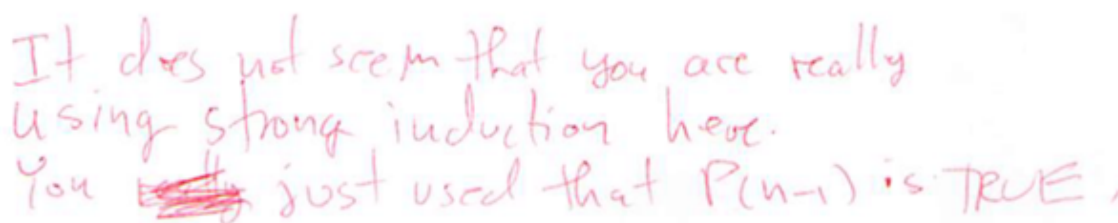

Figure 1. Professor's note about induction

We code this as *general proof feedback* (GPF) with a detail code of *proof framework* (PF) because the focus of the professor's comment is on the difference between strong and weak induction. We assigned the manner code of *explanation* because it cites a specific fact that distinguishes strong from weak induction, "you've just used that P(n-1) is TRUE" to justify the need for a change. Finally, we label it *global*, because the comment applies to the entire proof.

As a second example, consider the annotation shown in Figure 2, which edited the student's proof by inserting a symbol for union between *S* and *T*.

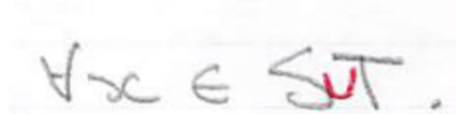

Figure 2. The professor inserted '∪' between *S* and *T*

The characteristic is *general proof feedback* (GPF) with a detail code of *mathematical language and notation* (MLN). We coded this as GPF because set theory is part of an introduction-to-proof course, and MLN because it focused on the symbolic language of mathematics. While we acknowledge that one could read this as a logical issue in that without the annotation, the sentence is not grammatical and therefore could not be interpreted, we argue that the professor appears to treat it as a "typo" where the student's meaning was clear, but missing a "word." We coded this as a *direct edit* (DE) because the professor edited the student's work by inserting the needed symbol, and as *local* (L) because it addressed a piece of content within a mathematical sentence.

Since the focus of the coding system is instructor feedback–as opposed to student errors–the coders attempted to divorce their choices of codes from the content of student proofs whenever possible. However, student content was considered in cases where the meaning of the feedback would be altered by the context of the proof. For instance, when an instructor added a symbol, such as in the above example, it would be impossible to identify the content without making an interpretation of the student's work. As a final example, we note that there are a number of annotations that professors commonly made that we did not feel we could assign content meanings to, such as a single question mark, a checkmark, or even the question "What?" and, as a result, we would code such annotations as *other* or *unclear*.

Checkmarks

One pattern shared by all professors across this study was the usage of annotations such as checkmarks which tended to make up 50% or more of the recorded comments for each coded assignment. These annotations are often informal grading tools which instructors use to tabulate scores rather than attempts at purposeful feedback to students, or they might simply be indications by the professor that she has read a particular exercise. For these reasons, this form of acknowledgement was specifically rejected as meaningful feedback for the purposes of this coding system.

Manner and Scope Types Make it Difficult for Annotations to Convey Information

The vast majority of all feedback made by the instructors from this study was *local.* No professor gave less than 78% *local* feedback, while *global* comments ranged from 3% to 17%. This may be indicative of the fact that most students had correct "big picture" ideas and structures, which might be because most items were on homework, allowing students to spend significant time and even ask for help from the professor and classmates. We further note that most local feedback did not convey information about why the professor made the annotation. The procedural was often emphasized over the conceptual, meaning that the content of annotations was, for example, focused on correct use of notation or the presentation of the proof, while explanations for why were generally absent. Even written comments focused on individual steps being taken or errors being made rather than broad feedback about how concepts were being understood. For each instructor *direct edit* was the most popular choice of manner, as exemplified in Figure 2. When *direct edits, prescriptions, and evaluations* were combined, they ranged from 43% to 71% of feedback manners, none of which explained why a change was needed. Additionally, nearly all feedback was made about incorrect work to convey a needed change. The most common positive responses were nonspecific comments such as "OK" or "Good," again without explanation of what caused the professor to evaluate it that way.

The most common characteristics were *proof presentation, validity,* and *operationalization*. We interpret the prevalence of these characteristics to be connected to the local, task-oriented nature of most of the coded comments. It is possible to make global, conceptual comments about the structure of how a proof should be written, or about the types of logical arguments that are valid. However, most often these characteristics were used for line-to-line error correction or identification.

Variations Over Time

In analyzing the occurrence of general characteristics over the course of a semester, we identified two patterns which we interpreted to indicate substantive differences between how feedback is given in algebra and analysis. The relation between *content specific feedback* and *general proof feedback* seemed to follow two subject specific patterns. All four courses began with a relatively small percentage of *content specific feedback* (less than 25%). We interpret this to be connected to early lessons designed to reintroduce students to format, structure, and logic of proofs rather than specific subject-based notation, theorems, and definitions. In both algebra classes the rate of content specific feedback rose to about 50% with a corresponding decrease in *general proof feedback*. In contrast, both analysis courses developed "bubbles" of content specific feedback which began in small percentages early in the semester, peaked around the middle of the semester, and shrank toward the end. Without making any specific causal links, these patterns could be interpreted as showing that future assertions about the quality of proof feedback for a given subject may not be immediately transferable to another proof-related class with different feedback needs. Further research is needed to better understand these patterns and whether faculty are purposeful about them. We note, for example, that the professor, Dr. T., in Weber's (2004) study wanted students to quasi-mechanically write proofs in the beginning of the semester in real analysis and only later develop the ability to write proofs with meaning. A professor like this might, at the beginning of a semester, purposefully make "behavioral" comments on proofs, editing or prescribing changes without explanation and as the semester progresses focus more on content and use explanatory concepts.

Discussion

The primary work here, of developing and implementing a coding scheme for professor proof annotations, has allowed us to note trends in annotating among four professors who teach abstract algebra and real analysis. In particular, we noted high rates of annotations that cannot meaningfully convey worthwhile information to students (e.g, checkmarks, question marks, ...). There were also a large percentage of annotations that indicated an error, and sometimes a correction (e.g, a *direct edit*), but without an explanation either of what the mistake was or why the correction was needed. Prior research by Byrne et al. (2018) suggests that students can correctly use these types of annotations to revise the given proof, but cannot explain why the corrections are needed. As a result, it seems unlikely that the annotations will lead to students changing their practice on future proofs, although this certainly warrants further study. Moreover, the trends identified were among faculty at a single university, so we should further explore how common they are. Similarly, we call for significant research exploring why faculty annotation students' papers, what they intend to convey to students, and what, if anything, they hope students will do in response.

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Using a Dynamic Geometric Context to Support Students' Constructions of Variables

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Using Thompson and Carlson's (2017) definition of a variable and the results of teaching sessions with two preservice secondary mathematics students, I describe the role of quantitative and covariational reasoning in constructing a formula with variables to describe a relationship between covarying quantities in a dynamic geometric context—the Parallelogram Problem. I report that although each student reasoned with a dynamic situation, their symbolic representations of that situation did not necessarily entail variables. I conclude that providing students with dynamic situations with which to construct formulas provides them opportunities to construct formulas with variables representing covariational relationships between quantities.

Keywords: Cognition, Precalculus, Preservice Secondary Teachers

Dreyfus (as cited in Izsák, 2000) claimed, "There must be some meaning association with a notion before a symbol for that notion can possibly be of any use" (1991, p. 31). At the time Küchemann (1981) had identified some different ways in which children interpreted "letters" (p. 110), but the pervasive difficulty of meaningful symbolization meant that he and other researchers continued to focus on students' understanding and construction of meaningful symbols (e.g., Izsák, 2000, 2003; Kaput, 1992; Kieran, 1992; Leinhardt, Zaslavsky, & Stein, 1990; Stephens, Ellis, Blanton, & Brizuela, 2017; Thompson, 1990, 1994b; Trigueros & Ursini, 2008). In this paper, I focus specifically on students' conceptions of symbols as variables within formulas. Thus, rather than students interpreting symbols as static unknowns (Dubinsky, 1991) or fixed, given referents (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), my goal for my teaching sessions (Steffe & Thompson, 2000) with preservice teachers was for them to conceptualize a symbol in a formula as representing a quantity whose value changes within a dynamic situation (Thompson & Carlson, 2017). I provide insights on how covariational reasoning—in which students conceive of situations as composed of quantities that vary in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002)—influenced students' construction of variables in formulas. I note that Thompson and Carlson (2017) argued that students' images of covarying quantities can differ dramatically, and moreover, that reasoning with a dynamic situation does not necessarily imply that a student conceives of smooth variation (see Castillo-Garsow, Johnson, & Moore, 2013). Both of these ideas meant that although a dynamic situation supported students' reasoning about variables, it alone was not sufficient for a student to construct a variable. Thus, I highlight the importance of having students attend to the roles of variables in representing covariational relationships, an important topic in calculus ideas (Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, 2017).

Background

Images and Theoretical Perspective

I adopt the radical constructivist perspective (von Glasersfeld, 1995) that individuals actively construct quantities and that an individual's image of a situation is projected from their mental organization of sensory data. The notion of image I am referring to stems from Piaget's (1967) descriptions of images as shaped by mental operations individuals perform. Thompson described the implications of this perspective on individual's viable images by noting while "the image is shaped by the operations, the operations are constrained by the image, for the image contains vestiges of having operated, and hence results of operating must be consistent with the transformations of the image" (Thompson, 1996). Moore and Carlson (2012) explored this relationship between images and operations when researching the role of images in the construction of a formula for the volume and height of a box. Travis's image of the situation differed from the image the researcher intended, but his resulting formula accurately represented his image of the situation. From this study, Thompson and Carlson (2017, p. 448) reemphasized the idea that students' constructions of symbolic expressions are constrained by the quantitative structures they construct about a situation. This idea is important to the notion of variables I use here because I argue students' images of a situation should be compatible with their formula.

Distinguishing Between Quantitative and Numerical Operations

Researchers have advocated for students' thinking of symbols as variables by involving a conception of varying values (e.g., Janvier, 1996; Kaput, 1994; Küchemann, 1978; Trigueros & Ursini, 1999). In an effort to support students' conceptions of a variable, I emphasize the role that differentiating between quantitative and numerical operations played in distinguishing between students' conceptions of variables/formulas and their images of a situation. Thompson (1994) summarizes this difference: "A quantitative operation is non-numerical; it has to do with the *comprehension* of a situation. Numerical operations are used to evaluate a quantity" (1994a, pp. 187-188, emphasis in original); a quantity is a measurable attribute (Thompson & Carlson, 2017). Thus, when I am referring to students reasoning quantitatively with a formula, I am referring to one of two cases. First, I refer to a student connecting the quantitative operations they have in a situation with the symbols in their formulas that represent numerical operations such that the symbols are conceptually tied to those quantitative operations. Second, I refer to a student constructing a formula from a given situation by considering the quantitative operations involved in relating the quantities with the anticipation that the formula also represents corresponding numerical operations.

Variables and Formulas

Symbols can serve different purposes to students depending on the meanings they attribute to them. Because of the focus on quantitative reasoning in regards to formulas, I rely on Thompson and Carlson's (2017, p. 425) three different meanings for symbols: constant, parameter, and variable. A person constructs a *constant* if the person envisions a quantity as having a value that does not vary. The symbol can take on different values, but these values do not change as the result of an image of variation. A person constructs a *parameter* if the person envisions the quantity as having a value that can change from setting to setting but does not vary within a setting. A person constructs a *variable* if the person envisions that a quantity's value varies within a setting. Unlike other researchers' approaches to variable conceptions, this study does not examine the construction of variables within a graphical setting (cf. Chazan, 2000), nor is it a more general description of a letter taking on different values (cf. Blanton, Levi, Crites, & Dougherty, 2011; Blanton et al., 2015; Izsák, 2003) or an attempt to identify non-quantitative meanings for "letters" (cf. Küchemann, 1981). This study does build on the work of others who are using contextual situations as a means to construct formulas at the elementary (Panorkou, 2017) and middle school levels (Matthews & Ellis, in press). The former reported successful covariational reasoning with a dynamic rectangular area context and the latter identified students' difficulties with reasoning about rates of change with the dynamic situation.

Methods

In an effort to understand the mental operations involved with constructing formulas through covariational reasoning, I conducted a study with four preservice teachers (two of which I report on here) from a large public university in the southeastern U.S. These participants were successful mathematics students (passed at least two upper level mathematics courses and Calculus sequence) who had experience with thinking critically about secondary and post-secondary mathematical ideas through their coursework at the university. Thus, they afford insights into how students with vast mathematical experiences conceptualize variables given dynamic situations. These students had just completed their first or second semester in a four-semester secondary mathematics education program during which they completed a secondary mathematics topics course designed from the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015). The study consisted of 3-5 exploratory teaching interviews (Steffe & Thompson, 2000) lasting 1.5-2 hours each over the course of four weeks. Each student answered the same sequence of pre-designed tasks. However, I encouraged the students to think aloud (Goldin, 2000) and I asked questions based on my understanding of their activity. The goal of my questioning was to build viable models of the *students' mathematics* (Steffe & Thompson, 2000). I conducted open (generative) and axial (convergent) analyses (Strauss & Corbin, 1998) to inform these second order models of the students' construction of formulas.

Task Description: The Parallelogram Problem

In this section, I describe some of the mental operations I hypothesize are involved in constructing a formula to represent the relevant covariational relationship. This description will provide insights into how to conceive of variables through covariational reasoning to construct formulas in the context of a novel situation. It is important to note that this description also involves constructing constants and a parameter as symbols for a formula, but my focus here is on how this dynamic situation supports the construction of variables.

The following is a description of the Parallelogram Problem, in which I presented students with the manipulative in Figure 1a and the following prompt: "Describe the relationship between the area inside the shape (shape formed by two pairs of parallel lines) and one of the interior angles of the parallelogram (up to a straight angle)." In a traditional construction of the formula to describe this relationship, a student need only work with a single static figure of a parallelogram, constructing *constants* in the situation and relating these constants to produce a formula; no images of variation are necessary to accomplish this goal beyond understanding that different states of the figure might correspond to different values. Although there is an underlying assumption that these symbols can take on different values, this construction of symbols in a formula do not fit with the notion of variable defined earlier. The goal of giving the students a shape they could manipulate was to support their construction of a variable by having them consider the covariational relationship of the area of the parallelogram *ABCD* and the openness of $\angle DAB$.

In order to motivate the construction of a variable for a formula via covariational reasoning, one needs to quantify two quantities in a situation and then determine the covariational relationship between them. One potential first step for the Parallelogram Problem is for a student to construct the relationship between the area of a parallelogram and a corresponding rectangle. A student can conceive a parallelogram's area as equivalent to the area of the rectangle constructed by translating a triangular region ($\triangle DEC$) alongside base $\overline{AD}$ (Figure 1d/e). Then, a student can anticipate that as $\angle DAB$'s openness increases, a rectangle with equivalent area can be produced for each instantiation of $\angle DAB$'s openness. At this point the student has determined the two quantities to covary: the angle of the parallelogram and the area of the rectangle constructed from translating the triangle shape in the parallelogram (Figure 1f). Reasoning about amounts of change, a student can then consider equal changes in $\angle DAB$'s openness and attend to the corresponding areas and changes in areas of the rectangle (and, equivalently, the area of the parallelogram) to make, for example, the following conclusion: For equal changes in $m\measuredangle DAB$ from 0 to π/2 radians, the measure of the area of the parallelogram is increasing by decreasing amounts with respect to angle measure (see shaded areas in Figure 1f).

To construct a formula with variables from this situation, a student can use a meaning for the sine relationship that involves understanding it as the height above the center of a circle measured in radii (Moore & LaForest, 2014). That is, the length of $\overline{AB}$ rotated around leaves the traces of a circle centered at point *A* with radius $\overline{AB}$ (Figure 1i), a non-trivial connection (Hardison, Stevens, Lee, & Moore, 2017). This connection between the covariational relationship identified in the situation and the sine relationship is what enables the construction of variables within a formula. That is, a student can represent the covariational relationship between the height of a parallelogram in the situation and $m\measuredangle DAB$ with the sine relationship (Figure 1g/h). Moreover, the student can use of symbols for the sine relationship (which entail numerical operations) to represent the covariational relationships identified in the situation. This reasoning is the second way of quantitative reasoning with formulas described previously.

From there, a student can construct a relationship between quantities to produce $EB = \sin(\theta)$, where $\sin(\theta)$, in this formula, represents the height's changing value in the situation. This magnitude is multiplied by the value of the length of the base of the parallelogram, *AD*. Thus, the final formula for the area of the parallelogram is $Area = AD\sin(\theta)$. Here, based on the student's image of the situation, the variables are *Area* and θ. I will not include a discussion of the role of units in this formula here, and assume that the student is constructing an area measured in *radii*[2]. However, see Alexandria's example in the results section to see how this image can be extended to consider other units, thus resulting in a normative formula.

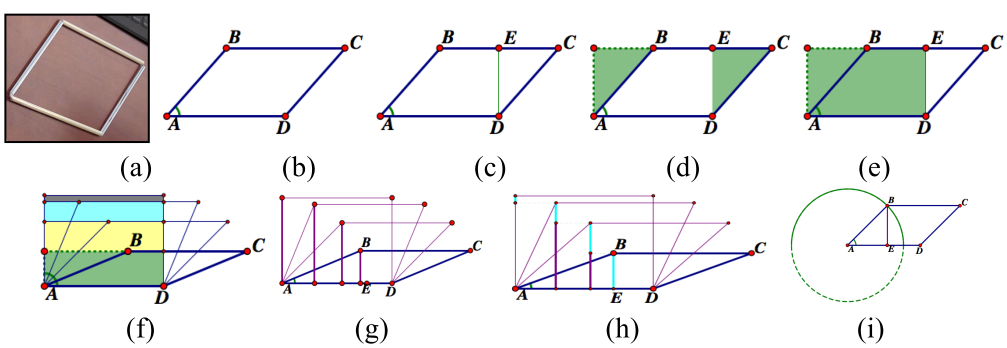


Figure 1. Image from Stevens (in press) (a) manipulative with changeable angles (b) labeled parallelogram (c) labeled height DE (d) triangular region in parallelogram translated to form rectangle (e) rectangular region with equivalent area to parallelogram (f) various colored areas indicate amounts of change in area for equal changes in angle measure (g) dark purple segments indicate various heights for equal changes in angle measure (h) light blue segments indicate amounts of change in height for equal changes in angle measure (i) height of the segment above $\overline{AD}$ *as the fractional amount of the radius AB of a circle centered at A (i.e., the sine relationship).*

Results

I now describe students' different types of formula construction as it relates to variables for the area of the parallelogram based on my analysis of Charlotte and Alexandria's interviews. To offer an indication of the difficulties the students initially had with the Parallelogram problem, I note that three of the four students (including Charlotte and Alexandria) initially attempted to justify that the area remained constant as they manipulated the object. When pushed on their justifications, they began to doubt their initial claims and were motivated to attend to the quantities to form new justifications. The remainder of the section focuses on the results of this reasoning as it relates to their formulas.

Constructing Multiple Systems of Measurement with Constants to Describe One Situation

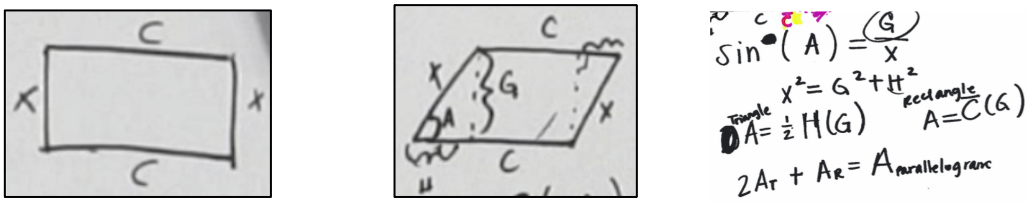
Before attempting to reason covariationally about the quantities, Charlotte had constructed a sequence of calculations (Figure 2c) to carry out in order to determine the measurement for the area of a specific parallelogram. This sequence was the result of reasoning with the static parallelogram in Figure 2b. This process was similar to the description for a traditional construction of the formula in that each of her symbols represented a constant from a static figure she drew. The process differed in that she did not combine all her sequences of actions to calculate the measurement into *one* formula.

Her goal for constructing this sequence of calculations was to compare its resulting value to the value for the area when the shape was a rectangle. She knew that to calculate a value for the area of a rectangle she should multiply the length of its base with the length of its height. She wanted to calculate these two measurements "because then I could compare-like I wouldn't be-I wouldn't be assuming based on like my eye, like changing- like trying to figure out how the area changed. I would know like-I would have concrete numbers." That is, she wanted to make gross comparisons between the numerical values to determine if the two areas' values were different.

Charlotte wanted to calculate the measure of the area for the shape when it was a rectangle by multiplying X and C together (Figure 2a). She then wanted to determine the measure of the area of a parallelogram with a given angle measure (Figure 2b) using her sequence of calculations (which would have resulted in the correct area measurement) (Figure 2c) and compare the resulting values. It is important to note that although this latter sequence of calculations could actually be used to find the area of every parallelogram in the situation, she viewed each of the symbols in her formula as unknown constants for that specific instance of the parallelogram. For instance, she referred to the angle as "the angle that I picked" and that she "would know the value of the angle" when she went to calculate the measure of the parallelogram's area with that angle.

The constructions of her formula and sequence of calculations themselves were insufficient for her to make a conclusion about whether or not the area of the parallelogram changes because she did not know what values to use. She realized this issue only after constructing her two systems of measurement, stating, "Okay. So, I don't know. If I-If I do algebra then I could see-I could tell you maybe." However, she did not continue trying to relate the two systems. In fact, by the end of her attempt with this strategy of comparing the two instances, she still anticipated that the areas for each would probably have the same value.

This conception of formulas differs from the one described in the first order model because she constructed two different systems of measurement to describe one situation. Thus, even though she identified changing angle measures and wanted to make conclusions about the directional covariational relationship between angle measure and area in the situation, neither her formula nor her sequence of calculations were the result of covariational reasoning. Rather, they were the result of analyzing static figures. Thus, all of her symbols represented constants.



(a) *(b)* *(c)*

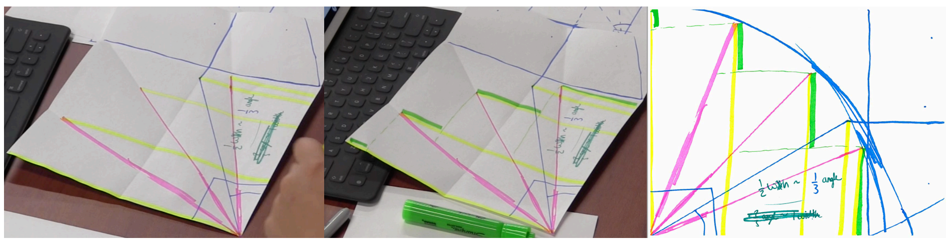
Figure 2. (a,b) The two instances of the situation Charlotte whose areas she wanted to measure and (c) her sequence of calculations in order to measure the area of the parallelogram in Figure 2b.

Constructing a Formula Disconnected from the Students' Image of the Situation

Later in the interview, Charlotte shifted to a different approach and went through steps similar to those described in Figure 1b-e. She used the manipulative in Figure 1a to describe how she wanted to calculate the area of the parallelogram. She wanted to multiply the "distance between yellow [side length] to yellow [side length]" in the manipulative by the "length of the yellow [side length]". Even with this new insight, Charlotte remained unsure whether the area changed. She concluded, "I reckon. It's really hard to tell because like in the back of my mind, I think, 'Oh, just because it's getting narrower-more narrow, I'll say it like that, it doesn't mean necessarily that the area is decreasing." At this point, Charlotte had a new way to calculate the area of the parallelogram by reasoning quantitatively about the situation. However, her image of the situation made it difficult to reconcile whether or not this new method was an appropriate way for her to reason about the directional covariational relationship between the quantities. Her uncertainty demonstrates that although she had constructed a way to measure area that involved her being able to make comparisons between different states using the same formula, the numerical operations she noticed that resulted from her new formula were not entirely connected to how she conceived of the quantities in the situation. One hypothesis for this uncertainty is that she had constructed this new calculation for the measurement by focusing on instances of the parallelogram to rectangle translation instead of imagining a smooth image of a growing rectangle she conceived. Regardless, the disconnect between her conclusions about the situation and her meaning for the symbols in her formula (i.e., the two distances between the side lengths) make it so that the latter do not fit with the aforementioned definition of a variable.

Constructing a Formula that Entails Students' Identified Covariational Relationships

Alexandria, similar to Charlotte, had gone through the process outlined in Figure 1b-e. At this point, Alexandria thought there was a linear relationship between the angle measure and the height of the parallelogram (which to her also implied a linear relationship between the angle measure and area of the parallelogram). To check, Alexandria constructed four equal changes in angle measure. She constructed the pink side lengths of the parallelogram (Figure 3a) and then, in yellow (Figure 3a), marked the corresponding heights for each marked angle measure. She then highlighted in green (Figure 3b) the segments that represented the change in height for each successive equal change in angle measure from a "full" right angle downwards. She claimed, "The change in height increases," and then concluded, "It's not linear." At this point, Alexandria had reasoned covariationally about the quantities in the situation, but she did not have a formula. Alexandria stared at her work for about 25 additional seconds, and then muttered, "Don't tell me this is sine?" She drew in a quarter circle in blue (Figure 3c). She then began to describe how she drew in her blue curve based on the pink lines, and then suddenly exclaimed, "Oh, it's a radius! No, duh. Ding ding ding ding ding! The pink line's a radius." Thus, Alexandria's meaning for the sine relationship entailed a covariational relationship between an angle measure and a height quantity that she was also able to identify in this situation (as outlined in Figure 1g-i). After this point, Alexandria quickly constructed a final formula to represent the relationship between the angle measure and area of the dynamic parallelogram. She wrote the following formula: "$L\sin(\theta)$=*Area*" and "L=*AD*", calling L her "length" and pointing to $\sin(\theta)$ saying that it "gives me my height." When I asked her what units she was using, she almost immediately wrote, "$Lr\sin(\theta)$" saying that $\sin(\theta)$ "alone [*circles sin(θ)*] gives us how much of a radius it is, so how-how much of this whole [*pointing along the edge of her paper corresponding to the height of the parallelogram with right angles*]...so I put the r [*pointing to the r in her formula*] in there to give me a typical measurement that we're used to." Thus, in this situation, she conceived of both angle measure and height changing in the situation, and she constructed the variables θ and *A*rea to describe her image of how quantities are varying within a dynamic situation.



(a) (b) (c)

Figure 3. Alexandria's (a) construction of heights (yellow) (b) construction of amounts of change in height (green) and (c) construction of a quarter circle (blue).

Discussion and Conclusions

I proposed two ways in which a student could reason quantitatively about a formula. The first involves connecting quantitative operations with numerical operations. Charlotte's activity demonstrated the difficulty of this reasoning. Her formulas were the result of quantitative operations using static images of the situation, which she then wanted to use to reason about the covariational relationship between quantities. However, in the first case, she was unable to use her two systems of measurement to draw a conclusion, and in the second case, she struggled to reconcile her image of the situation with the numerical operations resulting from her quantitative reasoning with the situation. These disconnects illustrate how important it is for a student to be able connect quantitative operations within a situation with the symbols that represent not only quantitative relationships but also numerical operations for measurement. Alternatively, Alexandria demonstrated the second way of reasoning quantitatively with formulas because she was able to identify a covariational relationship within the situation and construct variables, and ultimately a formula, that represented both the quantitative and numerical operations she conceived. Her reasoning illustrated a powerful conception of a variable in that it enabled her to construct a formula, which, unlike Charlotte's formulas, represented a covariational relationship between quantities. As a result of this study, I conclude that students should be provided with more opportunities to construct variables via reasoning covariationally with dynamic situations.

Acknowledgments

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An Analysis of a Mathematician's Reflections on Teaching Eigenvalues and Eigenvectors: Moving Between Embodied, Symbolic and Formal Worlds of Mathematical Thinking

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In this paper, we analyzed a mathematician's daily teaching journals of a 5-day series of teaching episodes on eigenvalues and eigenvectors in a first-year linear algebra course. We employed Tall's (2013) three world model, in conjunction with Tall and Vinner's (1981) concept images and concept definitions, to follow the mathematician and instructor's movements between Tall's worlds. The study showed that the instructor strived to build concept images that, while perhaps mirroring his own concept images, did not resonate with the students.

Keywords: Tall's Worlds, Concept images, IOLA, Reflections, Eigenvalues and Eigenvectors

Theoretical Background

How do mathematicians motivate mathematics concepts in teaching? As learners of mathematics, our past experiences bring to mind a variety of teaching styles. There were lectures where the professor only wrote definitions, theorems, and proofs on the board, followed by a number of examples, and in some rare occasions, professors motivated the idea with pictures. Building on Tall and Vinner's (1981) notions of *concept images* and *concept definitions*, Vinner (1991) claimed, "We assume that to acquire a concept means to form a concept image for it. To know by heart a concept definition does not guarantee understanding of the concept. To understand, so we believe, means to have a concept image" (p. 69). Vinner also examined the role of the definition; in his view, definitions help us to form a concept image. However, he noted that "the moment the image is formed, the definition becomes dispensable. It will remain inactive or even be forgotten when handling statements about the concept in consideration" (p. 69). Using the scaffolding metaphor, he compared this idea with building, saying "the moment a construction of a building is finished, the scaffolding is taken away" (p. 69).

Developing these two notions further, Tall's (2010; 2013) three worlds framework for mathematical thinking (embodied, symbolic, and formal) endeavors to lay out the individual mathematics learning journey from childhood to a research mathematician. According to Tall (2010), the embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns… and other forms of figures and diagrams" (p. 22). In other words, the various ways of thinking in the embodied world can also be characterized as giving body to an abstract idea. In Tall's (2010, p. 22) words, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number)". Finally, Tall defines thinking in the formal world as that which "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22).

Using Tall's model, Stewart, Thompson, and Brady (2017) investigated a mathematician's (and co-author) movements between Tall's worlds while teaching algebraic topology. In this study, the instructor reported that students experienced the most difficulty in moving from the embodied world into the formal world. Believing the struggle would stimulate mathematical growth in his students, this instructor "refused to give students proofs that were pre-packaged. More specifically, he wanted to provide students with intuitions and pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it" (p. 2262). Stewart (2018) created a set of linear algebra tasks designed to help students move between Tall's worlds. Stewart, Troup, and Plaxco (2018) examined a mathematics educator's (and co-author's) movements as well as decision-making moments while teaching linear algebra. All these studies indicate that movements between Tall's worlds are a rich research topic worthy of ongoing investigation.

As part of the first author's research program, the overarching goal of this study was to examine a mathematician's (the instructor and co-author) movements between Tall's worlds. Throughout this investigation, the instructor often emphasized that his goal was to reach the eigenvalues and eigenvectors section of the course, which motivated the focus of this paper. Furthermore, although research on students' difficulties and understanding of eigenvalues and eigenvectors has increased (e.g., Caglayan, 2015; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Thomas & Stewart, 2011), research on mathematician's voices and what goes on in the mind of the working mathematician while teaching the eigentheory is still scarce. The data analyzed in this study is from the instructor's reflections on teaching eigenvalues and eigenvectors. Most researchers maintain that reflection is an essential part of teaching mathematics (e.g., Davis, 2006; Fund, 2010; Moore-Russo & Wilsey, 2014;). According to Dewey (1933), reflection is "active, persistent, and careful consideration of any belief or form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). Fund (2010) adds that "teachers need to develop particular skills, such as observation and reasoning, in order to reflect effectively and should have qualities such as open-mindedness and responsibility" (p. 680).

The research questions guiding this study were: (a) How did a working mathematician convey to students a concept image of eigenvalues and eigenvectors? (b) What were some of the factors causing the instructor to move between Tall's worlds?

Methods

This qualitative narrative study (Creswell, 2013) is the second in a series of studies intended to examine the linear algebra instructor's mathematical thought processes while teaching a first course in linear algebra, as well as how they leverage Tall's (2013) three worlds. This study took place over the course of a semester at a Southwestern research university in the US. The analysis focuses on an instructor's observations, as recorded through journal entries, over a five-day period, while implementing tasks from the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro et al., 2013). The research team consisted of a mathematician specializing in differential geometry (the instructor, postdoctoral fellow, and co-author), two mathematics educators, and an undergraduate research assistant student.

Throughout the semester, the instructor recorded his observations on how his class reacted to a variety of teaching styles and ideas. He additionally met with the research team once a week throughout the semester and the following summer to discuss these experiences and reflections. This allowed the researchers to triangulate data via member checking with the instructor directly and additionally afforded him ample time to share a wide variety of teaching experiences, as well as his reasoning and thought processes while making these decisions. To collect additional data on the instructor's teaching from the student's perspective, the research team administered a survey, given as a worksheet, and conducted a student interview.

The research team converted the instructor's journal and the worksheet results into Excel spreadsheets to expedite coding and sorting the data to search for themes after coding. In keeping with a narrative study, the research team performed a retrospective analysis of the journal (Creswell, 2013) by iteratively coding the data. The team started with a combination of categories developed from the previous study (Stewart, Troup, & Plaxco, 2018) and an open coding (Strauss & Corbin, 1998) scheme to allow for the possibility of discovering new categories unique to this study. The main themes for this study were: Teaching, Students, Class Activities, Math (instructor's math, students' math), Reflection, and Tall's worlds. By instructor's math, we mean the math he was doing and talking about, and by students' math, we mean his reflections on students' mathematical abilities and conversations on math in class. For the purpose of this paper, we will only present the analysis from the instructor's journals.

Results

In this section, we will analyze the instructor's journals on five class periods of an introductory linear algebra course, during which the fundamentals of eigentheory were presented. The class met three times each week for a period of 50 minutes. The classes were structured around a sequence of four tasks designed by the Inquiry-Oriented Linear Algebra (IOLA) project (Wawro et al., 2013). The tasks use the ideas of "stretch direction" and "stretch factor" of a linear transformation to develop the formal notions of eigenvector and eigenvalue. Several of the requisite concepts, such as bases, coordinates and matrix representations of linear transformations, were covered earlier in the term so that the IOLA sequence could be used. In analyzing his 5-day teaching segments, we will examine the instructor's (a) movements between Tall's (2013) worlds, (b) pedagogical decision-making moments, and (c) reflections on self and students.

An Analysis of the Teaching Episode: Day 1 (March 30) - IOLA Task 1

The first IOLA task (see figure 1) describes a linear transformation geometrically, in terms of "stretch directions" and "stretch factors," and presents three questions related to it. First, the students are asked to sketch the image of a figure "Z" centered at the origin. In the second part, they are asked to sketch the image of two particular vectors and then compute the precise images. Lastly, they are asked to produce a matrix representation of the linear transformation.

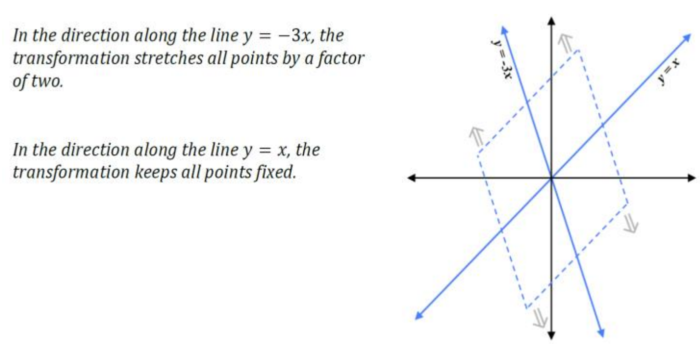

Figure 1. IOLA Task 1

This task is primarily situated in Tall's (2013) embodied and symbolic worlds. By withholding any matrix representation of the transformation, the task was meant to force students to interpret the action of the transformation on vectors via the embodied world. Ideally, this would build intuition and facility. The instructor very quickly noted that students were having difficulty with embodied thinking and decided to take a more active role in guiding them through the task on the board. His next intention was to move students to a more symbolic representation of an idea of stretching, which he wrote as a "mathematical one." The instructor mentioned in his journals that the students struggled again.

> We needed to iron out the common misunderstandings: for every linear transformation the zero vectors get sent to the zero vector, points are identified with vectors, etc. Then we needed to understand what stretching means. After one or two attempts and a geometric description, I asked for a mathematical one. Although no one could articulate it precisely, at least one student had the right idea: scalar multiplication.

In question 2, the instructor computed (symbolically) the images of vectors under the transformation, and had a feeling that students were able to follow. However, their understanding faltered when the instructor changed the vectors slightly. "So, in question 2, we converted the two vectors into linear combinations of vectors in the stretching direction, then used the linearity of the transformation to find their images. I'm not sure if this made sense to them." In question 3, students did not give much feedback. The instructor gave a handout—the preview of the next task— and hoped that "…perhaps the motivated student [would] see the connection of how to use it and then be more prepared for the next task."

An Analysis of the Teaching Episode: Day 2 (April 2) - IOLA Task 2

The second IOLA task continued to build the concept image in much the same way as the first, but instead of a figure "Z", there is a collection of discrete points (see figure 2). Moreover, both the standard coordinate grid (referred to as the "black" coordinates) and the one determined by the eigenvectors (referred to as "blue" coordinates) are overlaid on the collection of points.

At the start of the task, the instructor perceived that the students were not engaging with the tasks in a meaningful way. He remarked on having "difficulty getting the students to be active participants." As a result, he "decided to do the worksheet together," meaning that he would guide the class by doing the various parts at the board. He conjectured that "part of the reason that the worksheet took so long was because most students don't have a facility with coordinate vectors."

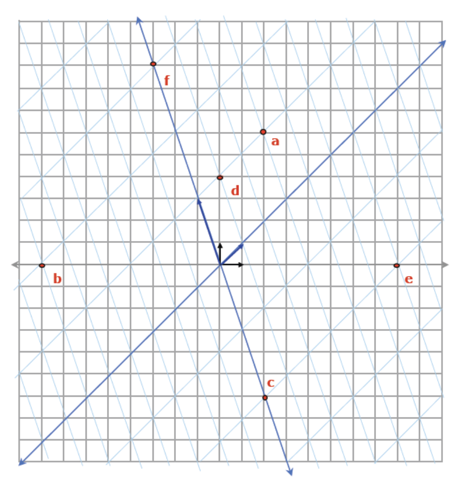

Figure 2. IOLA Task 2.

The instructor made the pedagogical decision before the class started to present the definition of eigenvalue and eigenvector after the first two tasks. Two class periods exploring the connection between coordinates and linear transformations would be sufficient as "a segue to define eigenvalues and eigenvectors." Introducing them halfway through gives some resolution to the first two tasks, while also providing a framework within which the last two tasks can be situated.

Despite recognizing the importance of everyday thought modes for developing concept images, the instructor still views the definition as the most important element in the concept image. Not only does he choose to present it after only two class periods, but he also expresses frustration at not arriving at the definition sooner. “Finally, I was able to define eigenvalue and eigenvector.” In fact, he makes the decision to cut short the discussion of Task 2, Part 3 in order to present the definition. He remarked, “Problem 3 was useful, and I wish I had more time to go through it.”

An Analysis of the Teaching Episode: Day 3 (April 4) Lecture

The instructor made the pedagogical decision to use Day 3 not for the next IOLA task, but instead to synthesize the various embodied, symbolic and formal aspects of eigentheory that the students have so far encountered. To do so, he used exclusively a lecture teaching style. First, he showed how the black and blue coordinate matrix representations of the transformation from those tasks are related by conjugation by the change of coordinate matrix. Next, starting with the standard coordinate representation of the linear transformation, he used GeoGebra to demonstrate visually the effect of the linear transformation on vectors in the unit circle, and in particular how it exactly stretches some, but not all, directions. At this point, he reiterated the eigenvalue and eigenvector definitions and derived the standard way of computing them from the characteristic polynomial and finding the nullspace of $A - \lambda I$. From here, he presented a series of examples including the transformation from the IOLA tasks, an eigenspace with more than one dimension, and the differentiation operator acting on function spaces.

The instructor did not make any remarks on how the students responded to the lecture. Instead, his journal entry was a rather clinical report of the content from the lecture, mainly including the instructor’s math and no mention of students’ math. From this, one could infer that the instructor was engrossed in conveying his own concept image and how he experiences the mathematical concepts of eigenvalues and eigenvectors.

An Analysis of the Teaching Episode: Day 4 (April 6) - IOLA Task 3

On Day 4, the instructor returned to the IOLA sequence with task 3. This task is the most similar to standard textbook exercises for eigentheory. For three distinct two-by-two matrices, the students are asked to 1) find the stretch factors given the stretch directions, 2) find the stretch directions given the stretch factors, and 3) find both the stretch factors and directions. After observing their work for the first part, the instructor noted that, even though “they had a WebWork assignment due the same day that was mostly about computing eigenvalues and eigenvectors,” he “was surprised to see how many were unsure where to start.” The WebWork assignment he mentioned contained only column vectors and matrices, while the IOLA task describes stretch directions. Hence, the instructor interpreted this as a lack of synthesis between the ideas of “direction” and “column vector.” This motivated the instructor’s pedagogical decision to use the blackboard to guide the class through the task, reinforcing certain connections in the image concept.

First, he “decided to go slowly through some fundamental concepts that might be getting in the way of using the eigentheory.” Among the fundamental concepts that the instructor covered were the embodied-symbolic connection between nonzero vectors and “directions” in the plane. Next, he reiterated how shapes in the plane could be thought of as collections of vectors. “I think it’s always worth repeating that a vector ‘lies in a shape or object’ if the tail sits at the origin and tip sits at a point in the shape.” Also, he showed the class how finding the stretch factor (given the stretch direction) is equivalent to solving a linear system with one unknown and usually more than one equation. The fact that the system is consistent is remarkable. With these fundamental notions in place, he proceeded with the work of completing the task. As on Day 3, there was no mention of students’ math in his journals.

An Analysis of the Teaching Episode: Day 5 (April 11) - IOLA Task 4

The fourth IOLA task aimed to introduce students to a subtlety, thus far hidden, of eigentheory: multiplicity. The entire task involved a single linear transformation of R^3, presented as a matrix. As in the previous task, the first two parts involved finding either a stretch direction or a stretch factor, given the other. In particular, it is found that a certain stretch factor has two stretch directions; i.e., the corresponding eigenspace is two-dimensional. The third and final part poses a rather provocative question: given that 2 and 3 are stretch factors and the former has two distinct stretch directions, could there be additional stretch factors? At the heart of this question is the observation that eigenvectors for distinct eigenvalues must be linearly independent. A counting argument then shows that we already have a basis of eigenvectors and hence there can be no other eigenvalues.

The instructor appeared eager for the class to spend time with this last part. He made the pedagogical decision to go through [the first two parts] together on the board. “My hope was that this would put everyone on the same page to try the third part.” Once the students had an opportunity to think about the third part, he observed:

> Every student’s work that I saw was the same. To decide if there was another eigenvalue or stretch direction they all computed the characteristic polynomial to see if there was another root. I anticipated this, so I then presented a solution that crucially uses the fact that all three eigenvectors form a basis for R^3. I did not get very much feedback from the class on whether they were internalizing this.

Although there were multiple ways to approach the third part, the students all reached for the most symbolic, computable solution. They found the characteristic polynomial in order to find all the eigenvalues; anticipating this, he presented a contrasting formal solution. In this way, the students would hopefully see alternatives to the symbolic world, and perhaps build a connection between the two concepts of basis and eigentheory. The final piece of eigentheory was diagonalization. After presenting an example with insufficiently many stretch directions, the instructor was in a position to explain diagonalization and when it can be done.

Discussion and Concluding Remarks

Throughout the course, the instructor tried to follow IOLA’s objectives designed for each task. His intention was to have the students work in small groups to complete each task first, and then come together as a class to discuss solutions. However, on many occasions when he noticed that progress among the students was much slower than anticipated, he often reverted to a more standard lecture format. While encouraging participation from the class, he would go through the tasks at the blackboard.

While the instructor’s decision to use the IOLA tasks shows that he values the embodied and symbolic worlds as part of the concept image, the instructor’s goal of reaching the formal world became apparent in many of his journal writings. For example, on Day 5, the instructor tries to speed through what he considers “rote” so that the class can get to something more formal that generates connections between concepts. In fact, his decision to present the definitions of eigenvalue and eigenvector at precisely the midpoint of the unit reflects the significance they hold for him. They represent, for the instructor, a single idea which unites the various notions from all three world that the students have been exposed to. A mathematical understanding of eigentheory (to him) involved primarily the definitions, but also how those definitions manifested themselves in the embodied and symbolic worlds. The instructor’s objective was a mathematical treatment of eigentheory, so he used IOLA to present a web of connections surrounding the formal definitions.

The instructor seemed to believe the more connections between eigentheory and other linear algebraic concepts that he could convey to the students, the more robust the concept image. He laments not showing how the linear system that must be solved to obtain the stretch factor would be inconsistent if it was set up with a non-stretch direction. "What I should have done, in addition, is to point out that when you choose a vector, not in one of the eigenspaces, then solving for a stretch factor will lead to an inconsistent system." Later, he regrets not connecting the formalism of solving linear systems to finding eigenvectors. "But perhaps I should have gone through the derivation of the nullspace of a matrix, rather than appealing to their experience with WebWork calculations." It was interesting to notice that even a mathematician that values all three worlds of mathematical thinking would still gravitate more toward the formal world as the most important part of a mathematical concept. Although he recognizes the necessity of embodied and symbolic concepts in the acquisition and understanding of mathematical concepts, they are merely the scaffold on which the formal notions are built.

In Tall's view, "formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used" (2013, p. 18). Due to this added power, Tall believes that the formal world can interact with and inform the embodied and symbolic worlds. In particular, in his view, "formal mathematics can reveal new embodied and symbolic ways of interpreting mathematics" (p. 18).

Lastly, one may speculate that the instructor underestimated the time necessary for establishing new connections between mathematical ideas. What appears "rote" and part of his "everyday" mode of thinking is completely foreign to the typical undergraduate linear algebra student. Hence, the connections between the formal definitions and surrounding concepts that appeared so strong to the instructor were quite tenuous with the students. Perhaps this accounts for the frustration that appears in his tone and the decision in multiple instances to explicitly guide the students to the connections.

The research team is in the process of analyzing the data from students' surveys as well as analyzing the instructor's journals on more linear algebra concepts. It will be interesting to know which connections, if any, were effective in developing the students' concept image. Do the connections between and within Tall's worlds benefit the teaching of eigentheory?

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PROMESAS SSC: Transforming the Teaching of Collegiate Mathematics

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In this study, we describe how a funded professional development program for collegiate mathematics faculty impacted their teaching philosophy. As a result of this program, participants felt an obligation to attend to students' needs as part of student-centered learning, found rich tasks useful to connect students' prior knowledge with new content, recognized the value of creating a sense of community in the classroom, and established a community among themselves. This newfound community was especially valuable for adjunct faculty. The participants expressed how they shifted and aligned their teaching beliefs with teaching practices, appreciated evidence-based teaching techniques, taught with intention, and realized how both they and their students had more fun in the classroom. Furthermore, the participants came to comprehend how these practical and philosophical transformations fostered equitable teaching practices in the mathematics classroom. The structure of this program may serve as a model for future professional development programs.

Keywords: Collegiate mathematics, Equity, Professional development, Transformation

Introduction

Professional training of collegiate mathematics instructors continues to gain momentum and research related to this realm of mathematics education has grown over the past decade. Some of the research stems from funded grants that support the training of collegiate mathematics instructors, particularly graduate teaching assistants (Deshler, Hauk, & Speer, 2015; Harris, Froman, & Surles, 2009; Speer, Gutmann, & Murphy, 2005). Our research is of the same nature; we explore collegiate instructors' experiences with engaging in *The STEM Service Courses Initiative of Project Pathways with Regional Outreach and Mathematics Excellence for Student Achievement in Science, Technology, Engineering, and Mathematics (STEM)*, which we refer to as *PROMESAS SSC*. This project is a regional STEM initiative where mathematics faculty from a 4-year Hispanic-Serving Institution (HSI) and three HSI community colleges collaboratively address systemic change in teaching collegiate mathematics. The aim of *PROMESAS SSC* focuses on transforming mathematics pathways into STEM and to strengthen the STEM student success pipeline. The project emphasizes faculty development on cultural competency, inclusive pedagogy, and renewing the collegiate mathematics curriculum. In an effort to address these topics, the lead researcher, in conjunction with five faculty from the various institutions, co-developed and co-facilitated a year-long professional development (PD) program for the participating collegiate instructors, who we refer to as fellows. The PD focuses on three themes: (a) building classroom community, (b) teaching with a student-centered lens, and (c) creating and implementing rich mathematical tasks, all aimed at promoting equity in the mathematics classroom. In this research, we address the research questions: (1) what is the nature of the fellows' experiences with *PROMESAS SSC* and (2) what is the nature of the fellows' transformation regarding their philosophical teaching practices as a result of *PROMESAS SSC*?

Literature Review

PD programs for collegiate mathematics instructors and graduate teaching assistants have slowly emerged around the globe, but a review of the literature related to collegiate PD for instructors resulted in only one study exclusive to mathematics (Barton, Oates, Paterson, & Thomas, 2015). Facilitators of PD workshops documented their specific institutions' unique, novel PD programs through case studies. There were several underlying similarities that contributed to successful PD programs, i.e., those that inspired a shift from teacher-centered instruction to student-centered instruction. Such characteristics included: promoting a community of educators, providing continued support for the instructors, and modeling evidenced-based teaching methods (Ash et al., 2009; Barton, et al.; Czajka & McConnel, 2016; Denecker, 2014; Ebert-May et al., 2011; Hadar & Brody, 2010; McCrickerd, 2012), which we incorporated into the PD. Although these common themes exist, there also remain conflicting views regarding PD content and participants such as discipline based versus interdisciplinary and novice teachers versus teachers with varying degrees of experience (Barton et al.; Denecker; Ebert-May et al.).

Although the aforementioned studies did not allude to the role of equity in PD, some programs promoted equitable teaching in mathematics. We adopt Gutiérrez's (2009) ideas on equity to guide our definition of equitable mathematics teaching as a student-centered classroom accessible to all students including various and diverse classroom environments, classroom discussions, and group work. Mathematics PD programs often contained these teaching practices, but it is unclear if and how equity fit into these teaching strategies because PD facilitators do not make this connection clear (Battey, Kafai, Nixon, & Kao, 2007). As such, collegiate instructors did not attend to equity issues in the mathematics classroom unless the PD addressed awareness of equity, subject-matter training, best practices, and inquiry as a whole package rather than as disjoint topics (Battey et al., 2007). Therefore, to encourage implementation of equitable teaching practices, PD facilitators are encouraged to explicitly connect content-specific training, such as student-centered learning, with equity issues (Battey et al.; Lee, 2004).

While it remains important to introduce collegiate instructors to research highlighting the benefits of equitable teaching practices, PD programs also need to expose instructors to practices that could promote inequity in the mathematics classroom. In general, equitable teaching practices nurture underrepresented students' mathematical self-confidence and mathematical agency (Deshler & Burroughs, 2013; Sax, Kenny, Riggers-Piehl, Wang, & Paulson, 2015). Similarly, phenomenon such as stereotype threat, teacher's subconscious bias, and classroom norms can create an important, but often unnoticed, negative impact on underrepresented students' confidence and independence as mathematics learners (Sax et al.; Steele, 1997). In the *PROMESAS SSC* PD, the facilitators routinely attempted to introduce the fellows to research connecting teaching practices to equity.

Theoretical Lens

Given our interest in exploring the fellows' experiences with *PROMESAS SSC*, we conducted a study with a phenomenological focus. Patton (2015) contends that such a lens is appropriate when researchers inquire into "how human beings make sense of experience and transform experience into consciousness, both individually and as shared meaning" (p. 115). The phenomenon of interest can include emotions, physical experiences, relationships, cultures, organizations, or participation in programs. Such descriptions and interpretations can prove difficult to differentiate because "interpretation is essential to an understanding of experience, and the experience includes the interpretation" (Patton, p. 116). Thus, in order to make sense of one's lived experience, researchers must place their focus on how research participants arrange the phenomenon that they experience, rather than whether it actually happened, how often it happened, or how the experience might be related to other conditions or events to define the *shared human experience*.

Methods

PROMESAS SSC Description

In creating the *PROMESAS SSC* PD, we took the aforementioned literature into account, as well as the *Instructional Practices Guide* (MAA, 2018). We created a year-long PD program for collegiate instructors scheduled to teach the first semester calculus course during the 2017-2018 academic year. The PD began with a one-week summer institute in 2017; a schedule of the summer institute appears in Appendix A. The first cohort consisted of 14 fellows, four were women, three were adjunct, six were of diverse backgrounds, and the fellows' teaching experience at the collegiate level as non-graduate students ranged from 0-42 years. Given that the themes of the PD were: (a) building classroom community, (b) teaching with a student-centered lens, and (c) creating and implementing rich mathematical tasks, all with an eye towards equity in the mathematics classroom, we asked the fellows to read six related journal articles prior to the summer institute.

Following the summer institute, the fellows participated in six monthly, day-long, follow-up meetings during the academic year. The monthly meetings provided an opportunity for the fellows to continue learning about the *PROMESAS SSC* themes and equity in the classroom, to share newly created teaching materials, and to discuss any successes and challenges that they encountered in attempting to transform their teaching of calculus. In both the summer institute and the follow-up meetings, the fellows worked on rich calculus tasks in a student-centered environment that promoted a sense of community. The first year of the project culminated with a 2-day workshop where the fellows began to develop an action plan for better transforming the teaching of their Calculus I course during year two of the project.

Data Collection and Analysis

This study examined two sources of data from the fellows: responses to journal prompts and audio-taped interviews. Within the journals, the fellows reflected on new readings, ways in which they could *adopt and adapt* what they learned during the PD, challenges with integrating new teaching strategies into their classroom, their goals for transforming their teaching, and offered suggestions for future PD meetings. The second set of data was audio-recorded clinical interviews that lasted 60-90 minutes. The questions gave the fellows an opportunity to reflect on their teaching prior to participating in the *PROMESAS SSC* PD along with their experience with the *PROMESAS SSC* PD. We used narrative analysis to explore the shared human experiences that the fellows had with *PROMESAS SSC*. The authors wrote stories from each transcribed interview which they compared to find common themes. These themes were further supported by fellows' journals. The comparison of stories allowed us to establish how each of the *PROMESAS*

SSC themes contributed to the essence of the fellows' shared human experience with the PD and how if at all this PD transformed their teaching philosophy.

Results

In terms of the *PROMESAS SSC* themes, our narrative analysis suggests that the fellows felt an obligation to attend to students' needs and interests as part of student-centered learning, found rich tasks useful to connect students' prior knowledge with new content, recognized that creating a sense of community in the classroom resulted in more classroom engagement, and established a community among themselves. In terms of transforming their teaching, the fellows believed that as a result of *PROMESAS SSC* they were able to shift and align their teaching belief systems with teaching practices, to better appreciate evidence-based techniques, to teach with intention, and to realize how both they and their students had more fun in the classroom. Furthermore, the fellows came to comprehend how these practical and philosophical transformations fostered equitable teaching practices in the mathematics classroom.

Student-Centered Learning

The fellows expressed three shared experiences related to student-centered learning. They felt an increased sense of responsibility and attentiveness to students' success, need to make mathematics relatable to each student, and desire for students to "do the mathematics." In terms of student success, Dillon wrote "my new experience [from *PROMESAS SSC*] has given me a wonderful sense of freedom and sometimes an overwhelming sense of responsibility." Filled with a sense of duty for their students' success, the fellows consciously attempted to promote a student-centered atmosphere in their mathematics classrooms. A typical way in which the fellows attempted to create a student-centered classroom entailed making the content relatable to each student. In his interview, Kyle said "I am bringing [examples from] . . . different fields just so there are things that kind of pique the interest of different students." Many of the fellows reported that as a way to maintain a student-centered classroom, they required that the students *do* the mathematics. Miguel explained, "the best practice is for [the students] to do it on their own . . . they're not going to understand it until they discover it themselves." Doing mathematics also often entailed conversing about mathematics. The fellows believed that requiring their students to discover, explore, and do mathematics on their own resulted in deeper understanding of the material.

Rich Tasks

The majority of the fellows perceived rich tasks as a novel concept. They routinely commented in both the interviews and the journals that they felt unsure about integrating rich tasks into their classroom. Yet, the fellows believed that through the rich tasks, they were able to activate their students' prior mathematical knowledge in order to fuel future learning. Megan wanted to implement rich tasks but found herself concerned about the quality of the rich task. Through the support of *PROMESAS SSC*, these worries faded. In reflecting about the richness of a task, Megan confidently asked herself, "is it getting [the students] engaged? Is it getting them thinking about [math]? Then it's fine, we're [going to] do it." Viewing rich tasks in this manner felt less overwhelming to Megan and she saw that the students were always very excited to work on such tasks. The fellows found it particularly beneficial to draw out the connections between mathematical topics through the rich tasks. Miguel explained how rich tasks allowed students to evoke prior knowledge and set the stage for him to foreshadow upcoming material. He said, “[The rich tasks use] something that [the students] know how to do, but it’s going to be related to what I’m going to teach [them next].” He realized that with rich tasks, his students were better prepared for the mathematics that came next because the rich tasks promoted deeper understanding by connecting mathematical ideas.

Sense of Community

Throughout the fellows’ interviews and journals, many reported witnessing the growth of a community. This sense of community manifested itself with two different populations, the fellows’ classrooms and between the fellows themselves.

Many of the fellows discussed the importance of creating a sense of community in their classrooms early in the semester. Dillon wrote, “the first opportunity to build a community is the best opportunity to build a community . . . my first order of business in the semester is to help the students identify as a group, to make a we.” The fellows also utilized sharing personal experiences in an effort to create a sense of community in their classrooms. Miguel decided to adopt this advice and he found that the students responded positively. He shared his journey as a first-generation college student and reported that “now [the students] just see me as a human and not their professor.” It seems that fellows’ purposeful intentions of creating a sense of community in their classrooms allowed them to witness their students feeling more comfortable in the classroom. For example, Max exclaimed, “[the students] started showing up to class early and talking with each other . . . about problems, and that doesn’t happen if you don’t create that kind of sense of community.”

Besides experiencing a community in their classrooms, the fellows also experienced their own sense of community within *PROMESAS SSC*. Before entering *PROMESAS SSC* some of the fellows described feeling isolated as educators and unheard by others. In his interview, Adam shyly admitted feeling overwhelmed, underappreciated, and invisible. He said, “if you’re just an adjunct . . . driving between schools, you are like a ghost going into different schools, teaching and leaving.” With *PROMESAS SSC* all the fellows agreed, they felt as though they had a community where they belonged. In his interview, Matthias talked about immediately feeling welcomed into the community of fellows. He said, “[The *PROMESAS* team] created this environment that made it safe for all of us . . . I was able to talk to [the other fellows] as if I knew them for a long time.” The fellows appreciated and relied on the support from their *PROMESAS SSC* community as motivation to try newly learned teaching techniques, thus, there existed a sense of fidelity to *actually* try the teaching techniques. Un-denounced to the facilitators, the community of fellows had become a way to hold the fellows accountable for their own teaching transformations.

Transformation

As a result of *PROMESAS SSC*, the fellows summarized how their teaching transformed in four different ways. This included: (a) shifting and aligning teaching belief systems with teaching practices, (b) increasing appreciation for evidence-based techniques, (c) teaching with intention, and (d) realizing how both they and their students had more fun in the classroom.

With varying degrees of magnitude, each fellow transformed their teaching by participating in the *PROMESAS SSC* PD. As a result of *PROMESAS SSC*, Max was able to align his teaching beliefs with his teaching practices. Miguel described this shift as “eye opening,” realizing that *PROMESAS SSC*’s greatest impact originated from the little, seemingly simple, things he had never even thought of doing in his own classroom before hearing them from the PD leaders or other fellows. Even Matthias, who had been teaching for 15 years before joining *PROMESAS SSC*, professed a dramatic transformation in his teaching style due to the PD. Matthias reported that he went from lecturing every day to believing that students learn best through proactive and "[high quality teaching that] can be done in the classroom and we [as teachers] can change the students' mentality" through active learning such as group work and group quizzes.

The fact that the PD was grounded in research especially encouraged this emerging mentality. Adam appreciated that, with *PROMESAS SSC*, "you know you're on the cutting edge of math education research … and we can share [this knowledge]." Megan exemplified how this evidence-based PD boosted fellows' confidence as educators. Even as a mathematics education PhD graduate, she felt more assured of the new tasks she presented in class, changing her mindset from "it has to be perfect," to "it's okay for [the students] to make mistakes, it's okay for me to make mistakes too." She expressed that just knowing that an idea had a name and "this is actually a thing that other people do," solidified her beliefs about her own teaching practices. While the PD centered around Calculus I, by the end of the program the fellows felt comfortable "adopting and adapting" these newly learned evidenced-based teaching techniques to better fit their teaching environment in any other mathematics course.

As a result of partaking in *PROMESAS SSC*, the fellows also felt more aware of their teaching and more intentional about their teaching choices. Adam synthesized many of the fellows' sentiments with his statement, "[I'm] really thinking about teaching, what [my students] are actually learning, and what they're going to leave the class with . . . [*PROMESAS SSC* is] making me just more conscious and intentional, on purpose and with a purpose." Beyond purposefully creating student-centered classroom activities, the fellows informed their students as to "why" they were implementing these activities. Miguel described a first day activity designed to promote community in the classroom, in which he explicitly told his students, "one of the reasons why I did this [activity] is because I want you guys to get comfortable . . . in this class." The fellows also found that providing such rationale to their students eased meeting teaching goals because the students understood the purpose of each task.

In addition to all of the above, the fellows discovered teaching to be "fun" again. Adam described it as "falling back in love with teaching," and with surprise Kyle asserted, "[I am astonished] how much more fun the lessons [from *PROMESAS SSC*] really are or how much more fun teaching is [now]." Max also noticed a more positive mindset displayed by his students, remarking that his students had developed a mindset of "just because you haven't succeeded at math before, doesn't mean that you can't at least do better." Integrating novel teaching techniques and activities appeared to liven up the fellows' classrooms and they not only felt themselves become better teachers, but also remembered why they loved teaching so much.

Equity

Every fellow noticed how the overarching theme of equity weaved into a majority of the PD components. As Megan put it, "I think [equity] is what *PROMESAS* is about" as she articulated how all the PD activities were supported by research as equitable practices. This theme of equity often presented itself in the form of providing access to every student in the mathematics classroom. After the summer institute, Kayla expressed that not only should every student feel like they can learn math, but "every student [should have] an opportunity for success with mathematics regardless of background. Every student should feel included, safe, and capable." Developing the philosophy that everyone can learn mathematics stemmed from the newly-found knowledge regarding student-centered learning in which how one presents the content often matters more than the content itself. As Adam described it, "[In *PROMESAS SSC*], you're addressing . . . different student populations and how to engage them. And then day to day, bring[ing] that into your work and try[ing] to create a better experience for all students." Adam believed that daily he could structure his classroom to be more equitable. For some fellows, the PD's focus on equity proved eye-opening. For example, David, realized that "even if a student doesn't understand the concept I'm currently teaching . . . there [are] things I can do in class to make it more accessible. Do some kind of activity or something that brings them into the conversation." Overall, the fellows came to value equitable teaching practices in the mathematics classroom. Max concisely conveyed how "equity basically brings everybody up, so I can't see how anybody could complain."

Discussion

The *PROMESAS SSC* project is a humble yet ambitious start into developing successful and sustainable PD for mathematics faculty. Yet, we believe the structure of the *PROMESAS SSC* PD could serve as a model for other PD designed for mathematics faculty. Offering ongoing support, focusing on a particular content strand, admitting faculty with diverse teaching experience, reading mathematics education research, and linking equity to content appeared to contribute to the fellows' teaching transformations. We found that the fellows expressed feelings beyond just learning new teaching techniques; they also expressed a sense of becoming a better and informed educator. The structure of the PD offered a safe space for the fellows to create their own sense of community, where they felt inspired to try new teaching techniques, which invigorated their teaching, and in turn engaged their students. Participating in this project resulted in a community where the fellows felt supported and accountable.

Outside of the classroom, this study also raised important issues related to equitable practices in collegiate mathematics departments. Many of the adjunct fellows painted a picture of isolation and aired a feeling of frustration because they did not have a voice. Furthermore, they believed that they did not have a right to complain about issues within the department. After *PROMESAS SSC*, the fellows reported feeling that they could speak up more in their departments with support from the other fellows. Similar to the students, these fellows became more comfortable and confident in their place of work and as a result gained mathematical voice and mathematical agency (Deshler & Burrough, 2013). This suggests that PD programs similar to *PROMESAS SSC* may not only create a more equitable environment for participants' students, but also for the participants themselves.

Future directions for this research includes exploring the impact that this program has on students' learning, retention, and continuation in their STEM field. In the future, we hope to collect video-tape data of the fellows' classrooms as they implement *PROMESAS SSC* teaching strategies. We also hope to continue investigating how *PROMESAS SSC* can further support and empower adjunct faculty. These research projects are currently underway.

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Generalizing Actions of Forming: Identifying Patterns and Relationships Between Quantities

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In this paper, we illustrate and discuss two undergraduate students' reasoning about quantities' magnitudes. One student identified regularities regarding the relationship between two quantities by focusing on successive amounts of change of one quantity (i.e., a pattern) while the other attended to relative amounts of changes in both quantities (i.e., a relationship). We illustrate that although reasoning about amounts of change is useful for making sense of the rate of change in quantities, reasoning about relative changes in identifying a relationship between quantities' magnitudes is likely more productive in developing the concept of rate of change.

Keywords: Quantitative and covariational reasoning, Generalization, Rate of change.

Quantitative and covariational reasoning is critical to supporting students in understanding major pre-calculus and calculus ideas (Ellis, 2007b; Confrey & Smith, 1995; Thompson, 1994, 2011; Thompson & Carlson, 2017). Moreover, Ellis (2007b) reported that quantitative reasoning plays a significant role in students' constructing productive generalizations. In this paper, we characterize two undergraduate students' generalizing actions during a teaching experiment focused on modeling covariational relationships. We give specific attention to how the students' engagement in covariational/quantitative reasoning differed and, in turn, how this difference led them to generalize different regularities regarding a covariational relationship between two quantities' magnitudes. We report the generalizing actions of two students, with one student operating with additive comparisons of amounts of change in one quantity, and the other student operating with additive and multiplicative comparisons of amounts of change of two quantities (i.e., relative changes and ratios). We also report the resulting identified regularities of these ways of operating.

Background and Theoretical Framework

Quantitative and Covariational Reasoning

This study focuses on students' generalizing actions involved in reasoning with relationships between quantities in dynamic situations. We use *quantity* to refer to a conceptual entity an individual construct as a measurable attribute of an object (Thompson, 2011). We also describe students' construction of *quantitative structures* by characterizing their *quantitative operations* when determining a *quantitative relationship*. By *quantitative operation*, we mean the conception of producing a new quantity from two others, and by a *quantitative structure*, we mean a network of *quantitative relationships* (i.e., the conception of these three quantities; Thompson, 1990, 2011). For example, someone can create a quantity as a result of *additive comparison* of two quantities by answering the question, "How much more (less) of this is there than that?", whereas someone can create a quantity as a result of *multiplicative comparison* of two quantities by answering the questions "'How many times bigger is this than that?' and 'This is (multiplicatively) what part of that?'" (Thompson, 1990, p. 11).

Furthermore, when students engage in a dynamic context that involve two quantities varying simultaneously, they need to coordinate quantitative operations with *covariational reasoning* (i.e., attending to how one quantity varies in relation to the other in tandem; Saldanha &

Thompson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). For example, in order to determine a pattern of *differences* in a quantity's variation in relation to the other, a student can coordinate the variation of two quantities values or magnitudes and the variation of the resultant *difference* quantity's values or magnitudes (e.g., as two quantities increase, the difference of these quantities decrease; see Mental Action 3 in Carlson et al., 2002).

Non-Ratio and Ratio-Based Reasoning

Many researchers have provided different ways of making sense of rate of change of one quantity with respect to another. For example, some researchers (e.g., Carlson et al., 2002; Confrey and Smith, 1994, 1995; Ellis, 2007b, 2011; Johnson, 2012, 2015b; Liang & Moore, 2017, 2018; Monk & Nemirovsky, 1994; Tasova & Moore, 2018) argued the importance of non-ratio based reasoning, which is reasoning about amounts of change in one quantity in relation to uniform changes in another quantity. For example, a constant rate of change in the perimeter of a square with respect to changes in side length can be conceived by determining that amounts of increase in the perimeter is "two centimeters" each time "if you increase both sides by point five [centimeters]" (Johnson, 2012, p. 322).

There are also researchers (e.g., Confrey and Smith, 1994, 1995; Ellis, 2007b, 2007c, 2011; Ellis, Özgür, Kulow, Williams, & Amidon, 2013, 2015; Johnson, 2015a) who have argued for the importance of ratio-based reasoning (i.e., forming ratios of one quantity's change to the other quantity's change) in making sense of the rate of change. For example, a constant speed of a Clown can be conceived as a ratio of distance to time (i.e., "5cm:4s"; Ellis, 2007b, p. 472). We note that these conceptualizations (mostly) included students' reasoning with numbers. In this paper, we expanded this body of literature by demonstrating ways in which students make sense of rate of change in dynamic events and in graphs by reasoning with quantities' *magnitudes* independent of numerical values (see Liang, Stevens, Tasova, and Moore [2018] and Thompson, Carlson, Byerley, and Hatfield [2014] for a detailed discussion on *magnitude reasoning*). Because reasoning with quantities' magnitudes does not necessitate reasoning with specified values of the quantities, we conceptualize "ratio-based reasoning" as reasoning with a "quotient [that] entails a multiplicative comparison of two quantities with the intention of determining their relative size" (Byerley and Thompson, 2017, p. 173). We aim at demonstrating students' generalizing actions by characterizing how they operate with magnitudes within a complex quantitative structure.

Generalizing Framework

Building on Ellis' (2007a) taxonomy of generalizations, Ellis, Tillema, Lockwood, and Moore (submitted) introduced a generalization framework involving three major forms of students' generalizing—relating, forming, and extending. Students' generalizing actions of *forming* occur within one context, task, or situation. This type of generalizing action includes students *searching for similarity and regularity* across cases, *isolating constancy* across varying features by *establishing a way of operating* that has the potential to be repeated, and *identifying a regularity* across cases, numbers, or figures. In this paper, we are using this framework to illustrate two students' generalizing actions of forming by focusing on their establishing ways of operating and identifying regularities as they relate to covarying quantities.

Method

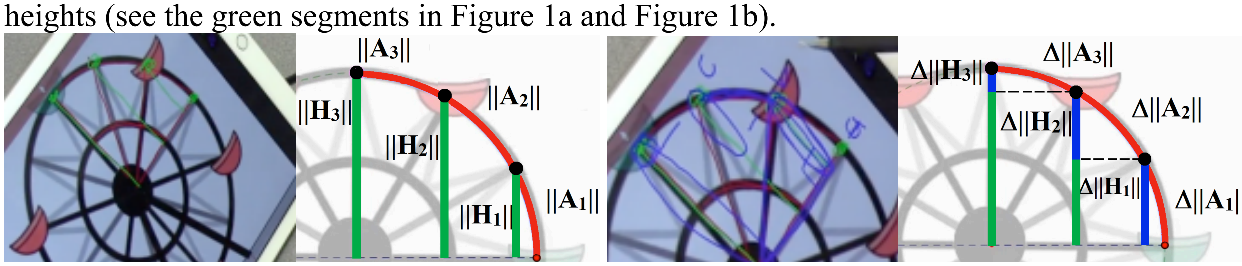
The data we present in this paper is from two semester-long teaching experiments (Steffe & Thompson, 2000) conducted at a large public university in the southeastern U.S. A common goal of both teaching experiments was to investigate undergraduate students' mental actions involved in reasoning with dynamic situations, magnitudes, and graphs from a quantitative and covariational reasoning perspective. In this paper, we focus on a student, Lydia, who at the time of the study, was a pre-service secondary mathematics teacher in her first year in the program, and another student, Caleb, who was a sophomore majoring in music education. Lydia participated in 11 videotaped teaching experiment sessions and Caleb participated in 14, each of which was approximately 1–2 hours long. We transcribed the video and digitized these students' written work for both on-going and retrospective conceptual analyses (Thompson, 2008) to analyze their observable and audible behaviors (e.g., talk, gestures, and task responses) and to develop working models of their thinking. We choose to present these two cases here because the students' generalizing actions including their established ways of operating and identified regularities are cognitively distinct, and thus are worth documenting and contrasting.

Analysis and Findings

In this paper, we illustrated two students' generalizing actions—by focusing on their *ways of operating* and *identified regularity* as they determined the covariational relationship between two quantities.

Lydia's Generalizing Actions

First, we characterize Lydia's activities in Taking a Ride to discuss her generalizing actions of *establishing a way of operating* (see Tasova & Moore [2018] for detailed account of her generalizing activity). To start with, we presented Lydia an animation of a Ferris Wheel rider that was indicated by a green bucket rotating counterclockwise from the 3:00 position (Desmos, 2014). Then, we asked her to describe how the height of the rider above the horizontal diameter changes in relation to arc length it has traveled. After reasoning about directional change in height in relation to arc length (i.e., height is increasing as the arc length increases in the first quarter of rotation), she engaged in *partitioning activity* (Liang & Moore, 2017, 2018) in order to investigate how height changes in relation to arc length. Namely, she used the spokes of the Ferris wheel (i.e., each of the black bars [see Figure 1a] connecting the center of the wheel to its edge) to partition the Ferris wheel into equal arc lengths, and then she drew corresponding heights (see the green segments in Figure 1a and Figure 1b).



(a) (b) (c) (d)

Figure 1. Lydia engaging the Taking a Ride task. Figure 1b and 1d were designed for the reader.

With support from the teacher-researcher's (TR) questioning, Lydia constructed successive amounts of change in height (i.e., circled in blue seen in Figure 1c and blue segments in Figure 1d) that corresponded to successive uniform incremental changes in arc length. That is, Lydia *established a way of operating* that involved the construction of a new quantity (i.e., amounts of change in height) and associated partitioning activity. We inferred from her activity that Lydia was constructing the *difference* of every two consecutive height magnitudes (i.e., $\Delta||H_1||$, $\Delta||H_2||$, and $\Delta||H_3||$, see blue segments in Figure 1d) corresponding to the magnitude of arc length that accumulates in equal increments; Smith III & Thompson, 2008; Thompson, 1990). This served as evidence that she was operating with additive comparisons among the accumulated height magnitudes at successive states (i.e., $||H_1||$, $||H_2||$, and $||H_3||$, see green segments in Figure 1b)). What's more, she additively compared the amounts of change magnitudes in height. Namely, she concluded $\Delta||H_1|| > \Delta||H_2 > \Delta||H_3||$.

We note that in her additive comparison, Lydia was not interested in measuring how much one quantity's magnitude exceeded (or fell short) of another quantity's magnitude. Instead, her quantitative operation included a gross additive comparison (Steffe, 1991) between the amounts of change within a quantity (e.g., $\Delta||H_3||$ being "smaller" than $\Delta||H_2||$). From this activity, therefore, we inferred that Lydia made a gross comparison of the differences, which is a more complex quantitative reasoning because this requires relating results of quantitative operations (i.e., an additive comparison of the results of two additive comparisons). After engaging in repeated additive comparisons, Lydia was able to *search* for pattern in those quantities' variation. With the recognition in the pattern of differences (i.e., decreasing change in height along with those equal partitioning in arc length as shown in Figure 1c and 1d), Lydia had *identified the regularity* in how height's magnitude changes in relation to arc length in the first quadrant, stating "as the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing."

Caleb's Generalizing Actions

We demonstrate Caleb's generalizing actions when engaging in the Changing Bars Task, which involved a simplified version of Ferris wheel situation (i.e., a circle) and six pairs of orthogonally oriented bars (see Figure 2). On the circle, the red segment represents the magnitude of the riders' height above the horizontal diameter and the blue segment represents the magnitude of the rider's arc length traveled from the 3 o'clock position. Caleb was able to move the end-point (i.e., the rider) along the circle between the 3:00 position to the 12:00 position. We asked Caleb to choose which, if any, of the orthogonal pairs accurately represents the relationship between the height and the arc length of the rider as it travels.

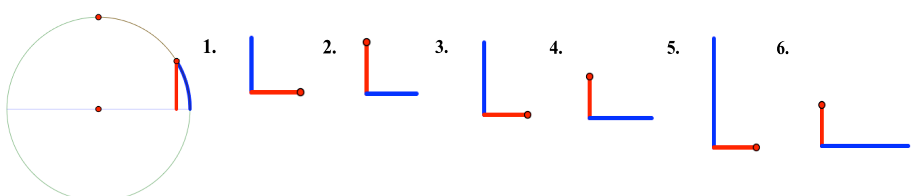
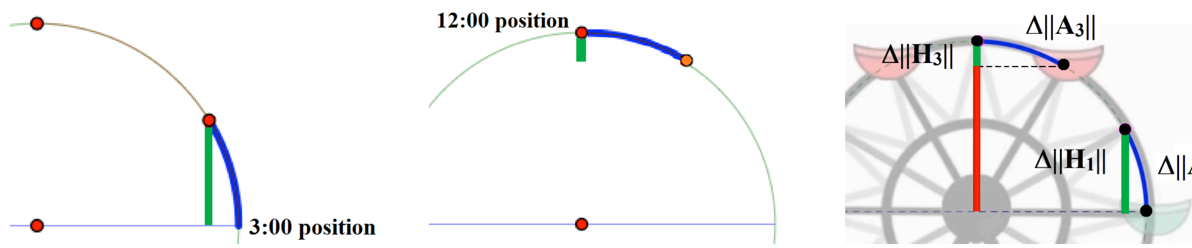


Figure 2. Changing Bars Task (numbering and locations of the six pairs was edited for readers).

In this section, we report Caleb's generalizing actions that involved him establishing ways of operating that entailed additive *and* multiplicative comparisons. We note that identifying these different operations does not imply that Caleb engaged in them in order. We believe that Caleb's reasoning involving additive and multiplicative comparisons was internally coherent and he could make claims about either one depending on the TR's questioning. Our goal of making such distinction was to characterize his different ways of operating and contrast his ways of operating with those of Lydia.

Additive comparison of amounts of change. Caleb started with comparing the amounts of change in arc and amounts of change in height as the dynamic point traveled a small distance from the 3:00 position. He stated that, "...at the very beginning, ... the height above the center and the distance traveled from 3:00 position should be similar." This *way of operating* was repeated several times during his generalizing actions with use of slightly different verbal statements. For example, in a later conversation, he stated "at the beginning of the path [*referring to 3:00, see Figure 3a*], ...the rate at which the height increases should be almost equal to the rate at which the distance it's traveled." We note that although he used the word of "rate," we infer that he meant amount of change in height and arc length. By repeating the same way of operating in 12:00 position (i.e., new case in the first quarter of rotation), Caleb further stated that:

> ...from this point [*pointing to the point denoted in orange in Figure 3b*] ... to this point [*pointing to 12:00 position in Figure 3b*], the height barely changes [*green segment in Figure 3b and Figure 3c (i.e.,* $\Delta||H_3||$*)*], but you're still traveling a fair distance around the circle [*blue annotation in Figure 3b and blue segment (i.e.,* $\Delta||A_3||$*) in Figure 3c*].



(a) (b) (c)

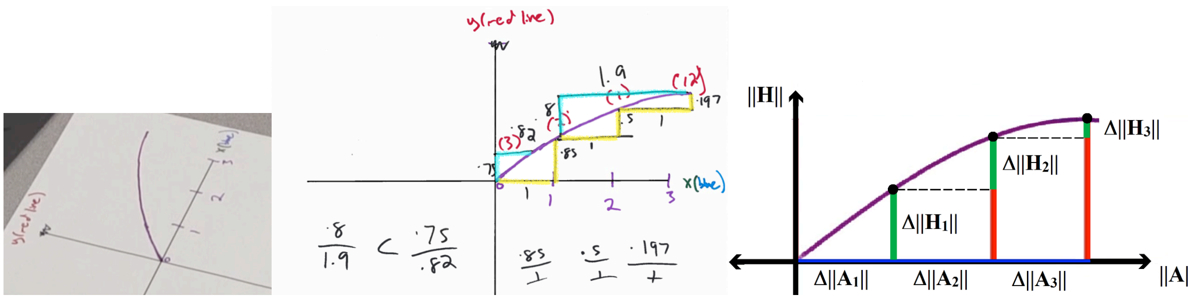
Figure 3. Recreation of Caleb's activity in the Changing Bars task.

From his activity, we infer that Caleb's established way of operating included an additive comparison of $\Delta||H_1||$ with $\Delta||A_1||$ near the 3:00 position (i.e., $\Delta||H_1||$ is almost equal to $\Delta||A_1||$) and of $\Delta||H_3||$ with $\Delta||A_3||$ near the 12:00 positon (i.e., $\Delta||H_3||$ is smaller than $\Delta||A_3||$). Similar to the case of Lydia, we did not have evidence that Caleb constructed the difference between amounts of change in two quantities (e.g., how much $\Delta||H_3||$ exceeded of $\Delta||A_3||$) beyond a gross additive comparison between the amounts of change in each quantity (Steffe, 1991).

As the teaching experiment proceeded, he *isolated a constant feature* of the relationship between the amounts of change in height's magnitude and the amounts of change in arc length's magnitude across the first quarter of rotation. He stated that "from any point to any other point along this stretch [*referring to the first quarter of rotation*], the amount that the red line [i.e., height's magnitude] changes should always be smaller than the amount that the blue line [i.e., arc length's magnitude] changes." Therefore, we infer that Caleb *isolated a constant feature* across varying features of the relationship between $\Delta||H||$ with $\Delta||A||$ without reaching the final stage of fully describing an identified regularity across the first quarter of rotation (e.g., $\Delta||H||$ becomes smaller relative to $\Delta||A||$ as the rider travels from 3:00 positon to 12:00 position). It is important to note that, however, Caleb knew that "when we're looking down here [*refers to 3:00 position*]" the relationship between $\Delta||H_1||$ and $\Delta||A_1||$ "should be vastly different from" the relationship between $\Delta||H_3||$ and $\Delta||A_3||$ (see Figure 3c).

Eventually, Caleb *identified a regularity* regarding the relationship between $\Delta||H||$ and $\Delta||A||$ across the all cases in the first quarter of rotation. He stated that "the further you move away from the 3:00 position, the more variance there would be between the red (i.e., $\Delta||H||$) and the blue lines (i.e., $\Delta||A||$)" and by "variance" he meant that $\Delta||A||$ became much bigger than $\Delta||H||$ as the dynamic point approached the 12:00 position.

Multiplicative comparison of amounts of change. Caleb also *established a way of operating* that involved multiplicative comparisons between $\Delta||H||$ and $\Delta||A||$. He stated that "As we approach this point right here [*refers to 12:00 position*], the ratio of the rate at which the height increases to the rate or to the distance we've traveled around...the circle, um, is at its smallest..." This way of operating was also repeated several times during his generalizing activity—both in the circle situation and in six pairs of bars. For example, near 12:00 position, he established that there is a "...1 to 2.5 or 1 to 3 ratio in the amount that you change the red line's length [i.e., height's magnitude] decreases to the blue line length [i.e., arc length's magnitude] decreasing." We infer that Caleb constructed a quantity as ratios of $\Delta||H||$ to $\Delta||A||$ across the first quarter rotation, and anticipated that the ratio gets smaller as the rider travels from 3:00 position to 12:00 position. His way of operating that entailed multiplicative comparisons of quantities and his identified regularity regarding the relationship between height and arc length became evident in his graphing activity, which we report next.



(a) (b) (c)

Figure 4. (a) Caleb's initial graph, (b) a resulting drawing of Caleb's partitioning activity, and (c) a recreation of Figure 4b for readers

Caleb's graphing activity. The researchers then asked Caleb to produce a graph that represents the relationship between height and arc length. He constructed the concave down graph shown in Figure 4a. To interpret his displayed graph in terms of amounts of change in height and arc length, Caleb engaged in partitioning activity (Liang & Moore, 2017, 2018) to construct incremental changes that represented amounts of change in height (i.e., $\Delta||H_1||$, $\Delta||H_2||$, and $\Delta||H_3||$ in Figure 4c; also see yellow vertical segments in Figure 4b) in relation to uniform changes in arc length (i.e., $\Delta||A_1||$, $\Delta||A_2||$, and $\Delta||A_3||$ in Figure 4c; also see yellow horizontal segments in Figure 4b). He then assigned estimated values for each segment to indicate its magnitude (i.e., $\Delta||A_1|| = \Delta||A_2|| = \Delta||A_3|| = 1$; $\Delta||H_1|| = .85 > \Delta||H_2|| = .5 > \Delta||H_3|| = .197$ and constructed ratios of each corresponding pairs, writing ".85/1", ".5/1", and ".197/1" (see Figure 4b). Caleb also operated on these ratios by additively comparing them, anticipating that these ratios should decrease—".85/1>.5/1>.197/1". This suggested that Caleb continued and generalized his ways of operating in the circle and bar situation to the graphical contexts and identified a regularity that the ratios of successive pairs of amounts of change in height and arc length should decrease as the rider travels in the first quarter of rotation. Caleb was also able to extend his ways of operating to non-uniform intervals. Namely, he anticipated that when increments of arc length are not equal (see his partitions in light blue in Figure 4b and his estimated values for each increment), the same regularity should hold, writing ".8/1.9<.75/.82" (see Figure 4b) without calculating the resulting value of the ratios.

Discussion

We focus on two students' generalizing actions by giving attention to their ways of operating and identified regularity. In establishing ways of operating, both Lydia and Caleb first constructed *differences (i.e., amounts of change)* in magnitudes of height (i.e., $\Delta||H_1||$, $\Delta||H_2||$, and

$\Delta||H_3||$) in relation to change of arc length's magnitude (i.e., $\Delta||A_1||$, $\Delta||A_2||$, and $\Delta||A_3||$). However, the way they operated on those *differences* differed. For example, they both engaged in quantitative operation of additive comparisons; however, the operands that they considered in their quantitative operations were different. That is, Lydia additively compared the successive amounts of change in height's magnitude, whereas Caleb additively compared amounts of change in height's magnitude with the corresponding amounts of change in arc length's magnitude. Therefore, the operands for Caleb were *differences* in height and *differences* in arc length (i.e., $\Delta||H_n||$ and $\Delta||A_n||$, where n=1, 2, and 3), as opposed to Lydia whose operands were *differences* of height in successive states (i.e., $\Delta||H_n||$ and $\Delta||H_{n+1}||$, where n=1 and 2). We conjecture that the way of additive comparison in Caleb's case might be more productive for generalizing the rate of change in height with respect to arc length since such comparison afforded him to anticipate a resultant ratio of *differences* in height and *differences* in arc length (i.e., $\Delta||H_n||/\Delta||A_n||$).

We also find that these two students' different ways of operating led them to identify different regularities regarding the similar situations. Lydia searched for the *pattern* (Ellis, 2007b) by making within-measure additive comparisons among heights in different states. Thus, she identified a pattern of how amounts of change in the height decrease as the arc length increases. Caleb searched for the *relationship* (Ellis, 2007b) by making between-measure multiplicative comparisons between height's magnitudes and arc length's magnitudes. Thus, he identified a regularity of relative change of the height with respect to the arc length decreases as the rider travels. We conjecture that this way of operating (i.e., multiplicative comparison between changing quantities' magnitudes) and the resultant identified regularity may afford students to develop productive understandings of rate of change.

We want to point out that, when additively comparing the ratios in justifying his identified regularity, Caleb's engagement with numbers does not imply that he performed arithmetic operations in a sense that he wanted to evaluate the quantities' values. We infer that the reason he assigned numbers to quantities' magnitudes is that he needed to "propagate information" (Thompson, 2011, p. 43) in order to deal with the complex quantitative situations. Thompson (2011) claimed that propagation can be made under the conditions of being *aware* of (i) quantitative structure and (ii) "numerical operations to perform to evaluate a quantity in that structure" (p. 43). Even though Caleb did not perform numerical operations to evaluate quantities, he satisfied the conditions of propagation. That is, he used numbers as intuitive measurements of quantities' magnitudes and he was aware of the quantitative structure. Moreover, Caleb's uses of numbers were necessary for him in order to compare the relative size of two quantities' magnitudes. Part of this necessity comes from the fact that there was no way for him to visually represent the magnitude of a quantitative ratio. That is, Caleb used estimated numbers to reason about the relationship between magnitudes of hard-to-visualize quantities, and then re-interpreted this relationship between *values* in the context of quantitative structure in order to propagate information about the relationship between quantities' *magnitudes*. To confirm if this is the case or to characterize the nature of this reasoning, we believe that future research is necessary.

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Intuition and Mathematical Thinking in a Mathematically Experienced Adult on the Autism Spectrum

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In this report, I examine the use of intuition by a mathematically experienced adult on the autism spectrum given a paradoxical mathematical problem involving infinity. I compare both his level of use of intuition and the importance he places on it against results from students in the general population. Interview results combined with previous data suggest that students on the autism spectrum are less likely to use approaches based in intuition, place less importance on intuitive ideas compared to other explanations, and may also have different views of the nature of intuition. Analysis of possible reasons for showing these differences and implications for teaching and further autism-related research are presented.

Keywords: intuition, mathematical paradoxes, autism

My research attends to mathematical problem solving by adults on the autism spectrum (with a formal diagnosis), particularly those with a relatively strong background in mathematics. In this report, I focus particularly on the case of one student's work on the Ping-Pong Ball Conundrum, a problem of infinity (Mamolo and Zazkis, 2008). I use this problem to highlight characteristics of intuition used in problem solving and how the use of intuition can differ for people on the autism spectrum in both nature and frequency. This can help to both examine the use of intuition in mathematics generally and to examine characteristics related to autism.

Brief Overview of Autism-Related Research in Mathematics Education

There is a wide range of conceptions of what being on the autism spectrum means, including various academic and clinical definitions. The Autistic Self Advocacy Network (2014), the leading autism advocacy group run by people who are themselves autistic (and identify as such) states that autism is a neurological difference with certain characteristics, each of which is not necessarily present in any given individual on the autism spectrum. These include differences in sensory sensitivity and experience, atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. They also list "different ways of learning" and particular focused interests (often referred to as 'special interests'), which are especially relevant for research in education. Of those characteristics, it is primarily the existence of special interests and the differences in language use and social interaction that are used as diagnostic criteria by the fifth edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-5).

Much of the research currently done on mathematics learning in people on the autism spectrum is focused on young children (e.g., Iuculano et al., 2014; Klin, Danovitch, Mers & Volkmar, 2010; Simpson, Gaus, Biggs & Williams, 2010) or looks at mostly arithmetic. There is also a notable strain of work done on the population of research mathematicians (e.g., Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007; James, 2003), but very little attention is paid to groups in the middle (mainly high school and college students, or adults other than career mathematicians). This is a gap which I have sought to help fill with my own research, including the particular selection which I present here.

Theoretical Framework

My theoretical framework is based partially in Vygotskian theory, particularly Vygotsky's (1929/1993) conception of overcompensation. Vygotsky explained this initially in a framework of physical overcompensation, such as a kidney or lung necessarily strengthening when the other one is missing or by analogy to vaccination. He argued that overcompensation also occurred in psychological development, both in its general course and in particular in the presence of various disabilities. While my views are informed by the Vygotskian framework, there are some issues with using it directly. Some parts that are particularly relevant in autistic people, such as the ideas about atypical development and concept formation, particularly concern things that have already must have occurred far before starting university coursework, and thus cannot be observed in my interview subjects. The examination of inner speech also has difficulties; Vygotsky himself used children whose inner speech had not yet fully developed in his clinical experimentation on the subject. Thus, while those ideas from Vygotsky inform my views, additional constructs were required for the data analysis; in this case, the main one is thinking regarding intuition.

In many contexts, the erroneous conclusions produced by students and the resistance to the mathematically valid solution are identified with forms of intuition. In Fischbein's (1979) use of the idea, intuition is separated into different categories, particularly "primary intuition" (developed outside of a systematic instructional setting) as opposed to "secondary intuition" (developed in a systematic instructional setting). The division of categories here has similarities to Vygotsky's distinction between everyday and scientific concepts, and I find it reasonable to consider the primary and secondary intuition used by Fischbein as identifying intuitive reasoning related to everyday or scientific concepts, respectively. Further exploration of intuition by Fischbein (1982) uses a similar division between "affirmatory intuitions" and "anticipatory intuitions", focusing primarily on the former. In this division, affirmatory intuitions are those that are "self-evident [and] intrinsically meaningful", which again stands outside the systematic instructional context.

In the context of other works, it is the primary and affirmatory definitions that are closest to what is typically meant when 'intuition' is named but not explicitly defined, which is useful for situating other work which mentions intuition but does not focus on it. Fischbein also argues for the importance of using intuitive ideas, which includes but is not limited to correcting those intuitive ideas which would otherwise lead to error. Here, his main focus is on developing intuitive ideas so that they are in accord with the analytic reasoning rather than in conflict. While there are still possible parallels between these theoretical constructs and Vygotskian concepts, the approach suggested by Fischbein, focused on development and adjustment of intuitive ideas, is more constructivist. Since these differences are reflected in neurological differences associated with autism, they would lead to contrasting predictions for the mathematical reasoning of people on the autism spectrum.

Ping-Pong Ball Problem

While I conducted interviews using a variety of problems, in the excerpt here I focus on results from a single problem described in Figure 1.

The Ping-Pong Ball Conundrum
Consider an infinite set of ping-pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are inserted and the '2' ball is removed, and so on. How many ping-pong balls remain in the barrel at the end of the minute?

Figure 1. Statement of the Ping-Pong problem used in the interview.

The accepted mathematical solution here is that there are no balls in the barrel, because for every possible ball, we can find a time after which it has been removed (this is because of the order the balls are removed in, and different orders can lead to different outcomes).

This problem was used by Mamolo and Zazkis (2008) with two groups of students, one undergraduate and one graduate. Both were in courses about fundamentals of mathematics at different levels which involved infinity. They were introduced to this problem after having seen the Hilbert Hotel problem, a simpler problem also involving infinity. In each case, after students' first responses to the problem, they were given the standard solution. Both groups initially gave responses that rejected things in the problem setup that seemed impossible, relating them to real-world facts such as the finite population of Earth. The undergraduate students, who were a more general population of liberal arts and social science students, continued to show resistance to the given mathematical solution in the Hilbert Hotel problem, while the graduate students (who were in a mathematics education program) did not. However, both groups continued to show disbelief in the mathematical solution for the Ping-Pong problem after instruction. One of the more common responses found in both student groups was that there were nine more balls at each step, often giving a 'nine times infinity' response. This highlights the importance of the numerical ordering in the problem, since if the balls were not ordered this way (if they were all simply generic and interchangeable balls, for instance), it would be correct to use the fact that at any step n, there were $9n$ balls in the barrel and to take the limit of that expression as n goes to infinity. In the numbered case, we can view it as essentially an arrangement of processes where a second process 'cleans up after' the first, but calculating the total at each step as $10n - n = 9n$ erases that ordering property. Without a numbering to provide order, that arrangement cannot be made; in that case, there is no information lost with the '$9n$ balls in the barrel' view.

Ely (2011) gave this problem (as the Tennis Ball Problem) to a range of participants from undergraduates who had finished college algebra to mathematics doctoral students and one mathematics professor. He used two versions, comparing the effect of asking "how many balls are left" to asking "which balls are left" (Ely, 2011, p. 8). He found that participants given the 'which' version were more likely to attend to the labeling (ordinal) rather than the total number (cardinal). The 'which' participants also showed more conflict from being presented the accepted mathematical solution of having no balls remaining, although they were still unlikely to accept it. No participants given the 'how many' version accepted the zero-ball solution, while the only participant given the 'which' version to accept it was the mathematics professor.

Disputes about the proper result of this problem have also been shown in publications where researchers discuss their own perceptions of and disagreements about the problem, rather than discussing the perceptions of students. Allis and Koetsier (1991) describe this paradox in terms of super-tasks, defined as "the execution of […] an infinite sequence of acts" (Allis & Koetsier, 1991, p. 189). They argue that this is possible not only in an abstract way, but also in a kinematic one. However, a later discussion by van Bendegem (1994) raises objections to both of these arguments. The objection to the abstract solution is an algebraic argument, while the objection to the kinematic one involves relativistic physical assumptions. The response from Allis and Koetsier (1995) points out that the algebraic argument from van Bendegem does make an assumption of continuity (although van Bendegem asserted that it did not), and raises multiple objections to the kinematic argument. Looking at what is disputed between the authors, it is notable that the two main strands closely parallel the 'nine times infinity' solution and the objection to the real-world possibility of the problem found with the students in the study from Mamolo and Zazkis.

Ultimately, from the range of mathematical experience in responses to this task, we can see that this is a paradox that can incite argument and confusion even at high levels of academic discourse, and that few people at any level are inclined to accept the standard mathematical solution. Also, at multiple levels the nature of disputes and confusion fits into two main categories, one directly related to infinity and continuity and another related to physical properties.

Methodology and Task Details

Given my interest in focusing in-depth on interviews with a small number of people, a method of case studies was a natural fit for my work. Case study focuses on in-depth understanding of the case in question, and only secondarily on generalizations from that understanding. Additionally, while generalization is possible, it is not of the same nature as generalization in other types of research (Stake, 1995). These are sometimes divided between embedded and holistic case studies, where an embedded case study is interpreted as examining a particular feature or subset of the case in question, while a holistic case study does not use such subdivisions (Yin, 2009). In this case, my decision for a holistic case study naturally follows from my neurodiversity-informed view that the nature of being autistic is not a discrete part of the person that can be separated, and thus an embedded design does not apply.

My participant here (who chose the pseudonym Cyrus) was recruited from the community and was in his thirties at the time of interview, holding a bachelor's degree in mathematics and working in computer programming. He received an ASD diagnosis at age 13. I conducted several interviews with him as part of my broader work; the one I focus on here involves the ping-pong ball problem, as described above.

Interview with Cyrus

In the beginning, the interviewer reads the problem to Cyrus and then shows him a printed version of the problem. There are multiple rounds of explanation as the problem is presented to Cyrus before he understands the problem correctly and addresses it. Once he does, this is his response.

C: Okay, but if that keeps going on, then, but we're eventually going to have an infinite number of balls in there, but it just depends on how many, if we go through n successions of this, we're just not going to have the first n in the basket. [*I*: Alright.] So, but it's still an infinite number of balls, we're just not going to have, 1 through n in there.

I: If n goes to infinity, what does not having the first n mean?

C: I don't know. That means we've got no balls in there whatsoever. That sounds like the intuitive answer, but I'm just not certain about that.

I: Hm. Ah, that's the intuitive answer for what reason?

C: Um, that still doesn't sound right, but you'd still, even if n went to infinity, you'd still have an infinite number in there. Yeah, that's what I would say. I don't know what my reasoning is for it, I think just like you could, just like in [the Hilbert] hotel problem you could keep placing more and more in there. Well, okay, something similar to that. In this case you could just keep adding more and more balls in there, even after, going through an infinite number of times. But except, oh my god it doesn't, yet it still seems to kind of contradict itself. Because if n now encompasses all the numbers, and you're taking them all away, then how can you have an infinite number of balls still in the basket? So, I don't know, I'm going to say there's zero balls in there, at that point, if you take the limit as n goes to infinity, I'm going to say there's zero balls. [*I*: Okay.] I know I sort of changed around on that one.

I: Alright. Mm, but at this point would you say that that is what you're sticking with?

C: Yeah, I'm going to stick with that, if n goes to infinity, then you have zero balls left.

I: Okay, and does that seem like, a reasonable sort of thing, or does it seem kind of weird, or, ah, is it something that you would accept?

C: I think it seems weird, but I would accept it.

At first, Cyrus looks at the time intervals, finding the length of the first four and suggesting that they diverge as a series. The interviewer clarifies that the focus of the problem is on the number of balls, and describes what is in the bin during the first two time intervals. Cyrus extrapolates from this that as each time interval progresses, more and more balls will be in the barrel, but the first n will be missing. The interviewer asks what this means as n approaches infinity (which may be considered either the number of steps performed or the number of balls removed: since these are always equal, it is not clear from the statements which conception Cyrus is focusing on), and Cyrus says that means there will be no balls in the barrel. At first he calls this an intuitive answer; thinking further, he switches to there being an infinite number and then back to zero (without any intervention). He describes it as weird, but is willing to accept it.

The brief characterization of the zero answer as intuitive is unusual and may suggest multiple layers of reasoning that Cyrus considers "intuitive". However, while this does not last, the intuitive conclusions are still not held by Cyrus to be particularly important for a final conclusion.

Next, the interviewer presents Cyrus with some alternative arguments made by other students:

I: Okay. And, there are, I think we have them here, ah, couple of arguments that, ah, different students had, ah, trying to work out this particular problem, and I'd like to tell you about a couple of those, and see what you think about them.

C: Okay.

I: Okay, so, one argument about this was that, for each chunk, you're essentially adding nine balls. [*C*: Mm-hm.] So the total amount at any time should be nine times, the amount of chunks you've gone through. [*C*: Mm-hm.] And, but that goes to infinity.

C: Right, okay, so, you add nine but, right, you add nine balls, okay.

I: So does that seem correct or incorrect, and, why? How does that argument sound to you?

C: So, it's like, nine times n, okay. And if n goes to infinity, from that one, the answer would clearly be it just blows up, if n goes to infinity, it would be infinite. You'd have infinite number of balls left in there. So, but it still doesn't seem to make sense to me when I try to actually predict in my head what's going on there, it doesn't really seem to make sense with that. I would say that answer doesn't make as much sense. To me it makes sense as long as you have a finite number of time intervals.

I: Okay. And why doesn't this work in the infinite case?

C: My only reasoning is somehow it doesn't make sense to me, once you've already taken away, essentially, once you've taken away every single natural number, then you can't have anything left.
I: Mm. Okay, well,
C: Yeah, now I'm really confused, I'm just not sure if there's a correct answer to this or not.
I: That's sort of, ah, the second argument, where we ask, okay, if there are balls remaining, ah, all our balls are numbered by natural numbers, [*C*: Mm-hm.] so, if there's some balls remaining, what are they? Name one.
C: Okay. So, if there are some balls remaining, then what are they. Name one. Okay. Um, so then, I don't know, my- my reasoning followed the case where you'd have nothing left at the end.
I: Right. Mm, which- yeah, that's- that's correct. And that's why that works, is it- there aren't any. [*C*: Mm-hm.] You can't name one.
C: Right.
I: This- this is sort of, the, ah, contradiction proof version of proving this. [*C*: Okay.] Where you go, okay, suppose by way of contradiction, that there are some balls left. [*C*: Mm-hm.] Then- since this is a set of natural numbers, it must have a least element. Call it n.
C: Right, okay.
I: But then, in the nth step, we've removed that. [*C*: Mm-hm.] So we don't have that.
C: Right, okay.
I: Contradiction.
C: That's a proof by contradiction, I see.
I: Yeah. Therefore, there are no balls.
C: Mm. Ah, okay. And, okay, but I wouldn't have really thought of that- not in that way, at least, but it makes sense, once you've- go to the next step, you've just removed the one that's remaining.

In this segment, Cyrus is first presented with the argument that, since nine balls are added each time, there should be infinitely many at the end, and asked what he thinks of it. He finds it to make less sense, but after being presented with it, is uncertain if there is a correct answer. He is then presented with the proof by contradiction argument for there being zero balls, and agrees that one cannot name a single ball remaining, though he says he would not have considered it in that way.

While Cyrus' conclusion agrees with the proof by contradiction conclusion, he says that he would not have viewed it that way. This may be related to the first intuitive answer he gave earlier, not having the 'infinity' conclusion as something to start off with as reasonable to contradict, which suggests that Cyrus may view the problem in an unusual way which is more conducive to the ultimately correct solution. In fact, not only does Cyrus not reach that as a conclusion, he unusually characterizes it as making less sense, while most typical students have the opposite view. However, he does agree that such a solution is valid for any finite case.

Analysis

In this particular interview, it appears that Cyrus may not have the unexamined continuity assumption that many students use to extrapolate to the infinite case in this problem as part of his intuition. Alternatively, he may have learned to ignore it. If it is ignored, this is also noteworthy; although Cyrus does have formal mathematical training, others at the same or higher level of formal mathematical training still did not have this response, as found by Ely (2011).

Here and in other interviews, Cyrus tends to have a high level of trust in the truth and consistency of mathematics, not displaying many of the typical objections to paradoxical tasks outlined in prior research. When faced with apparent contradictions, he is more likely to question an intuitive response rather than a formal mathematical result. Additionally, contextual considerations of problems phrased in a 'real-world' physical setting appear to be given less relevance. This also appears to be a logical result of an orientation toward structure, or in Vygotskian terms, systematic over intuitive reasoning.

One overall trend apparent from Cyrus' interviews is his inclination toward algebraic and formal methods of solution, and a distrust of or disinclination toward informal or intuitive methods. This can fit as another form of compensation, where stronger algebraic skills are used in place of weaker geometric or informal ones. However, this combined with interviews with my other participants (Truman, 2017) suggests that any effect of compensation related to autism is more complex and individual rather than people on the autism spectrum all fitting a certain type. Cyrus' forms of compensation here contain points of similarity to other participants, such as the mistrust of intuitive reasoning and inclination toward more systematic justifications. However, there were also differences fitting a broader pattern; in particular, Cyrus skews more heavily toward using algebraic methods and avoiding geometric methods or methods based in physical analogies. For Cyrus' reasoning in this problem specifically, the physical system of balls being removed was never a focus, and physical impossibility was not considered as a notable issue. This is in particular contrast to my first interview participant, who had the opposite pattern (Truman, 2018), although all of my participants showed a strong inclination toward a particular mode of problem-solving.

Conclusions

For many students, it is often difficult for them to trust in systematic over intuitive reasoning when they are in conflict, particularly in the sorts of problems typically called mathematical paradoxes. Thus, this tendency for students on the autism spectrum to rely more on systematic reasoning can be a particular advantage in such situations that most students find difficult, as we see in this particular case. However, instructional approaches that are designed to rely on students' intuitive reasoning or real-world concepts may be less successful for students on the autism spectrum for the same reason, or they may result in unusual responses that would require more instructional attention.

My research findings here are also consistent with the theory that people on the autism spectrum learn in a manner that relies less on prototypes (Klinger & Dawson, 2001) in favor of constructing concepts more systematically. I believe that the examples of problem-solving in the interviews show that this can produce positive results and does not need to be viewed as a deficiency. They also shed more light on cognitive differences of people on the autism spectrum in adulthood, which is particularly important because much of the research done related to autism is done with younger children. They are also consistent with the systemizing theory (e.g., Baron-Cohen, Wheelwright, Burtenshaw, & Hobson, 2007), where systemizing is viewed as an inclination to create or analyze a system based on the formulation of rules. However, this could also come from the mathematical inclinations already known about and sought in the participants. This should not be taken as support for the suggestion by many proponents of the systemizing theory of its opposition with empathizing (viewed here as the recognition of what someone else is feeling), since the nature of the interviews shows very little about any skills in that category, either positively or negatively.

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An Exploration of the Factors that Influence the Enactment of Teachers' Knowledge of Exponential Functions

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The undergraduate preparation of pre-service teachers requires attention to the factors that enable and constrain their application of mathematical knowledge to positive effect in the classroom. In this paper, we examine how the instructional decisions of three in-service secondary mathematics teachers were influenced by individually consistent patterns of such mediating factors. Acknowledging that such factors might be content-dependent, we focus on teachers' instruction of exponential functions, a topic foundational to both secondary and collegiate mathematics. We developed models of each teacher's implicit learning theory, professional identity, values and goals for students' learning, and beliefs about the nature of mathematics and about what constitutes genuine mathematical engagement. We illustrate these results by summarizing our analyses of a selection of mediating factors for each teacher. We conclude with a discussion of the implications of our findings for the preparation of pre-service mathematics teachers at the undergraduate level.

Keywords: teacher knowledge, mediating factors, mathematical knowledge for teaching, exponential functions.

Introduction

The content preparation of pre-service mathematics teachers is often based on the assumption that if they can construct powerful ways of understanding the content they will teach, then they will necessarily be positioned to leverage their mathematical and pedagogical knowledge to support students' construction of similarly sophisticated understandings. This assumption is itself based on implicit epistemological premises, namely the notion that knowledge resides in one's mind in the way books dwell on a shelf: readily accessible, waiting to be utilized whenever the individual possessing it wishes to do so. The lack of converging empirical evidence demonstrating a positive relationship between teachers' mathematical knowledge, or even *mathematical knowledge for teaching* (Thompson & Thompson, 1996), and their instructional quality calls these assumptions into question (Mewborn, 2003).

In the field of undergraduate mathematical preparation of future teachers, we often must make program design decisions without a detailed understanding of the contexts in which teachers will be required to apply their mathematical knowledge. This challenge stimulated our interest in the research question: What are the factors that mediate the enactment of in-service secondary teachers' knowledge of exponential functions? Our exploration of this question has provided insight into the complex processes that govern the enactment of teachers' content knowledge, and has begun to shed light on inconclusive findings about the relationship between teachers' mathematical knowledge for teaching and the quality of their instruction. Moreover, and of particular relevance to the RUME community, such insights have implications for the mathematical preparation of pre-service secondary teachers at the undergraduate level, which we discuss at the conclusion of this paper.

Research in the area of mathematical knowledge for teaching has prioritized the identification of knowledge categories that enable effective mathematics teaching (e.g., Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992; Rowland, Huckstep, & Thwaites, 2005). Few studies have documented the environmental, cognitive, and affective influences that mediate the enactment of mathematics teachers' content knowledge. Fewer still have done so for the purpose of informing the content preparation of pre-service teachers at the undergraduate level. For current scholarship on mathematics teacher knowledge to realize its intended effect of ensuring instructors are equipped to engage students in experiences that support their construction of productive mathematical meanings, it is crucial to determine the effect of those factors that condition the enactment of the knowledge teachers *do* possess in addition to characterizing the knowledge teachers *should* possess. Identifying the influences that mediate the knowledge that resides in teachers' minds and the knowledge they leverage to support students' mathematical learning is indispensable for designing well-informed teacher preparation programs and professional development initiatives that take seriously the effect of teacher knowledge and those influences that compromise it.

Theoretical Background

We do not intend our statement that enacted knowledge is afforded/constrained by situational influences to be interpreted as strictly deterministic. These influences are simultaneously reflective of characteristics of one's knowledge while also affording/constraining its enactment. Since knowledge is not invariantly accessible across time and space, and instead depends for its enactment on an individual assimilating stimuli to activate particular cognitive schemes, environmental circumstances per se do not constrain or support teachers' enacted knowledge (von Glasersfeld, 1995). Rather, teachers' *assimilation* of environmental circumstances affects the nature and quality of the knowledge they leverage in the context of practice. For this reason, particular circumstances do not maintain an objective designation as mediating factors, however consensual are teachers' construction and appraisal of them. Therefore, one category of influence that mediates the enactment of a teacher's knowledge is her subjective construction and appraisal of the *external* circumstances that impede or enhance her capacity to achieve her instructional goals and objectives. Another type of mediating factor includes the *internal* (psychological or affective) characteristics a teacher recognizes as affording or constraining her practice. These characteristics might include a teacher's image of his or her mathematical self-efficacy, social endowments, creativity, tolerance, attitude, perseverance, temperament, empathy, confidence, and professional identity. Finally, the influences that an observer notices as affecting the nature and quality of the teacher's enacted knowledge, but of which the teacher might not be consciously aware, constitutes a third category of mediating factor. These influences could include subconscious processes of emotional regulation, identity preservation/reformation, or the structure/organization of the teacher's mathematical knowledge itself.

Methods

Data Collection

In this paper we discuss three experienced teachers—an Algebra I, an Algebra II, and a Pre-Calculus teacher—recruited as part of a larger study. We collected data on multiple aspects of each teacher's practice related to their instruction of exponential functions. We selected this mathematical context because of its significance in both the secondary and post-secondary curriculum, and because its complexity affords teachers the opportunity to emphasize various meanings and to establish connections between them.

Our data corpus for each teacher consisted of ten to fourteen hours of video-recorded classroom observations and four types of interviews—initial, pre-observation, post-observation, and final—for each teacher. We conducted the initial interview prior to each teacher's instruction of exponential functions, pre- and post-observation interviews occurred prior to and following each lesson respectively, and the participants took part in the final interview after they had concluded their instruction of exponential functions.

We designed our experimental methods to reveal the motivations for teachers' instructional actions and decisions. These motivations include but are not limited to teachers' goal structures, beliefs, commitments, identities as teachers of mathematics, and theories of learning. We accomplished this in the initial interview by providing opportunities for the teachers to articulate their beliefs about the nature of mathematics and mathematics teaching, as well as their overarching, non-content specific instructional goals. The pre-observation interviews and classroom observations allowed us to identify the actions in which the teachers engaged during lesson planning and instruction so we could make inferences about the motivations that influenced them. Because individuals are often ineffective at articulating the motives for their actions, which might be implicit, the data we obtained through classroom observations were important for triangulating the interview data. The post-observation interview gave us the opportunity to elicit teachers' retroactive justifications for particular instructional actions. Lastly, through ongoing analysis of the interview data and classroom observations, we developed provisional hypotheses of each teacher's implicit learning theory, professional identity, beliefs about of the nature of mathematics and mathematical engagement, and their values and goals for students' learning, and we assessed the viability of these hypotheses by asking purposeful questions during the final interview.

Data Analysis

Our data analysis consisted of ongoing analysis and post analysis. We recorded memos during classroom observations that focused on documenting each teacher's instructional actions and articulating motives that might have informed them. We also documented the extent to which each teacher's classroom activity was consistent with the beliefs and instructional goals they reported in the pre-observation interview and the initial interview. We identified for discussion in the post-observation interview instructional actions that did not seem to align with the teacher's professed beliefs or assist the teacher in achieving his or her professed goals. After each lesson cycle—which consisted of a pre-observation interview, classroom observation, and post-observation interview—we articulated hypotheses about the motives for each teacher's activity. We focused on documenting each teacher's comments about what they prioritized, what considerations appeared to influence their instructional actions, and commonalities in their behavior during lessons. The essence of these commonalities and the design of the interview questions led to the hypotheses we generated regarding the teachers' instructional goals, professional identity, belief structures, and mathematical epistemology, which we refined after each lesson cycle.

In the post analysis, we focused on finding evidence that either supported or rejected the hypotheses we generated and refined during ongoing analysis. We also looked for any comments the teacher made that would suggest influences that mediated the enactment of his or her content knowledge that we did not identify in our ongoing analysis. For each teacher, we watched each interview multiple times and noted evidence that supported or refuted our hypotheses (we found no evidence that rejected our hypotheses) as well as any repeated instances that could lead to the generation of new mediating factors.

Results

To illustrate our results, we present statements of our validated and refined hypotheses for each teacher, Frankie, Mandy, and Molly. We also provide an abbreviated summary of evidence for these hypotheses. The evidence we provide as illustrative examples are meant to give detail to our characterizations of the personal and affective influences that mediated the enactment of the teachers' content knowledge. We also discuss the prevalence of such mediating factors and the range of influence they prominently manifest in the teachers' decision making.

Frankie

Frankie has an image of effective teaching methods that includes eliciting student thinking, leading whole class discussions, engaging students in group work, asking students to explain their thinking, encouraging them to defend their ideas, and having them critique other students' reasoning. Her image is grounded in the educational research that claims that students' recall of mathematical facts and procedures is enhanced, and that they are more likely to correct their misconceptions if other students are the ones identifying them. In practice, her commitments do not take into consideration a model of students' thinking or mathematical meanings. This lack of focus on student's conceptions impacts her enactment of these practices and consequently the development of her instructional goals. One of Frankie's primary goals is to promote her students' mathematical independence, which she views as their ability to attempt or to solve new problems without giving up. She focuses on avoiding repetition of problems in her assignments and presenting students with different representations of the same ideas (e.g., a sequence as a list of numbers versus in a table) stating, "I try to have something that's a little different, from what they're looking at."

Frankie also likes to have students voice multiple solution strategies, both in assignments and discussions, believing if only one method is presented a student who thinks differently will shut down. By eliciting multiple strategies, she also feels students will be more likely to recall a method when solving problems. Frankie started her first lesson by showing students a pattern of squares growing in a geometric progression and asked them to describe what they saw happening. As students responded Frankie recorded their thoughts on the board. While some of the students' descriptions focused on the quantitative pattern of tripling, others characterized the growth pattern in additive terms, by adding two more copies of the previous state. Several students focused on spatial descriptions such as, "The figure grows sideways, then up, then sideways" and another student confessed to only seeing squares and not understanding the other students' descriptions. Frankie masterfully elicited a rich variety of students' ideas, and the class was enthusiastically participating. Several of the students remained confused about the lack of resolution throughout this lesson and several subsequent lessons, however.

Frankie's commitment to eliciting students' contributions and to fostering their independence did not appear to be balanced by an attention to the meanings that the students were expressing, or how to support them from that foundation. As a result, she was often backed into a corner where she had to act counter to her commitments and declare a particular way forward, or move on from the discussion entirely despite students expressing confusion. Similar patterns of interaction repeated in Frankie's class discussions about (1) the difference between arithmetic and geometric sequences and between explicit and recursive formulas, (2) the multiplication in the formula $a_n = a_1+(n–1)d$ for arithmetic sequences despite supposedly being "additive," and (3) the role of the multiplicative factor in geometric sequences.

Mandy

Mandy is highly attuned to the amount of cognitive effort students are willing to expend at any particular moment, and she demonstrates mathematical skills and procedures in accessible chunks so as not to overburden students' cognitive resources. For example, in a pre-lesson interview while discussing why she teaches a method for simplifying radical expressions in a way that differs from her textbook, Mandy explained that she prefers algebraic procedures with the fewest number of steps. Even after acknowledging that the additional steps in the textbook more clearly convey the algebraic rationale for the simplification technique, Mandy thought it unnecessary and potentially confusing for her students. Although her instructional style is rather direct, Mandy often demonstrated her sensitivity to how her students interpret her instruction. For instance, in her third lesson on exponential functions, Mandy demonstrated how to solve exponential equations by expressing both sides of the equation with a common base. The way Mandy conveyed the steps for solving such a problem was heavily informed by her image of what students understood (or didn't understand) in the moment. Specifically, Mandy anticipated students' tendency to incorrectly interpret the equating of exponents with the same base as dividing both sides of the equation by the common base. She was aware that this algebraic overgeneralization certainly occurred to some students, and she addressed it in the process of demonstrating how to solve a particular kind of exponential equation. This is one of many occasions in which Mandy demonstrated her effectiveness at communicating steps for solving routine problems. Her explanations were accessible to students because she anticipated their in-the-moment interpretations.

Mandy is also committed to fostering students' mathematical self-efficacy by making the mathematics as easy for them as possible, even if this means knowingly compromising the conceptual rigor of the content. Indeed, Mandy feels it is her responsibility to shield students from unnecessary formalism, rigor, or conceptual depth. While planning her lessons, Mandy identifies the different kinds of problems, or variations of the same kind of problem, students will be expected to solve. She then thinks about how to present the material in a way that minimizes the mental effort students need to expend to become proficient at solving these problems.

Mandy believes that students learn best, at least initially, through rote practice of simple procedures. This view is integrally connected to her commitment to keep task difficulty to a minimum, as she anticipates the consequences of demanding too much mathematical reasoning or conceptual understanding from her students. For example, in the third pre-lesson interview, Mandy explained,

> They really need the rote. I know there's two trains of thought and some are that if they're doing rote are they really thinking? Or if I give them all, you know, twenty different problems then they're having to think on each one. Well, my students will shut down. They still do better with the rote. Let me get them used to the method and what we're doing here and maybe the understanding of why we're doing it, then we can go into different things if we have time. That's just how they learn better … I just still believe that there's a place for some rote learning—just some practice over and over and over.

Reflected in this quote is Mandy's belief that mathematics learning is a process of repeated exposure to mathematical facts, skills, and procedures. Often, when promoted by an interviewer to propose an instructional intervention for students who demonstrate a particular non-normative conception of exponential functions, Mandy proposed increased and more frequent exposure to the procedure required to solve specific kinds of problems. This repeated exposure has both a cognitive and affective rationale for Mandy: it has the obvious virtue of supporting students in remembering these facts, skills, and procedures while also reducing students' anxiety by enabling them to recognize something familiar in new content. She unapologetically eschews progressive pedagogies and is steadfast in her perspective that often the only way to learn mathematics is by rote. Mandy thinks this is especially true when introducing new topics, although she expects that students might be able to construct more sophisticated conceptions in more advanced courses. She defends her pragmatic, traditional, outcome-oriented view of teaching by appealing to the results she achieves.

Molly

Molly views mathematics as an enterprise of exploring new ideas and solving challenging problems. From her perspective, learning mathematics must have the same character. This belief about the nature of mathematics and mathematical engagement has the practical effect of compelling Molly to model for her students mathematical exploration and problem solving, as well as providing opportunities for them to leverage their creativity and intuition to solve novel problems. She articulates a strong growth mindset, readily identifies mathematical phenomena that she does not understand, and is open to sharing her own struggles and growth with her students, expecting them to be open to similar experiences, in turn. To Molly, course content is interpreted in terms of productive tools to apply to this larger enterprise. She selects/designs instructional materials to introduce, model, and reinforce these key conceptual tools. Allowing for genuine student engagement, Molly is willing to sacrifice significant class time—a commodity she finds scarce and precious—to allow productive student engagement about a problem.

Molly emphasizes specific content from both the topic of exponential functions and from the broader topic of function types throughout every class. Examples of the general strategies she models and emphasizes are drawing a graph with key points relevant to the function type, making a table of values, and applying function transformations. Specific to the topic of exponential functions, she emphasized strategies such as interpreting exponents as repeated multiplication to test and recall algebraic rules, attending to the horizontal asymptote, and using multiplication or division by the base to generate function values at nearby integer distances. While Molly often expressed the view that regularly and explicitly addressing these strategies helped reinforce those mathematical ideas, she placed greater and more frequent emphasis on framing and using them in terms of solving a larger problem. These strategies surfaced and were explicitly modeled by Molly in extemporaneous response to issues raised by the students in class and in planned activities.

Molly exhibited a consistent, strong commitment to promoting student exploration in her class. For example, during her third lesson on exponential functions, she announced that she was "going to demonstrate for you how to do this problem then I'll have you practice one that's similar to it," providing one of the more procedural framings she gave to any class activity we observed. She then displayed the equation $4^x + 2^x - 20 = 0$. As part of her "demonstration" she solicited ideas and debate from the students about how to rewrite the equation, justifications for their equivalence, discussions of what made the problem difficult, their questioning of her introduction of the variable $w = 2^x$ in terms of its value in the problem-solving process, and using their standard strategies (connecting graphs, tables, repeated multiplication, and asymptotes) to rule out solutions such as $2^x = -5$. Moreover, as students continued to explore different ways of solving $2^x = c$, they repeatedly raised tangential questions, such as how to demonstrate solutions using function transformations, what happens if the base of an exponent is negative, differences in notation between -2^x and $(-2)^x$, how to solve for x in cases such as $2^x = 6$ when the solution isn't an obvious integer. Throughout she encouraged student questions and new lines of investigation, allowing over 20 minutes for the activity. She did summarize the solution strategy to the original problem before asking students to work on a similar example of their own.

Discussion

Each of the participating teachers discussed in this paper maintained particular beliefs that, together with how they conceptualized themselves as mathematics teachers, informed the instructional goals they defined, which subsequently influenced the knowledge they leveraged to achieve them. Through our analysis across multiple teachers we have articulated hypotheses that explain our interpretation of the motives that underlie the teachers' planning, instruction, and assessment activity. We have found teachers' professional identities, instructional goals, learning theories, and beliefs regarding the nature of mathematics to be highly consequential influences that affect the enactment of their content knowledge. Elaborating these overarching characteristics of a teacher allows one to anticipate the mathematical content and pedagogical practices he or she will leverage when posed with an instructional situation. For example, each of the three teachers discussed in this paper made different decisions to foster students' engagement and to elicit their reasoning. Believing students are more receptive to critiques when posed by other students, Frankie elicited student contributions to foster discord between ideas. Similarly, Molly believes learning mathematics requires students to individually engage in exploration and problem-solving, but must also be challenged to communicate and test their ideas with others. Thus, she cultivated a communal expectation to question each other directly and to spend as much time as needed clarifying each other's understandings. Mandy, on the other hand, engaged her students by prompting them to state the step that comes next in a routine problem solving procedure.

The three cases discussed in this paper have implications for the preparation of pre-service teachers at the undergraduate level. Although teachers cannot leverage mathematical knowledge they do not possess, the enactment of their content knowledge is constrained/afforded by their implicit learning theories, beliefs about the nature of mathematics, instructional goals, and identities as mathematics teachers. Undergraduate teacher education programs should therefore devote greater attention to engaging students in mathematical learning experiences that (1) clarify the cognitive processes involved in mathematics learning, (2) enable their construction of particular beliefs about mathematics and mathematical engagement, and (3) form the foundation for their identities as future teachers of mathematics. Those of us tasked with the content preparation of mathematics teachers at the undergraduate level are not simply responsible for supporting pre-service teachers in developing productive conceptions of the content they will teach, but also for engaging them in mathematical experiences that establish the psychological and affective conditions necessary for them to leverage the sophisticated content knowledge that we are obligated to ensure they possess.

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Critical Features and Representations of Vectors in Student-Generated Mindmaps

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The purpose of this study is to investigate multivariable calculus students' communication of vectors by examining how their responses on a mindmap assignment change over time. A mindmap is a visual network of connected and related concepts often with one image or topic centrally located. Through this open-ended instrument, we conduct a qualitative analysis to explore the connections students make between different aspects and multiple representations of vectors.

Keywords: mindmaps, vectors, variation theory, multiple external representations

While basic vector concepts, representations, and operations are presented in both high school and college mathematics, students continue to have significant conceptual difficulties with them. Much research on student understanding of vectors explores students' misconceptions of physical concepts such as force and motion, but students' misconceptions regarding vector concepts, properties, and fluency in vector operations are not explored directly (Aguirre & Rankin, 1989; Barniol, Zavala, & Hinojosa, 2013; Flores, Kanim, & Kautz, 2003; Govender & Gashe, 2016; Hestenes & Wells 1992; Hestenes, Wells, & Swackhamer, 1992; Miller-Young, 2013). While some researchers provide more explicit consideration of students' understanding of vector concepts, representations, and operations outside of a kinematic context, the focus has not been on how students make connections between vector operations and between different representations of vectors (Barniol & Zavala, 2014; Knight, 1995; Kustusch, 2016; Nguyen & Metzler, 2003; Van Deventer & Wittmann, 2007; Wang & Sayre, 2010; Zavala & Barniol, 2010). The overarching research goal for this paper is to investigate multivariable calculus students' communication of vectors by examining how their responses on a mindmap assignment change over time. More specifically, the three research questions we consider are: what changes are noted with respect to the

1. critical features students address in the mindmaps?
2. connections that are made between these features in the mindmaps?
3. representations (e.g., graphical, verbal, symbolic, or numeric) used in the mindmaps?

Mindmaps and Concept Maps

In recent years, educators have begun using software mapping tools for a variety of pedagogical and research purposes (Govender & Gashe, 2016; Ayal, Kusuma, Sabandar, & Dahlan, 2016; Davies, 2011; Edmondson, 2005). These diagrammatic representations of ideas and their relationships "may not be the panacea … , but they do represent an approach that more effectively taps the dimensions of student thinking that many traditional assessment formats miss" (Edmondson, 2005, p. 36). Here, we identify what critical features and representations of vectors students present in a series of mindmap assignments.

The terms concept map and mindmap are used interchangeably by software developers and educators, but in the research literature there are distinctions. *Mindmaps* are networks of connected, related concepts often with one topic centrally located; they typically use line thickness, colors, and pictures to communicate ideas and connections (Davies, 2011). Mindmaps are used to study student understanding of a concept by providing a deliberately ambiguous central topic without suggesting relationships (Bandera, Eminet, Passerini, & Pon, 2018). A concept map can be thought of as a specific type of mindmap. *Concept maps* are more tightly structured, hierarchical networks with descriptive phrases such as "leads to," "results from," "is a part of," etc. characterizing the connections linking two ideas (Davies, 2011; Edmondson, 2005).

Because of the more formal structure of concept maps, automated and quantitative scoring rubrics are typically used to count the number and complexity of linkages, placing less emphasis on the content. Few guidelines and protocols exist for qualitative assessment, and most focus on the structure of the concept map independent of content (Keppens & Hay, 2008; Kinchin, 2000). While time consuming, a qualitative content analysis of concept maps can document change over time among a group of participants with varied backgrounds (Hough, O'Rode, Terman, & Weissglass, 2007).

Multiple Representations

Multiple External Representations (MERs) of mathematical and scientific concepts are commonly used to support learning by integrating and/or coordinating more than one source of information. However, this integration requires the ability to translate between different representations, which students often find difficult to do (Ainsworth, 2006; Kozma, 2003). How well an individual is able to move between different representations depends on several individual characteristics including domain knowledge and representational fluency (Ainsworth, 2006). In this study we not only consider the vector knowledge that students communicate in their mindmaps and how it is organized, but also which MERs they chose to include in their mindmaps. For the purposes of this study we use a modification of Shield and Galbraith's (1998) taxonomy of modes of representations of written mathematics: symbolic (i.e., algebraic), numeric, verbal, and graphical (Neira & Amit, 2004).

Theoretical Framework

Our work combines Simon's (2017) theoretical construct of "mathematical concept" with Marton and Booth's variation theory (Rundgren & Tibell, 2009). We begin with the assumption that effective mathematics instruction and assessment of student understanding requires clear articulation about the mathematical learning goal which is often too broadly described as "understanding a topic" (Simon, 2017, p. 128). Simon's construct of "mathematical concept" and the notion of "critical feature" from variation theory taken together have the potential to provide a way to more precisely define what it means to "understand vectors."

Simon defines a *mathematical concept* to be "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (Simon, 2017, p. 123). Like Simon's definition of mathematical concept, variation theory also focuses on intended and inferred student knowledge. In variation theory, the term *critical feature* refers to an aspect of or condition of a topic that is necessary for learning. According to variation theory, learning takes place when students perceive critical features, and students can only discern a critical feature if they experience variation of it (Runesson, 2006).

To specify a mathematical concept, Simon (2017) recommends observing contrasts in individuals' mathematical functioning, whether it be between: a student and an expert, two students, or observations at different times of a single student. The mathematical concept then arises as a specific explanation of differences observed. Simon (2017) cautions that further research or pedagogical activity will reveal modifications to the mathematical concept. As a first step towards developing a mathematical concept of vectors in multivariable calculus, critical features of the vector cross product have been identified: magnitude, direction, angle between two vectors, location of the vectors, and orientation of the cross product to the two vectors that form it (VanDieren, Moore-Russo, Wilsey, & Seeburger, 2017). Our study tests the validity of these critical features with different data and on a broader range of vector concepts. We begin the process of developing mathematical concepts for the nebulous goal of "understanding vectors" by contrasting work of students in a multivariable calculus class over time attending to differences in their communication of critical features.

Methodology

Context of the Study

The participants were 30 students in a multivariable calculus course at a private, regional university. On the first day of the semester, the first author introduced the mindmap activity to the students and explained its purposes to: (a) provide students the opportunity to organize their thoughts on vectors, (b) identify connections between different features, applications, and operations of vectors, and (c) serve as resource during the first exam. Students were allowed to include any items including images from textbooks, links to online tutorials, or photos of handwritten notes in their work. The first author suggested to students to use Inspiration and Lucid Charts software to create the mindmaps, but ultimately students could choose their preferred software. A sample mindmap on geometry content and a tutorial for creating a mindmap in Inspiration were offered to the students. Students were assigned to create a mindmap of what they knew about vectors at three points of time during the first three weeks of the semester during which the topics of vectors, vector operations, lines, and planes were covered in class. After each submission students were given feedback on their work including suggestions for adding graphical depictions, applications, or missing concepts in future submissions.

Data Analysis

Of the 30 students in the study, 24 students submitted at least one mindmap. One student was removed from the sample because his work was not in the form of a mindmap. Of the 23 students who submitted first and final drafts of the mindmaps, only 15 submitted an intermediate draft during the second week. Therefore, we report results from only the first and final mindmaps of the 23 students. An iterated coding analysis was conducted on the mindmaps.

Development of coding. Two days of discussions between the authors led to a first round of analysis, which was based on eight *a priori* content or topic categories (*vectors, scalar multiplication, addition, subtraction, dot product, cross product, projection,* and *other*). These categories were used to sort the content in each mindmap. These content categories were further refined according to critical features of vectors (*direction, magnitude, angle between vectors,* and *location*). Each category that was marked as present was then coded according to its representation on the mindmap (*graphical*, *verbal*, *symbolic* and/or *numerica*l). These were coded for presence and not for accuracy. For each of these categories, whether the mindmap included an application (e.g., force, work, etc.) was also coded. Finally, the researchers coded whether each concept was presented with two- and/or three-dimensional representations. A second set of codes was used to characterize the relationship between pairs of concepts and how they were depicted (by lines or words) and whether these connections represented declarative, procedural, and/or conceptual knowledge (Sarwar & Trumpower, 2015). We will not discuss the declarative, procedural, and conceptual knowledge coding further in this paper.

A sample of four student mindmaps was coded by the first author. Issues with this coding scheme were then discussed with the second author. A new coding scheme was proposed that added generic content categories. Based on emergent themes from this sample, three new codes were added to the scheme: "unit vectors" and "basis vectors" were added as subcategories of *vectors* and "orthogonal component of projection" was also added as a subcategory of *projection*. The *words* category for connections between content areas was split into three categories: "words," "multimedia static," and "multimedia dynamic" to distinguish verbal descriptions from graphs or images and video links. The category of 2D or 3D was only assessed over the entire mindmap and not on individual concepts because the previous level of refinement was deemed unnecessary. Similarly the *applications* code was evaluated at the level of topic and not critical feature. The original sample of four students plus two additional mindmaps were coded by the second author according to this new scheme. The authors then discussed some discrepancies in coding from these rounds. Clarifications were made in the codebook and the first author then coded the initial four and the additional two mindmaps with the new scheme. The codings of the two authors on the sample of six mindmaps were compared, discrepancies discussed, and clarifications to the codebook added. The dot product subcategories were eliminated from the codebook because these critical features did not apply to a scalar value.

Interrater reliability measures. Since the codes were not mutually exclusive categories, the measure Mezzich's kappa of interrater agreement for multivariate nominal data was used (Mezzich, Kraemer, Worthington, & Coffman, 1980; Eccleston, Weneke, Armon, Stephenson, & MacFaul, 2001). Mezzich's kappa statistic for this sample indicated 63% agreement. Most of the disagreement stemmed from the interpretation of the categories "verbal" and "declarative" in the fifth mindmap in the sample. The authors discussed these disagreements and came to a consensus that was then addressed in the codebook. Making adjustments to these codings based on the new consensus, brought Mezzich's kappa to 73%. Because this sample of six mindmaps did not exhibit every code in the codebook, two additional mindmaps were selected and coded by both authors independently. Results were compared resulting in Mezzich's kappa equal to 75%. At this stage, the codebook was finalized and the first author coded the remaining mindmaps.

The codebook. The codebook can be separated into two parts: <u>content</u> and <u>connections</u>. The <u>content</u> coding included the topic categories: *vector* (V), *scalar multiplication* (S), *addition* (A), *subtraction* (B), *cross product* (X), *dot product* (D), and *projection* (P). There was also an *other* (O) category to capture ideas (e.g., lines and planes) not directly fitting into these topics. Each topic category was marked for presence and whether the mindmap included an example of an application of each topic (V-app, S-app, A-app, B-app, X-app, D-app, P-app, and O-app respectively). In addition, any relevant critical features of these categories present on the mindmap were also coded. Table 1 below describes some of the subtopic codes that were observed along with examples. In addition to the subtopic codes listed in Table 1 and the application codes, the full list of subtopic codes included: Ag (general addition), Bg (general subtraction), Xg (general cross product), Xd (cross product direction or orientation in relation to the two vectors that form it), Xm (cross product magnitude), Xa (cross product as orthogonal to the two vectors that form it), Dg (general dot product), Pg (general projection), Pd (projection direction), Pm (magnitude of the projection), Po (orthogonal component of projection), and O (other). Finally, for all codes, except the application codes, it was noted whether or not the topic and/or critical feature was described *verbally, numerically, graphically,* or *symbolically.*

We used <u>connection</u> codes between topics and how those connections were represented. For example, the codes VS-W, VA-W, VB-W indicated the topic of vector was connected, respectively, to scalar multiplication, addition, and subtraction through words. An example was a bubble with the words "Vector operations" that connects to three bubbles with "scalar multiplication," "addition," and "subtraction." If these three are also connected through lines on the mindmap, then VS-L, VA-L, and VB-L were also coded. Other connections could be in the form of a static image (MS) or a dynamic multimedia clip (MD). For example, a video clip of an instructor working through a problem demonstrating **u** + (-**v**) = **u** - **v** was coded as AB-MD. The complete list of connecting codes included all pairs of topics (V, S, A, B, X, D, P) and all four types of connections (L, W, MS, MD).

Table 1. A sample of the commonly used codes, their descriptions, and representative examples.

Subtopic Codes	Description	Examples
Vg	General Vectors - any mention of vectors at all.	Different notations of vectors; graph of a vector; conversion between different representations of vectors; any of the examples under the "V" codes below; if there is very little information on the mindmap, this may be the only category coded.
Vd	Vector Direction	Picture of a vector with the direction marked; mention of change in x and change in y; image that marks the angle the vector makes with the x-axis; explanation of the process of how to find the angle that the vector makes with the x-axis
Vm	Vector Magnitude	Image of a vector with the length marked; formula or computation of the length of the vector; use of the Pythagorean Theorem for computing length
Va	Angle between Two Vectors	Image with angle between two vectors marked in a graph; image with angle between two vectors defined in a formula for dot or cross product (the presence of "theta" in a formula without a geometric or verbal definition would not be coded)
Vl	Vector Location	Statement that vectors can be moved or that location doesn't matter
Vu	Unit Vectors	Definition, formula or explanation for finding a unit vector in a given direction
Vb	Basis Vectors	Definition or graph of the **i**, **j**, **k** vectors
Sg	General Statement about Scalar Multiplication	Mention that scalar multiplication combines a scalar with a vector, Including the notation cv, any of the examples in the "S" codes below
Sd	Scalar Multiplication Direction	Indication that c**v** is parallel to **v**; mentioning the impact of -1 on the direction
Sm	Scalar Multiplication Magnitude	Demonstration the effect of the magnitude of c on the length of c**v**

Results

The vast majority of students included both two- and three-dimensional representations of vectors in their initial (87%) and final mindmaps (96%). On the other hand, very few students included applications of vectors in their initial mindmaps. Among the initial mindmaps, only seven students (30%) provided any application, but twenty students (87%) included an application of vectors in their final mindmaps. The distribution of the kinds of applications the students mentioned in their mindmaps appears in Table 2. Almost no students provided applications of scalar multiplication, addition, subtraction, and projection. Table 3 and Table 4 report the frequency counts of the other codes.

Table 2. Frequency comparison (counts) of application codes in the initial and final mindmaps (n=23).

Mindmap	V-app	S-app	A-app	B-app	X-app	D-app	P-app	O-app
Initial	7	0	1	0	0	0	0	0
Final	10	1	1	0	16	14	1	1

Table 3. Frequency comparison (counts) of topics and representations in the initial and final mindmaps (n=23).

Types of Representations	Vg	Vd	Vm	Va	Vl	Vu	Vb	Sg	Sd	Sm	Ag	Bg	Xg	Xd	Xm	Xa	Dg	Pg	Pd	Pm	Po	O
Initial Mindmap																						
Geometric	7	1	4	2	1	2	2	2	2	2	4	3	1	1	1	1	3	0	0	0	0	0
Numerical	17	5	5	3	0	4	12	9	0	0	8	10	0	0	0	0	3	0	0	0	0	0
Symbolic	20	4	12	4	0	6	13	12	1	2	12	12	1	0	0	0	6	0	0	0	0	0
Verbal	22	17	19	6	5	15	6	16	4	6	13	11	2	1	1	1	5	0	0	0	0	0
Final Mindmap																						
Geometric	13	6	7	5	1	4	4	5	4	4	11	9	7	9	4	6	8	6	1	2	3	2
Numerical	17	5	6	3	0	4	13	10	0	0	8	10	6	0	1	0	9	1	0	0	0	3
Symbolic	21	4	13	7	0	9	14	16	1	2	14	15	19	2	8	1	23	7	2	1	1	8
Verbal	21	17	20	12	7	16	6	17	4	6	12	10	17	8	4	6	19	6	1	1	4	8

Table 4. Frequency count of connections in the initial and final mindmaps (n=23).

Types of Connections	VS	VA	VB	VX	VD	VP	SA	SB	SX	SD	SP	AB	AP	BP	XD	DP
Initial Mindmap																
Lines	20	19	19	1	6	0	0	0	0	0	0	1	0	0	0	0
Words	9	8	7	0	1	0	1	0	0	0	0	6	0	0	0	0
Multi-media Static	3	1	3	1	0	0	1	1	0	0	0	0	0	0	0	0
Multi-media Dynamic	0	1	1	1	2	0	0	1	1	0	0	0	0	0	1	0
Final Mindmap																
Lines	22	20	21	21	22	7	0	0	0	0	0	1	0	1	0	2
Words	11	9	8	6	6	2	1	1	0	0	0	7	1	0	7	4
Multi-media Static	6	10	10	14	11	4	1	5	1	0	0	3	1	1	4	0
Multi-media Dynamic	0	1	1	1	2	0	0	1	1	0	0	0	0	0	1	1

Discussion

We first consider the topics and critical features in the mindmaps. Since the initial and final mindmaps were created three weeks apart and more material on vectors was presented in class during this time frame, it is not surprising to see more topics on the final mindmaps. Once students addressed a concept in the initial mindmap, they rarely made any changes or additions to that concept in their final mindmap. Therefore, the statistics reported below represent whether a category was coded on at least one of the initial or final mindmaps for each student.

Almost all students included the vector operations (scalar multiplication, addition, subtraction, cross product, dot product), but only nine (39%) of the students mentioned projection. Nearly all students described the direction and magnitude of the vector, but only 14 (61%) of the students mentioned the angle between two vectors. This echoes a study of pre-service teachers' concept maps of vector kinematics in which the most common code was "vectors have magnitude and direction," and only one concept map mentioned angle (Govender & Gashe, 2016, p. 331). Furthermore, in our study when discussing the cross product, less than half mentioned the direction or magnitude. Therefore, while students identified relevant vector operations, they did not communicate all critical features related to the operation.

Considering the connections between the topics, it is notable that students tended to treat the vector operations in isolation, especially scalar multiplication. Additionally, only eight students (35%) made a connection between addition and subtraction. Even straightforward connections between topics were not reported. For example, only five of the nine students who mentioned projection included the formula and/or connected it with the dot product. When students did demonstrate a connection between two of the operations, they did not explicitly draw a connecting line, but implicitly connected the ideas through a formula or a static image.

Students' representation of vectors changed over time. More students provided geometric representations of vectors and applications in their final mindmap than in their initial mindmap. This could be attributed to re-reading the assignment instructions and receiving instructor feedback after submitting the initial mindmap. However, geometric and numeric representations were sparsely used for every topic on both the initial and final mindmaps. Despite being given encouragement to graphically display information, students tended to rely on verbal and symbolic representations of vectors, and few initially reported applications of vector concepts. These observations are consistent with a think-aloud study of engineering students working through three-dimensional force problems (Miller-Young, 2013). Furthermore, the nature of the connections that students made and their use of representations supports research that shows that novices organize their groupings by surface-level features and use only one or two representations, while experts tend to cluster apparently different situations together into meaningful groups using a greater variety of representations (Kozma, 2003).

Limitations

Because the students were allowed to access class notes, the textbook, and online resources, the mindmaps created may not reflect the students' understanding of vectors. However, since students were allowed to use these mindmaps on their in-class exam, the mindmaps may reflect what students viewed as important or critical information about vectors. Also, since students were allowed to copy material from other sources, the representations that they added to their mindmaps may indicate what they found readily available versus the representation that they would have chosen to create on their own. Because the assignment was carried out on the computer, technological constraints may have influenced the representations that the students chose to include in their mindmaps. For instance, a student may have found it more convenient to type a verbal description rather than a symbolic description involving subscripts. Furthermore, the assignment instructions and instructor feedback to include multiple representations may have influenced students to include representations of vectors beyond what they would have chosen on their own. Finally, this study was limited to a sample of students from one section of multivariable calculus. A broader sample of students from different schools may provide varied results based on instructor emphasis and student background.

Conclusion

By examining how students communicate vector topics on mindmaps over time, this study contributes to the body of research on student understanding of vectors. Knowledge about which connections and which representations the students communicate can inform pedagogical practices and the development of technological environments to help students coordinate the ideas and representations (Kozma, 2003). Our study triangulates research on critical features of vectors (VanDieren et al., 2017). Additionally, our research serves as a testing ground for Simon's theoretical construct (2017) at the undergraduate level, since it was originally developed for K-12 content. Finally, critical features, and as Simon (2017) suggests, identifying student differences between one another and over time can both be used to articulate mathematical concepts that may later be used for assessment of student understanding.

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Calculus Variations as Figured Worlds for Mathematical Identity Development

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Calculus is often an essential milestone during a students' time in college and can be especially impactful for students wishing to major in in a math or science field. Given its relative importance, the ways in which calculus courses are delivered can have a lasting impact on a student's trajectory and relationship with mathematics. In this study we document the ways in which three calculus course variations at the same University operate to promote different mathematics identities for students. Drawing on the Holland et. al.'s (1998) framework of figured worlds we showcase the ways in which these course variations act as if they are different calculus worlds that constitute socially organized and produced realms of being. We highlight the ways in which these figured worlds position or fail to position students with the opportunity to refigure themselves and others as learners and doers of mathematics.

Keywords: Calculus, Math Identity, Figured Worlds, Course Variations

In the United States there is a national movement to increase the number of awarded STEM degrees in order to address the nearly 1 million additional STEM degrees needed to support the nation's growing research and technology economy (PCAST, 2012). Among the recommendations to address this need, the PCAST report recommended the adoption of empirically validated teaching practices, replacing standard lab courses with discovery-based research courses, addressing the mathematics-preparation gap, and diversify pathways to STEM degrees. Additionally, any efforts to improve the quality of undergraduate STEM education must also attend to fostering an environment that promotes diversity and inclusion in STEM classrooms (National Academies of Sciences Engineering and Medicine, 2017).

The vision and enactment of creating an equitable robust STEM education is a complex and multifaceted endeavor that will require continued research; however, one such promising approach in undergraduate mathematics that has been identified with successful calculus programs is the tailoring of calculus courses to meet the needs of individual students, which we refer to as course variations. Course variations have the potential for addressing the recommendations from the PCAST report since they can specifically address the preparation gap for students by incorporating prerequisite material in courses or stretching out the course content, can infuse labs and standard based teaching in courses tailored for science majors, and even provide diverse pathways into STEM for those that have taken a non-traditional math background through transition courses. Author (2016) documented how these variations to the standard course across the US have been associated with greater rates of passing calculus and put forth a call to future research to examine the ways that these courses may help promote a sense of community and identity development among students in the different variations. We take up this call for future research in this report, by examining *how the structures and activities of three different calculus courses at the same undergraduate institution impact the types of possible mathematical identities that emerge from those contexts?*

Framework and Literature Review

Mathematical identity has become a central and powerful concept in the analysis of mathematical learning, in part due to the recent social and political turn in education (Gutiérrez, 2010). Identity frameworks in math education have drawn largely from sociocultural perspectives that link identity and learning to one another and arise from social practices. Additionally, this research often utilizes positioning theory to account for identity as constructed through social interactions to construct storylines about who a person is in relation to others in a social context (Langer-Osuna & Esmonde, 2016). Holland, Lachicotte Jr., Skinner, and Cain (1998) sociocultural theory of identity and self, known as figured worlds, is useful as a tool for studying identity production in education, and how the context of education allows or does not allow the emergence of certain identities. Figured worlds are "socially and culturally constructed realms of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al.,1998, p. 52). Figured worlds are dynamic. They are constantly formed and re-formed in in relation to the everyday activities and events that occur within the realm of possible "as if" worlds. Figured worlds are thus situated in a social context and time period, and represent a reflexive relationship and negotiation of the possible identities that can be constructed and affirmed in the figured world. As cited by Urrieta (2007), Holland claims that figured worlds have four characteristics:

(1) Figured worlds are cultural phenomenon to which people are recruited, or into which people enter, and that develop through the work of their participants.
(2) Figured worlds function as contexts of meaning within which social encounters have significance and people's positions matter. Activities relevant to these worlds take meaning from them and are situated in particular times and places.
(3) Figured worlds are socially organized and reproduced, which means that in them people are sorted and learn to relate to each other in different ways.
(4) Figured worlds distribute people by relating them to landscapes of action (personae); thus activities related to the worlds are populated by familiar social types and host to individual senses of self.

Boaler and Greeno (2000) draw on the concepts of figured worlds to illustrate how two different types of high school classrooms afforded students different identities and storylines of mathematical learners. One such figured world of mathematics classroom, drew on the concept of "received knowing" and promoted a concept of doing mathematics as memorization and being able to quickly recall information. In contrast, the other figured world of mathematics classroom drew on the concept of "connected knowing" and promoted a concept of doing mathematics as making sense of mathematical concepts and procedures. The "connected knowing" world promoted as sense of agency among the mathematical learners since they were encouraged to draw on their own interpretations of mathematics to make sense of the concepts. Boaler and Greeno's study highlights how the context of education setting and approach to teaching impact students' identity production as learners and doers of mathematics, which impacts their choices for continuing or dropping out of further engagement with mathematics. This is especially problematic since there is some indication that educational contexts which promote decontextualized and abstract knowledge are more alienating for women and non-western students and therefore hindering the goals of educational equity in STEM.

Solomon, Croft and Lawson (2010) examined how mathematics support centers, which were intended to support skill development for engineering students, were dynamically co-opted by the students to support the development of group learning strategies which promoted a strong community identity among the participants. This study highlights the way the undergraduate STEM community of practice, which can often be highly competitive and individualistic, can refigure itself by reflecting on the positional identities that can be challenged in that space by drawing on the physical resources and artifacts to disrupt the available storylines. For instance, the physical space of the tutoring center, allowed students to refigure their relational identity to mathematics as a social endeavor of helping each other succeed.

Methods

The analysis presented draws on student focus group data from one University, which we refer to as Tree Line University (TLU). TLU offers three different calculus courses. In addition to the standard offering, TLU has a coordinated calculus-physics course for advanced students and a life science course, which includes a focus on biology. For each of these three calculus offerings, we conducted a focus group with three to five students currently enrolled in the target course. We use focus groups for this research since they help highlight the nature of figured worlds which are socially constructed among the members. Students for the focus group were sent an email invitation as well as recruiting efforts done in-person during the course. As such the students who participated represent a self-selected sample of student willing to participate in the focus group, and while this may not be representative of the entire course they highlight the realm of possible mathematical identities afforded in each of the courses. The focus group conversations centered on topics such as who they are, their experiences in the course, how and why they chose this particular course, what happens during a typical class period, how they relate to others in this course as well as to students in different calculus courses. All focus group interviews were audio recorded and transcribed for subsequent, thematic analysis (Braun & Clarke, 2006). Guided by our theoretical framing of figured worlds, we developed narrative accounts in a collaborative endeavor among the researchers by first producing a descriptive account of the focus group and then using within and cross-case comparison to develop themes related to the research focus. These narrative accounts centered around the themes of students' emerging mathematical identities, sense of community or belonging, and positional relationship to calculus as (ir)relevant to their major and career goals. We present how these themes are enacted as figured worlds in each of the three calculus courses along with illustrative quotations from the narrative accounts.

Results

Calculus for Life Sciences: A refiguring of productive mathematical identities

Calculus for Life Sciences at TLU functions as a combined differential and integral calculus course without topics in trigonometry. The course was originally designed at the request of the college of life sciences and agriculture for students majoring in the life sciences. The content remains fairly similar to the standard calculus course but has what faculty described as a "lighter approach" that emphasizes concepts and some application of topics. Our focus group in this course included five students enrolled with the same instructor (Dr. B) for the lecture session but who had different teaching assistants for the twice weekly recitation sections.

Many of the students in the focus group conveyed that prior to enrolling in this course they had identities as poor performers in mathematics, which made them anxious to take a university calculus course. One student shared that they had taken precalculus and had gotten a C- in the course, and stated that it, "was the lowest grade I had ever gotten for a college class," and as a result was worried about how well they would do in this class. Several of the other students in the focus group concurred with this sentiment, with one student stating, "I did so poorly in that class, and I just thought like I am not meant to pass calculus." Other students discussed how the gap between their last math course in secondary school and taking this calculus course made them less prepared, and that they were "nervous going into calculus." Students in the focus group had a personal social history (history-in-person) that positioned them outside of the world of learners and doers of mathematics. For example, one student stated that they were, “someone who is not naturally inclined to math,” while another stated, “I am not meant to pass calculus.” However, as we will show, the students conveyed that through their experiences in this course, they were able to refigure their identities as productive mathematical learners and doers largely as a result from positive interactions with their instructor.

Students in the focus group conveyed that as a result of this course they now viewed themselves as someone who was capable of learning and doing mathematics. One student said that “I feel like I'm not completely hopeless at all in math anymore.” This sentiment was supported by several of the students who recognized a shift from their prior conceptions and experience in mathematics. For example, one student said, “I can actually do this, rather than like, in many past courses where I really have no idea what's going on." Students discussed how they were really “understanding” what they were doing rather than memorizing formulas, which aligned with the goal and vision of the course from the faculty perspective. As a result, students were able to refigure their positionality towards learning mathematics, as exemplified in the following quotes: “I’ll be able to succeed in other math heavy courses” and it “boosts my confidence in that regard.”

One of the contributing factors that helped students refigure their mathematical identities was their relationship with the instructor. “I can't say enough about our professor, this is probably the only math class that's really felt like it made sense in my life.” Students described instructional practice that contributed to their positive experience such as the teacher breaking down concepts in a way that made sense, using anonymous polling to see how they were feeling about course concepts, and providing prerequisite information such as the quadratic formula without assuming the students had memorized this information. These practices seemed to convey to the students that the instructor cared about them and their learning, allowing for them to acknowledge their past mathematical identity while being supported in the negotiation of productive mathematical identities. The impact of the individual instructor versus other features can’t be isolated in this study; however, the instructor through instructional techniques allowed for the enactment of a figured world that aligned with the goals of the course to have students focus on understanding and connected knowing.

There were also ways in which the enactment of the course variation positioned the students outside the world of mathematics learners. For instance, while some of the students mentioned that they were unware of the difference between calculus for life sciences and the standard calculus course, some of them mentioned the ways in which it was “low base calculus” or “more basic algebra” compared to “real calculus.” One student even described how their friend who was studying physics teased them saying, “you're not taking calculus, calc for life sciences is just like classical math.” Additionally, many of the students felt that the stated goal of the course to serve life sciences students was too broad. This resulted in students feeling that the course was not tailored to their specific discipline identities, “I'm either getting pushed aside or pushed under the rug with everybody else by just saying, “Oh well, you're in the life sciences major, you got to do this.” In this figured world of calculus for life sciences, students were maintaining a strong discipline identity (equine science, zoology) which they viewed as not needing calculus.

Honors Calculus: A collaborative community of academically-minded students

Honors calculus at TLU is a unique course that it is designed to integrate topics in physics and calculus and takes a theoretical approach to the material. Our focus group consisted of three students majoring in mathematics. Students emphasized the difficulty of this course by the fact that they often have to rely on one another to finish the homework and study for exams. For example, an agreed upon sentiment is that “Collaboration is actually one of the strengths of the class…you know everyone in the class, you feel like you can trust that they're going to put in the effort, and you're going to put in the effort, and you're going to come together if you need to.” The word “trust” was often used by the students in this focus group interview. They felt that there was a need to trust each other in order to do well. It is important to note that the objective for these students was clear; it was not to just pass the class, but to do well in the course together.

All of the students in this focus group had AP calculus credit. They entered into a world where they viewed their peers as equals who enjoyed learning and doing mathematics as much as they did. From the start they described a course that positioned them in the figured world of calculus where they felt accepted and academically challenged. This is reflected by the students’ frequent reference to being surrounded by people who are the “same.” One student in the focus group reinforced this idea as follows: “In my calculus class, we have students who are all STEM. They are students who have the same mindset”. These students are in a space where they are comfortable to acknowledge that they are joined by, “intelligent people who have the same common objective”. This highlights how mathematical identity can be formed when students are surrounded by people who they perceive to be cut from the same cloth.

Students were able to relate to each other and work together based on the fact that they are all coming in with similar interests, similar class objectives and career goals. During lecture, they were required to work in groups, which was a point of contingency at the beginning of the semester. There was reluctance from some students to work with one another because they wanted to “motor through” the activities. However, once they created a world where they were able to share their ideas they came to view group work positively. As one focus group member put it, “I get to share my perspective, I get to hear their perspective,” which they felt created a class that was more enjoyable. The figured world of honors calculus that the students created for themselves allowed them to grow and form a mathematical identity that centers around succeeding, understanding the material, and supporting others.

In this figured world, full of high achieving STEM majors, the students in the focus group reported having an extremely strong sense of community. One student explained how close knit they are as follows: “If I have a concern about anything really, I feel like I can go and find someone from the class and talk to them about it and ask them what they think. And, you know, that’s something that I think might be more exclusive to the [Honors Calculus].” An important aspect to highlight regarding their community is that the students in the focus group felt a closer sense of community in the honors calculus class than they did in any of their other honors classes. One contributing factor as the students described it was the focus of the class being all STEM majors, who had a similar interest, didn’t “wince” at the word math, and had high academic engagement.

Standard Calculus: A realm of disconnected knowing and isolation

The standard calculus sequence at TLU is primarily a service course for engineering majors. The focus group consisted of three men, with majors in ocean engineering, mechanical engineering, and chemical engineering. The students in the focus group had varied secondary school mathematics experiences where one student took a non-AP calculus course, one took an AP calculus course, and the third student did not take calculus in secondary school.

The one student who had not taken calculus in secondary school described Calculus 1 as “fast-paced” and not well-organized. He also expressed some personal disconnect with the material when he said, “I didn't know what a derivative, like what is the definition of a derivative, till like two weeks after we had started them." The other two students who had taken calculus in secondary school also felt that the course was fast paced but were less concerned with the material. In general, the three students positioned themselves as external to calculus, where calculus was something they had to do, as opposed to something that they were excited about learning. For example, one student said, "it's a class and I have to do work for it. That's just normal college stuff" and another student said calculus was a course "they had to take." Thus, upon entering calculus as first year students, none of the three positioned themselves as particularly excited about mathematics or very interested in mathematics. As they progressed from Calculus 1 to Calculus 2, this feeling of being disconnected from mathematics was *not* refigured, but rather seemed to become entrenched and reified.

In both Calculus 1 and 2, the three students had similar experiences in lecture. One student explained that he felt so disconnected that he stopped going to his assigned lecture and attended a different lecture instead. He recounted that in class he felt, "nobody knows what's going on because you're just up there writing, and you won't answer the questions. So, this is very frustrating." Another student chimed in that "Everything that he just said that happens this semester, happened for me last semester." The feeling of being personally disconnected from their instructors and the course content was amplified in Calculus 2. In contrast to Calculus 1 where they felt the material was more applicable and useful, their experiences in Calculus 2 was on memorization. For example, one student contrasted his experience in Calculus 1 and 2 as follows: "The expectation [in Calculus 1] was that there would be understanding. The latter [Calculus 2] is memorization without any expectation of understanding." This was a common sentiment for all three students. In fact, one student explained that he was told that Calculus 2 is "really advanced math" and so there they are not expected to "understand what we are doing." Even his teaching assistant (TA) positioned the content as something that was not within their reach for understanding. "And like my TA has dropped a line similar to just saying like, 'You don't need to know further, this is what you need in order to do this. So, this is what you're given.'" Thus, their experience in calculus at TLU resonates with the figured world of "received knowing" described by Boaler and Greeno (2000).

TLU's no calculator policy seemed to further figure calculus as something that is disconnected from their interests and previous experiences. For example, one student explained that the no calculator policy in calculus stood in contrast to how he imagined his future self in the workplace as an engineer. "You're not going to be working in a laboratory somewhere and they're just having you do calculations derivatives and integrals like, in your head. Like you're going to have a calculator. Especially if you want to do real-world problems." They also contrasted their calculator experience in calculus with that in physics and chemistry, where calculators are used all the time. This positioned mathematics for them as outside the realm of connection with other disciplines.

When asked about the extent to which they felt they had formed bonds or connections with their classmates, the three students agreed that any relationships they formed were *not* the result of how class was structured or due to any effort on the part of their instructors. Instead, those that they do homework with are either friends or live in the same residence hall. Their ability to work with a wide range of students from different lectures was made possible because TLU has tightly coordinated curriculum, homework, and assessments. As these three students explained, "there's a lot of behind the scenes learning from kids explaining, or students explaining stuff to one another" and "there's a lot of, frankly, bonding over freaking out." Thus, at a system level, the course coordination allowed for considerable peer to peer bonding that otherwise might not have happened and allowed these three students to refigure their relational identities as helping residence hallmates survive calculus.

Discussion

Given the exploratory nature of this work we did not posit any hypothesis regarding how the different course variations would impact student mathematical identities, and instead our aim was to capture the salient features described by the students and how those related to their beliefs about knowing and doing mathematics. The enactment of these figured worlds considers the totality of the lived experience such as the role of the instructor, calculator policies, discipline-based problems, and the structures surrounding entry and pathways into the courses. These elements cannot be separated since they are fundamentally tied together. For instance, instructors for the calculus for life sciences are selected knowing the course should emphasize mathematical understanding and are aware that most of the students have had negative experiences with mathematics prior to starting the course. This results in assigning instructors who often are more student-centered in their teaching approach.

Comparing across the three variations in the study, we can see the ways in which the four characteristics of the figured world, as previously defined in the theoretical background section, contributed to the productive mathematical identity, mostly for students in the calculus for life sciences and the honors calculus. For instance regarding the first characteristic, the ways in which students were recruited or enter into the figured world, this helped promote a sense of academic-excellence and equivalency among the students in the honors sections because they had been invited to enroll in the course as part of the honors program as compared to students in the standard course which were “forced” to take the course as part of a degree requirement. Looking at the second characteristic, one can see how the social interactions and positionality of the instructor, Dr. B, for the life science students gave meaning to their sense of being cared for as a calculus student, allowing them to align and refigure themselves with positive mathematical interactions. Over the course of the semester, students in each of the three variations were socially organized, in some cases as peers actively supporting one another, and others as students living in the same residence working to survive in the figured world. Additionally, the fourth characteristic arose during students’ understanding of familiar social types and having a strong individual sense of self that was related to a discipline identity such as an engineer in the standard course or a zoologist in the life science that positioned the use of calculus as ancillary to their pursuits.

The course variations at TLU served as figured worlds for the students that seemed to impact their mathematical identity. The role of the instructor to either express care for their learning, to encourage peer collaboration, or to lecture the material at a quick pace was a paramount factor in how the students described their beliefs about being able to learn and do mathematics. The way the instructors approached teaching we speculate is tied with the programmatic features of the course variation. Whereby the standard course is content heavy and puts pressure on instructors to cover the material through lecture, the honors course has more contact hours and was designed with collaborative labs, and the life science course focus on understanding and less on computation which promotes instructor inquiry into students thinking.

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What Are You Looking At? Shape Thinking and Eye-Tracking

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Previous research has illuminated and defined meanings and understandings that students demonstrate when reasoning about graphical images. This study used verbal and physical cues to classify students' reasoning as either static or emergent thinking. Eye-tracking software provided further insight into precisely what students were attending to when reasoning about these graphical images. Eye-tracking results, such as eye movements, switches between depictions of relevant quantities, and total time spent on attending to attributes of the graph depicting quantities, were used to uncover patterns that emerged within groups of students that exhibited similar in-the-moment meanings and understandings. Results indicate that eye-tracking data supports previously defined verbal and physical indicators of students' ways of reasoning, and can document a change in attention for participants whose ways of reasoning over the course of a task change.

Key Words: Static Shape and Emergent Shape Thinking, Quantity, Covariation, Eye-Tracking

Many students entering calculus have been indoctrinated into a rule-based mathematics that uses rote memorization, but this can lead to struggles when students face problems that contain dynamic phenomena. To move students away from merely using memorized methods, they must be provided with tasks which require them to reason about individual quantities and how two quantities' magnitudes vary over time (Moore & Thompson, 2015; Stevens & Moore, 2017; Thompson, 2011).

In the past, researchers have been restricted to categorizing students' meanings and understanding of concepts based on verbal and physical cues. Recently, researchers have started using eye-tracking technology, which allows the addition of visual cues by tracking student fixations while they reason about tasks (i.e. Alcock, Hodds, Roy, & Inglis, 2015). Although eye-tracking studies have been conducted in undergraduate mathematics education, those studies have focused on the use of static images, such as describing how experts and novices read proofs (i.e. Alcock et al., 2015).

There has been much research on how students reason quantitatively and covariationally (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Monk, 1992; Moore & Carlson, 2012; Moore & Thompson, 2015). This research aimed to delve deeper into past results by re-creating quantitative and covariational graphical interview tasks synced with eye-tracking. The research questions addressed in our study are:

1. How do students fixate on various graphical attributes depicting quantities relevant to the corresponding task?
2. Is there a relationship between students' fixation patterns and observed meanings evidencing static or emergent shape thinking?

Background

Thompson, Hatfield, Yoon, Joshua, & Byerley (2017) presented statistical data about U.S. high school performance when asked to create a trace of a graph while watching an animation depicting covarying magnitudes (see Figure 1c). Only 23% of U.S. high school teachers were able to at least create a semi-accurate trace of the graph. They reported a high correlation ($p < .0001$) between the creation of the correct initial point and providing an accurate graph. Thompson (2017) noted that a potential limitation was that teachers could not simultaneously look at the animation and their paper on which they were sketching their graph.

Thompson et al. (2017) revealed problems, but insight into student reasoning about graphing was minimal. Moore and Thompson (2015) leveraged Piaget's notions of figurative and operative thought together with quantitative and covariational reasoning to better describe how students reason about graphs. Static shape thinking, according to Moore and Thompson (2015), is defined as seeing a graph "as an object in and of itself, essentially treating a graph as a piece of wire (graph-as-wire)" (p. 784). If a student interprets a graphical representation as "graph-as-wire," they see the wire as an entire unit with no individual components (multiplicative objects) making up the wire. Equations, function names and rules are "facts of shape" (p. 785). It is important to note that static shape thinking often suffices to evaluate procedural type problems. For example, memorizing shapes and rules, such as the first few terms of a Taylor series, can be a productive way to avoid re-inventing the wheel each time a new problem is presented (Martin & Thomas, 2017). However, static shape thinking becomes a problem when it inhibits a student's ability to reason about and conceive of the various aspects involved in dynamic graphical images.

In contrast to static shape thinking, emergent shape thinking "involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (Moore and Thompson, 2015, p. 785). This mode of thinking is rooted in students' abilities to reason in terms of quantities (quantitative reasoning) and how those quantities vary in tandem (covariational reasoning). By *quantity,* we are referring to a cognitive construct of a measurable attribute of an object or phenomenon (Thompson, 1994; Thompson, 2011). It is important to note that this type of reasoning is not an inherent feature of a situation; just because a student is immersed in a dynamic task that may seem to beg for covariational reasoning does *not* mean that the student will *conceive* of the situation in terms of covarying quantities.

Methods

Overall Study Design

Eleven student volunteers were asked to participate in two task-based, semi-structured, clinical interviews (Goldin, 2000) lasting no longer than two total hours. Since the prior mathematical knowledge of participants varied, the total amount of time to complete all tasks varied. Anticipated course grades were no lower than a C average for any participant, and students not recommended by their instructors based on inability to communicate were also not contacted. In total, eleven students completed twelve tasks presented on a computer monitor. For the purpose of this report we focus on three tasks presented in Figures 1a, 1b and 1c.

An over-the-shoulder camera captured students' note-taking and gestures. Tobii eye-tracking software (Tobii, 2018), collected eye fixation data. Audio from these sources was used to sync the camera and eye-tracking videos. Key moments were transcribed, including verbal utterances and relevant gestures. Raw data from the eye-tracking software consisted of coordinate points indicating participants' visual attention to specific locations on the monitor associated with a timestamp.

Areas of Interest (AOIs)

Areas of Interest (AOIs) were constructed prior to the interviews (see individual task protocol below for tasks and their corresponding AOIs). AOIs were not visible to the participants. Many AOIs were created to include single attributes depicted on the graph that students could conceive as relevant quantities. Task 1, for example, prompted students to identify *x* segment associated with a point, so the AOIs covered *x* attributes, *y* attributes, the point itself, and so on (see Figure 1d). Defining AOIs allowed researchers to collect information regarding eye movement, such as switches. A switch between AOIs is counted each time a student's fixation moves from one AOI to another provided that the student fixated within the second AOI within 0.5 seconds of their fixation leaving the first AOI. If the student fixated within one AOI, fixated within a second AOI in less than 0.5 seconds after leaving the first AOI, and then fixated within a third AOI in less than 0.5 seconds after leaving the second AOI, then that was counted as two switches, one switch from the first to second AOIs and another from the second to third AOIs.

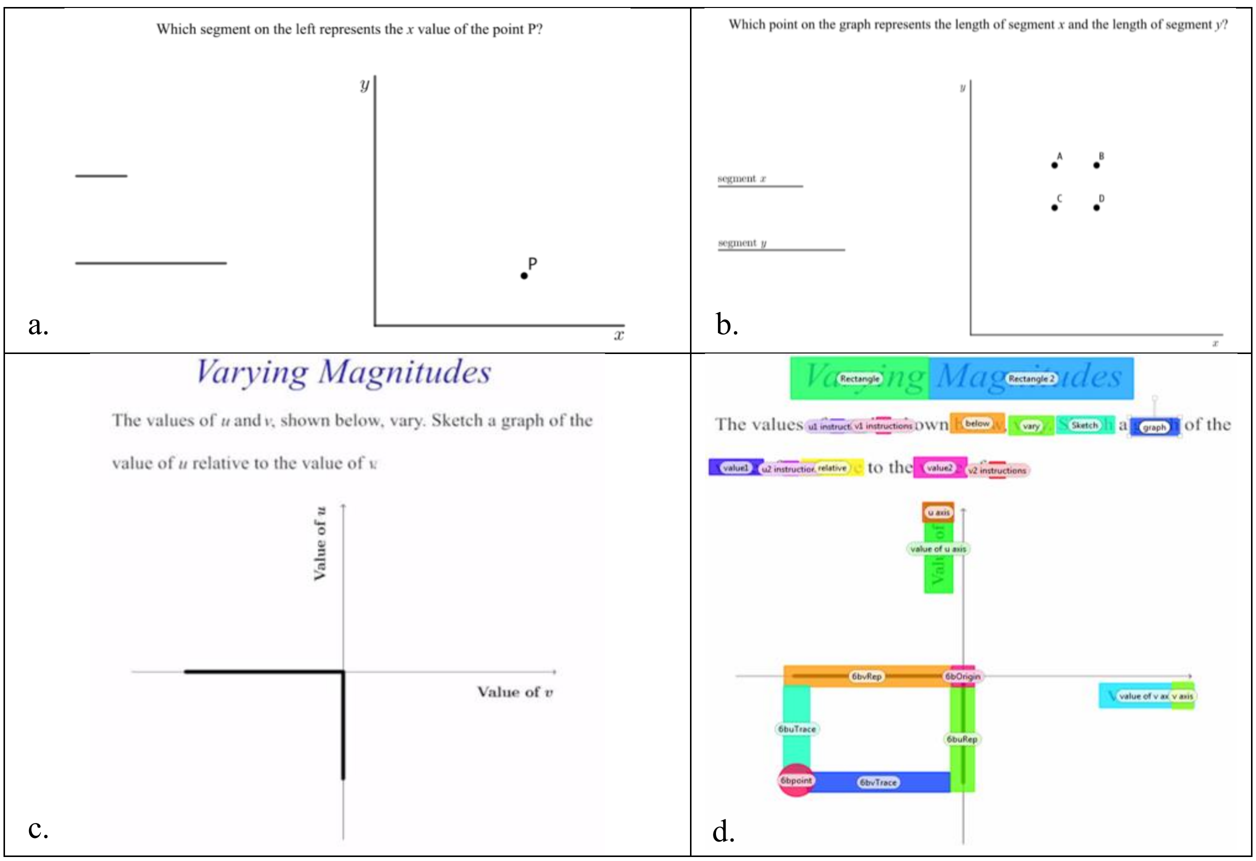

Figure 1. Screenshots of Tasks 1, 2, 6, and AOIs defined for Task 6.

Analysis and Coding

In Tasks 1 and 2 (Figure 1a and 1b, respectively), students were marked as correct if they indicated the bottom segment as corresponding to the *x*-value of point P, and indicated point C as representative of the two segments, respectively. Results for Task 6 were only coded as correct (see Figure 1c) if the students created a graph that closely resembled the correct trace (correct number of maximums, minimums, correct placement of initial point).

Potential indicators for quantitative reasoning included verbal and physical cues. Verbal cues for quantitative reasoning included words such as "length" or "distance." Gesturing with hands, such as spreading out thumb and index finger over a depicted segment, was also coded as a potential indicator of quantitative reasoning. Although it was technically possible that such a gesture might merely indicate visually transferring a line segment without explicit reference to measurement, such gestures were frequently paired with the participant acknowledging the "length" of the segment.

Indicators for quantitative reasoning coincide with emergent shape thinking. The lack of these indicators can be indicative of static shape thinking. In addition, the data was also coded for students' use of named shapes, such as "quadratic" as meaning graphs that increase and decrease. Finally, these potential indicators for static and emergent shape thinking were compared to eye-tracking fixation data (pulled from AOIs shown in Figure 1d) to determine if potential patterns emerged within different ways of thinking.

Results

Table 1 shows general results for each participant. Of the six participants who correctly responded to both Tasks 1 and 2, only P08 was unable to create an accurate trace in Task 6. This data supports Thompson's (2017) conclusions that a participant who correctly plots the initial point for Task 6, indicated by a participant's ability to correctly interpret and create a point in Tasks 1 and 2, respectively, is more likely to create an accurate graph.

Table 1
Overall Results for Correctness and Indicators of Potential Quantitative Reasoning (QR)

Participant	Task 1 Correct	QR	Task 2 Correct	QR	Task 6 Correct
P01	✓				
P02	✓		✓	✓	✓
P03			✓	✓	✓
P04	✓	✓		✓	✓
P05	✓			✓	
P06	✓	✓		✓	
P07	✓		✓	✓	✓
P08	✓	✓	✓	✓	
P09	✓	✓	✓	✓	✓
P10	✓	✓	✓	✓	✓
P11	✓	✓	✓	✓	✓

Neglecting Depictions of Quantities

P03 was the only participant who answered incorrectly on Task 1 but answered correctly on both Tasks 2 and 6. Eye-tracking data (Figure 2) offers a possible explanation. P03 incorrectly interpreted the task, which led to the choice of the top segment and the *y*

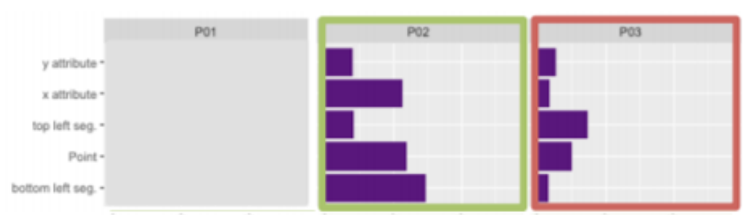

Figure 2. Total participant time on selected AOIs in seconds for Task 1.

attribute of the graph. As expected, Figure 2 indicates a lack of attention by P03 to attributes of the graph corresponding to the *x*-component of the point when compared to attributes of the *y*-component of the point. It is also apparent that almost all the remaining participants, similar to P02 shown, attended to the *x* attributes for a larger amount of time.

Figure 3 gives the number of switches between AOIs. It is evident that few switches from one *x* attribute of the graph to another occurred for P03. Figure 3 shows exactly this - very few switches other than from the top left segment to the point. In comparison, P02 indicates a satisfactory number of switches between relevant *x* quantities.

Figure 3. Task 1 switch count.

Changing Fixation Patterns

Eye-Tracking data also shows variance in participant's attention to graphical in real-time as their ways of reasoning about an image changed. For Task 2, we present an example of a fixation timeline produced by P07.

Figure 4. Fixation timeline for P07 during Task 2.

For Task 2, P07 had a large gap in his fixation timeline, with drastic difference in the color schemes before and after. The gap for P07 indicates that he was not fixated within any defined AOI. Video data indicated that he was attending to the interviewer while she added the word *relationship* to the task ("Which segment on the graph represents the *relationship* between the length of segment *x* and the length of segment *y*"). The color scheme on the left side shows P07 looking back and forth between segments *x* and y, then back and forth between x attributes (horizontal aspects of the graph) and y attributes (vertical aspects of the graph). The change in color scheme for the latter half of the fixation timeline shows attention to relevant aspects of the graph in a more meaningful order (segment *x* to *x* attributes and segment *y* to *y* attributes). His initial lack of awareness of coordination of quantities was resolved by the insertion of the word relationship in the task instructions. A look at P07's dialogue during the gap confirms this change in reasoning:

> I: Which point on the graph represents the relationship between the length of segment *x* and the length of segment *y*.
> P07: I'd say C because if you take the *y* segment and you match it up right here [right index at C, right thumb on *x* axis below C] it would be about that length and since the *x* is shorter it would probably be about at C [right index finger at C, right thumb on the projection of C onto the *y* axis].

When considering Task 6, P01 was incorrect, as shown by the red border in Figure 5. P02 was correct (green border in Figure 5). P03's total time spent on AOIs (yellow border in Figure 5) look very similar to P01's total time spent on AOIs. Yet, P03 was correct in his response to Task 6.

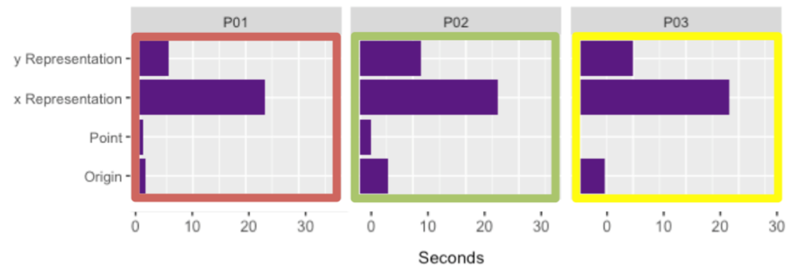

Figure 5. Total time spent on relevant AOIs for Task 6.

Unlike the total participant time, the switch count for Task 6 (Figure 6) provides two *very* different results for P01 and P03. Although the two participants spent a similar total amount of time on the *x* and *y* representations, we see from the switch count that P03 was actively switching between the AOIs (12 times) while P01 only made *one* switch between representations.

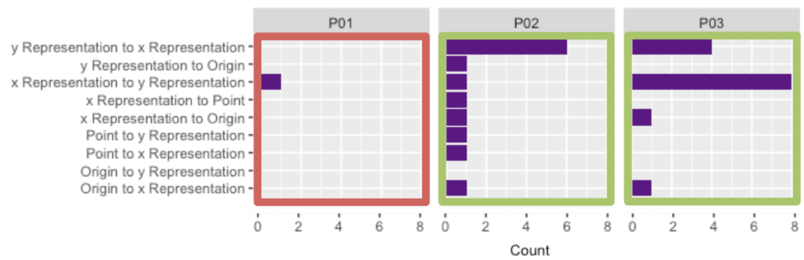

Figure 6. Switch count between relevant AOIs for Task 6.

Transition from Static Shape Thinking to Emergent Shape Thinking

During Task 6, P02's verbal and physical cues combined with his eye-tracking data yielded results evident of a possible transition from static to more emergent ways of reasoning.

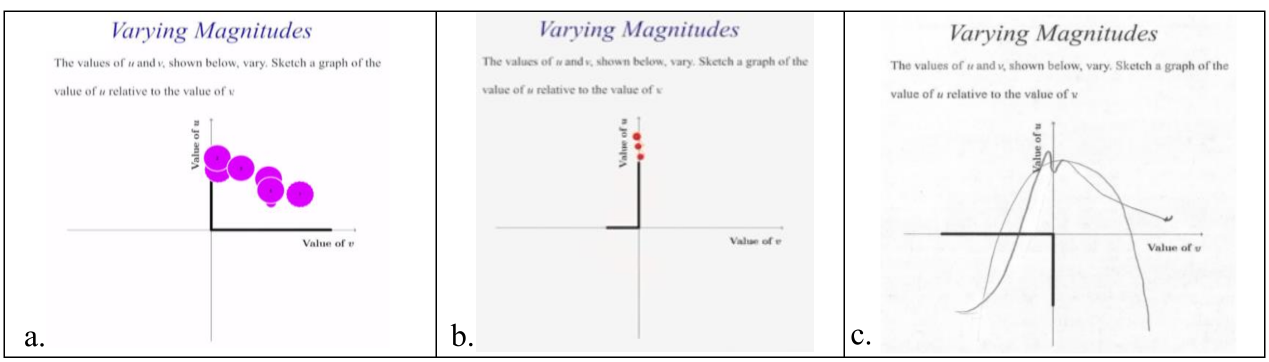

Figure 7. Screenshots of P02's fixation patterns, a. and b., and attempted Graphs in c.

P02's eye-tracking indicated that he was at moments following the moving point location resulting from the coordination of at least the endpoints of *u* and *v* line segments (see Figure 7a). Yet P02's dialogue indicated that the reasoning upon which he based his initial graph was more static in nature. "I think what it is is they… It's about like this [drawing parabola in Figure 7c] if we were to continue on as it would go on. I think it's just an upside down parabola, so *y* equals negative *x* squared is what I think…" He then drew the concave down parabola in Figure 5c.

A few moments later, however, P02 begin to follow the moving point location on the screen with his pencil. While P02 was still trying to attend to the perceived point created by *u* and *v*, he attended more so to the vertical attribute of *u* than before (Figure 7b). After making multiple up and down movements with the pencil on screen, the participant decide that his parabola was insufficient, and draw the more accurate image in Figure 7c over it.

Discussion

Conclusions

Participants' fixation counts alone were not necessarily indicative of whether they correctly interpreted an image, nor were they indicative of the ways of reasoning in which they were engaged. When paired with switch counts, as was the case when comparing P02 and P03, an ability to switch fixations between graphical attributes depicting quantities relevant to Task 1 appears to be related to the participants' ability to reason correctly about the task. Participants who correctly answered a given task generally had a higher volume of switch recordings, indicating a greater attention to the relationship between quantities represented in the task.

P07's timeline for Task 2 demonstrates an instance where the individual entered a state of disequilibrium through verbal cues that caused a change in fixation to relevant quantities and the relationships between them. Over the course of Task 6's animation, P02 engaged in static shape thinking to initially conceive of the shape of the graph. Even though he had attended to some variation, as demonstrated by the eye-tracking, he proceeded to assign a specific function to the graph, a parabola in this case. We anticipated that students who indicated emergent shape thinking might fixate on a moving point location resulting from the coordination of varying the values of the lengths of u and v. Yet, P02's shift to emergent shape thinking during Task 6, resulted after his fixations had transitioned primarily the value of u. When attending to the variation in u he was able to hold in mind the variation of v and eventually produce a more accurate trace of the graph. This demonstrates that students need not continually switch back and forth between varying depictions of quantities to successfully engage in emergent shape thinking.

However, the low switch count for P03 in Task 6 (Figure 6) shows that although the participant was attending to relevant quantities for an extended period of time (Figure 5), he was not actively moving his attention between the quantities. P02 did indeed switch (see Figure 6) and apparently engaged in enough relevant switching for him to produce an accurate graph with reduced switching while graphing. But, a lack of switch counts for students may be indicative of a lack of attention to the coordination of quantities.

Future Work

Eye-tracking software is a new tool that is emerging in mathematics education literature, which leaves a wide range of possibilities for further research on the aspects discussed in this study. One limitation of this study is that results need not generalize to other students, and therefore, a larger sample size is needed.

Eye-tracking results can also be used to develop instructional videos or tasks that better equip students to reason in terms of quantities and dynamic situations. Currently, a research team is working on an NSF funded project that is using eye-tracking to investigate how students are attending to the videos (see acknowledgment; calcvids.org).

Acknowledgements

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Student Reasoning about Eigenvectors and Eigenvalues from a Resources Perspective

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Eigentheory is an important concept for modeling quantum mechanical systems. The focus of the research presented is physics students' reasoning about eigenvectors and eigenvalues as they transition from linear algebra into quantum mechanics. Interviews were conducted at the beginning of the semester with 17 students at two different universities' during the first week of a quantum mechanics course. Interview responses were analyzed using a Resources (Hammer, 2000) framework, which allowed us to characterize nuances in how students understand various aspects of an eigentheory problem. We share three subthemes of results to illustrate this: interpreting the equations graphically, interpreting the equals sign, and determining solutions.

Key words: Linear algebra, Eigentheory, Resources, Physics, Student Understanding

In 2012, the National Research Council's DBER report stated, "The United States faces a great imperative to improve undergraduate science and engineering education" and advocated for more interdisciplinary studies to explore "crosscutting concepts ... and structural or conceptual similarities that underlie discipline-specific problems" (p. 202). In *Project LinAl-P* (NSF-DUE 1452889) we pursue research in this vein by investigating how students reason about and symbolize eigentheory in linear algebra and in quantum physics. For this paper, we explore the following research question: What ways of reasoning about eigenvectors and eigenvalues of real 2x2 matrices exist for physics students at the beginning of a quantum mechanics course?

Literature Review

Research on students' understanding of eigentheory has grown over the past decade, and it provides several insights into the complexity of the topic, students' sophisticated ways of reasoning, and pedagogical suggestions for overcoming the challenges students face. Thomas and Stewart (2011) noted students' difficulty with and need to understand both how matrix multiplication and scalar multiplication on the two sides of the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ yield the same result and how inserting the identity matrix is necessary when symbolically transforming $A\boldsymbol{x} = \lambda\boldsymbol{x}$ into the homogeneous equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. They also advocate for instructors to help their students develop a graphical conception of eigenvectors and eigenvalues, something they noted was weak in their study participants. Gol Tabaghi and Sinclair (2013) investigated students' visual and kinesthetic understanding of eigenvector and eigenvalue. The authors analyzed the results in terms of Sierpinska's (2000) modes of reasoning, finding that students' work with the sketch and their interaction with the interviewer promoted the students' flexibility between the synthetic-geometric and the analytic-arithmetic modes of reasoning.

Henderson, Rasmussen, Sweeney, Wawro, and Zandieh (2010) illustrated, prior to any instruction on eigentheory, various ways that students interpreted $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ (see Figure 1 part (a)). The authors parsed students' activity through their symbol sense, noting if they conducted superficial algebraic cancellation to conclude that $A = 2$ or if they interpreted the equals sign as a signifier of balanced results. The authors also found that of the students that were about to find the solution given a specific A, only some were able to interpret their results. This may relate to Harel's (2000) suggestion that the interpretation of "solution" in a matrix equation, the set of all vectors $\boldsymbol{x}$ that make the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ true, entails a new level of complexity than does solving equations such as $cx = d$ (where c and d are real numbers). Finally, in physics education research, Dreyfus, Elby, Gupta, and Sohr (2017) examined students' attempt to reconstruct the time-independent Schrödinger equation. The researchers focused on the relationship between the symbolic forms for eigentheory and the various meanings they could imbue for students, noting "parsing the conceptual meaning of mathematical expressions and equations can play a key role in mathematical sense-making" (p. 11). These particular aspects – the meaning of symbols and the objects they represent, graphical interpretations, and interpreting solutions – are all particularly relevant for our present study and help inform our analysis.

Theoretical Framework

To operationalize the research question, we assume a theoretical stance consistent with what Elby (2000) calls "fine-grained constructivism" in which "much of students' intuitive knowledge consists of loosely connected, often inarticulate minigeneralizations and other knowledge elements, the activation of which depends heavily on context" (p. 481). This is consistent with the Knowledge in Pieces theoretical framework (diSessa, 1993), which utilizes an assumption that students' intuitively held knowledge pieces are productive in some context. To conduct research on student understanding consistent with this theory, we characterize students' cognitive resources (Hammer, 2000) that are utilized when they engage in activity related to eigentheory in quantum physics. Sabella and Redish (2007) defined a resource as "a basic cognitive network that represents an element of student knowledge or a set of knowledge elements that the student tends to consistently activate together" (p. 1018). Resources are activated depending on how individuals frame a given situation, that is, how an individual unconsciously interprets what is happening around them (Hammer, Elby, Scherr, & Redish, 2005). Individuals may sometimes have the resources needed to solve a given problem but fail to activate them, activating instead other less-productive resources. However, all "resources are useful in some contexts, or they would not exist as resources" (Redish & Vicentini, 2004). Resources can be linked to other resources, in which activation of one resource can promote or demote activation of others. Furthermore, resources may internally consist of finer-grained resources linked in a particular structure (Hammer et al., 2005; Sayre & Wittmann, 2008). In our research, we seek to identify resources that characterize the knowledge elements quantum physics students activated when reasoning about eigenvectors and eigenvalues of a real 2x2 matrix.

Methods

The data consist of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 17 students enrolled in a quantum mechanics course. Nine were from a junior-level course at a large public research university in the northwest US (school A), and eight were in a senior-level course at a medium public research university in the northeast US (school C). Student pseudonyms are "A#" or "C#." Interviews occurred during the first week of the course, and questions aimed to elicit student understanding of several linear algebra concepts which they would use in the quantum mechanics course.

For this paper, we focus on students' reasoning on one particular interview question. There were additional follow-ups to check that the interviewer understood the students' points, but below the five main prompts to the question are in Figure 1. Parts (a)-(c) were introduced in Henderson et al. (2010) in their research on student thinking prior to any formal instruction on eigentheory. Because linear algebra was a prerequisite for the quantum mechanics courses in which our participants were enrolled, we knew they would have been exposed to eigentheory prior to the interview. By design, the terms "eigenvector" and "eigenvalue" did not appear until part (e); many students, however, immediately recognized the equation in (a) and brought up eigentheory ideas on their own in their responses to (a)-(e).

(a) Consider a 2×2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. How do you think about $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?
(b) Do you have a geometric or graphical way to think about this equation?
(c) How do you think about what the equals sign means when you see it written in the context of this equation?
(d) Now suppose that $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Now how do you think about $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$? What values of x and y would make the equation true?
(e) [If they hadn't already] Again consider the matrix A = $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Determine the eigenvalues and eigenvectors of A.

Figure 1. The main interview question prompts for the analyzed data.

Analysis was done through an iterative process of individual coding, group discussion, and codebook development. First, three members of the research team individually coded (Miles, Huberman, & Saldaña, 2014) transcripts of the student interviews, specifically assigning codes for what each researcher felt represented evidence of a student's resources that were activated as they answered the eigentheory questions. Next, researchers discussed their individual codes noting the specific evidence within the transcript used to mark that code. These codes and the evidence that identify them were extensively discussed, refined and solidified. Based on these discussions, a coding book was developed with labels and descriptions of the agreed-upon resources; this three-step process was repeated until the coding book was sufficient to characterize the thinking of all seventeen students. Finally, we individually coded each transcript one more time, achieving a high level of interrater reliability using the final codebook.

Results

In total, our analysis of student reasoning about eigentheory in this interview led to the identification of over 50 resources. When considered in small subsets or in total, these resources allow us to characterize nuances in how students understand various aspects of the concepts involved in an eigentheory problem. We share three subthemes of results to illustrate this: interpreting the equations graphically, interpreting the equals sign, and determining solutions.

Interpreting the Equations Graphically

Student responses to part (b) demonstrated a wide variety of ideas that are shown in Figure 2. Common among them were ideas about a matrix acting on one of its associated eigenvectors and scaling the eigenvector, while others stated that the matrix acts on one of its associated eigenvectors and stretches the eigenvector. Some students made comments about eigenvectors being "Anything on this line" referring to the line of the eigenspace defined by the eigenvector. Some students described that an eigenvector when acted on by its associated matrix provides a resultant vector on the same line.

Intriguingly, when students were first asked about geometric and graphical interpretations in part (b), nine students drew vectors on a plane and discussed the meaning of the eigenvector, the associated matrix, and eigenvalue; however, after giving students an explicit 2x2 matrix in part (d), five additional students engaged in this activity; we coded this as activating the *Vector Graphing* resource. Ultimately, 14 of 16 students that were asked this question activated a resource that connects the idea of an eigenvector, eigenvalue, and associated matrix with a 2-D plot of vectors. Having a specific example of a two-by-two matrix, and determining that a solution to the system of equations is also a solution to the eigenequations (which 14 students were able to do) seems to trigger this graphical drawing resource for an additional five students. Although most students activate geometric/graphical ways of thinking about the eigenequation, different students require different support and feedback to activate this idea.

As an example, responding to part (b), C3 said, “Not really because...I know. No. I wouldn't really say so. [*after another prompt from the interviewer*] Well I think about an eigenvector as this being a vector that when multiplied by something stays along the same path.” The student states that they don’t have any graphical or geometric way of thinking about this problem but eventually states an idea that we would code as Evec-Line.

After completing part (e) the interviewer asks: “Ok. Umm. Now that you have like actual numbers for A or numbers for that the vectors do you have any additional graphical or geometric ways you think about it?”

C3: “Umm... No. Not really. I would just say [draws two coordinate axes] that u 1 would look -- I really wouldn't, I wouldn't really think about it like this but I would say that u 1 looks something like this [draws vector into fourth quadrant] and u 2 looks something like this [draw another vector into second quadrant collinear with first vector].

Despite the insistence that the student “wouldn’t really think about it like this,” the student provides a clear vector graph consistent with the eigenvectors for the matrix.

Resource	*Resource Description*	*# activating resource*
Vector Graphing	Student talks about vectors as arrows on a Cartesian plane, or actually draws a graph with vectors or an “eigenline” on it.	14
Val-Stretch	Mentions that in an equation of the form A[x;y]=k[x;y], [x;y] is stretched by k. This captures any of the more geometric ideas like dilation, longer, etc.	8
Val-Scale	Mentions that in an equation of the form A[x;y]=k[x;y], [x;y] becomes k times that vector or is scaled by k. This captures any of the more algebraic ideas.	10
Evec-Line	Student explains some version of that an eigenvector of *A* lies along the same line or goes in the same direction after being acted on by *A*	6
Evec-Eigenspace	Student explains that vectors "in the same direction as" or "on the same plane as" or "the same line" as other eigenvectors of *A* would also be eigenvectors.	5

Figure 2. List of resources most related to graphical interpretations of the eigenequation.

Interpreting the Equals Sign

The seven main resources that were activated in response to part (c) are listed and defined in Figure 3. Although these resources were grounded in and grew from our data, our familiarity with the literature allowed us to notice when our students’ reasoning was consistent with a way of reasoning already documented in the literature. For instance, *RU*= and *OU*= are used to characterize student responses that seem to stem from either a relational or operational understanding of the equal sign. The terms “operational” and “relational” were used by Knuth, Stephens, McNeil, and Alibali (2006) to categorize students’ explanations of what the equal sign means (see also Behr, Erlwanger, & Nichols, 1980; Carpenter, Franke, & Levi, 1999; Kieren, 1981); those with an operational understanding view the equal sign as a signal to “compute” or “give the answer,” while those with a relational understanding view the equal sign as indicating a relation between the two sides of the equation, with one side being “the same as” the other. An example of *OU*= from our data is C5’s statement: “I think about it in terms of eigenvalue, I'm saying that with this matrix there is some eigenvalue that solves, that there is some unique value that corresponds to matrix *A* that solves this equation.” An example of *RU*= is below with C7.

The resources *Algebraic Cancellation* and *SOSE* are closely related. The former was introduced in Henderson et al. (2010), who used this term to describe overgeneralizing the notion of algebraic simplification to "cancel" the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ from both sides of the equation in part (a) and then trying to make sense of how the matrix A could equal the number 2. The resource *SOSE* characterizes student efforts to find a way to turn the matrix A into the number 2. Finally, the resources *PV-mult* and *OV-mult* are used to characterize student thinking that centrally considers the operations on either side of the equal sign and/or the resulting objects. The resource *PV-mult* indicates a student response fixated on matrix and scalar multiplication being different processes, whereas *OV-mult* indicates a student response highlighting that the result of matrix and scalar multiplication is the same object. We chose these resource names as a reference to the work by Thomas and Stewart (2011) who used the term "process-object obstacle" to describe "how the two sides of the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ represent different mathematical processes that have to be encapsulated to give equivalent mathematical objects" (p. 280).

Resource	*Resource Description*
RU =	Relational Understanding of Equal Sign means that entities on both sides of the equation must be "the same"
OU =	Operational Understanding of Equal Sign is a call to "do something" such as solve an equation or "compute."
SOSE	[Things have to be the Same Object to have the Same Effect] For the equation to make sense, there has to be a way to turn the matrix A into the number 2.
Structural Features	Student discusses the structure that the objects in the equations have. This often entails discussing or comparing one or more of the entities in the equation.
Algebraic Cancellation	If the same thing is on both sides of an equation in a structurally similar way, it is permissible "cancel" those things out of the equation.
PV - mult	[Process view of matrix and scalar multiplication] Student focuses on matrix multiplication being different from scalar multiplication (the student focus is on the operation).
OV - mult	[Object view" of matrix and scalar multiplication] Result of matrix multiplication and scalar multiplication is the same object (the student focus is on the objects created by the operation).

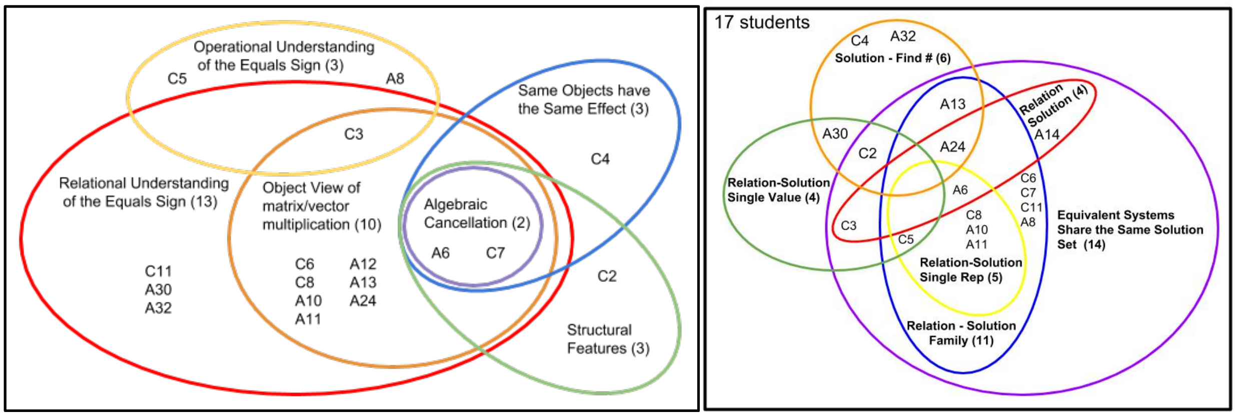
Figure 3. List of resources most related to interpreting the equals sign in $A\boldsymbol{x} = 2\boldsymbol{x}$.

Figure 4a illustrates how the various resource codes loaded across the 17 student responses in part (c) (only one student, A6, activated *PV-mult*, so it is not in Figure 2a). We note that this question is often difficult for students; some find it hard to describe their understanding without using the word "equals" (which they were prompted to do if needed), and many seem to be figuring it out as they respond. This latter aspect can be seen in students' responses such as C7, whose explanation was coded with 5 of the 7 resources in Table 2.

C7: Well it's weird cause it almost seems like A equals 2. You know what I mean? Like A has to equal to 2 for this to be equivalent, but A is not equal to 2. A is a matrix. So, that's what- I never thought of that but A does not equal 2. A is equal to a 2 by 2 matrix [draws brackets for a matrix]…which is not equal to 2 but it's like...The A on its, on its own does not equal 2 but the A operating on xy does equal 2 times xy. So, this group together [circles LHS in equation of problem statement] is equal to this group together [circles RHS]…But when you say, 'oh lets, let's divide both sides by xy vector' [makes air quotes]. That doesn't make sense linearly, I don't think. But, you- intuitively a lot of the time I guess in algebra- from algebra experience, you'd think A matrix is equal to 2.

The first four lines of C7's response, when he grappled with how to reconcile that A can't equal 2 even though it seems like it does, was coded with *SOSE* and *Structural Features*. He moved towards resolving this with the statement "the A on its, on its own does not equal 2 but the A

operating on xy does equal 2 times xy," which was coded with *OV-mult*, and by stating the two groups on either side of the equation were equal, which was coded with *RU*=. His conclusion, which brings up dividing both sides by a vector and how that is sensible in algebra, was coded as *Algebraic Cancellation.* We note that it is most likely the case, based on his activation of *OV-mult* and *RU*=, that C7 was confident that $A \neq 2$; however, we still also code his response with *Algebraic Cancellation* because this resource was activated for C7 during his thought process.



(a) (b)

Figure 4. Venn Diagram of main resources activated by students in Part (c) and in Part (d), respectively.

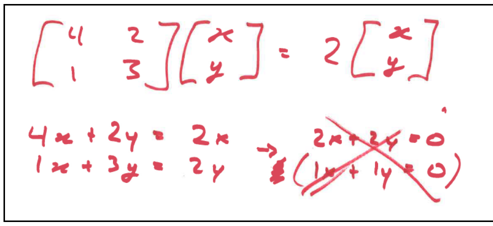
Determining Solutions to the Matrix Equation

In response to part (d), students activated six main resources to make sense of the solutions to the matrix equation, with some resources being more productive than others. We share these in Figure 5 and summarize the resource activation by the 17 students in Figure 4b.

Resource	*Resource Description*
Solution-Finds #	In a system of equations, the solution should be a single number for each variable.
ESS	Algebraically equivalent equations or systems of equations share the same solution set.
Relation-Solution	A relationship can define what values for the unknowns are solutions to the given equation(s).
Relation-Solution Single Rep	A single representative of a relationship can be used as a prototype or to check the solution.
Relation-Solution Single Value	A relationship that is a solution to a system of equations defines a single solution.
Relation-Solution Family	A relationship that is a solution to a system of equations defines an infinite number of possible solutions.

Figure 5. List of main resources for finding solutions to $\begin{bmatrix}4 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 2\begin{bmatrix}x\\ y\end{bmatrix}$.

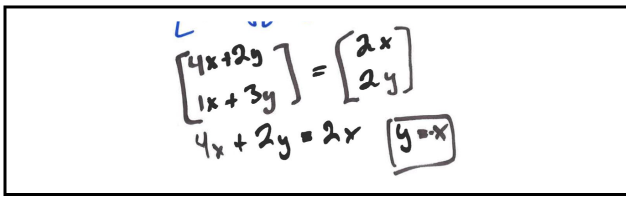
The resource *Solution-Find #* was activated by 6 students, implying they thought that the solution to the equation should result in single, specific numbers for both x and y. This was most often coupled with the students attempting to use the elimination or substitution methods for solving systems of equations. For example, consider C4's work and thoughts in Figure 6. C4 attempted to use the elimination method on the system of equations he had produced from the matrix equation but became stuck as the equations "cancelled" each other. In fact, C4, as well as A32, could not think of any other ways to approach the problem, and both were not able to find any solutions at all to the matrix equation.



"Could multiply that side by 2…no that doesn't work … I was thinking multiply that side, ya know, so that you'd get, so you could subtract one side from the other…But the fact that it's 2x + 2y = 0 for one equation 1x + 1y = 0 doesn't really. Ya know if I multiplied by 2 to cancel one of the variables and then subtract both variables are cancelled [crosses out system]. So obviously so that doesn't work in that in that sense."

Figure 6: C4's attempt to find solutions to $\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$.

In contrast, consider C3 who eventually realized that the equations define a relationship between x and y, which we coded as an activation of *Relation-Solution*: "Hmmm -- wait I think from here I can say that ... no ... What if I said -- So I could 2*y* equals minus 2*x* [writes 2*y* = -2*x*]. So now we're getting somewhere." While C3 did eventually realize the importance of this relationship, C3 was also one of the four students who activated the *Relation - Solution - Single Value* resource, thinking there should still only be one solution to the matrix equation, determined by the relationship. Another student who activated the *Relation - Solution - Single Value* resource, C2, recognized that he was trying to find an eigenvector, explained that eigenvectors must be normalized, and attempted to find this normalized vector. When the interviewer asked, "Is that the only one for $\lambda = 2$?" C2 replied, "Yes." We note this might be evidence that the *Relation - Solution - Single Value* resource could stem from students' nascent knowledge that quantum mechanical states (including eigenstates) are represented by normalized vectors due to the probabilistic nature of quantum mechanics.



C11: "Yeah. And since it's only *y* and *x*, I can just plug in any value of *y* and *x* that satisfies this equation [draws box around *y*= -*x*] and it will satisfy that same one [points to $\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$.]

Figure 7: C11's explanation of solutions to $\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$.

Impressively, 11 of the 17 students eventually concluded that the relationship $y = -x$ or $x = -y$ actually defines an infinite number of solutions, as any values of x and y which satisfy that relationship will be a solution to the matrix equation. For instance, consider C11's response in Figure 7. C11 also exemplifies an important resource that a large majority of the students (14 of the 17) activated as they worked through this problem, namely ESS. As students algebraically manipulated the matrix equation into other forms, it was notable most recognized that solutions to these new equations would also be solutions to the original matrix equation.

Conclusion

In this study, we identified a variety of resources that characterize students' thinking as they reasoned about eigenequations for 2x2 matrices during an interview at the start a course on quantum mechanics. The three themes presented here – reasoning about the equals sign, reasoning geometrically, and reasoning about solutions – represent a subset of the results that were obtained through our analysis. Our aim was to not identify incorrect reasoning but rather understand the various resources that students found useful at some point in the context of the interview question. Our analysis sheds light on both productive and occasionally unproductive resources for understanding eigentheory. These are helpful for instructors and curriculum developers to know so that they can help students build upon the common resources or seek to refine why certain resources aren't appropriate to activate in particular contexts.

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Instructor Perceptions of Using Primary Source Projects to Teach Undergraduate Mathematics Content

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This study investigates instructor perceptions of their teaching, as well as their students' learning, obstacles encountered, and methods of implementation from the use of Primary Source Projects (PSPs). PSPs are curricular modules designed to teach core mathematical topics from primary historical sources rather than from standard textbooks. In essence, they are a form of inquiry-based-learning that incorporates the history of mathematics through original sources. We provide an overview of results from two semesters of implementation reports and surveys administered at the beginning and end of the semester by instructors who implemented PSPs in their undergraduate mathematics class.

Keywords: Primary Source Projects, Inquiry-Based Learning, History of Mathematics

Background Introduction and Literature

Mathematics faculty and educational researchers are increasingly recognizing the value of the history of mathematics as an important means to support student learning. Primary sources have long been commonly used in teaching undergraduates in the humanities and social sciences (de Guzman, 2007; Klyve et al., 2011). Yet, while there has been some momentum for the use of primary sources to teach undergraduate mathematics, their use remains limited compared to other disciplines. Reading texts in which individuals first communicated their thinking offers an effective means of becoming mathematically educated in the broad sense of understanding both traditional and modern disciplinary methods (Fried, 2001; Laubenbacher et al., 2015). The use of original sources in the classroom promotes an enriched understanding of the subject, its creation, and its ongoing development for instructors as well as students (Jahnke, 2002; Jankvist, 2013).

Despite the benefits of primary source materials detailed above, and granting the wide availability of such materials via published collections and web resources (Calinger, 1995; Euler, 2015), there are significant challenges to incorporating primary sources directly into the classroom. Using secondary historical sources, such as (Katz, 1998), may suffice to reap some of the benefits of the original works. Yet the use of such sources carries its own difficulties, including the risk of placing too much emphasis on learning the history of mathematics per se, as opposed to using history to support the learning of undergraduate mathematics content.

One approach to addressing these issues is through Primary Source Projects (PSPs), which are curricular modules designed to teach core mathematical topics from primary historical sources rather than from standard textbooks. Each PSP is designed to cover its topic in about the same number of course days as classes would otherwise. With PSPs, rather than learning a set of ideas, definitions, and theorems from a modern textbook, students learn directly from mathematicians such as Leonhard Euler, Emmy Noether, or Georg Cantor. This distinction is crucial to PSPs: they are not designed to teach history; rather, they use history as a tool to better teach mathematics.

PSPs employ a selection of excerpts from primary historical sources that follows the discovery and evolution of the topic in question. Each PSP contains commentary about the historical author, the problem the author wished to solve, and information about how the subject has evolved over time. Exercises are woven throughout the project, requiring that students actively engage with the mathematics as they read and work through each excerpt. At appropriate junctures, students are also introduced to present-day notations and terminology and are asked to reflect on how modern definitions have evolved to capture key properties of solutions to problems posed in the past. Learning from the PSP via in-class activities and discussions replaces standard lectures and template blackboard calculations. PSP implementation helps promote more active learning via primary-source lessons, thereby making it an important form of inquiry-based learning.

Research Questions and Methods

To understand and evaluate the use of PSPs in the classroom, it is important to understand how teachers might implement them in their own classrooms, and how the implementation of PSPs may benefit teachers and students. This information is useful for educators who want to incorporate this new perspective in their teaching, and serves as an important contribution to the broader literature on inquiry-based learning.

To further explore these broader questions related to PSPs in the classroom, we recruited instructors of undergraduate mathematics to serve as "site-testers" through training and implementation of PSPs in their own classrooms. Teachers served as site-testers in either a fall or spring semester taking place over the course of an academic year. We surveyed teachers before and after the implementation period to further understand the efficacy of implementing these materials in real classroom contexts. We also gathered demographic information on teachers and asked them more broadly about their experiences implementing PSPs (the challenges they faced, the reaction of students, etc.). We aim to address several key questions designed to deepen our understanding of various aspects of faculty implementation of PSPs

1. Changes in Instructor Teaching Tendencies. How might the implementation of PSPs change instructors' perceptions about their own teaching behaviors and tendencies? How do instructors perceive/describe their implementation of PSPs as compared to their typical classroom teaching?

2. Instructors' Perception of the Impact of PSPs. Describe instructor's reported impacts of the implementation of PSPs on (1) perceptions of instructors concerning their students' knowledge of mathematics and its history, and perspectives and attitudes towards the subject; and (2) perceptions of instructors concerning the genre of their teaching and specific instructional practices.

3. Implementation of PSPs. Describe how PSPs are implemented, including modifications made to the PSP to meet the individual needs of their classrooms. What obstacles, if any, do instructors perceive to the successful implementation of PSPs?

Recruitment of Site-Testers

Site testers were recruited in a variety of ways. Seventeen of the site testers had attended a prior site-tester workshop, and we advertised using email listservs of groups likely to include people interested, such as the History of Mathematics Special Interest Group of the Mathematical Association of America (MAA), some geographic sections of the MAA, the MAA's Project New Experiences in Teaching (NExT), and the Americas Section of the International Study Group on the Relations between History and Pedagogy of Mathematics. Further recruiting was conducted through regional workshops, talks, and informal networks.

Data Collection and Analysis

We collected a variety of data from instructors before and after they implemented PSPs in their classrooms. Each of these data sources serves to address the specific research questions summarized above. Our data comes from four primary sources: the initial site-tester application, a pre-course survey, a post-course survey, and an implementation report from each PSP tested.

Pre-course survey. By the end of their first week of class, site testers completed a pre-course survey (through a series of Likert scale questions) that focused on instructors' perceptions of their own mathematics instruction (e.g. typical classroom structure, typical instructional goals, etc.), instructors' perception of their students (e.g. typical instructional assumptions made about their prospective students while lesson planning), and general descriptive information (e.g. professional rank, courses taught, etc.).

Post-course survey. During the last two weeks of their term, site testers completed a post-survey designed to gather information about instructors' perception of the effects on themselves of utilizing PSPs in the classroom (through a series of Likert scale questions), instructors' perception of the effects of utilizing PSPs in the classroom on the students and general information (e.g., which PSP was implemented, general classroom structure, etc.). The post-survey also contained a series of identical questions found in the pre-survey that targeted instructors' perceptions of their own mathematics instruction in order to assess any changes.

Implementation report. After the implementation of PSPs, instructors also completed implementation reports with a variety of open-ended questions that focused on the their experiences implementing PSPs in their classrooms.

To address Question 1 we first compared the identical items on pre- and post-surveys pertaining to teaching tendencies and behaviors by conducting a series of paired t-tests on each individual question. These comparisons will help reveal whether the implementation of PSPs had any influence on the types of teaching strategies that site testers use in their classrooms. Subsequently, addressing Question 2, we will examine the questions from the post-surveys that asked site-testers directly whether or not they believed the implementation of PSPs had any positive *impacts* on their own understanding/teaching of mathematics, as well as their students' learning. Finally, to address Question 3, we will provide a summary of the type of open-ended feedback instructors provided, along with some representative examples.

Results and Discussion

Results from the pre-course surveys show that 35 participants responded to the Fall 2017 surveys and 25 participants responded to the Spring 2018 surveys; 9 people site-tested in both semesters. These responses to the pre-course surveys indicated participants with a wide range of professorial ranks, teaching experience, current institutional incumbency, and PSP authorship status. We combined data from the two semester surveys for a total of 60 participants who completed both pre- and post-surveys.

Site tester applicants came from 39 different institutions, including public and private four-year universities, primarily research institutions, and community colleges. Site testers generally had between 0 and 35 years of mathematics teaching experience, with a noticeable grouping with 11-15 years of experience.

When asked about their experience with primary historical sources in mathematics, more than half of respondents (26 of 50 = 52%) indicated that they already possessed experience in using primary sources in their research. Significant fractions of respondents had prior experience using primary source materials in their teaching (18 of 50 = 36% in history of mathematics courses, and 13 of 50 = 26% in other mathematics courses) while about a quarter reported no such experience (12 of 50 = 24%).

Changes in Instructor Teaching Tendencies (Question 1).

To assess whether or not the implementation of PSPs changed instructors' teaching behaviors, we compared identical items on the pre- and post- instructor surveys. The respondents were asked to indicate on a 5-point Likert scale if each item corresponding to a specific teaching strategy was 'very descriptive of my teaching' (5), 'mostly descriptive of my teaching' (4), 'somewhat descriptive of my teaching' (3), 'minimally descriptive of my teaching' (2), or 'not at all descriptive of my teaching' (1). By comparing these identical items on pre- and post-surveys, our goal is to identify any of these tendencies that may have changed as a result of the implementation of PSPs.

Paired t-tests were conducted to identify any significant changes in perceived teaching behaviors before and after PSP implementation. One variable changed significantly before and after PSP implementation; specifically, instructors reported that the use of student questions and comments to determine the focus and direction of classroom discussions reflected their teaching tendencies *more so* after the implementation of PSPs than before, $t(59) = -3.37$, $p = .001$.

Although statistically insignificant, there were three other changes worth noting from the pre- and post-surveys. A noteworthy portion of instructors reported incorporating more time during class dedicated to student discussion of course concepts after PSP implementation ($p = 0.070$). A marginally significant portion of the instructors also reported that they allowed for more time dedicated to student reflection of their problem solving strategies ($p = 0.062$) and inter-student constructive criticism of ideas ($p = 0.057$).

Instructors' Perception of the Impact of PSP (Question 2).

A portion of the post-survey questions were 7-point, Likert-style questions designed to gather information regarding how PSP implementation *impacted* the instructors and their students' knowledge of mathematics, learning/teaching approaches, and beliefs about math. For example, instructors responded to items such as "To what extent do you feel that using PSPs in your class made you more/less open to using different teaching strategies?" Responses greater than 4 indicated favorable responses (in this example's case, 1 = extremely less, 4 = neutral, 7 = extremely more) while responses below 4 indicated non-favorable responses.

PSPs and Instructor Teaching Approaches. Implementers' perceptions of how their use of PSPs affected their own knowledge and beliefs about mathematics tended to be generally positive ($M = 5.12$). Although not all items are shown, Table 2 shows average responses to several questions focused around instructor's teaching abilities and tendencies.

Table 2
Instructor perception of how implementing PSPs affected their teaching. All 60 instructors responded to each question.

To what extent do you feel that using PSPs in your class…	M	SD
made you a more/less versatile teacher?	5.32	0.75
made you a better/worse teacher?	4.95	0.89
did/did not induce you to discuss course topics in a broader context?	5.17	0.91
made you more/less open to using different teaching techniques?	5.13	0.83
increased/decreased your confidence in incorporating history into your teaching?	5.30	0.93

PSPs and Student Knowledge and Learning Approaches. Instructor perceptions of how PSP implementation affected their students tended to relay positive (M = 5.29) trends in terms of their students' increase in knowledge, capacity and appreciation of the history of mathematics and mathematics in general. Table 3 shows average responses to questions focused around the impacts of PSPs on student knowledge and learning (all items are reported on).

Table 3
Instructor perception of how implementing PSPs affected their students. All 60 instructors responded to each question.

To what extent do you feel that using PSPs in your class…	M	SD
increased/decreased your students' knowledge of mathematics?	5.30	0.79
increased/decreased your students' knowledge of the history of mathematics?	5.72	0.64
increased/decreased your students' capacity for different ways of thinking?	5.33	0.71
increased/decreased your students' appreciation of the evolution of mathematics?	5.53	0.79
made your students more/less able to learn in a variety of ways?	5.05	0.77
did/did not provide a way of learning that better fit some of your students' needs?	4.68	1.07
did/did not induce your students to learn topics in a broader context in your classes?	5.32	0.91
made your students more/less open to learning in different ways?	5.07	0.80
made different areas of mathematics seem more/less unified to your students?	5.00	0.99
improved/worsened your students' attitude towards incorporating the history of mathematics into a mathematics course?	5.23	0.93
increased/decreased your students' understanding of the value of studying the history in mathematics for learning mathematics?	5.17	0.94
did/did not provide your students a different perspective on mathematics?	5.63	0.74
increased/decreased your students' appreciation of mathematics as a humanistic endeavor?	5.55	0.91
increased/decreased your students' appreciation of mathematics as a creative endeavor?	5.43	0.91

Implementation of PSPs (Question 3).

Describe in general terms how the PSP was implemented. Themes emerged pertaining to how PSPs were implemented in terms of student work both in and out of the classroom and also the role of the instructor during class time. Emergent themes revealed that instructors gave brief introductions to the material (28%), included instructor-led discussions (24%) and reconvened at the end of class with a debrief (29%) following student group work (40%). Although responses generally did not include these themes in succession, the most frequently reported codes from all responses communicated a general class structure where instructors introduced the material, led the class in discussion, allowed for group work throughout class time to work through tasks and ended by debriefing their students. PSP implementation generally constituted of one or some combination of the following classroom structural components:

1. Students were assigned preparatory work before PSP lessons were introduced. This work usually came in the form of assigned readings or initial attempts at introductory tasks in the PSP.
2. Some instructors opted to provide a brief introduction to the topics within the PSP as a mechanism to prevent confusion and promote efficiency of PSP completion.
3. Instructors also led class-wide discussions and other activities pertaining to PSP material.
4. Implementation reports also relayed that PSP implementation led to substantial group work on the material.
5. After a class period's work on PSP material, instructors reconvened the class for a short debrief on the material covered that day.
6. Unfinished PSP tasks were generally assigned as homework problems.

This classroom structure substantially echoes the intent of inquiry-based learning approaches that focus on group work, student discussion and less instructor lecturing without losing instructor guidance of their students through the material.

Comparison between PSP implementation and General Instructional Approach. Approximately two-thirds (64%) of instructors reported that their PSP implementation involved some deviation from their general instructional approach in the course. Increased use of group work (32%) and fewer instructor-driven activities (28%) were the most commonly reported differences. In addition, 19% reported "letting go" of their classes more. See the following quotes:

> Implementing PSP in class was very different than my general teaching approach in class. Definitely, more active learning was involved with the implementation and students seemed more interested in math which is not usually the case. It was more student centric and they seemed to join the class more and it was also enjoyable for me.
>
> I have never done something like this before...I have never used a lengthy project like this.
>
> Implementation of PSP showed me the importance of group work.

While 36% of the responses indicated that the PSP implementation did not deviate significantly from their general instructional approach in the class, some commented that they already lead a student-centered classroom. As two instructors noted,

> I would say that this PSP fit in very well with my teaching style, or at least the teaching style that I prefer to use (there are still lessons that are primarily lecture; I try to minimize lecturing, so I really liked having this PSP).
>
> This is not abnormal: I assign reading and exercises for each class. The students answer reading questions and reflect on their reading and questions they have, and they prepare the exercises for presentation..., and we spend class time with them doing presentations, discussing questions they had on the reading, and working in small groups on more problems. They also have an individual homework problem or two assigned after every class.

Overall, instructors reported less instructor-focused activity (e.g. lecturing), more group work and more instructional practices that would align with various active-learning strategies.

Modifications and Obstacles Experienced During Implementation. Instructors tended not to modify the PSP (32%) but still offered suggestions for future implementations (36%). When instructors did modify the PSP during implementation, they either omitted (15%) and condensed (14%) sections to fit into a smaller time frame. These findings suggest that the PSPs selected by instructors were sufficiently well-developed for a variety of classroom settings. When PSPs were modified, it was primarily due to time constraints and not due to material within the PSP. Both findings suggest that current PSP materials are useful.

Approximately 29% of instructors reported that they did not face significant obstacles during PSP implementation. When instructors did face obstacles, they were generally due to instructor inexperience with inquiry-based learning approaches (17%), students' inability or unfamiliarity with reading primary source material (12%) or general timing issues with regard to implementing the PSP (12%). The following quote(s) exemplify these findings:

> I also had to adapt to not lecturing. At times it felt like I wasn't helping them that much.
>
> The only obstacle is that I need to find a way to integrate it into the course more smoothly. I've had a standard syllabus for some years that doesn't really leave room for more student work. This semester (and last year) I simply imposed the PSP on top of the rest of the homework, which is unfair to the students, but I wasn't willing to make major changes until I was sure I would be continuing to use PSPs in this course. I will make those adjustments next year.

Conclusion

Overall, faculty seem to be reporting positive experiences with the implementation of PSPs in their classroom. Instructors reported perceived benefits for both themselves and their students as a result of PSP implementation. Notably, instructors consistently reported that the implementation of PSPs had numerous positive impacts on their teaching abilities and strategies, while they also consistently reported that implementation of PSPs increased their student's knowledge and understanding of mathematics. Instructors have reported that PSPs changed their perspectives on teaching, and opened their eyes to new approaches and techniques. Historical context also emphasized mathematics as a human endeavor, one full of struggle, perseverance and beauty.

Instructors also perceived that their students enjoyed the introduction of inquiry-based learning approaches as opposed to more traditional lecture-based formats. Many of the challenges that faculty face seem to be commonly reported challenges that faculty face when struggling to use active learning for the first time. Results suggest that we could improve upon providing ongoing support to faculty using PSPs. Future research should look into examining what types of ongoing support systems will most benefit instructors who choose to implement PSPs in their classrooms.

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Juxtaposing a Collective Mathematical Activity Framework with Sociomathematical Norms

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We utilize two analyses to confirm a multidimensional framework for analyzing contributions to classroom discourse, previous analysis using the framework and analysis of instances of sociomathematical norm negotiation juxtaposed with it (Cobb & Yackel, 1996). The framework considers social, epistemic, and argumentative activities exhibited in talk-turns during whole class discussion. In this study we show that collective and individual development occurred in an inquiry-oriented differential equations course and discuss patterns in ways learning partners participated in whole class discussions during sociomathematical norm negotiation.

Keywords: Discourse, Collective Mathematical Activity, Sociomathematical Norms

Investigating and understanding student learning as it takes place in mathematics classrooms is a challenging yet significant endeavor. This work has been advanced by researchers studying classroom discourse (e.g., Forman & Ansell, 2002; Lee et al., 2009; Stephan & Rasmussen, 2002), teacher questioning (e.g., Mesa, 2010; Roach, Noblet, Roberson, Tsay, & Hauk, 2010), and other pedagogical moves (e.g., Moyer & Milewicz, 2002; Nicol, 1998), which has yielded important results for researchers and practitioners to meaningfully reflect on teaching and associated classroom activity. Much of this work focuses on the language used by students and teachers, questioning, and discourse patterns (e.g., Mehan, 1979; Rasmussen & Kwon, 2007). However, there is still a need for research to understand how student contributions to discourse are related to how and what students learn in mathematics classrooms.

We (Keene, Williams, & McNeil, 2016; Williams, Keene, McMillian, & Lopez-Torres, in progress) adapted a framework from science education (Weinberger & Fischer, 2006) to analyze student participation in mathematics classrooms, the classroom argumentative knowledge construction (CAKC) framework, through three dimensions: epistemic, social, and argumentative. Individually, these dimensions allow contributions to classroom discourse to be understood in terms of how mathematical ideas are developed by multiple contributors (i.e. students and professor); whether contributions relate to specific problem(s), broader concepts and theories, or both; and how arguments are developed and advanced. Together, the framework allows for collective mathematical activity to be investigated through contributions to classroom discourse. The framework allowed us to characterize the nature of contributions to classroom discourse, identify socially constructed student- and teacher roles, and identify contributions to discourse which may demonstrate learning. The purpose of this investigation is to confirm the usefulness of the framework by juxtaposing an analysis of sociomathematical norm negotiation (Cobb & Yackel, 1996) evident in the same classroom data used in our original study with our original analysis. In doing so, we can identify patterns of participation during sociomathematical norm negotiation to further understand these important interactions.

Theoretical Framework

The emergent perspective (Cobb & Yackel, 1996) underpins the work in this paper. From this lens, social and psychological perspectives are used to analyze individual and collective activity at the classroom level and are understood to be reflexively related. That is, development in the social perspective is inextricably linked to individual development; one cannot occur without the other. Social and sociomathematical norms are fundamental notions of the emergent perspective from the collective lens. Social norms are content-irrelevant, normative aspects of classrooms. Sociomathematical norms are normative aspects of classrooms specifically associated with mathematics as the content. For example, that students explain their thinking is a social norm, while what constitutes sufficient mathematical explanations would be a sociomathematical norm (Cobb & Yackel, 1996). Sociomathematical norms are reflexively related to individual development of mathematical beliefs and values. In addition to what constitutes sufficient mathematical explanations, other norms are what constitutes: mathematically different solutions, sufficient mathematical justification, sophisticated mathematical reasoning, effective mathematical representations, and appropriate mathematical precision (Hershkowitz & Schwarz, 1999; Yackel & Cobb, 1996; Yackel, Rasmussen, & King, 2000). This notion is particularly important for our purpose, as we aim to extend the utility of the CAKC framework to better understand contributions to whole class discussions as a means for understanding collective mathematical activity *and* individual development.

Table 1 presents the CAKC framework, which consists of three dimensions each with various activities. Unfortunately, lack of space prohibits more detailed examples of the activities.

Table 1. The CAKC framework adapted from Weinberber & Fischer (2006)

Epistemic Dimension	
Activity	Brief Description
Construction of problem space (CPS)	Relating case information within the problem space with the aim to understand the problem
Construction of conceptual space (CCS)	Relating concepts with each other and explain theoretical principles to understand theory
Construction of adequate relations between conceptual and problem spaces (CAR+)	Applying relevant theoretical concepts adequately to solve a given problem. Relating theoretical concepts to case information
Construction of inadequate relations between conceptual and problem spaces (CAR-)	Applying concepts inadequately to a given problem by either selecting inappropriate concepts or not applying appropriate concepts according to principles dictated by theory
Construction of adequate relations between prior knowledge and problem space (CRP+)	Applying concepts adequately that stem from prior knowledge
Construction of inadequate relations between prior knowledge and problem space (CRP-)	Applying concepts inadequately that stem from prior knowledge rather than new theoretical concepts that are to be learned
Social Dimension	
Activity	Brief Description
Externalization (EXT)	Articulating thoughts to the group

Elicitation (ELI)	Questioning or provoking a reaction from learning partners
Integration-oriented consensus building (IOC)	Taking over, integrating, and applying perspectives of a learning partner
Conflict-oriented consensus building (COC)	Disagreeing, modifying, or replacing perspectives of a learning partner

Argumentative Dimension	
Activity	Brief Description
Argument (ARG)	Statement put forward in favor of a specific proposition
Counterargument (COU)	An argument opposing a preceding argument, favoring an opposing proposition
Reply/Integration (RPY)	Statement that aims to balance and advance a preceding (counter) argument
Non-argumentative moves (NAR)	Questions, coordinating moves, and meta-statements

Methods

The purpose of this project is to confirm the usefulness and power of the CAKC framework for analyzing students' participation in mathematics classrooms by juxtaposing analyses from the CAKC framework and instances of sociomathematical norm negotiation (Cobb & Yackel, 1996). Specifically, we address the question: What connections are present between the patterns of participation identified by the CAKC framework and patterns present for sociomathematical norm negotiation?

Setting and Participants

This study took place during a summer inquiry-oriented differential equations (IODE) course for teachers working to earn master's degrees in Mathematics Education. Twenty-one students participated in the course and study; most had experience as secondary mathematics teachers. Although the students had strong mathematics backgrounds, only some had previously taken undergraduate differential equations. Prior experience with differential equations was not prerequisite for the course. Most students indicated that they were starting with minimal or no knowledge of differential equations. The course was taught by an experienced professor.

The course met for three hours, three times each week for five weeks. The classroom was organized for students to work collaboratively in small groups, which were assigned and changed weekly. The class was taught using tenets of inquiry-oriented instruction (Rasmussen & Kwon, 2007). Course materials were informed by research on students' understanding of DE and give students opportunities to reinvent the mathematics through tasks. The research-based tasks involved using differential equations to model real world situations through analytical, qualitative, and numerical methods. Students cycled through small- and whole group discourse spaces while inquiring into the mathematics. The professor inquired into students' thinking, building on their thinking while keeping an eye on the mathematical horizon (Treffers, 1987).

Data Collection

For this report, we used the same transcripts from video recordings of whole class discussion from five class sessions used in our previous work (Keene et al., 2016; Williams et al., in progress). The dataset consists of transcripts from one hour of whole class discussion occurring in five different class sessions. Also serving as data for this study are the analysis and results from our original work. Note each contribution to classroom discourse received a code from each dimension in the CAKC framework.

Data Analysis

Data analysis for this study consisted of two phases: first, instances of sociomathematical norm negotiation evident from transcripts were identified; and second, results from the first phase were juxtaposed with results from the analysis conducted with the CAKC Framework. During the first phase, the first and second authors worked independently from each other and without referencing analysis with the CAKC framework to locate cases of sociomathematical norm negotiation within the transcripts. Then these instances were coded to document the specific sociomathematical norms being (re)established using *a priori* norms identified in the literature (e.g., Cobb & Yackel, 1996) as well as *a posteriori* norms that emerged from the data. The two coders discussed where instances of sociomathematical norm negotiation took place within the transcripts, and coding of these instances as examples of specific sociomathematical norms being established. Cases were discussed until consensus was reached when the two coders were not in agreement. Results presented reflect agreed upon codes.

After coding the same data independently using the CAKC framework and for sociomathematical norms, the two sets of codes were juxtaposed to look for patterns or relationships between them. In particular, we looked for patterns in social, epistemic, and argumentative activity identified by CAKC framework across all instances of sociomathematical norm negotiation and across examples of the same sociomathematical norm being (re)negotiated at different times.

Results and Discussion

Results are presented in two sections, reflecting the two phases of analysis conducted. First, we outline various sociomathematical norms negotiated during the five hours of whole class discussion. Then we share results from juxtaposing this analysis with our original work.

Sociomathematical Norm Negotiation

We first briefly discuss the sociomathematical norm analysis. Of note, we identified six different sociomathematical norms across 28 instances where norm negotiation took place. These norms include, what constitutes: (a) mathematically different – 7 instances, (b) sufficient mathematical justification – 11, (c) sophisticated mathematical reasoning – 4, (d) effective mathematical representations – 3, (e) appropriate mathematical precision – 1, and (f) valid mathematical assumptions – 2. The number of times it was negotiated does not necessarily mean that a sociomathematical norm was more important than others. For example, the fact that what constitutes valid mathematical assumptions was negotiated in only two occasions may reflect that the negotiations were effective and consensus about valid mathematical assumptions was reached. On the other hand, what constitutes sufficient mathematical justification was negotiated 11 times in five hours of discussion. This does not necessarily mean that negotiations were ineffective. Instead, recurrent negotiation may have been due to the fact that students were regularly engaging with new concepts and skills, while the teacher was prompting them for justification to support collective learning during whole class discourse. In fact, tenets of IODE include that students recreate the mathematics through real world problems (Rasmussen & Kwon, 2007), so it is desirable that learners would be establishing what constitutes effective use of recently developed concepts through justifications consistently throughout the course.

We offer the following example to demonstrate one instance of this class negotiating what constitutes valid mathematical assumptions. We chose this example as what constitutes a valid mathematical assumption emerged in this work. For context, the class was discussing a problem prompting students to create a rate of change equation from data of a cooling cup of coffee so that predictions could be made about other cups, and whether the equation would depend on temperature only, time only, or both.

Student 1: The one thing we talked about is that, we said, if you have this cup of coffee that is 160º or a huge mug that's 160º, they're probably not going to cool at the same rate because of volume. So, if you assume that you have the same volume then you don't need to depend on C [temperature]. We figured that this problem was assuming that you did, so-

Instructor: … So, here's the issue, how do you figure out a way to graph dC/dt against C? So most of you were just kind of trying it, and I enjoyed watching you because everybody had a different way of thinking about it. Student 2, could you come show us your graph?

Student 2: This will be the rate of change, so when the temperature is high- I kind of plotted a little bit, approximately- so if the temperature is low, the rate of change is small. And then if the temperature is high, the rate of change is large. So it's kind of, I mean if you look at it, 0.3 is a little bit- it gets a little bit bigger faster when the temperature is higher. So-

Student 3: What if it is really small and the room temperature is higher? Like in real life…

In this example, Student 1 is questioning whether a rate of change equation would be able to make meaningful predictions for other situations unless the future cup of coffee contains the same amount of coffee as that which generated the dataset. The instructor then interjects, refocusing students to think about variables in the problem – temperature and time. In doing so, the instructor communicated to Student 1, perhaps inadvertently, that assuming same volumes may not be necessary. However, citing "real life" as evidence to argue that the rate of change equation may not depend only on the coffee temperature, as explained by Student 2, suggests that consensus about mathematical assumptions from the problem may not have been reached. Using rate of change equations to model real-world situations is a key feature of IODE, so considering what constitutes valid mathematical assumptions is paramount to understanding course content. From the emergent perspective, participating in the collective process of sociomathematical norm negotiation simultaneously involves individuals learning how and when adhering to established norms is appropriate (Cobb & Yackel, 1996). In this way, sociomathematical norm negotiation occurs simultaneously with students' developing mathematical beliefs and values; thus, collective and individual development took place.

Juxtaposing Analyses

Relationships and patterns between the CAKC framework and sociomathematical norms emerged when these analyses were juxtaposed. We present the relationships found between each of the three dimensions: social, epistemic, and argumentative.

Results showed that an activity coded for the **social dimension** was not inherently necessary for sociomathematical norm negotiation to occur. That is, there was no code in the social dimension that appeared in every case of sociomathematical norm negotiation. However, although integration-oriented consensus building (IOC) and conflict-oriented consensus building (COC) constituted the least common social activities (combining for 21% of talk-turns), only three instances of sociomathematical norm negotiations did not involve these types of contributions. On the other hand, there were many instances where these types of social activities occurred where there was no negotiation taking place. Thus, the presence of sociomathematical norm negotiations points to some kind of consensus building, but consensus building does not imply sociomathematical norm negotiation.

A similar relationship between **epistemic dimension** activities and sociomathematical norm negotiation emerged. Again, there was no code from the epistemic dimension that appeared in each case of norm negotiation, so specific epistemic activities were not indicative of sociomathematical norms being (re)established. In fact, epistemic activities appeared equally often in sociomathematical norm negotiations as they did outside of these instances (table 2). Additionally, activities in the **argumentative dimension** occurred with equal frequency during instances of sociomathematical norm negotiation as they did outside of negotiations, except for contributions presenting arguments (ARG), which took place slightly more often when norms were not being (re)established (table 2).

Table 2. Counts of activities [not] during instances of sociomathematical norm negotiation.

Epistemic dimension						
In an instance of norm negotiation?	CPS	CCS	CAR+	CAR-	CRP+	CRP-
No	75	37	32	3	4	1
Yes	78	27	32	8	9	3
Argumentative dimension						
In an instance of norm negotiation?	ARG	COU	RPY	NAR		
No	66	36	162	61		
Yes	48	47	145	63		

The previous paragraphs suggest that activities within each dimension alone may not be particularly lively during instances when sociomathematical norms are being negotiated. However, a strength of the CAKC framework is that it offers a multidimensional approach to understanding talk-turns during whole class discussion. In our previous study, each contribution to classroom discourse received a code from all three dimensions. We used the term *code-string* to discuss the collection of these three codes ascribed to a single contribution. When examining this multidimensionality, a pattern emerged with talk-turns in which students constructed adequate relations between a given problem and mathematical concepts (CAR+) while participating in various social and argumentative activities (Figure 1). For example, contributions exhibiting CAR+ involving externalizations (EXT) used to establish an argument (ARG) or counterargument (COU) almost always took place during instances of sociomathematical norm negotiation. This pattern is intriguing considering that contributions presenting arguments tended to occur more often in cases not indicative of sociomathematical norm negotiation. In other words, when students were constructing adequate relations between mathematical concepts while presenting the first offered solution to a problem they were also almost always (re)establishing sociomathematical norms. On the other hand, talk-turns indicative of CAR+ involving consensus building (which were rare) occurred less frequently during sociomathematical norm negotiations.

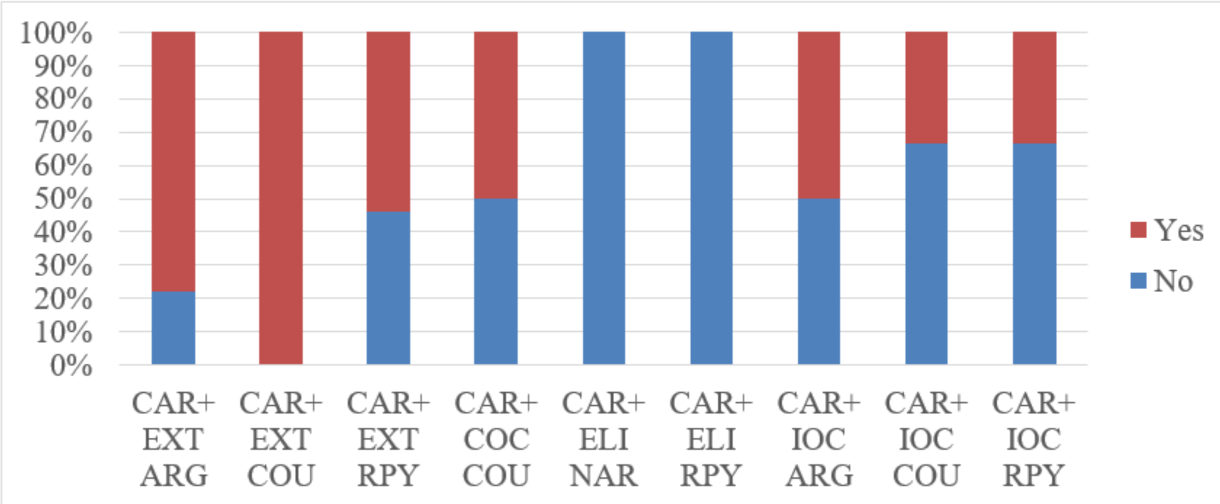

Figure 1. Proportions of contributions involving CAR+ during sociomathematical norm negotiation.

Conclusion

Confirming the utility of the CAKC framework involves demonstrating that meaningful understanding of contributions to discourse can be uncovered through its use. In this study, we showed that collective and individual mathematical activity took place in this IODE course through our analysis of sociomathematical norms (Cobb & Yackel, 1996). Then, we examined patterns in epistemic, social, and argumentative codes exhibited while sociomathematical norm negotiation occurred. The patterns we presented suggest that the multidimensionality of the CAKC framework can be used to better understand relationships between participation in classroom discourse and *both* collective and individual mathematical activity.

Specifically, cases in which students were making connections between mathematical concepts and a given problem while articulating an argument or counterargument may be essential for collective and individual development as these types of contributions rarely occurred outside of sociomathematical norm negotiation, even though talk-turns presenting arguments took place more frequently during instances outside of sociomathematical norm negotiation. This result demonstrates the utility of the multidimensionality of the CAKC framework. Mathematics education researchers can utilize the CAKC framework to understand how individual talk-turns contribute to collective mathematical activity. Additionally, consensus building was shown to be an integral component of collective mathematical activity. Many have demonstrated the significance of sociomathematical norms and the importance of active-learning in undergraduate mathematics classrooms (e.g., Rasmussen, Apkarian, Dreyfus, & Voigt, 2016; Rasmussen, Wawro, & Zandieh, 2015; Yackel et al., 2000). This study furthers that body of work by delineating significant ways to analyze participation in mathematical activity.

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Leadership and Commitment to Educational Innovation: Comparing Two Cases of Active Learning Reforms

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Several studies have shown that student-centered instruction can help improve student success and persistence in STEM-related fields (e.g., Freeman et al., 2014). Despite this, institutional change can be difficult to enact. Accordingly, it is important to understand how departments both initiate and sustain meaningful change. For this paper we use interview data collected in Spring 2017 to examine how institutional and departmental factors affected reform efforts at two different institutions. In particular, we compare how two universities' leadership and commitment to educational innovation contribute to the initiation, implementation, and sustainability of active learning in the undergraduate calculus sequence (Precalculus through Calculus 2).

Keywords: Active Learning, Institutional Change, Calculus Reform, Undergraduate Mathematics, Case Study

Introduction

Universities are increasingly concerned with student retention, graduation rates, and overall student success. While much more is known now about effective instructional practices and campus structures to support student success, institutes of higher education are slow to change (Kezar, 2014) and faculty have not widely adopted such research-based practices (Stains et al., 2018). Student-centered instructional practices that address not just student learning but also attitudes, beliefs, motivation and goals, are connected with increased student success and persistence in mathematics and related fields (e.g., Freeman et al., 2014). However, some faculty and some universities are changing, exhibiting culture shifts that value instructional improvement efforts.

We present two cases of large land-grant universities that have transformed instruction in lower-level mathematics courses via a comprehensive approach to cultural and instructional change. In both cases, these reforms started with a focus on Calculus 1, and then grew to encompass Calculus 2, Precalculus-level courses, and other multi-section courses. The changes included attention to instruction and instructors; this case study focuses specifically on the department and institution level changes. These cases are drawn from a larger set being developed by a collaborative National Science Foundation project: Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL). SEMINAL is studying how mathematics departments successfully incorporate active learning into their calculus sequence courses and how to guide other departments looking to institute similar reforms.

Literature and Theoretical Framework

Change efforts to improve student outcomes necessarily include a classroom instruction focus. However, to achieve cultural change, instructional improvement efforts also need to have components at the department, campus, and community levels (Elrod & Kezar, 2016). When faculty seek to improve instructional practices, they rightly tend to focus on instructional materials, activities and tasks, assessments, mathematical coherence and structures that allow students to communicate their reasoning (e.g., MAA, 2017). They may also focus on developing norms for mathematical discussions (e.g., MAA, 2017; Smith & Stein, 2018). Departments focused on instructional improvement and equitable student outcomes may initiate or refine course coordination efforts (Bressoud, Mesa, & Rasmussen, 2015) and provide instructional training and mentoring. All of these changes require significant investment of time and other resources, along with a commitment to improvement; lack of widespread support for such efforts will undermine them (Kezar, 2014).

The foundation of effective change efforts is the development of a common vision among stakeholders (Elrod & Kezar, 2016). Stains et al. (2018) summarize effective instructional practices as ones that focus on actively engaging students. The heart of the transformation efforts enacted by these two departments of mathematics is the effective use of active learning strategies, defined as: (1) students learn mathematics by engaging in challenging, cognitively demanding tasks; (2) students routinely communicate (orally and in writing) their own reasoning and engage with the reasoning of others; (3) instructors attend to and make use of student thinking to advance the mathematical agenda; and (4) instructors are explicitly attending to issues of diversity, equity, and inclusion (Laursen & Rasmussen, 2018). While not explicitly labeled "active learning," these principles are also embodied in the recommendations of the MAA's recent *Instructional Practices Guide* (2017).

At both the department and campus levels, when the culture supports instructional innovation, the environment is more favorable for faculty and departments to invest in course improvements (Kezar, 2014). Bergquist and Pollack (2008) suggest culture is a lens through which faculty members understand their universities: "A culture provides a framework and guidelines that help to define the nature of reality - the lens through which its members interpret and assign value to the various events and products of this world" (Bergquist & Pawlak, 2008, p. 7). Culture as lens can be a useful framework, but to capture the dynamic aspects of culture, additional dimensions are necessary.

Apkarian and Reinholz (2018) provide a higher education adaptation of four frames through which to understand institutional culture: *people*, *power*, *symbols*, and *structures*. The symbolic frame of culture includes the values, beliefs, and attitudes of the various stakeholders in the system. By also considering the power dynamics, the people involved, and the structures of the institution, this framework can support understanding of educational cultures and cultural shifts.

In this paper, we focus specifically on two cross-cutting dimensions of institutional culture: leadership and commitment to educational innovation. Both of these dimensions span the four frames (Apkarian & Reinholz, 2018). Leadership includes the people in formal and informal positions (*structures*), their beliefs and values (*symbols*), and the interrelated *power* dynamics of leadership relationships. The value placed on instructional improvements by a campus includes the values and beliefs related to the importance of improving teaching and learning (*people* and *symbols*) and resources to support instructional improvements (*power* and *structures*).

Purpose and Research Questions

The SEMINAL project's overall research question is: What conditions, strategies, interventions and actions at the departmental and classroom levels contribute to the initiation, implementation, and institutional sustainability of active learning in the undergraduate calculus sequence (Precalculus through Calculus 2—P2C2) across varied institutions? The purpose of this research is to compare the commitment to reform efforts focused on active learning strategies, and the particular leadership roles of departmental and campus administrators in the initiation, implementation and sustainability in improvements in P2C2 courses. Thus, the research question guiding this study is:

> *How do leadership and commitment to educational innovation contribute to the initiation, implementation, and sustainability of active learning in the undergraduate calculus sequence (Precalculus through Calculus 2) compare between Big State University 1 and Big State University 2?*

Methods

SEMINAL is a 5-year NSF-funded mixed-methods research project studying the initiation and sustainability of active learning in mathematics in two phases. Phase 1 focused on retrospective case studies of institutions that have sustained active learning reforms for at least three years. Phase 2 focuses on incentivized case studies of institutions in the midst of reforms. Data for this paper draw on two of the Phase 1 institutions: Big State University 1 (BSU1) and Big State University 2 (BSU2).

Data was collected at site visits in Spring 2017. During these visits four researchers collected qualitative data including audio-recorded interviews with campus administrators, tenure track and non-tenure track faculty within the math department, postdocs, graduate students, course coordinators, faculty from client disciplines, and undergraduate students. Each interview was transcribed and coded in MAXQDA 12. The initial framework for code categories (e.g., coordination, department leadership, professional development, etc.) was drawn from the grant proposal, which in turn was informed by Bressoud, Mesa and Rasmussen (2015) and institutional change literature. This same framework was used to design the project's data collection plan, including interview protocols. Researchers used an iterative process to generate sub-codes for each category. Each transcript was individually coded by at least 3 people, followed by reconciliation (Creswell & Poth, 2018).

After coding, individual researchers were assigned categories of codes (e.g., coordination) and constructed reports of facts and emerging themes (Creswell & Poth, 2018). Researchers then exchanged reports and codes for additional reconciliation. Using these reconciled reports, and other documents provided by each site, researchers drafted thick descriptions for both institutions to make a side-by-side comparison for this comparative case study (Stake, 1995).

Findings

In this section we describe the initiation, implementation and sustainability of reforms through the lens of leadership and commitment to educational innovation.

Initiation of Change and Implementation of Reforms

Stimulus for Change. In order to incorporate active learning into their P2C2 programs, both math departments began with Calculus 1. BSU1's motivation for change came internally: two department leaders wanted to change the structure of the Calculus 1 recitations after observing graduate students solving problems in front of disengaged students. Therefore, these two leaders initiated reform efforts focused on increasing coordination of recitations; making recitations more meaningful by transforming them into sessions with active learning where students would work cooperatively on common projects; many of these projects focus on building conceptual understanding by incorporating high cognitive demand tasks (Stein et al., 2000).

Unlike BSU1, the BSU2 department received top-down pressure to "fix" Calculus 1 due to student complaints and low pass rates. While motivation was, in part, external, the reforms were initiated in large part because the department was willing to change; one campus administrator noted other departments had been similarly pressured without comparable positive results. The department chair and a faculty member who was interested in technology planned the changes they wanted to make, and purposefully sought external resources to support their plans. The latter worked with two other faculty to apply for an NSF grant, which they received in the early 1990s. The grant was used to pilot Hughes-Hallett et al.'s (1994) *Calculus* and introduce new technology into the classroom.

Leadership and Commitment Following the Initiation. Since BSU2's initial reform efforts, the math department has benefited from "departmental support [which] has been unwavering." At the time of initial reforms, other departments on campus were not trying to "fix" their courses in the same ways as BSU2's math department. After early pilots, leaders in the reform became "vigilant" in attending workshops on effective implementation with technology and active learning. These leaders were described as "evangelists," people who were able to articulate and defend positive outcomes of this type of model for teaching. When state appropriations for higher education declined, these "evangelists" were able to help "sway" the department and college to keep reform changes in place. Thus, while the college and department were supportive, early leaders had to make a strong effort for this support.

Prior to BSU1's changes to Calculus 1, science departments on campus had successfully implemented similar educational reforms. As a result, campus administrators were already on board with supporting change. As described above, this context was different from BSU2's neighboring departments and colleges. Leaders in the math department at BSU1 had a particularly positive relationship with one campus administrator, who was formerly an academic dean within the college. In their role as a dean, the administrator supported reform efforts by approving and allocating resources for the math department to hire a full-time coordinator to oversee the calculus sequence and protecting resources to maintain small class sizes. One mathematics faculty member stated that this administrator "in essence made the resources available to us for everything we've done in the last 4 years." This administrator supported leaders' reform efforts not only because Calculus 1 was designated a gateway course, but also because the leaders had a well-developed proposal and were "truly dedicated to improvement." Thus, while leaders from BSU1 and BSU2 received support from their departments and colleges, the difference in institution innovation at the time of changes may have impacted how leaders were able to obtain that support.

Sustainability

For this study, we operationalize the concept of sustainability as evidence of maintaining and extending reforms, institutionalizing change, and addressing ongoing issues related to these reforms. Table 1 is a brief summary of findings related to sustainability.

Table 1. Comparing two universities' reforms

	Extending Reforms	Institutionalizing Reforms	Facing Challenges
BSU1	Calculus for Life Science	refining coordination system	mixed value of teaching
both	Calculus 1-3, Precalculus	hiring more coordinators, instructor meetings	initial buy-in, leadership turnover
BSU2	other multi-section courses (e.g., Differential Equations)	adding pedagogical focus to instructor meetings	efforts ahead of campus shift to value teaching

Extending Reforms. Reforms at both universities began in Calculus 1 then extended to other courses. Both universities followed a similar trajectory of reforming Calculus 2 next. For BSU1, leaders received an external grant which allowed them to extend active learning in Calculus 1 from one day a week to all class periods, which motivated them to change Calculus 2 in similar ways. One interviewee mentioned that some students who experienced active learning in Calculus 1 and 2 expressed a desire for similar experiences in Calculus 3, and at the time of data collection leaders were in the process of extending active learning to Precalculus and Calculus 3. BSU2 followed a similar trajectory as BSU1; at BSU2 some upper level courses were already taught with the Moore method, which perhaps allowed active learning strategies to infuse other multi-section courses more quickly than at BSU1.

Institutionalizing Reform. Leaders at both universities have helped implement lasting structural changes to ensure the uptake of active learning reforms. At BSU1, coordinating classes was essential to sustaining reform efforts because coordination makes it much harder for any one individual to undo reforms. The first full-time coordinator became a key leader in structuring and implementing the coordination system, and was given free rein to do so. Multiple interviewees cited them as a leader in implementing the reforms, going above and beyond what was originally envisioned for the position. This coordinator took charge of the professional development for GTAs, making it "pedagogically sound", and helped educate faculty members about active learning through an inquiry-based learning (IBL) workshop. Eventually the department hired multiple coordinators to help support the P2C2 courses.

BSU2 has also hired additional coordinators since beginning the reforms. At first, there was only one director, and regular faculty members served as coordinators on a rotating basis. "There was no official team," and "it wasn't a dedicated job, so those courses that had rotating coordinators were more variable." Eventually, more permanent coordinators were hired, which helped with the continuity and consistency of the courses. These coordinators were given significant autonomy, and one coordinator described the coordination in P2C2 courses as a "self-sustaining system."

Facing Challenges

Both departments have faced challenges to sustaining active learning reforms. Both departments experienced pushback from GTAs and other instructors when reforms were initiated. At this point, BSU2 experiences very little pushback compared to BSU1, perhaps due to the longer duration of reforms. In addition to buy-in, nearly all leaders at both universities have changed since reforms began. Throughout the leadership turnover, critical aspects of the reforms have been sustained and expanded. The core reformers and outgoing leaders actively worked to ensure the sustainability of efforts with the new leaders.

Another challenge in sustaining reform is creating a culture that supports and rewards leaders in educational innovation. At BSU1, the promotion process for instructors, including the coordinators, is based on years of experience rather than merit. Unlike BSU1, BSU2 does have a path to promotion for full-time coordinators and instructors based on teaching excellence. Consequently, the coordinators at BSU2 have stayed in their roles longer, providing continuity and institutional memory. Coordinators at BSU1 mentioned feeling like "second-class citizens" at times, yet overall have maintained enthusiasm for the departmental mission.

Discussion

Application of the Four Frames

The institutional changes at both universities can be viewed through Apkarian and Reinholz's (2018) four frames: *people*, *power*, *symbols* and *structures*. When the math departments initiated changes, they did not just make one or two changes, but sought to understand the larger system and improve it. Such a view of the change process is aligned with what is known about effective and sustainable changes (e.g., Kezar, 2014). Both universities have exhibited strong commitments to educational innovation; such commitment is embodied in the symbolic dimensions of culture: the beliefs and values of those involved. This commitment translates into support of *people* and *structures* that perpetuate and refine the reform strategies.

Reforms at BSU1 and BSU2 started for different reasons, but it was the leadership (*people*) at both universities who utilized their *power* to create *structures* incorporating active learning in P2C2 courses. These *structures* embody leaders' personal values and commitment to educational innovation (*symbols*), and perhaps challenge other *people*'s values (*symbols*) related to teaching and learning. As mentioned in the findings, the contexts for the initial implementation of reforms were different at the two universities. BSU2's math department was one of the first departments on campus to incorporate active learning. Therefore, their commitment to educational innovation (*symbols*) was not yet widely shared with other departments. In contrast, BSU1's math department benefitted from other departments' prior efforts to improve education (*structures*) and shared values (*symbols*). Thus, it is possible that the *people* at BSU2 had to utilize their *power* and *structures* differently than at BSU1 in order to implement reforms.

We contend that sustainability is a careful balance between *people*, *power*, *structures*, and *symbols*. In particular, a reciprocal relationship between *symbols* and *structures* is apparent in the process of extending reforms. Extending reforms to multiple courses (*structures*) could influence common values (*symbols*) of students and department members. Yet, the influence of initial reforms on common values (*symbols*) could, in turn, prompt the extension of *structures* supporting reform. For example, students at BSU1 expressed interest in Calculus 3 having the same *structures* as those present in Calculus 1 and Calculus 2, perhaps because they developed a shared value (*symbol*) of those *structures*.

When institutionalizing reforms, leaders (*people*) must focus on creating lasting *structures* which embody their values and commitment to educational innovation (*symbols*), and empower others to support those *structures*. For example, at BSU2, leaders (*people*) gave coordinators the *power* to support and maintain coordination *structures*, which makes lasting change possible. Challenges to sustainability may arise and create conflicts between the four frames. When making changes to *structures*, those in *power* must carefully consider the needs of the *people* supporting and participating in reform efforts to avoid conflicts in values (*symbols*). For instance, when instructors at BSU1 feel like "second-class citizens," despite the department's belief that they are valuable, there is a conflict in values (*symbols*), which has the potential to derail reform efforts.

Limitations

One limitation to our study is that BSU1 and BSU2 are not representative of all institutes of higher education. However, the experiences of these two departments who have sought to improve student outcomes via implementing active learning strategies can still be informative to other departments considering similar changes. Another limitation is that reforms at BSU2 happened in a different decade than BSU1, so it is important consider the differences in external contexts when making direct comparisons between the two departments.

Implications

In our analysis we focused on how leadership and commitment to educational innovation influenced reform efforts at both universities. From this discussion, it is clear that their effects cannot be fully understood by focusing on just one of the four frames. Leadership is not just about *people*. Commitment does not relate only to *symbols*. Departments seeking to make similar reforms need to broadly consider the complex systems that created the current state of affairs, as well as the interplay among *people*, *structures*, *symbols*, and *power* inherent in these systems. Effective change strategies address **all** of these dimensions, particularly at the initiation of reform efforts, and careful consideration of sustainability from the start can help ensure the long-term success of reform efforts.

Acknowledgment

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Developing Algebraic Conceptual Understanding: Can procedural knowledge get in the way?

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In this study we use latent class analysis, distractor analysis, and qualitative analysis of cognitive interviews of student responses to questions on an algebra concept inventory, in order to generate theories about how students' selections of specific answer choices may reflect different stages or types of algebraic conceptual understanding. Our analysis reveals three groups of students in elementary algebra courses, which we label as "mostly random guessing", "some procedural fluency with key misconceptions", and "procedural fluency with emergent conceptual understanding". Student responses also revealed high rates of misconceptions that stem from misuse or misunderstanding of procedures, and whose prevalence often correlates with higher levels of procedural fluency.

Keywords: elementary algebra, conceptual understanding, concept inventory

Elementary algebra and other developmental courses have consistently been identified as barriers to student degree progress and completion. Only as few as one fifth of students who are placed into developmental mathematics ever successfully complete a credit-bearing math course in college (see e.g., Bailey, Jeong, & Cho, 2010). At the same time, elementary algebra has higher enrollments than any other mathematics course at US community colleges (Blair, Kirkman, & Maxwell, 2010).

There is evidence that students struggle in these courses because they do not understand fundamental algebraic concepts (see e.g., Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010). Conceptual understanding has been identified as one of the critical components of mathematical proficiency (see e.g., National Council of Teachers of Mathematics (NCTM), 2000; National Research Council, 2001), and many research studies have documented the negative consequences of learning algebraic procedures without any connection to the underlying concepts (see e.g., J. C. Hiebert & Grouws, 2007). However, developmental mathematics classes currently focus heavily on recall and procedural skills without integrating reasoning and sense-making (Goldrick-Rab, 2007; Hammerman & Goldberg, 2003). This focus on procedural skills in isolation may actually be counter-productive, in that students may often attempt to use procedures inappropriately because they lack understanding of when and why the procedures work (e.g., Givvin et al., 2011; Stigler et al., 2010).

In this paper we explore student response to conceptual questions at the end of an elementary algebra course in college. We combine quantitative analysis of responses (using latent class analysis and distractor analysis) with qualitative analysis of cognitive interviews in order to better understand different typologies of student reasoning around some basic conceptual questions in algebra, and to explore the relationship between conceptual understanding and procedural fluency in this context.

Conceptual understanding

The definition of conceptual understanding (and its relationship with other dimensions of mathematical knowledge, particularly procedural fluency) has been much debated and discussed (e.g., Baroody, Feil, & Johnson, 2007; Star, 2005), with as yet no clear consensus. This study recognizes the interrelatedness of conceptual understanding with other mathematical skills (e.g., Hiebert & Lefevre, 1986; National Research Council, 2001), and defines it this way: An item tests *conceptual understanding* if a student must use *logical reasoning* grounded in mathematical definitions to answer correctly, and it is *not* possible to arrive at a correct response *solely* by carrying out a procedure or restating memorized facts. We define a *procedure* as a sequence of algebraic actions and/or criteria for implementing those actions that could be memorized and correctly applied with or without a deeper understanding of the mathematical justification. Using this definition, no question is wholly conceptual or procedural, but rather falls on a spectrum, with more conceptual questions at one end, and more procedural questions at the other. For more details about how conceptual understanding was operationalized during the creation of the questions analyzed here see (Wladis, Offenholley, Licwinko, Dawes, & Lee, 2018).

Methods

This study focuses on student responses to the multiple choice questions on the Elementary Algebra Concept Inventory (EACI). For details on the development and validation of the EACI, see Wladis et al., (2018). In this paper we focus on 698 students who took the inventory at the end of the semester of their elementary algebra class in 2016-2017. Ninety-one percent were students of color (half black and a third Hispanic), and roughly two-thirds were women. Roughly half were first-semester freshmen, and one-third were repeating the course because they failed or dropped it previously. The mean GPA for returning students was 2.47. Participants earned a mean score on the (entirely procedural) university final exam of 58%, and roughly one-third passed the course. In order to supplement quantitative data, 10 cognitive interviews conducted towards the end of the semester with students who were enrolled in an elementary algebra class were also analyzed using grounded theory (Glaser & Strauss, 1967), although a full qualitative analysis is not presented here due to space constraints.

In this paper we pursued latent class analysis (LCA) of the binary scored (right/wrong) multiple-choice items on the inventory. LCA is a latent variable model that is based on the principle of local independence but presumes that the items to be locally independent conditional on a discrete nominal latent variable (e.g., Collins & Lanza, 2010). As such, it does not assume an underlying continuous latent trait. We used Stata 15.1 (StataCorp, 2017) to fit the analysis via the EM algorithm using random starts to protect against local optima. No convergence problems were observed during the process of fitting. Both a two-class and a three-class analysis were explored. The three-class analysis fit the data better (AIC 7122 and 7114, respectively). For the three class versus the saturated model, $G^2(482) = 8.79$, with $p < 0.001$, suggesting that it fit the data well. We did not use covariates or the nominal item responses in LCA models to ensure that we could use the classes to examine the relationship with external variables (e.g., end-of-class standardized test scores, course outcome) and distractor responses.

Description of the classes

The latent class analysis revealed three groups that we characterize in the following way:

- C1 (27%): Students whose answers to most items are indistinguishable from random guessing, likely due to low procedural/conceptual knowledge and/or low motivation.

- C2 (28%): Students who likely have some good procedural skills but limited conceptual understanding.
- C3 (45%): Students who likely have good procedural skills and emergent conceptual understanding.

These class descriptions emerged from looking at the data in a number of different ways. Firstly, we consider the response patterns of students from each of the three classes, and we see some clear trends (see Figure 1).

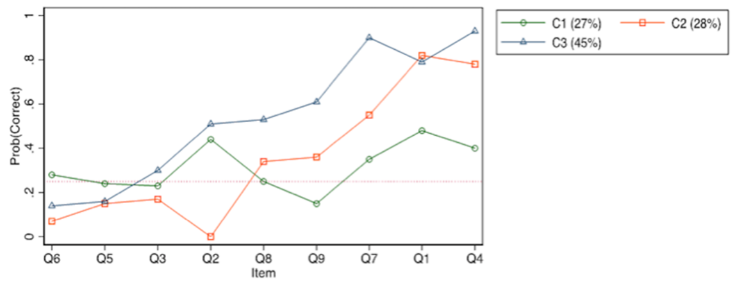


Figure 1. LCA profiles of student responses in each class[1]

Student responses in class 1 do not vary much from what would be expected for random guessing on four-option multiple choice items. These students only answer correctly at rates that are higher than chance on questions 1, 2, 4, and 7, with 1 and 4 being the two easiest questions of these nine for all classes. Their performance is never better than 50% on even their best item. While all of the items on the test were designed to test conceptual understanding, some of them are closer to traditional procedural questions or ways of thinking than others. Questions 1, 4, 7, 8 and 9 are more similar to standard problems and procedures than questions 2, 3, 5, and 6, which use more abstract or non-standard formulations of algebraic ideas. Responses on these more procedural questions are precisely what primarily distinguish class 2 from class 1. Class 2 answers significantly worse than chance on questions 2 and 6 because of the presence of attractive distractors that likely tap into misconceptions related to the misuse of procedures. Classes 2 and 3 are distinguished by improved performance on the items overall as well as different proportions of key misconceptions. We see also that students who passed the class were most likely to be in class 3, then class 2, and least likely to be in class 1 (see Figure 2). An end-of-course standardized assessment that measures procedural fluency showed a similar outcome.

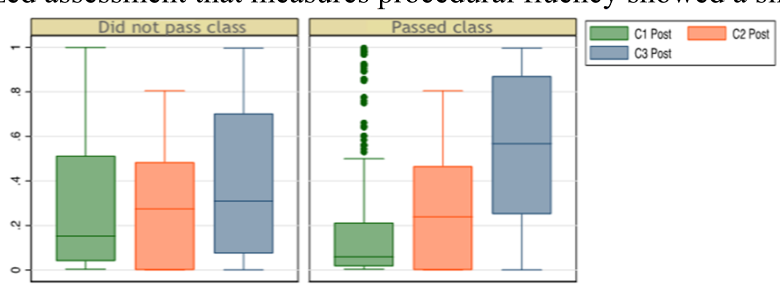


Figure 2. Posterior probabilities of class membership by course outcome (passing the class)

In order to illustrate how different response patterns might distinguish these three classes, we performed a distractor analysis and analyzed cognitive interviews for three exemplars: items 2, 4,

[1] For reference, in the plot above a somewhat conservative approximate 95% margin of error for a given point is .15. That is to say, points that differ by .15 are outside the 95% confidence interval.

and 6. We used the Bayes modal assignment to determine class membership. The median of the modal membership probabilities was 0.73. Examining the normalized entropy within each class suggested that class 2 was the best distinguished although no class was so poorly distinguished as to make classification useless.

Three example questions: illustrating different class response patterns
Item 4: First we consider Item 4:

Which of the following is a result of correctly substituting $x - 4$ for y in the equation $3y - 2 = y^2 + 1$?

a. $3x - 4 - 2 = x - 4^2 + 1$
b. $3x - 4 - 2 = x^2 - 4^2 + 1$
c. $3(x - 4) - 2 = (x - 4)^2 + 1$
d. $3x - 3 \cdot 4 - 2 = x^2(-4)^2 + 1$

The correct answer is c. We would expect students who understand that substitution means to substitute $x - 4$ in for y, but who do not completely understand how the underlying structure of substitution works would select options a and b with high frequency.

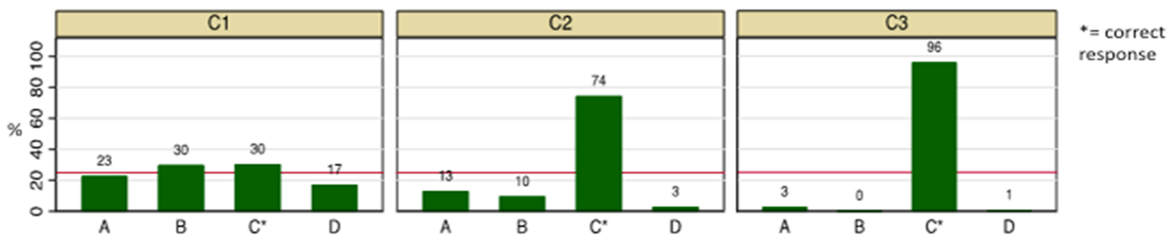


Figure 3. Item 4 Distractor Analysis

C1's option selection is clearly scattered in a pattern consistent with random guessing (see Figure 3). By contrast, classes 2 and 3 have a high probability of choosing the correct response, with C3's probability being significantly higher than that of C2. Selecting option c is highly correlated with student scores on the procedural exam, corresponding to a score that is higher by 10.8 percentage points ($p = 0.000$).

Looking at student interview responses reinforces our interpretation of the three classes.

C1 (chose B): It says $x - 4$ for y, this is what I think like because y^2 . It could be like changed to a 4^2. I put together like $3x - 4 - 2 = x^2 - 4^2 + 1$. [I didn't pick c or d because] they [pointing to the $x - 4$ in the item stem] didn't have no bracket around them. [I picked B with the x^2 in it instead of A, which doesn't have the x^2] because x equals y^2 so it has to have an x^2 in it because the y is squared there.

C2 (chose C): So usually like when a math question says, "substituting" that's like basically putting the numbers that they give you into x or y that they say to put it. And then I automatically substituted it in, and my correct answer was $3(x - 4) - 2$… I didn't pick any other answer, because I didn't see the parentheses.

$3y - 2 = x - 4^2 + 1$
$3(x-4) - 2 = x - 4^2 + 1$

C3 (chose C): I didn't choose A because when trying to multiply the y, which is $x - 4$, you have to put the parenthesis behind 3, unless you already multiplied 3 times $x - 4$… it [answer choice D] does have the parenthesis on -4, but then, it will be missing the complete equation for y because -4 is not the only equation that equals to y is $x - 4$.

In these examples, the C1 student shows a conception of substitution that involves putting $x - 4$ in where the y is in the equation, but does not show an awareness of the equation structure (e.g. that the $x - 4$ needs to be treated as a single unit when substituting). The C2 student shows an awareness of the procedure of putting in a set of parentheses around whatever is being substituted, but doesn't execute this procedure completely correctly on both sides, and doesn't demonstrate any awareness of why the parentheses are necessary. In contrast, the C3 student shows both an awareness of the need for the parentheses and an understanding of *why* the parentheses are necessary—because without them, the structure of the equation will be altered.

Item 6: Now we consider item 6, which shows a different pattern of responses:

A student is trying to simplify two different expressions:

i. $(x^2y^3)^2$
ii. $(x^2+y^3)^2$

Which one of the following steps could the student perform to correctly simplify each expression?

a. For both expressions, the student can distribute the exponent.
b. The student can distribute the exponent in the first expression, but not in the second expression.
c. The student can distribute the exponent in the second expression, but not in the first expression.
d. The student cannot distribute the exponent in either expression.

The correct answer to this question is b. Classes 2 and 3 were strongly attracted to option a (see Figure 4), likely because they are familiar with procedures associated with the distributive properties but do not recognize the critical difference between distributing multiplication versus exponents—likely because they lack a deeper conceptual understanding of why these properties work. We note that all three classes were strongly attracted to this distractor, likely for similar reasons. Examinees in class 1 do best on this item. Unlike item 4, selecting the correct answer for this item was negatively correlated (and selecting the distractor a was positively correlated) with scores on the procedural exam, with students who selected this distractor on average scoring 7.1 percentage points higher ($p < 0.000$). This suggests that in this context (where procedures are typically taught in isolation from concepts) procedural fluency in standard problem contexts is inversely related to conceptual understanding of the distributive properties.

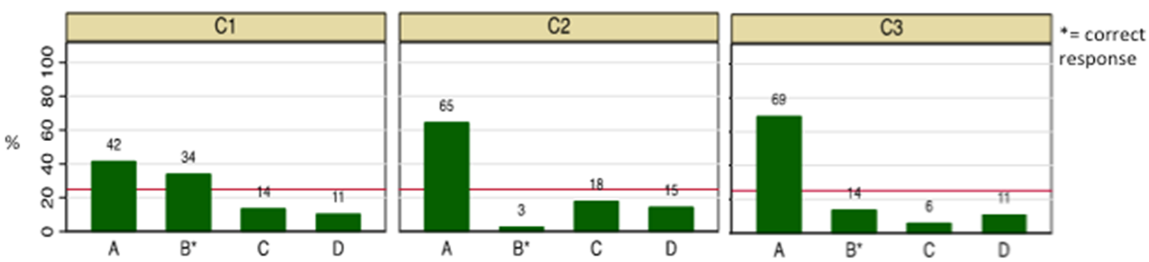
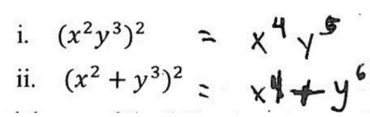


Figure 4. Item 6 distractor analysis

Looking at student interview responses reinforces our interpretation of the three classes.

C1 (chose B): [The difference between the first and second equation] is that there's a plus right there [pointing to the second equation]. I think for this one [pointing to the second equation], you have to add and for this one [pointing to the first equation] you don't…. Actually, I think like over here [pointing to the second equation] you add a 3. 3 plus 2. [For the first one] you do x^2 times x^2 and y^3 times y^3.

C2 (chose A): I feel like that's correct because in order to solve x^2 and y, you have to distribute…. Because I've seen problems like this before and it's like you have to solve it, there is no not solving it because there is no … there is no solution.



C3, but close to C2 (chose A): Since [both equations] are in parenthesis and they have exponents, the first thing that came into my head was PEMDAS…so after parenthesis will be exponents. So with the exponent, I know you would have to distribute and then you'll be able to solve the rest.

C3 (chose A): Both expressions the student can distribute the exponents because for the parenthesis you do multiply.

C3 (chose A): That's how you kind of get rid of the parenthesis and get rid of the outer exponents by distributing it in the inside. Whether it's with another exponent or with a number… You want to add or multiply that exponent [outside the parentheses] to the ones inside the parentheses but I can't remember whether you add or multiply…

In these examples, the C1 student notices that there is a difference between the two equations and suggests that it is important, but doesn't actually know how to perform the distribution correctly. For the C2 and C3 students, we see a number of ways in which students are incorrectly employing procedures or experience with procedures—we have only listed a few of them here, but every interviewee cited a different, procedural explanation for why the exponent could be distributed, including: reciting a procedure for distributing; citing standard problem contexts based on surface structure; stating that parentheses always mean that one should multiply; citing the order of operations. None of the students we interviewed in any class showed a deeper understanding of what distributing means or when it is possible. While C2 and C3 students may have shown evidence of understanding what exponents mean, they did not provide any evidence that they understood what distributing means in this case, beyond a basic citing of procedures (often inaccurately) that they had learned in class.

Item 2: Next we consider item 2, which reveals another interesting pattern of responses:

Consider the equation $x + y = 10$. Which of the following statements must be true?

a. There is only one possible solution to this equation, a single point on the line $x + y = 10$.
b. There are an infinite number of possible solutions, all points on the line $x + y = 10$
c. This equation has no solution.
d. There are exactly two possible solutions to this equation: one for x and one for y.

For this question, the correct answer is b, which was the most popular answer chosen by students in classes 1 and 3, but no examinee in class 2 chose it (see Figure 5). They were strongly attracted to option d, which was also the second most popular choice of students in both other classes, although at a much lower rate. Answer option d is a common response from students who are used to finding a solution to two linear equations in two variables; thus many students may select this answer because of an inappropriate application of procedural knowledge based on surface features of the equation. Interestingly, both the correct answer b (+4.0 percentage points, $p = 0.039$) and the popular distractor d (+5.5 percentage points, $p = 0.017$) are correlated with higher scores on the standardized procedural exam, although choosing the distractor is more strongly correlated with higher procedural skills as measured by the exam.

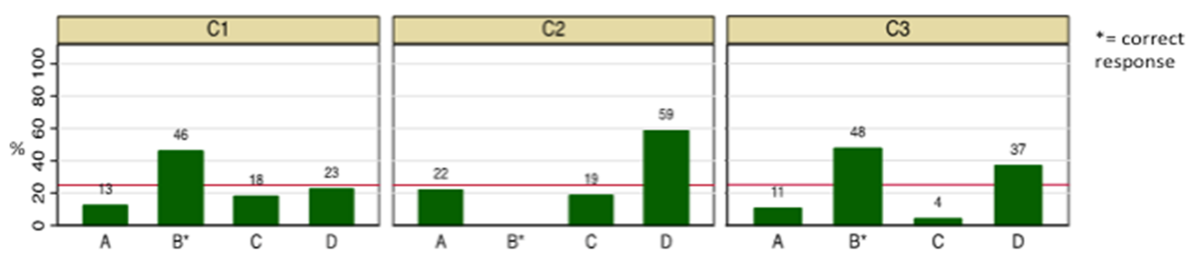


Figure 5. Item 2 distractor analysis

Looking at student interview responses reinforces our interpretation of the three classes.

C1 (originally chose C, but drifted towards B in the interview): $x + y$ equals nothing so it can't be 10. Right?... [Maybe infinite means] what could be like possible? I don't know. Like equal number maybe? $x + y = 10$. It could be possible like it equals 10. [Option D isn't correct] maybe because x and y could be equal to anything?

C2 (chose D): What I assumed was the x term and the y term, you would have to substitute. And I know there are certain numbers that will add up to ten, so there could be two solutions, since there's only a x term and a y term… Like x could equal 5, y could equal 5… since it is two terms, so you could say two different solutions.

C3 (chose B): Ten could equal to many things. Like five plus five could equal ten. Nine plus one could equal ten. Seven plus three. That's why I chose that, because it could be any number that will equal to ten. It's not just one certain number.

In these examples, the C1 student chose "no solution" because they didn't know what x and y could be, but as they explained more, they started to relate this to the idea that x and y could be "anything". While their reasoning is not strictly correct, they are beginning to explore the idea that x and y may have many possible values. The C2 student seems to understand enough about what the equation means to find a single solution, but once they find one solution they stop there, not exploring whether there might be others. Further, they confuse the number of solutions with the number of variables in the solution set, showing that they do not understand that a solution set is a collection of all possible combinations of variables that make the statement true. The student from C3 describes how this equation could have multiple solutions, demonstrating some conceptual understanding of how solution sets for equations work. They also demonstrate understanding that the values for both variables are related and that the solution set describes this. Other students in C3 cited the graphical representation of a line to describe similar ideas.

Discussion and Limitations

The patterns of student responses and explanations in cognitive interviews suggest that many students, including those who pass the course, are consistently using procedures inappropriately and without understanding; thus, these results suggest that instruction which stresses procedures divorced from conceptual understanding likely worsens a number of misconceptions. As we saw on two of the more conceptual questions, higher procedural fluency on standard problems actually corresponded to lower conceptual understanding of certain concepts. This suggests that the widespread use of instruction through repeated procedural practice, when isolated from any systematic attempts to practice interpreting and understanding these procedures, may actually be worsening fundamental algebraic conceptual understanding.

We note that this study did not attempt to directly link student responses to specific types of instruction—there is a pressing need for future research to examine the relationship between instructional characteristics and pre-post response patterns on validated concept inventories, in order to determine which kinds of instruction have the most positive or negative impact on student growth in conceptual understanding. In the meantime, this research reinforces existing research that suggests that teaching procedures in isolation, without concomitant conceptual understanding, may have negative consequences (Givvin et al., 2011; Stigler et al., 2010).

One limitation of this analysis is that we have only examined traditional four-option multiple choice items. Other items were administered that use a "choose all the apply" format, but are not analyzed here. This format may be very helpful in providing more information about student thinking than can be obtained from single option multiple choice; however, scoring these items and managing the local dependence is complex and is thus left for further research.

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Mathematicians' Perceptions of their Teaching

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Recent research in mathematics education has uncovered a host of teaching behaviors that are commonly enacted by instructors of advanced mathematics courses. While these descriptive accounts of math teaching are useful, little investigation has been conducted into the reasons for why these practices are so prevalent. In this study, we interviewed seven mathematicians about regularities that have been observed in the literature on the teaching of advanced mathematics. In this report we discuss whether mathematicians view these findings as accurate (they often did), whether they thought these regularities were productive or problematic teaching practices, and why mathematicians engaged in these teaching practices. We discuss how these themes may elucidate the practices of instructors, and later propose implications of the methods of the present study for changing how advanced math courses are taught.

Keywords: proof-based courses, teaching practices, formal content, informal content

In the last decade, numerous researchers in undergraduate mathematics education have sought to understand the pedagogical practice of mathematicians by observing how advanced mathematics courses are taught (e.g., Artemeva & Fox, 2011; Gabel & Dreyfus, 2016; Fukawa-Connelly et al., 2017; Mills, 2014; Pinto & Karsenty, 2018). The results of these studies give researchers insights into what practices are typical in the teaching of proof-based courses. Despite our advances in knowledge about what kind of moves and habits are used by instructors of these classes, little investigation has been carried out into mathematicians' motives and reasons for adopting the practices that are identified specifically with respect to these findings. (Other scholars have investigated mathematicians' motives and rationality for teaching advanced mathematics in general—e.g., Alcock, 2009; Hemmi, 2010; Nardi, 2008; Weber, 2012).

The aim of this study is to shed light on this largely unexplored area of pedagogy by entering into a conversation with mathematicians explicitly about these findings. If we believe that mathematicians are reflective about their own teaching and the teaching that goes on around them, then they have important knowledge to help explain the classroom behaviors that education researchers have documented. We shared the results of some research on teaching in proof-based math courses with mathematicians who have been instructors of these kinds of courses and asked them to reflect on whether these results were an accurate depiction of their experiences, how they felt about the practices described in the results, and what reasons they saw for engaging in or avoiding these practices.

We believe that the contribution of this research is threefold. First, it is a continuation of the dialogue between mathematicians and math educators whose significance several members of our community have extolled (e.g., Iannone & Nardi, 2005; Alcock, 2009). More importantly, it develops this conversation in a direction that has been neglected by recognizing the value that mathematicians bring to interpreting research in which mathematics instructors are themselves the subject of study. Finally, from a practical standpoint, if the education community wishes for its research to effect change in the way that proof-based math courses are taught then how mathematicians feel about the research will suggest different ways of working with instructors to bring about that change. For instance, if mathematicians were surprised and unsettled by the research findings of our field, this suggests a pivotal way toward changing instruction is to disseminate our results to make mathematicians more aware of their teaching practices. However, if mathematicians are aware these teaching practices are common and feel that they are productive or necessary, then understanding mathematicians' rationality for engaging in the teaching practices is pivotal if mathematics educators hope to change them.

Literature Review

Speer et al. (2010) noted the lack of what they referred to as "descriptive empirical research on teaching practice" (p. 100) in collegiate mathematics and called for more work that elaborated the decisions and actions that instructors make when they teach college-level math classes. This sparked an increase in the amount of research that focused on the facets of instruction that are witnessed in advanced math classes (e.g., Fukawa-Connelly, 2012; Pinto, 2013; Gabel & Dreyfus, 2016; Mills 2014). While many researchers focused on case studies (Fukawa-Connelly, 2012; Fukawa-Connelly & Newton, 2014; Lew et al., 2016; Pinto, 2013), other studies analyzed and compared a relatively larger amount of instructors simultaneously. One general finding from this work is that mathematicians' instruction is nuanced and deviates from the "definition-theorem-proof" formalist caricature that is found in the literature (cf. Weber, 2004), but there are nonetheless some commonalities in how mathematicians teach advanced mathematics. Our present research involved presenting the findings of five of these studies (Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Paoletti et al., 2018; Moore, 2016; Miller et al., 2018) to mathematicians and asking them to speak about them. We briefly describe these studies here, with a focus on the work by Fukawa-Connelly and his colleagues in 2017 as this is the study that the mathematicians considered in the results we have chosen to include in this report.

Artemeva and Fox (2011) observed 33 college-level math lectures across seven countries with the goal of noticing which elements of instruction were shared by their participants and which elements differed. Prominent among their findings was a pedagogical genre they called "chalk talk," in which an instructor (a) wrote mathematics on the board, (b) narrated aloud what was being written along with her thought processes, and (c) occasionally took a break to present a metanarrative that discussed broader themes with the class. Paoletti and his colleagues (2018) used data obtained from 11 upper-level math instructors' teaching to draw conclusions about the types of questions instructors asked to their classes and how they used these questions to invite participation from the students. Their results showed that instructors often used a large amount of questions per lecture, most of which asked students to provide the next line in a proof, recall a fact, or perform a calculation, but that very often less than three seconds were provided for students to respond to these questions. Moore carried out a task-based study in 2016 to see what considerations went into how four math instructors graded student-written proofs. He found that there was a sizeable variation in the scores that his participants gave to the same proofs and that all of his participants assigned scores to a proof based on what they believed the student was thinking when he or she wrote it. Similarly, Miller, Infante, and Weber (2018) asked nine mathematicians to assign grades to proofs, half of which were designed to contain logical gaps. In addition to confirming Moore's findings, they noticed that several participants assigned less than perfect scores to proofs that they still deemed "correct."

A study performed by Fukawa-Connelly et al. in 2017 sought to clarify the extent to which informal content plays a role in advanced math classes and how instructors present it to their students. Their definition of informal content included any information that could not be conveyed in formal symbolic language, such as heuristics for thinking about a mathematical concept or for producing a proof. In their analysis of the lectures they observed, the researchers identified when informal content was displayed to the class, whether or not the instructor wrote it on the board, and when the content made it into each of the students' notes. They discovered that while informal content is used frequently in advanced math classes, this information is usually only delivered orally and is not written on the blackboard. Moreover, they found that informal content that was only presented orally and not written on the board only appeared in students' notes in less than 3.2% of possible instances. This was contrasted with both formal and informal content that was written on the blackboard, which was almost always found to be recorded in the students' notebooks.

Despite the progress made in detailing widespread regularities in collegiate teaching practice, little has been done to share these results with mathematics instructors and to understand how they make sense of them. As Fukawa-Connelly, Johnson, and Keller (2016) lamented, "there has been little research attempting to explore [the extent of the adoption of reform practices] from the perspective of the instructors who are the ones being asked to change practice" (p. 276). Consequently, this has impeded mathematics education reform efforts as mathematics educators seek solutions to teaching practices that mathematicians do not find problematic. We share their belief that mathematics instructors possess a unique corpus of knowledge that can bring more light to the findings on collegiate teaching than the findings alone are able to convey themselves. The work presented in this report is our attempt to begin a conversation with mathematicians that utilizes this special knowledge and positions the results of mathematics education research according to their viewpoints.

Theoretical Perspective

In the current study, we largely wanted to understand the issue from the perspective of mathematicians. Consequently, we sought to provide accounts of mathematicians' rationality that was grounded in the data that we collected and avoided applying a theoretical perspective on the data at an early stage (Glaser, 1998). Nonetheless, our study was inspired by Herbst and Chazan's (2003) notion of practical rationality and their dictum that teachers do not engage in traditional teaching practices "from a lack of knowledge or a paucity of vision" (p. 3). Rather teachers are reflective and rational; their pedagogical actions are reasoned attempts to fulfill their goals, obligations, and desires, which can involve a complex constellation of disciplinary, institutional, and ethical considerations (Chazan, Herbst, & Clark, 2016). In analyzing our data, we sought to understand what goals mathematicians had and how they thought their goals could best be achieved.

Methods

Participants

The participants for this study were seven mathematicians (one female and six male) from a large, public research university in the northeastern United States. Each participant had taught at least one proof-based mathematics course within the last five years.

Data Collection

Each participant took part in an approximately hour-long semi-structured interview with the first author. These interviews were audio recorded and subsequently transcribed. Questions for the interviews were pre-written in a protocol that focused on each of the five sets of findings described in the literature review of this report. The questions were designed to investigate each mathematician's general impressions of the results, if they believed the results were typical of teaching in advanced math courses, reasons for why they or others engaged in the teaching practices discussed in those results, and strengths and weaknesses of the practices. Follow-up questions were posed by the interviewer to clarify participants' responses or to encourage the participants to expand upon an idea they had shared.

Data Analysis

Data analysis consisted of a separate round of coding for each of the five sets of findings that were the focus of the interviews. Coding of these sections of the interview transcripts was carried out using thematic analysis (Braun & Clarke, 2006). The authors made an initial pass through the data, highlighting excerpts of the participants' responses that exemplified interesting ideas they had shared about the findings that were guiding the discussions. A descriptor for each idea was entered into a Word file along with a more detailed explanation of this idea. After these ideas had all been generated, the authors sought for commonalities among the ideas and arranged them into larger themes that preserved the general spirit of the ideas while capturing the notable similarities between them, along with specific criteria for when an utterance would be coded as a member of that category. For each theme, we sought to understand if mathematicians were expressing that they perceived that they had a goal or obligation to meet in their instruction (cf., Herbst & Chazan, 2003) and if they had a belief about whether a specific teaching practice would be productive or counterproductive for achieving that goal. When these larger themes were created, a second pass through the data was made to code the corresponding sections of the transcripts with them. After, a Word document was created for each of the larger themes and interview excerpts that were coded with that theme were copied and pasted into the corresponding document.

Results

For the sake of brevity, we report only the results pertaining to the portion of our interviews with mathematicians that concerned the portrayal of formal and informal content in upper-level math classes (Fukawa-Connelly et al., 2017). Of the seven mathematicians interviewed, all agreed that the finding that formal content is written down on the blackboard and informal content is usually only spoken orally is an accurate portrayal of advanced math classes. All seven also agreed that these are generally good teaching practices, although five expressed reservations according to a sentiment that good teaching would display a better balance of formal and informal content being written on the blackboard. Five of the mathematicians stated that these findings were generally representative of their own teaching while two denied so, stating that they also often made informal ideas and processes explicit in writing.

During coding, eight broader themes emerged from the interviewees' commentary on these findings. Five of these themes dealt largely with the practice of writing formal content on the blackboard whereas the other three spoke more to the practice of presenting informal content exclusively verbally. In what follows of this results section we describe the ideas that were expressed in these themes, giving examples from interview transcripts to illustrate when appropriate. Each theme is also shown in Table 1, next to the list of mathematicians who had at least one utterance coded with that theme.

Table 1. A table that lists each of the eight themes that emerged from the data, and which mathematicians' utterances comprised those themes.

Mathematicians' rationales for their use or disuse of the blackboard	
Theme	Interviewees that Contributed to this Theme
Blackboard allows for deeper processing and comprehension	M1, M3, M5, M6, M7
Written content is given permanence and importance	M2, M3, M4, M5
Blackboard enables and requires precision	M3, M5, M6, M7
Writing on the board slows the instructor down	M5, M6, M7
Writing formal content emphasizes the language, notation, and nature of mathematics	M4, M7
Oral presentation is needed to hold students' attention	M1, M2, M3, M5, M6
Informal content is conversational in nature	M1, M2, M6
Content should be repeated to be noticed	M2, M3

Writing Formal Content on the Blackboard

Blackboard Allows for Deeper Processing and Comprehension Five of the mathematicians interviewed expressed that writing content on the blackboard allows for it to be carefully processed and aids students in understanding it. As M6 said, "your visual cortex is extremely powerful and somehow seeing words written down on - somewhere, anywhere, a blackboard being a good place, really clarifies a lot of things. So I think writing things down on a blackboard is extremely important."

Within this theme, the mathematicians pointed out that formal content is likely to be less relatable or more unfamiliar to students. M2 said "the technical nitty-gritty is what's least gonna be in the students' minds and so it's most important to have…that written down." On the other hand, informal content is unlikely to need such a high level of processing, and so it is more acceptable to forgo writing it on the board. M5 illustrated this notion when he said "when I'm conveying intuition, the oral words convey that intuition, I don't need to analyze that intuition, intuition is sort of part of the analysis in a way." These mathematicians had the goal that students understand (or at least "process") the technical mathematics; to do so required students having a specific object for this reflection.

Written Content is Given Permanence and Importance Four of the participants spoke to this theme, which deals with two related properties of information that is conveyed on the board. One of these properties is permanence. Writing things such as formal content down on the board preserves them so that they may be checked and referred to later, and so that students can make their own records of them. To this latter end M4 said "what's written on the board students would take notes of, and what's not they might remember, they might not remember. So if you really want to present something you want to write it on the board." The other property that participants mentioned about written content is a higher level of perceived importance of content that is written down compared to content that is not. M5 said "when you write something on the blackboard you are emphasizing that it's important," and M2 remarked that students think "what I have to pay attention to is what's been written down." Implicit in this commentary is that the informal mathematics that is not written down might be less important.

Blackboard Enables and Requires Precision Four of the participants mentioned the significance that the level of precision has on which content is written down during class. Within this theme, the mathematicians noted that formal content is often very precise, and the blackboard is a crucial tool for displaying this precision properly. M3 shared that when discussing a definition, theorem or proof, "it does have to be written on the blackboard because it has to be precise, notation has to be set up, things have to be checked." Informal content usually lacks this degree of detail, and indeed can sometimes be difficult to portray accurately via a written medium. M3 explained this with her comment that "if you write something informal [students] can often misinterpret what you've said and write something entirely different in their notes, so it's a bit problematic, it's a bit tricky to convey this extra information." In addition to the board being useful for expressing precise content, some interviewees noted that some instructors may view the blackboard as being reserved for precise content. When discussing why informal content is usually only delivered orally, M6 related that "I know of people who want to be extremely precise…and that's why they will only write the things that are absolute certainties."

Writing on the Board Slows the Instructor Down Three of the interviewees remarked that a virtue of writing things down on the blackboard is that it slows the pace of instruction and gives the students a chance to comprehend what is being taught to them. Especially when it comes to formal content, the time it takes to utter a definition, theorem, or proof may not be enough for a student to properly analyze it. Writing these things down in addition to speaking them allows students extra time to process them. M7 exemplified this notion with his comment that "the proofs also should be written down…otherwise it would just be too fast to follow."

Writing Formal Content Emphasizes the Language, Notation, and Nature of Mathematics Two mathematicians gave responses that illustrated this theme. M4 mentioned that writing formal content down on the board helps students understand a key fact about the nature of the subject matter, "the idea that mathematics consists of definitions, theorems, and proofs." M7 also noted that mathematics requires a commonly established language and notation, and that writing formal content down helps to achieve this classroom goal. The goal here for these two participants is that students understand the general nature of how formal mathematics is expressed which (naturally) requires seeing the expression of formal mathematics.

Not Writing Informal Content on the Blackboard

Oral Presentation is Needed to Hold Students' Attention Five participants contributed to this theme, which contrasts the level of engagement students have with written versus oral content. The mathematicians noted that sharing informal content is often a matter of telling students how you would like them to be thinking and reasoning. This is best achieved orally, and not through writing. Along these lines M1 stated

> When you're facing the students and you're talking to the students they're more engaged. So when you're trying to explain something that's not formal but you're trying to give an idea of what's going on and how they should be thinking about it, then you want to have them engaged.

M3 expressed a similar opinion that oral, non-written content has the ability to awaken students from a seeming daze.

> Well, in my experience when the instructor maybe puts down the chalk and turns to the front of the class and addresses the class with some anecdote or some informal way of thinking about a concept…that's when students start paying attention.

Written content, on the other hand, can lead to students mindlessly copying down what they see without giving it thorough consideration. Speaking about written content, M5 worried that "they're just transferring it onto each of their notebooks, and whether they actually are getting anything out of it it's not clear." One goal expressed here is that students should be engaged during their advanced mathematical lectures; the stilted process of writing points down can diminish this engagement. This point is interesting as we imagine many mathematics educators would question the assumption that speaking to students more informally would be sufficient to obtain meaningful engagement.

Informal Content is Conversational in Nature Three interviewees gave responses that suggested informal content is itself conversational, and therefore is more naturally conveyed to students in an oral and non-written fashion. M1 described informal content as a "flow of ideas" and said that "if I'm trying to have a discussion with somebody about how you think about this informally, then writing it down on the board converts a discussion into a stilted process." M6 further characterized informal content as a conversation when he stated that "the very nature of informal discussion is that it's not precise." M2 noted the full power of informal content must be exchanged orally, saying "I don't think you're going to inspire people as to the importance of the big ideas without giving a verbal…description of those big ideas." Here the participants are expressing the importance of informal content, but felt that oral presentation is the best way to present this content, which often lacks the precision of formal mathematics.

Content Should Be Repeated to Be Noticed Two participants contributed to this final theme. M3 noted that "research shows that in any room for any presentation, regardless of the topic, at one moment in time, at any given moment in time, one third of the people are not actually listening. And so…to get across an important point, you have to repeat it three times." M2 hypothesized that the big ideas that are usually contained within informal content are repeated enough for students to take notice of them, whereas formal content that is not often repeated can use the blackboard to garner students' awareness. He shared that "I feel like I have to write those technical ones down far more often and the broader ideas get repeated just continually so that I don't."

Discussion

The results we have presented support our claim that instructors of advanced mathematics have valuable observations to make regarding the regularities that mathematics education researchers find in their research. While education researchers can be said to have given a descriptive account of the teaching of mathematicians, discussion with mathematicians can reveal the beliefs and goals that account for the prevalence of the practices we see.

The perspectives that mathematicians bring to math education research have a considerable practical implication. If we wish to change how advanced math classes are taught, it is imperative that we understand how instructors view the practices that comprise their instructional activity. For example, if mathematicians view a particular practice as undesirable and are unaware that it is common among instructors, then to change it may simply require bringing it to the attention of the larger teaching community. If mathematicians see a practice as undesirable but are aware of its widespread use, then this suggests that they may need assistance in the development of teaching practices that can take its place. However, if mathematicians express reasons for viewing a practice as desirable despite math education researchers' aversion to it, then to change such a practice it may be most fruitful to explore alternative practices that possess the qualities that mathematicians find useful and favorable about the current one.

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Bridging the Gap: From Graduate Student Instructor Observation Protocol to Actionable Post-Observation Feedback

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In this study, two universities created and implemented a student-centered graduate student instructor observation protocol (GSIOP) and a post-observational Red-Yellow-Green feedback structure (RYG feedback). The GSIOP and RYG feedback was used with novice graduate student instructors (GSIs) by experienced GSIs through a peer-mentorship program. Ten trained mentor GSIs completed 50 sets of three observations of novice GSIs. Analyzing 151 GSIOPs and 151 RYG feedback meetings longitudinally provided insight to identify what types of feedback informed and influenced GSIOP scores. After qualitatively coding feedback along multiple dimensions, we found certain forms of feedback were more influential for GSI development than others with respect to change in GSIOP score. Our results indicate contextually-specific feedback leads to more observed changes and improvement across multiple observations than decontextualized feedback.

Keywords: Graduate Student Instructors, Feedback, Observation, Mentoring

Introduction

Mathematics graduate student instruction significantly impacts undergraduate courses and students (Belnap & Allred, 2009). Graduate student instructors (GSIs)[1] have been identified as a key component of success for collegiate mathematics departments for teaching undergraduate mathematics (Bressoud, Mesa, & Rassmussen, 2015, p. 117). As a result, mathematics departments and research in undergraduate mathematics education continue to focus on supporting and improving GSIs' student-centered instruction (Rogers & Yee, 2018; Speer & Murphy, 2009; Yee & Rogers, 2017). There are multiple methods of student-centered pedagogical support for GSIs (e.g. professional development, mentoring, pedagogically-focused courses; Speer, Gutmann, & Murphy, 2005; Yee & Rogers, 2017), but there is currently limited research on GSI teaching observation protocols and even less research on post-observation feedback (Reinholz, 2017). Multiple observation protocols exist to assess undergraduate mathematics instructors' classrooms (e.g. MCOP[2], RTOP, C-LASS, etc.), often with scalar metrics such as 1-4, but many do not discuss how to make that assessment actionable so that it can be beneficial for the teacher.

To this end, we created a GSI observation protocol (GSIOP, Rogers, Petrulis, Yee & Deshler, under review) and a post-observation feedback structure at two universities to provide ongoing support for novice GSIs. Together, the GSIOP and feedback were implemented for two years as part of a peer-mentorship model[2] where novice GSIs were mentored by experienced (two or more years of experience) GSIs who had completed a mentor professional development (PD) seminar. This mentor PD included training with the GSIOP and post-observation feedback (See Rogers & Yee, 2018 and Yee & Rogers, 2017 for more information on peer-mentorship). The

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[1] GSI was used instead of TA (Teaching Assistant) because GSI references graduate students who are full instructors of record.

[2] Supported by a Collaborative National Grant

purpose of this paper is to help bridge the research gap between observations and post-observation feedback by identifying how feedback within this peer-mentoring model informed and influenced future observations. Our research questions for this study are:

- RQ1: In what ways (if any) did the feedback structure lead to changes in teaching observations throughout a semester?
- RQ2: How do those changes inform (if at all) methods for providing actionable feedback to influence observed teaching?

It is important to note that our study focused on GSIs but the observation protocol, feedback, and results are applicable to undergraduate mathematics instructors, not just GSIs.

Related Literature

Feedback

For over a century, psychology has long researched the importance of feedback as a means to change performance, cognition, and understanding in many professions (Kluger & DeNisi, 1996). Hattie and Timperley's (2007) meta-analysis looked at 500 articles of teachers providing feedback to students and found assessment-based feedback was one of the most dangerous forms because "rarely does such enhance the processes and metacognitive attributes of the task" (p. 101). White's (2007) research on 16 pre-service teachers showed that clear, concise, specific, and encouraging feedback were the most valuable forms of feedback. White's research also emphasized what Hattie and Timperley (2007) identified, that feedback (and thus observations) needs to be done regularly, not intermittently.

Although K-12 mathematics education research has extensively studied feedback within practicum courses (e.g. student teachers are observed regularly by their master teacher and university supervisor as a critical means of ongoing teacher development) our review of the literature has found few studies focusing on mathematics GSI peer feedback (Reinholz, 2017; Rogers & Steele, 2016; Yee & Rogers, 2017; Rogers & Yee, 2018). One exception is a recent study by Reinholz (2017) that explores peer feedback with mathematics graduate students as equal peers. Reinholz had six GSIs provide peer-feedback to one another and found that feedback not only helped the novice, but enhanced teacher noticing (Sherin, Jacobs & Philipp, 2011) and reflection in the observer, aligning with Reinholz's previous work (2016) where peer assessment led to improved self-assessment. Rogers and Steele (2016) concluded that novice instructors struggle to discuss teaching methods, which Reinholz (2017) argues could be aided by peer feedback. Thus, Reinholz's and Rogers and Steele's (2016) research supports post-observation feedback as a means of improving GSIs' teaching through discourse and reflection.

Complexities of Observations and Feedback

Reinholz (2017) reminds us that "how instructors engage with peer feedback is complicated" (p. 7) due to GSIs' beliefs about mathematics and its often-assumed relationship to innate intelligence. Kluger and DeNisi's (1998) meta-analysis of 607 studies on feedback interventions (i.e. providing people with some information regarding their task performance) showed that while overall feedback improves performance, it can also sometimes reduce performance, depending on the type of feedback and means by which it is delivered. Certain feedback was helpful for improving performance as long as attention was directed towards task-motivation and task-learning rather than praise, negative criticism, or focus on the person because deviating from the focus on the task requires effort that was found to decrease performance.

In light of the complexity that links observations and feedback, we questioned what type of feedback is most effective for GSIs. Cannon and Witherspoon (2005) provide a framework to navigate this complexity effectively using *actionable feedback* "that produces both learning and tangible, appropriate results" (p. 120). Actionable feedback provides a framework for examining undergraduate mathematics classrooms and providing feedback to help novices make changes to improve their teaching. We use this frame in our data analysis to determine how feedback affected the tangible result of novices' GSIOP scores over a semester.

Framework of Study

Our peer-mentorship research (Yee & Rogers, 2017; Rogers & Yee, 2018) and current literature (Reinholz, 2017) has found observational protocols need to have complementary feedback structure where novices are able to reflect more openly about how they can modify their teaching to achieve their goals. Hence, our design emphasizes post-observation feedback as reflective to complement the more evaluative observation protocol.

GSIOP

The initial goal of our peer-mentorship model was to provide feedback and facilitate discussions among novice GSIs around student-centered teaching strategies to improve undergraduate mathematics instruction (Rogers et. al., 2018, under review). The Mathematics Classroom Observation Protocol for Practices (MCOP^2, Gleason, Livers & Zelkowski, 2017) is an observation protocol designed for K-12 that originates from the STEM-based Reformed Teaching Observation Protocol (RTOP, Sawada et al., 2002), but unlike the RTOP, includes a means to observe student-centered investigations and collaborative learning environments focusing on mathematics. Thus, we modified the MCOP^2 to be applicable for use when observing GSIs and developed the GSIOP which focuses on both student and instructor actions. Similar to the MCOP^2, the GSIOP contains questions on an ordinal scale from 0 to 3 for four sections: classroom management, student engagement, teacher facilitation, and lesson design. A more thorough explanation of the GSIOP design can be found in Rogers et al's validation study (under review).

RYG Feedback

Mentors were educated through the mentor PD to use the GSIOP during their PD program (see Yee & Rogers, 2017) and to facilitate post-observation conversations using a Red-Yellow-Green feedback structure. Using this structure, mentors identify key points from the GSIOP that they could summarize for the novice in three categories: methods the novice is doing well (green), methods the novice could work on (yellow), and methods the novice needs to address (red). The mentor would summarize points of discussion from the GSIOP and keep the feedback manageable by discussing at most two concerns within the yellow and red categories. Scenarios, role playing, and live observations helped prepare mentors to provide feedback in each category during post-observation meetings that occurred within a week of the observation. We refer to this post-observation feedback as the *RYG feedback*.

Methods

In this mixed-methods study, we quantitatively analyzed changes to GSIOP scores to answer our first research question. We then qualitatively coded the RYG feedback for types of actionable feedback and compared the types of feedback with the changes in GSIOP scores to answer our second research question.

Participants & Observations

This study included 10 mentor GSIs and 32 novice GSIs from two universities in the United States over two semesters. New novices were added between semesters while other novices completed their training after one semester. For this reason, we focused on sets of semester-long observations, which consisted of three observations with feedback for each novice on average (two novices were observed only twice while three novices were observed four times). This generated 50 sets of semester-long observations with feedback, totaling 151 observations with feedback. Mentors submitted novice teaching notes, videos of the novice's class, observation summaries, completed GSIOPs, and RYG feedback for analysis.

Data Analysis

As our research study emphasized student-centered instruction and RYG feedback, we focused only on the two sections of the GSIOP that emphasized student-centered instruction, the *student-focused* (student engagement) and *teacher-focused* (teacher facilitation) sections. One research assistant at each university longitudinally analyzed the GSIOP scores from both the student- and teacher-focused sections for each novice over an entire semester. Similarly, each research assistant analyzed the RYG feedback and observation summaries for student-focused feedback and teacher-focused feedback that aligned with the questions from appropriate sections of the GSIOP. This created 100 longitudinal data sets of semester-long observations and 100 data sets of semester-long feedback (50 student-focused and 50 teacher-focused).

To answer our first research question, we summed the questions on the GSIOP student-focused section (4 questions) and the GSIOP teacher-focused section (5 questions) separately. Thus, for each observation of each novice each semester, there was a teacher-focused GSIOP score and a student-focused GSIOP score. We looked at change in GSIOP scores over a single semester by looking for trends and subtracting novices' final GSIOP score from their initial GSIOP score for both the student- and teacher-focused sections.

To answer the second research question, we looked at the data collected by the mentor during each observation and the feedback each novice received from the mentor. We analyzed feedback through an *advice and improvement* framework. We looked at RYG feedback, GSIOP comments, and mentor observation summaries for suggestions that provided the novice with *advice* on teaching that focused on student learning or teacher facilitation. We then looked through the data sets at each novice to see if the mentor noted any observed *improvements* related to advice given previously in the semester.

Next, we coded each piece of advice and each noted improvement as *broad* or *specific*. To frame broad versus specific objectively, we used Nilsson and Ryve's (2010) definition of contextualization where the context of an event must be given to make a situation specific and not referencing a context or event (often referred to as decontextualized) would be considered broad. Looking at feedback as advice or improvement concomitantly as broad or specific provides a categorization demonstrated on Table 1 with prototypical examples.

The last two categories, Advice Without Improvement (AWI) and No Advice Nor Improvement (NANI) took into account if advice and improvement were not given. AWI implied advice (broad or specific) was given, but improvement was not noted in subsequent observations. NANI lacked advice and therefore no improvement could be noted in subsequent observations.

To triangulate the qualitative coding of advice and improvement as broad or specific, after each research assistant qualitatively coded the results according to Table 1, two additional researchers went back and verified their work by comparing 75 of the 151 observations and post-observation feedback artifacts for both teacher-focused feedback and student-focused feedback. Interrater agreement was initially 94% and after discussion of the coding discrepancies, researchers agreed on the appropriate coding for the remaining 6%.

Table 1. Qualitative Coding Scheme for Feedback across an Entire Semester

Code	Description	Example

SASI	**Specific Advice Specific Improvement**: Feedback included at least one contextualized suggestion the novice could take to improve their teaching. In subsequent observations, the mentor noted that the novice had addressed the issues through particular contexts, actions, and/or strategies.	"Elaborate with the material and explain the importance of the concept. For example, one instance in which you could give a little more insight and explanation was when the student used P(A U B) = P(A)+P(B) - P(A cap B)"...(*later observation*) "You elaborated more than last time.. I felt that this was the perfect amount of elaboration. Also, you asked well thought out questions, and you rarely missed good opportunities to ask further questions."
BASI	**Broad Advice Specific Improvement**: Feedback included suggestions without context on when or how to improve the novice's teaching. In subsequent observations, the mentor noted that the novice had addressed the issues through particular contexts, actions, and/or strategies.	"Have tiny bits of student involvement through to keep students engaged" … *(later observation)* "Student questioning chosen was very effective in engaging students [with 2^x and log_2(x)]"
SABI	**Specific Advice Broad Improvement**: Feedback included at least one contextualized suggestion the novice could take to improve their teaching. In subsequent observations, the mentor noted that the novice had improved upon previous issues, but without referencing specific contexts.	"I encourage you to give more wait time before answering the questions yourself, this can have them participate more" … (*later observation*) "I saw great improvement since last time with student engagement….(*later observation*) "Great student interaction".
BABI	**Broad Advice Broad Improvement**: Feedback included suggestions without context on when or how to improve the novice's teaching. In subsequent observations, the mentor noted that the novice had improved upon previous issues, but without referencing specific contexts.	"Student engagement should be addressed" … (*later observation*) "Even though she ask[ed] many questions, students are not really active in this particular class"…(*later observation*). "She did not just answer but encourage[d] students to respond".
AWI	**Advice Without Improvement**: Feedback included suggestions, but the suggestions did not appear to be noted throughout the subsequent observations.	"For the next time, I hope that he can get more active participation during his lecture portions" *No follow up.*
NANI	**Neither Advice Nor Improvement**: Feedback was either statements extolling the novice's instruction or platitudes on teaching. Mentor did not provide advice nor improvements.	"He did a great job in his lesson of engaging the students, explaining material adequately and also giving his students problems to work on at the end of class". *No advice.*

Results

Longitudinally, each novice's three GSIOP scores from both the student-focused and teacher-focused sections determined how each set of three scores varied. We categorized the changes as *decrease* (each observation was at least two points less than the previous one), *steady* (each observation was within one point of the previous one), *moderate increase* (each observation was at least two points higher than the previous), *substantial increase* (each observation was at least three points higher than the previous), *hill* (middle score is at least two points higher than the other scores), and *valley* (middle score is at least two points lower than the other scores). Table 2 shows how many student-focused and teacher-focused sections (changes across a semester) fell into each category.

Table 2. Longitudinal Semester-Long Changes in GSIOP Scores by Student- and Teacher-Focused Sections

GSIOP Change Categories	Substantial Increase	Moderate Increase	Steady	Decrease	Hill	Valley	Grand Total
Number of Student-Focused Sections	9	12	15	10	2	2	50
Average GSIOP Change Per Student-Focused Section	5.00	2.50	0.20	-3.90	-1.00	-0.50	0.72
Number of Teacher-Focused Sections	9	14	18	5	3	1	50

Average GSIOP Change Per Teacher-Focused Section	5.11	2.21	0.28	-3.60	0.67	-1.00	1.30
Number of Student- and Teacher-Focused Sections	18	26	33	15	5	3	100
Average Change Per Student- and Teacher-Focused Sections	5.06	2.35	0.24	-3.80	0.00	-0.67	1.01

Results show that for both the student- and teacher-focused sections, on a 0-3 point scale, there was an average positive change of 1.01 points per section. We see that the number within each category had a fairly equal distribution between student- and teacher-focused sections, with the student-focused sections showing more decreases and the teacher-focused sections showing more steady or moderate increases. Although a majority of the GSIOP scores remained steady (33 out of 100), there were significantly more novices whose score increased moderately or substantially (44) than those that decreased (15) over a semester. Thus, our results indicated there was an observed change in teaching throughout a semester via the GSIOP score showing an overall increase in point value.

To answer our second research question, we wanted to understand the feedback at a more contextual (Nilsson & Ryve, 2010) level to determine how the feedback was actionable. We tallied the total change in score for all novices during a semester by taking the final GSIOP score for each section and subtracting it from the initial GSIOP score for that section. We then divided the total change by the number of novices to get the average change per novice.

Table 3. Inductive Analysis of Feedback Types Cross-Referenced with Change in GSIOP score

Feedback Types	SASI	BASI	SABI	BABI	NANI	AWI	Grand Total
Student-Focused Feedback	4	2	7	12	11	14	50
Average GSIOP Change Per Student-Focused Section	4.50	3.50	3.57	0.58	-0.73	-0.93	0.72
Teacher-Focused Feedback	10	4	4	8	5	19	50
Average GSIOP Change Per Teacher-Focused Section	3.40	3.00	-0.25	2.38	0.80	-0.16	1.3
Student and Teacher Feedback	14	6	11	20	16	33	100
Average GSIOP Change Per Student- and Teacher-Focused Feedback	3.71	3.17	2.18	1.30	-0.25	-0.48	1.01

Table 3 shows that of all 100 data sets of semester-long feedback, the one with the highest average change in GSIOP score was when mentors provided and noticed Specific Advice and Specific Improvement (SASI, $M=3.71$). SASI feedback also resulted in the highest change in GSIOP scores for both student and teacher sections. BASI feedback provided high changes as well, but with fewer student-focused feedback ($N=2$) and teacher-focused feedback ($N=4$) sections. SABI feedback influenced the student-focused section more ($M=3.57$) than the teacher-focused section ($M=-0.25$) while BABI feedback influenced the teacher section ($M=2.38$) more than the student section ($M=0.58$). Both Advice Without Improvement (AWI, $M=-0.48$) feedback and No Advice and No Improvement feedback (NANI, $M=-0.25$) had the least change in GSIOP scores.

Discussion

In answering our first research question, we see from Table 2 that RYG feedback in our study led to both increases and decreases in GSIOP scores associated with student engagement and teacher facilitation, but that there were more increases than decreases in GSIOP scores over semester-long observation-feedback iterations. In answering our second research question, our coding of feedback (advice/improvement and broad/specific) illustrated how GSIOP scores on the teacher and student sections would change relative to the type of feedback. Moreover, feedback that included specific advice and specific improvements had the largest positive change in GSIOP observation score indicating that contextualizing feedback leads to more actionable feedback.

Limitations

The structure of the post-observation feedback and the overall design of the peer-mentorship model could have influenced the results of this study. Specifically, the training of mentors and the use of the peer-mentorship model may be critical factors in the results of this study. This in no way voids the results but is a limitation of implementing RYG feedback with another observation protocol or using the GSIOP with a non-RYG feedback structure.

Implications for Research and Practice

Tables 2 and 3 support Kluger and DeNisi's (1998) theory of feedback being "a double-edged sword" because Table 2 demonstrates overall growth to both the student and teacher sections, but it varies according to the type of feedback. Table 3 verifies Kluger and DeNisi's argument that change depends on the type of feedback. When mentors provided specific advice and noted specific improvement, or provided broad advice and noted specific improvement, novice GSIOP scores improved on observation questions focusing on student engagement and teacher facilitation of student-centered learning. However, if the mentor's feedback provided no advice nor improvements, or advice without improvements, there was a minor positive or negative change in GSIOP score for both student engagement and teacher facilitation of student-centered learning.

Our research provides undergraduate mathematics education with a framework for looking at post-observation feedback using a tested observation protocol (Rogers et al., under review) and a post-observation feedback structure. Our results (Table 3) indicate providing specific improvements had the most actionable (Cannon & Witherspoon, 2005) results with respect to the observation protocol. Consider Roberto's yellow feedback and following green feedback which had a substantial increase in his novice's student- and teacher-focused GSIOP scores.

> *(Yellow Feedback)* Engage more with the students. Particularly, ask more questions. I see that you are using the PowerPoints…I will do a demonstration for you in the one-on-one for a slide that was in your lecture. The main thing is to actively think if this is a moment I can ask a constructive question to engage with the learning… *(Following Green Feedback)* You are asking more questions to your students and you are getting more participation! This is great. Keep it up but remember that you can also…

The specific advice to engage through questioning, followed by specific improvement that promoted continued development demonstrates actionable feedback that can positively frame post-observation feedback.

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A Calculus Teacher's Image of Student Thinking

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This paper focuses on how a teacher's image of student thinking influences the meanings she conveyed to students. I observed a calculus teacher's lessons and interviewed the teacher and her students. By exploring the data, I see the extent to which the teacher attention to student thinking has an impact on (1) the ways she expressed her meanings during instruction, (2) the ways she interpreted students' understandings that they expressed, (3) the ways she decided to adjust instructional actions. My analyses suggest that teachers need to think about how students might understand their instructional actions so that they can better convey what they intend to their students.

Keywords: Derivative, Conveyance of meaning, Key Pedagogical Understanding

There has been substantial interest in teachers' mathematical understanding needed for the practice of teaching. Shulman (1986) has distinguished between *pedagogical content knowledge* (PCK) and *content knowledge* (CK). According to Shulman (1986, 1987), PCK is the knowledge of how to make the subject comprehensible to others whereas CK refers to understanding of the subject per se.

On the basis of Shulman's distinction between CK and PCK, researchers have investigated teachers' knowledge for teaching mathematics. For example, Ball and her colleagues developed a theoretical framework "mathematical knowledge for teaching (MKT)" including knowledge dimensions such as specialized content knowledge and knowledge of content and students (Ball, Thames, & Phelps, 2008). Researchers have focused on identifying the particular understanding in teaching mathematics such as how to explain why a problem-solving procedure is valid. I acknowledge that identifying teachers' understandings is important in the context of teaching, but my stance accords with Silverman and Thompson (2008). They asked the question "What mathematical understandings allow a teacher to act in these ways spontaneously? How might these understandings develop?" (Silverman & Thompson, 2008, p. 500).

This study focuses on a calculus teacher's image of student thinking and how it influenced the meanings she conveyed to students. I used the constructs of *meaning*, *conveyance of meaning* and *Key Pedagogical Understanding* (KPU) to guide observations and analyses of classroom observations and interviews with the teacher and her students. I present a subset of data by focusing on student understandings of the difference quotient in the definition of derivative, and describe the consequences of the teacher's image of student thinking in her teaching of the definition of derivative. I seek to answer the following research question: How does a teacher's image of student thinking influence the meanings she conveyed to students?

Literature Review

Researchers have attempted to identify types of mathematical knowledge needed in the practice of teaching. In an attempt to measure teachers' CK and PCK separately, some studies reported difficulties of distinguishing between CK and PCK. Krauss, Baumert, and Blum (2008) described that constructing PCK items that are not affected by CK is hard because PCK and CK are closely related. Kahan, Cooper, and Bethea (2003) also pointed out that CK and PCK seem to inseparable.

Ball (1989, 2002) claimed that knowledge for teaching is more than knowing mathematical ideas. Her claim suggests that focusing only on teachers' knowledge is limited because teachers' knowledge is meaningful only when it helps students build powerful understanding. Her research team, which investigates MKT, has been interested in relationships between teacher knowledge and student learning.

> "As a result, although many assume, on the basis of the educational production function literature, that teachers' knowledge as redefined in the teacher knowledge literature does matter in producing student achievement, exactly what this knowledge *is*, and whether and how it affects student learning, has not yet been empirically established"(Hill, Rowan, & Ball, 2005, p. 377).

Hill et al. (2005) pointed out that what teachers' knowledge for teaching is and relationships between teacher knowledge and student learning have not been empirically demonstrated. In an attempt to show relationships between teacher knowledge and student learning MKT group tried to validate the assumption that their measures of teachers' mathematical knowledge for teaching are related to teachers' instruction and student learning (Hill, Ball, Blunk, Goffney, & Rowan, 2007; Hill et al., 2005).

Ball's research group tried to find statistical relationships between what teachers know and what students performed as well as between what teachers know and what teachers teach. However, the group focused on whether or not students solved mathematical problems correctly rather than focusing on the concepts students formed from teachers' instruction. Moreover, the group did not try to explain what they mean by "knowledge" nor how teachers' knowledge (whatever that means) influences student performance through their instruction because they did not ask what a student understands of what his or her teacher said. Thus, Ball's research group did not explain why what teachers know led them to do what they did in their instruction, nor how their instruction led students to learn what they learned. In the following, I present a new lens to investigate how a teacher's pedagogical understanding influences her instructional actions and meanings she conveyed to students.

Theoretical Framework

Consider a teacher who teaches mathematical ideas to his students. A teacher has his meanings for the mathematical ideas. The teacher intends to convey the mathematical ideas to his students. In doing so, the teacher and his students are interacting and making an attempt to interpret others in class. Thompson (2013) proposed a theory to explain how two people (person A and person B) attempt to have a conversation that leads to mutual understanding.

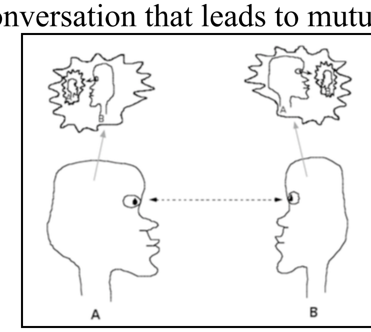

Figure 1. Person A and B attempting to have a conversation

According to Thompson (2013), person A in Figure 1 holds something in mind that he intends Person B to understand. Person A considers not only how to express what he intends to convey but also how person B might hear person A. In doing so, person A constructs his model of how he thinks person B might interpret him. Person B does the same thing in the conversation. Person B constructs her understanding of what person A said by thinking of what she might have meant were she were to say it. Thus, person B's understanding of what person A said comes from what she knows about person A's meanings, thereby person B's understanding of person A's utterance need not be the same, and likely is not the same, as what person A meant.

Silverman and Thompson (2008)'s framework explains for a teacher to develop knowledge that supports conceptual teaching of a particular mathematical idea when she has an image of how her students might hear her statements. Silverman and Thompson referred to Key Pedagogical Understanding (KPU) to discuss teachers' image of students' thinking. Thompson (2008) described a six-phase model of teachers' development of a KPU.

Table 1. KPU phases (Thompson, 2008)

Phase	Description
1	Teacher develops an understanding of an idea that the curriculum addresses. Student thinking is not an issue.
2	Oriented to student thinking, but tacitly assumes that information is all that students need, Projecting oneself by default. That is, person A presumes unthinkingly all students are A' (on the road to being A)
3	Teacher becomes aware that students think differently than teacher anticipates they do, but teacher is overwhelmed by seeming cacophony of student thinking (students in her head are B1, B2, B3, …)
4	In dealing with students' (B1, B2, B3, …) contributions: 1. Teacher begins to imagine different "ways of thinking" (epistemic students) 2. These ways of thinking are still grounded largely in teacher's ways of thinking.
5	Teacher begins to imagine how different ways of thinking among students will lead to different interpretations of what she says and does. Begins to develop a mini-theory of actions that might help students think the way teacher intends
6	Teacher adjusts: 1. Her understanding of the mathematical idea as she adjusts her image of ways students think about it. 2. Her understanding of how students might think about the idea as she adjusts her understanding of it

Thompson (2013) theory of conveyance of meaning and the KPU phases are useful to explain what occurs in class. A teacher will express his meanings to his students by saying or doing something. Then, his students try to understand what the teacher says and does. Whatever meanings his students construct by attempting to understand what the teacher intends is the meaning that the teacher conveyed to the students. The conveyed meaning might or might not be the same as the teacher's meaning, and most likely is not. Moreover, the extent to which a teacher envisions how his students might understand what he says and does affects both meanings conveyed to students as well as his interpretations of students' actions and utterances.

Methodology

I observed 15 calculus lessons taught by Terri (pseudonym). I asked her to select two middle-performing students who, in her judgment, pay close attention during lessons. Terri selected Amy and Alex for interviews. Terri told me her lesson goals in each Pre-Lesson Interviews. She also expressed her meanings and her image of student thinking in the Pre- and Post-Lesson Interviews and her lessons. For example, one pre-lesson interview question was, "Do you think your students might understand slope differently than what you intend?" I asked this question to discern how a teacher thinks about student thinking before the lesson. Pre-Lesson Interviews with Terri took 10-15 minutes. Post-Lesson Interviews with Terri were two types: 5 minutes Post-Lesson Interviews right after the lesson and one hour Post-Lesson Interviews every three lessons.

Amy and Alex also expressed their meanings in Pre- and Post-Lesson Interviews. I conducted a Pre-Lesson Interview prior to every lesson, so the next Pre-Lesson Interview sometimes served as a Post-Lesson Interview for the previous lesson. For example, I was able to see what they understood in lesson 1 during Pre-Lesson Interview 2. Every Pre-Lesson Interview with students took approximately 5 minutes and each Post-Lesson Interview with students took about 30 minutes. The schedule for interviews with Terri, Amy and Alex is shown in Table 2.

Table 2. The schedule for the first four lessons (repeated for 11 more lessons)

Lesson 1	Lesson 2	Lesson 3	Lesson 4
Pre-Lesson 1 with Terri	Pre-Lesson 2 with Terri	Pre-Lesson 3 with Terri	Pre-Lesson 4 with Terri
Pre-Lesson 1 with Amy & Alex	Pre-Lesson 2 with Amy & Alex	Pre-Lesson 3 with Amy & Alex	Pre-Lesson 4 with Amy & Alex
Lesson observation	Lesson observation	Lesson observation	Lesson observation
5min Post-Lesson with Terri	5min Post-Lesson with Terri	5min Post-Lesson with Terri	5min Post-Lesson with Terri
		Post-Lesson 3 with Amy & Alex	
			Post-Lesson 4 with Terri

I audio recorded pre-lesson interviews with the teacher and the two students and post-lesson interviews with the teacher. However, I video recorded the lessons and post-lesson interviews with students because I showed video clips of students' post-lesson interviews to the teacher. The purpose of sharing students' video clips with the teacher was to provide opportunities to think about students' understandings and to reflect on her teaching and meanings by showing excerpts from the two student interviews that revealed how they understood central ideas of the lesson.

I met the two students selected for interviews after the pre-lesson interviews with the teacher. The purpose of the pre-lesson interviews for students was to see their understanding of the topic to be covered in the upcoming lesson. I compared students' meanings demonstrated in pre-lesson interviews to their meanings demonstrated in post-lesson interviews to infer what they understood from the lesson. After the lesson, I asked each student to describe what he or she learned from the lesson.

Results

On the first day of observation Terri introduced the definition of derivative. The definition of derivative includes the difference quotient, so students' meanings for rate of change or slope influenced how they made sense of Terri's lesson. The Pre-Lesson Interviews with Amy and Alex suggest that they had different schemes for slope and rate of change. Amy's meaning for slope was "going up and over" and did not involve any changes whereas Alex's meaning for slope was the relationship between the change in x and the change in y. When Terri taught the definition of derivative, her lessons (Lessons 1-4) had generally focused on how to find formulas using the definition of the derivative with algebraic procedures. Amy and Alex tried to make sense of what Terri said in the lessons, but their different schemes allowed them to understand Terri's lessons differently. After watching the students' video clips Terri decided to adjust her instructional actions in order to help their understandings. Terri's adjustments happened in Lesson 6. I will first describe Terri's lessons and the two students' understandings, and then discuss Terri's thinking about students' understandings and her adjustments.

Terri's Interviews and Lessons

In the Pre-Lesson 1 Interview, Terri said she would introduce the definition of derivative in lesson 1. She explained that her goal for lesson 1 was for students to be able to "find a derivative", by which she meant to use the definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and rules for limits to derive a closed form definition for f'. Terri mentioned the idea of slope to explain the difference quotient in the definition of derivative by saying, without elaborating, that the definition of derivative without the limit was just a slope between two points, but her main focus was how to find formulas with algebraic skills.

Amy's Story

After Amy experienced Terri's Lesson 1, I conducted Pre-Lesson 2 Interview with Amy. When I asked Amy to find the derivative of $f(x)=1/x$ she first said "Terri told us plugging it in yesterday [referring to Lesson 1]". Then, Amy wrote $f'(x)=\lim_{h\to 0}\frac{(1/x)+1/x}{h}$, which provides strong suggestion that "$1/x$" was the "it" in "plug it in". Amy then said, "I kind of forgot how to". When I asked her meaning of $f'(x)=\lim_{h\to 0}\frac{(1/x)+1/x}{h}$ (see **Error! Reference source not found.**, note that there is the sum of $(1/x)+1/x$ in the numerator and *h* only appears once in the denominator), she said "use the original function and plug it in into the new formula to find the limit of it and once you factor it like take out the elements of it". This suggests Amy's understanding of Terri's Lesson 1 was "use the original function and plug it into the new formula", which expanded to substitute $\frac{1}{x}$ into the blanks in $\lim_{h\to 0}\frac{(\square)+\square}{h}$. Amy first got the idea of "plugging in" which seemed to mean "writing something in place of". This tells us that her meaning of the original function was just the inscription $\frac{1}{x}$ that consists of 1, "—", and *x*. Amy said $\lim_{h\to 0}\frac{(\square)+\square}{h}$ was the new formula, which means she already had a formula that was what she called the original function: the inscription "1 fraction bar *x*". Amy understood the definition of derivative as a new formula that gave her a new inscription.

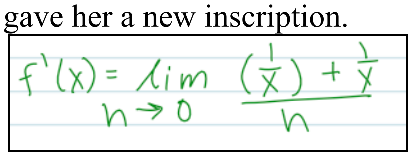

Figure 2. Amy's derivative formula in Pre-Lesson 2 Interview

Alex's Story

In Pre-Lesson 2 Interview I asked him how to find the derivative of $f(x)=1/x$. Alex said " I am trying to remember" and wrote $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)+f(x)}{\Delta x}$ saying "I think it's the limit as delta *x* approaches zero of f of x plus delta x then f of x over change in x". He tried to remember what Terri wrote in Lesson 1 and wrote what he remembered (see Figure 3).

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta_x)+f(x)}{\Delta x}$$

Figure 3. Alex's work in Pre-Lesson 2 Interview (Note that he used "+" in the numerator)

When I asked him about the meaning of $f(x+\Delta x)+f(x)$ he said "It's the change in something". He continued to say the difference quotient was finding the slope. He seemed to connect the difference quotient with the concept of slope. Unlike Amy who only wanted to know the correct actions to take, Alex realized that he was missing something about his meanings for derivative and wanted clarification. Finally, during Post-Lesson 3 Interview Alex said

> I was saying um that it's plus. And then that led me into some difficulty during the Pre-Lesson Interview (*referring to the Pre-Lesson 2 Interview where he wrote "+"*) because you asked me to explain the top part (*circling* $f(x+\Delta x)+f(x)$). And then I realized now that once it's a minus um… it's just the difference between the values on the points. (Alex, Post-Lesson 3 Interview)

His statements show that he realized that he had not represented what he intended and he wanted to represent changes, as in the Pre-Lesson Interview. The nature of his realization was that he could not represent changes with sum of functions' values in $f(x+\Delta x)+f(x)$.

Terri's Thinking about Amy and Alex's Understandings

After Terri saw Amy and Alex using a plus in the denominator of the difference quotient of the definition of derivative Terri said that made no sense because the numerator stands for change in y. Terri was surprised because she remembered Amy said the slope formula correctly in the previous lesson. Terri only focused on Amy's action (speaking of the slope formula and using a plus) and put her own meaning for the difference quotient into her model for Amy's understanding, which led Terri away from a productive understanding of Amy's thinking. Then, Terri said she would adjust her lesson so that she could teach the numerator has a minus by writing Figure 4.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(\cancel{x}+\Delta x)-\cancel{x}} \qquad h = \Delta x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \text{"difference quotient"}$$

$$\frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$$

Figure 4. Terri's adjustment for the next lesson to put more emphasis on "—" operation in the difference quotient

Terri said using Δx instead of h and eliminating x in the denominator would help her students remember that that there was a minus in the numerator by making a connection to the slope formula. She thought emphasizing a minus in the denominator would address Amy's problem. Although Terri said she would adjust her instruction, it seemed she thought to adjust it based on Amy's writing "+" instead of "-" and not on a broader understanding of Amy's orientation to write expressions without thinking of their meanings. Her future plan still focused on actions such as eliminating x in the denominator and lacked a plan to convey what the numerator and the denominator of the difference quotient mean.

Terri's Adjustments and Amy's Understanding of the Adjustments

Terri's goal of the adjustments was to help students remember how to use the formula correctly. Terri's behavior in Lesson 6 for adjustments confirms that she never identified the difference between a student's ability to use a formula, and the student's meaning for that formula.

After Amy experienced Terri's adjustments in Lesson 6 she was still using a plus in the numerator of difference quotient and said $f(x+\Delta x)+f(x)$ was a change. Amy ended up with the conclusion that the notation $f(x)$ represents y, so the numerator is about y. Her meaning for slope allowed her to think $\frac{f(x+\Delta x)+f(x)}{\Delta x}$ as rise over run because she thought rise over run was y values over x values.

After watching Amy's Post-Lesson 6 Interview where Amy expressed her understanding Terri said she wanted to call the usage of a plus "habit" that is hard to undo, but she believed Amy would write the difference quotient correctly. It seems that Terri thought Amy meant the numerator is a change although Amy wrote $f(x+\Delta x)+f(x)$ because Terri called it "habit". This interview indicates that Terri put her own meaning into her model of Amy's statement, so she thought Amy knew the numerator represents change in y, but Amy accidentally used a plus.

Conclusion

The analysis show that Terri's lack of orientation to her students' mathematics played a significant role in her instructional actions, adjustments, and the meanings she conveyed to the students. Terri taught the ideas in ways that were obvious or clear to her, which led to miscommunication between Terri and the two students. When Terri detected miscommunication she focused on what Amy and Alex did not understand instead of what they understood. Moreover, Terri did not see Amy's meaning for slope as "going up and over" as an inadequate foundation for thinking of the difference quotient of the definition of derivative. Terri did not think of Amy's way of thinking as the *cause* of her difficulty. In addition, Terri's decisions to adjust her lessons were unrelated to what Amy understood, thus her adjustment did not address the sources of Amy's difficulties.

Terri slowly moved from KPU Phase 1 to 2 and then 3 as she watched students' understandings in the video clips with me over the observations. After watching two students' video clips she began to say "Students all think differently", but she did not talk about how the two students think differently. As Terri found that her students' understandings were not consistent with her intention, her anticipation about students' difficulties seemed to become concrete. However, Terri still did not imagine different ways of thinking that Amy and Alex construct. A teacher in KPU Phase 4 tries to imagine students' different ways of thinking. Terri did not enter KPU Phase 4 during my time with her. It seems that Terri arrived at phase 3 for the idea of derivative at the end of the observation.

In this study, I presented a subset of my data as an illustration of my method for exploring how a teacher's image of student thinking influences her conveyance of meaning. The results point to a breakdown in the conveyance of meaning from Terri to students because Terri had no image of how students might understand her statements and actions. This study indicates that teachers need to think about different ways of thinking that his or her students might have in order to convey what they intend.

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Understanding Students' Achievement and Perceptions of Inquiry-oriented Instruction

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Inquiry-oriented instruction (IOI) has been shown to increase students' cognitive outcomes, but the relationship between students' cognitive and affective outcomes in IOI remains unclear. Furthermore, students' perceptions of the usefulness of and their success in an academic setting are related to their engagement and achievement. The present mixed methods study seeks to understand students' perceptions of IOI and whether those perceptions are related to their achievement. Results of qualitative analysis reveal four themes related to developing conceptual understanding, connecting ideas, feelings of helplessness, and a preference for traditional instruction. Results of a mixed linear model show positive or neutral perceptions of IOI are related to higher achievement. The relationship between these results is discussed, and is framed in motivational theory.

Keywords: Inquiry-oriented instruction; Student engagement; Mixed Methods

Inquiry-oriented instruction (IOI) is a narrower focus on instruction taken from inquiry-based learning (IBL; Kuster, Johnson, Andrews-Larson, & Keene, 2017). There are four guiding principles at the heart of IOI: (1) generating student reasoning, (2) building on student reasoning, (3) developing a shared understanding, and (4) formalizing the mathematics. Typically, an inquiry-oriented classroom is a collaborative setting in which students communicate with one another by providing explanations and justifications for their methods and solutions. Tasks are designed to help students develop more effective problem solving skills with the goal of developing conceptual understanding of mathematical topics. By using such methods, instructors can better engage students in mathematics classroom and offer them mathematical authority to construct their own understanding (Kuster et al., 2017). This definition of IOI makes a purposeful and clear break from traditional instruction, which focuses on the teacher's mathematical authority, delivery of instruction through lectures, and the student's passive acceptance of the teacher's knowledge (Lampert, 1990). Thus, it stands to reason that such a distinctive change in instructional philosophy and practice will both affectively and cognitively affect students. The present study explores the relationship between these effects.

Literature Review

Most often research is focused on students' cognitive outcomes as a way to gain insight to performance or achievement. With this focus, researchers investigate students' mental actions and application of those actions, including "remembering, understanding, applying, analyzing, evaluating, and creating" (Burn & Mesa, 2015, p. 47). This would include such outcomes as student performance, achievement, grades, knowledge, and even skills (Dochy, Segers, Van den Bossche & Gijbels, 2003; Freeman et al., 2014; Johnson et al., under review; Laursen Hassi, Kogan & Weston, 2014; Lazonder & Harmsen, 2016). However, by only focusing on student performance, researchers lose insight to students' full abilities and participation. For example, some students may participate in class and have high level thinking not reflected on assessments. Others may not participate in class but have high performance on assessments. Only considering performance limits researchers to paint half of the complete picture of student learning.

Students' cognitive outcomes in lecture and non-lecture courses have been well documented. To synthesize these results, Freeman et al. (2014) investigated the effect of active learning on student performance through meta-analysis of lecture and non-lecture classes. Student performance increased with active learning environments versus lecture, and students were one and a half times more likely to fail in lecture classes than active classes. These results held for STEM, non-STEM, introductory, and advanced courses. Results also show that active learning is most effective in classes of about 50 or fewer students, but can be beneficial to students' learning in larger classes as well. These findings demonstrate different levels of cognitive achievement for students in active learning and lecture classes, regardless of course or class size.

In a comparison of cognitive outcomes in IBL courses, Laursen et al. (2014) measured students' cognitive, affective, and collaborative gains in IBL and non-IBL classes. Students in IBL classes in this study had higher cognitive gains, as indicated by their grades and self-report of learning gains. This suggests that the inherently different experiences of students in IBL classes promote higher cognitive achievement than do non-IBL courses. In combination, this research demonstrates that active and inquiry learning environments are likely to positively influence students' cognitive outcomes. However, these studies do not take into account students' affective outcomes. Affective outcomes provide insight into students' "beliefs, attitudes, and perceptions" (Burn & Mesa, 2015, p. 97), as well as confidence, enjoyment, persistence, and interest (Laursen et al., 2014; Sonnert & Sadler, 2015). Affective outcomes can also include factors such as self-efficacy (Bandura, 1997, 2006) and motivation (Jones, 2009, 2018). Although the study of affective gains can be difficult, as these factors are internal and difficult to observe, investigating affective gains allows researchers to better understand student learning.

Theoretical Framework

As one of the strategies of IOI is to increase students' engagement with mathematical activity through active inquiry (Henningsen & Stein, 1997), it is relevant to consider how students' motivation to engage with mathematics is affected by IOI. The MUSIC model of motivation (Jones, 2009, 2018) is a macro-theory of motivation, that describes how instruction can be designed to increase students' motivation by fostering students' empowerment (M), increasing students' perceptions of usefulness (U), success (S), and interest (I), and ensuring the students feel cared about (C) by their instructor. In particular, Jones reports that students' perceptions of their ability to be successful with course content, or their self-efficacy (Bandura, 1997, 2006), and their perceptions of the usefulness of learning activities to their success, which is part of expectancy-value theory (Wigfield & Eccles, 2000), are influencing factors in their motivation to engage in the learning process.

Students' self-efficacy is one affective construct that is useful in understanding students' "motivation, achievement, and self-regulation" (Schunk & Pajares, 2009, p. 26), as well as students' choices, interest, effort, and persistence (Bandura, 2006; Schunk & Pajares, 2009). Self-efficacy also has a higher correlation to achievement than do other affective constructs (Jones, 2018). A related affective construct, expectancy-value theory, indicates that students' engagement in academics is related to their expected outcomes (e.g., success or failure) and the value they place on the activity; moreover, the value students assign to math has been used to predict the likeliness that they will persist in math classes (Wigfield & Eccles, 2000). Thus, as IOI seeks to engage students in authentic mathematical activity through the process of inquiry, students' perceptions of the IO classroom may be related to their engagement and subsequent learning. Therefore, the purpose of this study is to employ a mixed methods approach to investigate how students' perceptions of IOI are related to their academic achievement in an undergraduate inquiry-oriented differential equations (IODE) course. These perceptions will be framed within the MUSIC model of motivation (Jones, 2009, 2018). Specifically, the study will examine what students' perceptions of IOI are, whether those perceptions are positive or negative, and whether students' perceptions of IOI are related to their achievement.

Methods

The mixed methods research design is concurrent (Creamer, 2018) and utilizes mixing for the purpose of achieving complementarity (Greene, Caracelli & Graham, 1989). This work was supported by Teaching Inquiry-oriented Mathematics: Establishing Supports (TIMES).

Research Context and Participants

Data from this study comes from a survey administered to 16 IODE classes. The instructors were TIMES Fellows, which means that they were involved in continuing professional development focused around the implementation of IOI (Keene, Fortune, & Hall, under review). In a typical IODE class students work in small groups on tasks as well as participate in whole class discussions. IODE is not a theorem driven course, thus tasks focus on building conceptual understanding of various topics (e.g., Euler's method, eigentheory, modeling with autonomous differential equations)[1]. For example, in an IODE class instructors may generate student ways of reasoning about varying step sizes of Euler's method compare and contrast, build on those various contributions to develop a shared understanding, and ultimately formalize the student's conceptual understanding and connect it to the standard Euler's formula. In total, 226 students from IODE classes completed the survey. Of the 226, 36 did not complete the short answer question on the survey and were removed. Additionally, 9 students' responses were removed because they were not related to the course, leaving 181 participants in total.

Instrument and Data Collection

Qualitative data were collected from one short answer question on the Student Assessment of their Learning Gains (SALG) survey, originally developed by Laursen, Hassi, Kogan, Hunger, and Weston (2011), but modified by TIMES to fit the context of our project. The short answer item used for qualitative analysis asked students to "please comment on how the way this class was taught affects your ability to remember key ideas." In response to this item, students were able to respond with as much or as little detail as they deemed fit. The students were not asked to comment specifically on IOI, but because only students who were in IO classes were included in the study, any responses related to their instructor's teaching style or the activities of the course were considered indicators of the students' perceptions of IOI. Students' responses were quantitized (Sandelowski, Voils, & Knafl, 2009) by coding them as indicating a positive (3), neutral (2), or negative (1) perception of IOI; this coding facilitated quantitative analysis, the purpose of which was to understand the relationship between students' perceptions of IOI and their content assessment (CA) scores. The CA is from the work of Hall, Keene, and Fortune (2016), who created a common multiple-choice assessment of undergraduate students' understanding of differential equations. This multiple-choice assessment consisted of 15 questions spanning topics from first to second order linear differential equations and systems of differential equations. Students' CA scores are reported in percentages.

[1] For more information on IODE visit: https://iode.wordpress.ncsu.edu.

Data Analysis

Students' written responses were first analyzed by two researchers, who developed open codes (Charmaz, 2014). Codes were generated for any detail that indicated students' sentiment toward IOI, or their learning in the IO course. Some written responses generated only one code, but others generated as many as four open codes, depending on the length and detail of the response. The researchers then grouped the open codes into categories. Finally, themes were created by axial coding, which defines the relationships between categories, and relates them to one another around the theme's "axis" (Charmaz, 2014, p. 147). The resulting themes are representative of the qualitative data set as a whole.

After themes were defined, each student's written response was coded by the researchers as indicating a negative (1), neutral or mixed sentiment (2), or positive (3) sentiment toward IOI. Examples of how students' responses were binned into each category are described in the results. Finally, after quantitizing students' written responses, the results of a mixed linear model were used to understand the relationship between students' perceptions of IOI and their CA scores. A mixed linear model accounts for the nesting of students within instructors, thereby controlling for the effect of particular instructor or classroom characteristics.

Results

Results of the qualitative analysis indicate four themes: (1) Engaging with Math and Each Other, (2) Less to "Know," (3) Feelings of Helplessness, and (4) Resistance to Change (Table 1). The first and second themes that emerged were generally positive, with students noting more conceptual understanding and the simplicity of the concepts when they are connected to other concepts. The third and fourth themes were more negative, with students reporting frustration with IOI and preference for direct instruction.

Table 1. Emergent Themes from Qualitative Analysis

Definition	Categories
Theme 1: Engaging with Math and Each Other Students report learning more or having deeper understanding due to engaging with the mathematics, or with their peers.	Learning is a Process; Teaching is Learning; Learning from Peers; Engaging with Material
Theme 2: Less to "Know" Students describe learning "less material" than in other courses as a result of viewing concepts as interconnected. Some students described this as a focus on key ideas.	Real-world Applications; Learning by Connecting; Key Ideas; Less Content
Theme 3: Feelings of Helplessness Students report that they did not feel they could learn because of components of IOI.	Unhelpful groups; No study materials; Lack of consolidation during class; Pacing issues; Not sure of right answers
Theme 4: Resistance to Change Students explicitly state the desire to have more traditional class structure.	Prefer lecture; Need examples

Students whose responses were included in the first theme, *engaging with math and each other*, indicated that IOI helped them to have a deeper understanding of the key ideas of the course because they were engaged in group work, discussions, sharing responses and justifying their work. In particular, some students noted that explaining their ideas to their peers, and listening to their peers explain different ideas was beneficial to their learning and helped them understand how and why the ideas work. One student said that "it's like we are training our minds [to] work on its [*sic*] own rather than following some steps." Another reported that, "I remember because it took me a lot of time to figure it out on my own but when I did figure it out it was like a light came on and it all made sense." These quotations are characteristic of the first theme, and responses that fit within this theme described generally positive experiences with IOI in which they learned through the processes of teaching, engaging, and working with others.

The second theme was that there was *less to "know"* than in other courses. Of course, this is unlikely to be true, but to the students whose responses formed this theme, that was the perception. One student stated that "This class helped us remember key ideas by connecting them to other ideas." This idea resonated with many students, all of whom reported an ability to remember and focus on key ideas more because the mathematics at hand was being connected to other ideas. Some responses within this theme were paradoxical, however. For example, one student said "I feel like I remember more than I would in a lecture based course. However, not sure if this is because it was less material to learn." Another stated, "The intuitive discussion and openness of the class is incredibly helpful" but went on to say that there was less to learn. For the students whose responses fit within the second theme, the connectedness of ideas facilitated their learning by allowing them to feel as though there was less to know.

The third theme was *feelings of helplessness*, and includes students' responses related to struggling to learn due to qualities of IOI that were beyond their control. These responses attended to group members that could not or would not help others, lack of organized notes or study materials to reference outside of class, or lack of consolidation of learning during class. Although one student noted that discussions about "wrong" methods improved their understanding, many more students noted that discussion of the "wrong" methods hindered their ability to remember the "right" ideas. For other students, group members made learning difficult either because the "spread of math abilities was a little too wide for this [IOI] to have worked exceptionally well", because group work was too time consuming, or because their group members simply weren't helpful. These students' perception of the course was that their learning was not supported due to failures within their group or from the instructor's organization of class time and course materials.

Finally, a pervasive theme in students' written responses was a *resistance to change*. Many students simply stated that they prefer lecture over IOI. One student's response in particular was, to the researchers, contradictory; the student indicated that he preferred lecture and that group work was not helpful to his learning, but concluded that "In terms of learning how to solve, it worked though." The implication seems to be that although the student was successful in his coursework, he would prefer the instructor return to a traditional style of instruction. Others indicated that they wished the instructor would have gone over more examples with the class, given more organized notes or learning materials, or that the instructor would just directly indicate whether their answers were correct. Often these students did not give a reason, or even indicated that IOI was helpful, but they were insistent that traditional instruction was better.

In general, the responses of students that fit within the first two themes were coded as indicating a positive sentiment toward IOI and the responses that fit within the third and fourth themes were coded as negative. Some responses were coded as neutral or as indicating a mixed sentiment if they made both positive and negative statements about IOI or their learning. Any indication of not liking, hating, not learning, or being confused by methods of IOI were coded as a negative sentiment toward IOI. Mentions of enjoying, liking, or the methods of IOI enhancing students' learning or enjoyment of the course were coded as positive. If students made mention of both positive and negative emotions, they were coded as neutral. For example, one student stated that the group work was good but that they moved too slowly. Another type of response that was coded as neutral were those that didn't indicate a positive or negative sentiment. One student stated that the course was "fine" for helping them remember key ideas, for example.

In total, 56 students (30.8%) indicated a negative sentiment, 26 were neutral (14.4%), and 99 (54.7%) were positive with regard to IOI and learning in an IOI classroom. Furthermore, results of a mixed linear model indicate that students' sentiment toward IOI were significantly related to their CA scores ($F(165.63)=5.95$, $p=.003$). Also, there is a statistically significant difference in CA scores between students who had a negative sentiment compared to a positive sentiment toward IOI ($t(130.66)=2.97$, $p=.004$). Students with a positive sentiment toward IOI are predicted to have a CA score of 58.29, and students with a negative sentiment are predicted to score 8.06 points lower, all other things held constant. Students with a neutral sentiment toward IOI did not score significantly different than students with a positive sentiment ($t(176.82)=.811$, $p=.42$). The predicted CA score of 58.29 for students with a neutral or positive sentiment is in comparison to the mean CA score for all students, 56.04 (SD=16.34). This indicates more than half of students in these IO classes were positive about the instruction and their learning, and these students are predicted to score slightly better than average on the CA. In comparison, roughly 30% of students had a negative sentiment, and are predicted to score approximately 8 points lower than their peers with a positive sentiment.

Discussion and Conclusions

Students' perceptions of IOI in this study indicated deep understanding of course content (theme 1), simplified learning due to connections (theme 2), feeling helpless due to components of IOI that were outside their control (theme 3), or a preference for traditional instruction (theme 4). Responses within theme 1, *engaging with math and each other*, indicated deeper conceptual understanding, and learning by acting as a teacher to their peers or as a pupil to their peers' teaching. Students' responses within this category were generally positive. Similarly, responses within theme 2, *less to "know,"* noted that making connections between concepts simplified the amount of material that they needed to learn. Framed within the MUSIC model of motivation (Jones, 2009, 2018), the first two themes relate to students' feelings of the usefulness of learning activities and their success in learning course content. In other words, students' perceptions that they are developing deeper understanding through group work can be interpreted as the student placing value on group work (Usefulness) because it leads to their academic success (Success).

Conversely, characteristic responses within theme 3, *feelings of helplessness*, indicated that students felt as though they were not successful in group work for a variety of reasons. For example, group members could not or would not help them, homework was difficult without well-organized notes from class, and understanding correct ideas was muddied by discussion of incorrect ideas, to name a few. Interpreted within the MUSIC model of motivation (Jones, 2009, 2018), these feelings of helplessness will affect students' motivation to engage in IOI because they do not perceive group work or discussion to be useful if they do not lead to correct insights. As a result, students disengage from IOI. It is impossible to tell whether the students' perception of IOI as not useful to their learning causes them to feel that they cannot be successful, or whether feelings of not being successful led them to view IOI as not useful. Regardless, self-efficacy (Bandura, 1997, 2006) and expectancy-value (Wigfield & Eccles, 2000) theories of motivation indicate students with these perceptions are less likely to engage or be successful in the course (Jones, 2009, 2018). Similarly, the fourth theme that emerged in this research was a *resistance to change*; responses in this theme indicated students preferred traditional instruction. Many of these responses did not provide justification for preferring lecture, but the cycle of not assigning value to IOI and not perceiving one's success in IOI is similar to that of theme 3. The MUSIC model of motivation predicts that if students do not value IOI or believe that they can be successful as a result of IOI, then they will be less motivated to engage in the class activities.

In addition to analyzing the themes that emerged from this research in light of motivational theory, students' responses can also be related to the main purposes of IOI put forth by Kuster and his colleagues (2017). They assert that the guiding principles of IOI are (1) generating student reasoning, (2) building on student contributions, (3) developing a shared understanding, and (4) formalizing mathematics. While the fourth principle is teacher-driven, the output is students connecting their informal ways of reasoning to formal mathematics (Kuster et al., 2017). This is often done by "linking student-generated solution methods to disciplinary methods and important mathematical ideas" (Jackson, Garrison, Wilson, Gibbons & Shahan, 2013). This is a guiding principle for IO because without formalizing the mathematics, students could miss the connection between their informal ideas and formal mathematics. However, many students' negative perceptions of IOI can be related to the possibility that this formalization did not happen for them. Examples of students' responses in this research study that demonstrate students' negative perceptions of IOI as they relate to not formalizing mathematical ideas include how discussing both incorrect and correct solutions left them confused, or that course content was disconnected. In other words, many students' negative perceptions of IOI hinged on having an unclear vision of the formal mathematics; this is a point of consideration for IO instructors who wish to help their students construct more positive perceptions of IOI. Future research should consider whether the improvement of students' perceptions of formalizing mathematics is related to more positive perceptions of and achievement in IOI.

Thus, it can be concluded that students perceptions of IOI are polarized, and relate to students' perceptions of the usefulness of IOI to their learning, and their perceptions of success in an IO course. Additionally, the MUSIC model of motivation (Jones, 2009, 2018) indicates that students are likely to experience increased motivation when instructors incorporate elements of empowerment (M), elements that interest students (I), and demonstrate caring for their students (C). The students' written responses in this research study were not indicative of elements relating to empowerment, interest, or caring. That is not to say that these elements were not present within the IOI classrooms, but they were not the focus of the students' reflections. The manner by which students' perceptions of IOI are influenced by these three components of motivation requires future research.

Motivational theory also indicates that students' perception of their ability to be successful, or their self-efficacy, is highly predictive of academic success (Jones, 2018). This finding is supported by the present results, which indicate that students with positive perceptions of IOI are related to higher achievement as measured by their CA scores. Future research should consider how positive and negative perceptions of IOI can be more clearly refined, facilitating a better understanding of whether students' positive perceptions in general are related to higher achievement, or whether aspects of positive perceptions related to usefulness or success are more closely related to achievement. In conclusion, while it appears that the IO instructors included in this study were generally successful in helping their students develop a positive disposition toward IOI and mathematics, it is worthwhile to ensure that as many students as possible perceive the course content to be attainable and the IO activities to be useful to their learning, so as to support students' achievement.

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Mathematical Knowledge for Tutoring Undergraduate Mathematics

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Undergraduate math tutoring is an important venue for student learning, yet little empirical work has been done to study tutoring interactions and few theories specifically address tutoring interactions. Drawing upon literature from problem solving, peer learning, and mathematics teaching, this report proposes a schema for Mathematical Knowledge for Tutors (MKTu). The proposed framework expands Ball's (2008) Mathematical Knowledge for Teaching by adding dimensions of affect and self-regulation. This additional depth reflects the individualism, immediacy, and interactivity which are unique to the tutoring setting where problem solving and mentoring take place between an advanced undergraduate tutor and an undergraduate student.

Keywords: Undergraduate Mathematics Tutoring, Affect, Self-Regulation, Problem Solving

Tutoring has long been recognized as an excellent form of education. The results of Mathematics Association of America's national study of college calculus indicate that 97% of the 105 American institutions surveyed had a tutoring center for students to receive help for Calculus, and 89% of the institutions offered tutoring by undergraduate students (Bressoud, Mesa, Rasmussen, 2015). While undergraduate mathematics peer tutoring is common, the research community is just beginning to focus on this critical out-of-classroom learning context. Several quantitative analyses indicate tutoring is associated with higher final grades (Byerly & Rickard, 2018; Rickard & Mills, 2018; Xu, Hartman, Uribe & Menke, 2014). To understand why tutoring is effective must include a better understanding of the mathematical knowledge necessary for effective tutoring. In this paper, we consider how Ball's (2008) Mathematical Knowledge for Teaching (MKT) framework might be adapted to apply to undergraduate mathematics tutors. Like Ball we question, "What do [undergraduate tutors] need to know and be able to do in order to [tutor] effectively. Or, what does effective [undergraduate tutoring] require in terms of content understanding?" (Ball et. al., 2008, p.394) In our contribution, we describe how the components of Ball's MKT construct translate to tutor knowledge, and we add dimensions to reflect the knowledge specific to an undergraduate tutoring context.

We focus on the knowledge of undergraduate tutors because it is ubiquitous, but also unique. Undergraduate math tutoring in this paper refers to peer tutoring in which a more experienced (typically upper-class) undergraduate student provides tutoring to another undergraduate math student. Peer tutors' knowledge differs from both mathematics instructors and fellow classmates; their experience bridges the gap between those with substantial subject matter knowledge and those of peer learners.

Tutoring is not Teaching

Mathematical Knowledge for Teaching (Ball et. al. 2008) is well established for elementary students and has been extended to secondary and undergraduate teaching (Speer, 2015; Hauk, 2014). Still, a different type of mathematical knowledge is needed for tutoring undergraduate mathematics. As Mills, director of NSF funded mathematics resource center workshops, regularly reminds the tutor research community, "The application of teaching theories to tutoring likely results in a deficit model" (2018, unpublished manuscript). Unlike teaching, tutoring is not a profession; undergraduate tutors typically work for 1-3 years. Teachers typically have extensive pedagogical training; tutors may have experience teaching, but enter the job with no formal teacher training. Undergraduate math tutors have different math backgrounds from each other as well as different math backgrounds from trained teachers. Lastly, tutors' understanding of mathematics curriculum commonly differs from instructors. While undergraduate math instructors have a good sense for the math content which they teach, undergraduate math tutors have a unique sense of how their math courses connect to courses in their particular major.

In addition, the tutoring context is substantially different from the classroom context: the instructional goals of each context may differ, and the knowledge required to meet those goals also differs. In the classroom, an instructor is responsible for teaching new material to many learners at once. In the tutoring context, the learner has some previous familiarity with the content, and the tutoring interaction typically takes place in an individualized setting with a focus on solving problems. The individualized tutoring context also allows for immediate, individualized feedback, while a classroom context typically cannot allow immediate feedback to all learners. A key role of the tutor is to help the student become an independent learner; thus prioritizing the development of self-learning skills over the mastery of content (Marx, Wolf, Howard, 2016). Although self-learning is valued in the classroom, most formative assessments in mathematics prioritize proficiency with content (Burn & Mesa, 2015). The power dynamic between a tutor and student is also likely different than that between a student and instructor. While undergraduate peer tutors are not peers in the strictest sense: they typically have slightly more math background than the students they are tutoring, and they relate more closely than an instructor does to a student.

Given these differences between tutor and teacher, an extended model for the Mathematical Knowledge for Tutoring Undergraduate Mathematics (MKTu) is needed and is relevant to the RUME community. To advance a research agenda aimed at describing and improving tutoring practices and tutor training, a theoretical model is needed to describe the mathematical knowledge necessary for undergraduate tutoring. The schema for MKTu proposed here builds on Ball's MKT and is based on tutoring observations (McDonald and Mills, 2018; James and Burks, 2018), tutoring literature, problem solving theory, and peer learning methodology. The proposed framework is a theoretical contribution grounded in existing literature. This framework will require ongoing refinement based on empirical studies, and will help guide the focus of qualitative analysis of tutor actions and interactions.

Mathematical Knowledge for Teaching

Ball describes and illustrates this theory of Mathematical Knowledge for Teachers (MKT) using the well-known "egg" diagram seen in Figure 1. Following Shuman's (1986) analysis, Ball divides MKT into Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK is then further divided into Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). Common Content Knowledge is math knowledge which teachers use in ways similar to the way it is used in other occupations. Horizon Content Knowledge is cognizance of how mathematical concepts are related across the curriculum. Specialized Content Knowledge (SCK) is content knowledge specifically used by teachers. Similarly, PCK is subdivided into Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC). Knowledge of Content and Students (KCS) includes recognition of student misconceptions and reasoning how to build new understanding on student's current thinking. Knowledge of Content and Teaching (KCT) is knowledge of teaching moves. Knowledge of Content and Curriculum (KCC) indicates awareness of when a particular topic is first covered and then revisited within the elementary curriculum. While Ball's focus has been on elementary math teaching, others have applied it to secondary and tertiary math teaching (Speer, 2015; Hauk, 2014).

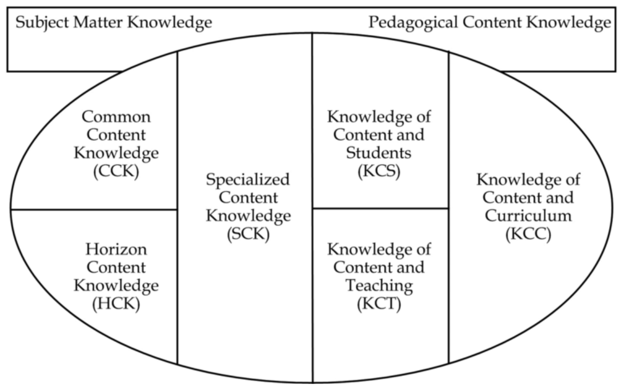

Figure 1. MKT from Ball, Thames, Phelps (2008, p.403)

Mathematical Knowledge for Tutoring

Based on supporting literature, Ball's original MKT model is modified to reflect the mathematical knowledge uniquely representative of tutors. Our theoretical proposal extends Ball's egg to include affective and self-regulatory components. Ball's planar egg becomes a cross section of a 3-dimensional ellipsoid that forms the MKTu framework as seen in Figure 2. The 2D cross section looks very similar to the Ball framework; however, a lower supporting affective arc is laid underneath and a guiding overarching edge of self-regulation is added above. The planar cross section is discussed first, followed by the lower and upper arcs of affect and self-regulation. The three divisions of SMK for teaching remain in the planar cross section of SMK for tutoring: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK).

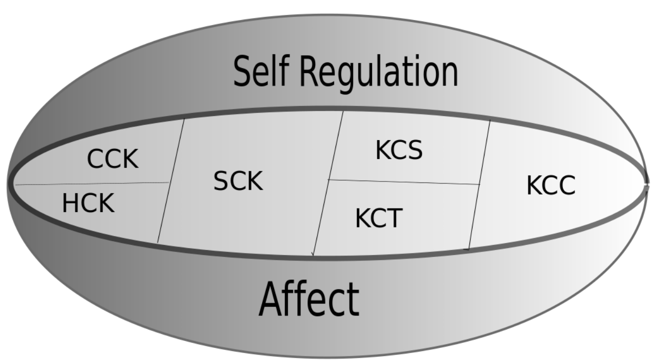

Figure 2. Mathematical Knowledge for Tutors (MKTu)

Subject Matter Knowledge for Tutoring

Within SMK, a tutor's Common Content Knowledge tends to focus primarily on *knowing how* (Mason, 1999), which includes identifying the approach needed to solve the problem and subsequently carrying out the appropriate computations correctly. In contrast, a classroom instructor must draw many knowledge types -- knowing *how*, *why*, and *that* (Mason, 1999)-- while explaining concepts, carrying out procedures, and solving problems in the course of classroom instruction.

In addition, a tutor's CCK may differ significantly in scope compared to a teacher. Undergraduate tutors cannot be expected to have the depth and breadth of understanding of an undergraduate curriculum common to instructors; however, Common Content Knowledge may not be as critical to the tutoring experience. Tutors do not introduce new material; instead their primary role should be to encourage students to make use of their own resources (such as the textbook and class-notes), and guide students through the process of articulating their own self-explanations (Chi, 2008). Since the role of the tutor is different from a teacher, the type of content knowledge required for effective tutoring differs as well.

Common Content Knowledge and Specialized Content Knowledge for tutors is commonly characterized by the dominate role of problem-solving within the undergraduate tutor context. Students may be aware of concepts taught in class and may also have mastered topics from previous courses, but it is the tutor who helps students refine problem solving skills and build connections between prior knowledge and current knowledge. Whereas HCK for teachers considers how a current math topic fits in context with *math* curricula from prior and future years, HCK for tutors connects current math work with aspects of a specific *major* curriculum. For example, a tutor engineering major has knowledge of how integration theory is used in junior-level engineering courses in a way that a math instructor might not.

Pedagogical Content Knowledge for Tutoring

Within Pedagogical Content Knowledge for tutors (PCKtu), Knowledge of Content and Students (KCS) for tutoring includes identifying and understanding student mathematical contributions to progress the mathematical agenda. This type of knowledge is very similar to the KCS for teaching described by Johnson (2012), which focused on the listening needed for instructors who enact inquiry-oriented mathematics. Like inquiry-oriented instructors, tutors also draw on SCK to make mathematical sense of students' contributions. However, unlike instructors who leverage student contributions toward a specific mathematical goal for the whole class, the tutor is interested in redirecting the student's ideas in a way that allows the student to engage in self-reflection to solve a specific mathematical task.

Additional components of KCS for tutors include *cognitive conflict*, *scaffolding*, and *error management* (Topping, 1999). In drop-in tutoring sessions, tutors must know how to effectively balance cognitive conflict; it is beneficial for a student to be productively confused, but harmful for a student to be hopelessly confused (Graesser, 2011). In addition, tutors must use error management to identify student conceptions and tailoring questions to lead a student to reflect upon and reform those conceptions when needed (Topping, 1999). Knowledge of effective scaffolding is also an important component of a tutor's KCS (Chi, 1996); it differs from classroom scaffolding primarily because of the individualization required to adapt to a specific student's problem-solving approach, rather than a whole-class scaffold.

Communication, organization, and engagement are all critical components within a tutor's Knowledge of Content and Teaching. Topping (1999) identifies communication as critical to effective tutoring, which includes listening, explaining, and questioning. This type of communication differs significantly from the communication found in a classroom: while an instructor must facilitate dialogue among many voices (Gay, 2002), the tutor must manage one-on-one interaction. *Organization and engagement* (Topping, 1999) captures the importance of active learning in the tutoring session. Topping includes goal setting, planning, time on task, the opportunity for individualization of learning, and immediacy of feedback within organization and engagement. In the mathematics tutoring setting, tutor and student goals for the tutoring session and selection of appropriate problems are included in the process of *organization and engagement*. In the context of problem solving, *organization and engagement* also includes the tutor-student dialogue taking place at each phase of problem solving: orienting, planning, executing, checking, monitoring.

Since tutors are more of a mentor than a peer, KCC comes from the tutors' extended experiences of courses and university culture. Tutor knowledge will arise from personal experience with the curriculum. Tutors have first-hand experience with the curriculum in their major and department; whereas math instructors have a knowledge of the content in context of the mathematics curricula.

Affective Knowledge for Tutoring

In this theoretical framework, we utilize Philipp's (2017) definition for *affect*: affect is "a disposition of tendency of an emotion or feeling attached to an idea or object. Affect is comprised of emotions, attitudes, and belief" (p. 259). Understanding how affect relates to mathematical learning is indeed an important component of mathematical knowledge for teaching as well as tutoring. However, affect plays a different, more prominent and more fundamental role, in MKTu. In particular, motivating students and helping them to cope with frustration are two key components of tutoring (Topping, 1999). This role of managing student affect, unique to tutors, is seen in motivation and emotions. Affect is so critical to the tutoring context that it is displayed as an arc underlying both Subject Matter Knowledge and Pedagogical Content Knowledge for Tutors in Figure 2.

Experienced problem solvers effectively work through an intense emotional cycle as they simultaneously work through a cognitive problem-solving cycle (McLeod, 1989). Awareness that affective elements are part of the mathematical problem-solving process is part of CCK for tutors. Confident students begin to work on a problem with enthusiasm. If they get stuck carrying out a plan, they may get tense and grow more frustrated with each attempt that leads nowhere. If they reach a solution, they experience the satisfaction, and possibly even delight, of an 'Aha' experience (Schoenfeld, 1992; Carlson, 2005). In the less ideal situation where students do not reach a solution, their frustration may turn to anger. If simmering, this anger may interfere with the effectiveness of a tutoring session. Knowing how to support students in their emotional responses is part of both KCS and KCT.

Motivation is another meaningful part of the affect arc supporting MKTu. Tutors need to have knowledge of what elements of the mathematics are motivating for students, which is another type of KCS. Whereas a teacher provides motivation in the enacted problem-solving process found in classroom, the individualized context of tutoring means tutors have the opportunity to uniquely motivate a particular student (Lepper and Woolverton, 2002).

Since understanding and practicing mathematics can raise heightened emotions (Beilock & Maloney, 2015), a math tutor may need to handle intense feelings from a student. To do so adequately, a trusting relationship between tutor and student is essential. A competent tutor models enthusiasm and confidence, which the student notices, either directly or indirectly. Tutors help students move from anxiety and fear to perseverance, persistence, and resilience; this is part of PCK for tutors.

Math anxiety is a different than other anxieties; it is uniquely related to the discipline of mathematics (Dowker, Sarkar, Looi, 2016). Math anxiety bridges both the cognitive and affective domains. Tutors need to manage the relationship between what the student needs to motivate them and what the student needs to develop mathematical understanding (Lepper & Woolverton, 2002). In doing so, the tutor coordinates mathematical subject knowledge with pedagogical content knowledge.

In MKTu, the affect arc undergirds SMK as well as PCK. Evidence of affect is seen in almost every tutoring session (James and Burks, 2018; Graesser, 2011). In fact, tutors with no training can be effective (Leary et. al., 2013). This observation suggests that the focus, persistence, and affirmation, each of which a tutor naturally gives a student, are key elements of student success. And so, affect is represented as a supporting foundation of Mathematical Knowledge for Tutors.

Self-Regulatory Knowledge for Tutoring

Self-regulation, which includes metacognition and skills for self-control and decision-making, overlays both Subject Matter Knowledge and Pedagogical Content Knowledge for Tutors. Metacognition, which is the ability to think about one's thinking, is a particularly important while problem-solving (Schoenfeld, 1992). Effective problem solvers spend time understanding the problem, designing a plan to solve the problem, carrying out the plan, and reflecting back. As they progress through each problem-solving phase, efficient problem solvers are aware of their position in the problem-solving process and cycle back to a previous phase when needed. Once a solution is reached, the problem solver looks back at the solution, checks the work, reflects on its validity and makes connections to other work.

This knowledge is not unique to tutoring; however, because problem solving forms the basis of most tutoring interactions metacognition is particularly important for tutoring. Tutors not only need to have metacognition about their own problem solving (a type of Common Content Knowledge), they need to understand which components of the metacognitive process are challenging when problem solving (SCK). In addition, they must evaluate where a student is in their metacognitive process (KCS) and know ways to move the student to the next step (KCT). The individual and immediate nature of tutoring (Lepper & Woolverton, 2002) makes this type of knowledge of metacognition for tutors distinct from the knowledge typically found in teaching.

Because problem solving permeates SMK for tutors, the rich discussions of the importance of metacognition in problem solving, peer learning, and tutoring seem relevant; however, little evidence of metacognitive moves are observed in tutoring sessions (Graesser, 2011). Topping (1996) specifies metacognition as key component of peer learning. Graesser (2011) describes the metacognition of the tutor with respect to teaching the student; however, more work is needed to help tutors share the metacognitive aspects of problem solving with their students.

Metacognition is one component of self-regulation. Other components such as teaching students to be more self-regulated with respect to study skills indicate that self-regulation is also a critical covering of PCK. Self-regulatory study skills help students acquire knowledge, connect knowledge, and apply knowledge. Within the context of mathematics, self-regulation may include general study skills such as setting goals, planning to reach those goals, and assessing whether those goals have been obtained. The elements of individualization, immediacy, and interactivity, distinctive characteristics of tutoring, suggest that there exist techniques of self-regulation which are unique to tutoring (Lepper and Woolverton, 2002). In addition, peer tutors have a different power dynamic with the students they tutor compared to a teacher, and they may have unique, less evaluative ways to relate to students regarding self-regulation and study skills.

The relevance of individualization, immediacy, and interactivity in the tutoring session leads to a greater distinction between teaching and tutoring. An effective tutor makes a crucial connection between a student's cognitive model and motivational model. This connection may be congruent, independent, or conflicting (Lepper and Woolverton, 2002). If cognitive and motivational diagnoses are congruent and lead to the same approach, the situation is ideal. If cognitive and motivational diagnoses are independent, an approach used to address either cognitive or motivational needs will not affect the other. When cognitive and motivational diagnoses lead to conflicting approaches, the tutor needs to discern which approach is most appropriate at a given time. This astute decision-making process, which follows the complex assessment of cognitive and motivation needs of the students, is unique to tutors and validates the placement of self-regulation as an awning overlaying Mathematical Knowledge for Tutors.

Conclusions

This report proposes an initial schema for Mathematical Knowledge for Tutoring of undergraduate mathematics; the schema is based on literature in problem solving, peer learning, and mathematics teaching and tutoring. Ball's (2008) model of Mathematical Knowledge for Teachers is deepened to include the important role of affect in MKTu and raised to highlight the particular role of self-regulation in MKTu. It is important to note that many of the types of knowledge proposed in this framework are already a part of the responsive, personalized teaching that takes place during one-on-one discussions during classroom interactions. However, the tutoring context is necessarily unique due to the context, goals of the interaction, and the breadth of tutor experience. Imposing a model of teaching knowledge onto tutors results in a deficit evaluation: in contrast, this model highlights the critical types of knowledge necessary for tutors while expanding the framework to capture types of knowledge outside of the typical role of a teacher.

The adapted egg raises research questions and lays the basis for formal observations of undergraduate mathematics tutoring. As findings from research studies of tutoring interactions emerge, the MKTu egg will evolve into a more complex theoretical framework integrating subject matter knowledge, pedagogical content knowledge, self-regulation, and affect. The modified framework will serve as an incubator of new research questions and further studies.

Ultimately, our goal is to leverage future research built on this framework toward the development of training materials for undergraduate tutors. These materials will be implemented, assessed, and tested. Updates to the MKTu theory will in turn generate new sets of observations and research studies. The momentum of this continuing cycle will propel our work in identifying, understanding, and implementing effective practices as well as developing and testing training materials for undergraduate math tutoring.

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Leveraging Cognitive Theory to Create Large-Scale Learning Tools

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At the 21st Annual Conference on Research in Undergraduate Mathematics Education, Ed Dubinsky highlighted the disparity between what the research community knows and what is actually used by practicing instructors. One of the heaviest burdens on instructors is the continual assessment of student understanding as it develops. This theoretical paper proposes to address this practical issue by describing how to dynamically construct multiple-choice items that assess student knowledge as it progresses throughout a course. By utilizing Automated Item Generation in conjunction with already-published results or any theoretical foundation that describes how students may develop understanding of a concept, the research community can develop and disseminate theoretically grounded and easy-to-use assessments that can track student understanding over the course of a semester.

Keywords: Assessment, Automated Item Generation, Technology in Mathematics Education

At the 21st Annual Conference on Research in Undergraduate Mathematics Education, Ed Dubinsky highlighted the disparity between what the research community knows and what knowledge is actually applied by practicing instructors. This disparity is not unique to mathematics education and exists even in education research in general (Van Velzen, 2013). One method to bridge this disparity is to involve mathematics instructors in research projects (Vidakovic, Chen, & Miller, 2016). Even outside of funded efforts, the RUME community in general has been encouraging practicing instructors to participate. However, there are a plethora of instructors that this approach cannot be applied to as they have limited time and resources. For these instructors, administering and providing feedback for open-response classroom activities is not feasible, especially for those who coordinate large-scale courses such as Calculus. This burden renders the knowledge of the research community moot, as it is too resource-expensive to collect and implement the knowledge practically. We propose an alternative method to engage these instructors: utilize the mathematics education literature and programming languages to dynamically generate multiple-choice questions that can form easy-to-use assessments throughout a course. To set the stage for this method, we briefly summarize how research has been presented to the community at large.

Disseminating Research Results

Let us consider how mathematics education research is disseminated from the perspective of a mathematics instructor. First, we need access to journal articles on the results from research - articles that may or may not be available through our school's library. Assuming we get access, we read through the results and notice the majority of assessments used are free-response assessments. This is not unexpected as qualitative research primarily use free-response assessments to gain as much insight into student thinking as possible. However, these results and assessments are difficult for instructors to utilize. The result is an isolating effect where instructors rely on their own experience and knowledge to develop instruments that may or may not be based in how students develop their mathematical conceptions.

One avenue for potential instructor use of research results is *concept inventories* - multiple-choice assessments designed to explore students' conceptual knowledge of a specific topic. One of the earliest such assessments is the Force Concept Inventory (Hestenes, Wells, & Swackhamer, 1992), which outlines the conceptions necessary to understand Newtonian force. More recently in mathematics education, Carlson, Oehrtman, and Engelke (2010) introduced the Precalculus Concept Assessment (PCA). We quickly summarize how they developed this assessment below.

Through numerous studies, Carlson et al. (2010) developed a taxonomy of the necessary conceptions students should develop before taking calculus. Each assessment item is linked to one or more of these conceptions and went through multiple phases of refinement and validation:

- Phases I & II: Series of studies to identify and analyze how students understand the central ideas of precalculus and calculus. Open-ended questions were refined and common student responses were identified.

- Phase III: Validated multiple-choice items based on the open-ended questions from Phases I & II. This phase went through eight cycles of administering the assessment, conducting follow-up interviews, analyzing student work, and revising the taxonomy and assessment based on the results.

- Phase IV: Widespread administration of the revised 25-item multiple-choice assessment.

Based on this short explanation of their generation and validation process, it is no wonder few concept inventories have been developed to date – these assessments are time-consuming and costly to develop and validate properly. It is likely the primary reason qualitative research does not present more easily-accessible materials for instructors to implement in their classroom. Yet, these assessments are crucial as they provide an avenue to efficiently assess students' conceptual understanding of a mathematical topic. These research-based multiple-choice assessments provide the practical application of the RUME community to mathematics instructors. We feel that technology can aid researchers in transforming more open-ended questions and qualitative results into multiple-choice assessments that can be used by instructors. The next section will describe how we can dynamically generate quality multiple-choice items.

Automated Item Generation

We use a typical College Algebra item (Figure 1) to introduce multiple-choice item terminology. A *multiple-choice item* consists of a stem and options. The *stem* includes the context, content, and problem for the student to answer. In the example in Figure 1, this includes the instructions (context) and the problem. By *problem*, we refer to the content issue that must be solved. In the example in Figure 1, this would be solving the linear equation. Solving this problem leads to the *solution*. Plausible, but incorrect, answers to the problem are referred to as *distractors*. The solution and distractors are used to create the *options*, or choices presented that the student must choose from. Of these, the *correct option* corresponds to the option that correctly solves the problem in the stem (solution) while the *distractor options* correspond to the incorrect, distractor solutions.

There are currently two general strategies to generate distractors. The first strategy focuses on similarities between the solution and distractors. For example, a numeric solution could be manipulated in some form: being negated, divided by a factor, or shifted a small amount. Manipulating the solution in some way to make similar responses does not require a great deal of time and resources, and thus is commonly utilized (Gierl, Bulut, Guo, & Zhang, 2017). The disadvantage to this method is that distractors may not reflect actual student thinking. Students with incomplete

[Stem]

Solve the linear equation below.

[Problem]

$$\frac{-3x-6}{3} - \frac{-8x-8}{5} = \frac{7x+6}{2}$$

[Options]

A. $x = -\frac{40}{29}$ **[Distractor]**

B. $x = -\frac{34}{29}$ **[Solution]**

C. $x = -\frac{66}{29}$ **[Distractor]**

D. $x = -\frac{17}{10}$ **[Distractor]**

Figure 1: Example of a typical multiple-choice item.

knowledge may be able to eliminate these types of distractors and thus arrive at the solution (or, at least, more easily guess at the solution), thereby rendering the goal of assessing student knowledge moot. In short, multiple-choice items developed with these types of distractors would not provide feedback on students' potential cognitive processes.

The second method focuses on common misconceptions in student thinking while they reason about the problem. These misconceptions can be recalled and utilized by experienced content specialists reflecting on the common errors they have seen in the past or identified through evidence-based research on students' work during open-ended items (Gierl et al., 2017). This approach creates high-quality distractors that mirror responses students may make during an open-ended assessment.

In addition to the two methods above, we could consider how a student's conception of a mathematical topic would influence their response to the question. This strategy would enable the instructor to link certain multiple-choice responses to the student's conception at the time of the test. Carlson et al. (2010) utilize this method in the PCA to great success. By creating distractors based on a student's conception as it develops over time, instructors can more accurately assess and improve student understanding.

One avenue for creating quality distractors based on all three methods above is Automatic Item Generation (AIG). AIG utilizes computer technologies and content specialists (or evidence-based research) to automatically generate problems, solutions, and quality distractors. Few examples of AIG currently exist, even in the context of mathematics (Gierl et al., 2017; Gierl, Lai, Hogan, & Matovinovic, 2015). We will now illustrate how to leverage the knowledge of the research community to automatically generate distractors, and in doing so, generate ways to assess student knowledge as it develops.

Methodology

Dubinsky and Wilson (2013) investigated low-achieving high school students and their understanding of the concept of function. In their research assessment, they asked the following typical questions about composition of functions:

1. Suppose f and g are two functions. Find the compositions $f \circ g$ and $g \circ f$.
2. Suppose $h = f \circ g$ is the composition of two functions f and g. Given h and g, find f.
3. Suppose $h = f \circ g$ is the composition of two functions f and g. Given h and f, find g (Dubinsky & Wilson, 2013, p. 97).

Correct answers to these questions can provide some knowledge about students' understanding of functions in general. In fact, the authors state:

> In both the written instrument and the interviews, we asked students questions, some of which we considered to be difficult, about composition of functions. Our intention was to investigate the depth of their understanding of the function. We also felt that success in solving these problems was an indication of a process conception of function and in some cases, an indication of a process conception that was strong enough so that it could be reversed in the mind of a participant in order to solve a difficult composition problem (Dubinsky & Wilson, 2013, pgs. 96-97).

While open-response items would provide more information about students' understanding, this illustrates how correct answers to multiple-choice items could suggest students' conceptions of a particular concept. It is this belief that allows even multiple-choice questions to be used as learning tools in the classroom, as they can shed light on what students understand and allow instructors to challenge misconceptions. In order to be successful, multiple-choice items should include the common conceptions students may have. We illustrate how to develop quality distractors in the context of composition of functions below.

Consider the typical College Algebra exam item in Figure 2. The question requires students to compose two functions and evaluate the composition at a given point $x = a$. A student with adequate procedural understanding of function composition will compose the new function and evaluate it at the point to obtain $f(g(5)) = \left(\frac{1}{3}(5)^2 + 1\right)^2 = \frac{784}{9}$ which is answer choice A. in Fig. 2. Two other common responses Dubinsky and Wilson (2013) observed students made when solving function composition problems of this type were (a) composing the functions in an opposite order *(answer choice C.)* and (b) conflating the composition notation with multiplication notation *(answer choice B.)*. These responses would correspond to (a) a student recognizing composition as a new operation yet not performing the action correctly and (b) a student not recognizing composition as a new operation, similar to multiplication having multiple representations: $\times, \cdot$, and the absence of an explicit operator such as with $4x$.

To be clear - this question does not assess a student's conceptual understanding of composition of functions. It is however necessary students can illustrate adequate procedural knowledge of composition *before* moving on to develop a conceptual understanding of the operation. We now illustrate an automated question meant to assess a student's conceptual understanding of function composition.

Specific: Suppose $f(x) = (x+1)^2$ and $g(x) = \frac{1}{3}x^2$ are two functions. Find the composition $(f \circ g)(x)$ at the point $x = 5$.

A. $\frac{784}{9}$

B. 300

C. $\frac{1296}{3}$

Generalized: Suppose $f(x) = (x+c)^2$ and $g(x) = \frac{b_1}{b_2}x^2$ are two functions. Find the composition $(f \circ g)(x)$ at the point $x = a$.

A. $f(g(a)) = \left(\frac{b_1}{b_2}a + c\right)^2$

B. $(f \cdot g)(a) = \frac{b_1}{b_2}a^2\,(a+c)^2$

C. $g(f(a)) = \frac{b_1}{b_2}\,(a+c)^4$

Figure 2: Typical College Algebra function composition exam item and generalized template.

The second and third types of function composition questions used by Dubinsky and Wilson (2013) required students to take a composed function $h(x) = (f \circ g)(x)$ and isolate the functions composing it. For example, given $h(x)$ and $g(x)$, a student would then be asked to find the expression for $f(x)$, or find the value of the expression at a given point a. While Dubinsky and Wilson (2013) did not provide alternative student responses, we constructed a question and two "potential" student responses in Figure 3. In this example, the student is given a table representation of the functions and asked to consider reversing the function composition to evaluate f at. Rather than reversing the composition, a student could evaluate $h(g(2))$ and "solve" the problem using similar steps to question 1. This would suggest the student has memorized a procedure to evaluate composition of functions, but does not recognize the need to reverse the process. Alternatively, a student could evaluate h at 2, then find the corresponding x value to when g is 1. This would suggest the student recognizes the need to reverse the composition process but the order of the function composition $f(g(x))$ was inverted. Finally, a student could state that without knowledge of the function f, they cannot evaluate any point. This would suggest the student views a function as a single algebraic formula.

The ability to reverse the composition and isolate the functions composed, as well as describe this process in general, would be growth of a conceptual understanding of composition. Now, a single multiple-choice question cannot provide an instructor with strong evidence of a student's procedural and/or conceptual understanding. However, by combining a series of questions linked to common conceptions on composition, instructors can identify where a student is in their conception and provide targeted feedback. With the added technological component, this identification

Specific: Given **only** the information in the following table, find $f(2)$ *(if possible)*.

x	$h(x)$	$g(x)$
2	1	-3
-3	4	1
-2	0	2

A. $f(2) = 4$

B. $f(2) = 0$

C. $f(2) = 1$

D. It is not possible to find $f(2)$ based only on the information in the table.

General: Given **only** the information in the following table, find $f(a_1)$ *(if possible)*.

x	$h(x)$	$g(x)$
a_1	c_1	b_2
b_2	b_3	c_1
a_2	a_3	a_1

A. $f(a_1) = b_3$

B. $f(a_1) = a_3$

C. $f(a_1) = c_1$

D. It is not possible to find $f(a_1)$ based only on the information in the table.

Figure 3: Example and template for function composition problems type 2 and 3.

can be automated and provided to the student without the instructor combing over the student's work. It is this fine-grained assessment and feedback that can improve how students develop their understanding throughout a course. In short, quality multiple-choice assessments can remove the time-burden of free-response assessments while (theoretically) providing similar results on student thinking.

Discussion

Generating multiple-choice assessments that can potentially indicate a student's level of understanding is attractive for a variety of reasons. From a practicality standpoint, these assessments would be cost-effective (both in time and resources to develop) and quick to grade. Providing the linked distractors for each item choice can also draw students' attention to their conception, allowing them to modify their thinking. It is in this way - explicitly challenging student conceptions in a cost and time effective manner - that these assessments can be used as practical learning tools. The mathematics education research community has the knowledge needed to create these quality multiple-choice assessments. By combining this knowledge with automatic item generation, the mathematics education research community can provide instructors with practical results based on empirical data.

In addition, theoretical frameworks such as APOS Theory posit learning trajectories students may take to learn a concept. By creating multiple-choice questions aligned to the various levels as students' knowledge develops, instructors can track a student's progress to provide individual feedback. With the ease that multiple-choice assignments can be graded, this individualized feedback can be scaled to large courses such as College Algebra and Calculus.

Automatically generated multiple-choice assessments can also serve as a research tool. We noted a paper by Dubinsky and Wilson (2013) in which they asked students to answer common college algebra questions. By converting these types of questions to multiple-choice items, researchers can widen their sample size to provide greater certainty of the results. Some authors in the RUME community have begun to tap the potential of multiple-choice assessments in research, such as Carlson et al. (2010) with their use of multiple-choice assessments in a precalculus concept inventory. A wide-spread use of multiple-choice assessments based on empirical evidence can provide the sample size needed to produce robust results.

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A Comparison of Frameworks for Conceptualizing Graphs in the Cartesian Coordinate System

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The use of the Cartesian Coordinate system (CCS) pervades secondary and tertiary mathematics curriculum, as the dominant convention for displaying graphs of functions. The CCS in two dimensions may be framed as a conceptual blend of two number lines and a Euclidean plane (Lakoff & Núñez, 2000). Within the concept of a number line is a conceptual metaphor uniting numerical values with points on a line. While such a description of the CCS may describe a shared understanding of the convention among the mathematics community, it may not account for the ways in which individual students interpret graphs presented in the CCS. Other theories, such as David et al.'s (2017) constructs of value-thinking and location-thinking, have been proposed to account for students' graphical interpretations. In this paper, I outline these two ways of framing conceptions of graphs, the uses of each framework, and their relation to each other.

Keywords: Cartesian Coordinate System, Graphing, Conceptual Blend, Conceptual Metaphor, Value-Thinking and Location-Thinking

Across numerous mathematics courses at the secondary and undergraduate level, students are asked to interpret and reason with graphs that are represented in the two-dimensional Cartesian coordinate system (CCS). In the U.S., the CCS is typically the first coordinate system in which students are expected to graph points (5.G.A.1-A.2) and the standard coordinate system used in curriculum from Algebra through Calculus (e.g., Stewart, 2012). The CCS, like other coordinate systems, follows certain conventions. In two dimensions, the Cartesian plane consists of two axes with specified units that meet at a right angle. Pairs of values are represented given distances from the intersection of these axes, referred to as the origin. Due to its fundamental role in the teaching and learning of mathematics at the secondary and undergraduate level, researchers have examined ways in which both individuals and the mathematics community use, reason with and interpret graphs in this coordinate system. Through various modes of research from different perspectives, several theoretical frameworks have been proposed to explain some ways in which graphs are understood in the CCS.

In this paper, I will describe two theoretical frameworks from different theoretical traditions, explain how they may be used, and offer some examples of how researchers have used these to frame their data analysis. I will also compare the purposes and benefits of adopting and utilizing each of these frameworks. One framework comes from the work of Lakoff and Núñez (2000), whose perspective offers insight into the underlying cognitive structure of the CCS as developed and used by the mathematics community. In their description, the CCS relies on a conceptual metaphor of numbers as points to understand. While Lakoff and Núñez's (2000) framework offers one view of the CCS as a conventional system, rooted in an embodied cognition perspective, their theory may not readily apply when describing the ways in which individuals may interpret graphs represented by such a system. For instance, instructors teaching students content that includes graphs in the CCS may use different understandings than those proposed by Lakoff and Núñez (2000). Furthermore, the way in which students interpret graphs presented to them in their courses may differ from the mathematics community as well as their instructors. Thus, I will also describe David, Roh, and Sellers (2017) framework to characterize students' graphical interpretations. Their framework also recognizes the role of both values and locations of points in interpreting graphs, namely that students may attend to one aspect of points rather than the other in their interpretations. Both David et al.'s (2017) framework, as well as Lakoff and Núñez's (2000) theory offer valuable insight into cognition related to graphs. The adoption of one theoretical frame for conceptualizing graphs rather than another ought to be guided by a researcher's purposes.

Conceptual and Ideational Mathematics

To frame my discussion of the content and purpose of these frameworks for studying conceptions of graphs, I adopt two considerations offered by Schiralli and Sinclair (2003) in their commentary on the work of Lakoff and Núñez (2000). The first is the distinction they make between conceptual mathematics and ideational mathematics. In their explanation, Lakoff and Núñez (2000) offer a description of *conceptual mathematics*, which refers to the discipline of mathematics as a collective subject matter, negotiated by participants in the mathematics community who hold a shared meaning. In contrast, they use the term *ideational mathematics* to refer to the ways in which individuals interpret or reason about conceptual mathematics. Ideational mathematics includes the ways that mathematicians may use and conceptualize mathematical ideas and the ways students may interpret and understand ideas. In defining these terms, Schiralli and Sinclair seek to clarify whether the mathematical concept to be studied is shared knowledge in the field of mathematics or lies in the mind of an individual engaged in mathematical thinking. Schiralli and Sinclair (ibid) also emphasize that the way in which a particular group or individual is engaged with the mathematics, "whether one is *learning, doing,* or *using* mathematics" may influence their cognitive processes and should be considered (p. 81). For the purposes of discussing the theoretical frameworks in this paper, I follow these two considerations posited by Schiralli and Sinclair (ibid): I situate each framework and its use based on (1) "which mathematics" aims to be studied and (2) the nature of the goals of the individual or group conceptualizing the mathematics. These considerations of Schiralli and Sinclair (ibid) help to make explicit certain underlying assumptions within each theoretical framework as well as offer insight into how these frameworks may serve researchers in their purposes of investigating various conceptions of mathematical ideas.

Cartesian Plane as a Conceptual Blend

Lakoff and Núñez (2000), who operate from a perspective of embodied cognition, view mathematical thinking as fundamentally rooted in humans' sensorimotor experiences, influenced by their neural biology. In their work to describe "where mathematics comes from," Lakoff and Núñez (ibid) seek to reveal and untangle the underlying cognitive structures that serve as the foundation of mathematics as a discipline. Their framework derives from a method of "mathematical idea analysis," a linguistic approach in which they uncovered underlying metaphors from the language used in central concepts in mathematics (Schiralli & Sinclair, 2003). Thus, the frameworks they propose for making sense of the cognitive structure of mathematical ideas describe *conceptual mathematics* used by mathematicians in doing mathematics.

Relative to graphs in the Cartesian Coordinate System, Lakoff and Núñez (2000) describe the Cartesian Plane as a conceptual blend, a combination of conceptual domains. In their description, number-lines make up the axes of the Cartesian Plane, which rely on the conceptual metaphor "Numbers are Points on a Line." Lakoff and Núñez (ibid) describe conceptual metaphors as a cognitive tool to make concrete concepts which are inherently abstract, such as those in mathematics. In a conceptual metaphor, an object is mapped from a source domain to another object in a target domain in such a way that preserves inferences. In the Numbers are Points on a Line" metaphor, numbers are the target, abstract domain described by points on a line, a more concrete concept. Table 1 contains Lakoff and Núñez's (ibid) description of the key correspondences in the "Numbers are Points on a Line" metaphor.

Table 1. Numbers are Points on a Line (for Naturally Continuous Space) (Lakoff & Núñez, 2000, p. 279)

Source Domain Points on a Line		*Target Domain* A Collection of Numbers
A Point *P* on a line	→	A Number *P'*
A Point *O*	→	Zero
A point *I* to the right of O	→	One
Point *P* is to the right of point *Q*	→	Number *P'* is greater than Number *Q'*
Point *Q* is to the left of point *P*	→	Number *Q'* is less than Number *P'*
Point *P* is in the same location as point *Q*	→	Number *P'* equals number *Q'*
Points to the left of *O*	→	Negative numbers
The distance between *O* and *P*	→	The absolute value of number *P'*

In this metaphor, points on a line are the source domain, the concrete object, to which a collection of numbers is mapped. Ordering is one of the inferences preserved in this mapping, with points to the left defined as numbers with smaller values. Through what they refer to as the "Number-Line Blend," new objects are created which they refer to as number-points, at once numbers and points on a line.

Moving from one to two dimensions, the Cartesian Plane is described by Lakoff and Núñez (2000) as comprised of a conceptual blend. A conceptual blend, distinct from a conceptual metaphor, refers to a blending of "two distinct cognitive structures with fixed correspondences between them" (Lakoff & Núñez, ibid, p. 48). Table 2 shows the correspondences that comprise the Cartesian Plane Blend.

Table 2. The Cartesian Plane Blend (Lakoff & Núñez, 2000, p. 385)

Conceptual Domain 1 Number Lines		*Conceptual Domain 2* The Euclidean Plane with Line X Perpendicular to Line Y
Number line *x*	↔	Line X
Number line *y*	↔	Line Y
Number *m* on number line *x*	↔	Line *M* parallel to line *Y*
Number *n* on number line *y*	↔	Line *N* parallel to line *X*
The ordered pair of numbers (*m*, *n*)	↔	The point where *M* intersects *N*
The ordered pair of numbers (0, 0)	↔	The point where *X* intersects *Y*
A function *y*=f(*x*); that is, a set of ordered pairs (*x*, *y*)	↔	A curve with each point being the intersection of two lines, one parallel to *X* and one parallel to *Y*
An equation linking *x* and *y*; that is, a set of ordered pairs (*x*, *y*)	↔ ↔	A figure with each point being the intersection of two lines, one parallel to *X* and one parallel to *Y*

The solutions to two simultaneous equations in variables x and y	The intersection point of two figures in the plane

In this conceptual blend, the cognitive structures of number lines and the Euclidean plane are combined. Each element of one domain combines with an element from the other domain. For instance, the x-axis in the CCS is a blend of both a number line 'x' as well as a Line 'X' in the Euclidean plane. Similarly, points in the CCS are at once ordered pairs (m, n) and locations of intersections of two lines related to m and n, parallel to the y-axis and x-axis respectively.

Use of Cartesian Plane as Conceptual Blend Framework

Describing the mathematical use of number lines as relying on a conceptual metaphor and the Cartesian Plane as a conceptual blend may offer insight into the emergence of these conventions in the development of the field of mathematics. Namely, the mental act of ascribing geometric notions of locations and distances offers a powerful conceptual tool to conceptualize abstract ideas of number, ordered pairs, and functions. However, this framework characterizes ideas that have developed into shared meanings within the mathematical community, rather than individual differences in working with such ideas.

Although the nature of Lakoff and Núñez's (2000) framework is designed to characterize conceptual mathematics, the construct of a conceptual metaphor offers a lens to consider the ideational mathematics of individuals. For instance, Font, Bolite, and Acevedo (2010) investigated the metaphors that Spanish high school instructors used in their classrooms while teaching graphs of function. In their study, Font et al. (2010) were interested in investigating instructors' *ideational mathematics* while engaged in the act of teaching. They found that instructors used various metaphors to communicate properties of graphs. These metaphors included the graph as a path, orientational metaphors, and object metaphors, in addition to the ones identified by Lakoff and Núñez's (2000) description of *conceptual* mathematics related to graphs. Furthermore, instructors were found to be unaware of their use of language related to these metaphors in their instruction. When asked to consider their own metaphorical language, instructors commented that their purpose in using it was to support their students in understanding a certain principle. While this study examined how instructors interpret and use ideas related to graphing while teaching, other studies have focused on characterizing *students'* ideational mathematics.

Interpreting Graphs via Value-Thinking or Location-Thinking

In contrast with a perspective of embodied cognition, David et al.'s (2017) framework to characterize conceptions of graphs is situated in a constructivist perspective. This framework, shown in Table 1, details two ways *students* may interpret aspects of graphs, referred to as *value-thinking* and *location-thinking*. This framework was developed to describe phenomena that were observed in undergraduate students' interpretations of graphs in the CCS in the context of the Intermediate Value Theorem (IVT) (David et al., ibid). To be clear, their framework was not a priori theory; rather, this framework emerged from their data analysis (David et al., ibid). In this framework, if a student attends to the pairs of *values* that points represent, this way of thinking is referred to as *value-thinking*. On the other hand, if a student focuses on the *location* of points in the Cartesian plane, this way of thinking is referred to as *location-thinking*.

Table 3. Comparison of Characteristics of Value-Thinking and Location-Thinking (David, Roh, & Sellers, 2017, p. 96)

<table>
<tr><th colspan="2"></th><th colspan="2">Value-Thinking</th><th colspan="2">Location-Thinking</th></tr>
<tr><th colspan="2"></th><th>Interpretations</th><th>Evidence</th><th>Interpretations</th><th>Evidence</th></tr>
<tr><td rowspan="3">Aspects of a Graph</td><td>Output of Function</td><td>The resulting value from inputting a value in the function</td><td>▪ Labels output values on output axis
▪ Speaks about output values</td><td>The resulting location in the Cartesian plane from inputting a value in the function</td><td rowspan="3">▪ Labels outputs on the graph
▪ Labels points as outputs
▪ Speaks about points as a result of an input into the function (e.g., “an input maps to a point on the graph”)</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs
▪ Speaks about points as the result of coordinating an input and output value</td><td>A specified spatial location in the Cartesian plane</td></tr>
<tr><td>Graph as a Whole</td><td>A collection of coordinated values of the input and output</td><td>A collection of spatial locations in the Cartesian plane associated with input values</td></tr>
</table>

These two ways of thinking characterize students’ interpretation of graphs. The framework explains each way of thinking by detailing how a student engaged in that way of thinking thinks about three aspects of a given graph: outputs of the function, points on the graph, and the graph as a whole. Each of these aspects of graphs is described from the perspective of a researcher using conventional interpretations of the Cartesian coordinate system. The output of a function is conventionally represented as a magnitude of length in the direction of the y-axis. A point conventionally represents a pair of both input and output values, located a distance of the input value to the right of the origin, and a distance of the output value above the origin. Conventionally, a graph as a whole represents the set of all ordered pairs that satisfy the equation of the function. The framework also describes observable evidence indicative of thinking about aspects of the graph in a particular way. Using these descriptions of observable evidence in the framework, students’ words, gestures, and markings on the graph can be used to characterize their way of thinking about graphs as either value-thinking or location-thinking.

Value-Thinking

In this framework, value-thinking refers to an attention to the values represented by a point in Cartesian space. Students whom David et al. (2017) classified as engaged in value-thinking treated outputs as values associated with corresponding input values. These students may have indicated their thinking by labeling output values on the output axis, or speaking about output *values*. In their description, students engaged in value-thinking think of points as coordinated pairs of input and output values. These students may indicate this way of thinking by labeling points as ordered pairs, and speaking of simultaneous pairs of values when referring to points on a graph. Thus, students engaged in value-thinking treat graphs as a collection of points, each of which represents a pair of input and output values.

Location-Thinking

In contrast, location-thinking refers to an attention to the locations of the points in space. Students whom David et al. (2017) classified as engaged in location-thinking treated points on the graph as outputs, confounding outputs of the function with points on the graph. These students may have indicated that they were thinking in this way by referring to points solely as outputs or describing the output of a function as the location of the graph itself (e.g., “each input is mapped to a point on the graph”). Additionally, students engaged in location-thinking may label a point with an output value only, thus placing the output label at a point, rather than on the output axis. Thus, students engaged in location-thinking treat graphs as a collection of points that represent locations in the plane that correspond with input values.

Use of Value-Thinking and Location-Thinking Framework

To highlight the distinction between value-thinking and location-thinking, consider the two examples of sample student labeling on the same graph indicative of each of these ways of thinking, shown in Figure 1.

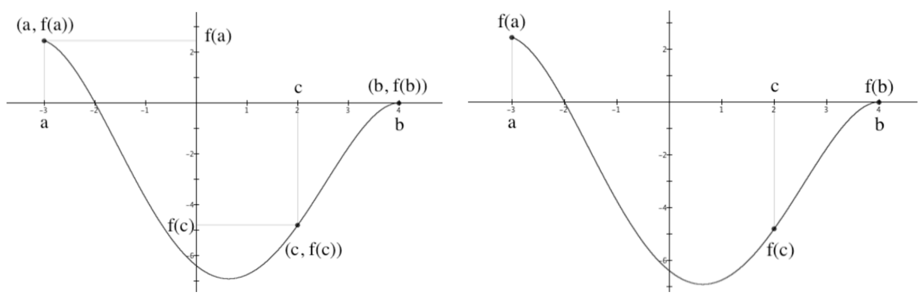

Figure 1: Example labels indicative of value-thinking, left, or location-thinking, right. (David et al., 2017 p. 97)

The labels on the graph in Figure 2, left, may indicate value-thinking. In this graph, output values are labeled on the output axis, and points are labeled as ordered pairs. In contrast, the labels on the graph in Figure 2, right, may indicate location-thinking. Output labels are not placed on the output axis but rather at the locations of points. Consequently, points are not labeled as ordered pairs but solely as outputs. While a student’s gestures and words should be examined in addition to the labels on a graph, these examples highlight distinctive characteristics of value-thinking and location-thinking.

David et al.’s (2017) framework emerged from analysis of a data set from interviews of nine undergraduate math students who were asked to evaluate and interpret statements related to the Intermediate Value Theorem using graphs. Their final coding scheme involved classifying students as engaged in value-thinking or location-thinking throughout episodes of their interviews. In their later work, David et al. (2018) report details of a student, Zack, whose thinking was characterized as location-thinking. See his graph labels in Figure 2.

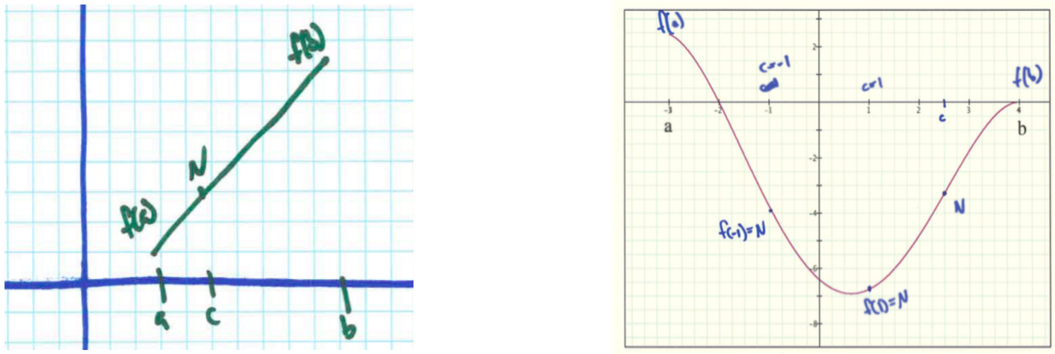

Figure 2. Zack's labels of points as outputs, a common characteristic of location-thinking (David et al., 2018)

David et al. (2018) point to several pieces of evidence support the claim that Zack was engaged in location-thinking when reasoning with these graphs and statements related to the IVT. First, Zack placed output labels at locations on the graph, rather than on the y-axis and referred to the endpoints of the graph as "$f(a)$" and "$f(b)$." Additionally, Zack labeled N's between $f(a)$ and $f(b)$ along the graph of the function, rather than along the y-axis. In addition to the context used in David et al.'s (2017) study, this framework may also be applied by researchers to characterize student thinking in other contexts. Instructors may even find such a framework useful in attending to their students' reasoning when teaching graphs of functions. In my view, this framework best supports the goal of characterizing the thinking of students engaged in the learning of mathematics.

Conclusion

The study of the conceptions and uses of graphs in the Cartesian Coordinate System is a valuable line of research in mathematics education. In this paper, I have compared two such theoretical frameworks related to the study of graphing in mathematics in terms of its relation to conceptual or ideational mathematics and the activity of those engaged with the mathematics at hand. By framing the Cartesian plane as a conceptual blend built on metaphors, Lakoff and Núñez's (2000) framework supports researchers in uncovering the cognitive processes involved in considering graphs in the CCS within the domain of conceptual mathematics. By extension, researchers have begun to use their metaphorical framing to capture ways in which instructors conceptualize graphs while teaching (Font et al., 2003). Characterizing students' ways of interpreting graphs (their ideational mathematics), David et al.'s (2017) framework of value-thinking and location-thinking highlights previously undocumented phenomena in students' graphical interpretations. Both of the theoretical frameworks illustrated in this paper acknowledge two aspects of points on graphs in Cartesian coordinates: the values represented by points and the locations of these points spatially. The way in which these frameworks view this duality differs due to the perspective adopted. For Lakoff and Núñez (2000), a practitioner of mathematics uses this duality, even if subconsciously. From the perspective of David et al. (2017), the extent to which students conceptualize this duality varies; in fact, students may be more likely to focus on one aspect of a point rather than both simultaneously. Going forward, researchers should take careful theoretical consideration in deciding how to frame investigations of graphing. Such attention may yield extensions of the current frameworks, further delineations of these ways of thinking, or other characteristics that have yet to be identified. In this way, theory on graphing as part of mathematical activity will continue to be built and refined.

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Why Don't Students Check their Solutions to Mathematical Problems?
A Field-based Hypothesis

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This theoretical paper introduces a field-base hypothesis, according to which the intensity and type of an intellectual need that students can experience for checking their solution to a problem might be related to the epistemological status of methods that they employed for solving the problem. The hypothesis emerged from the analysis of a final exam in a first-year course where 421 students worked on four problems in linear algebra. In one of them, 33 students provided evidence of checking their solutions, all of which appeared as educated guesses. No written evidence of checks was indicated in the deductive solutions, in which the students utilized algorithms, procedures, and theorems that were introduced to them in the course. Thus, it might be proposed that problem-solving methods with a low epistemological status (e.g., educated guesses) may instigate the need for checking a solution as a means to compensate for their status.

Keywords: checking solutions, DNR-framework, epistemological status, intellectual needs, problem solving.

Introduction and Literature Review

Let us assume that in her final exam in linear algebra, Rina was assigned with the problem in Figure 1. The solution is far from easy in this case, as it requires fitting together a considerable number of topics that the course covered: systems of linear equations, bases of vectors spaces, column spaces, and that just for the first part! Now, let us assume that Rina has put her course studies to use, which created an opportunity for her to check her own work. Hence come the questions whether she will do the checks, and if yes, how.

Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ 0 & 1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$, and let S be the set of all vectors $\vec{b} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ has a solution.

(a) Find a basis for S (no need to show that S is a subspace of $\mathbb{R}^3$).
(b) Find the least square solution to $A\bar{x} = \vec{c}$.
(c) Find the corresponding least square error.
(d) Give a non-zero vector that is orthogonal to every vector in S.

Figure 1. An assigned problem.

Research has been approaching such questions through the lens of metacognition and problem solving, when the lion's share of studies have been conducted in the context of school mathematics (e.g., Cai, 1994; Lucangeli & Cornoldi, 1997; Pugalee, 2004; Schoenfeld, 1992). One line of this research might propose that it would be rather atypical if Rina attempted to check her solutions. For example, in his study with twenty ninth-graders, Pugalee (2004) found that verification – evaluating decisions and checking calculations – was the rarest behaviour compared to the ones that the students exhibited at the orientation, organization, and execution phases in their problem solving. Another line of research might advise Rina to undertake the checks due to the recurrent findings on the relation between verification of solutions and successful problem solving (e.g., Cai, 1994; Lucangeli & Cornoldi, 1997). Malloy and Jones (1998), for instance, found a moderate correlation between problem success of twenty-four students of ages 12-14 and their verification behaviors. The verification in their study was associated with rereading the problem, checking calculations, checking the plan for solution, using another method, and redoing the problem. However, the findings of Mashiach Eizenberg and Zaslavsky (2004) may confuse Rina's decision-making. The participants in this study were fourteen undergraduate students, who initiated a verification of their solutions in nearly two thirds of the cases. Despite students' attempts, however, every second solution remained incorrect.

From the metacognitive point of view, the question of "to check or not to check" a devised solution pertains to how one allocates cognitive and affective resources during problem solving (Schoenfeld, 1992). Verschaffel (1999) maintains that such checks are especially important at the final stages of problem-solving cycles, where solvers need to interpret the outcomes of their work. In Schoenfeld's (1992) terms, checking can be viewed as an instance of monitoring since it is part of one's reflecting on the effectiveness of her problem-solving processes and products. Overall, the acknowledgement of the importance of checking can be traced back to the classical work of Pólya (1945), specifically to the "carrying out the plan" step for solving a problem and "looking back" at the devised solution. When carrying out the plan, Pólya recommends the solver to check and prove the correctness of each move that she undertakes. The "looking back" step, in turn, is instigated by such questions as "can you check the result?", "can you derive the result differently?", and "can you use the result, or the method, for some other problem?" In this way, despite its title, this step is targeted at preparing the solver for the next problem, the solution of which might be easier if she would take the time to critically reflect on the problem that has been solved already.

In this theoretical paper, I present a *field-based hypothesis* on possible relations between contextual affordances that can emerge when one solves a problem and consequent moves that she might undertake for checking her solution. Harel (2017) posits that a field-based hypothesis is

> "suggested by observations of learners' mathematical behaviors in an authentic learning environment, and is explained by cognitive and instructional analyses oriented within a particular theory of learning, but has not, yet, been proved or disapproved by rigorous empirical methodologies in large scale settings" (p. 70).

The DNR-framework is used as a theory of learning in this paper, when its selected constructs are reviewed in the next section. This is followed by a description of an authentic learning environment, in which observations of a large cohort of students were made. An analysis of these observations gives rise to the hypothesis in the last section.

Intellectual Need and Epistemological Justification

In Harel (2008a, b), Guershon Harel introduced a comprehensive conceptual framework "which seeks to understand fundamental problems of mathematics teaching and learning" (Harel, 2013a, p. 3). This epistemologically solid framework has already exhibited its analytical power and usefulness for designing teaching environments (e.g., Harel, 2013a, b, 2017). The framework has been termed with the acronym DNR, which stands for three pillar principles: duality, necessity, and repeated reasoning. The full brunt of DNR goes beyond the scope of this paper, hence, I provide a brief overview of its central constructs that are utilized later on.

The necessity principle grows from the work of Piaget (1985), in which learning is viewed as occurring in situations where one attempts to resolve a mental disequilibrium. Harel

(2017) encapsulates the principle as follows: “For students to learn what we intend to teach them, they must have a need for it, where ‘need’ refers to intellectual need” (p. 75). *Intellectual need* is conceived as a contextualized construct that comes into being in a situation which one experiences as problematic in the sense that her current state of knowledge is insufficient or incompatible and additional piece of knowledge should be acquired in order to reach an equilibrium. Specifically, Harel (2013a) distinguishes between five categories of intellectual needs: *the need for certainty* can emerge when a learner has doubts about the trueness of a particular assertion; *the need for causality* is the need to determine a cause of a phenomenon (i.e. to explain); *the need for computation* pertains to quantifying numeric values that are missing; *the need for communication* is manifested through formulating and formalizing for the sake of conveying and exchanging ideas; finally *the need for structure* is the need to reorganizing one’s knowledge into a logical structure. Kontorovich and Zazkis (2016) offered to enrich this categorization with Koichu’s (2008) principle of intellectual parsimony, which states that when solving a problem, a person can avoid investing more intellectual effort than the needed minimum for obtaining a solution. This principle may be positioned as an intellectual *need for parsimony*, the need which might explain why a particular piece of knowledge has not been constructed.

Harel (2013a) maintains that the notion of intellectual need is tightly connected to *epistemological justification*, which “refers to the learner’s discernment of how and why a particular piece of knowledge came to be. It involves the learner’s perceived cause for the birth of knowledge” (p. 8). In his later work, Harel (2018) offers a typology of epistemological justifications, where one of the types is *apodictic*. This justification pertains to one’s viewing the proving process of the logical implication $\alpha \rightarrow \beta$ either in causality or explanatory terms. An apodictic justification manifests itself when one is interested either in the consequences of α, or in the possible causes of β. Accordingly, α and the whole apodictic chain that leads to β endow a *high epistemological status* in relation to β.

Harel (2013a) emphasizes that intellectual needs are ingrained in all aspects of mathematical practice, which allows the application of his framework to the purposes of this paper. Indeed, α can be associated with an assigned problem, where the solution process constitutes an apodictic epistemological justification that causes and explains the emergence of the final answer β. Thus, the checking of β turns into an act of knowledge construction, through which one might fulfil her intellectual needs.

Observational Environment

The data illustrated comes from written solutions that 421 students submitted as part of their final exam in a first-year mathematics course. The course was delivered at a large New Zealand university and it was intended for undergraduates majoring in computer science, economics, statistics, and finance. For students enrolled in the course, it was their second encounter with university mathematics with a focus on two-variable calculus, differential equations, and topics in linear algebra where the necessary methods for solving Figure 1 were introduced. The course instruction can be described as mostly traditional and lecturer-centred, with some emphasis put on students’ reasoning. For instance, the guidelines for the exam in which Figure 1 was assigned stated, “You must give **full** working and **reasons** for your answers to obtain full marks” (bold in the origin).

The analysis of the solutions that the students submitted consisted of an iterative process with deductive and inductive components (Denzin & Lincoln, 2011), that corresponded with two questions: (i) What types of mistakes do students make in their solutions? (ii) What characterizes the solutions, in which the students provided written evidence of checking their final answers? The analysis started with a review of the correctness of students' submissions, where the ones with mistakes were classified according to the steps that distorted the problem-solving chain. This classification was informed by Movshovitz-Hadar, Zaslavsky and Inbar (1987), who explored common errors that students make in their matriculation exams. After analyzing 860 scripts, the researchers came up with six categories: distorted theorem or definition, technical error, misused data, misinterpreted language, unverified solution, and logically invalid inference. Due to the similarity of the analyses and types of data, these categories were used as a baseline for analyzing students' solutions in this study. At the next stage, a constant comparison technique (Glaser & Strauss, 1967) was employed for characterizing those solutions with written checks. The comparisons were targeted at delineating similarities between the solutions submitted by different students. The emergent similarities were applied for all data corpus to validate that they are characteristic indeed.

Overview of Students' Solutions and Their Checks

Table 1 provides an overview of students' submissions, and it shows that obtaining a final answer cannot be taken for granted in the cohort under scrutiny. Clearly, the written solutions that the students submitted captured only a part of the problem-solving journey that the students undertook. Hence, a lack of a check of a solution provided no evidence of whether and how a student monitored her work (some of the checks could have been carried out mentally, for instance). However, students' submissions of mistaken solutions point at the struggle to check the work, or a missed opportunity to do so. In turn, instances where students provided written checks deserve a special attention.

Table 1. Overview of students' solutions.

	Part (a)	Part (b)	Part (c)	Part (d)
<u>Final Answers Submitted</u>	263 (62.5%)	309 (73.4%)	217 (51.5%)	171 (40.6%)
Correct	131 (49.8%)	145 (46.9%)	151 (69.6%)	40 (23.4%)
Incorrect	132 (50.2%)	164 (53.1%)	66 (30.4%)	131 (76.6%)
<u>Sources of Incorrect Answers</u>				
Mismatch between a problem and employed method	44 (33.3%)	5 (3.05%)	13 (19.7%) -	56 (42.75%)
Methods with distorted steps	63 (47.73%)	6 (3.66%)	20 (30.3%)	13 (9.92%)
Computation mistakes	-	149 (90.85%)	-	19 (14.5%)
No solution process	25 (18.94%)	-	-	19 (14.5%)
<u>Written Checks</u>	-	-	-	33 (19.3%)

Table 1 shows that *all* written checks that the participating students submitted as part of their solutions to Figure 1, appeared as a response to Part (d). These checks encompassed computations of the dot products of the vectors from the basis in Part (a) with the vector which was a candidate for an answer. While every four out of ten computations contained a mistake (see Figure 2 for example), all the checks maintained that the dot product is zero. Accordingly, these checks can be viewed as enactments of an appropriate strategy, in which vectors' orthogonality has been attempted to be verified with a critical attribute that was used in the course for defining the concept.

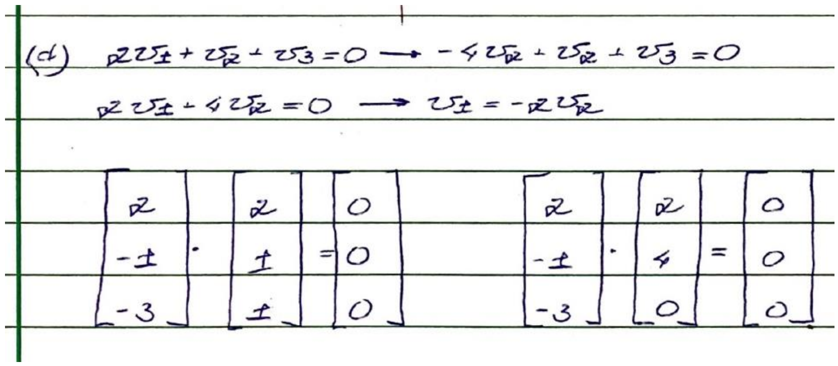

Figure 2. Example of an educated guess in Part (d).

One notable characteristic that was identified among all the solutions with written checks is that the candidates for orthogonal vectors were not devised with structured problem-solving methods that were studied in the course. Figure 2 exemplifies one third of such solutions, where the students started with a system of linear equations that the coordinates of the orthogonal vector were expected to satisfy. Yet, the equations were not solved fully and an orthogonal vector was introduced at some point. In the remaining solutions, the students started by declaring which vector is orthogonal (see Figure 3 for an example).

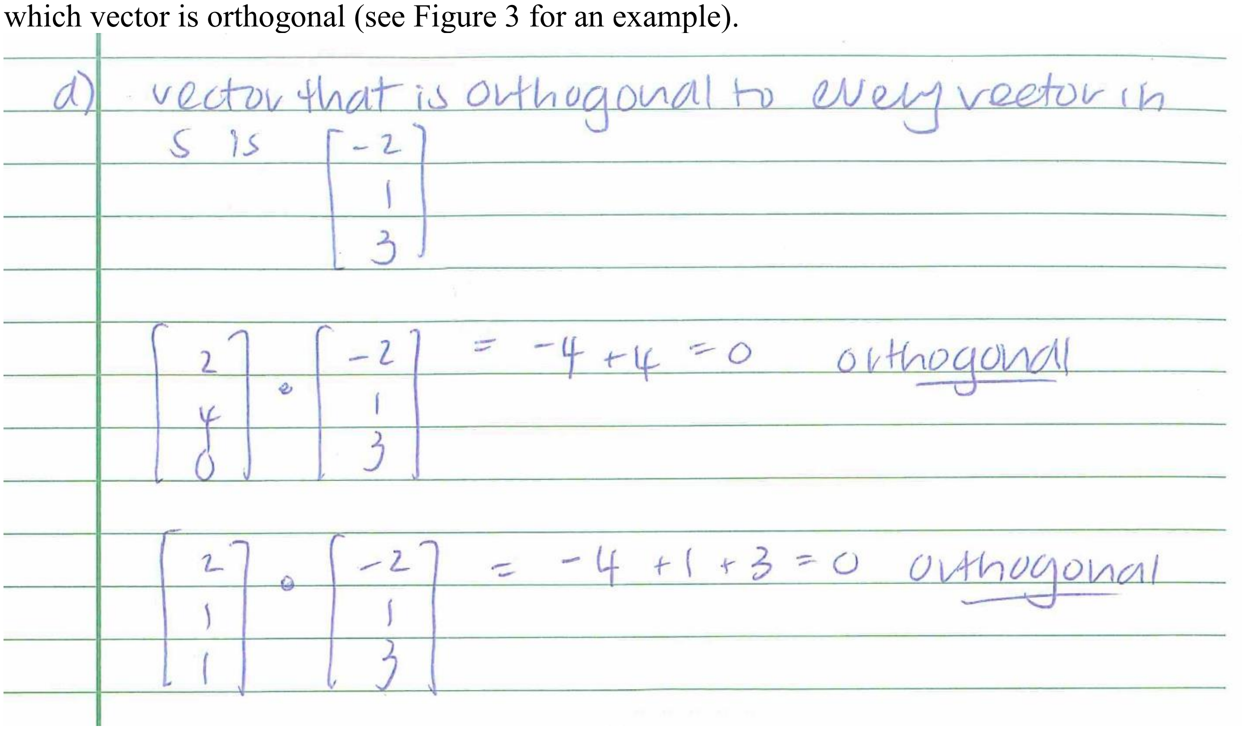

Figure 3. Example of an educated guess in Part (d).

To an external analyst who reviews students' submissions, the described introductions of orthogonal vectors appear as an act of *guessing*. My informal conversations with seven students who submitted a written check corroborated this impression. For instance, when reflecting on her solution in Figure 2, Rina (pseudonym) said,

> This vector [in Part (d)] must be perpendicular to my vectors from the first question. So I made a system of equations first, but then I kind of guessed what vector will work. It turned out to be correct.

Rina's words resonate with Mahajan (2010), who views guessing as a valuable problem-solving approach that releases one from "the fear of making an unjustified leap" and allows her to "shoot first and ask questions later" (p. xiii). Since the checks led none of the students to the conclusion that their introduced vectors were invalid, it seems justifiable to refer to their guesses as *educated*.

In terms of Harel (2013), capturing the act of checking educated guesses in writing can be viewed as fulfilling students' intellectual needs: to ascertain the correctness of the guessed vector, to use computation as a means to show orthogonality, and for communication with the assessor, whose corresponding needs in regard to the vector should also be fulfilled. One need that this act is incapable of fulfilling is the need for causality. Indeed, guessing can be contraposed to deductive reasoning, which NCTM (1989) defines as "a careful sequences of steps with each step following logically from an assumed or previously proved statement and from previous steps" (p. 144). Many students demonstrated deductive reasoning when row-reducing matrices in Part (a), applying the standard method $A^T A\bar{x} = A^T\vec{c}$ for devising the least square solution in Part (b), using the formula $\|A\bar{x} - \vec{c}\|$ in Part (c), and stating that the vector from the second part will solve Part (d) as well.

Field-based Hypothesis

It has been repeatedly reported that students rarely bother to verify the outcomes of their mathematical doings (e.g., see Kirsten, 2018 for proving; Kontorovich, Koichu, Leikin & Berman, 2012 for problem posing; Pugalee, 2004 for problem solving). Therefore, it is notable that without being engaged in any special course of instruction, nearly a fifth of the students submitted written checks of their final answers to Part (d) in Figure 1. Some may argue that there is nothing really to notice about this as the check in this part was easier than in the other three. This argument is incommensurable with the theoretical standpoint of this paper, which operates with students' mental acts (Harel, 2008a, b) and does not ascribe cognitive properties to inanimate artefacts. Indeed, the data analysis associated students' decisions to capture their checks in writing with situations where educated guesses were involved; no checks were documented in the cases of deductive problem solving, i.e. where students operated with structured procedures, algorithms, and theorems that were taught in the course.

Within Harel's (2008a, b, 2013a, b, 2017) theory of learning, an application of a conventional procedure, algorithm, or theorem provides an apodictic epistemological justification for the emergence of a solution to a problem (see $\alpha \rightarrow \beta$ in the second section). Furthermore, in a typical learning environment, such deductive methods are purposefully promoted among students through teachers' epistemological efforts that vary in their degree of explicitness. Explicit efforts can be associated with devoting time and space to these mathematical instances during the lesson, explaining and proving them, requesting students to use them for solving problems, et cetera. More covert efforts can also be indicated. For instance, the conventional name "Gram-Schmidt orthonormalization process" promises that the process indeed orthonormalizes. At the end of the course, Rina's usage of these mathematical instances in problem solving seems inseparable from her solid belief in their high epistemological status, the one that vouches for the instances' capabilities to produce the outcomes that they were positioned as producing.

With the principle of an intellectual parsimony in mind (Koichu, 2008), it seems reasonable to propose that when Rina is convinced by a match between the assigned problem and a mathematical instance with a high epistemological status, she is unlikely to experience an intellectual need to check her solution. Indeed, the usage of the mathematical instance for devising a solution, an instance that has been actively promoted by the same authoritative figures who assigned the problem, seems "to tick many boxes" of needs, especially for certainty, causality, communication, structure, and in many cases, also for computation. If there are still doubts about the obtained solution, it seems more reasonable for Rina to review how she applied the promoted mathematics rather than to verify her final answer as a stand-alone candidate for a solution. In turn, if Rina's recollection of the mathematical instance is distorted or mismatched to the problem in hand (something that happened frequently among the participating students), it is unlikely that she will benefit from such a review.

On the other hand, as educated as guessing can be, it creates a disruption in a deductive sequence of problem-solving steps and gives birth to an outcome that comes almost "out of nothing". This disruption can not only perturb the intellectual needs of the solver but it also clashes with the usual indoctrination in a "good" mathematics classroom where no claim is accepted without being shown to be a necessary entailment. As a result, the act and the outcome of guessing can be ascribed with a low epistemological status that summons a compensation. It has been demonstrated in the previous section that this compensation can appear in the form of a special type of an epistemological justification, where a solver shows that the candidate for an answer fulfills the requirements of the assigned problem (i.e. $\alpha,\beta \nrightarrow \emptyset$).

The presented interpretation of the checking tendency that the participating students demonstrated can be framed by a chain of hypotheses as follows:

> When solving a problem, Rina can apply pieces of knowledge that she endows with different epistemological statuses. For instance, guessing and applying mathematical instances that were promoted in a classroom can be positioned at opposite ends of an epistemological scale. As a result, Rina may experience intellectual needs to check her solution that differ in terms of intensity and type. These different needs entail different checking behaviors, which predetermines to some extent Rina's chances of indicating mistakes in her own work.

Hopefully, the mathematics education community will experience these hypotheses as educated guesses that provoke a need for rigorous explorations. The potential value of this hypotheses is in linking the act of checking to the contextual affordances that emerge when a solver puts particular mathematical knowledge to use. On the theoretical level, this positioning might be viewed as an extension of previous approaches, according to which the act is driven by a solver's familiarity with verification strategies (e.g., Mashiach Eizenberg & Zaslavsky, 2004) or a matter of habits of mind (e.g., Goldenberg, 1996) that she developed. On the practical level, the hypothesis summons a search for pedagogies that are capable of provoking students' intellectual needs for checking their own solutions; the ones that are often obtained with epistemologically solid mathematics. Accordingly, I believe that explorations of the hypothesis will lead to interesting conclusions that will find their way into Rina's classroom.

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Theoretical Report: A Framework for Examining Prospective Teachers' Use of Mathematical Knowledge for Teaching in Mathematics Courses

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This theoretical report addresses the challenge and promise of improving prospective secondary mathematics teachers' experiences in undergraduate mathematics courses through tasks embedded in pedagogical contexts. The objective of this approach, used by multiple nationally-funded projects, is to enhance the development of teachers' MKT. We report on the construction of a framework for observing and analyzing the development of teachers' MKT. This framework is the result of integrating several existing frameworks and analyzing a sample of prospective secondary teachers' responses to tasks embedded in pedagogical contexts. We discuss the methods used to build this framework, the strengths and weaknesses of the framework, and the potential of the framework for informing future work in curriculum design and implementation.

Keywords: Mathematical knowledge for teaching, Secondary teacher preparation, Educative curriculum

Recent years have seen multiple nationally-funded efforts to improve the mathematical preparation of teachers by developing materials for undergraduate mathematics courses.[1] Underlying these projects is recognition that mathematics courses are an opportunity to develop mathematical knowledge for teaching (MKT) in ways that are connected to undergraduate mathematics. This opportunity is all too often missed (e.g., Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Wasserman, Weber, Villanueva, & Mejia-Ramos, 2018; Zazkis & Leikin, 2010).

Scholars have proposed that an important strategy for bridging the gap between mathematical preparation and teaching practice is the use of tasks embedded in pedagogical contexts (Lai & Howell, 2016; Stylianides & Stylianides, 2010; Wasserman et al., 2018). Pedagogical contexts can support teachers' learning of mathematics in ways that are more meaningful and accessible than pure mathematics tasks when it comes to developing MKT (Stylianides & Stylianides, 2010). We conceive of such tasks as approximations of mathematical teaching practice (cf. Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009), which show promise for helping teachers transfer notions of upper level undergraduate mathematics to secondary mathematics teaching (Wasserman et al., 2018). Wielded skillfully by mathematics faculty teaching university courses, approximations of mathematics teaching practice provide opportunities "absent in fieldwork, [that allow] novices greater freedom to experiment, falter, regroup, and reflect" (p. 2076) when applying mathematical knowledge to the work of teaching.

Problem Addressed

This theoretical report addresses a potential obstacle to enacting approximations of mathematical teaching practice with prospective secondary teachers. Enacting approximations of

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[1] These include NSF DRL #1050595, and DUE #1504551, #1726624, #1726707, and #1524739.

practice has shown promise in small-scale studies involving mathematics education researchers (e.g., Lischka et al., 2017; Wasserman et al., 2016); however, we must take into account contextual differences among preparation programs when scaling up their use. Mathematics education researchers can draw on their field-specific expertise when analyzing teachers' knowledge and providing feedback on teachers' responses to approximations of practice. In contrast, mathematics faculty teach mathematics content courses for teachers in many preparation programs, particularly at the secondary level (e.g., Murray & Star, 2013). Although their experiences may include years of teaching at the undergraduate level, mathematicians' backgrounds are more likely focused on doing mathematics and providing purely mathematical feedback to students. Indeed, even when mathematicians want to use tasks with pedagogical context because they value developing teachers' MKT, they may feel stymied by not knowing how to evaluate prospective teachers' work on such tasks, let alone provide constructive feedback to the teachers (Lai, 2018). The background of mathematics faculty positions them to analyze and observe *mathematics*, but not necessarily MKT.

In sum, instructional improvement efforts face the problem of simultaneously supporting learners (the prospective secondary teachers) and instructors (the mathematicians) in developing MKT at the secondary level. This kind of simultaneous support is the signature characteristic of educative curriculum materials, which have in the past been used as a resource to shift mathematics instruction in sustained, meaningful ways (Davis & Krajcik, 2005).

In this report, we propose a novel integration of existing observational frameworks for examining MKT and its development for the purpose of examining prospective teachers' use of MKT in mathematics courses. We discuss why existing frameworks alone do not suffice for this purpose. Finally, we argue that our proposed integration supports the process of developing educative curriculum materials for undergraduate mathematics courses that feature approximations of mathematical teaching practice. Indeed, we hold that mathematicians and mathematics educators alike can utilize the integrated framework as useful tool when considering how they might provide opportunities for teachers in their courses to develop MKT.

Conceptual Foundations and Proposed Framework

We interlace theory and practice in the improvement work of creating and enacting educative curriculum materials for developing prospective secondary mathematics teachers' MKT in mathematics content courses. This work includes constructing a framework for observing the development of prospective teachers' MKT in their responses to approximations of mathematical teaching practice tasks.

Method

To do this improvement work, we follow a Networked Improvement Community model, with multiple plan-do-study-act cycles (Gomez, Russell, Bryk, LeMahieu, & Merjia, 2016). In this model, the following processes are mutually informing: developing materials, enacting materials, and constructing a framework for observing and analyzing development of MKT. Upon completing three plan-do-study-act cycles focused on observing the development of MKT based on prospective secondary teachers' responses to pedagogically embedded mathematics tasks—specifically approximations of mathematical teaching practice—three principles have emerged to guide our development of a framework. Namely, the framework must: (1) be grounded in theory for how MKT develops; (2) apply to a range of actions that good teaching entails; and (3) be consistent with what is known about observing ways in which MKT is activated in good teaching practice.

Theory for characterizing the development of MKT

Following Ball and Bass (2003), we construe mathematical knowledge for teaching in broad terms—as the knowledge used in recognizing, understanding, and responding to mathematical situations, considerations, and challenges that arise in the course of teaching mathematics. Moreover, we take MKT to include coherent and generative understandings of key ideas that make up the curriculum (Thompson, Carlson, & Silverman, 2007). In alignment with this principle, Silverman and Thompson (2008) used Simon's (2006) idea of key developmental understandings (KDUs) in combination with Piaget's notions of decentering and reflective abstraction to propose a framework for examining how MKT develops. We take Silverman and Thompson's work as a working theory for characterizing the development of MKT.

One principal characteristic of KDUs is that they are "conceptual advances." That is, when a learner (e.g., a prospective teacher or a K-12 student) has a KDU of a mathematical idea, the learner can perceive of and use mathematical relationships to build new understandings in a way that a learner without the KDU cannot. Simon contended that learners acquire KDUs from multiple experiences and reflection. An important implication is that teachers' possession of a KDU, does not ensure that they will create opportunities for students to acquire KDUs (e.g., Silverman, 2004). Indeed, for a teacher do to so, they must not only use or explain personal KDUs, but also envision instructional activities that promote students' learning of KDUs.

Hence, Silverman and Thompson argued developing MKT involves two abstractions, where abstraction aligns with Piaget's notion of reflective abstraction (1977/2001). The first abstraction results in a teacher's personal KDU for a mathematical idea. The second abstraction is on *learners'* thinking and results in multiple models of how learners may understand the idea and how one may come to such an understanding. Silverman and Thompson conceptualize this second abstraction as Piaget's notion of decentering, resulting in: (1) an image of instructional activities and conversations that would produce these understandings and (2) whether these understandings empower students to learn subsequent related ideas, as Table 1 summarizes.

Table 1. Silverman and Thompson's (2008) Characterization of the Development of MKT

Component	Description	
1	Personal KDU:	Teachers have developed a personal KDU for a particular mathematical idea
2	Decentering: Understanding	Teachers have constructed multiple models of student understandings of the idea
3	Student Thinking:	Teachers have an image of *how* a student may come to these understandings
4	Activities:	Teachers can envision instructional activities and conversations that would result in these understandings
5	Potential for Student KDU:	Teachers can analyze how and whether students who have come to think about the mathematical idea in these ways are empowered to learn other, related mathematical ideas

Applying types of knowledge to teaching actions

It follows from Simon's (2006) and Silverman and Thompson's (2008) theory that teachers need to grapple with experiences that promote the abstractions needed to develop MKT. Moreover, instructors of prospective secondary teachers need opportunities to comprehend how teachers understand MKT. For instance, in an example provided by Grossman et al. (2009), prospective teachers responded to two second grade students, coming up with questions to ask the students, reflecting upon the kinds of responses these questions might elicit, and determining extent to which these responses were productive. Through approximations of practice, teachers have the opportunity to engage with student thinking and mathematics to develop MKT.

Although Silverman and Thompson's work describes components of MKT development, it does not elaborate on where to observe these components in teaching practice or in an approximation of practice. To understand where MKT is activated during teaching, we turn to the Knowledge Quartet, which identifies dimensions of teaching in which knowledge is revealed (Rowland, Thwaites, & Jared, 2016). The Knowledge Quartet's purpose resonates with that of Silverman and Thompson's characterization of MKT development, while its focus is complementary. Both acknowledge that instruction should be informed by coherent mathematical knowledge and predictions about learners. Silverman and Thompson focus on mental actions where the Knowledge Quartet identifies visible actions due to teachers' MKT. The four dimensions in the Knowledge Quartet each pair with contributory codes—descriptions of actions that manifest the dimension. The first dimension, (1) Foundation, includes knowledge of mathematics and its nature. The remaining three are contexts in which Foundation knowledge is brought to bear. They are (2) Transformation, the presentation of ideas to learners in the form of illustrations, examples, and explanations; (3) Connection, the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks; and (4) Contingency, the ability to respond to unanticipated events in the work of teaching.

Silverman and Thompson describe the development of MKT, and the Knowledge Quartet describes actions possible due to MKT. When teachers have a personal KDU and engage in the decentering needed to develop MKT, they can design instructional activities to be more responsive to student thinking as well as analyze students' knowledge more acutely. We interpret Foundation to include a teacher's personal KDUs and Transformation, Connection, and Contingency to be actions informed by decentering, understanding students' thinking, and analyzing students' potential KDUs. We summarize the Knowledge Quartet and its relationship to Silverman and Thompson's work in Table 2.

Table 2. Knowledge Quartet in Correspondence with the Development of MKT

Dimension	Example Contributory Codes	Correspondence to MKT Development Components
Foundation	Awareness of purpose; overt display of subject knowledge; use of mathematical terminology	Personal KDU
Transformation	Choice of examples; choice of representation; use of instructional materials; teacher demonstration (to explain a procedure)	Decentering, Potential for Student KDU
Connection	Anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness	Decentering, Activities, Potential for Student KDU
Contingency	Responding to students' ideas; use of opportunities; deviation from agenda	Decentering, Potential for Student KDU

Activation of MKT in teaching practice

Although Silverman and Thompson provide a theory for how MKT develops and the Knowledge Quartet provides a description of how different types of knowledge are applied to aspects of teaching, neither elaborate how one instance of an application of MKT to teaching practice may be more sophisticated than another instance. For this, Ader and Carlson's (2018) work provides a mechanism for distinguishing levels of the sophistication of activation of MKT by describing patterns in observable behaviors during teaching practice that indicate the extent to which the teacher has decentered. We describe our view of the correspondence of these levels and observable behaviors to Silverman and Thompson's characterization of MKT in Table 3.

Table 3. Ader and Carlson's Framework in Correspondence with MKT Development

Level	Observable behaviors	Correspondence to MKT Development Components
Level 1: Interested in getting students to say correct answers but not in students' thinking	Asks questions to elicit students' answers; listens to students' answers; does not pose questions aimed at understanding students' thinking	*Decentering*: lack of decentering, uses only a first order model; *Understanding Student Thinking*: only elicits student answers, not thinking; *Activities*: constrained by thinking only with first order model
Level 2: Interested in students' thinking, but only in order to get students to think like the teacher	Poses questions to reveal student thinking but does not attempt to understand students' thinking; guides students toward his/her own way of thinking.	*Decentering*: lack of decentering, uses only a first order model; *Understanding Student Thinking*: only elicits student thinking, does not utilize that thinking in a response; *Activities*: constrained by only thinking with a first order model
Level 3: Makes sense of students' thinking and makes general teaching moves based on that thinking	Asks questions to reveal and understand students' thinking; follows up on students' responses in order to perturb students in a way that extends their current ways of thinking; attempts to move students to his/her thinking or perspective	*Decentering*: evidence of first and second order models; *Understanding Student Thinking*: utilizes student thinking when formulating responses; *Activities*: uses second order model to make decisions about activities and conversations; *Personal KDU*: draws on personal KDU when responding to students
Level 4: Seeks to understand students' thinking, and builds on that thinking during instruction	Poses questions to gain insights into students' thinking; draws on students' ways of thinking to advance students' understanding of key ideas in the lesson	*Decentering*: evidence of first and second order models; *Understanding Student Thinking*: utilizes student thinking when formulating responses; *Activities*: uses second order model to make decisions about activities and conversations; *Personal KDU*: draws on personal KDU when responding to students; *Potential for Student KDU*: seeks to provide opportunities for students to develop KDUs

Working Framework for Observing and Analyzing the Development of MKT in Approximations of Mathematics Teaching Practice Used in Content Courses

In Networked Improvement Community work involving multiple rounds of coding prospective secondary teachers' responses to approximations of mathematical teaching practice, we began by using the dimensions and components in Tables 1 and 2. We found it difficult to determine the *development* of prospective secondary teachers' MKT over time. To remedy this issue, we incorporated and generated hypothesized extensions of the correspondence of levels (Table 3) for each dimension (Table 2).

Method for extending levels. To construct extensions of levels, the six authors analyzed the responses of 15 prospective secondary teachers to approximations of mathematical teaching practice that were has been used in mathematics courses at 3 different institutions in different states in multiple years; the responses analyzed were representative of the responses across these sites. We adapted a two-stage coding process (Miles, Huberman, & Saldana, 2013), using the dimensions of the Knowledge Quartet as *descriptive codes* and Ader and Carlson's levels as initial *process codes* in a first cycle of coding, and then used a second cycle of coding to consolidate codes for structure and unity. To do so, we drew on critiques of episodes of teaching found on the Knowledge Quartet's website (Rowland, 2017) and observation protocols that have been validated as measuring quality of teaching (Junker et al., 2004; Learning Mathematics for Teaching, 2011).

Results. We interpret our work has contributing several results. Our first result is theoretical: the framework to which our analysis led. This framework is presented in Table 4.

Second, as a practical result, we report which aspects of the framework led to the most and least reliably coded approximations of mathematical teaching practice.

We describe this second result in brief here, for the sake of space limitations, and provide more elaboration in our presentation. The first-cycle descriptive codes for the dimensions of the Knowledge Quartet, as well as the process codes for the levels for Transformation, were most reliably coded among the research team. The least reliable codes were Levels 2 and 3 within Connection, as well as the Level 4 codes for Transformation and Connection. We see reliability of codes as an important result to report because it bears on interpreting the framework as well as pointing to future work in validating this framework for observing the development of MKT.

Table 4. Framework for Observing and Analyzing the Development of MKT in Approximations of Practice

Developmental component	Knowledge dimension	Mental actions	Level (L), in terms of observable behaviors
Personal KDU	Foundation	Reflective abstraction on personal mathematical knowledge	*Note: Levels here depend on the KDU of the topic. This is just one possible example of how levels may appear.* L0: Specific reference to mathematics is not present OR Performs procedures incorrectly and describes underlying concepts incorrectly (lacking in CCK) L1: Performs relevant procedures correctly L2: Describes relevant procedures accurately, with mathematically precise and appropriate language L3: Connects isolated features of procedures to underlying concepts L4: Connects structure of procedure to underlying concepts
Decentering applied to Activities and Analyzing Potential for Student KDU	Transformation	Reflective abstraction on student thinking	Gives explanations, representations, and examples to students that: L0: Does not provide any explanations, representations, or examples to students L1: Describe only procedures or echo key phrases L2: Describe own way of thinking of the mathematics L3: Attempt to change students' current thinking L4: Build on and respect students' understanding toward the intended KDU
	Connection		Prompts students to say or do things in ways that: L0: Does not ask students to say or do anything L1: Focus on procedures or echoing key phrases L2: May reveal student thinking, but then teacher gives explanations while not asking students to provide reasoning L3: Attempts to change students' current thinking L4: Build on and respect students' understanding toward the intended KDU
	Contingency		Uses student thinking in ways that: L0: Do not act in any visible way upon the thinking L1: Evaluate the mathematical validity of the thinking but do not use the thinking in teaching L2: Reference the thinking to guide students toward teacher's way of thinking L3: Follow up on students' responses to perturb students to change their thinking L4: Frame questions or explanations in terms of students' thinking to help move students' understanding toward the intended KDU. Students are positioned as decision-makers.

Discussion

Our work is built from conceptual foundations to elaborate how and where teachers' development of MKT can be observed and analyzed, both by all those who teach teachers as well as by mathematics education researchers. Silverman and Thompson's framework provides a characterization of how MKT develops, in terms of mental actions of teachers, leaving open where in teaching to observe these mental actions and what observable behaviors those mental actions might produce. The Knowledge Quartet describes where the results of teachers' mental actions show up in teaching practice, and Ader and Carlson's framework characterizes the relative sophistication of those mental actions in terms of observable behaviors. Approximations of mathematical teaching practice engage prospective teachers in teaching actions and provide opportunities for teachers to engage in the mental actions needed to develop MKT.

The framework we present supports mathematics faculty and teacher education researchers in discerning knowledge use in approximations of practice. The dimensions of Foundation, Connection, and Transformation emphasize places where prospective teacher might display personal knowledge, provide explanations to students, and pose questions that elicit student reasoning. Although faculty may not traditionally provide feedback on these distinctions in a mathematics course, these distinctions are ones that may be familiar to faculty and may well be educative for their own teaching practice (e.g., Bass, 2015; Pascoe & Stockero, 2017). Our data suggest that the dimensions of knowledge were independent, providing evidence that pathways through development of MKT may well proceed along these dimensions in different ways. For instance, one prospective teacher in our dataset explained the connection between a definition and procedure as a rationale for a task they would assign to their students (Foundation, L4), but only posed questions that focused on echoing key phrases (Connection, L1), proposed only explanations of procedures to the students (Transformation, L1), and did not acknowledge any of the sample student thinking provided by approximation of practice (Contingency, L1). Another teacher began with using the provided sample student work to make a specific mathematical point about a definition (Contingency, L4) then did not provide any subsequent examples or explanations to connection of procedure and definition (Transformation, L1).

Upon receiving initial feedback from mathematics faculty regarding how observables may correspond to knowledge dimensions, our work writing approximations of mathematics teaching practice for use in content courses shifted to address more clearly the specific opportunities for learning that those approximations provide. For instance, in an early draft of an approximation of practice, we asked teachers to respond to student thinking, but we did not give a clear mathematical goal for the teaching situation. This left unspecified the Foundation knowledge we were aiming to elicit, which impacted the Transformation and Connection knowledge visible in teachers' responses.

Finally, the framework supports validating and refining the articulation of the development of MKT. We view this framework as a set of testable hypotheses grounded in known results. Given the relative nascence of research on developing MKT (Hoover, Mosvold, Ball, & Lai, 2016), such hypotheses can contribute to advancing understanding of MKT.

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Multiple Representation Systems in Binomial Identities: An Exploration of Proofs that Explain and Proofs that Only Convince

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Abstract: In the mathematics education literature on proof, there is a longstanding conversation about proofs that only convince versus proofs that explain. In this theoretical report, we aim to extend both of those ideas by exploring proofs in the domain of combinatorics. As an example of an affordance of the combinatorial setting, we explore proofs of binomial identities, which offer novel insights into current distinctions and ideas in the literature about the nature of proof. We demonstrate examples of proofs that can be explanatory or convincing (or both), depending on how a person understands the claim being made (which we refer to as their preferred semantic representation system). We conclude with points of discussion and potential implications.

Keywords: Proof, Proofs that explain and convince, Combinatorics, Binomial identities

Introduction and Motivation

An interesting algebraic question to ponder is why the sum of the binomial coefficients equals 2^n (that is, why $\sum_{k=0}^{n}\binom{n}{k} = 2^n$). For us (and perhaps for others), if we consider actually expanding and summing the left-hand side of the equation, the fact that it simplifies so nicely to the expression 2^n feels a bit like an algebraic miracle. If we ask for a justification of this equation, someone may give a straightforward counting argument, noting that both sides count the same set – namely all possible subsets of all sizes from a set of *n* distinct elements. We may find that counting argument to be convincing and also explanatory in terms of why each expression represents a process that counts the same set of outcomes. Following such a combinatorial argument, we could be *convinced of the truth* of the algebraic relationship without gaining the desired insight into the *algebraic* mystery that we originally observed.

The proof literature has long articulated such a distinction between proofs that only convince and proofs that explain (e.g., Hanna, 1990; Hersh, 1993; Steiner, 1978; Weber, 2010), and it has been pointed out that this distinction is not a simple dichotomy (e.g., Hanna, 2000; Raman, 2003; Stylianides, Sandefur, & Watson, 2016;). Generally, proofs that only convince are characterized as proofs that demonstrate that a proposition is true but without necessarily providing particular insight into *why* it might be true. Proofs that explain are characterized as proofs that do give some indication as to why a particular proposition is true. In this theoretical report, we aim to extend both of those ideas by exploring proofs in the domain of combinatorics. We believe that, generally, the combinatorics can provide an insightful context in which to study questions related to the practice of proof. To demonstrate an affordance of the combinatorial setting, we explore proofs of binomial identities, which offer unique insights that extend useful ideas about the nature of proof. By exploring mathematical examples in a combinatorial setting, we offer examples of proofs that can be explanatory or convincing (or both) depending on how a person understands the claim (which we refer to as a preferred semantic representation system).

Background Literature and Relevant Theoretical Perspectives

How are we taking proof?

Weber and Alcock (2004) say, "When asked to prove a statement, professional mathematicians and logically capable mathematics students all share the same goal – to produce a logically valid argument that concludes with the statement to be proven" (p. 210). We draw on this statement and use a definition of proving as *the process of producing a logically valid argument that concludes with the statement to be proven*. We follow Stylianides, et al. (2016) in distinguishing between proof and proving in the following way: "we consider *proving* to be the activity in search for a proof, whereby proof is the final product of this activity that meets certain criteria" (p. 20). In this paper, we are interested broadly in both proving and proof, and we will clarify if we are exclusively referring to one or the other. In the examples we explore in this paper, the statements to be proven are statements that relate expressions involving binomial coefficients. These expressions are known as binomial identities.

Multiple purposes of proof

The mathematics education literature reports a number of purposes that proof and proving play in the domain of mathematics. One primary reason for proof in mathematics is to convince a reader that a theorem is true. This is typically proposed as a main purpose for proof, especially for research mathematicians. For example, Hersh (1993) notes that "in mathematical research, [proof's] primary role is convincing" (p. 398), and he points out that for the mathematics community, "proof is *convincing argument, as judged by qualified judges*" (p. 389, emphasis in original). Here we interpret that convincing means that one understands the necessity of the conclusion following from the premises, but without the additional constraint that the tools and relationships one wants to see employed are necessarily the only tools and relationships used.

Even though convincing is an important purpose of proof, researchers (e.g., Hanna, 2000; Hersh, 1993; Weber, 2010) are quick to note that simple formal deduction, which may technically prove a theorem, is not why mathematicians value proof and is not what they view as the sole purpose of proof. For instance, Weber (2010) argues that mathematicians value proofs not just because they show that a statement is true, but because they provide additional insight into mathematical content or into the practice of proving. As another example, Hersh (1993) says, "More than whether a conjecture is correct, mathematicians want to know why it is correct. We want to understand the proof, not just be told it exists" (p. 390). These sentiments suggest that proof may be useful for additional reasons than demonstrating the veracity of a theorem.

These multiple purposes of proof highlight a distinction in the literature between proofs as explanatory and proofs as convincing. Hanna (1990) reports that a proof is valued for bringing out essential mathematical relationships rather than for merely demonstrating the correctness of a result. She distinguishes between proofs that *prove* and proofs that *explain*. She points out that a proof that proves "shows only *that* a theorem is true; it provides evidential reasons alone" (p. 9), while a proof that explains "also shows *why* a theorem is true; it provides a set of reasons that derive from the phenomenon itself" (p. 9). A similar dichotomy is also articulated in Hersh (1993), and he distinguishes between proof in a research setting and proof in a classroom setting.

Defining proofs that explain

We now discuss the literature on what it might mean for a proof to explain. There are several ways in which researchers characterize proofs that explain. Hanna (1990) clarifies that she prefers "to use the term *explain* only when the proof reveals and makes use of the mathematical ideas which motivate it," (p. 10). She follows Steiner (1978) by saying that "a proof explains when it shows what "characteristic property" entails the theorem it purports to prove" (Hanna, 1990, p. 10). According to Weber and Alcock, a proof that convinces is "an argument that establishes the mathematical veracity of a statement. Such proofs are typically highly formal, and their function is to remove all doubt that a statement is true" (p. 231). A proof that explains, on the other hand, is "an argument that explains, often at an intuitive level, why a result is true" (p. 231). In another approach, Weber (2010) conceptualizes a proof that explains as one that "allows the reader to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (p. 34). Common to all of these characterizations is the idea that a proof that explains offers some insight into *why* a statement is true (or false). In addition, Stylianides, et al. (2016) refer to literature that defined what it meant for a proof to be explanatory for a prover, "namely, whether the proof illuminated or provided insight to a prover into why a mathematical statement is true (Bell, 1976; de Villiers, 1999; Hanna, 1990; Steiner, 1978) or false (Stylianides, 2009)" (p. 21). Stylianides, et al. (2016) consider proving activity to be "explanatory for the prover (or provers) if the method used in a proof provided a way for the prover to formalize the thinking that preceded and that illuminated or provided insight to the prover into why a statement is true or false" (p. 21).

Stylianides, et al. (2016) is particularly relevant to our work, as they challenged and extended the typical distinction between proofs that convince and explain, especially questioning the assertion that proofs by mathematical induction are necessarily not explanatory. They explore ways in which proofs by mathematical induction may be explanatory for students, and they frame what conditions might best facilitate this phenomenon. We hope similarly to further the conversation about proofs that convince and explain by using proofs of binomial identities.

Our characterization of proofs that explain

We follow Weber (2010) in using Weber and Alcock's (2004) distinction between semantic and syntactic proof production as a way of conceptualizing proofs that explain. Weber and Alcock (2004) identify two qualitatively different ways in which someone might produce a correct proof. They define a *syntactic proof production* as "one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In a syntactic proof production, the prover does not make use of diagrams or other intuitive and non-formal representations of mathematical concepts" (p. 210). In contrast, they define a *semantic proof production* to be "a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). The authors clarify that an instantiation refers "to a systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual […] What is crucial is that the prover use these instantiations in a meaningful way to make sense of the statement to be proven and to suggest formal inferences that could be drawn" (p. 211). We interpret that in semantic proof productions, students meaningfully draw on some instantiation of a mathematical object or idea that may be external from the situation at hand.

Even more specifically, Weber (2010) draws on these ideas of intuition and instantiations to provide a definition of an explanatory proof. He notes that, "often, students and mathematicians will use [semantic] reasoning as a basis for constructing a formal proof" (p. 34). In this way, the informal, meaningful semantic reasoning might guide the development of a formal proof. Weber says, "I conceptualize a proof that explains as a proof that enables the reader of the proof to reverse the connection – that is, this proof allows the reader to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (p.

34). We interpret, then, that a proof that explains allows for a prover to make meaning of whatever formal representation system he or she may be working with in order to connect ideas to some semantic system.[1] We thus follow Weber in using instantiations and the notions of semantic proof production (and comprehension) as we define proof that explains.

Finally, as Weber's (2010) definition suggests, he takes a reader-centered perspective on explanatory proof. Indeed, this approach resonates with us, as what constitutes a meaningful semantic system could vary from person to person, according to the content or robustness of their particular concept image (Tall & Vinner, 1981). We thus follow Weber (2010) who emphasizes that proofs that explain are from the perspective of the reader (or the prover).

Mathematical examples

Semantic representation systems

Weber (2010) discussed the semantic representation systems (SRS), which he attributes to Weber and Alcock (2009). An SRS is the system in which a reader's (or a prover's) semantic reasoning may take place, and we interpret an SRS as a mathematical perspective in which a person is interpreting a claim being made. Ultimately, we argue that to ask whether or not a proof is convincing or explanatory, we ought to consider in which SRS(s) a proof is being produced or comprehended. Broadly, these semantic contexts represent the particular perspective in which a prover is proving (or a reader is comprehending) a proof. Often, the statement that is meant to be proven is expressed in a particular symbol system, which may be interpreted in a number of ways. The main idea we are proposing is that proofs (and proving activity) exist within a particular SRS, each of which represents different ways of interpreting and making meaning of the same (symbolically identical) statement to be proven.

We are using Weber's (2010) notion of SRSs as a way to make sense of a variety of proofs of the binomial theorem, and we use the notion of SRSs to understand two important phenomena related to proofs that only convince and proofs that explain. First, in terms of proofs that explain, we use SRSs as a way to articulate *what is being explained* in a proof that explains. By specifying in which SRS we are working, we can gain clarity about what is being mathematically explained. Second, we use SRSs as a way to consider a mechanism by which a proof may be convincing but not explanatory. Specifically, there may be some translation that occurs between SRSs in order to complete a proof. And if a person is trying to prove a claim in one SRS (SRS1), but then translates to another SRS (SRS2) to prove the statement, the proof may be *convincing* to the prover in SRS1 while it is explanatory in SRS2. Thus, a proof may be convincing (but not explanatory) depending on the SRS in which it is proved and the SRS in which the prover is considering the proof. We explore these ideas further in the following sections.

We envision that different proof-related activities occur within a given SRS. Proofs may be direct or indirect, and it may be the case that multiple different proofs could exist within each SRS. Further, proofs within a given SRS may be formal or informal, and the given SRS determines what rules, tools, approaches, and conventions apply to the given SRS. As noted, we also view that there is potential movement between SRSs.

Insights from proofs of binomial identities

[1] Also, while Weber (2010) defined an explanatory proof from the perspective of proof comprehension (talking about a reader of a proof), we could similarly consider an explanatory proof from the perspective of proof production. That is, the proof that has been produced may be explanatory if it enables the *prover* of the proof to translate the argument that he or she is formulating to an argument in a separate semantic representation system.

In order to elaborate these ideas, we provide examples from combinatorics, specifically proving binomial identities. There is nothing in the notion of SRS that specific to combinatorics (which we address in the Discussion Section). But, we contend that proofs of binomial identities are particularly enlightening because combinatorics naturally lends itself to moving between semantic domains. Indeed, as we will describe below, it is commonplace to use another SRS (perhaps an algebraic system) to prove a relationship in a given SRS (perhaps an enumerative system). In the following section we will provide an algebraic and an enumerative proof of the statement $\binom{n}{k} = \binom{n}{n-k}$, acknowledging that we could also explore additional SRSs of this same expression (such as induction or block-walking). We will present these each of these proofs through the lens of a different SRS.

An explanatory proof in the algebraic SRS

In the algebraic SRS, $\binom{n}{k} = \binom{n}{n-k}$ can be interpreted as a statement about (nonnegative) integers, and valid tools include properties of integers and algebraic rules. Substituting the definition of binomial coefficients ($\binom{n}{k} = \frac{n!}{(n-k)!k!}$) into the identity and applying rules of algebra yields the following proof:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{\big(n-(n-k)\big)!\,(n-k)!} = \binom{n}{n-k}$$

Since we can use rules of algebra to manipulate one expression into the other, both sides of the equation are equivalent, and so the statement is true. We call this an explanatory proof in the algebraic SRS (or an algebraic proof of the identity) because it follows algebraic rules to demonstrate why the identity is true.

An explanatory proof in the enumerative SRS

There are also enumerative, or combinatorial, proofs to this identity. In an enumerative proof, we argue that the two sides of the identity each represent two different counting processes (in the sense of Lockwood, 2013) that either a) count the same set of outcomes (a direct combinatorial proof) or b) count two different sets of outcomes between which there is a bijection (a bijective combinatorial proof). For the sake of simplicity, we give one example of a direct combinatorial proof, noting that there are many other enumerative proofs we could introduce. Note that these enumerative proofs look quite different than the algebraic proof presented above, and sentence descriptions of counting processes and sets (rather than manipulation of algebraic symbols) comprise the proof.

We show that both sides of the identity count the number of k-element subsets of an n-element set. That is, we interpret $\binom{n}{k} = \binom{n}{n-k}$ as being a statement that relates different kind of subsets of n-element sets. The left-hand side counts this set by selecting k elements from n distinct elements that should be included in the subset, and this process reflects the left-hand expression of $\binom{n}{k}$. The right-hand side counts this set by using the notion of a complement of the set – by selecting the n-k elements from n distinct elements that should *not* be included in the subset. Therefore, because both sides of the identity count the same set, they represent expressions that are numerically equal, and thus the equality holds.

Explaining and convincing in algebraic and enumerative proofs – what is being explained, and what is convincing?

We take these two proofs to further our discussion about proofs that explain versus only convince. The perhaps "easy" way to interpret these two proofs in terms of proofs that explain and proofs that only convince is to say that the algebraic proof convinces but does not explain, while the enumerative proof is somehow more explanatory. However, we argue that there is a deeper story to tell, and each of the above proofs could be considered to be explanatory and/or convincing depending on which SRS we are considering.

In particular, we contend that the question *What is the proof explaining?* is not a simple inquiry. We argue that the enumerative proof is explanatory in the SRS of enumeration because it demonstrates *why* both sides of the identity counts the same set of outcomes. Further, following our definition of proofs that explain (which we borrow from Weber, 2010), there is a particular instantiation to properties that we know about sets and choosing elements of sets that makes a meaningful connection between the expressions, the counting processes described in the proof (Lockwood, 2013), and what we know about what it means for sets to be equal. Thus, this proof satisfies our need for understanding what is happening enumeratively. However, the enumerative proof is *not* explanatory in the SRS of algebra. That is, the enumerative proof does nothing to explain why the identity holds algebraically.

Conversely, it is true that the algebraic proof does not provide any explanation for why the identity is true in the enumerative domain. However, we claim that the algebraic proof *is* explanatory in the SRS of algebra. Specifically, using the definition of binomial coefficients and the rules of algebra we can see the logical, algebraic steps that justify why that relationship is true. Thus, we could say that that proof explained why, algebraically, the relationship holds.

We claim that if a proof is explanatory in a given SRS it is necessarily convincing, but a proof may be convincing but not explanatory for a different SRS. Returning to our examples of algebraic and enumerative proofs of $\binom{n}{k} = \binom{n}{n-k}$, we would say that the algebraic proof may be convincing in the enumerative system, even if it not explanatory in that enumerative system. Similarly, the enumerative proof may convince someone that the algebra must be true, even if the enumerative proof offers no insight into why the algebraic steps are true.

We commonly use this relationship between SRSs in proving theorems and identities in combinatorics. To emphasize this point, consider the relationship $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, which we mentioned in the introduction. This is an identity that is quite natural to prove enumeratively.[2] However, it is not immediately apparent why the algebra should hold. Summing all of the terms, finding common denominators, canceling, and simplifying for the general value of *n* require considerable work, particularly by hand. Here, then, the enumerative proof may *convince* us of the algebra, even if we cannot actually describe and list out all of the steps that would satisfy the identity algebraically. If all we needed was to be convinced that this identity holds, it would make sense to use a combinatorial argument to prove the result, rather than an algebraic one.

More commonly in combinatorics research, we go in the other direction – we use algebra to convince us of identities that are difficult to prove combinatorially. For example, generating functions (e.g., Wilf, 2005) offer a well-established technique of translating difficult combinatorial questions into more manageable algebraic settings. In this technique, we encode combinatorial objects as coefficients of polynomials, and we use rules of polynomials and algebra to derive results that are then translated back to the combinatorial context. A proof of an

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[2] Both sides count the total number of subsets of any size from a set of *n* elements. The left-hand side counts this by summing up all possible numbers of k-element subsets for values of k from 0 to n. The right-hand side counts this by considering, for each of the n elements in the set, whether or not it is an element of a subset.

identity involving generating functions is explanatory in an algebraic domain, as it demonstrates clearly why the algebra holds to establish the relationship, but it does not explain why the relationship holds from an enumerative perspective. The fact that we have different SRSs in which to have proofs convince or explain is a wonderful aspect of mathematics, as it opens up many opportunities for us to develop convincing proofs even if one SRS is particularly difficult.

Our point, then, is that it sells proof short to simply characterize a proof as being convincing or explanatory without further specifying what precisely is being explained. Further, it is misleading to dismiss algebraic or inductive proofs as being necessarily not explanatory. Certainly, progress is being made in this regard (e.g., Stylianides, et al. (2016)), and we want to contribute to these conversations about what it can mean for a proof to be explanatory.

Further, returning to an important distinction in the proof literature, SRSs also allow us to address another dimension of this conversation – the importance of who the prover (or the reader) is. As noted above, we particularly appreciate Weber's (2010) viewpoint in clarifying that these ideas must be considered from the prover's/reader's perspective, and we also adopt this framing. That is, different proofs may be explanatory or convincing in different SRSs depending on the perspective of the prover. Weber's (2010) notion of SRSs is in line with this perspective, and we note that for an individual prover (or reader), he or she may naturally tend toward a particular SRS. Based on a person's background or familiarity with ideas (their concept image), they may be more or less inclined to be able to deem a certain proof as explanatory or convincing, depending on which system they are examining.

Discussion and Conclusion

In this paper, we have argued that combinatorics (and proofs of binomial identities) offers a novel mechanism by which to investigate proofs that explain versus proofs that only convince. In this section, we highlight points of discussion and implications related to this conversation.

Combinatorics as a rich domain in which to study proof

Combinatorics is a fertile domain in which to study proof. In particular, binomial identities (and combinatorics more generally) are characterized by translation between SRSs, and this has repercussions for elaborating the ideas of proofs that only convince and proofs that explain. We hope that more proof researchers will explore this domain, as it may potentially shed light on other interesting aspects of proof.

The discussion in this paper extends to other mathematical domains

And yet, even though we want to make a case for the value of combinatorics in studying proof, our findings and discussion are not unique to combinatorics. Although we have primarily focused on combinatorial examples of proving binomial identities to discuss SRSs and proofs that convince and explain, these ideas also extend to non-combinatorial contexts. For instance, we could consider different proofs of the Pythagorean theorem. A proof without words (Nelsen, 1993) of the Pythagorean theorem may be explanatory in a geometric SRS, but it may not be explanatory in the algebraic SRS. Similarly, in an algebraic proof, if the numbers are viewed only as integers and not as side lengths with some dimension, then the algebraic manipulation is explanatory in the algebraic system but not in the geometric system.

Pedagogical suggestions

We have demonstrated the value of translating between SRSs, and we have shown that in some fields like combinatorics this is a natural thing to do. However, we want to emphasize that we should be careful when translating between SRSs. For example, when translating between an algebraic and an enumerative SRS when proving a binomial identity, one must consider what assumptions can be made within a given SRS.

The notion of SRSs in proof also allows us to reframe how we think about students' proving activity. The idea that students might be working from different SRSs gives a useful lens through which to consider student activity in discrete math classes, perhaps giving students more credit than simply interpreting their activity as meaningless and purely syntactic. When a student tends toward algebra when proving a binomial identity, it is easy to assume that the student is being shallow (and we admit to adopting this perspective at times). However, such a student may be viewing the statement to be proven through an algebraic SRS, which may be meaningful to them. Thus, this perspective on proofs that convince versus explain may give agency to the prover.

Finally, for those teaching discrete math, we suggest to keep in mind that in teaching counting and binomial identities specifically, we are asking students to coordinate multiple SRSs. The notion of SRSs highlights the fact that any discussion of explaining and convincing must be considered from the perspective of the prover and/or reader. In combinatorics, there are many different perspectives from which to interpret/view the same symbolic binomial identity. The fact that so many different SRSs exist (particularly in proving binomial identities) highlights that it is important to consider who is proving or reading a proof.

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The Relational Meaning of the Equals Sign: a Philosophical Perspective

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While there has been research on students' understanding of the meaning of the equals sign, there has yet to be a thorough discussion in math education on a strong meaning of the equals sign. This paper discusses the philosophical and logical literature on the identity relation and reviews the math education research community's attempt to characterize a productive meaning for the equals sign.

Keywords: Equals Sign, Equality, Identity, Equation

The study of students' understanding of the equals sign has been a long-standing theme in mathematics education research. This line of inquiry predates mathematics education's existence as a well-formed field, and the philosophical literature addressing the meaning of the equals sign dates to the late 19th century or earlier (Renwick, 1932; Noonan & Curtis, 2014). There is extensive discussion on the meaning of the identity relation that predates the equals sign and continues to this day. This discussion appears as early as the 4th century BCE in Plato's *Parmenides* and as late as Williamson in the 20th century (Noonan & Curtis, 2014). In spite of the long history of the research and development of ideas associated with identity, math education literature is plagued with vagueness and carelessness in characterizing the meaning of the equals sign. Despite this lack of rigor, the field of math education has produced compelling research revealing that students have weak understandings of the equals sign (Baroody & Ginsburg, 1982; Behr, Erlwanger, & Nichols, 1980; Byrd, McNeil, Chesney, & Matthews, 2015; Denmark, Barco, & Voran, 1976; Kieran, 1981; Oksuz, 2007; Sáenz-Ludlow & Walgamuth, 1998).

The Importance of the Identity Relation

Identity statements, and their assessments of sameness, are an important part of mathematics. This is especially evident when we consider the prevalence of the equals sign, and there is a body of math education literature addressing students' understanding of it. This paper serves as a survey and critique of that existing literature, together with a philosophical discussion on the meaning of the equals sign. The importance of the equals sign in mathematics cannot be understated. There is not one branch of mathematics that does not rely on it. Consider, for example, the equation "2+2=4" or "$cos(x) = \Sigma^{\infty}{}_{0} \ \ (-1)^n x^{2n}/(2n)!$".

One way to view identity statements is to see them as giving multiple representations of the same thing. "The evening star" is a representation of the planet Venus (in the sense that it refers to Venus), as is "the morning star", and the statement of identity tells us that in fact these phrases refer to the *same* thing, despite being *different* representations.

Having multiple representations of the same thing plays an important role in mathematics – for example, in combinatorial arguments where we find two ways to calculate the same thing, resulting in an identity statement. A quick examination reveals that represents the number of ways to choose k items from a group of n items, and represents the number of ways to leave out n-k items from a group of n items. Hence, we see that =, which is indeed an informative and useful statement.

When we deal with mathematical statements, we run into another conundrum: what *thing* does the name of a function or number refer to? Clearly, the name of a function refers to a function, and the name of a number refers to a number, but what is a number, and what is a function? Numbers are not physical objects *out there* that we can easily point to. When we say that 7+2=8+1, what is it that we are saying is the same? That is, what are the referents of each side of the equation? Mathematicians and philosophers cannot agree on the existence of numbers as objects, let alone what numbers *are* (Horsten, 2016). Yet, despite there being an age-old question of what an abstract mathematical object *is* (if anything), mathematicians continue to reason productively using equations, identity statements, numbers, and functions.

Consider the notion of a set; in Zermelo-Frankel (ZF) set theory, which is arguably the basis of much of mathematics, the word "set" is undefined (Bagaria, 2016). We may *think* of it as a collection of objects. However, we infer *sameness* of sets via the axiom of extensionality, which says that two sets are identical if and only if they have the same elements. This requirement of sameness is what allows mathematicians to identify the set {3,2} as the same as the set {2,3}, By coming up with this criterion of sameness, mathematicians can define the set {3, 2} as the same as the set {2,3} while remaining agnostic about what a set *is* ontologically. We can see that criteria for sameness can give math inferential power.

The Meaning of the Equals Sign

For mathematicians, the equals sign means *is the same as* or *is identical to.* We must be clear about how we use the word "is". There are at least two distinct ways that we use the word "is". The word "is" can refer to identity (i.e. mean the same as "="), but it can also be used as a means of predication ("To Be," n.d.) [1] For example, in the sentence "the morning star is the evening star," the word "is" denotes the identity relation, since the object "the morning star" refers to is the same object as "the evening star". Contrast this with the sentence "Socrates is mortal," in which "is" refers to a property of Socrates. In this paper, I use the word "is" to refer to predication, and for identity, I use "equals", "is the same as", and "is identical to" for emphasis.

There is a standard modern criterion for truth of identity statements: "a=b" is true if and only if the object named by "a" is the same object as the object named by "b" (Frege, 1879/1967; Mendelson, 2009; Noonan & Curtis, 2014). For example, the "president of the United States inaugurated in 2017=Donald Trump" is true because the phrase "the president of the United states inaugurated in 2017" names the same object as "Donald Trump". Similarly, "2+3=4+1" is true because the object named by "2+3" is the same object (the same number) as 4+1 (provided that numbers are objects that exist). This is the criterion that Frege held for truth of equality sentences (Zalta, 2016).

But what do identity statements *mean*, and why are they informative? What does "as the same as" mean? Gottlob Frege addresses these questions using his infamous puzzles in *On Concept and Object* and *On Sense and Reference* (Frege, 1892/1948). One of his puzzles discusses the meaningfulness of the sentence

(1) "The morning star is the evening star".

If nouns mean no more than their referents, then since "the morning star" and "the evening star" both refer to a physical object (the planet Venus), (1) just means

(2) "Venus is Venus".

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[1] Frege wrote a letter to Wittgenstein that his opening line to the Tractatus, "The world is everything which is the case" ("Die Welt ist alles, was der Fall ist") is ambiguous due to not specifying whether the first use of the word "is" ("ist") is used as predication or as identity.

However, (1) is clearly an informative statement, whereas (2) is not. In other words, if noun phrases do nothing more than refer to objects (in this case, the planet Venus), statements of identity are not informative.

Although this is the accepted view, Frege himself struggled with the nature of the equality relation (the "is" of identity). While he never doubted the criterion for truth of "a=b", as described above, he initially considered two possibilities for the *meaning* and nature of the equality relation: (a) that the equals sign expresses a relation between names and (b) that the equals sign expresses a relation between objects. He decided that the equality relation is a relation between names, with the rationale that some identity statements (e.g. "The Morning Star=The Evening Star", 2+2=4) are indeed informative (Dejnozka, 1981; Frege, 1879/1967; Makin, 2010). That is, he decides that "The morning star=the evening star" just *means* that the object that the name "the morning star" refers to is the same object that the name "the evening star" refers to, and that "2+3=4+1" means that "2+3" and "4+1" are names for the same number. However, he later rejects his assertion that the equality relation is a relation between names, on the grounds that the meaning of identity statements would then be statements about arbitrary linguistic convention, rather than expressing what he calls "objective knowledge" (Frege, 1879/1967; Makin, 2010). As a result, he creates a notion of s*ense* as an aspect of a name's meaning (in addition to its *referent*). A name *expresses* a sense and *refers to* or *denotes* its referent. He characterizes a name's sense as a "mode of presentation" of its referent (Makin, 2010). Roughly, a name's sense is what *picks out* its referent. Frege also called sense "cognitive value". A sense is something that we *grasp*: "We relate to a sense by *grasping* it, which is what understanding the attached name consists in" (Makin, 2010). The terms "The morning star" and "the evening star" express different senses but denote the same referent (the planet Venus), and the sentence "The morning star = the evening star" means that the *senses* expressed by "the morning star" and "the evening star" pick out the same referent (the planet Venus). Similarly, "2+3" and "4+1" express different *senses* and denote the same referent, and the sentence "2+3=4+1" means that the *senses* expressed by "2+3" and "4+1" pick out the same referent (the number 5, which is the same as the number 2+3 and 4+1).

To emphasize what I mentioned earlier, equality represents true identity, not merely an equivalence relation: a=b if and only if a *is the same thing* as b. It does not suffice for a to be *equivalent* or *isomorphic to* b. Taking another example from set theory, it is not the case that Z/2Z=Z2. Mathematicians might casually refer to them as "the same group," but they are actually different groups (members of Z/2Z are *sets* of integers*,* whereas members of Z2 are integers). Z/2Z and Z2 are of the same *isomorphism class,* but they are not *equal* to each other. This is not to say that that are unequal simply because we write members of Z/2Z one way and members of Z2 another way; indeed we can have two different names for the same thing. For example, we can write *the same* group with additive or multiplicative notation, we have the same group, not merely isomorphic. Similarly, we can call the same function both "f" and "g". So long as the set for the group, together with its operation, are identical (despite different names), then the groups are identical.

Hodges (1997), a model theorist, attempts to make a similar point that I am making here: "A group theorist will happily write the same abelian group multiplicatively or additively, whichever is more convenient for the matter in hand. Not so much for the model theorist: for him or her the group with '*' is one structure and the group with "+" is a different structure." (p.1). The main point Hodges makes is that isomorphism and identity ought not be conflated. However, he conflates the notion of isomorphic structures with structures of the same name. By reducing the notion of isomorphism to the notion of naming, there's an important nuance he's missing: that we can have isomorphic, non-identical structures, yet *also* have a structure and name it two different ways. Let's return to the example of groups with two elements: the groups Z/2Z and Z2 are isomorphic but *not* identical, for the reasons described above. Yet, if we wanted, we could write Z2 the typical way, as the set {0,1} together with the function {(0,0,0),(0,1,1),(1,1,0),(1,0,1)},or we could simply say "let a=0 and a=1", and write Z2 as the set {a,b} together with the function {(a,a,a),(a,b,b),(b,b,a),(b,a,b)} and indeed these would be the *same group* while still remaining distinct from Z/2Z.

This may seem like mindless pedantry, however, there are good reasons to be careful about equivalence versus equality. If we conflated equivalence and equality, Galois theory (which involves counting isomorphisms) would be a lot less interesting (Burgess, 2015; Hodges, 1997). Despite the distinction, equivalency and identity do have some properties in common - namely, identity (equality) is a particular equivalence relation in the sense it is reflective, transitive, and symmetric.

I choose to precede the literature review on the equals sign with a discussion of the philosophy of identity to emphasize that appraising the meaning of identity statements is a non-trivial activity. Frege himself considered at least *three* different meanings for the equals sign (relation between objects, relation between signs, relation between senses). It is thus entirely possible that students can also have multiple, nuanced meanings for equations. We can adapt the meanings that philosophers and mathematicians have considered, since they are ways of thinking of the equals sign that may pertain to students' particular mental models

Here is summary of the major philosophical and mathematical points the reader should keep in mind when reading the literature review on the equals sign. The sentence "a=b" is true if and only if the object named by "a" is the same as the object named by "b". While this is a criterion for *truth* of statements of the form "a=b," it does not account for the *meaning* of "a=b," as can be seen by Frege's work. As Frege shows us, discerning the meaning of such statements is nontrivial, and even an esteemed philosopher and mathematician such as himself considered more than one possible meaning. Another important thing to keep in mind is that in mathematics, "=" expresses the relation of identity: that is, "is equal to," "is the same as," and "=" all refer to the same relation. This relation is a specific example of an equivalence relation, but there are other equivalence relations that are not this specific relation.

Understandings of the Equals Sign: the Relational View

"Operational" is the word in the literature used to characterize these weak understandings. Roughly, an operational understanding involves viewing the equals sign as involving a performance of an operation. The authors contrast an operational understanding with a "relational" understanding, which is characterized in various ways. Although the authors vary in their meanings for "operational" and "relational," they are consistent in that an "operational" view always describes an incorrect or unproductive understanding, and a "relational" view describes a correct or productive understanding. I will adopt this terminology throughout the remainder of this paper. Due to space constraints, we only discuss characterizations of the "relational" view.

In the literature, the operational view is contrasted with the relational view. Authors tend to treat a relational view as anything that is not an operational view, but they are not explicit about this dichotomy and are sometimes imprecise or narrow about what they mean by a "relational" view. In all cases, the relational/non-operational view is what the authors endorse as the desired view for students to hold. Despite the attention that philosophers and mathematicians have paid to the nuanced meanings of the equals sign, math educators focus only on the misunderstandings of the equals sign. That is, they focus on students' operational understandings and do not consider the varied understandings within the relational characterization. As discussed, even within the mathematical community, there is a lack of consistency and clarity about what the equals sign means. While the nuances of the relational view vary across authors, what remains consistent is that the relational view involves viewing the equals sign as expressing an equivalence relation. Exactly what this equivalence relation is and what it applies to is not apparent or consistent in the literate.

Several authors characterize a relational view of the equals sign in such a way that is tantamount to expressing a relation between names of numbers. Denmark et al. (1976) and Kieran (1981) describe a relational view in a precise but narrow way: a student has a relational meaning for the equal sign when she sees whatever is on each side of the equals sign as names for the same number. Sáenz-Ludlow and Walgamuth (1998) and Oksuz (2007) do not clearly state that a relational view means "names for the same number," but do seem to imply it; Seans-Ludlow and Walgamuth (1998) describe a relational understanding as "quantitative sameness of two numerical expressions" and Oksuz (2007) describes it as a view that the equals sign is "a relationship expressing the idea that two mathematical expressions hold the same value" (p.3). Unfortunately, the authors do not define what "quantitative sameness" or "same value" means, but they indicate that the equals sign expresses a relation between symbols or signs by their use of the term "expressions". Notice that characterizing the equals sign as expressing a relation between signs or linguistic objects is a view that Frege originally espoused and then later rejected in favor of the notion of *sense*.

Some authors suggest a relational meaning of the equals sign as involving sameness as an *attribute* of expressions. For example, Oksuz (2007) refers to the expressions having the same "value" but does not elaborate further on what he means by "value". McNeil and Alibali (2005a), Knuth, E., Alibali, M., Hattikudur, S, McNeil, N., & Stephens, A (2008), and Seans-Ludlow and Walgamuth (1998) refer to "quantity" but in slightly different ways. McNeil and Alibali (2005a) say that the expressions actually *are* the "same quantity," Seans-Ludlow and Walgamuth (1998) refer to the "quantitative sameness of expressions," and Knuth et al. (2008) refer to the equals sign as "representing a relationship between two quantities. " Notice that McNeil and Alibali refer to the *expressions* as being the same quantity, whereas Knuth et al. do not explicitly refer to expressions and instead refer to the equals sign as a relation between quantit*ies* (plural). McNeil and Alibali's characterization is a bit odd - if an expression is a quantity, then different expressions (which is usually what is on either side of the equals sign) should indicate different quantities, yet they refer to the "same" quantity. Knuth et al's characterization is also a bit odd - if the equals sign is referring to *two* quantities, then what is it that is the same? None of the authors define what they mean by "quantity," nor do they explain what the equals relation *says* about quantities. Behr et al. (1980) are more explicit than the other authors about the relational meaning of the equals sign. They say that "the most basic meaning is an abstraction of the notion of *sameness.* This is an intuitive notion of equality which arises from experience with equivalent sets of objects. This is the notion of equality which we would hope children would exhibit" (p.13).

In other words, Behr et al. seem to be suggesting a meaning of the equals sign that alludes to the sameness of numerosity of equinumerous sets of objects, that perhaps, "3+4" refers to the cardinality of the set resulting from forming the union of a set of 3 elements with a set of 4 elements, and a sentence such as “3+4=5+2” has the meaning that such a set is equinumerous to the set resulting from the union of a set of 5 elements with a set of 2 elements.

Several authors (McNeil and Alibali, 2005 a and b; Byrd et al., 2015; Behr et al., 1980, McNeil et al, 2006; Kieran, 1981) ambiguously refer to the relational view of the equals sign as expressing “equivalence”. Unfortunately, the authors are not always clear about what particular equivalence the relation is on -- perhaps expressions or numbers -- nor are they clear about what the equivalence relation is. In fact, the authors do not explicate the relevance of the properties that make an equivalence relation an equivalence relation -symmetry, reflexivity, transitivity.

Kieran describes the relational view as an “equivalence view” of the equals sign and synonymous with “another name for.” She opens her paper with a quote from Gattegno (1974) that *“equivalence* is concerned with a wider relationship [than identity or equality] where one agrees that *for certain purposes* it is possible to replace one item by another. Equivalence being the most comprehensive relationship it will also be the most flexible, and therefore the most useful” (p.83, emphasis added). Gattegno is not contrasting equivalence with computational or operational understandings - instead, he is contrasting equivalence with identity. He is making the point that identity is a special *kind* of equivalence. In other words, he is emphasizing the difference between identity and equivalence, not treating them as one-and-the-same. Gattegno elaborates on what he means by “equivalence” by discussing an analogy between mathematical statements and natural language; he describes equivalence as a sort of linguistic replaceability that is a consequence of identity. In his view, “2x16” and “32” name the same thing and hence, are equivalent in the sense that in mathematical computations it is permitted to replace one with the other. He compares this sort of permitted replacement with replacing “he is on my right” with “I am on his left”. In other words, for Gattegno, equivalence is a consequence of identity that allows for linguistic replacement in certain contexts. This sort of permitted replacement -- the “equivalence” -- is what Gattegno suggests gives identity its power. Gattegno is not explicit about what this equivalence relation is, but he seems to hint that it is an equivalence relation between signs - e.g., “a” and “b” are equivalent if and only if “a” is replaceable by “b”. Gattegno hints at a relationship between viewing terms as “equivalent” and viewing them as “names for the same thing” - one is a consequence of the other. He explains that “4+1” and “5” are names for the same thing, and that therefore they are “equivalent” in the sense that they are interchangeable linguistically. Kieran, in citing Gattegno, does not seem to notice the relationship between “name for same thing” and replaceability. She instead muses that *another name for* is an equivalence relation on ordered *pairs* of numbers, which she calls “R”: “(a,b)R (c,d) iff a+b=c+d”. This relation R applies to statements like “4+5=3+6,” but not “4+5=9” or “9=9”. Moreover, it is not an equivalence relation on names - it is an equivalence relation on actual ordered pairs of numbers and only applies in narrow contexts. Moreover, it is odd that she seems to define this equivalence relation in such a way that it allows only for equations with two summands - why not instead say “aRb if and only if ‘a’ and ‘b’ name the same thing”?

Kieran is not the only author ambiguously using the word “equivalence”. Other authors also refer to the equals sign as expressing an “equivalence relation” but are unclear about what this relation is or what the relation is on (Knuth et al. 2008; McNeil & Alibali 2005; Byrd et al. 2015).

Despite the ambiguous characterizations of the relational understanding, we can tease out some significance of viewing the equals sign as an equivalence relation. It is important that students conceive of the equals sign in such a way such that it expresses an equivalence relation.

Further, having the properties of an equivalence relation is what allows for substitutability. Let us return to the "rule violations" that characterize an operational (non-relational) understanding:

	Rule Violations	Rule Violated
(i)	$5 = 2 + 3$	The "answer" comes to the right of the "problem". Here, the "answer" is on the left.
(ii)	$2 + 3 = 4 + 1$	The "answer" should follow the "problem". Here, the answer is "5," but no answer is written.
(iii)	$5 = 5$	There needs to be a "problem". Here, there is no problem.

Figure 1: three equations that students with operational views of the equals sign frequently reject

Viewing the equals sign as an equivalence relation accounts for a way of thinking in which (i), (ii), and (iii) are not rule violations, and, relatedly, the power of replaceability. Suppose = is an equivalence relation. So long as 2+3=5, it follows from symmetry that 5=2+3, in which case (i) is no longer a rule violation. Similarly, it follows from reflexivity that 5=5, in which case (iii) is no longer a rule violation. Concluding (ii) is a bit more involved but can be easily obtained through symmetry and transitivity: we have that 2+3=5, and by symmetry, 5=4+1. Hence, it follows from transitivity that 2+3=4+1. In other words, it is permissible in a mathematical context to use "5", "2+3", and "4+1" interchangeably - i.e. we can replace one term with another and have used the properties of equivalence to do so.

Discussion

The equals sign literature discussed above indicates that many students are *not* conceptualizing the equals sign as an equivalence relation. If someone were to view the equals sign as representing an equivalence relation, then the rule violations commonly discussed in the literature would not be rule violations; students would accept "5=2+3", "2+3=4+1," and "5=5" as true assertions. The most common misconception manifested itself in a rejection of statements like "2+3=4+1" with an acceptance of equations like "2+3=5+1=6" (Rule Violation (iii)). Renwick (1932) found this misconception amongst 8-14 year old girls of varying abilities. Oksuz (2007) found this misconception amongst middle school students. Fifty 5th graders and sixty 6th graders were asked what goes in the blank in "6+7=___+4", and 38% of 5th graders and 24% of 6th graders answered with "13". Behr et al. (1980), through performing individual interviews with 6-12 year old students, found that students viewed (i), (ii), and (iii) all as rule violations. When asked to give a definition of the equals sign, most children's responses could be summarized as "when two numbers are added, that's what it turns out to be". In other words, students have a conception of the equals sign in such a way that it does not express an equivalence relation, and therefore do not have a mathematically normative or productive understanding.

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Making Implicit Differentiation Explicit

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This paper discusses the conceptual basis for differentiating an equation, an essential aspect of implicit differentiation. We explain that implicit differentiation is more than merely the procedure of differentiating an equation and carefully provide a conceptual analysis of what is entailed in understanding the legitimacy of this procedure. This conceptual analysis provides a basis for discussion of the literature, as well as empirical justification for the importance of this topic.

Keywords: Implicit Differentiation, Related Rates, Derivative Operator, Calculus

The topic of implicit differentiation has been identified as "missing" from RUME research (Speer & Kung, 2016). This theoretical paper aims to begin to fill that gap by addressing the legitimacy of applying the differential operator to each side of an equation, an essential aspect of implicit differentiation. We take as axiomatic that understanding implicit differentiation involves understanding why it is legitimate to perform the procedure of differentiating each side of an equation.

In this article, we provide a conceptual analysis of what it means for someone to understand the legitimacy of differentiating both sides of an equation. By carefully examining an implicit differentiation problem and a related rates problem, we explain how, despite a procedural similarity, implicit differentiation is not merely the procedure of "taking the derivative of both sides," a conflation that exists even within the math education literature. We discuss literature *after* presenting the conceptual analysis, as the discussion is through the lens of our conceptual analysis. Finally, student data illustrates that understanding the legitimacy of this operation is nontrivial for students.

The Normative Solution to a Ubiquitous Problem

We begin with a pair of ubiquitous problems as well as their standard solutions, which can be found in the implicit differentiation and related rates sections of most calculus curricula. This illustrates how the conceptual basis for implicit differentiation can easily be lost in the implementation of its procedure.

> Suppose a 3-meter ladder, starting flush against the wall, begins sliding down the wall, the top sliding down at 0.1 meters per second.
> **(a)** Find the rate of change of the distance of the top of the ladder from the base of the wall with respect to the distance of the bottom of the ladder from the base of the wall.
> **(b)** Find the rate of change of the distance of the bottom of the ladder from the base of the wall with respect to time.

Figure 1. The ladder problems

A prototypical solution of problem (a) involves letting x be the distance the bottom of the ladder is from the base of the wall and y be the distance the top of the ladder is from the base of the wall, both in meters. Then we get:

(1a)	$x^2+y^2=9$	From the Pythagorean theorem
(2a)	$2x+2y(dy/dx)=0$	By differentiating with respect to x
(3a)	$dy/dx=(-x/y)$	Solving for dy/dx

This equation yields the relevant rate of change at any point in the ladder's motion. The solution for (b) is similar:

(1b)	$x^2+y^2=9$	From the Pythagorean theorem
(2b)	$2x(dx/dt)+2y(dy/dt)=0$	By differentiating with respect to t
(3b)	$dx/dt=(-y/x)(dy/dt)$	Solving for dx/dt
(4b)	$dx/dt=(0.1y/x)$	Substituting -0.1 m/s for dy/dt

The most commonly used calculus texts (e.g. Rogawski, 2011; Stewart, 2006; Weir, M. D., Hass, J. R., & Thomas, G. B., 2011) present solutions to related rates and implicit differentiation problems similarly to what is above, often without an explanation of the legitimacy of the procedure (Broussoud, 2011). This treatment overlooks why the procedure works and treats the derivative like a basic algebraic operator that can be applied to both sides of an equation (Thurston, 1972, Staden, 1989). However, the derivative cannot simply be applied to both sides of any equation. To illustrate this point, consider the equation x=1. Taking the derivative of both sides of this equations leads to 1=0, an absurdity, whereas applying any basic arithmetic operation yields a related legitimate equation (2x=2, x+2=3, x-1=0, etc.). So clearly there is more to why the derivative of both sides procedure works and when it can be applied than is evident from the prototypical examples above. We explore this in the following section.

A Conceptual Analysis

In order to address why the procedure for solving implicit differentiation problems is valid, we begin with a *conceptual analysis* (under the epistemological perspective of radical constructivism (Thompson, 2008)). This conceptual analysis is intended to put the reader's understanding of the relevant mathematics on solid footing and explicitly lay out conceptual operations involved for one to understand implicit differentiation and differentiating equations robustly. It can thus enhance any framework that addresses problems in which students differentiate equations, such as related rates problems (Engelke, 2007; Martin, 2000). This conceptual analysis facilitates our discussion of student struggles with the validity of the implicit differentiation procedure later in this manuscript.

Let us revisit **(a)** from Figure 1 above. We start as before with the equation $x^2+y^2=9$, with $y\geq0$. **Treating y as a function of x, define f(x) as the unique y such that $y\geq0$ and**

$$\textbf{(1)} \quad x^2+y^2=9.$$

Hence, for $|x|\leq3$,

$$(2) \quad x^2+(f(x))^2=9.$$

Let's call the function defined on the left hand side of equation (2) 'm' and the function on the right hand side 'r'. So $m(x)= x^2+(f(x))^2$ and $r(x)=9$. Notice that m(x) and r(x) are both functions of x and that (2) states that they are equal on the interval $0\leq x\leq3$. From this statement of function equality, we can conclude that r and m have the same rate of change on this interval. So when we take the derivative of both sides we maintain equality on this interval. That is:

$$(3) \quad m'(x)=2x+2f(x)f'(x)=0=r'(x) \text{ for } 0\leq x\leq3$$

In the above case we get that:

$$(4) \quad f'(x)=-x/f(x)$$

The conceptual steps involved in legitimately making the inference of taking the derivative of both sides (transition from (2) to (3) above) appear below in Figure 2:

> **1.** Defining *f* by using (1).
> **2.** Viewing both sides of the equation as functions (of x).
> **3.** Recognizing that the functions defined by the left hand side and the right hand side are equal on the relevant interval.
> **4.** Recognizing that, since the functions are equal on an interval, the respective derivatives of the functions are also equal on that interval.

Figure 2. Conceptual steps involved in implicit differentiation, solving Fig 1 part a.

It bears mentioning that both Thurston (1972) and Staden (1989) noted that the legitimacy of the derivative of both sides procedure is rooted in function equality, although, Staden (1989) referred to these statements of function equality as "identity statements." However, as discussed in the introduction, much of the current education research on related rates and implicit differentiation problems overlooks this issue.

We pause briefly to discuss notation usage. At this stage in the manuscript we have solved problem (a) twice. The introduction used Leibniz notation (d/dx) in its solution to (a). This is consistent with popular calculus textbooks, where "taking the derivative of both sides" is often equated with "taking d/dx of both sides" (e.g., Rogawski, 2011; Stewart, 2006; Weir, et al., 2011). However, as we illustrated above, the reason taking the derivative of both sides is a legitimate procedure stems from the equation under consideration expressing function equality on some interval. Viewing the equation this way requires viewing the equation as (implicitly) defining a function; in the above, f(x) is implicitly defined in terms of its relationship to x^2 and 9. Hence the label "implicit differentiation". The role of function equality - in fact, the role of functions - is obscured by the procedural emphasis and use of Leibniz notation. With Leibniz notation, there are no functions explicitly under consideration. With the standard function notation used in the conceptual analysis, it is more apparent which functions are being differentiated and that f(x) is being implicitly defined. Further, some research suggests that students need to see equations written in standard function notation before differentiating (Engelke, 2008).

When Does the Equation Serve as a Function Definition?

The two problems in Figure 1 look very similar to each other; they have similar solution procedures that involve taking the derivative of both sides of the same equation,

(*) $$x^2+y^2=9,$$

and then applying derivative rules accordingly. However, the underlying reasoning that justifies the validity of performing a derivative operator on an equation differs between the two. Specifically, it is more involved to conceptualize (*) as a statement of function equality in **(a)** than it is in **(b).** In **(a),** the equation (*) not only asserts equality of functions, *but also "implicitly" defines a function* (the function f, discussed **in bold** above). In order to make sense of (*) as asserting a statement of function equality in terms of functions of x, one must conceptualize (*) as defining f and viewing y as equal to f(x).

The relevant functions in **(b)** are functions of time (not of x), since the task is to find a rate of change of distance with respect to time (t). Unlike with **(a),** Conceptual Step 1 is unnecessary; one does not need to conceptualize (*) as defining a function in order to view it as asserting a statement of function equality.

Given our previous discussion of the limitations of Leibniz notation, we use standard function notation in the remainder of this discussion. The letters *x* and *y* are shorthand for functions of time, x(t) and y(t), respectively. So for all t:

(**) $$(x(t))^2+(y(t))^2=9.$$

Similar to our earlier discussion, if we give the functions on the left and right side of the equation labels, say $m(t)=(x(t))^2+(y(t)^2)$ and $r(t)=9$, respectively, then (**) simply asserts that the functions *m* and *r* are equal for all values of *t.* Using similar reasoning to that of the previous problem this statement of function equality implies that m'(t)=r'(t). So:

(***) $$2x(t)x'(t)+2y(t)y'(t)=0$$

which, since we know y'(t)= -0.1m/s, yields:

(****) $$x'(t)=(0.1y(t)/x(t))$$

Notice that unlike (a), (b) entails only conceptual steps 2-4 from Figure 2, as there was no function of *t* implicitly defined by the equation. In other words, in problem (a), (*) simultaneously serves the purposes of both asserting a statement of function equality *and* implicitly defining a function. In problem (b), (*) only serves the purpose of asserting a statement of function equality. In this sense, only (a) truly involves implicit differentiation. In both situations, students must conceive of an equation as asserting function equality; however, to conceive of (*) as a statement of function equality involves first conceiving (*) as defining a function. Hence, it seems reasonable that problems like (a) might be more conceptually difficult for students than problems like (b). Defining the function *f*, as in (2), although perhaps trivial to mathematicians, could be a conceptual obstacle for students. Notice that (2) takes the form of "f(x) is the unique *y* such that the proposition P(x,y) is true." Being able to conceive of a function definition that involves outputs according to whether or not a proposition is true requires a process conception of function, which many students lack (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Unfortunately, it is common for educators to treat the two types of problems in Figure 1 synonymously as applications of taking the derivative of both sides of an equation, without attending to the meaning of the equation or the legitimacy of such an operation (discussed later).

To summarize, in order for one to understand the legitimacy of differentiating an equation, one must have a robust understanding of the equation itself. This robust understanding should involve viewing the equation as asserting a statement of function equality (Conceptual Step 3), which requires viewing each side of the equation as defining a function (Conceptual Step 2). As argued above, in the problem in Figure 1b, Conceptual Step 2 is *easier* than in 1a, as Conceptual Step 1 is not involved. Our reason for carefully contrasting the two types of problems in Figure 1 is to emphasize that, while these problems have similar procedural solutions, when attending to the legitimacy of differentiating the equation, they are not the same.

We are *not* claiming that the only conceptual work involved in understanding implicit differentiation and related rates problems is in understanding the legitimacy of differentiating an equation. This is just the conceptual aspect that we choose to focus on in this paper, as it has largely been ignored so far in the literature. Now that we have provided the reader with a conceptual basis for understanding implicit differentiation/related rates problems, including the conceptual steps required to make sense of the legitimacy of these procedures, we shift to discussing the literature.

Literature

We searched the literature thoroughly by reviewing every available RUME paper that included the words "implicit differentiation" or "related rates", as well as every paper in the online archives of The Journal of Mathematical Behavior, Mathematics Teacher, Journal for Research in Mathematics Education, Mathematics Education Research Journal and all other National Council of Teachers of Mathematics publications. A Google Scholar search was also performed, but the methodology for that search was not recorded. Despite this expansive search, only two articles (Thurston, 1972; Staden, 1989) address the legitimacy of differentiating both sides of an equation. In both articles, the topic is only mentioned in passing, and there is no discussion of student understanding. Staden (1989) specifically argues that students are "mistaught" by being told that they can differentiate each side of an equation (when, as discussed above, this does not work for *any* true equation), and suggests that students might have resulting misunderstanding.

The remainder of the literature tends to treat differentiating an equation as only a procedural aspect of implicit differentiation or related rates problems. In fact, many authors appear to treat "implicit differentiation" to mean something like "using Leibniz notation while differentiating an equation", not distinguishing true implicit differentiation (like **(a)**) from differentiation in related rates problems that's not truly implicit (like **(b)**) (Jones, 2017; Martin, 2000; Engelke, 2007; Garcia & Engelke, 2013). This is unsurprising when we consider that, when viewed procedurally, differentiating with respect to x and with respect to t is almost identical. Hare and Philippy (2004), for example, write a lesson plan outline that includes the assertion "Implicit differentiation must be used whenever the differentiation variable differs from the variable in the algebraic expression (p.9)" and stresses use of the chain rule. If one is not attending to the rationale for differentiating, then attending to the "differentiation variable" and when to use the chain rule is similar in problems like **(a)** as in problems like **(b)** (in **(a)**, the "differentiation variable" is x, and in **(b)** it is t).

Martin (2000) provides a "problem-solving framework" characterizing the steps in solving related rates problems similar to **(b).** She not only conflates "implicit differentiation" with "taking d/dx of both sides," but also overtly labels differentiating each side of an equation as "procedural." Engelke (2007) utilizes Martin's framework to develop a "mental model"; this mental model further de-emphasizes the conceptual aspect of differentiating equations by consolidating Martin's "implicitly differentiate" step with another step to create what she calls a "phase". When we consider how Martin created her framework, it is unsurprising that she does not address the legitimacy of "implicit differentiation"; she created the framework by observing written solution *procedures* to related rates problems. Since conceptualizing a justification for differentiating an equation is not a procedure, it makes sense that it would remain unaddressed. This is not to suggest that Martin's model is not useful, only that it leaves this particular matter unaddressed.

Student Confusion

We have established, that both common textbooks and the majority of mathematics education literature ignore the conceptual basis for implicit differentiation. However, we realize that some might view this as unproblematic. In this section we take a brief look at some data to establish that the lack of student understanding of the conceptual basis for implicit differentiation. A search of popular online student help forums, Khan Academy and Stack Exchange, suggests that students are unclear of the validity and meaning of applying the differential operator to each side of the equation (Anonymous, n.d.; Frank-vel, 2015; Jon, 2013; Klik, 2013; Mathematicsstudent1122, 2016; Ryan, 2016; Wchargin, 2013). Further, the work of one of the authors of this manuscript suggests that a strong understanding of function equality may be absent in a number of calculus students (Mirin, 2017a; 2017b).

In order to learn more about students' understandings of the conceptual steps involved in implicit differentiation (Figure 2), a student, John, was interviewed by the first author of this manuscript. He was enrolled in Calculus II at Anonymous State University (ASU) and had taken Calculus I, which includes a unit on implicit differentiation, the semester prior. The interview was a semi-structured clinical interview and lasted an hour (Hunting, 1997). Throughout the interview, John was asked to think about ideas regarding implicit differentiation and function equality that he had perhaps not reflected on before. John might have never considered these matters, and might therefore have made on-the-spot explanations.

The interview centered around four prompts:

Prompt 1. What is your meaning for implicit differentiation? How do you interpret the word "implicit" in this situation?

Prompt 2. Find dy/dx for $x^2+y^2=1$ when $y>0$

Prompt 3. A 10-foot ladder leans against a wall; the ladder's bottom slides away from the wall at a rate of 1.3 ft/sec after a mischievous monkey kicks it. Suppose h(t)= the height (in feet) of the top of the ladder at t seconds, and g(t)=the distance (in feet) the bottom ladder is from the wall at t seconds. Then $(h(t))^2-100=-(g(t))^2$. How fast is the ladder sliding down the wall?

Prompt 4. True or false: Suppose f(x)=g(x) for all values of x. Then f'(x)=g'(x).

Figure 3: The prompts that formed the basis for the clinical interview.

The interview lasted an hour. Due to space constraints only the most pertinent highlights are reported here.

John expressed that he did not remember exactly what the procedure of implicit differentiation was, but that it was something that must be done when there is no function (i.e., due to failure of the vertical line test). He did not have an idea of what the *implicit* referred to in implicit differentiation. John did not have an idea of how to approach Prompt 2, so the interviewer reminded him of a procedure that was done in his Calculus I class: replacing *y* with f(x) before differentiating the equation and that $x^2+y^2=1$, $y>0$ defines the top half of a circle, and asked him to elaborate on what $x^2+(f(x))^2= 1$ means. He explained that 1 is the radius, and having f(x) [in place of *y*] "makes the computation easier". He was then asked him explicitly what it means for the right hand side of $x^2+(f(x))^2=1$ to equal the left hand side, and he responded "It's a circle. I just see a circle." When prompted to explain what the circle has to do with the equation, he graphed two parabolas - a sideways parabola (representing y^2) and an upright parabola (representing x^2) and asked "how is that a circle?". In this situation, it seems that John was not thinking of y (or f(x)) as a function of x; instead, he seemed to be thinking of "y^2" as denoting the parabola that he associates with "$x= y^2$". After reasoning with a graph was unhelpful to John, he began considering specific values of x and y, observing that "as they change together, in this equation here, they have to change together in such a way that it always equals 1."When asked about the legitimacy of taking d/dx of both sides, he drew an analogy to algebra: "If I have x=1, I multiply by 2 and get 2x=2, it would be the same thing." He related the procedure of taking d/dx to inferring equal rates of change: "if you take the rate of change of this [left hand side], it is the rate of change of this [right hand side]. They're equal to each other, so the change in one is gonna be the change in the other." Since John believed the inference of equal rate of change came from *something* being equal, to get at what that *something* was, the interviewer asked him what happens if he differentiates each side of x=1. He noticed that it results in 0=1, which he said did not make sense.

The interview then shifted to Prompt 3. John was reminded that he could take the derivative of both sides of the equation, and he did so with some minor errors. He explained that the distance the ladder is from the wall, g(t), and the distance the ladder is from the floor, h(t), "change together". Even when pushed, he did not say why taking the derivative of both sides is a valid procedure. Instead, John continued to express an understanding of the two distances as changing together as time changes, and failed to mention each side of the equation as representing a function:

> "We take the derivative of both sides because...you need to have the two rates change together, in order for this scenario to work. Because if they don't with respect to each other, then uh...it just doesn't hold true. So we do it on both sides in order to have the scenario change together and everything stay true to itself...maybe."

Since John was not using the language of functions on his own, the interviewer decided to move to Prompt 4 in order to see if he could relate taking the derivative of each side of an equation to an inference from function equality. John almost immediately provided what he viewed as a counterexample to the assertion that if two functions are equal, then their derivatives are also equal. By misapplying the quotient rule, he argued that f(x)=x and g(x)=2x/2 are equal for all values of *x* but have different derivatives. He explained that, if he were to simplify g(x), he would end up with the same derivative as that of *f*, but that simplification before finding derivatives is not permitted. This highlights that John had a fundamental misunderstanding of how the derivatives of two equal functions relate, a key aspect in understanding the legitimacy of applying the derivative operator. We believe this misunderstanding contributed to his struggles with making sense of why the implicit differentiation procedure is legitimate.

Discussion

We have performed a conceptual analysis of the implicit differentiation procedure. We have established the conditions under which taking the derivative of both sides of an equation is legitimate, why it is a legitimate procedure under these conditions and when a function is implicitly defined. In the conceptual analysis this process is broken down into 4 conceptual steps, which may form the basis of instruction aimed at better student understanding of implicit differentiation. Only 3 of these steps are needed to make sense of related rates problems. We showed that the way of understanding described in the conceptual analysis is largely absent from the mathematics education literature, which in turn bolsters the need for this analysis**.** This points to the fact that the understanding developed in the conceptual analysis may be non-trivial to develop in students. Finally, the study reports a brief excerpt from a successful calculus student John, which establishes that the understanding developed in the conceptual analysis is not present in some students and is non-trivial to develop. In future research we aim to explore this issue in more detail by conducting a multi-student teaching experiment aimed at developing rich student understanding of implicit differentiation.

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Spreading Evidence-Based Instructional Practices:
Modeling Change Using Peer Observation

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Evidence to date that active, student-centered learning in mathematics classrooms contributes to desired student outcomes has now accumulated to compelling levels. However, promoting and supporting widespread use of alternative practices is challenging, even amongst practitioners open to such changes. One contributing factor is the fact that a majority of instructional change efforts focus on only a small portion of the instructional system, while true transformation requires systemic reform. Successful institutional change initiatives have been shown to involve common features: they involve ongoing interventions, align with individuals' beliefs, and work within the existing landscapes of institutional values. Here we propose a theory to support instructional change in undergraduate mathematics by adding a new dimension – instructor peer observation– to an existing model for institutional change (the CACAO model), thereby aligning with evidence regarding what promotes effective change. An exemplar is given to illustrate how this theory might be realized in practice.

Keywords: Institutional change, instructional change, active learning, peer observation

Introduction

Significant evidence supports active learning as critical for student success in mathematics classrooms. Freeman and colleagues' 2014 meta-analysis of 225 studies found significant improvement in student grades and pass rates in classrooms with active elements compared to those with only lecture. Many other studies have identified additional benefits in active learning classrooms, such as demonstrable conceptual learning gains (e.g., Kogan & Laursen, 2013; Kwon, Rasmussen, & Allen, 2005; Larsen, Johnson, & Bartlo, 2013), reduced disparity between dominant and historically marginalized groups (President's Council of Advisors on Science and Technology, 2012; Kogan & Laursen, 2013; Riordan & Noyce, 2001), continued student gains in future classes (Kogan & Laursen, 2013), and improved STEM retention rates (Rasmussen & Ellis, 2013; Seymour & Hewitt 1997). In light of the mounting evidence, the National Science Foundation has called for wider propagation of interactive instructional methods (NSF, 2013).

Despite the clear evidence of effectiveness and the national mandates for change, classroom instructional practices do not reflect a prevalence of student-centered approaches. Lecture still dominates (Johnson et al., 2017; Hora, Ferrare, & Oleson, 2012) and, according to the National Science Foundation, "highly effective teaching and learning practices are still not widespread in most institutions of higher education" (NSF, 2013, paragraph 67). Change is difficult: even when instructors want to change and believe they can, lecturing persists in undergraduate mathematics (Johnson, et al., 2017). Providing people with evidence that active learning works is not enough to motivate change (Foertsch, Millar, Squire, & Gunter, 1997; Reese, 2014; Dancy & Henderson, 2010), and even disseminating research-based "best practice" curricular materials is insufficient to support meaningful shifts in undergraduate STEM instruction (Henderson, et al., 2011).

Some researchers have been working to provide ongoing supports to practitioners trying out innovative curricula, in the hopes of helping them successfully implement and sustain new practices (Lockwood et al., 2013; Johnson, Keene & Andrews-Larson, 2015). Because instructors must develop new knowledge and skills in order to teach in student-centered ways (e.g., learning how to elicit and understand student thinking, lead effective whole-class discussions, and build on student conceptions and strategies to advance course content goals), these professional development efforts focus on combating the finding by Henderson et al. (2011) that interventions lasting less than one semester were ineffective. These change initiatives focusing on ongoing supports have had some lasting success (Lockwood et al., 2013), but the cost is high. In one model, instructors are provided with comprehensive curricular materials and asked to engage in regular, ongoing (virtual) collaboration with other colleagues, thus the commitment required is extremely large. While effective, this type of intervention is feasible only for those with advanced practice who are interested in significant reform. This paper offers a more accessible alternative to these advanced-practice interventions, which targets instructors at any stage of the adoption spectrum. Namely, we propose peer observation as a research-aligned tool for supporting and sustaining systemic change of teaching culture.

Institutional Change Theory and the CACAO Model

Theories of and models for organizational change are myriad and, though often contextualized for the business and non-profit sectors, pertain to all types of organizations, though their adaptation to the educational context is relatively recent (Reinholz, 2017B). Change models range from "top-down" strategies that rely on policy set by organization leaders to "middle-out" and "bottom-up" approaches that initiate change from the starting point of individuals, or that target departments or other small teaching units. In 2011, Henderson, Beach, and Finkelstein conducted a meta-study of 191 published reports of organizational reform efforts that specifically addressed instructional change in STEM higher education. The authors identified four broad categories that captured the salient differences across the collection of systemic change initiatives. All approaches could roughly be characterized as focusing on one of the following: (1) disseminating curriculum and pedagogy, (2) developing reflective teachers, (3) enacting policy, and (4) creating a shared vision. These four headings emerged from their observations that change strategies tended to fall along one spectrum that measured intended outcomes (wherein efforts to achieve outcomes could be either *prescriptive* or *emergent*) and along another spectrum measuring the aspect of the system being targeted (either *individuals* or larger *environments and structures*). Taken together, these two axes suggest quadrants that distinguish types of approaches. For example, a project in the "disseminating curriculum and pedagogy" quadrant is characterized by a prescriptive effort that targets individuals rather than overall institutional structures (i.e., *"here are some teaching materials that you should use"*).

Amongst the various approaches implemented, it was found that two change strategies commonly used in education – providing teachers with "best practice" curricular materials, and enacting top-down policies intended to levy new practices – were "clearly not effective" (Henderson, Beach & Finklestein, 2011). These authors, however, also distilled three features common to all the systemic change programs deemed successful with respect to realizing some portion of the intended outcomes. The salient take-home messages for change agents are as follows: effective projects (1) align with the beliefs of the individuals involved (or seek to change their beliefs), (2) include long-term interventions (beyond one semester), and (3) are compatible with the broader institutional culture and structure (Henderson et al., 2011). These findings communicate an important message for the RUME community: disseminating research-based curricular materials may be a necessary component for widespread instructional change, but is insufficient to promote lasting change. Moreover, agents of reform must balance change efforts targeted at individuals with those addressing the larger systems at play.

The CACAO model for institutional change, described below, is one theoretical paradigm that has been adapted for use in higher education and has been applied to a variety of programs in which change agents want to promote an institution-wide shift in teaching practices (Maker, et al., 2015). A synthesis of models previously developed by Kotter (1990) and Rogers (2003), the CACAO model was introduced by Dormant in 2011 and integrates top-down and bottom-up approaches in order to leverage existing institutional supports and mitigate barriers to change. The model is flexible enough to allow change agents to weigh the benefits and drawbacks of the proposed change, incorporate the beliefs of adopters and their relative stages of adoption, and consider the institutional context in recruiting a diverse project team and developing a customized plan. As such, the CACAO model is naturally well positioned to include multiple of the “necessary” conditions observed by Henderson and colleagues (2011).

There are four dimensions addressed by the CACAO change model: Change, Adopters, Change Agents, and Organization. The *Change* dimension considers the proposed change itself – in our case, more widespread adoption of evidence-based instructional practices in undergraduate mathematics classes -- and guides an examination of the likelihood that a proposed change will be adopted by key stakeholders (i.e., instructors) by identifying existing incentives to change while anticipating and mitigating potential impediments. The *Adopters* dimension considers the audience – those poised to consider making the change – as well as the various “stages of adoption” that may describe a potential adopters' current mindset relative to change (awareness, curiosity, mental tryout, actual trial, sustained adoption). For example, the CACAO model would suggest that someone who is merely curious about the change but not yet ready for an actual trial is presented with the “2-minute elevator pitch” on the proposed change, rather than the one-hour, in-depth presentation that might be motivating for an adopter in the “mental tryout” stage. *Change Agents* is the dimension in the model that offers recommendations for building an effective leadership team with diverse expertise and broad influence with regard to proposed adopters. Finally, *Organization* is concerned with identifying and leveraging the complex organizational hierarchy and appropriately matching personnel with important roles within the change implementation plan. This dimension of the CACAO model is critical in identifying agents who can act as exemplars, early adopters, and opinion leaders, and who can provide perspective to new members about why the proposed changes are important to the overarching project goals and to an individual’s personal goals.

Peer Observation

Peer observation among instructors has been shown to be an effective tool for promoting and sustaining instructional change. In particular, structured peer observation has been shown to (a) stimulate reflection on one’s own teaching practice (Bell, 2001; Cordingley et al., 2005; Cosh, 1999; Reinholz, 2015), (b) improve collegial relationships and collaboration (Carroll & O’Loughlin, 2014; Shortland, 2010; Reinholz, 2017A), and (c) provide on-going support for shifts in teaching (Byrne, Brown, & Challen, 2010; Martin & Double, 1998). An unexplored outcome of peer observation is its potential to transform teaching culture across an institution. We describe how previous literature on peer observation fits with this theory of local instructional change, and posit a theoretical contribution of how peer observation can be leveraged toward sustained institutional transformation.

Peer Observation to Support Reflective Practice

The model of peer observation we consider is one that relies on personal reflection and close-knit cohorts, rather than external judgment, as the catalysts for change. Gosling (2002) characterizes this type of peer observation model as *collaborative*: rather than focusing on training or evaluation outcomes, the collaborative model focuses on developing teaching through dialogue, reflection, and collaboration.

The role of the observer is radically different in the collaborative model than in an evaluative observational approach. Rather than observing with the intention of making judgments upon others, the observer seeks active self-development. As Cosh (1998) elegantly explains, "the rationale of the observation here [is] to make us aware of different approaches, to encourage an open-mind and questioning attitude, and to provide an environment in which we can reassess our own teaching in the light of the teaching of others" (p. 173). Thus the observation serves as a mirror: the observer can more readily see themselves in the reflection of others. The observer is also freed from the cognitive constraints of teaching to notice elements of instruction more aptly (Reinholz, 2017A).

Peer Observation for Sustained Individual Change

Peer observation has the potential to impact instructional change in a way that some professional development programs do not: it targets an instructor's beliefs about mathematics instruction. Many researchers have noted that instructional change is extremely difficult, in part because our teaching practices stem largely from our beliefs about mathematics and mathematics instruction (Ambrose, 2004; Cooney, 2002; Stipek et al., 2001). Understanding learning theories or improved curricular materials typically has little lasting impact on instructional practices (Silverthorn, Thorn, & Svinicki, 2006). Peer observation offers the opportunity to expand one's breadth of teaching styles and approaches. As part of peer observation, the peer is immersed in the classroom and takes part in the visceral experience similar to that of a student. We posit that this experience can be more powerful than a video club-style professional development program (which has been shown to have lasting impacts on instructional change) (Sherin & Han, 2004).

Peer Observation to Improve Collegial Relationships and Foster Community

Ongoing peer observation initiatives also have the potential to develop communities of practice, a critical element in sustained change. Rather than being evaluated by a more senior mentor, graduate teaching fellows who participated in peer observations noted greater camaraderie with their fellow peers (Reinholz, 2017A). However, the expectations for peer observation must be carefully managed: exposing one's teaching practice to a colleague and inviting feedback makes one vulnerable. If achieved, though, this vulnerability can lead to trust and deeper professional relationships.

Linking Peer Observation to Institutional Change

As discussed, collaboratively-oriented peer observation programs have been shown to impact individual instructional practice. This theoretical report describes how peer observation can be leveraged toward institutional change as well. Sustained, low-stakes (i.e., non-evaluative) peer observation aligns with all three of Henderson and colleagues' (2011) elements of successful institutional change programs.

Alignment with Beliefs

As described previously, peer observation has been shown to develop and promote self-reflective practice. In the process of self-reflection, instructors have the opportunity to articulate their beliefs more clearly; only when made explicit can beliefs be examined and possibly changed. The experience of being in a class and observing from the point of view of a student may create productive conflict within the observer's beliefs (*What does it mean to engage students?*) and help problematize their own classroom practice (*Are my methods as effective as I hope? Could my students benefit from what's being modeled here?*). Additionally, when the observed instructor hears feedback about their teaching that conflicts with their own self-image, it creates an opportunity for them to reflect on their beliefs from this alternate perspective. Putting beliefs in direct conflict with one another is how beliefs change (Gill, Ashton, & Algina, 2004). Thus, peer observation can be helpful in drawing out and formalizing one's own beliefs, engaging with the beliefs of others, and potentially shifting beliefs as a result of experiencing different teaching practices. Of course, there is a risk that peer observation will reinforce an instructor's existing beliefs in an unintended way: upon seeing another instructor struggle to implement a student-centered activity, an observer predisposed to lecture might conclude that lecture is indeed the preferred way to teach. The hope in this case is that the self-reflection and peer-to-peer conversation built into this collaborative observation model are enough to compel participating instructors to reflect on how change might serve them and their students.

Long-Term Intervention

Peer observation (as formulated here) is also a long-term intervention. In our proposed model, instructors collaborate for an entire year, mutually observing one another (in pairs or trios) at least twice a semester. Unlike some interventions that can be extremely costly to implement, peer observation has very little cost in terms of curricular adoption (though there is the cost of instructor time). It can "grow with" the participants, supporting instructors at various stages along the adoption spectrum. For example, those merely curious about active learning are given the opportunity to try-out the practices mentally when observing student-centered teaching in action. At the other end of the adoption spectrum, instructors with advanced evidence-based practices will benefit from the sustained support from regular peer observations and the relationships developed therein.

Alignment with Institutional Culture

Finally, peer observation is a practice that can be molded to fit within virtually any institutions' culture. Presently, formative peer observation is not widely employed within collegiate instruction. Many instructors of mathematics enjoy the privacy and autonomy of their classrooms: opening up one's classroom can be uncomfortable and potentially invasive. However, even though formative peer observation is not currently a part of the teaching culture at many institutions, it can be leveraged as a tool that addresses other institutional concerns. For example, if used conscientiously and carefully, it can provide a platform for better informed insights into peoples' classrooms and, in turn, benefit teachers, students, and administrators alike. It has been well documented that student evaluations are systematically biased (e.g., Centra & Gaubatz, 2000), and yet reviews from students form the primary assessment measure for teaching at most institutions. Many faculty are dissatisfied with the "consumer-based" model of education this implies. On the other hand, having specific feedback from colleagues who can attest to the reflective growth one's practice has undergone could be especially helpful in awarding teaching accolades or offering informed perspectives that supplement student evaluations in letters for promotion cases. Thus, while peer observation can be introduced to shift teaching culture, its potential to address other institutional needs could add both to the longevity of the change effort and its fit with the institution.

Instantiation

The REFLECT project is an example of how change agents are applying the CACAO model together with collaborative peer observation to encourage systemic change across STEM departments on the campus of one small, private, comprehensive institution in the Pacific Northwest. The goal of the REFLECT project is to increase the awareness and use of evidence-based and student-centered practices by STEM faculty on campus, while helping shift the teaching culture to one that widely embraces active learning and views peer observation as a valuable and regular part of reflective teaching practice. Project organizers considered all facets of the CACAO model and identified a number of affordances specific to the institution that could be leveraged in support of the project (e.g., growing interest among faculty for evidence-based practices, an administration that supports reflective and innovative teaching, a desire among faculty for teaching feedback that is not student-based). They also identified ways to mitigate potential barriers to change (e.g., avoid top-down pedagogical prescriptions, work with entire departments to build support for change, compensate participants for their time), and identified key players who could support and enhance institutional change (e.g., the university provost and president, regional experts, respected faculty opinion leaders).

The REFLECT project has three major components: (1) a week-long "innovation institute" designed to expose participants to rationale and techniques for implementing active learning, forge collaboration between new adopters, and provide planning time; (2) a one-day peer observation training, wherein participants examine, refine, and practice applying a protocol focused on student-centered teaching (via a customizable rubric); (3) monthly lunch gatherings to discuss teaching practice; and (4) an ongoing peer observation cohort consisting of both participants and project leaders, intended to provide continuing support for adopters by fostering reflection on participants' teaching through conversation and shared experience. An overview of the REFLECT project components and how they align with the Henderson et al. (2011) findings is shown in Figure 1.

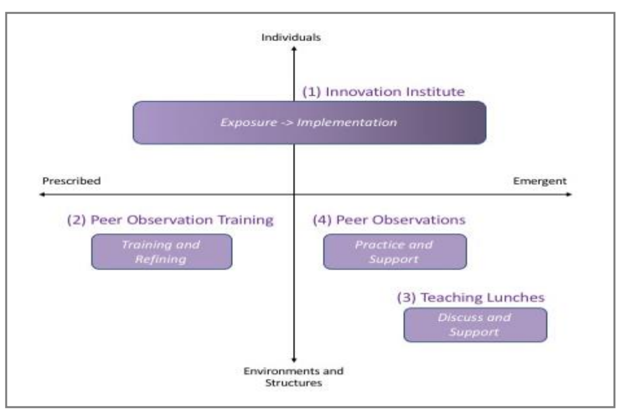

Figure 1: Summary of REFLECT project components mapped to Henderson et al. (2011) framework.

The observational protocol includes three components: pre-observation discussion prompts, a customizable rubric for observation, and a post-observation discussion and reflection. The pre- and post-observation meetings are intended to build trust, establish instructional context, and provide formative feedback (post-observation) from the observer's perspective regarding topics of the observee's choosing. The rubric is designed to provide both guidance for the observer and individualization for the observee. For example, the observee is asked to select one dimension of practice they would like the observer to focus on during the observed class (such as responding to student thinking, use of technology, goal-oriented instruction, or others). The observee then reflects on where various aspects of their current practice fall within the rubric, which in turn provides aspirational examples for advanced practice without implying judgment or inviting summative external evaluation. While the rubric targets specific components of effective instruction, any new teaching practices implemented are determined primarily via self-reflection and cohort feedback. As such, specific changes being adopted by instructors are emergent, rather than prescribed, and can align more effectively with participants' beliefs. Further, beliefs are made explicit and then examined in the pre-and post-observation discussions. Since these conversations are necessarily conducted by instructors immersed in the ambient teaching culture on campus, they imply an understanding of the broader institutional context. Thus, used with other elements in the CACAO framework, the peer observation protocol helps achieve balance between the emphases on individual and community, and avoids the main pitfalls of the unsuccessful efforts identified by Henderson and colleagues (2011) (namely, disseminating specific pedagogical materials and enacting top-down policies for change). Furthermore, this model allows change agents to incorporate the three dimensions evidenced as necessary for success: the proposed changes align with participants' beliefs (or seek to change them via self-reflection and community conversation), involve ongoing supports (a year or more of collaborative peer observation), and are consistent with the broader institutional context (in which reflective and innovative teaching are celebrated).

Conclusions

Peer observation is demonstrably effective for increasing self-reflection and promoting *individual* instructional change. In this theoretical paper, we propose peer observation as a powerful tool that will enhance an existing model (CACAO) for systemic *institutional* change by helping it address the dimensions common to successful change efforts identified in Henderson et al. (2011). To date, it appears that this particular combination of a CACAO-based program for organizational change with a formal peer observation framework is untested, and thus represents a new theoretical contribution. We believe it offers a promising direction for change agents who wish to promote instructional change at scale, particularly in cases where the institutional context is similar to that in the REFLECT project. Moreover, by tying work already being done by the RUME community (in developing research-based curricular materials, and examining what supports are needed to help instructors reshape their teaching practice) to further evidence about how to achieve institutional change, this offers compelling invitations for RUME researchers who wish to accelerate the uptake of student-centered practices.

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Providing Undergraduates an Authentic Perspective on Mathematical Meaning-making: A focus on Mathematical Text Types

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A disciplinary literacy perspective suggests that the goal of instruction in any discipline is to apprentice students into increasing participation in the disciplinary community. In this paper we explore four distinct types of mathematical texts and the critical role each plays in mathematical meaning-making. We argue that understanding the nature and uses of mathematical text types moves undergraduate students closer to the goal of approximating/engaging in mathematical practices, resulting in greater access to powerful mathematics.

Keywords: mathematical practices, disciplinary literacy, reasoning and proof, algebra and algebraic thinking

Mathematics with Purpose

When mathematics is used in the world it is always *for a purpose*. An engineer writes mathematical text in order to make structures more resilient to the elements (Grayson, Pang, & Schiff, 2012). A computer scientist writes mathematical text in order to predict the next unrest event (Dopson, Lowery, and Joshi, 2014), while an applied mathematician writes mathematical text in order to create better tools for image compression (Roach, 2010). Therefore, understanding the nature and uses of mathematical text types moves undergraduate students closer to the goal of approximating/engaging in mathematical practices that make use of text in varied ways.

A Disciplinary Literacy Perspective on Uses of Mathematical Texts

A disciplinary literacy perspective suggests that the goal of instruction in any discipline is to apprentice students into increasing participation in the disciplinary community. That is, to engage students in the practices of those who generate, critique and disseminate knowledge in a given field. As Gabriel & Wenz (2017) note, members of a disciplinary community utilize "agreed-upon conventions that guide the production, communication, and critique of disciplinary knowledge. The central goal of disciplinary literacy instruction is to help adolescents develop 'insider status' in these communities." We began our inquiry by working to uncover the nature of texts used in mathematics by interviewing six pure and applied mathematicians across the US to explore the ways in which members of the field of mathematics orient to and engage with texts.

It should be noted that the notion of mathematical text types is distinct from multiple representations, in that utilizing multiple representations is an essentially pedagogical technique in which showing something multiple ways is intended to promote student understanding. In contrast, within the discipline of mathematics, one would usually not create multiple representations of a single idea without a rationale. Instead, each mathematical text type is utilized with a specific mathematical purpose in mind.

Mathematical Text Types

Within disciplinary literacy circles, the phrase "mathematical text" often conjures up one of two visions: a math textbook (Feng and Schleppegrell, 2010; Shanahan & Shanahan, 2008) or a mathematical proof (Moje, 2007; Moje, 2008). With respect to the former, we agree with Fang and Coatoam (2013) that "school subjects are disciplinary discourses recontextualized for educational purposes" (p. 628). This is not to discount the important and thoughtful pedagogical work that goes into creating a school textbook. Our argument is that the texts students interact with most often are not necessarily representative of text types used in the doing of mathematics outside of classroom settings (applied or pure mathematics). As a pedagogical text, its main purpose is to instruct, and its main audience is outsiders to the discipline's community. This is analogous to the way in which biology students study frogs. When a frog is recontextualized for educational purposes, the resulting corpse is more easily studied, yet lacks much of its inherent frog-ness (hopping, eating flies, croaking, etc.) that made it of interest in the first place. In a mathematics textbook, mathematical ideas are explained and demonstrated, but perhaps not communicated in ways that are authentic to the discipline's everyday work of generating, critiquing and sharing knowledge.

Mathematical proof, on the other hand, is an example of an authentic mathematical text written to convince the reader of a mathematical claim (existence proof), or explain why that claim is true (constructive proof). In fact, this type of text is so synonymous with the discipline of mathematics that some literacy researchers view it as *the only* type of mathematical text (Moje, 2007; Moje, 2008). In our view, to limit mathematical text to only proof text would be to further privilege pure mathematics over applied fields of mathematics, fields which rely much more heavily on the other three text types (algebraic/symbolic, algorithmic, and visual).

In the following paragraphs, we briefly introduce the other text types, providing not only a description of each, but also supporting evidence for the text types from a diverse set of fields including linguistics (O'Halloran, 2005, 2015; Pimm, 1987), literacy (Draper and Broomhead, 2010; Feng and Schleppegrell, 2010; Moje, 2007; Shanahan & Shanahan, 2008), history of mathematics (Cajori, 1993; Maur, 2014), and mathematics education (Kaput, Blanton, and Moreno-Armella, 2008). For each text type we highlight the purpose and the text features which are used to accomplish that purpose.

Technologically Driven, Algebraic/Symbolic Text

The purposes of algebraic/symbolic texts are to *generalize* and *condense* (see Kaput, Blanton, and Moreno-Armella, 2008). The key feature of algebraic/symbolic text is specialized notation (Pimm, 2015).

Algebraic/symbolic texts have developed as a natural outgrowth of mathematical work, as necessary tools for mathematical meaning-making, especially when it comes to increased levels of abstraction (for full descriptions of the history of mathematical symbols, see Cajori (1993) or the more recent and less terse work of Mazur (2014)). Since the early Renaissance, developments in mathematics are inextricably linked to developments in mathematical writing and associated technologies. For example, algebra can be thought to have developed in three stages: rhetorical, syncopated, and symbolic (O'Halloran, 2005). Rhetorical algebra is mathematics in which unknown quantities are referred to using words instead of symbols. As the printing press increased access to mathematical texts, mathematicians began to develop new arithmetic algorithms, while simultaneously standardizing mathematical procedures and symbols. This would set mathematics on a path to syncopated algebra (a combination of words and symbols), and finally symbolic algebra, with the first algebra text *Summa de Arithmetica* printed in 1494. (O'Halloran, 2005). By the time of Descartes, symbols were able to "liberate algebra from the informality of the word" (Mazur, 2014, p.xvii).

Notice how algebraic text allows us to treat complex relationships as a single object (or collection of objects). In order to work with algebraic text in any meaningful way, we must be able to unpack that complexity at will, attending to aspects of form relevant to the mathematical task at hand. Undergraduate students should not be expected to read mathematical text in this way as a simple byproduct of engaging in mathematics (Ferrari, 2004). The cultivation of such disciplinary habits of mind requires explicit instruction. Moreover, attending to the historical development of symbolic texts is important since "causes of the success or failure of past notations may enable us to predict with greater certainty the fate of new symbols which may seem to be required, as the subject gains further development" (Cajori, 1993, p.196).

Mathematical Processes, Algorithmic Texts

The purpose of algorithmic texts is to provide access to problems that have *no known analytic solutions* or are *too large in scale* for hand computation (e.g., large datasets or a large number of cases to consider). The key features of algorithmic texts are control structures.

Along with symbolic text, algorithmic text is likely one of the two oldest text types, occurring anywhere that a person needed to perform a mathematical procedure repeatedly. While today we often associate algorithmic text with computer code, computers are a sufficient but not necessary condition for the use of these texts. For example, the classic description of how to approximate the square root of a number by repeatedly averaging successive guesses (Joseph, 2010) is an algorithmic text that can be read and implemented on paper. Entire fields of mathematics now rely heavily on algorithmic text, including numerical analysis, numerical linear algebra, discrete mathematics, and computational algebraic topology to name a few.

From Cannon Balls to Pendulums, Visual Text

The purposes of visual text are to *highlight relationships* and *appeal to mathematical intuition*. The key features of visual text are static aspects of a functional relationship such as domain, range, maxima, or minima, or more dynamic aspects, including end behavior or average rate of change.

Visual texts "enable mathematicians to represent the linguistically and symbolically encoded information in ways that are tangible to the human perceptual sense" (Fang, 2012, p.26). Historically, visual text has been spurred on by the advent of new technologies. Initially, advances in printing allowed for diagrams to be included in mathematical texts. The first of such texts in western mathematics included diagrams as *additions to* the mathematics being discussed (O'Halloran, 2005). But, in the hands of Newton and Leibniz, visual text *became the mathematics.* For example, the method of integration by parts undergraduates learn in second-semester calculus is based on visual text and an accompanying geometric proof which relates the distance between an axis and the geometric center of a figure (i.e., the moment of the figure) (Suzuki, 2002). In fact, the calculus of Newton and Leibniz is largely the study of curves and the various types of change they encompass (including slope of the tangent line to a curve and area under a curve). More recently, since the 1980s, computers have allowed for the further integration of visual text into mathematical work, both inside and outside of the classroom. Notice that, by our definition, everything from graphs of functions, to tables of values, to interactive diagrams in DESMOS all count as visual texts. For the current discussion we focus on graphs of functional relationships due to their prominence within the undergraduate curriculum.

Feedback from disciplinary insiders: Initial results from a Delphi study. In order to further refine our understanding of the nature and use of these mathematical text types, we are currently undertaking a modified Delphi study (Green, 2014) of research mathematicians from several universities, representing multiple mathematical subdisciplines. While a complete discussion is beyond the scope of this paper, we present preliminary results from the second round of our modified Delphi, a focus group of 6 current research mathematicians (2 pure and 4 applied), representing 4 different colleges and universities from the southeast, and 5 subdisciplines of mathematics. In the following paragraphs we highlight specific facets of the text types that these researchers were able to highlight, providing us with new insights into how these types of texts are created and negotiated within the discipline of mathematics.

Visual Text: "...Not just a Cheap Cartoon Version of Proof"

Defining the role and legitimacy of visual text as a means of mathematical meaning making has been a long running debate within mathematical circles. Davis (1974) provides one of the more ardent defenses of visual text, with his notion of "mathematical theorems of perceived type":

> The analytic program [algebraic/symbolic text], then, is a prosthetic device, acting as a surrogate for the 'real thing.' The unit circle as perceived by the eye and acted on by the brain is a very different thing from the symbol string $x^2 + y^2 = 1$...The visual circle is the carrier of an unlimited number of theorems which are instantly perceived. (p. 119)

Algebraic text and its inherent malleability allow for the proof of mathematical results in ways that are not as readily possible with the often static visual text (O'Halloran, 2005). But for applied mathematicians who often create graphs *as* a mathematical result, visual text can play a much more central role in mathematical meaning making, a perspective which naturally bleeds over into their classroom teaching.

> *Participant*: In my world of applied math I try to get students to realize that that graph might be the answer. The whole solution text *is* that visualization. That's not just a cheap cartoon version of a proof. It is itself in fact a mathematical object. Now I don't know if I have convinced a lot of people of that…Everything else is secondary to proof, I know that that's the way it is, but I don't think I would teach like that. I don't teach like that.

This mathematician is aware of "the way it is" (visual text as secondary within the wider disciplinary community) and chooses to push back against prevailing disciplinary norms by providing an alternate perspective to his students. The separation between texts used by and for students and those used by mathematicians was a frequent theme across the interview. This signals a distance between the textual practices of math students and those used by insiders in the disciplinary community. Advocacy for the use of visual text with students was also echoed by another mathematician:

> *Participant*: Visual text is by far the most powerful. It's what I use in my classes. I use it instead of proof, and it seemed ok, but there is a part of me that's thinking I'm only representing an example, I can't draw all graphs that are decreasing or increasing. But, I can show them one graph that is decreasing, and sure enough, the tangents are all negative. I mean, it feels like a little bit of a cheat, but by far the students get it. Whereas,

you do the algebraic/symbolic text, they might not get it, or it might not be as meaningful to them.

The notion of visual text as pedagogically powerful and meaningful was a recurring theme. Not only did mathematicians have this view when working with their students, students had this view when thinking forward to their future interactions in the workplace.

> *Participant*: I had a conversation about that point just the other day with my students in an applied math class. We're doing Fourier series in two variables. You solve a problem and there's the Fourier series. I asked them, and they were essentially all engineers, would you present this to your boss if your boss gave you that problem? Everybody said "Well God no! The boss would fire me!" Well what would you do? And the thoughtful students said "Well I'd show the surface. You know, I'd show the picture of the surface to the boss. That would give him the information he wants. That's the sort of text, it's a visual sort of text, not an algebraic text."

Again, this underscores the importance of purposeful choice of mathematical text type. When you want to convey mathematics with meaning, either from expert to novice or across disciplinary boundaries, you'll likely choose visual text. Understanding how and when that communication is necessary is part of the work of engaging in mathematics purposefully or using it to do work outside of a school setting. Moreover, understanding the implications of certain choices for different audiences can contribute both to students' understanding of mathematical texts types and their decisions to make use of one or another. The quotes above also demonstrate the orientation towards different text types within the discipline - with an implied hierarchy of purity and legitimacy on the one hand, and a hierarchy of utility and accessibility on the other. Unlike other disciplines where the use of a certain text type and purpose for writing may be more neutral or may be defined entirely in relation to a certain sub-discipline (e.g., nonfiction in journalism), decisions about text type in mathematics convey something about the identity and purpose of the creator and their intended audience.

Algorithmic Text: "...What is a Number?"

Initially, algorithmic text made the list of text types as a clear rebuttal of the notion that proof text was the only type of mathematical text. During the focus group interview our participants discussed the ubiquity of algorithmic text across all fields of mathematics. For example, one participant underscored the strong mathematical relationships that tie together numbers and algorithms.

> *Participant*: One thing we deal with in numerical analysis is "what is a number?" I mean, do real numbers even exist, ones that can't be represented in a computer, that can't be constructed? So even like $\sqrt{2}$. That is a nonterminating decimal. Now it is the diagonal of a square with sides one by one. But what *is* $\sqrt{2}$? …one argument we do talk about in my field is numbers are things that can be represented either as the conclusion of an algorithm, like Newton's method, or numbers that are the limit point of something that can be described.

These comments helped the authors to better understand algorithmic text as it is viewed within a community of mathematicians. To mathematicians, algorithmic texts are not simply code on a computer, but also include any well-defined process that terminates. As a result, algorithmic texts are an essential part of both pure and applied mathematics that facilitate certain kinds of mathematical processes. When students are introduced to these mathematical texts, the mathematical purpose (not just the communicative features) should be conveyed, so that students learn to make use of mathematical text *in the work of mathematics* rather than viewing them only as a way to capture or record mathematics after the fact. In other words, knowing what such texts can be used to do in the processes of mathematical thinking interrupts the idea that mathematical texts are created as an end product for others - an idea that is reinforced if the main exposure to mathematical texts is within a textbook. Students should be aware that mathematical texts have a key role in facilitating the process of doing of mathematics so that they can engage with them in this way.

Proof Text: **"...Mathematics is so Much Bigger Than That Now"**

Proof text has traditionally been viewed as the pinnacle of mathematical rigor and achievement, by both disciplinary insiders and the layman alike. Such a perspective can often be at odds with how mathematicians go about their daily work, especially for those in more applied fields.

> *Participant*: It is tough. I think proof has a privileged status in mathematics. As a numerical analyst myself…there are things you can't prove. The best wavelet compression for a fingerprint? It depends on the fingerprint! So, some things you cannot prove. But there is definitely a bias towards things that are not followed up with a proof or knowing when this formula works, that's important in mathematics.

Such bias has direct implications not only for practicing mathematicians, but also for the next generation of scholars.

> *Participant*: I just advised a PhD student and there wasn't a single proof in his dissertation, right? And those papers are getting published, applied papers about models and results, numerical algorithms, optimization, parameter estimation…so, I think it's a much more broad spectrum of things that are published and acceptable.

This participant is making the case that the mathematics she and her students create, mathematics without proof text, ought to be viewed as disciplinarily "acceptable," where here acceptable is synonymous with mathematical rigor. The argument would be that as the discipline of mathematics changes, so too must the nature and role mathematical texts in mathematical meaning making. This is nowhere more apparent than in the increasingly interdisciplinary nature of mathematics itself.

> *Participant*: And all the connections that mathematics has made, I think biology has been a big driver in terms of change. All the sciences and social sciences have expanded what is meaningful, right? A biologist doesn't care about a proof, necessarily, right?

While the discipline may be ever-changing, the perspective on the discipline that is enacted in the K-12 classroom is highly resistant to change. As a result, proof text can act as a gatekeeper to future mathematical engagement, providing an "inauthentic" perspective on what it means to do mathematics.

> *Participant*: I would say an undergraduate major doesn't really fully understand the concept of proof and able to produce at any kind of maximal level, maybe junior or senior and I think there are students who not until they are a graduate student. To be pushing that into the high school, and holding that up as a standard of achievement? I mean most students are not mathematically mature, and to say to them "you are not going to be a mathematician," or whatever you are saying, to a 10th grader who can't do proofs in geometry? If you say: this is what a mathematician does. You can't do it. Therefore, this path is not available to you? Inauthentic! It's so narrowly defining mathematics that you are eliminating the option for so many people. Because mathematics is so much bigger than that now.

Thus, proof writing is only one of several disciplinary practices used by those who engage with mathematics. Moreover, it is a sufficient, but not necessary practice for full engagement with the discipline. In contrast, disciplinary literacy practices play a fundamental role in the work of today's mathematicians (both applied and pure), and are a necessary for any real type of mathematical meaning making. As a result, they represent a high leverage opportunity for teachers.

Conclusion

Discussion with our participants made us aware of notions of access to powerful mathematics that we would not have previously associated with mathematical text types. This further underscores the multifaceted role that mathematical text types play regarding not only the enactment of mathematical practices, but also issues of equity, including "students' development of a sense of efficacy (empowerment) in mathematics together with the desire and capability to learn more about mathematics when the opportunity arises" (Cobb and Hodge, 2010, p.181).

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Points of Connection to Secondary Teaching in Undergraduate Mathematics Courses

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Prospective secondary mathematics teachers frequently take as many (or more) mathematics courses from a mathematics department as they do methods courses from an education department. Sadly, however, prospective secondary teachers frequently view their mathematical experiences in such courses as unrelated to their future teaching (e.g., Zazkis & Leikin, 2010). Yet there is some optimism that having instructors alter their instructional approaches in such mathematics courses can enhance such experiences to be a positive part of their preparation for teaching. This theoretical report elaborates on four points of connection to secondary teaching that can be made in undergraduate mathematics courses, illustrated via examples from abstract algebra, and organized along a spectrum of intended implications on secondary teaching. The purpose is to provide a theoretical bridging between instructional approaches in undergraduate mathematics and aspects of secondary teaching practice.

Keywords: Secondary teacher education, Content knowledge, Teacher's mathematical education

Over the past century, mathematicians and mathematics educators have weighed in about the preparation of secondary teachers. On the one hand, secondary mathematics teachers need a sufficiently deep and robust knowledge of mathematics to teach secondary content; on the other hand, (strictly) mathematical ideas are not the only aspect for which secondary teachers need preparation. Teaching is a notoriously complex profession; teacher education, then, is that much more complex.

In this theoretical paper, we explore some of these issues as they relate particularly to instruction in undergraduate mathematics courses. We do so because secondary mathematics teachers are frequently required to be mathematics majors; the main point being that a significant portion of their teacher preparation program consists of content courses in a mathematics department. These include courses such as abstract algebra – which is the primary course we draw on in our examples in this report. This paper aims to explore the following questions: What are ways in which instruction in undergraduate mathematics courses such as abstract algebra, historically, have made connections to secondary teaching? What are other ways in which instruction in undergraduate mathematics courses such as abstract algebra might make connections to secondary teaching? We consider the first question by synthesizing extant literature; we explore the second through the use of intentionally selected examples from current teacher education efforts.

Two Connections to Secondary Teaching in Mathematics Courses in Extant Literature

In exploring and synthesizing extant literature, we attempt to make clear from the outset one of our assumptions. Namely, we aimed to identify common ways in which undergraduate mathematics course instructors have attempted – explicitly or implicitly – to make their content relevant to secondary teacher preparation. As Wasserman (2018) described: to make their *nonlocal* content relevant not only to the *local secondary mathematics* but (in some way) to the *teaching* of *local secondary mathematics*. That is, what we report on below could be conceived of as potential actions an undergraduate mathematics instructor might take that could serve as a point of connection to teaching secondary mathematics. Essentially, the two points of connection described below – content connections and modeled instruction connections – stem from broad syntheses of literature from secondary teacher education and from research in undergraduate mathematics education studies.

Content Connections

One of the most influential, and innovative, scholars to consider content courses for secondary teachers was Felix Klein. Amongst other things, Klein (1932) pointed out what he described as a "double discontinuity" for secondary teachers. The first discontinuity was that the study of university mathematics did not develop from or suggest the school mathematics that students (i.e., future teachers) knew. That is, the teaching of, say abstract algebra, did not draw on or remotely resemble the algebra they had learned previously, which made learning it more difficult. Klein's second discontinuity was a disconnect for these future teachers in returning back to school mathematics, where the university mathematics appeared unrelated to the tasks of teaching school mathematics. That is, the abstract algebra they learned did not seem useful for teaching algebra to secondary students. Despite the fact that his observation goes back about 100 years, it still rings true today. Undergraduate students, including prospective teachers, often find their experiences in university mathematics courses difficult (e.g., Dubinksy, Dautermann, Leron, & Zazkis, 1994), and secondary teachers find them disconnected from their future classroom teaching (e.g., Zazkis & Leikin, 2010). Klein's primary resolution to this dilemma was to make explicit the mathematical connections that existed between school and university mathematics – an approach he coined as "elementary mathematics from an advanced perspective." Klein's approach – content connections as a point of connections to teaching – is still important today.

The *Mathematical Education of Teachers (I and II)*, more recent reports published by the Conference Board of the Mathematical Sciences (CBMS, 2001; 2012) that outline recommendations for mathematical content and courses to be included in teacher education programs, adopts a similar stance to Klein. They suggest, for example, that "[i]t would be quite useful for prospective teachers to see how $\mathbb{C}$ can be "built" as a quotient of $\mathbb{R}[x]$ and, more generally, how splitting fields for polynomials can be gotten in this way" (CBMS, 2012, p. 59). Mathematicians and secondary teacher educators agree that these mathematical connections are important; textbooks about mathematics for high school teachers (Bremigan, Bremigan, & Lorch, 2011; Sultan & Artzt, 2011; Usiskin et al., 2003) frequently explore such connections between the content of undergraduate mathematics and how it relates to the mathematics studied in secondary school.

The general premise is that studying undergraduate mathematics serves to deepen, and more rigorously confirm, the specific mathematical ideas secondary teachers will teach. In terms of teaching, though, the intended implication is that secondary mathematics teachers will have a normatively correct understanding of secondary mathematics topics and be able to convey these concepts accurately to their students. Such development is particularly important in mathematics writ large, given that mathematical ideas explored earlier in school are often re-explored later with increasing mathematical sophistication. That is, mathematical ideas build on themselves. Secondary teachers need to do a sufficiently good job teaching school mathematics to secondary students since, in undergraduate mathematics, these ideas will continue to be developed.

Coherent concept development and points of mathematical connection, at least ostensibly, serve a specific purpose in teacher education – a point of connection to secondary teaching.

Modeled Instruction Connections

Perhaps less explicit in the literature, but no less powerful, is a point of connection that might be described as modeled instruction. Undergraduate mathematics instructors have the opportunity to take advantage of the age-old adage, "we teach how we were taught," by teaching in ways they would want their students to teach secondary mathematics. This is likely (and perhaps rightly) not at the fore of an undergraduate mathematics instructor's mind when teaching; but it nonetheless provides another point of connection to teaching. Especially given the observation in teacher education (e.g., Brown & Borko, 1992) that "methods" courses are often insufficient to shift a prospective teacher's future practice to more reform-oriented instruction; many revert to teaching in ways they themselves were taught. In the literature, we see much of this notion of modeled instruction of a point of connection to teaching, implicitly or explicitly, as part of the work of the RUME community. In this literature base, scholars have studied and redesigned undergraduate courses to be more in accord with how students learn and develop mathematical ideas, which aligns with more inquiry- and reform-oriented mathematical instruction.

Frequently steered by the notion of guided reinvention from the instructional design theory of Realistic Mathematics Education (RME) (e.g., Gravemeijer & Doorman, 1999), the RUME community has provided many examples of, and resources for, instruction in undergraduate mathematics courses that align with reform-oriented instruction. (In abstract algebra, see Larsen, et al. (2013); in linear algebra, see Wawro, et al. (2013); in calculus, see Oehrtman, et al. (2014); etc.) Often, by building on student thinking, these instructional approaches help alleviate aspects of the first discontinuity Klein observed. But also, as argued by Cook (in press), such instructional approaches, which build on student thinking, provide a model of good pedagogical practices for secondary teachers. This portion represents another connection to secondary teaching via modeled instruction. By instructing in particular ways, students learn mathematics in new ways, which potentially shapes the way they believe that mathematics instruction should occur.

Identifying Two Other Connections to Secondary Teaching in Mathematics Courses

Essentially, the literature has pointed out two sides in what might be regarded as a spectrum of connections (Figure 1a). On one side are connections that are "mathematical" in nature – content connections which primarily aim to influence the mathematical aspects of one's instruction. On the other side are those that are "pedagogical" in nature – modeled instruction connections which primarily aim to influence the pedagogical aspects of one's instruction. Indeed, mathematics and pedagogy are two important, perhaps obvious, lenses through which to view mathematics teaching. The purpose in placing these two on different sides of one spectrum is not to claim they are disjoint, or even easily separable; rather, it is to highlight that content connections and modeled instruction connections – two "means" by which an undergraduate mathematics instructor might make a point of connection to teaching – have *different* "ends" when it comes to teaching, and also to situate the two other connections discussed in this paper as being *between* these two sides of the spectrum – that is, as having intended "ends" that aim to have influence partly on mathematics and partly on pedagogy. Indeed, part of the premise of this paper is that elaborating on different points of connection to teaching that could be made in undergraduate mathematics instruction is good because having an arsenal of "means" (not just two – or three or four, for that matter, but many) and a variety of "ends" both expands and gives substance to the complexity of mathematics teaching.

Figure 1a. Connections to secondary teaching in mathematics courses in extant literature

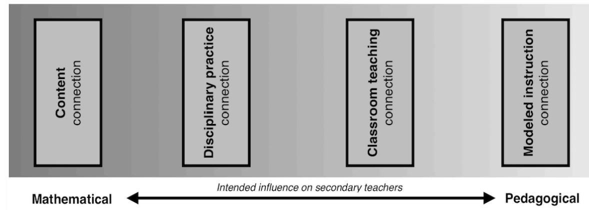

Figure 1b. Four connections to secondary teaching in mathematics courses along a spectrum of implication

In recent work, Wasserman (in press) elaborated on two other kinds of connections to secondary teaching that might exist – ones that fill in areas on the spectrum above, serving part mathematical and part pedagogical ways of connecting to secondary teaching (Figure 1b). Now, the purpose in elaborating on these other two kinds of connections is not to speculate some as better than others, but rather to add to the list of different points of connections to secondary teaching and to organize them along a spectrum of intended influence to be more explicit about their role in connection to teaching. In what follows, we elaborate on these two other kinds of connection to secondary teaching, using examples from abstract algebra: i) disciplinary practice connections; and ii) classroom teaching connections.

Disciplinary Practice Connections

By a *disciplinary practice connection* being the point of connection to secondary teaching, Wasserman (in press) meant that the same kind of disciplinary practice that one engages in while studying undergraduate mathematics can also be engaged in while studying secondary mathematics. Such practices might include defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, et al., 2005), or what Cuoco et al. (1996) termed mathematical habits of mind. Indeed, the processes that one engages in while "doing" undergraduate mathematics are related to some of the important mathematical practices that have been identified and stated as explicit learning goals for school mathematics – e.g., NCTM's (2000) process standards, or CCSSM's (2010) mathematical practice standards.

Hence, these kinds of connections serve a dual purpose. First, they serve a mathematical purpose. By becoming better "doers" of mathematics, secondary teachers have a better grasp on the discipline itself – i.e., the epistemological nature of mathematics, etc. Second, though, these connections also serve a pedagogical purpose. That is, by learning more about what doing mathematics means, there is a hope that secondary teacher's pedagogical choices will, in fact, engage their own students in these forms of thinking and doing. Thus, while these may be primarily about an improved mathematical sensibility (more on the mathematical end of the spectrum) there is also an embedded pedagogical implication (at least partially toward the pedagogical end of the spectrum). Indeed, one of the three perspectives of the Mathematical Understanding for Secondary Teaching (MUST) framework (Heid & Wilson, 2015) is mathematical activity; that how one is engaged in doing mathematics can be a point of connection to the practice of teaching mathematics. The MUST framework highlights mathematical noticing, reasoning, and creating as activities whereby one's experience in undergraduate mathematics courses can parallel the work of teaching school mathematics (Zbiek & Heid, in press).

An example of disciplinary practice connections from an abstract algebra course. In a recent study, Baldinger (in press) used a multiple case study approach to describe four pre-service secondary teachers' learning of mathematical practices from an abstract algebra course.

The abstract algebra course was designed specifically for an audience of secondary teachers, and, although there were certainly content connections (e.g., fundamental theorem of algebra) and modeled practice connections (e.g., problem solving), one of the primary instructional approaches in the course revolved around disciplinary practice connections. That is, the instructor was explicit in describing disciplinary practices, such as, "That's one teaching tactic I have for a challenging proof. I try to come up with a simple example where all the reasoning for the general case is right there. A generic example... A generic example illustrates a line of reasoning that generalizes." Indeed, students were provided intentional opportunities to practice using such generic examples as they solved problems during the course.

Using a pre-post analysis from task-based interviews, Baldinger (in press) found that the pre-service secondary teachers had become more expert in engaging in mathematical practices. That is, when given a novel mathematical problem, the mathematical activities and lines of reasoning they engaged in better reflected such disciplinary practices after having taken the abstract algebra course. Furthermore, the specific disciplinary practices they engaged in reflected those that the instructor had made very explicit during the course. Although one would hope that taking undergraduate mathematics courses would improve students' mathematical activities, students often emerge unable to engage in core practices such as proving (e.g., Weber, 2001). In this study, being explicit about disciplinary practices, with opportunities to practice using them in class, seemed to help the pre-service teachers incorporate such practices into their own mathematical activity. Additionally, three of the four participants also reported that they saw specific connections between the course and their own (future) teaching. The connections they described primarily suggested that they intended to incorporate such disciplinary practices into their own instruction.

Classroom Teaching Connections

In terms of a *classroom teaching connection* being the point of connection to secondary teaching, Wasserman (in press) meant that some connection regarding the content of undergraduate mathematics was being applied to a specific secondary teaching situation. That is, the undergraduate mathematics served as a means to motivate particular and specific kinds of pedagogical actions in the classroom. For example, Wasserman and Weber (2017) explored how the study of proofs of the algebraic limit theorems can be applied to situations when secondary teachers interact with secondary students about rounding and operating on rounded values.

The primary implication in these kinds of connections is about shaping a teacher's pedagogical response to a specific teaching situation – which may be about designing problems with particular characteristics, about responding to students, about sequencing activities, etc. However, such situations are also mathematical, in the sense that the intended point to exploring the teaching situation also includes applying and incorporating mathematical (and not strictly pedagogical) ideas. That is, one's pedagogical response to a situation is explicitly informed by some mathematical idea or mathematical analysis.

An example of classroom teaching connections related to abstract algebra content. In a recent paper, Zazkis and Marmur (in press) elaborated on several instructional situations in secondary mathematics where teachers' knowledge of group theory could serve to shape teaching – namely, their responses to situations of contingency.

School mathematics requires that students understand different sets of numbers (i.e., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) as well as basic operations on those numbers, (i.e., $+, - \times, \div$). In particular, one goal of school mathematics is to help students understand that, as the sets of numbers "expand," the ways in which we conceptualize the operations might also need to expand. That is, while multiplication on the natural numbers can be viewed as "X groups of Y," this idea makes less sense with rational, real, and complex numbers. So although students might "know" multiplication, their notion of multiplication must also adapt somewhat to take into account the kinds of numbers under consideration. Responding to student questions about "What does $\frac{1}{2} \times \frac{3}{5}$ mean?" or "What does $(2 + 3i) \div (5 + i)$ mean?" takes paying attention to, and pointing out, the differences in meaning of an operation depending on the numbers involved.

Here, an experience with programming may be a useful source for understanding the pertinence of group theory. In MAPLE, the command *isprime* tests for whether the input is prime. Yet, in an earlier version of MAPLE, the command *isprime*(14/2) returned "false" (i.e., not prime) – a strange conclusion indeed. It turns out, *isprime* was defined for integer inputs and division was defined for rational inputs. Individually, both of these are sensible: all primes belong to the integers; division makes the most sense with rational numbers because it then maintains the property of closure – dividing rational numbers yields rational numbers. Yet, in combination, MAPLE took 14/2 to mean the rational number 7.0, and not the integer 7 – and it reported the rational number 7.0 to be not prime since it was not an integer. (This bug has since been corrected in MAPLE.) Experiencing this sort of dissonance from a programming environment, and as connected to ideas in abstract algebra, can help teachers develop the ability to attend to ideas of mathematical importance in situations of contingency – e.g., recognizing the importance of different number sets in conceptualizing multiplication with rational numbers or pointing out the importance of closure in defining complex division.

Discussion

The aim of this theoretical report is to provide some initial organizational framing to different points of connection to secondary teaching – especially ones in which undergraduate mathematics instructors might incorporate into their own instruction. We see this as contributing in two aspects. First, although extant literature in the field has explicitly emphasized content connections and, more implicitly, underscored modeled instruction connections, we have identified and exemplified two others: disciplinary practice connections and classroom teaching connections. Second, organizing these four points of connection along a spectrum helps indicate what kind of influence these might have with respect to prospective teachers' instruction. In particular, they provide an ability to be more explicit about how attempted connections made in undergraduate mathematics course might relate to teaching.

We also offer some insights based on the specific examples used in this report. First, from the disciplinary practice connections example, we see that an instructor's choice to be explicit about disciplinary practices during their instruction, and to give students the opportunity to engage in those disciplinary practices during class, appears to have been critical to helping the teachers in the study become more expert in incorporating such practices into their own mathematical thinking and problem solving. We regard being explicit as an important consideration for disciplinary practice connections: without such naming of particular activities, students may miss the generality of a disciplinary practice and the ways in which it gets enacted across a multitude of settings. Second, from the classroom teaching connections example, we see that problems which intentionally mix things may be particularly productive for learning. The cognitive conflict that stemmed from *isprime*(14/2) being "false" required interrogating issues of definition and of closure; not only might we use similar strategies in helping secondary teachers develop additional mathematical awareness in situations of teaching, but we might also discuss pedagogical strategies that leverage cognitive conflict in similar ways to help students themselves attend to (and appreciate) such mathematical nuances and complexities.

Lastly, we discuss some limitations and further reflections. In particular, our theoretical framing has paid particular attention to "mathematical" and "pedagogical" aspects of secondary instruction. This is some ways is a natural starting point – mathematics and pedagogy are intrinsically important. However, there are certainly other important areas of instruction that merit consideration as well – including affective implications, belief systems, issues of equity, etc. How instruction in undergraduate mathematics courses can intentionally make points of connection to other aspects of instruction is an interesting question, worthy of further consideration. In addition, it may be that the four points of connection described in this report also inherently attend to some of these other areas of instruction as well. Regardless, identifying and leveraging theory that merges instructional choices that can be made in the teaching of undergraduate mathematics, with the kinds of implications for secondary teaching that are related to such choices, is an important step in helping to make secondary teachers' experiences in undergraduate mathematics a more meaningful component of their teacher preparation and development process.

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Investigating Student Understanding of Self-Explanation Training to Improve Proof Comprehension

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Self-explanation is a reading strategy in which readers explain a text to themselves as they encounter new information. Hodds, Alcock, and Inglis (2014) reported proof comprehension gains on students who had been trained to self-explain, when compared to students who had not received this training. We report a multiple case study in which we interviewed undergraduate students in introductory and advanced proof-based courses, to examine their understanding of self-explanation training and their use of this strategy throughout one semester. Preliminary findings indicate that self-explanation made students examine each line of the proof more deliberately, because they knew they would have to hold themselves accountable for figuring out how to explain each line of the proof. However, some students reported almost never using the technique, either because they prioritized the proof techniques demonstrated by their professors, or because they only felt the need to do so with particularly difficult proofs.

Keywords: Self-Explanation, Proof Comprehension, Proof Reading

Introduction

Mathematicians and mathematics educators have stressed that comprehending mathematics text is fundamentally different to comprehending traditional text, and that we need to address the reading of mathematics in undergraduate, proof-based mathematics courses if we want to improve students' understanding of the mathematics texts they are asked to read (Cowen, 1991; Fuentes, 1998; Österholm, 2006). Students have traditionally struggled with proof comprehension in undergraduate math classes. Such difficulties have been documented extensively in the literature and include difficulties attending to the logical structure of mathematical statements (Selden & Selden, 1995), and distinguishing between valid and invalid arguments within proofs (Alcock & Weber, 2005). In response to such difficulties, researchers have proposed various strategies. In this study, we focus on the strategy of self-explanation.

Literature Review

Students spend the majority of their time in undergraduate mathematics classrooms taking notes on lectures in which theorems and their proofs are presented to students by their professors (Fukawa-Connelly, 2012). In turn, professors expect their students to study these proofs outside of class. However, in a survey with 175 mathematics majors and 83 mathematicians, Weber and Mejia-Ramos (2014) found that students had vastly different ideas about the expectations of their proof reading behavior. Most mathematicians expected that students needed to spend more time reading proofs compared to the time expected by students. Additionally, when students did read proofs, they did not report engaging with them in ways that aligned with the reading behaviors of mathematicians that are expected for comprehension (p. 19-20). This implies that there is work to be done on the part of mathematics educators to promote proof reading behaviors that encourage productive reading strategies, such as avoiding attending to surface features in favor of attempting to infer implicit warrants between consecutive lines of proof (Inglis & Alcock, 2012). One option for addressing proof comprehension is to change the format of the proof.

Notable examples of such techniques are Leron's structured proofs (Leron, 1983), Alcock's e-Proofs (Alcock, 2009), and Mason and Pimm's generic proofs (Mason & Pimm, 1984). However, these techniques have been met with little success (Fuller et al., 2015; Roy, 2014; Weber et al., 2012) in terms of gains in student proof comprehension.

Another option for addressing proof comprehension is to change the behavior of the reader. The most prominent example of such a technique in the proof-based literature is called the self-explanation strategy in which readers explain lines of texts to themselves as they encounter new information. It is hypothesized that self-explanation improves comprehension by promoting active integration of new knowledge with existing knowledge, the reevaluation of accuracy and usefulness of mental models, and the coupling of the relationship of actions in a text to overall textual goals (Chi et al., 1989; Chi et al., 1994). The genre of mathematical proof lends itself well to self-explanation due to the importance of logical connectives between lines and the principle-based writing style (Rittle-Johnson & Loehr, 2017). Self-explanation is sometimes accompanied by a training which encourages specific types of explanations and discourages other types of comments. In the case of mathematical proof, preferred explanations are those that promote the integration of prior knowledge with the information in the text, the inferencing of warrants to justify the conclusions drawn in specific lines, and the inferencing of goals and sub-goals of the proof. Self-explanation training discourages non-explanations such as paraphrasing and statements about the reader's affective state ('This is confusing' or 'I get this').

Several studies have promoted the use of self-explanation training (Rittle-Johnson, Loehr, & Durkin, 2017), particularly for participants with low levels of domain knowledge (McNamara & Scott, 1999; McNamara, 2004) because it encourages behaviors that align with the hypothesized benefits of self-explanation. Hodds, Alcock, and Inglis (2014) showed that students who received self-explanation training specific to mathematical proof produced more explanations vs non-explanations and received greater proof comprehension scores when compared to an untrained control group. They also showed, using eye-tracking, that self-explanation training changed students' proof reading behavior. Students who received training spent more time fixating on each line of the proof, and more time focusing on between-line transitions than those who did not. However, these successful students did still produce non-explanations. Thus, although self-explanation training has been promoted in the literature and shown to increase proof comprehension, little is known about the ways in which the training is interpreted by students, how those interpretations impact readers' goals while producing explanations, and what material students retain about the training over time. This study addresses these gaps in the literature by having students describe the ways they used their training to create self-explanations in real time, rank explanations in terms of quality, and describe the features that impact the quality of an explanation. This study provides information about how students consciously use the training to create explanations, and how they interpret information about the types of explanations that theoretically promote understanding and those that should be avoided.

Finally, little is known about the effects of self-explanation training over time. Hodds, Alcock, and Inglis (2014) found that readers retained the benefits of self-explanation training after a few weeks after going through the training only once. Arguably some of the greatest benefits of the training proposed by Hodds et al. (2014) are that it takes up no time on the part of the instructor due to the online format, and that it only takes one 20 minute session of a student's time at home. However, in a meta analysis of self-explanation literature, Rittle-Johnson, Loehr, & Durkin (2017) found a large degree of variability with respect to the longevity of the self-explanation effect. This suggests that more research is needed on the degree of initial scaffolding and the frequency of training required to sustain the self-explanation effect over a long period of time. To date, there are no studies of trained students' self-explanation behavior over the course of an entire semester. Thus, although the students' proof comprehension gains in Hodds et al.'s (2014) study were retained after a couple of weeks from initial training, it is unclear exactly what was their self-explanation behavior. It is possible that students did not consciously self-explain (or that they did it rather poorly compared to right after training), yet retained the benefits of the training in other ways (e.g. through the increased between-line transitions found by Hodds et al.). This study aimed to address this issue by interviewing both novice and advanced students immediately after their self-explanation training at the beginning of a semester (and again at the end of a semester) about their self-explanation behavior and their degree of retention of the training material. This information can help us determine how frequently self-explaining training should be done/discussed throughout introductory and advanced proof-based courses in order to see maximum benefits in student proof comprehension.

Research Questions

The goal of this study was *not* to establish whether self-explanation training is effective and leads to increased proof comprehension. These goals would necessitate an experimental study, and have been addressed by Hodds, Alcock, and Inglis (2014). Instead, the goals of this study were to detail the ways in which self-explanation training is used and understood by participants, and to use those data to generate hypotheses as to the ways in which self-explanation training could be made more effective for different student populations. In particular, the questions motivating this study are: How do novice and advanced students who have received self-explanation training (i) use their training to make decisions when self-explaining a proof (including decisions about the quality of individual self-explanations), and (ii) retain information about their self-explanation training (including how often they report using self-explanation over an entire semester)?

Methods

In order to answer these research questions, we conducted a multiple case study (Bromley, 1986). The descriptive and in-depth nature of these goals necessitate a qualitative interview study, while the desire to address nuances between and within various student populations necessitates a method with students in both introductory and advanced proof-based courses.

Participants

Four students were interviewed at the beginning of the Summer 2018 semester. We were able to bring three back for follow-up interviews. Two students were enrolled in an introductory proof-based course, and two in a real analysis course for which the introductory course is a prerequisite. The real analysis course will be referred to as an advanced course for the sake of clarity. Neither of the researchers were teaching these courses in the Summer 2018 semester.

All four participants were men in their second or third year of study at a four-year institution in the United States. Throughout this report, pseudonyms will be used to discuss each participant. Andrew and Brandon were second-year students enrolled in the introductory course, while Colin and David were third-year students enrolled in the advanced course. Both Colin and David had previously taken the introductory course at the same four-year institution.

The four participants were chosen from a list of students that had expressed interest in the study after it was discussed by the researchers during one class session. In the session, the researchers invited all students to indicate their interest in participating in two paid interviews about mathematical proof reading techniques during the semester.

Procedure

The first interview had four phases. In the first phase, students were asked about their current reading strategies and behaviors when reading proofs. In the second, students completed the online self-explanation training used in Hodds et al. (2014). In the third phase, students self-explained a proof involving concepts used recently in their respective math classes (e.g. students in the introductory course self-explained a proof about rational and irrational numbers). Self-explanations were followed by a series of questions that asked students to describe how they did or did not use their training to produce their self-explanations, how the training did or did not impact the way they read and understood the proof, and the degree to which they thought their explanations were of high quality. In the fourth phase, students were given pre-written self-explanations for Proof B[1] from the Hodds et al. (2014) study. The explanations were written to intentionally focus on specific features of self-explanations that were either promoted or discouraged during the training. For example, one explanation would involve both inferencing of connections between consecutive lines and paraphrasing. Students were asked to comment on the quality of these explanations and the features they believed increased or decreased their quality.

The second interview had three phases. In the first phase, students were asked to describe their current proof reading habits and whether those had changed over the course of the semester. In the second, students described what they remembered about their self-explanation training, how often they had used self-explanation over the course of the semester, and what factors either promoted or inhibited their use of self-explanation. Advanced students were also asked to describe the ways in which the self-explanation training did or did not change their established proof-reading behaviors. Students were reminded that saying ‘I don’t know’ or ‘I don’t remember’ was an acceptable answer. Students were also reminded that their use of the training did not impact the success of the study, so they could be honest in their responses. In the third phase, students self-explained a proof involving concepts used recently in their math classes. Students were asked the same questions about their self-explanations from the first interview.

Interviews were transcribed and we are using thematic analysis to generate claims about their proof reading behaviors over time, and the effectiveness and impact of the self-explanation training. Namely, we describe how students reported using the training to generate explanations, to form their ideas about the desirable and undesirable qualities of self-explanations, and the degree to which those qualities were present in their own explanations.

Preliminary Results

We are in the process of analyzing these data, and briefly discuss two of the themes that have emerged from our analysis.

On Perceived Impact of Self-Explanation Training

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[1] Proof B is a proof of the statement: n is even if and only if $3n^2+8$ is even for n in N.

Every student interviewed indicated that a main effect of self-explanation training on their behavior when reading proofs was that it made them examine each line of the proof more deliberately than they might have before, because they knew they would have to hold themselves accountable for figuring out how to explain each line of the proof. Andrew, for example, said the while explaining,

> I think it's being honest with yourself because it forces you to say 'am I actually learning things, am I actually retaining information in class, am I doing what I have to do?' because it kind of holds yourself accountable.

Andrew found that self-explanation made him more likely to question his own understanding of the proof. David echoed this statement by saying,

> I guess it makes me not try to skip over lines too quickly. Like I was like 'okay I have to explain this I better read it carefully'. So it basically makes sure that you're reading every line and if I don't know something you won't be able to explain it yourself.

Here, David emphasized that self-explanation motivated him to thoroughly examine each line to ensure that he would be able to explain each part of the proof well.

On Using Self-explanation Over Time

In the follow up interviews, professor influence and proof difficulty were large determining factors for the use of the self-explanation technique. Brandon and Colin both expressed that while the training was influential in the moment, its influence over their actions when reading proof weakened when they returned to class. Colin, for example, stated, "How the professor teaches is always being pounded into me whereas what you mentioned, I only talked with you once." Colin considered the possibility that self-explanation was taught in person by his professor and stressed throughout the course,

> It would be something that would always be there in your mind because you might think of "how might the professor want me to do this?" [Students] probably [think] "this is how I'm going to be tested, this is what [the professor] would want on a piece of paper."

Colin felt that he would be more likely to use techniques endorsed by his professor, because he would assume these techniques would increase his chances of doing well on exams. Brandon, on the other hand, said he rarely used the technique with proofs in class because "If I'm reading a proof for the first time I don't generally use the technique unless I'm confused or something" which occurred with about 20% of the proofs he read. For Brandon, the technique was a resource that was only necessary when he didn't understand part of a proof, but this did not occur often.

Questions for Audience

1. All students emphasized that high quality explanations should explain the logic behind each line of the proof *and* why the line is necessary. However, not all students produced explanations that included both of these qualities. How should this be interpreted?
2. Many students often conflated statements about reading and writing proofs. How should we handle claims in which a student is discussing the benefits of self-explanation for proof writing rather than proof reading?

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Course Coordination Patterns in University Precalculus and Calculus Courses

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In this report we present findings from a preliminary investigation aimed at describing models of course coordination systems currently in place within university precalculus and single variable calculus courses. Hierarchical cluster analysis was used on national survey data to identify homogeneous clusters of courses based on the intended use of uniform course elements across sections. The analysis revealed eleven clusters of courses, nested within five larger groups. We briefly describe each of the eleven clusters in terms of the uniform course elements and the five larger groups in terms of the clusters nested within them. We then characterize these groups with respect to department type (Masters- versus PhD-granting), course level (Precalculus, Calculus 1, and Calculus 2), regularity of instructor meetings, and type of course coordinator.

Keywords: Course coordination; precalculus/calculus; course structure

With enrollments increasing in courses across the Precalculus to Calculus 2 (P2C2) sequence (Blair, Kirkman, & Maxwell, 2018), many mathematics departments have begun implementing course coordination systems in an attempt to create consistency in student learning opportunities within and across these courses (Apkarian & Kirin, 2017; Rasmussen et al., in press). While creating consistency in student learning opportunities broadly underlines such efforts, the exact reasons for implementing such a system vary across institutions. Rasmussen and Ellis (2015) point out that course coordination systems can be used to ensure uniformity in certain course elements (e.g., textbook, exams) for all sections of a particular course, and they suggest that building and implementing coordination systems can engender a sense of community among regular instructors of a coordinated course. Other studies suggest that these systems are important program components for enacting and sustaining change in P2C2 courses (Apkarian, Bowers, O'Sullivan, & Rasmussen, 2018; Pilgrim & Gehrtz, 2018) and that such change has the potential to positively impact student learning (Rasmussen, Ellis, Zazkis, & Bressoud, 2014). Taken together these studies provide some evidence that developing and implementing such a structured system for P2C2 courses has the potential to positively impact both instructor and student experiences within this sequence. Yet surprisingly little is known about how P2C2 coordination systems are currently organized and structured within mathematics departments across the United States. Thus, we ask: *What, if any, patterns of usage exist among different aspects and components of course coordination systems in undergraduate P2C2 courses*?

Background

Theory regarding course coordination systems, or how the coordination of particular course elements affects student experiences in introductory STEM courses is scarce in mathematics education literature. However, it is something many schools are interested in, it is important to begin that conversation (Apkarian, Kirin, Vroom, & Gehrtz, under review). One resource that stands out for thinking about course coordination of introductory mathematics courses is Rasmussen and Ellis's (2015) work, which suggests that in addition to controlling some of the variation in student experiences by controlling course elements, coordination systems may have a social component which engenders a sense of community among instructors. This sense of community affects the development of norms, including norms for teaching, which has an effect on student experiences in the classroom. A related body of work is that regarding *curriculum*. The constructs of written, intended, enacted, assessed, and learned curricula are particularly useful as different dimensions of course structure (Porter & Smithson, 2001; Stein, Remillard, & Smith, 2007). While postsecondary education is not governed by national standards and/or assessments as are the lower grades, a coordination system should, in principle, affect the curriculum. For example, a common course syllabus in Calculus 1, which includes a common textbook and a common set of topics to be covered, can be thought of as a common written curriculum; common exams relate to commonality in terms of the assessed curriculum. The transformation of written curriculum into intended and enacted curricula is affected by instructors' identity and situational context, which are likely impacted by regular course meetings and conversations about instruction.

Methods

Data for this analysis comes from a national survey aimed at investigating P2C2 programs across the country. The survey was completed by 223 of the 330 university departments offering an MA, MS, and/or PhD in mathematics in the United States of America. The survey covered many aspects of these programs and departments, as well as details about the P2C2 courses themselves. From this survey, we gathered detailed information about course delivery and management for 889 courses, 261 of which were categorized as Precalculus, 327 categorized as Calculus 1, and 301 categorized as Calculus 2. These details include what, if any, course elements are uniform across sections; regularity of instructor meetings; primary instructional approach for regular course meetings and recitations; role of coordinator; and the regularity with which the course is taught by differently ranked members of the university (e.g., research faculty, teaching faculty, graduate students). Our knowledge of the literature related to course coordination led us to select particular items from the survey to investigate as part of modeling course coordination systems. In particular, Rasmussen and Ellis (2015) point out that *systems* of coordination include both superficial aspects, such as coordinating course elements across specific course sections, and departmental features, such as the presence and role of course coordinators. In the analysis presented here, we consider the items related to the presence of uniform elements across course sections, regularity of instructor meetings, and the presence and role of a course coordinator for each of the 889 courses.

We began our analyses by grouping courses based on their response to the question about uniform course elements, specifically whether or not each of eleven[1] course elements were indicated as uniform across different sections of the course. Our intention with this analysis was twofold. First, to explore conjectures about what course elements are coordinated together or separately. Second, to reduce the uniform course element data for further analyses. Grouping was done using agglomerative hierarchical cluster analysis. Due to the binary nature of the data (e.g., coordinated/not coordinated) we used complete-linkage (or *farthest neighbor*) clustering and the Jaccard distance measure (Choi, Cha, & Tappert, 2010; Hastie, Tibshirani, & Friedman, 2009). Agglomerative hierarchical clustering begins by assigning each observations (here, 889 11-tuple course responses) to its own cluster, then sequentially combining the two closest clusters. The result of this process is a sequence of cluster fusion indicating at what step the clusters joined, and from what distance. This sequence preserves nested relationships between clusters and

[1] The survey contained 15 such items. Three were dropped to avoid confounding results based on the rarity of those elements; a fourth was dropped due to improper wording which make responses uninterpretable.

indicates relationships between the groupings. From the clustering sequence, decisions must be made to determine the appropriate number of clusters at which to “cut” the results.

An ideal cluster sequence cutoff is one which minimizes the distance between observations in each cluster and maximizes the distance between clusters, thereby creating distinct clusters of similar observations. With this in mind, we employed the elbow and average silhouette methods to inform the number of clusters selected (Hastie et al., 2009). The elbow method considers within-cluster sum of squared error, and recommends cutting the clustering sequence at a point where increasing the number of clusters corresponds to relatively small decrease in that error. The silhouette method considers the quality of clusters by comparing the relative similarity of each cluster element to others in its cluster as compared to observations outside the cluster. Using this method, high silhouette values indicate an appropriate clustering configuration. Using these two methods and taking advantage of the hierarchical structure of the clusters, we identified eleven clusters of courses nested within five larger groups.

Once the cutoffs were determined, we considered the responses which made up each cluster and group. Table 1 shows the proportion of observations within each cluster which selected each item. This allowed us to characterize the clusters based on the uniform course elements. Having satisfied ourselves that the clusters and groups were sensibly coherent, rather than just noise, we assessed the aggregated responses of courses in each cluster and group to the other course-coordination related items. While future work will expand on describing existing patterns within and across the eleven clusters, for the purpose of this report we focus on describing and comparing the five larger groups.

Preliminary Results

The clusters (1-11) and groups (A-E) from our analysis are shown in Table 1, alongside the proportion of observations in each cluster coordinating each element.

Table 1. Groups and clusters from hierarchical agglomerative clustering methods along with proportion of courses within each cluster that coordinate a particular course element. Reported values are those over 0.5 and true zeros, so that a blank entry carries a value between 0 and 0.50.

Group	A	B	C			D		E			
Cluster #	A4	B7	C3	C9	C11	D2	D5	E1	E6	E8	E10
Number of Courses	99	13	255	8	1	82	64	243	65	49	10
Textbook	0	1	1	0.88	1	0.99	1	1	1	1	0
Topics	0	0	1	1	0	1	1	1	1	1	1
Pacing	0	0	0.87	0	0	1	1	0			0
Midterms	0	0	0.90	0	1	0		0			0
Final Exam	0		1	1	1	0	0.78		1	0	0
HW (Online)	0	0	0.78	0	1	0	0.56	0		1	0
HW (Written)	0	0	0.56	0	0	0		0	0		0
Quizzes	0	0		0	1	0			0	0	0
Grading (Course)	0	0	0.92	1	1	0			0		0
Grading (Exam)	0	0	1		0	0		0		0	0
Approach	0	0		0	0						0

Group A consists of only a single cluster, A4, which consists of 99 courses for which no elements were identified as intended to be uniform across multiple sections. Group B is also a single cluster, B7, which consists of 13 courses which all coordinate their textbook, 2 of which also coordinate final exams. Group C consists of three clusters, C3, C9, and C11, which coordinate a lot of elements. Group D consists of two clusters, D2 and D5, which primarily coordinate textbook, topics, and pacing. Group E is the largest group, consisting of four clusters, E1, E6, E8, and E10. These courses coordinate topics, most also coordinate textbooks, and are then delineated by the few other items which are coordinated. In the following section, we discuss other aspects of course coordination systems as they interact with these groups. For brevity, in this paper we omit a discussion of Group B.

Group A (Cluster A4) is the set of 99 courses with no reported uniform elements across sections, 11% of the total courses reported. This cluster includes 71 (13%) of the 554 courses from PhD-granting universities and 28 (9%) of the 314 courses from MA/MS-granting universities. This cluster includes 22 (8%) of the Precalculus, 43 (13%) of the Calculus 1, and 34 (11%) of the Calculus 2 courses reported. Of the courses in this cluster, only five indicated that there are regular instructor meetings at least once per term of instruction. There were 60 courses with no response to this item (80% of the blanks) and 34 which reported never, 13% of that set. Additionally, Group A accounts for 86% (25) of the courses which skipped this item and 56% (53) of those who responded with "N/A." These findings corroborate the expectations one might have for courses with no uniform elements.

Group C includes a total of 264 courses (30% of all), 255 of which are in cluster C3. Group C includes 249 (45%) of the courses reported by PhD-granting departments, and only 15 (5%) of those from MA/MS-granting departments. This group also accounts for 33% of the Precalculus, 32% of the Calculus 1, and 24% of the Calculus 2 courses reported in the survey. The courses in this group have the most instructor meetings, accounting for 72% (94) of the courses which report weekly instructor meetings, 62% (29) of the biweekly meetings, and 45% (72) of the courses which report meeting 2-4 times per term. Additionally, Group C accounts for 24% (46) of the courses which meet only once per term, 6% (15) of those who report never meetings, and 11% (8) of the courses which left this item blank. 69% (181) of the courses in Group C indicated that the person responsible for maintaining the uniform efforts was someone who took on this role for multiple years, and these 181 courses are 49% of all courses with a similar role for their coordinators. Though smaller in their respective proportion of Group C, courses in this group account for 34% (32) of courses with a one-year rotating coordinator, 37% (42) of those where the coordinator is one of the instructors on a term-by-term basis, and only 2% (4) of the courses coordinated by committees. Of the remaining courses, four were marked "other" and one "N/A;" there were no blanks.

Group D consists of 146 courses, which is 16% of all those reported. This group includes 19% of the courses reported by PhD departments and 13% of those reported by MA/MS departments. These 146 courses include 16% of the PC courses, 15% of the C1 courses, and 18% of the C2 courses. The majority of these courses have lower frequency of instructor meetings. The largest pool is a set of 52 courses for which instructors meet once per term, and this accounts for 27% of all such courses and 34 courses in which instructors meet 2-4 times per term, which is 21% of that set. Group D also contains 15% (38) of the courses which never meet, 14% (18) of those that meet weekly, and 4% (2) of those which meet biweekly. Group D includes 22% (40) of the courses for which a committee is responsible for uniform course elements; 20% (73) of those for which there is an individual coordinator who oversees the course for multiple years; 18% (17) of the courses with a rotating coordinator structure; and 10% of those where one of the instructors in the term manages those elements. Of the remaining, there was one blank, one “N/A”, and 3 “other” responses. The clusters within this group, D2 and D5, are similar in size, differentiated primarily by the high rate at which courses in D5 have common exams, which are wholly absent in D2. Each accounts for a similar proportion of courses, with the following exception(s). While D2 includes 13% of the reported C2 courses, D5 includes only 5% of them.

Group E is the largest group, including 346 (39%) of the reported courses. This group includes 25% of the courses from PhD departments and a whopping 72% of the courses reported by MA/MS departments. By course level, this group includes 41% of all reported PC courses, 39% of C1, and 44% of C2. Instructor meetings are fairly rare in this group as well, including 64% (164) of all courses which never have instructor meetings and 45% (87) of those with one meeting per term. This large group also includes 33% (160) of those courses which meet 2-4 times per term; 34% (16) of those with biweekly instructor meetings; and 11% (15) of those which meet weekly. There were also five courses in Group E which did not provide an answer to the instructor meeting question. Group E accounts for 71% (127) of the courses which have a committee overseeing the uniform course elements, which is a large overrepresentation. This group also has 46% (52) of those courses overseen by one of the instructors and 45% (42) of those overseen by a rotating coordinator, 39% (37) of the courses which indicated “N/A”; and 28% (102) of those which are overseen by a multiyear coordinator. There were three blanks and four “other” responses.

Discussion & Questions for the Audience

Our initial analysis of data related to coordination systems in university-level P2C2 courses reveals some structure. Recall that the clustering and grouping was done *only* using the course elements, not responses to other coordination items. Thus, over- or underrepresentation of other components is not due to the delineation of groups and clusters. There appear to be associations, as one might expect, between the number of coordinated course elements and the frequency of instructor meetings. There are also some suggestive patterns in the type of coordinator and groupings. In particular, coordination by committee is overrepresented in Group E, Calculus 2 courses are underrepresented in cluster D5, courses from PhD-granting departments are overrepresented in Group C, and courses from MA/MS-granting departments are overrepresented in Group E. These patterns are suggestive of associations, but as yet we do not have evidence of the strength of these associations and only very preliminary conjectures about potential causes. We continue to analyze this data set and review related literature to strengthen these conjectures (e.g., taking into account school size, class size, number of sections). To further our work, we present the following questions for our audience:

1. Are there other relevant bodies of work that we should be leveraging in order to understand/situate our results more appropriately?
2. What questions do you have about our results, which further exploration might reveal?
3. What conjectures (with what basis) exist about the nature of course coordination systems which are testable with our data and analysis?

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Investigating Student Understanding of Rate Constants in Chemical Kinetics: When is a Constant "Constant"?

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The concept of rate constants is important for developing a deep understanding of chemical kinetics, an area of chemistry that models the rates of reactions. Reaction rates are modeled mathematically, typically using an equation called a "rate law". One of the terms in this equation, the rate constant, embodies important variables that affect rate, such as temperature-dependence, Our primary research focus in this work is investigating the question: How do students reason about rate constants in chemical kinetics? Preliminary analysis reveals that students often conflate ideas from chemical kinetics and equilibrium, such as rate constants and equilibrium constants. Furthermore, students demonstrated varying levels of sophistication regarding the distinction and relationship between rate and rate constants. Finally, students conveyed different ideas about the mathematical nature of the rate constant quantity.

Keywords: Constants, Parameters, Variables, Rate, Chemistry

Introduction and Rationale

One of the core ideas in the discipline of chemistry is change and stability of chemical systems (Cooper, Posey, Underwood, 2017; Holme, Luxford, & Murphy, 2015; Holme & Murphy, 2012; Laverty et al., 2016). This foundational idea that "energy and entropy changes, *the rates of competing processes* [emphasis added], and the balance between opposing forces govern the fate of chemical systems" takes many shapes and forms (Laverty et al., 2016). One area of study called chemical kinetics models rates of reaction, often utilizing rate law equations. For example, a generic chemical reaction, $x\,\mathrm{X} + y\,\mathrm{Y} \rightarrow P$, the rate law would be $\text{rate} = k[X]^m[Y]^n$ (concentration of a reactant is represented by surrounding the reactant with square brackets). Rate laws are empirically derived and demonstrate the dependence of reaction rates on the concentration (or pressure) of reactants and other parameters, typically a coefficient (k) and reaction orders (m and n) (Holme et al., 2015). In the case of elementary reactions or reaction steps, the order is also empirically derived and relates to the molecularity, or the number of molecules that react. The coefficient that appears in the rate law is typically termed the rate constant (k). The temperature dependence of reaction rate is contained in the rate constant and is typically modeled by the Arrhenius equation, $k = Ae^{-E_a/RT}$, where E_a is the activation energy of the reaction, A is a preexponential or frequency factor, R is the gas constant, and T is temperature (Holme et al., 2015). As temperature is controlled in an experimental setting, rate constants are generally held constant during a reaction.

Understanding the information encoded in rate constants is an important part of understanding the chemistry being modeled by kinetics equations (Holme et al., 2015). However, studies of chemistry students at both secondary and tertiary levels demonstrate that students have difficulty with this. Students often have an incorrect understanding of the relationship between reaction rate and temperature, a relationship that is contained in the rate constant (Bain & Towns, 2016). Students also often falsely relate temperature and activation energy or mischaracterize the mathematical nature of rate's time-dependence (Bain & Towns, 2016). While these studies give some insight into the nature of student thinking in this area, more work is needed at the undergraduate level (Bain & Towns, 2016; Singer, Nielson, & Schweingruber, 2012).

A robust understanding of rate constants would, among other things, include mathematical resources related to constants, parameters, variables, and functions. Mathematical symbols, like those present in a rate law, encode meaning. These quantities, represented by different symbols, could represent a constant (does not vary ever), a parameter (does not vary within a given setting), or a variable (varies with a given setting) (Thompson & Carlson, 2017). A rate constant would typically be considered a parameter, or a "generalized constant" (Philipp, 1992; Thompson & Carlson, 2017). As discussed by colleagues from the physics education research community (Redish, 2005; Redish & Gupta, 2009), the labeling and use of constants, parameters, and variables is very different in scientific communities, such as physics or chemistry, compared to mathematics communities. Further, scientists also load meaning onto these symbols, which can lead to different interpretation of equations and changes how equations are viewed; such differences arise because the goals and purposes for the use of mathematics are so divergent (Redish, 2005; Redish & Gupta, 2009). These distinctions and differences are often not apparent to students, who are concurrently enrolled in math and science courses (Redish, 2005; Tuminaro & Redish, 2007). Considering student reasoning from both a chemistry and mathematics perspective, this work was guided by the following research question: *How do students reason about rate constants in chemical kinetics?*

Theoretical Perspectives

We have framed our data analysis and discussion of results in terms of the resources framework, which is a model of cognition that defines knowledge as a network of fine-grained *resources*, or cognitive units, that are activated and constructed in response to a task or prompting (Hammer & Elby, 2003; Hammer, Elby, Scherr, and Reddish, 2005). The resources perspective builds on diSessa's (1993) *knowledge-in-pieces* conceptualization, which accounts for the observed inconsistency of student responses, since different resources or groups of resources may be activated when reasoning about different contexts (Hammer et al., 2005).

The resources perspective is in contrast to an alternate model of cognition that presupposes student understanding as composed of unitary, stable conceptions that are applied generally across contexts (Hammer & Elby, 2003; Hammer et al., 2005). This has implications for understanding the role of instruction in relation to how student ideas change over time; instead of targeting and replacing large entities or conceptions, conceptual change involves adding fine-grained resources and modifying connections between resources, ultimately restructuring students' local cluster of ideas to create a more coherent network of meaningfully connected resources (Wittmann, 2006). We are interested in identifying the resources students used to reason about rate constants, and we are particularly interested in understanding the connections between these resources. One useful representation of resources discussed in the literature is a *resource graph*, which visually indicates the links between different resources activated in a specific context (Wittmann, 2006; Sayre & Wittmann, 2008). Ongoing analysis involves determining the utility of such a representation for our work. A better understanding of how students cognitively organize resources would provide insight regarding which resources need to be targeted and which connections between resources need to be emphasized.

Methods

The study that we discuss in this preliminary report is part of larger project interested in investigating how students integrate chemistry and mathematics when solving chemical kinetics problems. For this project we have previously reported on student engagement in modeling (Bain, Rodriguez, Moon, & Towns, 2018), student conceptions regarding zero-order systems (Bain, Rodriguez, & Towns, 2018), productive features of problem solving (Rodriguez, Bain, Hux, & Towns, 2018), and student use of symbolic and graphical forms (Rodriguez, Santos-Diaz, Bain, & Towns, Submitted); here we focus on student reasoning related to rate constants. Our primary data source for this study is semi-structured interviews involving students working through a series of prompts (Table 1), with data collection involving the use of a Livescribe™ smartpen to digitally synchronize audio and written data (Linenberger and Bretz, 2012; Harle and Towns, 2013; Cruz-Ramirez de Arellano and Towns, 2014). Participants were undergraduate chemistry students from a second-semester general chemistry course (n=40), an upper-level physical chemistry course (n=5), and an upper-level reactions engineering course (n=3).

Table 1. Second-order and zero-order math and chemistry prompts.

Second-Order Math Prompt	**Zero-Order Math Prompt**
Here is another equation you've probably seen in class: $\frac{1}{[A]} = kt + \frac{1}{[A]_0}$ How would you explain this equation to a friend from class? How would you explain this on an exam?	Here is another equation you've probably seen in class: $[A] = -kt + [A]_0$ How would you explain this equation to a friend from class? How would you explain this on an exam?

Second-Order Chemistry Prompt

A second-order reaction

$2\ C_4H_6(g) \rightarrow C_8H_{12}(g)$

was run first at an initial concentration of 1.24 M and then again at an initial concentration of 2.48 M. They were run under the same reaction conditions (e.g. same temperature). Data collected from these reactions are provided in the table. Is the rate constant for reaction 2 (1.24 M) greater than, less than, or equal to the rate constant for reaction 1 (2.48 M)?

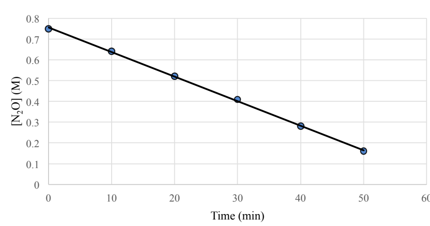
Time (hrs)	$[C_4H_6]$ (M) Rxn 1	$[C_4H_6]$ (M) Rxn 2
0	1.24	2.48
1	0.960	1.55
2	0.775	1.13
3	0.655	0.89
4	0.560	0.73
5	0.502	0.62
6	0.442	0.54
7	0.402	0.48
8	0.365	0.43
9	0.335	0.39
10	0.310	0.35

Zero-Order Chemistry Prompt

Below is a zero-order rate plot for the reaction

$N_2O(g) \rightarrow N_2(g) + \frac{1}{2}O_2(g)$

where $[N_2O]_0 = 0.75$ M and $k = 0.012$ M/min. The reaction is conducted at 575 ˚C with a solid platinum wire, which acts as a catalyst. If you were to double the concentration of N_2O and run the reaction again, how would the half-life change? At the half-lives for each reaction run, how do the chemical systems compare?



Student interviews were transcribed and open coded using constant comparison (Bain et al., 2018; Strauss and Corbin, 1990). Data analysis involved two researchers coding in tandem, discussing coding discrepancies and requiring 100% consensus for code assignments (Campbell, Quincy, Osserman, & Pederson, 2013). The coding scheme for the larger project had three primary themes, where one was comprised of codes that characterized the type of chemistry and mathematics content resources expressed. The codes primarily related to rate and rate constants were further analyzed for themes surrounding student understanding of rate constants.

Preliminary Results

Our preliminary analysis reveals three primary themes: (1) conflation of rate constants with equilibrium constants, (2) potential levels of sophistication in differentiating the concepts of rate and rate constants, and (3) various types of understanding regarding the mathematical nature of rate constants.

Conflation of Rate Constant (k) with Equilibrium Constant (K)

One commonly used idea is in the study of chemical equilibrium is the equilibrium constant, *K*. It is used to determine the extent of a reaction and the amount of reactants and products present at equilibrium from a given initial state; it is also a function of temperature and change in free energy (Holme et al., 2015). As reported in prior research, students often confuse kinetics and equilibrium concepts (Bain & Towns, 2016; Becker, Rupp, & Brandriet, 2017). In light of this, it was unsurprising to see that almost a quarter of our participants demonstrated rate constant (*k*) and equilibrium constant (*K*) conflation, a finding similar to Becker et al. (2017). The reason for this appears to be two-fold. First, the symbols for each constant are represented by the letter "k", which are only distinguishable by capitalization (or lack thereof). Second, from the perspective of Sherin's (2001) symbolic forms, the pattern of terms in the equations (symbol templates) is somewhat similar (Figure 1).

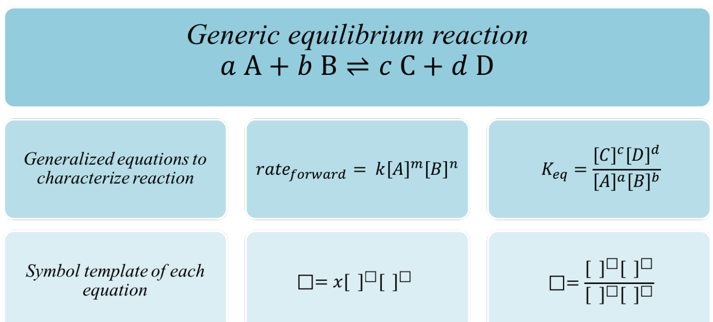


Figure 1. Side-by-side comparison of two mathematical equations and their corresponding symbol templates that model various aspects of this generic equilibrium reaction.

The sentiment that symbols and topics in general chemistry are similar and difficult to differentiate is summarized in a statement made by a general chemistry student, Nelly:

> Nelly: *"That's like equilibrium* [constant]. *Not rate constant. I don't know. That's also another thing that's hard about chemistry. It just seems that everything is the same almost, and it's hard to distinguish each equation and each principle."*

This discussion stemmed from her reasoning about if and how rate constants change for different reactions. She began reasoning about rate constants as equilibrium constants, but realized that she was thinking about the inappropriate constant, correcting herself. The similar nature of the symbols and equation structure caused temporary conflation of the ideas during her interview.

Another general chemistry student, Georgina, demonstrated conflation of equilibrium and rate constants as well, utilizing an an equilibrium-like expression to solve for reaction order.

> Georgina: *"I remember from zero order, you didn't have to do anything to do the concentration of a for it to be a straight line."*
>
> Interviewer: *"Why do you think that is?"*
>
> Georgina: *"I know it has something to do ... I kinda remember vaguely that ... Say that your equation would be A plus B equals C plus D.* [writing chemical equation, top of Figure 2] *Concentrations of your products go over your concentration of the reactants.* [writing variation of equilibrium expression, bottom of Figure 2] *I know it has something to do with whatever exponents you ended up with here."*

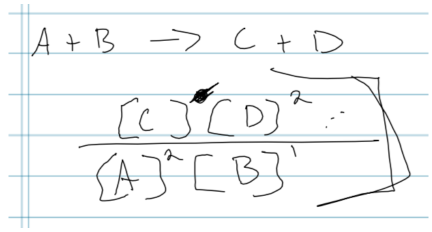


Figure 2. Chemical and mathematical equations written by Georgina (general chemistry student). The mathematical equation is structured like that of an equilibrium expression.

In this passage, Georgina was using the inappropriate equation to solve for order; she should have been using rate law equation, which contains a rate constant term, rather than the equilibrium constant. As shown in Figure 1, the symbol templates of the two equations are similar in structure. In general, each equation contained a variable related to the product of bracketed quantities, each raised to a power (Becker & Towns, 2012; Dorko & Speer, 2015; Rodriguez, Bain, & Towns, 2018; Rodriguez et al., 2018; Sherin, 2001). It is this similarity that often caused participants to activate the inappropriate resource for this context.

Possible Levels of Sophistication in Student Understanding of Rate Constants

There were a wide variety of resources characterized regarding student understanding of rate constants. Analysis revealed varying participant understanding of the relationship between rate and rate constant. Some students expressed conflation of these ideas, while others conveyed distinctive understanding of these two concepts with differing degrees of sophistication. The exact nature of these ideas is presently being explored.

Levels of Sophistication in Understanding the Mathematical Nature of Rate Constants

When analyzing participant understanding of rate constants among students who did conceive of rate and rate constants as distinct, there were three levels of understanding conveyed with respect to what type of quantity rate constants were. First, participants sometimes conveyed the idea that rate constants were like *universal constants*, that is quantity was the same at all times. This is distinct from other participants who stated that rate constants were only constant for a given reaction, demonstrating a more *parameter-like* understanding. Finally, some participants went further to describe on what rate constants depend. These participants cited specific variables, such as temperature, or provided the Arrhenius equation, demonstrating an even more sophisticated *parameter-like* understanding.

Conclusions and Questions

While the analysis for this work is ongoing, the preliminary findings for this project indicate that an important instruction target for undergraduate chemistry (and likely other science and mathematics courses) is a nuanced understanding of the distinction between constants, parameters, and variables. While terms like “rate constant” and “equilibrium constant” may be misleading for students, explicit discussion of the mathematical nature of equation terms is important in developing deep understanding of the chemistry being mathematically modeled.

Further analysis involves addressing the following questions:

(1) What insight into students’ knowledge structures can be gained using resource maps?
(2) What is the relationship between participant understanding of rate and rate constants?
(3) Are there other lenses in the RUME community that would be helpful for investigating students’ mathematical understanding in chemistry contexts?

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Exploring College Geometry Students' Understandings of Taxicab Geometry

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Non-Euclidean geometries are commonly used in college geometry courses to highlight aspects of Euclidean geometry. Scholars have theorized that working in non-Euclidean geometries requires thinking at the highest van Hiele level of geometric thinking, which was developed by investigating students' learning of Euclidean geometry, but few have pursued this empirically. This empirical study seeks to develop levels of geometric thinking for students in Taxicab geometry, which is the non-Euclidean geometry that is closest in structure to Euclidean geometry. Students in a college geometry course that included prospective secondary teachers were audio-recorded in group discussions as they completed tasks about congruence and transformations in taxicab geometry, and their written work was collected. Portraits of participants' thinking about Taxicab geometry were developed, leading to a proposed structure for the levels of geometric thinking for Taxicab geometry.

Keywords: College Geometry, Student Thinking, Preservice Teachers

Introduction

College geometry courses can include a wide variety of students, from mathematics majors to preservice middle-grades and secondary teachers. These courses commonly introduce some form of non-Euclidean geometry, either as a worthwhile topic in its own right or as a way to highlight subtleties or assumptions in Euclidean geometry. The prevalence of non-Euclidean geometry in these courses makes student thinking about non-Euclidean geometries a useful topic for exploration in research. The van Hiele model was developed to characterize the different levels of geometric understanding and can be used to assess students' levels of geometric thinking. The van Hiele model has generally been researched with applications to Euclidian geometry, but non-Euclidean geometries may also be introduced to students in a college geometry course. However, limited research has been done on how the van Hiele model can be applied to non-Euclidean geometries. A non-Euclidian geometry that can be found in the curriculum for an undergraduate geometry course is Taxicab geometry. Due to its inclusion in curriculum, we conducted a preliminary exploration of the levels of thinking in Taxicab geometry.

Background

The van Hiele Levels

The van Hiele levels of geometric thought are a way of identifying a student's level of geometric thinking (Crowley, 1987). The van Hiele levels are as follows:

1. *Visualization:* This is sometimes considered the base level. In this level, students can name figures judging by their appearance, but their properties are not understood.
2. *Analysis:* In this level, figures are bearers of their properties and students can reason that they are classified based on their properties. However, the properties have no logical order to them.
3. *Informal Deduction:* In this level, students can deduce that one property precedes or follows another property. Definitions are introduced. Students are also able to give informal arguments to justify their statements and follow formal proofs, but cannot construct a formal proof from a different or unfamiliar premise.

4. *Deduction:* Students can now construct proofs and see relationships between definitions, axioms, and theorems and use them to establish further theory and can distinguish between statements and their converses.
5. *Rigor:* Geometry can now be seen abstractly by students. The students can work with different axiomatic systems to further study non-Euclidean geometries.
(van Hiele, 1959/2004; Crowley, 1987)

One of the properties that accompanies the levels is the *sequential* property which states that to achieve one level, one must have achieved all prior levels too (van Hiele, 1959/2004; Crowley, 1987). Mayberry (1983) displayed that the sequential property did take effect on her study of the van Hiele levels. Research conducted on college students demonstrates that most students only achieved level 3 thinking during a college geometry course (Mayberry, 1983; Wang, 2011). Research suggests that if students do not progress through to the fourth level, then they are not likely to succeed in a college based geometry course where they are generally expected to reason deductively (Mayberry, 1983; Wang, 2011).

Taxicab Geometry

Students in college geometry courses are most likely to have been exposed to Euclidean geometry in school, and most prospective teachers are only required to teach Euclidean geometry in their future careers. Other geometries can be constructed through changes in the axioms or the metric of Euclidean geometry. Taxicab geometry is created by changing the Euclidean metric to the Taxicab metric. In Euclidean geometry, the distance between two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is measured by the length of the straight line connecting the two points (analytically, $d_E(A, B) := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$) while Taxicab geometry measures distance by taking the sum of the horizontal and vertical distances between the two points ($d_T(A, B) := |x_2 - x_1| + |y_2 - y_1|$). This is demonstrated in Figure 1. A common analogy that is used pedagogically to describe this new metric is of a taxicab driving in a grid-like city such as Manhattan (Krause, 1986).

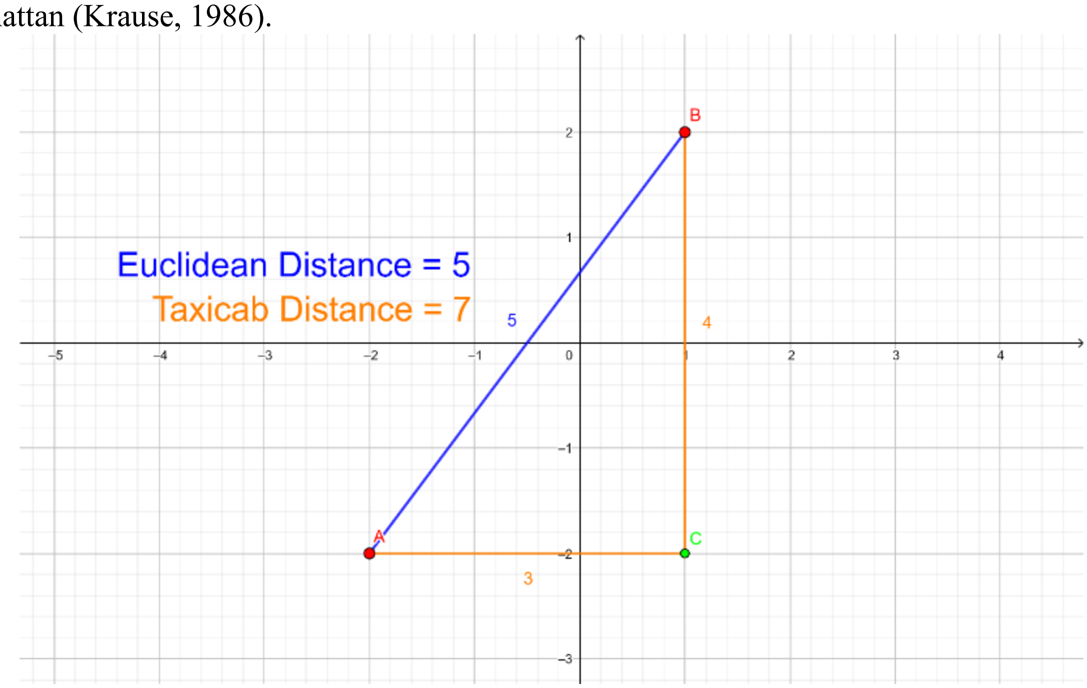

Figure 1: Euclidean distance vs. Taxicab distance.

Research on Understandings of Non-Euclidean Geometries

There is a lack of research concerning how levels of understanding, like the van Hiele levels for Euclidean geometry, would look for non-Euclidean geometries. Kemp & Vidakovic (2017) mention that despite over two decades of research on the van Hiele model, few participants have been classified to be at the fifth level of geometric thinking, which could be why the Rigor level has not been studied, resulting in a lack of research in levels of understanding for non-Euclidean geometries. While there is limited research on how levels of understanding in non-Euclidean geometries would be described, Guven and Baki (2010) have developed levels of understanding in spherical geometry known as Transition, Definition Comparison, Pre-Deductive, and Deductive, which develop sequentially. This work was very influential to this particular study.

Research Question

Since Taxicab geometry has the potential to improve the understanding students have of Euclidean geometry, particularly with respect to ideas about congruence (Boyce & Prasad, 2018), it becomes useful to understand how students develop their thinking about Taxicab geometry. This study posits a preliminary results in response to the following question: What are students' levels of geometric thinking in Taxicab geometry?

Methods

The setting for the data that was collected for this study was a college geometry course at a large southwestern university. Prospective middle and high school teachers are required to take this course and made up around 50% of the class. A course in introduction to proof writing was not a prerequisite. Students participated in a week-long exploration of Taxicab geometry. After the students were introduced to the Taxicab metric, they studied relationships between Euclidean and Taxicab geometry. The following tasks were assigned to the students in the course to gather information of how they understand Taxicab geometry.

1. Come up with examples of each, or explain why such an example is not possible
    a. 2 triangles that are congruent in both Euclidean geometry and Taxicab geometry
    b. 2 triangles that are congruent in Euclidean geometry but **not** Taxicab geometry
    c. 2 triangles that are congruent in Taxicab geometry but **not** Euclidean geometry
2. Identify all the Taxicab isometries.
3. How can we define congruence in Taxicab geometry?

Students worked on these problems in assigned groups of roughly four students; these groups were audio recorded and their written work was also digitized and synced with the audio recording using LiveScribe dot paper and smart pens ("Dot Paper", n.d.). The researcher took notes over all group recordings to search for key pieces of dialogue and writing in order to identify group recordings for transcription. The authors transcribed the aforementioned groups' recordings and took notes to find recurring patterns in the thought process of the students. Using these patterns and following Guven and Baki's (2010) levels of understanding in spherical geometry and van Hiele's levels of geometric thinking (van Hiele, 1959/2004), the authors created preliminary levels of geometric thinking for Taxicab geometry.

Results

Students' group work generally followed a few particular trajectories of thought, prompted by these tasks. These helped the authors propose the following preliminary levels of thinking in the Taxicab geometry.

Level 1: Transition

The student is now aware that they are studying a geometry different to Euclidean geometry because of the difference in metric. The student names observed figure in a manner consistent with Euclidean geometry. Most groups displayed this when they answered the first bullet point of the first task, spending time discussing the manner in which to measure the length of the hypotenuse in a right triangle (see Figure 2).

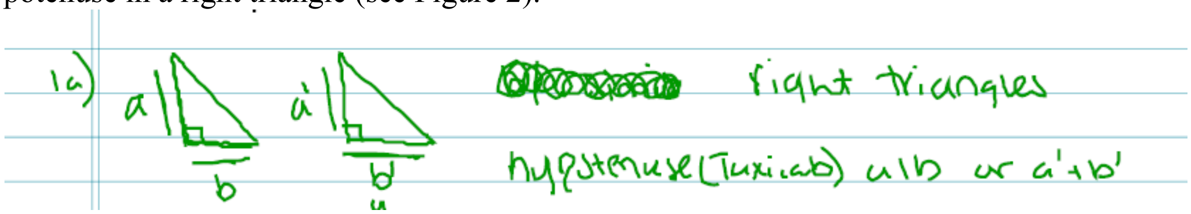

Figure 2: Students' work as they discuss the change in metric.

Level 2: Geometry Comparison

The student can focus on definitions of concepts and figures and learns to represent them visually on the Cartesian plane and can compare what they are and do in either geometry. While the student understands what the concepts are, they do not know how to use them for problem solving. For example, students demonstrate that the orientation of a figure will affect distance on that figure. The group whose work is shown in Figure 3 changes the orientation of a triangle so that the sides would be calculated differently, comparing in either geometry after changing the orientation.

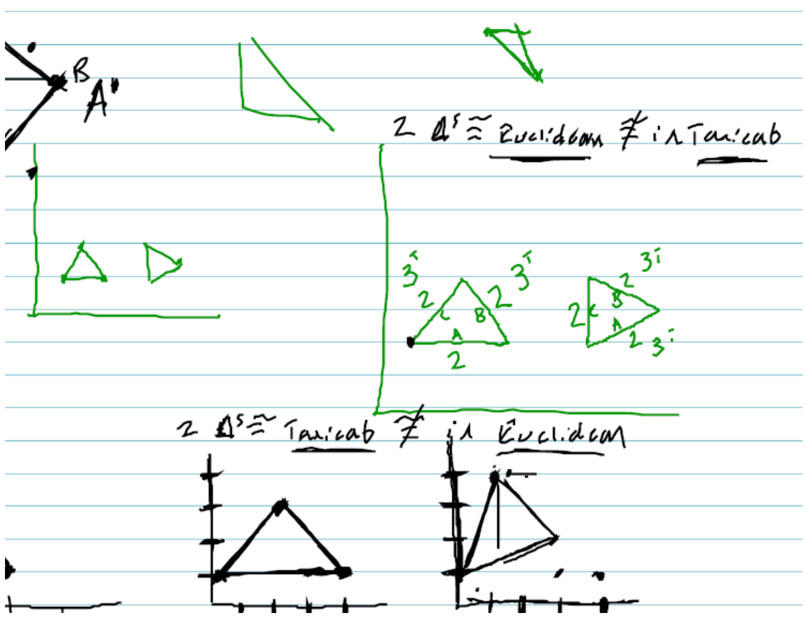

Figure 3: Students' exploration of how orientation affects distance.

Level 3: Pre-Deductive

The student can solve problems using Taxicab geometry constructions but does not yet solve problems involving deductive reasoning. The student can follow formal proofs but cannot alter the logical order of the proof or deduce a new proof. Only one of the groups studied attempted to use a Taxicab circle to justify a transformation where they placed one point of a triangle that was on a Taxicab circle to another point on that circle. They made the assumption that this would preserve distance since the distance from the center would not change but they did not account for the side of the triangle that did not share a point with the center of the circle.

Discussion

As expected by the Sequential property that both the van Hiele levels of geometric thinking and the levels of understanding in spherical geometry share, there were many examples of Level 1 thinking throughout the student discussion while they worked on their assignments. Naturally, due to this being an early exploration, there was much discussion about the Taxicab metric. In discussing the second level of thinking in Taxicab geometry, a recent study by Kemp and Vidakovic (2017) shows a student having an understanding of the definition of a circle and recognition that the Taxicab circle will appear in a different manner to the traditional Euclidean circle but fails to construct such Taxicab circle. This situation shows growing thinking in our proposed second level of thinking in Taxicab geometry, Geometric Comparison, since the student was aware of the definition of the circle and showed development in the visualization of the Taxicab circle. The group data collected in this research displayed the groups using the notion of orientation to affect distance in Taxicab geometry. This gives insight to their understanding of the rotation transformation and the effect it had in Euclidean geometry. Only one group contributed to the development of Level 3. Due to the students studying Taxicab geometry for only a week, it was not expected that there would be many results to provide further analysis of this level. While they were not completely correct, this type of work helped highlight the characteristics of our proposed third level.

The results in this paper do not include a Deduction level as previous findings do. This is primarily because none of the groups displayed characteristics expected of a student that can make deductions in Taxicab geometry. However, based on the Deduction levels proposed by van Hiele (1959/2004) and Guven and Baki (2010), the following was hypothesized as a candidate of the fourth level of thinking in Taxicab geometry (also called Deductive): The student can prove propositions deductively and support theorems through more universal definitions rather than specific geometry definitions. The student can make deductions from such definitions that are universal to other geometries. A hypothetical example of a student thinking at this level of would be able to give a proof or a counterexample to the following statement, "All rotations by $\frac{k\pi}{2}$ are isometries in both Euclidean and Taxicab geometry. $k \in \mathbf{Z}$."

Next Steps

The assigned tasks used to gather data for this research were not designed with the intent of developing levels of thinking in Taxicab geometry; thus, the levels proposed here are preliminary. Future plans for this research include conducting individual interviews in order to get a better understanding of the characteristics that can form the levels of understanding in Taxicab geometry, as Guven and Baki (2010) did to develop levels of thinking in Spherical geometry.

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Writing out a group: Interpreting student generated representations of the group concept

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In this presentation I will explore the various ways in which a pair of students used representations of the group concept to explore the structure of symmetry groups. During a series of teaching experiments, a pair of mathematics education graduate students were asked to develop an algorithm for classifying chemically important point groups beginning with an investigation of a few ball and stick models of molecules. The progress the students made through the use of each symbolization of the group concept is framed with the Realistic Mathematics Education design heuristic of emergent models.

Keywords: Abstract Algebra, Group Theory, Realistic Mathematics Education, Emergent Models, Representations

Group theory, specifically the study of symmetry groups or symmetry theory, is particularly prevalent in undergraduate chemistry curriculum. Symmetry theory is especially powerful in inorganic chemistry and provides an important foundation for other concepts such as point-group notation, spectroscopy, and molecular orbital theory (Luxford, Crowder, & Bretz, 2011). Once a chemist knows the particular symmetry group of a given molecule it gives them an idea of the various types of experimental techniques which will be most appropriate to perform on the molecule. Results of these experiments can then give insight into structural properties of the molecule including, but not limited to, information on bonding including bond lengths and angles, and the various modes of vibration (rotation, vibration, translation) the molecule may possess. The need for a firm understanding of symmetry is important in order for students to successfully understand so many topics within inorganic chemistry it has been outlined by the American Chemical Society in the guidelines for accreditation for bachelors programs in chemistry (Larive & Polik, 2008).

Abstract algebra, the larger field of mathematics in which group theory is studied, is an essential part of undergraduate mathematics curriculum (Gallian, 2009; Hazzan, 1999; Selden & Selden, 1987). Unfortunately, much of the abstract algebra literature highlights students' difficulties in learning fundamental concepts in group theory in particular (Larsen, 2010; Leron, Hazzan, & Zazkis, 1995). As noted in Dubinsky, Dautermann, Leron, and Zazkis (1994), and so "Mathematics faculty and students generally consider it to be one of the most troublesome undergraduate subjects" (p.268). Many student difficulties within abstract algebra can be partially attributed to the abstract nature of the course's content. The course is built around objects such as groups that are discussed and argued about abstractly (Hazzan, 1999, 2001). Abstract algebra is often the first-time students are asked to go beyond 'imitative behavior patterns' for solution finding and instead reason about mathematical concepts and consequences from interpreting formal definitions (Dubinsky et al., 1994). In this new abstract approach, concepts are defined and presented by their properties and by an examination of 'what facts can be determined just from (the properties) alone' (Dubinsky & Leron, 1994, p. 42).

The preliminary results reported here are from a recent effort to engage students in the richness of group theory *and* to gain experience with its applicability, through a design research experiment aimed to develop a local instructional theory for student reinvention of the classification of chemically important symmetry groups. The primary goal of this research project is to explicate a way in which students *can* reinvent a classification system for differentiating various shapes of molecules by engaging in the group theory that is used to differentiate and describe various shapes of molecules. However, instead of the traditional approach of giving students definitions of symmetries, groups, and a ready-made flow chart to find specific groups, I want to provide students an opportunity to articulate their own algorithm for classifying symmetry groups based on their experiences with select ball and stick models of molecules, *specifically* so that the activity is experientially real to them.

Theoretical Framework

Realistic Mathematics Education (RME) serves as the underlying instructional design theory for this experiment and is built on the theoretical perspective that mathematics is first and foremost a process, a human activity (Gravemeijer & Terwel, 2000). From its very beginnings Freudenthal has described this activity as, "an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be matter from reality which has to be organized according to mathematics patterns if problems from reality have to be solved" (1971, p. 413). Through the activity of *mathematizing* students can be guided in reinventing particular mathematical concepts, as opposed to learning the topic as a ready-made or previously discovered theorem. Therefore, overarching goals of RME include discovering how to provide students an opportunity to reinvent mathematics and also how to support them throughout the activity of mathematizing so that the mathematics that they develop is experienced as developing common sense (Gravemeijer, 1998).

The theoretical framework of RME has three accompanying design heuristics, which can serve as both guiding principles for instructional design and as a guide for further analysis. These heuristics include the reinvention principle, emergent models, and didactic phenomenology (Gravemeijer, 1998). A local instructional theory (Gravemeijer, 1998) describes a generalized roadmap for student reinvention of a particular mathematical concept, in which students feel ownership over the mathematical concepts they investigate. The first heuristic informed the development of the tasks used in teaching experiment. A context was chosen that offered an opportunity for the students to begin using their own intuitions and experiences to develop informal highly context-specific solution strategies (Gravemeijer & Doorman, 1999) which can later be used in a more formal mathematical reality. Didactical phenomenology focuses on the relationship between a mathematical content and the "phenomenon" it describes and analyses, or, in short, organizes (Gravemeijer, 2004). In this sense the heuristic helped at a global level to inform a good starting point for the reinvention process. Didactical phenomenology was also used at a more local level during the teaching experiments to drive the study by helping to identify ways in which I as the researcher could support the students transform their informal approaches to the molecules into more powerful formal arguments about molecules in general (Larsen, in press).

Lastly, the design heuristic of emergent models is used to describe both the character and the process of evolution of student's formal mathematical knowledge form an initial informal understanding. Emergent models can also be helpful in describing the progression of the students' mathematical activity from contextually situated to a more formal mathematical activity in a new mathematical reality (Gravemeijer, 1999). While the overarching emergent model that I am investigating is *a classification algorithm for chemically important point groups*, this global model took on various manifestations and the meaning of the label *model* is in fact much broader. In Gravemeijer's description of emergent models he highlights "…three interrelated processes. Firstly, there is the overarching model, which first emerges as a model of informal activity, and then gradually develops into a model for more formal mathematical reasoning. Secondly, the model-of/model-for transition involves the constitution of some new mathematical reality – which can be called formal in relation to the original starting points of the students. Thirdly, in the concrete elaboration of the instructional, there is not one model, but the model is actually shaped as a series of symbolizations" (2002, p. 3).

The inscriptions the students produce during their mathematical activity can serve as indicators of their emergent model. Each symbolization, and the purpose the student associates with the symbolization, can offer the researcher insight on the current state of the student's model. Inscriptions also help the student make progress in their mathematization and organization activities. This preliminary report focuses on these inscriptions and symbolizations, specifically those of the group concept, that the students created as evidence of their emergent model. The discussion includes the ways in which the students use of each model varied with their mathematical activity and how their representations of the group concept helped them make progress toward their goal of classifying symmetry groups.

Methods

In order to gain insight on the mathematical activity of the students as they reinvent an algorithm for classifying chemically important symmetry groups, I have chosen to conduct a series of teaching experiments (Steffe, 1991). In total the design experiment includes three series of teaching experiments including a pilot study. The results shared in this preliminary report are from the pilot study recently conducted with a pair of students. The goal of the pilot study was to explicate a way in which a pair of students successfully classified chemically important point groups. In some sense the pilot study served as an existence proof, or an initial model of success that could ultimately inform an instructional sequence supporting this reinvention. To better ensure that the students would be successful for the initial attempt I chose to conduct the pilot study with a pair of mathematics education graduate students at a large public university on the west coast. The students, referred to by pseudonyms Emmy and Felix, had both completed a yearlong graduate sequence in abstract algebra including a term in which they classified various groups of finite order. They had worked together as partners in a previous class and were extremely supportive work partners, especially in particularly difficult settings. The pilot study consisted of four 60 to 90-minute episodes. Data consisted of video recordings of each episodes and all written work was collected. The participants were compensated monetarily for their time.

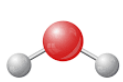
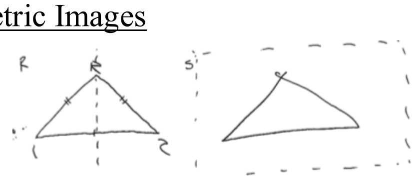
An important aspect of this study overall was to avoid using mature, conventional symbolizations of mathematical starting points for instruction. Often in traditional instruction, both formal mathematical definitions and rich molecular representations are often presented from the perspective of an expert. These artifacts are representative of concepts that are meaningful to the expert given in a relevant representation, which the novice, the student, is meant to extract particular meaning from (Gravemeijer & Doorman, 1999). In the attempt to avoid such an anti-didactical inversion, the students had no apriori experience with a conventional classification flow-chart and instead were given a set of three ball and stick model representations of water, ammonia, and ethane, in an eclipsed configuration (as seen in Figure 1). They were then asked to *develop and describe a procedure for efficiently and comprehensively finding all the symmetries of any given molecule.* The molecules chosen for the initial task were conical examples often used to introduce symmetry groups as they contain most, but not all of the symmetry elements present in 3-space (they were lacking an inversion center).

Results

The pilot students began the experiment by determining and describing the symmetry groups of specific molecules. The students shared a common strategy; 1) identify all symmetries, 2) distinguish which symmetries could/should be considered as generators, 3) determine the relations between each pair of generators, and lastly 4) decide to which 'familiar group' the new found group was isomorphic. The pair never wavered from is approach and it proved to be very powerful for them as they were able to successful identify the unique symmetry group of each of the molecules. In this first phase the students' mathematical activity was situated in the task setting as they were focused on the symmetry groups of specifically selected molecules. Later in the experiment, the students reflected on their own activity of finding specific symmetry groups to create their model of how to find classify the symmetry group of any given model. Throughout the experiment the students used multiple representations of the group concept for various purposes.

The group concept can be represented by many different symbolizations. An in-depth textbook analysis of the most frequently used introductory abstract algebra texts identified at least 11 different representations for the group concept (Melhuish, 2015). Some of these representations include group names, verbal descriptions, Cayley tables, lists of elements, group presentations, etc. Not only did the students in the pilot study use different kinds of representations of the group concept they used them with different purposes. A subset of the students group representations used while determining each of the ball and stick models they were asked to consider can be found in Figure 2 along with a short description of how the students used the representation.

The preliminary presentation proposed here will discuss the progress the students made from the use of each of their representations towards their overall goals of classifying symmetry groups. Furthermore, I would like to discuss the affordances and limitations of each of the representations, in particular how the students often produced a complete description of a group, say in group presentation form (eg. |R|=2, |S|=2, RS=SR) yet never felt "done" until they had the group name (eg. $\mathbb{Z}_2 \times \mathbb{Z}_2$), and what implications this may have on student understanding of a group.

Molecule	Group representation / symbolization	Student's use of representation
 Water	Geometric Images 	Determining various group elements, determining equivalence of group elements, checking relation between generator

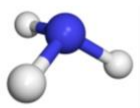
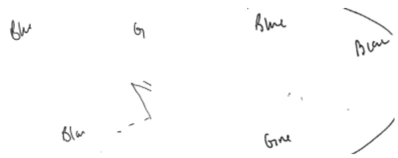
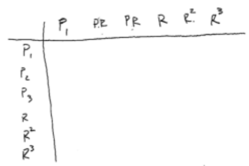
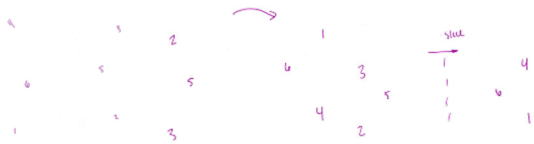
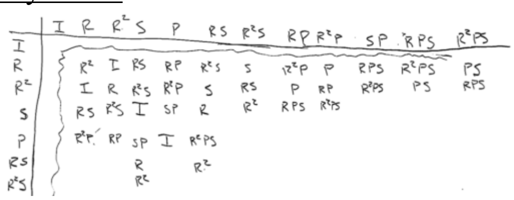
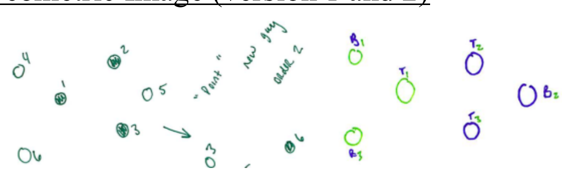
Molecule	Group representation / symbolization	Student's use of representation
 Ammonia	Geometric Images 	Determining various group elements, determining equivalence of group elements, checking relation between generator
	Cayley Table 	List elements
 Ethane (eclipsed)	Geometric Images 	Determining various group elements, determining equivalence of group elements, checking relation between generator
	Cayley Table 	List elements, count elements, record products of symmetry combinations
	Group Presentation $3=\|R\|$ $2=\|P\|$ $2=\|S\|$ $RS=SR$ $RP=PR^2$ S commutes	Organize generators and their relations
 Ethane (staggered)	Geometric Image (version 1 and 2) 	Determining various group elements, determining equivalence of group elements, checking relation between generator
	Function mapping $T_1 \leftrightarrow B_2$ $B_3 \leftrightarrow T_2$ $B_1 \leftrightarrow T_3$	Determining order and equivalence of group elements
	Group Name $D_6 \times \mathbb{Z}_2$	Encapsulating the symmetry group they had determined through investigation

Figure 1 Student produced group representations and the ways in which the students used them.

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Mathematical Errors When Teaching: A Case of Secondary Mathematics Prospective Teachers' Early Field Experiences

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The construct of mathematical knowledge for teaching (MKT) has transformed research and practice regarding the mathematical preparation of future teachers. However, the measures used to assess MKT are largely written tasks, which may fail to adequately represent the nature of content knowledge as it is used in instructional decision making. This preliminary report shares initial findings into one measure of MKT in practice – mathematical errors made during planning and enactment of mathematics instruction. We analyzed lesson plans and classroom video from prospective secondary mathematics teachers (PSTs)' supervised field experiences in college algebra course. We found that there tended be more errors related to understanding of functions (especially logarithmic), but relatively few errors happened overall during instruction. Of the errors made during planning, the majority of these errors were issues of mathematical precision. Implications for the mathematical preparation of secondary PSTs, as well as research on MKT in practice, are discussed.

Keywords: Pre-Service Teachers, Mathematical Knowledge for Teaching, Content Knowledge, Early Field Experience, Secondary Teacher Preparation

Introduction

Although the knowledge of mathematics teachers has been a widely discussed and researched topic for decades, surprisingly little empirical research has examined evidence of teachers' mathematical knowledge from teaching episodes. The mathematical errors teachers make during instruction, particularly when consistent, may reveal aspects of their content knowledge that need further development. Certainly, anyone who has taught mathematics knows that making mathematical errors, when unintentional, is inevitable. Yet, surprisingly little empirical evidence, especially when compared against the extensive research on students' mathematical errors, exists regarding the nature of mathematical errors made by teachers during mathematics instruction. Such work could shed light on the robustness of novice teachers' content knowledge as they engage in the complex decision making inherent to classroom teaching, and suggest areas where novice teachers' mathematical knowledge might be further developed prior to completing teacher preparation.

In this paper, we present exploratory research extending existing work measuring the nature of teachers' content and pedagogical content knowledge using written assessments (e.g., Hill, Schilling & Ball, 2004; McCrory, Floden, Ferrini-Mundy, Reckase & Senk, 2012) that investigates the kinds of mathematical errors secondary PSTs make when planning and teaching mathematics. The results suggest not only interesting directions for future research on PSTs enactment of their content knowledge while teaching, but also implications for content and methods courses in terms of topics where PSTs may need reinforcement of their knowledge prior to being certified and, more importantly, how to support PSTs in managing moments where the inevitable mathematical errors will happen.

Theoretical Framework

Much of the contemporary work in teacher education is founded upon the assumption, which some research has established empirically (Wilson, Floden & Ferrini-Mundy, 2002; Hill, Umland, Litke & Kapitula, 2012) that teachers' knowledge influences their teaching practice. As a result, a number of projects to improve novice secondary mathematics teachers' practice have aimed to develop prospective teachers' content knowledge for teaching (Garet et al., 2016; Sevis, Cross & Hudson, 2017). However, much of the existing empirical research to understand and measure teachers' content knowledge for teaching have involved the use of specifically designed written tasks rather than attending to how knowledge is used during practice. While written measures are certainly easier to implement and analyze at a large scale, they are imperfect measures of how a teacher might use or draw upon their content knowledge during instruction (see Shechtman, Rochelle, Haertel & Knudsen, 2010). Through an analysis of secondary PSTs' planning and enactment of instruction in an early field experience, the research question addressed by this study was: *What characterizes the kinds of mathematical errors made by novice secondary mathematics teachers when planning and enacting mathematics instruction*?

Methods

To address the study goal, we analyzed data collected as part of a larger study to investigate the opportunities to learn about mathematics teaching through an early field experience planning and teaching lessons in a college algebra course. This experience was a required component of a secondary mathematics methods course participants were concurrently enrolled in. All participants were in their senior year of a 5-year, university-based, secondary mathematics teacher preparation program, which requires candidates to complete a Bachelor of Science degree in Mathematics, along with education coursework and a full-year student teaching placement in their fifth year of the program. A total of 14 PSTs (n=14) agreed to allow members of the project team to analyze the videos of their teaching in the college algebra class, as well as analyze their lesson plan artifacts (mathematics pre-planning worksheet (P1), initial lesson plan (P2), and revised lesson plan (P3)).

To code the enacted lessons for mathematical errors, we first assembled the collection of instances where mathematical errors had occurred as captured on video of the 14 lessons taught by pairs of PSTs (each pair taught a lesson twice in the course). The first step to building this collection was to isolate all of the episodes where a mathematics teacher educator (MTE) who observed all lessons in the college algebra course intervened in the lesson to provide in-the-moment coaching to the PSTs. The second step was to have a trained rater on the project team use the Mathematical Quality of Instruction rubric to identify moments where PSTs made a mathematical error regardless of whether this resulted in an intervention by the MTE. This resulted in an initial collection of 5 possible episodes where PSTs had made mathematical errors. We then reviewed each of these instances to develop open codes to describe the error that had been made. In addition, we reviewed feedback that the MTE had provided to the P1, P2 and P3 lesson planning artifacts and isolated all instances (n=21 comments) where the MTE commented on mathematical content issues.

Two iterations of refinements to the coding categories resulted in four codes to describe the types of content-related errors PSTs were making in their planned or enacted instruction. Instances coded as *Content Error Correction* required PSTs to have made an explicit mathematical error that needed correction. For instance, one PST pair had written in their lesson plan that "negative exponents create fractions." The instructor was quick to point out, however, that "negative exponents invert fractions," making sure the PSTs understand that non-whole numbers also can be taken to a negative exponent. Instances coded *Mathematical Precision* included feedback or interventions that reminded PSTs to be careful about the language they use or the instructor asked clarifying questions to clear up parts of the lesson plan that were not immediately clear mathematically. The code *Knowledge of Content and Students Suggestion* included instructor comments suggesting alternate phrasing or terms in order to avoid confusion for the students while also providing justification by connecting the comment to students' prior knowledge or broader knowledge of the content as it is taught in schools. Lastly, *Typo/Other* included comments that corrected a simple typographical error or comments that were otherwise different from the rest.

In addition to assigning these codes, we accounted for the mathematical topic of the lesson, and whether the error during enactment was during content presentation or originated in response to a question from students.

Results

Table 1 shows the mathematical content addressed during the implementation interventions or in the lesson plan feedback. The most common mathematical areas where content errors occurred were in the areas of Functions and their Inverses (n=5), Composition of Functions (n=4), and Solving Exponential Equations (n=3). At first glance, one could see high error numbers as being a result of particular mathematical content being more difficult. It might also likely be a result of particular pairs finding difficulty in planning or teaching the content. The data suggest both of these conclusions are plausible; errors working with inverse functions spanned across three PST teams, whereas all four errors in composing functions happened with one particular team of PSTs.

Table 1
Mathematical content addressed by interventions during planning and enactment

Content	**Number of Instances**
Functions and their Inverses	5
Composition of Functions	4
Solving exponential equations	3
Properties of Logarithms	2
Simplifying logarithmic expressions	2
Exponential vs Logarithmic Functions	2
Place Value for large numbers	1
Transformations, parent functions.	1
Perfect Squares	1
Definition of Logarithmic Function	1
Slope of a Line	1
Interpreting Variable meaning	1
Even, Odd functions	1
Exponential Growth	1
Negative exponents	1

Kinds of Content-Related Errors in Planning

All seven teams of PSTs received mathematical content error feedback on their lesson plans. Three teams of PSTs received feedback on their round two pre-instruction lesson plan documents, while six teams received feedback on their round three documents. Of the 22 feedback items coded as errors, 10 were coded as Mathematical Precision, 6 were coded as Content Error Correction, 4 were coded as Knowledge of Content and Students Suggestion, and 2 were coded as Typo/Other. Select examples of each error can be seen in Table 2.

Table 2
Sampling of lesson plan errors and feedback given by Teaching Assistant

PST Lesson Plan Error	Teaching Assistant Feedback	Code
Make observations about how logarithmic and polynomial functions are different.	Exponential functions – there are no logarithmic functions included in the number talk.	Mathematical Precision
A student may incorrectly generalize from their classes on exponents, that "negative values are impossible.""	I'm not sure what you mean by this – they haven't learned that negative values are impossible as exponents… Or do you mean that 10 to a power can never give you a negative value?	Mathematical Precision
This [standard] is used when students recognize that the x values and y values are constant when looking for slope (linear)"	The change in x values and y values is constant, not the values themselves.	Content Error Correction
The inputs of the first function equals the outputs of the second function and hence are inverses.	And vice versa – without also looking at the outputs of the first function and the inputs of the second function, you don't have enough information to say they're inverses.	Content Error Correction
Properties of Even/Odd Functions: Symmetric over the y axis (even) or origin (odd)	I would be careful with how you describe this – if the students are only thinking of reflective symmetry and not rotational symmetry, this could be pretty confusing.	Knowledge of Content and Students Suggestion
Also, just before the explore activity, we plan on having a "bridge" activity to list what they think log problems are.	Should [problems] be "properties"?	Typo/Other

Kinds of Content-Related Errors During Instruction

Very few mathematical errors occurred during instruction (n=5), and no errors were repeats of those addressed during the lesson planning phase. The low number of errors and lack of repetitive errors indicates that receiving feedback during the lesson planning phase was successful in preventing many instructional errors. Of the five errors requiring intervention from the mathematics teacher educator (MTE) observing their instruction, four were coded Content Error Correction and one was coded Mathematical Precision. There were also two styles of interventions that occurred. In three of the error instances, teaching assistants made inquiring questions or comments to assist the PSTs in recognizing their error and worked with the PSTs to correct themselves and move on more naturally. In the other two instances, however, PSTs had to take a more direct intervention approach where the MTE took over instruction in order to avoid student confusion. In both instances, PSTs resumed instruction when the MTE finished the explanation, continuing their instruction as planned. Intervention sequences were brief, with the longest being only 3 minutes and 20 seconds (and that one sequence included two separate errors requiring intervention).

Discussion and Conclusion

Although our sample size is small, the results suggest the need for further inquiry into fundamental conceptions that secondary prospective teachers hold about the mathematics they will be teaching. Existing literature documenting the nature of secondary mathematics' PSTs content knowledge for teaching is sparse, with a few studies in areas such as geometry (Herbst & Kosko, 2012) and rational number (Depaepe et al., 2015), yet nearly all of the existing work has focused on capturing knowledge through written assessment measures rather than assessing knowledge as it is used in instruction. However, this study, along with work by Snider (2016), begins to unpack the nature of secondary mathematics' PSTs content knowledge for teaching as it is used in instruction.

The findings suggest at least two areas worthy of further inquiry. First, given the prominence of algebra in the secondary curriculum, it is important to acknowledge that participating PSTs needed further support in developing their understanding of topics such as invertible functions, composition of functions, and properties of exponential and logarithmic functions. The fact that these topics are difficult for secondary PSTs is not surprising as these are traditionally topics that pose difficulties for students in college algebra. However, our findings show that the additional coursework the secondary PSTs completed to prepare them for teaching mathematics did not resolve their misunderstandings or, for instance, add to their awareness of using mathematically precise terminology when discussing these topics in instruction.

Second, relatedly, our research raises the question of how best to develop PSTs content knowledge for secondary mathematics instruction. If, ultimately, strengthening PSTs content knowledge as used during instruction is the goal, then more attention should be paid to both researching knowledge as it is being used as well as strengthening knowledge within the context in which it is being used. For example, many of the interventions by teacher educators in this case involved issues of using mathematically precise terminology, because being precise contributes to clear communication with students and minimizes opportunities for confusion. Yet, it is no surprise that PSTs might not have received feedback about mathematical precision in their mathematics coursework if the work they produced resulted in a valid answer. The key obligations of mathematics teaching (Herbst & Chazan, 2012), such as managing the learning needs of a classroom of individuals, that may elevate particular aspects of content knowledge as especially important for teaching. The design and implementation of "content-focused" methods courses might be particularly promising for not only addressing the question of developing content knowledge for teaching as it is used in teaching but also serving as a productive site for collaborations between mathematics educators and mathematics teacher educators.

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"Derivative makes more sense with *differentials*": How primary historical sources informed a university mathematics instructor's teaching of derivative[1]

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In this brief research report, we address the recent calls to improve undergraduate mathematics instruction through our investigation of an instructor's teaching of derivative in a Calculus course. Considering his efforts to modify the presentation of derivative in the textbook as attempts to improve his teaching as a result of his engagement with primary historical sources, we analyze his teaching to identify the changes in his practice by using Speer, Smith, and Horvath's (2010) framework for "teaching practice." With our analysis of instructor interviews and video-recordings of classroom sessions, we observe that Leonhard Euler's use of differentials in defining derivative had responded to his pedagogical concerns, and had convincing power as a method, which, in turn, led him to make significant changes in how he selects and sequences content for his teaching.
Keywords: Primary Historical Sources, Derivative, Calculus, Teaching Practice

Introduction

When called upon improving their instruction, instructors of undergraduate mathematics are suggested to "present key ideas and concepts from a variety of perspectives, employ a broad range of examples and applications to motivate and illustrate the material, promote awareness of connections to other subjects, and introduce contemporary topics and applications" (Saxe & Braddy, 2015, p. 1). Given the importance of classroom mathematics teaching for student learning (Hiebert & Grouws, 2007) and the many challenges that an instructor has to deal with regarding, how can an instructor of mathematics ensure the completion of tasks that are expected from her? How can she manage to create learning environments that are meaningful to her students for every class that she is supposed to teach?

In this regard, a more important question to ask is about the support that instructors receive, rather than expecting them to meet the needs of students, departments, and institutions on their own. Instead, given the complexity and difficulty of teaching mathematics in itself along with all the logistics that an instructor has to deal with to create a learning environment, regardless of instructor's philosophical or theoretical orientation towards pedagogy of mathematics, instructors should get, to list a few, logistical, curricular, and motivational support. In this preliminary report, we are arguing for the ways that primary historical source can encourage, inform, or guide teaching practice for the teaching of undergraduate mathematics.

Problem Domain and Purpose

The primary motivation of this article is Speer, Smith, and Horvath's (2010) call for more empirical research on collegiate mathematics instructors' teaching practices. To better situate their discussion of instructors' practices, Speer et al. distinguish instructional activity and

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teaching practice. "*Instructional activities* are the organized and regularly practiced routines for bringing together students and instructional materials" (emphasis added, p. 101). Lecture, small group work, or whole-class discussion are some of the commonly used instructional activities at the undergraduate mathematics education. Accordingly, teaching practice refers to "teachers' *thinking*, *judgments*, and *decision-making* as they prepare for and teach their class sessions, each involving one or more instructional activities" (p. 101). Furthermore, we are interested in a situation where history of mathematics was used in a specific way. In Jankvist's (2009) terminology, this report is concerned with the "hows" but not the "whys" of using history of mathematics.

The purpose of this article is to share some preliminary findings of our research where we investigated the ways that primary historical sources can inform, guide, or inspire the teaching practices of a university mathematics instructor in his teaching of derivative. Through sharing the story of an instructor and making the changes in his teaching practices explicit as a result of his engagement with primary historical sources, we are aiming to contribute to "our understanding of collegiate mathematics teaching and of the resources that collegiate teachers, especially beginners, might access to learn about the work of teaching" (Speer et al., p. 99). The research question that our investigation was based on is the following:

In what ways, do primary historical sources, inform, guide, or support a university mathematics instructor's teaching practices for the teaching of derivative in the first course of the Calculus sequence?

Theoretical Framework

Our use of a theoretical framework in this report is to explore the teaching practices of an instructor and how they change as a result of his engagement with primary historical sources, rather than to discuss the effectiveness of such an engagement for student learning. In particular, we use Speer et al.'s (2010) framework on teaching practices to describe the practices of an instructor in his attempts to teach derivative as a result of his engagement with primary historical sources. Due to the space considerations, we only share results for only one component of the framework. There are seven dimensions of teaching practice that are identified by Speer et al. The one that we are interested in this proposal is italicized (a) Allocating time within lessons, (b) *Selecting and sequencing content (e.g., examples) within lessons*, (c) Motivating specific content, (d) Posing questions, using wait time, and reacting to student responses, (e) Representing mathematical concepts and relationships, (f) Evaluating and preparing for the next lesson, and (g) Designing assessment problems and evaluating student work. In this brief report, we are able to analyze our data for one aspect.

Selecting and sequencing content. This component of the framework refers to the content to be taught for a course, the order of topics through the semester, and examples/exercises to be shared with the students are some of the aspects of how an instructor selects and sequences content. As Speer et al. (2010) noted, although instructors mostly rely on textbooks for this aspect of their teaching practice, there are times that instructors may decide to consider, for instance, omitting some parts of a chapter in the textbook, provide students with examples from a different source, or create her own set of exercises for her students.

Methodology

Our research is a result of our interest in an instructor's attempts to modify his teaching of derivative based on his engagement with primary historical sources. Our goal is to provide in-depth description of the experiences and views of the instructor to better demonstrate his interactions with the primary historical sources, and how such interactions led him to reconsider his teaching of derivative. Therefore, our inquiry in this research is qualitative in nature and descriptive by purpose. In Stake's (1998) terms, we identify our research as an intrinsic case study: a result of our interest in the story of an instructor, rather than trying to understand a phenomenon.

The participant of our study is a male mathematics instructor, from now on we call T, who was at his first semester in teaching at a tenure-track faculty position at a university located at Central region in the United States of America. Our data is on his teaching of derivative in the first course of Calculus sequence. Although this was his first semester in teaching Calculus as a faculty member, he had three semesters of experience in teaching Calculus as a doctoral student. His first interaction with primary historical sources is through one of the instructional materials known as Primary Source Projects (PSPs). To describe briefly, a PSP is a curricular material aiming to guide students' reading and study of selected excerpts from primary historical sources. (see Barnett (2012) and Barnett, Lodder, and Pengelley (2014) for detailed information on PSPs.)

The PSP that T used for his teaching is *The derivatives of the sine and cosine functions* (Klyve, 2017), which is designed for two class sessions of teaching. This PSP includes excerpts from Leonhard Euler's *Foundations of Differential Calculus*. Through some excerpts from Euler (1755) and tasks related to these excerpts, Klyve, first, aimed to familiarize students with how Euler used differentials. Consequently, the goal was to share an alternative approach to the limit definition of derivative, where Klyve, eventually, provided how Euler used *differentials* to calculate the derivative of the sine function. Our decision to conduct research on T's teaching on derivative began with his decision on extending the idea of using differentials for the entire derivative chapter of the course. The following quote is used in the PSP as an excerpt from Euler's original text to demonstrate how differential was calculated, and derivative was defined using differentials.

> From this fact there arises a question; namely, if the quantity x is increased or decreased, by how much is the function changed, whether it increases or decreases? For the more simple cases, this question is easily answered. If the quantity x is increased by the quantity ω, its square x^2 receives an increase of $2x\omega + \omega^2$.
> Hence, the increase in x is to the increase of x^2as ω is to $2x\omega + \omega^2$, that is, as 1 is to $2x + \omega$. In a similar way, we consider the ratio of the increase of x to the increase or decrease that any function of x receives.
> Indeed, the investigation of this kind of ratio of increments is not only very important, but it is, in fact, the foundation of the whole of analysis of the infinite.
> In order that this may become even clearer, let us take up again the example of the square $x^2$with its increment of $2x\omega + \omega^2$, which it receives when x itself is increased by ω. We have seen that the ratio here is $2x + \omega$ to 1. From this it should be perfectly clear that the smaller the increment is taken to be, the closer this ratio comes to the ratio of $2x$ to 1. (Klyve, 2017, p. 2)

We collected data through pre-semester and post-semester surveys, interviews, and video recordings of selected classroom sessions. If the instructor believed that he would spend time on differentials, we decided to include that session for video-recording. Each class session was 75 minutes, and ten out of 30 sessions were video-recorded. We conducted two interviews with the instructor: One at the beginning of the semester, where our goal was, basically, to develop an understanding of him regarding his perspective on mathematics, pedagogy, Calculus and its teaching, and his experience with history of mathematics. Second interview was conducted at the end of the semester. We asked him to reflect on his experience by asking some specific questions on his teaching practice. In our analysis, we mainly rely on interviews and video recordings to describe and understand his teaching practice.

We analyzed data, primarily video-recordings in this case, to observe the changes in the teaching practice based on instructor's description of his regular and planned teaching of derivative. In this analysis, we also paid attention to discovering the potential role of his engagement with the PSP and Euler (1755) on the changes in his teaching practice. We discussed our observations and interpretations with the instructor for the validation of findings.

Findings

Although T was about the begin teaching as a faculty member for the first time, he had observed extensively in his prior experience in teaching Calculus that students used to struggle in understanding the concept of derivative. As he stated in the pre-interview, one of the most notable challenges that students experienced was the limit definition of derivative, which was also a challenge for him when he was a mathematics major in his undergraduate program.

T's initial decision to use a PSP for his teaching relied on his interest in the history of mathematics. However, he had never used any primary historical document for his teaching prior to his experience with PSPs. When he decided to make use of the opportunity of using PSPs for his teaching, T's goal was primarily to supplement his teaching with the textbook, which was supposed to take two class sessions. However, as we share in our further analysis, T ended up making fundamental changes in his teaching after his engagement with the PSP.

When asked about his first reaction after his first reading of the PSP, T stated that he was very surprised with the emphasis given on differential in defining derivative concept since differential was mentioned in the last section of the nine sections in the derivative chapter of the Calculus textbook. In his words during the pre-interview, "but when you look at the textbook, there are nine sections in derivative chapter and the differential section is the last section." He continued as follows to describe his reaction to the importance given on the differential in the PSP:

> When I look at my previous experiences, students cannot really learn the definition of the derivative, limit definition of the derivative and they just memorize the formula. […] They do not really learn what is going on, why that formula works, what that *dx* means in the formula. But […] when I define the derivative using differential and when I first explain the idea of using differential to them, and then using that idea to computing and defining the derivative, I believe and I expect […] they will really understand what is going on in the definition of the derivative and what the derivative is.

In this regard, it is important to note that differential as a mathematical idea central to the definition of derivative provided the instructor with a vocabulary so that he believed that he had the tools to communicate commonly used symbols in derivative, *dx* and *dy*, with students in meaningful ways. For instance, using differentials in defining the derivative allowed T to provide a justification for why Leibniz's notation in chain rule makes sense, and why it works, first of all, for himself as an instructor of mathematics. To us, Euler's approach, using differentials to define derivative, allowed T to produce narratives on derivative that, first, convinced him as a learner of mathematics. Therefore, he proceeded with modifying his teaching practice expecting that Euler's approach would also support a meaningful conceptualization of derivative.

Next, we share the significant changes in T's teaching practices as a result of his engagement with the PSP and Euler's approach for defining derivative using Speer et al.'s (2010) framework. Due to the space considerations, we report our findings on *selecting and sequencing content* aspect of that framework.

Selecting and Sequencing Content

In the pre-interview, when asked about how he used curricular materials, in particular textbook, informed his teaching, T told that textbook would be the main guide for his teaching in planning and delivering his lectures. The textbook used to dictate, as he expressed, almost all of his teaching practices, including how he defined the concepts, the examples he used to explain mathematical ideas, and the exercises that he asked students to work on in his prior teaching experiences, and would dictate if he did not meet Euler's approach.

Following his interaction with the PSP (Klyve, 2017), and Euler (1755), T did not only replace the limit definition of derivative with Euler's approach using differentials, but also, he redesigned each section in the derivative chapter of the textbook. For instance, he used differentials while introducing the differentiation techniques for the derivatives of constant and identity functions. As another example, in the product and quotient rule section, he said in the class, "we will go back to 1700s and visit Euler in his office and ask him how we can take the derivative of product of two functions. Let's see what he is doing" and showed what those rules are and why they work while using the differential approach. He used the Leibniz notation as the primary representation for derivative in his teaching. Associating it with the phrase "crucial word," he used to call "the ratio of $\frac{dy}{dx}$," as the "magical ratio."

Therefore, we conclude that T's engagement with the PSP (Klyve, 2017) and Euler (1755) provided him a different perspective on conceptualizing, defining, and introducing derivative, which ended up with significant changes in how the content was presented to students.

Discussion Questions for Further Analysis

For our work in this report, we found Speer et al.'s (2010) framework as an effective tool to explore the teaching practices of a mathematics instructor and investigate the changes in his practice as a result of his engagement with a PSP (Klyve, 2017) and Euler (1755). Clearly, T's interest in the history of mathematics was influential on his interest in using Klyve's PSP. However, based on what our data suggests, we argue that it was Euler's approach that triggered the changes in the teaching practice. Although the effectiveness of these changes in teaching practice on student learning is a question of interest, we believe that finding a motivation for instructional change is noteworthy.

In this regard, we highlighted the role of primary historical sources in this proposal, but we also believe that further research needs to consider instructor characteristics as an aspect of investigation to deepen our understanding on the dynamics of change in teaching practice.

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Business Calculus Students' Understanding of Marginal Functions

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Business majors represent a significant proportion of the population of students enrolled in calculus at the college level. However, there is a lack in research literature that tackles the teaching and learning of business applications at this level. This pilot study represents the beginning phases of a project that aims to investigate business students' reasoning through tasks pertaining to marginal analysis (derivatives in a business context), accumulation functions and Riemann sums. A preliminary analysis of interviews with two pairs of students is presented, with an emphasis on their thought process while answering questions related to cost, revenue and profit functions as well as their marginal counterparts. The context-based activities were designed with a realistic mathematics education perspective, motivated by guided reinvention.

Keywords: Business Calculus, Marginal, Derivatives, Realistic Mathematics Education.

The teaching and learning of calculus, relative to the theoretical advances in education, is a research area that has only recently gained the interest of mathematicians, educators and psychologists. Researchers such as Warnock, Orton, Tall, Vinner are considered the founders of the aforementioned field, and it was their work in the early 1980s that created the foundations for future research and a need for curricular reform (Rasmussen, Marrongelle, & Borba, 2014). Standard calculus topics such as limits and derivatives have been typically researched over the past couple of decades, but it was not until the mid-2000s that the teaching and learning of Riemann sums and definite integrals became an area of interest for some researchers. When it comes to the teaching and learning of business calculus, there is a scarce amount of research that deals with the cognitive obstacles that students face. In fact, business students represent about 45% of the students that are enrolled in a first semester calculus at the university where this study was conducted. The lack of representation of this population of students in research studies is problematic.

Motivation and Relevant Literature

Originally, the pilot study the authors-researchers had in mind was designed to tackle students' understanding of accumulation and Riemann sums in business contexts. The activity we are considering in this report was intended to prime them on rate of change in a business setting through the context of marginal cost, revenue and profit. We assumed that students came in with the knowledge since they had already covered it in class. When analyzing their work on that introductory task, we found that their understanding of derivatives and marginal cost, revenue and profit was not as fully developed as we had expected. Our research focus thus shifted to analyzing what the participants understood in order to build a new activity centered around marginal functions.

Throughout the interview, students struggled to express their understanding of marginal quantities relatively to the context. This issue seemed to support previous theories that students' struggle with the concept of integration is strongly related to a poor understanding of rates of change (Kouropatov & Dreyfus, 2014; Thompson, 1994; Thompson & Silverman, 2008). Some studies documented the main issues that arise with students' interpretation of derivatives and noted that they generally perform derivative computations without paying close attention to what the values represent (Bressoud, Ghedamsi, Martinez-Luaces, & Törner, 2016). When it comes to studies that analyze how students reason through problems involving applications of derivatives in business contexts, research is very limited. To our knowledge, only Mkhatshwa and Doerr (2016, 2017) first investigated context-based opportunities to learn for business calculus students and then focused on revenue maximization applications. In addition, students primarily view integrals as a tool to calculate areas of unconventional shapes using the antiderivative of the integrand. This is not enough for them to thoroughly understand the multiplicative summation structure and thus utilize it in non-routine situations (Jones, 2015; Sealey, 2006). Therefore, we took this opportunity to first analyze business calculus students' understanding of marginal functions, which would eventually reinforce their understanding of accumulation functions, Riemann sums and definite integrals conceptually rather than algorithmically.

Because this study represents the beginning phases of a bigger project tailored towards the aforementioned topics, our analysis revolved around the following questions:

- How do students interpret and analyze the cost, revenue and profit functions as well as the relationship between them?
- What are some of the observations that can be made with regards to student interpretation of marginal cost, revenue and profit values on optimal business strategies?

Theoretical Perspective

The original tasks, including the one we focus on in this report, were designed under a Realistic Mathematics Education (RME) lens through guided reinvention. RME is a teaching and learning theory developed at the Freudenthal Institute in the Netherlands. Historically, mathematics instruction is typically done through formal definitions, theorems and occasional proofs. Contextual applications are usually given as concluding activities to relate the formal theory to real life examples. RME advocates argue that mathematics should be viewed as a *human activity* (Hough & Gough, 2007). To this end, guided reinvention is utilized so that students engage in their own learning and "reconstruct" the mathematics that they are expected to learn (Freudenthal, 1978; Stephan, Underwood, & Yackel, 2014). In addition, the context and models should be *experientially real* to the students, in the sense that students need to connect what they are doing to the ultimate goal of the lesson (Stephan et al., 2014). According to Treffers (1987), teaching from an RME perspective requires the use of contexts and models, allowing students to construct their own mathematical understanding through interactive learning. Our sequence of tasks was designed to help students reinvent the big ideas within accumulation in a Business Calculus context. Due to the obstacles that appeared during students' interpretation of marginal values, we decided to limit this study to an analysis of the latter topic.

Context wise, marginal cost (or revenue or profit) is the instantaneous rate of change of cost (or revenue or profit) relative to production at a given production level (cite book). Hence, if x represents the quantity of items produced and sold in a hypothetical business context, the marginal revenue $R'(x)$ is the derivative of the revenue function $R(x)$, the marginal cost $C'(x)$ is the derivative of the cost function $C(x)$, and the marginal profit $P'(x)$ is the derivative of the profit function $P(x)$. For instance, a value of $P'(50) = 24$ means that the marginal profit at a production level of 50 items is 24 dollars/item. This implies that if the company produces and sells one additional unit, thus at the sale of the 51^{st} item, it is expected to gain about \$24. The same reasoning applies to marginal cost (estimate for the cost of production of the $(n + 1)^{st}$ item) and marginal revenue (estimate for the revenue generated from the sale of the $(n + 1)^{st}$ item) at a production level of n items.

Methods

During a summer semester at a large university, students from a business calculus course were given the opportunity to participate in a 90-minute recorded session while they work through all the tasks that were originally designed, and answered some questions asked by the researcher. The task that is the focus of our report situated students in a hypothetical jacket manufacturing company. Given the fixed and variable costs of production, as well as a quadratic revenue function, they were asked to develop a model for both the cost and revenue functions, as well as the corresponding marginal functions. Then, students had to evaluate those functions and their marginals at two different levels of production in order to decide whether or not it would be a good business move for the company to produce that many jackets. The last question prompted students to find the production level that would maximize the company's profit.

Two pairs of students volunteered to participate. They were split in two groups: Piper and Jay (Group 1), Mo and Ty (Group 2). The students had diverse ethnic backgrounds (African American, Hispanic and white). The researcher allowed students to get comfortable working together first and limited his interaction with them until it was clear that they were collaborating and sharing ideas. For both pairs, the first task (the one we are analyzing in this report) took about 45 minutes to complete, which included discussion time with the researcher. Hence, around 90 minutes of audio-video footage were analyzed using an open coding process in which annotations and comments were split into the two major themes that are elaborated in the next section. Below is a reproduction of the questions presented to the students in this first task.

A company manufactures jackets. The costs of rent and utilities are fixed at \$2800 per month, and each jacket costs \$4 to produce. In addition, suppose that the revenue function is given by $R(x) = 320x - 0.08x^2$ where x represents the number of jackets produced monthly.

a) Find the cost function and the profit function.
b) What is the cost of producing 1000 jackets monthly? Find the corresponding revenue and profit.
c) Do you think it is a good idea for the company to produce 1000 jackets monthly? Elaborate.
d) What is the cost of producing 3000 jackets monthly? Find the corresponding revenue and profit.
e) Do you think it is a good idea for the company to produce 3000 jackets monthly? Elaborate.
f) Find the marginal cost, marginal revenue and marginal profit functions.
g) Evaluate the marginal cost, revenue and profit when producing 1000 jackets. Interpret your answers.
h) Evaluate the marginal cost, revenue and profit when producing 3000 jackets. Interpret your answers.
i) Are you still certain from your answers to parts c) and e)? Explain.
j) How many jackets do you think the company should produce and sell in order to maximize profit?

Figure 1. Questions from the first task of the interview.

Preliminary Findings

After a careful examination of the conversations between the participants, as well as the participants and the researcher, it seemed like students' have uncertainties in how the cost, revenue and profit functions are related as well as what are the implications of the marginal function values at specific levels of production.

Analysis and Interpretation of Business Functions

The first half of the task tackled students' familiarities with functions that model the cost, revenue and profit. Given fixed costs for production and variable unit prices, students were asked to model a cost function and were expected to utilize a linear model. After some guidance from the researcher, group 1 correctly modeled the cost function. When asked to write an expression for the profit function, Piper noted that "profit equals revenue minus cost because revenue is more, and cost is less", which may indicate she believes that profit always represents a positive quantity, a gain. Given a production level of 1000 jackets where profit is positive, the students were asked if it is a good idea for the company to produce that many jackets. Students in this group thought it was indeed beneficial because “the profit is substantially greater than the initial cost”, seemingly thinking of those as two comparable quantities. When asked to find the maximum profit, Piper and Jay utilized Desmos to plot the graph of the profit function and were able to locate the maximal value. Students in group 2 also compared cost to profit at a production level of 1000 jackets. Remarkably, when production level changed to 3000 jackets, they compared the cost and profit at that level to those at 1000 jackets “[the company ends] up making the same amount of money, but it costs more to produce”. When asked to find the maximum profit, Mo and Ty started by equating the revenue to the cost function. This reflects a confusion between the concept at hand and the break-even points, where the profit is in fact null. After discussing this idea with them, they took a graphical route. Surprisingly, they plotted the cost and revenue functions, but not the profit function and tried to estimate the production level (x value) that “maximizes the distance between the cost and the revenue […] biggest positive difference since you can have a bigger difference down there [pointing at regions of loss] but then [the company] would be losing money.” Their analysis of the difference between revenue and cost in lieu of analyzing the profit function directly indicates they may not have a robust understanding of the relationship between the three quantities.

Analysis and Interpretation of Marginal Functions

The second half of the task prompted students to answer questions pertaining to marginal cost, revenue and functions. Both groups had no issues in connecting marginals to derivatives and were able to find them using the power rule for polynomials. Group 1 was also familiar with the linearity property of the derivative, since they subtracted the marginal cost from the marginal revenue as a shortcut to finding the marginal profit. However, all four students seemed to face some obstacles when asked to interpret the marginal values they obtained at given levels of production. The hindrances started during a conversation about what “derivative” means to them in any context. One particular answer reflected students’ association of derivatives with an algorithmic process without paying close attention to its connection with rates of change: “it’s kind of like taking the original form and then transforming it or condensing it into something else”. In addition, For the given levels of production, students struggled to utilize correct terminology while interpreting the marginal values they obtained. For instance, Piper noted that “at [a production level of] 1000 jackets, the ROC for profit is 156, and there is no ROC for the cost”. Besides not using units with their values, it seemed like a constant rate of change for a function was mistaken with the function itself being constant, thus having a null rate of change.

Perhaps the most notable observation, after redirecting the students to derivative values being slopes of tangent lines, is that students used linear approximations of functions to approximate the additional revenue and profit for any other level of production. The excerpt below showed us elements of a productive understanding of liner approximations and how they relate to rates of change, but the robustness that is needed for a more advanced interpretation was yet to be developed. While the reasoning below applies to the case of linear cost functions (the marginal cost is constant thus each additional unit costs the same to produce), it cannot be extended to the case of marginal revenue and marginal profit. The following is an excerpt of Mo’s interpretation for the marginal values he found at a production level of 1000 jackets.

Mo: So, for marginal cost, it's like per unit or whatever so for cost, it's going to be each unit costs $4 to produce on top of the last one. And then same for [the revenue and profit], each unit nets us $160 more revenue than the last one we made, and each unit gives us a profit $156 more than the last one. *(Ty agreed with Mo's statement)*

Interviewer: What do you mean by "each unit"?

Ty: Like each additional unit

Interviewer: So, if I'm at 1000 [units] and I [produce] one more unit, then […] the revenue is going to be $160. If I produce 10 more units after, is my revenue going to be $1600?

Mo: I think that's the assumption that it will do that as long as you're basing it on that 1000

Interviewer: Your starting point is 1000 [jackets] and after that you can take any value […] and estimate the additional revenue?

Mo: Yes, so like when we base it on 3000 [jackets] now we have a negative number because we're starting to lose money on each one that we make additionally.

Interviewer: Okay so your "each" means that starting with 1000 or 3000, you can [increase] by as much as you want, say by 10, 100, 500 units… and then that would tell you what your additional cost, revenue and profit are?

Ty: Yea for each one, I guess.

This presents evidence to support our next claim that Mo and Ty understand the effect of a marginal value's sign (positive or negative) on gain or loss. However, the use those values as *local* approximations of the additional revenue or profit is a skill that is yet to be acquired.

Implications

Our preliminary findings suggest that students did not master additional preparation in order to give correct interpretations of the marginal cost, revenue and profit functions. Finding derivatives by hand is a skill that is typically focused on during traditional calculus courses but with all the software available to do that in practical applications, it would be more beneficial for them to demonstrate strong analytical skills through interpreting the meaning of the marginal values. Taking our observations in this study into consideration, our next step would be to conduct a teaching experiment that emphasize on ideas that are not typically focused on, such as the profit being a quantity that could be positive or negative, the revenue and cost being comparable and how they relate to profit, as well as using correct vocabulary and units to describe the meaning of the derivative in any context. Thus, our ensuing goal is to create a sequence of tasks that guide students through the theoretical underpinnings of business functions and their marginal counterparts as bases for optimal business strategies. We have posited some initial learning goals for the next tasks that will be tailored towards students being able to:

1. Write and evaluate the cost, revenue and profit functions using given information (fixed and variable costs, price-demand equation…)
2. Interpret and analyze the cost, revenue and profit functions at a given level of production
3. Derive the marginal cost, revenue and profit functions.
4. Interpret and analyze the marginal cost, revenue and profit at a given level of production.

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How Do Students Interpret Multiply Quantified Statements in Mathematics?

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We presented introduction to proof students from five different US universities with multiply quantified statements to assess and interpret. The survey was designed to allow us to compare the influence of syntax, semantics, and pragmatics in student interpretation. We analyzed the ways students interpreted the statements both before and after instruction. Current analysis suggests that students became more sensitive to syntax (reversing quantifier order) after instruction and became better able to construct a semantically odd construal (e.g. the distance between two points is equal to multiple numbers). Our analysis of pragmatics suggests that students were more likely before instruction to construct a relevant construal, but we did not find evidence that truth-value influenced students' interpretation of the given claims.

Keywords: Logic; Multiple Quantification; Introduction to Proof

Advanced mathematical language involves a number of very particular conventions of syntax and interpretation because mathematicians strive to communicate intended meanings with fidelity. Many previous studies have particularly investigated how students make sense of statements that combine universal (∀) and existential (∃) quantifiers, what we shall call multiply quantified (MQ) statements. Such statements appear quite frequently in advanced mathematics and experts almost always use them in a consistent manner, though the precise nature of the relationships conveyed varies in important ways (c.f. Durand-Guerrier & Arsac, 2005). These studies have assessed students' naïve readings of such statements (Dubinsky & Yiparaki, 2000) and have proposed and evaluated certain methods of teaching students to interpret MQ statements as mathematicians do (Dubinsky, Elterman, & Gong, 1988; Dubinsky & Yiparaki, 2000; Durand-Guerrier & Arsac, 2005; Roh & Lee, 2011). This study seeks to extend our insights into student interpretation of MQ statements by contributing a conceptual analysis of the interpretation process that is evaluated through survey instruments administered to introduction to proof students both before and after instruction. We investigate roles syntax, semantics, and pragmatics each may play in the ways students construct meaning for MQ statements.

Interpreting MQ statements in mathematics resides at the interface between mathematical logic and mathematical language. We concur with previous authors that while there exist formal rules for trying to render mathematical language purely syntactic (able to operate by precise rules ignorant of subject matter), mathematicians rarely operate in this manner and teaching novices will almost certainly require some balance between syntactic rules and semantic sense-making (Durand-Guerrier, 2003; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012).

Conceptual Analysis and the Research Tasks

Previous studies have used the language AE ("for every-there exists") and EA ("there exists-for every") to alternatively refer to 1) the structure of a mathematical statement, 2) the normative interpretation shared among mathematicians, and 3) a student's interpretation of those statements. While we continue to use those two-letter codes for convenience, we adopt a different terminology to distinguish these constituents of the analytical process. Figure 1 presents the four statements (these are technically predicates, but we shall reserve that term for something else) that we asked students to interpret, each regarding two different referents. The *wordings of* *the statements* clearly exhibit AE or EA structure. We refer to the meaning an individual makes for any such wording as their *construal* of the statement. The construal shared among mathematicians – AE means "each to some" and EA means "one to every" – we call the *normative construal*. Each student then construes each statement in ways tantamount to "each to some," "one to every," or something else.

S1. "There exists a real number M such that for all real numbers x, $f(x) < M$." S2. "For all real numbers x, there exists a real number M such that $f(x) < M$."	referents	$f(x) = 3x + 2$ $f(x) = \sin(x)$
S3. "For every positive real number s, there exists a point C on the segment [ray] such that $d(A, C) = s$." S4. "There exists a point C on the segment [ray] such that for every positive real numbers s, $d(A, C) = s$."	referents	segment $\overline{AB}$ ray $\overrightarrow{AB}$

Figure 1. The four statements and four referents comprising the study tasks.

We parse the elements that students may use to construct meaning in the following way: *quantifiers*, *predicate*, and *referent*. For instance, for S1 the quantifiers are "There exists a real number M such that for all real numbers x," the predicate is "$f(x) < M$," and the referent is "$f(x) = 3x + 2$" or "$f(x) = \sin(x)$." To observe the influence of each, our survey alternated the order of quantifiers, the mathematical context and predicate, and the referent within each context. Students may alternatively give meaning to a statement like S1 by constructing a meaning for the syntax of quantification (one M that satisfies the predicate for all x) or using their semantic knowledge of the boundedness property of the referent $f(x) = \sin(x)$.

To assess the role of pragmatics, we operationalized two of Grice's (1975) pragmatic maxims. Grice's maxims express rules by which interlocutors in discourse may draw reasonable implications from another's statements (possibly beyond the express meaning). We consider two: a Maxim of Quality "Try to make your contribution one that is true" (p. 46) and a Maxim of Relation "Be relevant" (p. 46). If this maxim were operative, then we would expect students to attempt to construe a false statement in some way that made it true. This maxim would be inert in interpreting a true statement. We expect this effect is preconscious, and we looked for its effect on the first statement students read in each context. The normative construal of both Statements 2 and 4 are semantically uninteresting (the former is always true and the latter is patently false), which we consider violations of the Maxim of Relation. Thus, if students avoid such a construal, this is evidence of the role of pragmatics in interpretation.

Methodology

We designed a survey that consisted of four pairs of tasks using the MQ statements and referents in Figure 1. Each task presents a pair of MQ statements differing only by the order of quantifiers. We refer to the task as follows: S1 – EA function, S2 – AE function, S3 – AE geometry, and S4 – EA geometry. The task presentation follows for two requests for response: (1) the truth-value (true or false) of each statement for the given referent and (2) an explanation of what each statement says about the given referent.

We created two versions of the survey instrument: True-first version and False-first version. These two versions of the survey instrument contain the same tasks – four function tasks first followed by four geometry tasks – presented in different orders. For each task group (either function or geometry), T-first version presents a referent first that makes the first MQ statement in the pair to be true (EA – sine/ AE – ray), whereas F-first version presents a referent first that makes the first MQ statement in the pair to be false (EA – line/ AE – segment).

Six instructors of introduction to proof courses from five different universities in the United States allowed their students to participate to our research study in Spring 2018. We randomly assigned the student participants into two groups. To facilitate the multi-site data gathering, students completed the surveys online through an emailed link. Students were invited to complete the survey both before and after their class covered topics related to MQ statements. In this paper, we report our results from the 77 students who completed both pretest and posttest.

We first compared students' responses to the determination of the truth-value for each statement and their explanations about what each statement says. We coded a student response as EA if it exhibits "one to every" structure, AE if it exhibits "each to some" structure, and OTHER if the student construal conveyed neither such relationship or it appeared the student construed a different predicate or referent. Once we coded all student responses to each statement the tasks in terms of the three codes, we calculated how frequently students construed each statement-referent pair in a normative way. For instance, AE-sin refers to the percentage of students who interpreted S2 with reference to the sine function as an "each to some" relationship. The next section presents our preliminary analysis of these frequencies of normative construal.

Results

Figure 2 presents the rates of normative construal by group and time. These data show two initial trends: students more frequently construed the first statement in each pair normatively and the AE sine task resulted in the lowest percentages of normative construal overall. The first pattern results in the jagged appearance of each graph. This reflects on our hypothesis about pragmatics, namely that students were less likely to construct the normative construal when its contextual meaning was either obvious (the EA function statement) or patently false (the AE geometry statement). A possible alternative explanation has to do with the order of appearance, since students always saw the more "natural" (according to normative construal) statement first.

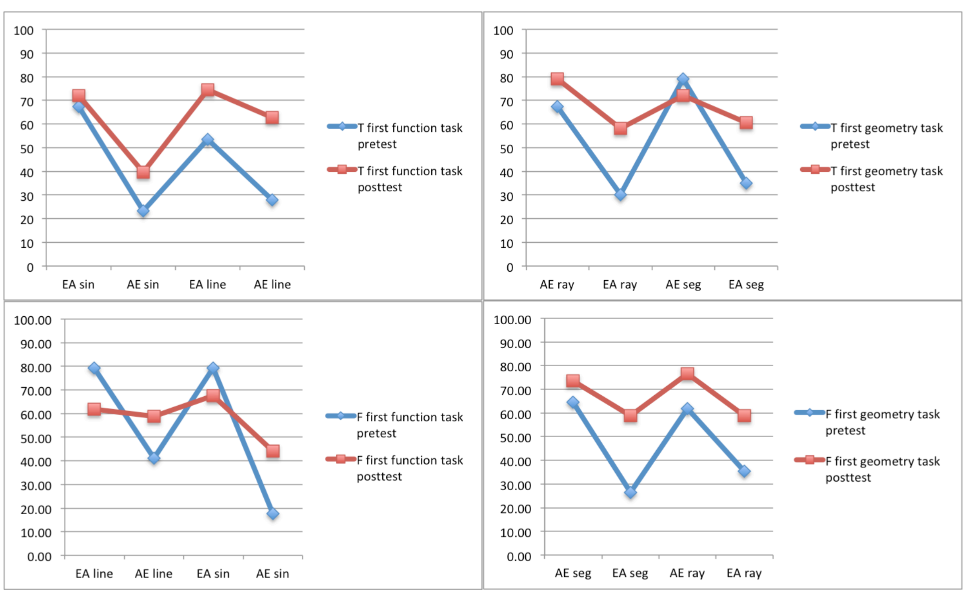

Figure 2. Percentages of normative construal, organized by mathematical context and group.

It may be that students construed the second statement less normatively because they had to develop a new construal for a very closely related statement, and this was more challenging. Our current study design does not allow us to fully distinguish these two explanations.

The AE sine task's low normative construal rate should be viewed in part as a byproduct of gathering data in surveys rather than interviews. While S2 entails a slightly different construal than S1 (e.g. M could be .5 when $f(x) = 0$), both statements can be verified by a single M. Under either construal S2 is true, and students declared it so 88% of the time. When a student explains their interpretation of the AE sine task by noting that $M = 2$, this is insufficient evidence to indicate whether the student held a "one to every" or "each to some" construal. Without evidence that students thought that M could vary with x, we did not code their responses as an "every to some" construal. It is likely that more students responded to the AE sine task according to a normative construal, but their explanation did not provide enough evidence for us to discern it. Many other explanations provided clearer evidence of either a "one to every" or an "every to one" construal, but we had to choose a system for coding ambiguous responses.

If one ignores the AE sine tasks, a third pattern arises from the data in Figure 3: instruction greatly increased the rate of normative construal for the more difficult statements (function EA and geometry AE) and resulted in a much more consistent rate of normative construal across group and context. Indeed, the rate of normative construal was above 58% (and below 80%) on all of the posttest tasks (data points marked with squares in Figure 2) except the AE sine task. We currently do not have a clear explanation for why the rate of normative construal actually decreased after instruction for some groups on some tasks.

Influence of Task Order

One of our primary hypotheses regarded the influence of Grice's Maxim of Quality that students might be prone to interpret statements to render them true. Operationally, would reading false statements first make students more likely to search for a (non-normative) construal that rendered the claim true? Comparing the two group's construal of each task above, the rate of normative construal differed by 10% or more on the following tasks: EA sine pre (F-First +12%), EA line pre (F-First +25.9%), AE line pre (F-First +13.3%), AE segment pre (T First +14.4%), and EA line post (T-First +12.7). Thus, the strongest evidence that the order of presentation affected student construal appeared on the function tasks prior to instruction. In this case, we see that the F-first group (who read a false statement first) were much more successful in constructing the normative construal of both EA function tasks. This suggests that in this context, reading the statement with reference to the linear function first aided students in construing the definition of bounded above with reference to both functions. This disconfirms our hypothesis that reading the definition of bounded above with reference to a bounded function would aide in developing a normative construal.

However, the geometry tasks caution against a simple explanation that seeing a false statement first is better. The T-First group fared better than the F-First group on three of the four geometry pretest tasks, with differences ranging from 5.1% to 14.4%. This means the group who saw the ray first (of which S3 is true) more frequently construed the geometry tasks normatively than did the group who saw the segment first (of which S3 is false). So, while there seemed to be some effect due to order of presentation, it varied with semantic content and not merely with the truth-value of the statement. This suggests that semantic content was more salient in student interpretation than was Grice's Maxim of Quality.

Influence of Quantifier Order

We assessed the influence of syntax, focused on the quantifier part of the statement, by comparing each student's construal of statements that varied only in the order of quantifiers. Figure 3 presents the percentage of students who construed corresponding EA and AE statements with the same construal. The sine task showed the greatest frequency of invariant construal at both times. Prior to instruction, students construed the other three pairs of statements the same between one third and one half of the time. After instruction, this rate dropped from between one tenth to one third of the time. Thus, reversing quantifier order frequently did not elicit a novel construal before instruction and instruction made students more sensitive to quantifier order.

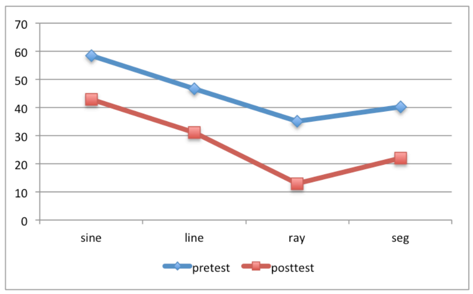

Figure 3. Frequency of students construing different quantifier order statements the same way.

Discussion

This study investigates the resources that students use to give meaning to MQ statements. Our study design helped us to compare the relative roles of syntax, semantics, and pragmatics. Initial analysis of the data suggest that all three played some role in interpretation, and each aspects was at times inoperative in interpretation (for at least some students).

Syntax clearly influenced student interpretation, inasmuch as students normatively construed the various statements with at least modest frequency, especially after instruction. However, before instruction students also constructed the same construal for AE and EA statements at least 30% of the time. Semantics clearly played a role inasmuch as the pattern of interpretation was quite different between function and geometry settings. As was expected in the study design, it appeared that "one to every" relations were easier to construct in the function context and "each to some" relations were easier to construe in the geometry context. Instruction seemed to shift interpretation from semantics toward syntax inasmuch as the posttest rates of normative construal were less varied.

Regarding pragmatics, we did not find support for the claim that Grice's Maxim of Quality influenced students' interpretations. On the function items, students fared better when they first read the definition of bounded above with reference to an unbounded function. A possible explanation is that since they could not give meaning to the statement based on their understanding of the sine function's prominent property of being bounded, they had to attend more closely to the quantifier structure. The data may support the role of the Maxim of Relation in explaining why certain statements were uniformly harder to construe normatively, but we cannot rule out that order of appearance explains this pattern instead.

Ongoing analysis will attend to other details of student construal such as how explicitly they explained quantification and the dependence between variables. We also plan to conduct statistical analysis on the data presented here. In our presentation, we will discuss the following:

1. How can we explain the reduced rate of normative construal on some tasks?
2. What other comparisons and analyses should we conduct on the data?

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Mathematics on the Internet: Charting a Hidden Curriculum

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I report on a pilot study for an explanatory multi-method (Creswell & Plano Clark, 2011) research project that examines how undergraduate students in mathematics courses make use of online resources in order to assist with their studies. A survey of 42 students in a diverse undergraduate institution along with 4 semi-structured follow-up interviews were used to collect preliminary data on how these undergraduates make use of the internet as well as to test the data collection protocol. Initial findings suggest that students make use of online resources (beyond those assigned by the instructor) extensively and to a greater extent than in other subject areas. I also report on which resources are being used by students and find evidence of two distinct ways in which these resources are being employed. Questions will be posed about how an expanded follow-up study can best be of service to the mathematics education research community.

Keywords: Information literacy, Information seeking behavior, Multi-Method

Introduction

Students in mathematics courses spend the bulk of their time working outside of class and much of that work takes place in an online environment. Students may post textbook questions to mathematics forums, trade advice regarding the best YouTube tutorials, make use of online graphing calculators, or even sometimes go cold turkey when they realize that they are relying on the internet too much for help. These are all anecdotes related to the author by his students and, while of limited use for making generalizations, they serve to illustrate the variety of hidden work that often occurs out of the instructor's sight. The invisible nature of this work is of particular concern when we consider those student populations (e.g., nontraditional, first generation, minoritized) for whom mathematics traditionally serves as a gatekeeper (Gainen, 1995; Atanda, 1999; Eagan & Jaeger, 2008; Martin, Gholson, & Leonard, 2010).

While there is a growing body of research devoted to undergraduate students' general online searching behavior (Rowley & Urquhart, 2007; Urquhart & Rowley, 2007; Nicholas, Huntington, Jamali, Rowlands, & Fieldhouse, 2009; Lai & Hong, 2015), there is very little work that looks at how students are employing online resources for to help themselves learn specific disciplines. Undergraduate students make use of the internet as a supplement to their learning with increasing frequency (Selwyn, 2008; Lai, Wang, & Lei, 2012). This usage cuts across all demographic groups (Selwin, 2008) including those nontraditional, first generation, and minoritized students who are already marginalized by the education system (Stone, 1998; Stein, Kaufman, Sherman, & Hillen, 2011). Unfortunately, this informal use of the internet is not reflected in teacher training or professional development. Practicing mathematics instructors' knowledge of which resources students are using and how they are using them is idiosyncratic because there exists no readily available knowledge base about such usage. Accordingly, this proposal will report on a pilot study that employs an explanatory multi-method approach (Creswell & Plano Clark, 2011) in order to a) describe the extent and type of online resources that students are using to help them study for mathematics classes and b) describe the strategies that students are employing as they make use of online resources.

Review of the Literature

Mathematics Education and the Internet

Much of the research on the online aspects of mathematics education explores how to teach mathematics on the internet (Timmerman, 2004; Engelbrecht & Harding, 2005; Foster, Anthony, Clements, Sarama, & Williams, 2016) or how students interact with novel web-based interventions (Rosa & Lerman, 2011; Biehler, Ben-Zvi, Bakker, & Makar, 2012). However, some researchers have looked at how students interact with online resources that are not part of assigned classroom activities. For example, Van de Sande (2011) studied 200 student interactions in a free online help forum and discovered that meaningful learning sometimes took place there. Similarly, Puustinen, Volckaert-Legrier, Coquin, and Bernicot (2009) report on a study in which they observed how 206 French middle school students sought out help with the mathematics that they were learning in school. These researchers also looked at submissions by students to a help forum. Notably, in both cases, the traces of student search activities in a particular website were examined without providing any analysis of what proportion of students relied on the website or whether the use they made of it was a typical way for students to seek out help online. For research along those lines, we must turn towards work in the information sciences.

Information Seeking Behavior

If we direct our attention to studies of how students seek out information online more generally, the field is much larger. Researchers in the information sciences have been attempting to model how people seek out information for decades now (Bates, 1989). One line of inquiry explores students' pathological use of the internet (Anderson, 2001). In the course of this survey study ($n = 1300$), the researcher found that mathematics majors use the internet significantly less than many other groups, particularly those in the hard sciences. While not addressing the use of the internet for academic purposes, Sin and Kim (2013) provide an example of a study that examines how a particular population of students looks for information – in their case, the everyday information seeking of international students on social networking sites. The study was conducted with surveys and found that the use of social networking sites was a positive predictor of the perceived usefulness of information for everyday life. Torre, Reiser, LePeau, Davis, and Ruder (2006) used a grounded theory approach to describe the academic information-seeking behavior of 24 first-generation Latino/a students. However, this study did not actually address questions about how these students use internet resources to help themselves academically but rather focused on how students sought out academic advising-related information, such as the requirements for different majors or possible career fields. Some research looked at differences in internet use between different demographic groups (Odell, Korgen, Schumacher, & Delucchi, 2000), but that work focused on macro-behaviors, e.g., how many online hours spent per week on games versus homework, without attempting to unpack exactly which resources students are accessing and how they are making use of them.

Research Questions

Mathematics educators need a better understanding of how students are making use of the internet. Current research either focuses on specific interventions or takes a broad look at student information-seeking without providing insight into how students in mathematics classes are making use of the internet. Thus, the goal of this pilot study is to address the following research questions:

1. Which online resources do undergraduate students in mathematics courses rely on in order to help them with their mathematics courses and to what extent do they make use of these resources?
2. How are students seeking out and interacting with these resources?

Method

An explanatory mixed methods study (Creswell & Plano Clark, 2011) begins with the collection of quantitative data via surveys with follow-up interviews designed to explore the initial quantitative findings. An online survey was administered to students taking summer courses on a voluntary basis with the understanding that the students would receive a $25 incentive if they were randomly chosen to participate in a 30-minute follow-up interview. The surveys collected basic information about the frequency with which students make use of the internet while also asking them specifically about a series of internet resources that had previously been brought up frequently in the course of informal surveys and conversations with students and colleagues. These questions had a dual purpose, they provided data on the resources that students are using and they also primed participants for the subsequent open-ended questions asking them more details about which information resources that they use and how they use them. The students were told to only refer to internet use that was not part of the curriculum provided by their instructor, thus they did not report on their use of online course management software.

The sample (N = 42) in this pilot reflected the institution's demographics with respect to race and gender[1]. They were enrolled either in the Calculus sequence, Differential Equations, and/or Linear Algebra. This constrained selection of courses is a limitation of the current pilot and the subsequent larger study will draw students from mathematics courses from all different stages. Follow-up interviews were conducted with four students as part of this preliminary study. These interviews were semi-structured (Arksey & Knight, 1999) in order to build off of our findings from the initial survey. Given that our goal with these interviews is to come to a better understanding of how students seek out information, the students were asked to describe the most recent situation in which they sought out information to help with their mathematics class, to elaborate on the use of those resources that they stated that they used frequently in the survey, invited to reflect on how their use of the internet as a resource for mathematics courses had changed over the years, and to describe how they found out about the online resources they used.

Findings

Survey Results

The surveyed students reported using online resources, beyond those assigned by the instructor, extensively. All of the students made use of online resources at least a few times each week and almost half of them made use of those resources every day (see Table 1). Further, most of the students reported using online resources more frequently in mathematics courses than in

[1] This information is available upon request, I am only leaving it out of the present proposal due to space constraints.

other subject areas with only one student reporting that they used online resources less often in their mathematics courses.

Table 1
The Use of Online Resources by Students in Mathematics Courses

How Often	Frequency	Percentages
Every Day	19	45.24%
A Few Times a Week	23	54.76%
About Once a Week	0	0%
A Few Times a Quarter	0	0%
Less Than Once a Quarter	0	0%
Compared to Other Courses		
More Often	29	69.05%
About the Same	12	28.57%
Less Often	1	2.38%

Table 2
Percentage of Students Using Different Online Resources

Online Resource	More Than Once a Week	Several Times a Quarter	Once a Quarter or Less	Do Not Recognize
Google	85.72%	14.28%	0%	0%
Youtube	66.67%	28.57%	2.38%	2.38%
Wolfram Alpha	38.09%	28.57%	16.67%	16.67%
Khan Academy	33.33%	50%	14.29%	2.38%
Desmos	21.43%	42.85%	14.29%	21.43%
Chegg	19.04%	9.52%	52.38%	19.05%
Mathematica	11.9%	14.28%	9.52 %	64.29%
Wikipedia	9.58%	26.19%	64.29%	0%
Stack Exchange	4.76%	19.05%	14.29%	61.9%

The most commonly used resources (see Table 2) were Google and Youtube, a finding that highlights the limitations of the survey format as these resources could be used in any number of ways by students. The follow-up interviews, as reported below, provided an opportunity to learn more about what students meant when they reported using those two resources. The next most commonly used resources were Wolfram Alpha and Khan Academy with around a third or more of students using these more than once a week and a strong majority of students using them several times a quarter or more. The use of these resources is consistent with two strategies for using online resources that arose in all of the interviews, namely the use of online instructional videos and the use of online calculators / answer engines. Students also had the opportunity to volunteer resources that were not mentioned by name in the survey. This elicited resources such as Symbolab, Geogebra, Mathway, Slader, IntegralCalculator and DerivativeCalculator as resources that they had used in their latest course. Symbolab, an answer engine, was brought up by over a quarter of the students who took the survey.

Interview Results

There were two primary ways that these students described using online resources in their interviews: solidifying concepts by interactively viewing online lectures and making use of online calculators and/or answer engines in order to double-check problems that were causing the students difficulty. Crucially, all four of the students were frequently (i.e., more than once per week) using the internet for both purposes. The students all elaborated on how they made use of these resources. For example, students would choose lectures based on their popularity (both in terms of the total viewers and the number of "likes" received) and scan the comments in order to determine if they should skip to particular points in the lecture or to identify whether the lecturer made any mathematical mistakes. They generally settled on specific lecturers on Youtube – these were often encountered in the course of their searching, although they sometimes had been recommended by peers or even their instructors. The students also elaborated on their use of answer engines such as Wolfram Alpha and Symbolab. They stated that they would always keep them on hand in order to check their work if it was marked wrong (in the case of online homework) or if they suspected that they had done something wrong (in the case of written homework). One of the students said that they would use these resources if they were not confident in the steps that they took to arrive at their answer, even if they had gotten the correct answer.

The student responses affirmed the importance of online resources for their studies. Indeed, beyond the fact that all four students used the internet extensively, three of the four students stated that they were uncertain whether they would have passed college-level mathematics courses without online resources. This is put in greater relief by the fact that none of these four students reported using the internet for help with mathematics when they were in high school. Two students stated that they wished that they had known about these resources when they were in high school because they may have been more successful, while another student said that it was the greater demands, particularly with respect to the amount of material being covered within an abbreviated timeframe that necessitated the use of online materials. This last student highlighted the role of answer engines as a way of checking their work, stating that they would spend much more time on their mathematics homework if they were not able to immediately confirm whether they had answered the problems correctly or not.

Conclusion

This pilot study serves to demonstrate that students are making extensive use of the internet to study for their mathematics courses. In particular, the interviews suggest that these students may believe that they owe their success in mathematics courses to the judicious use of internet resources. I will be following up this work with a large-scale follow-up study. In particular, such a study will help provide a general model of students' information seeking and information use that can help support equitable mathematics instruction by making effective strategies for the use of online resources available to all students and instructors.

Questions for the Audience:

- As researchers and as educators, what would you most like to know about how your students are making use of the internet in order to aid with their studies?
- What are your experiences with your students' use of the internet? Do you adjust your instruction in order to take their internet use into account?
- Would student diaries or screen-capture sessions be a useful supplement to the surveys and follow-up interviews?

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Bringing Social Justice Topics to Differential Equations: Climate Change, Identity, and Power

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Recently, Adiredja and Andrews-Larson (2017) challenged the field to consider and recognize the political and contextual nature of teaching and learning postsecondary mathematics education including its power dynamics and social discourses. In this preliminary report, we discuss the early stages of a classroom teaching experiment to bridge research and practice by bringing social justice topics into a differential equations course. Our iterative research process consists of using theory that informs our instructional design and theory that informs our classroom analysis. Here we discuss preliminary results from the classroom analysis through Gutiérrez's (2009, 2013) four dimensions of equity. Preliminary results show that identity and power emerge from student portfolios after engaging in a climate change problem but more work is necessary in our instructional design to draw out those dimensions more explicitly.

Keywords: Differential Equations, Social Justice, Equity, Teaching Experiment

Issues of equity have come to the forefront in postsecondary mathematics education. Recently, scholars have argued that a sociopolitical perspective shift is necessary for the advancement of critical postsecondary mathematics education research (Adiredja & Andrews-Larson, 2017). Adiredja and Andrews-Larson (2017) challenge the field to consider and recognize the political and contextual nature of teaching and learning postsecondary mathematics education including its power dynamics and social discourses. Further, issues of identity and power are of critical importance in today's climate and must be considered at every level of our research (Gutiérrez, 2013). One avenue to include such topics in postsecondary mathematics education is to bring in relevant discussions through the mathematical content itself.

In this preliminary report, we discuss how undergraduate mathematics students in inquiry oriented differential equations (IODE) courses confronted the social justice and environmental topic of climate change. The ultimate goal of this work is weave research and practice through conducting a teaching experiment (Cobb, 2000; Confrey & Lachance, 2000). Namely, we aim to conduct an iterative research process by using a Realistic Mathematics Education (RME) theory that informs our instructional design and an equity theory that informs our classroom analysis (in this case, the dominant and critical dimensions of equity (Gutiérrez, 2009, 2013)). In this report, we discuss only the preliminary analysis of the first iteration of our classroom analysis through Gutiérrez's (2009) lens of the critical dimensions of equity (i.e., identity and power) as a means to inform future instructional designs. We aim to answer the research question: *How are students' identities and conceptions of power shaped and/or influenced by engaging with a differential equations task on climate change?*

Literature Review

Here, we first briefly discuss the framing of equity and how it aligns with the framing of this study. We then discuss literature related to climate science to root our students' mathematics exploration in current climate science research findings.

Framing Equity

Gutiérrez (2009) argues that equity must be framed from four dimensions: access, achievement, identity, and power. Access considers the resources that students have available to them (e.g., technology, curriculum, teachers), but often does not consider that simply giving access to a resource at a time point in a student's education does not account for the fact that this resource may never have been available to them in the past (Gutiérrez, 2009). Achievement refers to various student outcomes measured in many, inconsistent, ways. Oftentimes, achievement is tied to the idea of closing the achievement gap (Gutiérrez, 2008). However, Gutiérrez (2009) argues that moving from access to achievement is important considering the various levels of access of students. These two dimensions are the dominant dimensions of equity, that is, they prepare "students to participate economically in society and privileg[e] a status quo" (Gutiérrez, 2009, p. 6). Here, access is a precursor to achievement and moving from access to achievement measures "how well students can play the game called mathematics" (Gutiérrez, 2009, p. 6).

Identity refers to focusing on students' pasts including how they have been racialized, gendered, and/or classed. "The goal is not to replace traditional mathematics with a pre-defined 'culturally relevant mathematics' in an essentialistic way, but rather to strike a balance between opportunities to reflect on oneself and others as part of the mathematics learning experience" (Gutiérrez, 2009, p. 5). Lastly, power considers social transformations such as who has the voice in the classroom or if students have opportunity to use mathematics to critique society (Gutiérrez, 2009). These two dimensions are the critical dimensions, where identity can be seen as a precursor to power, "ensur[ing] that students' frames of reference and resources are acknowledged in ways that help build critical citizens so that they may change the game" (Gutiérrez, 2009, p. 6).

In this preliminary report we focus on how differential equations students may 'change the game' (i.e., student identity and power systems) in reference to studying climate change and its impact on the world and society.

Climate Change Background

In studying climate, scientists are often concerned about positive feedback loops: two or more processes that magnify each other, creating a system of amplification that leads to an enhanced cycle (Kellogg & Schneider, 1974). One example is the interaction of water vapor with global temperature. As the global temperature increases, the capacity of the atmosphere to contain evaporated water vapor also increases. Continued relative humidity levels would result in an increased amount of water vapor in the atmosphere. Water vapor is a greenhouse gas. Thus, if a climate system has more water vapor in the atmosphere, the global temperature will elevate due to the increased insulation of the atmosphere. These positive feedback loops will eventually equilibrate at a higher temperature. In a high emission scenario, scientists predict that a global increase in average temperature would be enough to kick off a system of positive feedback loops that would equilibrate, by the end of the 21st century, relative to 1986-2005, to a temperature between 2.6 and 4.8 degrees Celsius higher (Intergovernmental Panel on Climate Change [IPCC], 2014). The result of this increase would be enough to melt ice caps, completely shift ecological systems, and contribute to species extinction due to significant changes in temperature, precipitation, and ocean acidification (IPCC, 2014). It may even redistribute the areas of the world that can support human life, making previously uninhabitable places like the northern reaches of Siberia and Canada habitable (though they may not support agriculture), and previously habitable places, like coastal zones (McGranahan, Balk, & Anderson, 2007) and southwest Asia (Pal & Eltahir, 2016), uninhabitable.

Climate Change Problem

This environmental phenomenon can be studied in a first course in differential equations using bifurcation diagrams. A bifurcation diagram is a plot of equilibrium solutions as a function of a parameter in a differential equation. The climate change problem has important mathematical concepts, namely bifurcation analysis (i.e., the effect of varying a parameter in a differential equation) and practical implications related to understanding societies' and governments' impact on the climate. Specifically, this problem highlights how it may be the case that damage done to the environment by a small change cannot be reversed merely by undoing that small change. Instead, reversing the damage may require dramatic changes in policy. The problem sequence is as follows:

- A group of scientists came up with the following model for this global climate system: $\frac{dC}{dt} = \frac{1}{10}(C - 20)(22 - C)(C - 26) - k$, where C is the temperature, in Celsius, and k is a parameter that represents governmental regulation of greenhouse gas emissions. Assume the baseline regulation corresponds to $k=0$, increasing regulation corresponds to increasing k, and the current equatorial temperature is around 20 degrees. To what equatorial temperature will the global climate equilibrate?
- Sketch a bifurcation diagram and use it to describe what happens to the global temperature for various values of k.
- Suppose at the start of a new governmental administration, the temperature at the equator is about 20 degrees Celsius, and $k=0$. Based on the model and other economic concerns, a government decides to deregulate emissions so that $k=-0.5$. Later, the Smokestack Association successfully lobbied for a 5% change, resulting in $k=-0.525$. Subsequently, a new administration undid that change, reverting to $k=-0.5$, and eventually back to $k=0$. What is the equilibrium temperature at the equator after all of these changes?
- Use your bifurcation diagram to propose a plan that will return the temperature at the equator to 20 degrees Celsius.

Methods

The climate change problem is part of a full course on differential equations taught from an inquiry-oriented perspective. By inquiry-oriented we mean mathematics learning and instruction such that students are actively inquiring into the mathematics, while teachers, importantly, are inquiring into student thinking and are interested in using it to advance their mathematical agenda (Rasmussen & Kwon, 2007, Rasmussen, Marrongelle, Kwon, & Hodge, 2017). An inquiry oriented differential equations (IODE) course is problem focused, with problems being experientially real, meaning students can utilize their existing ways of reasoning and experiences to make progress (Gravemeijer & Doorman, 1999), and class time is devoted to a split of small group work and whole class discussion. Whole class discussion is facilitated by the instructor who focuses on generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation (Kuster, Johnson, Andrews-Larson, & Keene, 2017).

Data for the first iteration of this classroom teaching experiment comes from student work submitted as part of an end of the semester portfolio. The portfolio consisted of the complete responses to three problems from the course (with the climate change being one of them) and a rationale statement that explains the personal significance of their work on that problem. Future iterations of this cycle will videotape students engaging in the tasks for a deeper look at in-the-moment discussions of identity and power, and discussions of how the bifurcation diagram can functions in the RME inspired instructional sequence.

We collected data from one IODE class resulting in 12 portfolios. These data were deidentified during analysis. Consequently, while we acknowledge the importance of reporting on student demographics in equity research (Adiredja & Andrews-Larson, 2017), we cannot do that in this first preliminary iteration. Future iterations of this classroom teaching experiment that collect video will be able to report on student demographics. The student works were coded using a constant comparative method (Strauss & Corbin, 1998) with particular attention given to how students discussed the critical dimensions of equity (identity and power). We acknowledge the dominant dimensions is also of importance but in this preliminary work we are not focusing on access nor achievement.

Preliminary Results

Recall our research question is: *How are students' identities and conceptions of power shaped and/or influenced by engaging with a differential equations task on climate change?* This analysis is important in the iterative process of a teaching experiment as our prompting and facilitating of this task in future iterations will be shaped by the results here tacitly tied to the critical dimensions of equity. As stated, future iterations will also consider students' mathematics engagement from an RME perspective. Overall, students gave a general thesis that this problem was important for them to work on because it showed them "mathematics in the real world." These responses were not as deep as we would have liked to see, as educators. However, there were several instances of identity and power that emerged from students' portfolios.

Identity. Our climate change problem did not seek to draw out students' pasts directly. Rather, through engagement with the task students sometimes positioned themselves and their identities within the context and spontaneously referenced such issues in their rationale statements. Two tentative themes emerged from the analysis of rationale statements: empowerment and future teaching practices. For example, one particular student discussed the pride they felt while engaging with this problem. In particular, they discuss how they knew they were truly learning:

> …it was the first problem that I completely understood the topic the entire way through. Even though that idea seems basic for people in an upper division class, for myself this was a very prideful moment. It made me realize that these difficult topics aren't as daunting as they seem. Since the first day it was introduced to our class, my table seemed to click with what the question was asking. … Then I felt as if my brain went onto autopilot, it was exhilarating. The concept of each question became clear and I understood the path to finding each upcoming answer. It was the first time that I knew I was learning, with my group able to bounce ideas off each other as if we were discovering bifurcation for the first time, there was no stopping our progress. … The most rewarding part, was that I knew my mathematical ability was growing, and all this led to me getting my highest grade on an exam in my entire college career of a 99%. … It [the climate change problem] ignited my curiosity for mathematics.

Some of the students in these courses were pre-service teachers. As an example of the second theme, one student highlighted how this problem probed them to consider their identity as a future teacher.

> To me, as a future teacher it is a great reminder of how if you can make the material relevant and slightly more interesting to the student it can make a big difference on how well the student understands the material. … What this problem taught me is that working with problems in a context that the student is interested in is very beneficial to allow the student to truly own and understand that material in their own way.

This preliminary analysis highlights how this context holds promise for situated students within the context of the mathematics they are studying. However, if we are to gain deeper insights our future iterations must more directly inquire into students' pasts and identities.

Power. In our analysis we found two ways in which power was discussed. First, it was discussed in the context of the mathematics (i.e., who has the power to do something about climate change). Second, it was discussed in the context of who has the power in the class. While different in their scope, both are important aspects of power.

For example, one student said "I remember there was a time in Mexico City where people could not go outside because there was a lot of pollution in the air. The city got so polluted that the government placed oxygen tanks on the streets." Here this student showcases that the power in this context lies with government. Another student said, "this is because after deregulation, you get stuck at the repeller [unstable equilibrium point]." Similarly, this student shows that the power to do something about climate change is tied to regulation/deregulation. Of course, this is tied to the problem we constructed, but it is important to reiterate that this problem is rooted in current climate change science research.

Lastly, many students discussed how it was important for them to talk to their peers about this problem. In particular, one student said "for all of us, it took listening to other students' ideas to really understand what was happening." This power is related to the structure of an IODE class (i.e., focused on group work). While we are not explicitly analyzing the instruction here we believe this to be a critical aspect of power (i.e., whose voice is being heard in the classroom (Gutiérrez, 2009)).

Questions for Audience

We conclude with three questions for the RUME community:

1) How can we better analyze the four dimensions of equity? In that same vein, is that lens appropriate here?
2) How might we leverage the existing heuristics of the instructional design theory of Realistic Mathematics Education (guided reinvention, emergent models, and didactical phenomenology) to disrupt current teaching and learning practices?
3) What other social justice contexts lend themselves to modeling with differential equations?

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Mathematics Graduate Teaching Assistants' Development as Teachers: Complexity Science as a Lens for Identifying Change

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Mathematics Graduate Teaching Assistants (MGTAs) are both current and future teachers of college mathematics, but there is limited research investigating their growth as teachers. To create better professional development for training MGTAs, we first need to understand how they learn to teach. This study aims to identify why MGTAs change their teaching practices and what factors influence their development as teachers. Survey, group interview, and individual interview data from seven MGTAs at a doctoral-granting university were analyzed deductively using complexity science as a framework.

Keywords: Graduate Teaching Assistants, Professional Development, Teaching Practices

Background

Improving instruction in undergraduate mathematics courses has been a rising priority for mathematics education researchers and professional mathematics organizations. Research repeatedly shows that lecture-based teaching contributes to students leaving STEM fields (PCAST, 2012; Saxe & Braddy, 2015; Seymour & Hewitt, 1997), while active learning is linked to improved student performance (Freeman et al., 2014). In an effort to increase retention in STEM and better support student learning, college mathematics teachers are being urged by the mathematics community to incorporate active learning into their instruction. As a notable example, the Conference Board of the Mathematical Sciences released a statement in 2016 advising the adoption of active learning practices:

> We call on institutions of higher education, mathematics departments and the mathematics faculty, public policy-makers, and funding agencies to invest time and resources to ensure that effective active learning is incorporated into post-secondary mathematics classrooms. (p. 1)

In an effort to support this change in instruction, mathematics educators and education researchers have looked to Mathematics Graduate Teaching Assistants (MGTAs). MGTAs are both current and future teachers of mathematics. During their time as graduate students, MGTAs have a significant role in the teaching and learning of mathematics for undergraduate students through their work as instructors, discussion and laboratory leaders, tutors, and graders (Belnap & Allred, 2009; DeFranco & McGivney-Burelle, 2001; Ellis, 2014). After completing their graduate programs, MGTAs continue to impact undergraduate learners: in 2016, over 60 percent of new Mathematics PhDs hires were employed in academic positions (Golbeck, Barr, & Rose, 2016). Thus, MGTAs development as teachers impacts how mathematics is and will be taught.

Most graduate programs in mathematics offer some form of professional development for MGTAs (Deshler, Hauk, & Speer, 2015; Speer, Murphy, & Gutmann, 2009). There is wide variation in the duration and setting of these programs (Belnap & Allred, 2009), and most take place exclusively during a student's first year as a MGTA (Deshler et al., 2015). To develop and assess teaching training for MGTAs, mathematics education researchers have drawn from the literature base in K-12 professional development. Although researchers have been able to document a change in beliefs about teaching and learning from participating in professional development, this alone has not been sufficient for a change in MGTAs' instruction (Belnap, 2005; Defranco & McGivney-Burelle, 2001; Speer, 2001).

Previous studies have identified multiple factors influencing MGTAs' decisions about teaching, including previous classroom experiences as a student (Deshler et al., 2015), perception of faculty attitudes about teaching (Harris, Froman, & Surles, 2009), social context of the department (DeFranco & McGivney-Burrelle, 2001), and types of teaching required (Beisiegel & Simmt, 2012). However, it is still unclear how to impart lasting changes in MGTAs teaching practices, and there is currently no consensus in the research community for how MGTAs learn to teach. In particular, a recent literature review revealed that MGTAs' "growth as teachers is a largely unexamined practice" (Miller et al., 2018, p. 2). That is, there is little research attending to MGTAs development of teaching practices over time (Beisiegel, 2017; Miller et al., 2018). If we want to provide professional development that has a lasting impact on MGTAs' teaching, we first need to understand why MGTAs teach the way they do. Thus, this study is guided by the following research questions:

1. What do MGTAs cite as reasons for changing their teaching practices?
2. What factors influence MGTAs development as teachers?

For the purposes of this research, *teaching practice* refers to the definition explicated by Speer, Smith, and Horvath (2010). That is, teaching practices are the "instructional judgments, decisions, and actions employed by instructors inside and outside the classroom" (Miller et al., 2018, p. 3).

Theoretical Framework

Complexity science has been used in previous studies of teacher learning when considering both mathematics classrooms and professional development for mathematics teachers. For example, Davis and Simmt (2003) conducted a teaching experiment in a seventh-grade classroom, viewing the class as a complex system in an attempt to foster a mathematics learning community. The authors later used complexity science as a lens for investigating the mathematical knowledge for teaching (MKT) of a group of K-12 teachers attending monthly professional development sessions (Davis & Simmt, 2006). Both studies viewed a group of learners as a complex system.

Similarly, MGTAs can be viewed as a complex system. A complex system is both self-organizing and adaptive. *Self-organizing* means that the group establishes norms and expectations without a specific plan or single leader. *Adaptive* indicates that the group is not rigid and can change over time (Davis & Sumara, 2001). To put these characteristics in context, consider the structure of a MGTA's work as a teacher and graduate student. MGTAs are situated within an academic department, which is also part of the larger university. Each MGTA likely has multiple supervisors, such as a research advisor and the department chair, and they may also look to a graduate coordinator or a teaching committee advisor as a point of authority. Without a central leader or specific instructions about how to be a teacher, MGTAs *self-organize* and develop an understanding of "how things are done around here." Also, as MGTAs continue their graduate programs, they learn and thus *adapt*.

Complexity science places a focus on "collective learners rather than collections of learners" (Davis & Simmt, 2006, p. 309). In the context of MGTAs, this notion implies that a MGTA's teaching development influences, and is influenced by, the growth of their MGTA peers. Previous research indicates that a change in beliefs is not sufficient for an individual MGTA to change their teaching practices. A complexity science lens views MGTAs as a group rather than as individuals and thus offers a means of considering what they need as a collective in order to grow as teachers.

The complexity science framework presented by Davis and Simmt (2003, 2006) includes five necessary but not sufficient conditions for a complex system to learn: a balance of *internal diversity* and *internal redundancy* of beliefs, attitudes, and understandings; *decentralized control* where authority is distributed among members; *enabling constraints* that provide guidelines for behavior but space for exploration and experimentation; and opportunities for *neighbor interactions* where beliefs, attitudes, and understandings can be shared between members.

Methodology

The Mathematics Graduate Teaching Assistants (MGTAs) at a large doctoral-granting university in the United States were recruited to participate in a year-long study. Seven of the MGTAs contacted agreed to participate in the study. The participants' ages range from 22 to 36, while five of the participants are first-year graduate students and two are sixth-year students. One first-year student identifies as female, while the other participants identify as male. The MGTAs have varying trajectories that brought them to graduate school: some participants started the program immediately after completing their undergraduate degrees, while the others taught high school, worked outside of academia, or completed masters degrees before attending graduate school. One participant is an international student, while the others are domestic.

At this university, MGTAs typically serve as the instructor of record. Most classes have 25-35 students, and later-year MGTAs are frequently assigned to teach upper-division courses. Occasionally, a MGTA is assigned a grading position for a graduate course or serves as a teaching assistant for a lecture section of business mathematics. All first-year graduate students are assigned to teach a pre-calculus course during their first term of teaching. During this first 10-week quarter, the MGTAs also attend a weekly teaching seminar. There are limited opportunities for formal discussions of teaching outside of this first-year, first-term seminar.

The data collected for each participant include an entrance survey, three focus group interviews, and two individual interviews. The survey and interview instruments were designed with the intention of capturing the participants' experiences as teachers and learners of mathematics, with an emphasis on how their teaching changes over time. Although the data presented here is from one academic year, the study will repeat at the same university for a second year after another round of recruitment.

Analysis of the surveys and interview transcripts uses a thematic analysis approach (Braun & Clarke, 2006). In the initial stages of analysis, complexity science is being used as a deductive tool for identifying particular themes, namely the five necessary conditions for a complex system. Later analysis will shift to a less-structured, inductive coding approach to capture MGTAs' growth as teachers more broadly. At this stage of the study, preliminary analyses have been conducted to begin exploring how the complexity science framework captures changes in MGTAs' teaching practices.

Preliminary Results

Talking to other graduate students, a *neighbor interaction*, is frequently cited by MGTAs as a resource for making decisions about their teaching practices and for finding support. For example, MGTAs will look to their peers for support in designing assessments for their students: "When we're doing things like writing tests or whatever, I'll just go up to other people and say, 'Hey, can you look at this test and make sure it seems reasonable?' And, you know, I'll do the same for them and that way I get ideas about what other people are doing and I get other people's ideas on what I'm doing." *Internal redundancy* helps this MGTA find reassurance that they are creating a reasonable assessment, while *internal diversity* allows them to share ideas with their peers.

Additionally, MGTAs talk about teaching as a way to find guidance and support for their teaching choices. In the words of one first-year MGTA, "I was lucky enough to have three officemates, all who have been here at least two years, and so they have experience teaching numerous classes. Anytime something comes up I don't know about, they're just like, do this, do that, and oh I had that happen, don't worry, it'll happen again, it'll be okay." A sixth-year MGTA describes a similar experience: "I do spend a lot of time talking to other graduate students about their teaching experiences. I think that definitely it helps to make sure you're on the same page as your peers." In both of these examples, finding *internal redundancy* among others helps MGTAs to feel more confident in their teaching.

In another case, a MGTA explains the benefits of hearing differing ideas and how it helps inform their decisions about how they want to teach:

> Just like, hearing other people's perspectives on things and how they deal with certain situations. I mean, sometimes it's positive things, like, "Oh, that's great. I should be doing more of that." And also sometimes, even though you might not say it to their face, it's kind of like, "Eh, I don't know. I don't know about that." I think the more you can hear and see, the more you can kind of decide for yourself what you think is right and what you think is wrong. And so that has been really good for my development.

This MGTA is relying on both *neighbor interactions* and the presence of *internal diversity* to hear multiple perspectives and then make their own judgment. Both when the MGTA is skeptical about someone's decision and when they would like to adopt a particular teaching practice, it is the diversity of the ideas from the MGTAs own that make the interaction impactful.

However, not all MGTAs are having conversations about teaching that they feel are productive or helpful to their growth. As one MGTA describes in a Fall term group interview, "It'd be nice if we had more venues for productive discussion about teaching. Cause right now, at least for me, it's mostly sort of Band-Aid kinds of things." They reiterate this again the following term, stating, "Most of the shop talk is just kvetching about students, which is cathartic but not useful." Here, examining these *neighbor interactions* illustrates that not all talking about teaching is impactful for MGTAs teaching practices. It seems that having *internal redundancy* in a conversation may make it seem more cathartic than useful. Instead, MGTAs perceive conversations that rely on *internal diversity* as more productive for their teaching development.

In an individual interview at the end of the year, a first-year MGTA recalls that they had shown up late to one of their graduate courses because they were finishing lecture notes for later that day. The instructor of the course approached them after class and said that while preparing for teaching is important, it was disruptive to come in late. From this, the MGTA felt conflicted about how they were expected to balance their coursework and their teaching duties:

> I have studies, but I also have 30 people who I am responsible for. And you can't have a class of 30 students absolutely learning nothing. What are our priorities here? Am I lecturer, or am I not? I don't understand. If it is a second priority, then tell me that up front, "Hey, if your studies are lacking, then procrastinate on your lecturing." Oh, okay. I will do that, if that is from the top the message. But if we are going to get contradictory messages, I'm going to do what I feel is right. If I'm told you need to study and you need

> to be good at lecture, then I'm going to do what I feel is right. And my obligation to those 30 students takes priority.

In this case, the MGTA was not experiencing *enabling constraints*. The MGTA believed they were doing something wrong by prioritizing preparing for lecture, and they felt restricted in how they should spend their time. However, the MGTA also did not know where to find guidance about how to balance their studies and their teaching, and so this constraint was not enabling to them. It also seems that the *decentralized control* of the system was too present; the MGTA was looking for a message "from the top" to provide directions about how they should manage their time and thus felt the absence of a central leader and explicit instructions.

The sixth-year MGTAs both discuss the amount of freedom they were given when teaching their own classes. As one MGTA explains, "After the first year, like starting the second year, I thought it was almost comical how little direct oversight there is of us. I was just like, I can't believe they give me this much trust to do this. I feel like I'm just let free." The sixth-year MGTAs appreciate the space to make their own decisions about teaching, but they also acknowledge that more involvement would have been valuable for their development: "It's nice that they're kind of hands off. You have some room to kind of explore and have some academic freedom to figure out how you want to do things. But I wouldn't have minded a little more check-in over the years." Having freedom in their teaching is *enabling* for the MGTAs because it offers them space to try different teaching methods and gives them, as one MGTA puts it, "free rein to fail." However, it does not serve as any type of *constraint*, thus leaving the MGTAs wanting more feedback. For example, a MGTA describes their concern that the lack of direction is negatively impacting the quality of teaching in the department:

> I've been observed once. More than once every six years would be nice. I don't mind that they're not observing me, because of course I care and am trying to do a good job. But if I didn't, and wasn't, there'd still be no oversight. And so I don't know. It seems a little irresponsible. It's not hurting me, but I think it's hurting some graduate students.

The MGTA has identified that having their teaching observed would be a helpful *enabling constraint* for them, and they also believe that it would be beneficial for other MGTAs.

Discussion

The five necessary conditions of the complexity science framework were helpful in identifying some areas where MGTAs are missing support for their teaching. Additionally, complexity science highlights how MGTAs are influenced by their peers and the context of the department they work in. However, it seems that there are some factors for change that were observed in the data but are not captured by the framework, such as MGTAs changing how they structure class time based on observations of their class while teaching. This prompts the following questions for discussion:

1. Can the complexity science framework describe changes a MGTA makes to their teaching that are influenced by the individual rather than the collective?
2. How might the results of this study be effective for informing professional development for MGTAs? Are the results applicable in other departments' contexts?
3. What types of professional development are fitting for supporting each of the necessary conditions?

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Learning Mathematics through Service Learning

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This study describes a service learning-based mathematics course for non-math majors at a private liberal arts university in the Midwest. Thirty-six undergraduate students participated in the course and developed lesson plans from the content taught in class. Students then taught the lessons to third graders at a local public elementary school. Undergraduates wrote self-reflections that were collected after the service and analyzed. Data reveal students felt an increase in value and more confident learning mathematical concepts because of its real-world application in the community. We conclude that including a service learning component in teaching mathematics is valuable. Service learning can help students understand mathematics beyond numbers and equations and see its importance in societal reform.

Keywords: non-math majors, service learning, mathematical anxiety, communication

Today many undergraduate students experience math anxiety. Math anxiety is defined as a "feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in ordinary life and academic situations" (Hopko et al., 2003, p. 648). Math anxiety is cyclic in nature. More anxious students display a strong tendency to avoid learning mathematics, which results in students being mathematically unprepared, which in turn increases their math anxiety (Nagy et al., 2010). This often leads to failure of passing the graduation requirements for a four-year bachelor's degree (Bound, Lovenheim, & Turner, 2010) as well as avoiding STEM fields and careers. Various studies (Maas & Schloeglmann, 2009; Philipp, 2007) suggest there is connection between students' attitudes and their beliefs in their capability to learn. Attitudes are mental concepts representing favorable or unfavorable feelings, and beliefs are perceived information about an object (Koballa, 1998). Students with favorable feelings and beliefs about a subject are more likely to act in favor of it and in turn will see more value in learning it. On the contrary, students with high math anxiety develop negative attitudes toward mathematics and hence are less likely to engage in mathematical learning. This problem has encouraged educators to consider ways to help undergraduates in mathematics courses better learn, especially those students who experience math anxiety such as non-mathematics majors.

Educators can use service learning as a tool in mathematics courses to lessen math anxiety and to demonstrate that mathematics is useful and applicable in students' daily lives. Studies (Soria & Thomas-Card, 2014; Soria, Nobbe, & Fink, 2013) suggest service learning opportunities positively affect students' self-confidence and sense of community responsibility. Schulteis (2013) discusses a service learning project for a non-major mathematics course at Concordia University and suggests service learning can be an "excellent way to enhance the extent of student learning" and help students develop "greater mastery of classroom material and an increase in civic values and skills" (p. 582). Here, we describe a study of a service learning-based mathematics course for non-math majors at a small, private liberal arts university in the Midwest. The approach to service learning discussed in this study is novel because undergraduate students directly applied the mathematical concepts learned in class to teaching the concepts to elementary children. We also discuss the effect of service learning on undergraduate students' views of mathematics in particular and education in general.

Theoretical Framework

Our study's approach to service learning was grounded in a feminist community engagement framework (Iverson & James, 2014; Novek, 1999). This framework embraces consciousness raising, connectedness, and empathy by centering opportunities for dialogue and reciprocal collaboration (Rojas, 2014). In this study, undergraduate students communicated and collaborated with one another and the instructor to develop effective lesson plans; they then reciprocated their knowledge-making by teaching it to elementary school students. Ultimately, our study practiced "emancipatory feminist teaching" (Novek, 1999), which allowed students to practice concepts they learned in math class while "working cooperatively for the greater good" (pp. 230-231), in this case, by raising mathematical literacy in the elementary school students through service learning and community engagement.

Methodology

This study was conducted in a mathematics course at a small, private liberal arts university in the Midwest. This course is for non-math majors and counts toward undergraduate students' general education graduation requirement. This course was offered during a short-term semester for one month, Monday through Friday for 3.5 hours each day. Thirty-six undergraduate students were enrolled in the course and participated in the study. Students used the course textbook, Heart of Mathematics by Burger & Starbird (2012), and were provided with supplementary activities. Additionally, the course had a service learning component that counted as 10 percent of the final course grade. Three service learning activities were conducted at a local public elementary school. Undergraduate students were divided into 18 groups, with 12 groups of teachers and six groups of observers for each of the activities. This allowed each student two teachings and one observation opportunity. The course instructor helped undergraduate students develop lesson plans based on hands-on activities about topics taught in class, including laws of reflection, fractals, and symmetry and quilting. Students used class time to prepare and practice the lesson plans before going to the elementary school. Class time also was used to visit the elementary school and teach the lesson plans to about 50 third graders. Undergraduate students spent one hour with the elementary students teaching them the lesson plans while observers gave feedback after the lesson. Based on feedback, subsequent lessons were adjusted accordingly.

After each visit to the elementary school, undergraduate students were asked to write a self-evaluation and self-reflection. These were adapted from the Campus Compact's "The What? So What?? Now What??? Reflection Model" (A guide to reflection, n.d.). The WHAT component describes the event, i.e., teaching lesson plans to elementary students. The SO WHAT component examines the significance of the event in terms of classroom concepts as well as personal experiences. The NOW WHAT component reflects on future actions that relate to the "big picture" of using mathematics in the "real world."

Data Analysis

We conducted data analysis from a qualitative, mixed methods approach. Drawing from Creswell's (2007) description of qualitative research methods, we articulate our analysis as a combination of grounded theory and narrative approaches. Analyzing data from a grounded theory approach allowed researchers to code the undergraduate students' self-evaluations and self-reflections according to emerging "major categories of information" (Creswell, 2007, p. 64).

Then, analyzing data from a narrative approach allowed researchers to use the emerging major categories to "re-story" the undergraduate students' service learning experiences. This "re-storying" organized the emerging categories from the self-evaluations and self-reflections into a general framework (Creswell, 2007, p. 56) that provides overarching insights about the role of service learning in mathematics education and in reducing mathematical anxiety.

Researchers read each self-evaluation and self-reflection, paying particular attention to the language undergraduate students used to express the SO WHAT and NOW WHAT of their service learning experiences (the WHAT descriptions were similar, as to be expected). Researchers used open coding to document initial findings (codes), which are listed alphabetically (left to right) in Figure 1.

attitude	career	civic duty	collaboration	communication	community
confidence	connection	difference	diversity	education	embrace challenges
enjoy	excited	flexibility	future	growth	impact
interaction	logic	passion	patience	power	privilege
problem solving	responsibility	role models	service	tool	understanding

Figure 1. Initial open codes. Researchers identified 30 codes from the data collected.

Next, researchers examined the initial codes and grouped them together under four emerging categories, which are listed alphabetically (top to bottom and left to right) in Table 1.

Table 1. Emerging categories. Researchers identified four categories emerging from the 30 initial codes analyzed.

Emerging Categories	Codes supporting categories
Community engagement	civic duty, collaboration, communication, community, connection, diversity, education, embrace challenges, enjoy, impact, interaction, passion, power, privilege, problem solving, responsibility, role models, service, tool, understanding
Facing adversity	attitude, communication, confidence, difference, education, embrace challenges, flexibility, growth, logic, patience, problem solving, understanding
Looking forward	attitude, confidence, diversity, education, enjoy, excited, future, growth, impact, passion, patience, privilege, responsibility, role models, service
Relationship building	civic duty, collaboration, communication, community, connection, embrace challenges, enjoy, flexibility, future, impact, interaction, passion, power, problem solving, role models, understanding

Researchers then narrowed and focused these categories into three themes. These themes described the data at the latent level, or the "underlying the phenomenon" being analyzed (Boyatizis, 1998, p. vii), which for our study were students' reactions to learning mathematics through service learning. Because of their service learning opportunities, undergraduate students [1] viewed education a more joyful, purposeful and less anxious experience; [2] became more self-aware about the role of mathematics in the world; and [3] became aware of community ties and responsibilities to community. Below, we provide representative student comments that support each theme.

[1] viewed education a more joyful, purposeful and less anxious experience

- [I enjoyed] "working with others to teach them something rather than doing something for others."
- [I witnessed] "unexpected moments of joy that this interaction brought to both the students in our class and the young 3rd grader students."
- "What's the point of learning anything if we don't share that knowledge with anyone else? Knowledge should be a conversation, and that's something you can clearly see when you are working with the kids and they understand it."

[2] became more self-aware about the role of mathematics in the world

- "[I] learned applicability of math/math communication to everyday life."
- "In class we talk about not always having the same strategies or ending up with the same answer so I used this knowledge to be able to talk with the students about their different approaches."
- "[I] Re realiz[ed] math can be learned and service shows a way to think outside of the box."
- "Math is a very important thing in our world, and being able to use it in my writing could prove very important in inspiring change in the world. It is very important to embrace things that are difficult, that is how you learn and grow as a person."

[3] became aware of community ties and responsibilities to community

- "Community involvement is crucial to a well-developed future ... and my civic responsibility is to make sure no one is left behind."
- "[S]ocietal change can happen when we communicate and help each other as we did in this project."

Researchers used these themes and students' supporting comments to "re-story" a framework about non-math major undergraduate students' engagement in service learning. This framework is discussed in the Results and Conclusions section.

Results and Conclusion

The framework that developed from our data speaks to the growth mindset (Dweck, 2015) undergraduate students developed and fostered as they engaged in mathematics through a practical lens of service learning. First, students expressed more confidence in mathematics communication and a better understanding of its role in society. Service learning helped them reduce their math anxiety and realize learning mathematics is an ongoing process that takes time and practice. Second, undergraduate students found teaching through hands-on mathematical activities more applicable to the real world, which was different than prior experiences learning in a traditional university classroom setting. In this way, service learning helped undergraduate students think of math beyond numbers and equations and see the "real world" value that studying and applying mathematical concepts can have in others' and their own lives. Third, undergraduate students reflected on becoming more aware of future generations of young(er) students; they shared hopeful statements that these elementary children would grow up to make a difference in the world because of educational opportunities like this course/study. Service learning, therefore, was viewed as an important community investment. This helped undergraduate students develop a strong sense of civic responsibility. Ultimately, our data demonstrate that service learning opportunities can transform mathematics from something scary and disconnected to a more meaningful and civically engaged area of study for undergraduate students, particularly those who do not identify as math majors.

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First and Final Year Undergraduate Students' Perceptions of University Mathematics Departments

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In many countries, concerns have been raised regarding the lack of participation of students in mathematics at the university level due to a dearth of skilled professionals to meet the needs of an increasingly technological, and thus mathematical, world. In this paper, we report on a study in which we are comparing first and final year undergraduate students' experiences in mathematics departments. We focus on students' conceptions of the supports and challenges that they experience in mathematics departments, using a multimodal data collection method, photovoice. We will share findings from this ongoing research project focusing on comparisons between first and final year students' perceptions of their learning environment. The knowledge that will be gained from this research is crucial in understanding students' lived experiences and thus making suggestions to address university mathematics pipeline issues.

Keywords: Undergraduate Mathematics, Lived Experience, Learning Environment, Pipeline Issues, Photovoice

Declining numbers of undergraduate students graduating in mathematics impact innovation in a world that is "becoming increasingly technological and significantly more mathematical" (Australian Academy of Science [AAS], 2016, p. 37). In Australia, a very small proportion (0.4%) of students enrolling in tertiary education plan to pursue degrees in the mathematical sciences (AAS, 2016). Consequently, several government and scientific organizations have stressed the need for increased participation in the mathematical sciences at the tertiary level (AAS, 2016; Australian Mathematical Sciences Institute [AMSI], 2017). Additionally, women remain a minority of students in university programs in the mathematical sciences, and women's proportion of the enrolments has been declining in recent years (AAS, 2016; AMSI, 2017; Johnston, 2015). To address these issues, we report on findings from a study focusing on issues of student experience in mathematics degree programs, via a multimodal methodology, photovoice, combined with individual interviews.

Literature Review

Tertiary mathematics education is an expanding field of research, and experts have suggested that research is needed about students' experiences, as existing research often focuses on the teaching and learning of specific mathematical topics (Coupland, Dunn, Galligan, Oates, & Trenholm, 2016). While there are many studies (e.g., Hernandez-Martinez et al., 2011; Wade, Sonnert, Sadler, & Hazari, 2017) about the transition to university, there is a paucity of research about students' progressions throughout mathematics degree programs. Rather, most studies of mathematics majors (e.g., Piatek-Jiminez, 2015) tend to focus on a particular year level.

Previous researchers have identified issues in the tertiary mathematics pipeline that have contributed to attrition, such as a lack of understanding of career pathways, poor teaching, the demands of course load, and loss of interest in content (Fenwick-Sehl, Fioroni, & Lovric, 2009;

Piatek-Jimenez, 2015). While students of all abilities appear to lose mathematical confidence as they progress through introductory calculus (Ellis, Fosdick, & Rasmussen, 2016), students' self-beliefs warrant further examination (Sheldrake, Mujtaba, & Reiss, 2015).

Gender-specific issues have also been highlighted. Reasons for women's attrition include a lack of support from faculty members, feelings of being invisible or not fitting in, a low proportion of women in the program, and a loss of interest in the subject area (Damarin, 2000; Herzig, 2004; Mastekaasa & Smeby, 2008; Rodd & Bartholomew, 2006). In contrast, protective aspects include social and academic support from friends and family, encouragement from educators, and personal characteristics such as determination and competitiveness (Gill, 2000; Hall, 2010; Robnett, 2013, 2016; Rodd & Bartholomew, 2006).

Theoretical Framework

This study is framed by a feminist and social constructivist epistemological stance (e.g., Butler, 1999; Fosnot, 2005). We view knowledge as a human construction that is gendered and culturally, socially, and historically situated. Furthermore, we view disciplinary knowledge of mathematics, as well as views of mathematics and mathematicians, as socially constructed, gendered, and linked to the specificities of time and place. With regard to the context of the study, we apply this lens to the students' experiences in mathematics degree programs by viewing their learning as "both a process of active individual construction and a process of enculturation into the mathematical practices of the wider society" (Cobb, 1994, p. 13).

Objectives

The current situation in undergraduate mathematics warrants further investigation to better understand the experiences that contribute to students' perseverance in the field. By providing students with an opportunity to discuss their experiences, both supportive and challenging, we hope to develop an understanding of the issues that they face. In so doing, we will inform mathematics faculties and other stakeholders of ways to address students' concerns.

The aims of this project are: (1) to understand the experiences of undergraduate students enrolled in mathematics degree programs, in order to explore how mathematics departments support or challenge them, and (2) to examine how gender may play a role in students' experiences of studying mathematics at the undergraduate level. Our project is guided by the following research questions:

1. What are mathematics majors' experiences of university mathematics departments?
    a. What aspects of the departments do students find supportive?
    b. What aspects of the departments do students find challenging?
    c. How do students' experiences in the mathematics departments influence their career aspirations?
2. Are there differences in experiences by:
    a. Gender?
    b. Year level (first year versus final year)?
    c. Institution?

To address these questions, we will utilise qualitative research methodologies, namely comparative case study and photovoice.

Methodology

The research project is a comparative case study of two Australian universities. Specifically, we are investigating the experiences of first year and final year students, with a focus on gendered aspects of students' experiences. Experiences in the first year of university have been shown to be critical in supporting students, particularly women and gender minorities, to continue in the field (Herzig, 2004; King, Cattlin, & Ward, 2015). The final undergraduate year is the time when students need to finalise decisions about future careers and/or further studies. By having participants from different year levels, genders, and institutions, we will be able to examine how these aspects may influence students' experiences. In the following sections, we provide an overview of the study's methodology, namely comparative case study and photovoice, and discuss the data sources, participants, and analysis methods.

Comparative Case Study

As a case study, our research involves "the study of an issue explored through one or more cases within a bounded system" (Creswell, 2007, p. 73). According to Stake's conception (1995, 2005), our research project is an instrumental case study, as we are focusing on a broader issue of which the case is representative, and a collective case study, as it is an instrumental case study extended to several cases (i.e., multiple case study design). The broader issue is the differential experiences and participation by gender and year level in studying university mathematics, as illustrated by the cases of the first year and final year students at each institution. To investigate this issue, we are using a modified version of photovoice (Wang & Burris, 1997).

Photovoice

Photovoice involves participants taking photographs that are relevant to their lives in order to "promote critical dialogue and knowledge about important community issues through large and small group discussion of photographs" (Wang & Burris, 1997, p. 370), with the goal of reaching policymakers. The participants both create and discuss the data, increasing their autonomy in the research process, as they can "identify, define, and enhance their community according to their own specific concerns and priorities" (Wang & Burris, 1997, p. 374).

The use of photovoice has grown exponentially in the past few years, presumably due to the widespread use of smartphones. Several researchers (e.g., Cook & Quigley, 2013; Wilkinson, Santoro, Major, & Langat, 2012) have used photovoice to learn about post-secondary students' experiences. However, we only know of three examples of photovoice in mathematics education research (Chao, 2012; Harkness & Stallworth, 2013; Tan & Lim, 2010), and these studies were not conducted at the post-secondary level.

We are using a modified version of photovoice that begins with individual semi-structured interviews, focused on each participant's educational pathway into the mathematics degree program, experiences in the program, and career aspirations. Then, per the photovoice process (Wang & Burris, 1997), each participant takes photographs to represent the supportive and challenging aspects of the mathematics department. In focus group interviews, participants discuss the photographs. Photographs can focus and encourage discussion in focus group interviews, as well as provide a different mode in which participants can express themselves (Whitfield & Meyer, 2005). Supported by the interview facilitator, participants discuss themes that they see across the photographs.

Data Sources

We have data from three sources – individual interviews, focus group interviews, and photographs – with the latter two intertwined. The individual interviews are audio-recorded, the focus group interviews are video-recorded, and the photographs are provided electronically to the researchers for further analysis. Multiple data sources allow for triangulation, "a process of using multiple perceptions to clarify meaning… [that] helps to identify different realities" (Stake, 2005, p. 454). Moreover, by comparing the data sources, "various strands of data are braided together to promote a greater understanding of the case" (Baxter & Jack, 2008, p. 554).

Participants

Data are currently being collected from two comparable, prestigious Australian universities (herein referred to as University X and University Y), both of which have large mathematics departments. We will involve 20 participants per institution, 10 first year and 10 final year students. Data collection will be completed by October of 2018 (i.e., the end of the semester). In Table 1, we provide information about the participants to date. These participants have completed individual interviews and are in the midst of undertaking the photovoice process.

Table 1. Participant information.

University X	Women	Men
First year	4	4
Final year	2	0
University Y	**Women**	**Men**
First year	1	2
Final year	2	1

The focus groups will be comprised of five students each (first year or final year students only in each focus group), as this group size is ideal in terms of providing space for all participants to share their photographs and contribute fully to the discussion.

Analysis

The multiple data sources and participant groups necessitate a complex and multi-stage approach to data analysis. The individual interviews are currently being analyzed through a process of emergent coding (Bogdan & Biklen, 2007; Creswell, 2014). Due to the importance placed on the participants' explanations of the photographs, they will be analyzed within the context of the focus group interviews. After the focus group interviews, we will further analyze the photographs using content analysis (Riffe, Lacy, & Fico, 2014), in order to provide additional detail and description that may not be evident in the focus group interview videos.

With regard to the comparative case study methodology, analyses will occur at multiple levels. To begin, thematic analyses of the entire dataset will take place, in order to understand key themes regarding supports and challenges in mathematics departments for all participants. Then, the thematic findings will be further considered with regard to the year level and gender of the participants, to see if there are any trends specific to these groups. Each institution will be considered separately to develop a holistic understanding of each mathematics department.

Results

In the presentation, we will share findings from all aspects of the project – individual interviews, focus group interviews, and photographs – focusing on comparisons by year level. Here, we share initial findings from the individual interviews that have been completed.

The participants have taken many different pathways into the field of mathematics, such as transferring from other programs or working prior to beginning their studies. While most participants were traditional-age undergraduate students (18-22 years old), five participants were mature-aged students. Surprisingly, some participants reported that they have failed mathematics classes in past, and several described themselves as “slow learners” or “not that good at maths.” However, they reported that they persisted due to personal interest in mathematics or because they were close to completing their degree. All participants reported that peers were a major support in their academic success and persistence. Hence, a desire for more social dimensions to their programs was commonly reported, rather than changes to pedagogical/structural elements.

The first year participants have had quite a positive experience. They reported that the faculty members are passionate and helpful, and the tutorials are interactive and focus on collective understanding. While unaccustomed to discussing mathematics in groups, the collaborative style of learning allowed the participants to meet other students and develop a better understanding of the content. The participants attributed any lack of achievement in mathematics to the challenge of transitioning between high school and university, rather than any distinct factors in the mathematics departments. The participants explained that if they had any difficulty adjusting, they have felt supported and encouraged by staff to continue participating in mathematics. Concerns raised by the participants related to the distribution of marks across assessments, the quantity of work to complete, and the level of scaffolding in some classes. Additionally, the participants reported an even gender distribution in their classes and noted that they have not felt that there has been any gender-related differential treatment of students.

The final year participants echoed that they generally feel supported, but that the quality of support varied by faculty member. Students were very attentive to the implicit educational values of staff (i.e., whether the staff care about students and prioritize their teaching) as demonstrated in even minor interactions, such as providing email addresses and noting their availability for support. As with the first year students, the final year students did not report any gender-based discrimination. However, most were acutely aware of stereotypes regarding lower participation/interest of women in mathematics. The final year women participants reported feeling significant internal pressure to justify their own continued participation and felt that any personal failure would contribute to gender inequity. For instance, Participant 2 from University X stated, “Failing at a concept often feels like failing as a girl. Or, as a female in mathematics, I feel very representative of that.” Final year participants also identified issues regarding pedagogy and assessment (e.g., mandatory attendance, informal learning opportunities); in these discussions, the participants demonstrated their understanding of institutional constraints.

Implications

In this paper, we provide an example of the use of photovoice, a novel methodology in mathematics education, particularly at the university level. By sharing our experiences, we may assist colleagues in expanding their methodological repertoires. Photovoice allows participants to express their feelings about their experiences in a democratic, participatory manner. Visual representations allow participants to share their ideas in a mode that may be more accessible, thus providing unique insights into their experiences.

Our findings will inform practice at the participating universities and hopefully increase retention of mathematics majors, thus addressing the National Innovation and Science Agenda aim to boost the number of Australian graduates in science and mathematics as a strategy to build Australian capabilities for innovative economies (Australian Government Department of Industry, Innovation and Science, 2015). An important aspect of meeting this goal is addressing the significant decline in students graduating in mathematics. This study will contribute to a greater understanding of the mathematics pipeline by investigating students’ (especially women and gender minorities’) reasons for remaining in (or leaving) mathematics departments.

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Riemann Summary: An Investigation of How Instructors Summarize Group Work Activities to Build the Structure of the Riemann Sum

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In this preliminary report, we present data and preliminary findings on the instruction that follows active learning activities designed to introduce first-semester calculus students to the definite integral. We are particularly interested in the two to three days of class that follow these group work activities to see how instructors leverage the content of the activities to summarize and build the structure of the Riemann sum and definite integral. Video data of five instructors has been collected, and we present preliminary analysis focused on the ways in which one of the instructors introduced the definite integral as a sum of products.

Keywords: Riemann sum, definite integral, calculus, classroom instruction

In this preliminary paper, we report on progress from our analysis of five instructors teaching the definite integral in a first-semester calculus class. Prior research on student understanding of definite integrals in both physics and mathematics education literature emphasizes the importance of viewing a definite integral as a sum of products (Jones, 2015; Meredith and Marrongelle, 2008; Sealey, 2014). While viewing the definite integral as area between a function and the *x*-axis is a common and prevalent way of thinking about definite integrals (Jones, Lim, & Chandler, 2017), many studies have shown that students need to be able to understand *why* area under a curve can be computed with a definite integral (Orton, 1983), especially in contexts where the goal is not to find area (Meredith & Marrongelle, 2008; Sealey, 2006).

Jones, Lim, and Chandler (2017) report that even when classroom instruction emphasized the multiplicative structure of the Riemann sum through area under a curve, students still did not view the integral as a way to add up quantities that were multiplicative in nature (i.e. a sum of products). These authors as well as Sealey (2006, 2014) recommend that instruction on definite integrals begin with context problems that are not directly related to area under a curve. Sealey (2014) and Sealey and Engelke (2012) report on classroom activities that were designed to guide students towards developing a conceptual understanding of the definite integral as a sum of products. These activities were designed for students to work in groups to approximate various quantities, all of which result in Riemann sums. The *Gorilla Problem* provides students with a table of velocities of a gorilla and asks students to approximate the distance the gorilla fell, the *Water Problem* has students approximate the force of water on a rectangular dam, and the *Spring Problem* has students approximate the energy (work) required to stretch a spring. All of these activities are designed to be the first piece of instruction on definite integrals in a first-semester calculus class.

Numerous researchers have outlined the pedagogical practices that best support student learning. For example, Kuster, Johnson, Keene, & Andrews-Larson (2017) outline four principles of what they call inquiry-oriented instruction, which includes developing a shared understanding and connecting to standard mathematical language and notation. Similarly, Smith and Stein (2011) state that instructors of mathematics should help “students draw connections between their solutions and other students’ solutions as well as the key mathematical ideas of the lesson” as one of their five practices for orchestrating productive mathematical discussions (p. 11). In this vein, we are particularly interested in the whole-class discussion and/or lecture that follows group work. We seek to identify parallels, if they exist, between the way that students approach the group work activities and the ways that the instructors use them to introduce Riemann sums and definite integrals.

In our study, we analyze classroom video data from first-semester calculus courses that used Sealey's activities at the beginning of instruction on Riemann sums and definite integrals. In our work, we examine the two to three class periods following the group work using Sealey's activities to see how the instructors leverage these activities to highlight the structure of Riemann sums or definite integrals. Specifically, we seek to answer the research questions: How do instructors wrap-up student-centered, active learning tasks concerning accumulation? How do the topics of instruction align with Sealey's Riemann Integral Framework, and how are the instructors emphasizing the sum of product structure of a Riemann sum in these student-centered, active learning tasks?

Theoretical Framework

Structuralism (Piaget, 1970, 1975), and constructivism more broadly, serves as the theoretical foundations for our beliefs about the teaching and learning of calculus and the construction of the Riemann Integral Framework. With such a perspective, understanding definite integrals includes not only understanding the constituent pieces (e.g. a series of products and their sum) but how those pieces relate to one another. The Riemann Integral Framework itself outlines four layers (product, summation, limit, and function) and one pre-layer (orienting) of mathematical components of Riemann sums and definite integrals. Sealey (2014) discussed this framework and how it can be used to analyze how students engage with word problems involving definite integrals in a way that encourages them to construct the structure of the definite integral as the sum of products. This serves as our analytical framework for data analysis.

The calculus courses in which we collected data were designed to utilize Oehrtman's approximation framework (2008), which describes a way of conceptualizing the limit concept that is both intuitive to students and also aligns with the formal epsilon-delta definition of limit. Within this framework, students are expected to answer five approximation questions across several contexts.

1. What unknown value were you approximating?
2. What were your approximations?
3. Describe what the error for each approximation was. Why is the exact value of the error impossible for you to determine?
4. How did you bound the error?
5. Explain a procedure for getting an approximation with error smaller than any pre-determined bound. (Oehrtman, 2008, p. 74)

Oehrtman (2008) describes a method of instruction in calculus that relies on "layers of abstraction" throughout the course. Students complete several tasks related to the concept of limit, answering the five approximation questions, and then students are able to abstract the similar structure of the mathematics across these tasks. In our data, the instructors were using Sealey's (2014) curriculum, which was based on Oehrtman's (2008) framework. As such, in our analysis, we seek to identify areas in which students or instructors are referencing the five approximation questions, as well as how instructors guide students in this process of abstraction.

In summary, Piaget's structuralism and Oehrtman’s approximation framework guided the development of the three group work activities that were designed by Sealey (2014). Sealey's Riemann Integral Framework is the framework through which we will analyze our own data, looking for instances of each layer of the definite integral in the classroom instruction. Sealey developed this framework by analyzing videos of groups of students working through the

Gorilla, Water, and Spring Problems, and in our analysis, we will look for ways in which the instructor leverages these activities to build the structure of the definite integral during the class sessions that follow these days of group work.

Methods

Data from this study is from a larger data set which includes videos of five instructors teaching first-semester calculus for two semesters. We use the term "instructor" to refer to the instructor of record for the course, regardless of position (graduate student, faculty, etc.). One of the authors coordinated the course, which used common exams across all sections. It was also a requirement that all instructors used three student-centered lessons throughout the semester, one to introduce limits, one for derivatives, and a version of Sealey's *Gorilla, Water, and Spring Problems* to introduce definite integrals, mimicking Oehrtman's (2008) recommendations for building opportunities for layers of abstraction in the limit context. Other than the three units to introduce limits, derivatives, and definite integrals, instructors were free to cover other material in any way he or she chose. In the unit on Riemann sums and definite integrals, instructors were required to use the *Gorilla, Water, and Spring Problems* with their classes, but could then continue their own instruction however they chose. However, instructors were provided notes from the coordinator explaining how she intended to leverage the group work activities to build the concept of the definite integral. All of the instructors used these notes to some degree, but everyone inserted his or her own information as well.

Videos were taken of all five instructors during the first semester of data collection, starting when the instructor introduced the derivative. Select topics, including the definite integral, were recorded during the second semester of data collection for the same five instructors. All instructors except the author received a stipend for his or her participation in the study. Students in the classes signed media release forms to allow us to use the data for research purposes. In each class, the video camera was set at a wide angle in the back of the classroom, zooming in as necessary, and following the instructor if he/she walked around the room. Class size in each of the classes was between 30 and 35 students.

Data analysis is on-going. Initially, both authors watched videos from the days following the *Gorilla, Water, and Spring* group work activities. Summaries were written for each video in order to get a global view the data set. Videos were transcribed, and we have now begun a line-by-line analysis of the transcripts using Sealey's Riemann integral framework. To do this, we have coded each line of one of the transcripts according to which activity was being referenced (gorilla, water, or spring), which layer of the Riemann integral was being discussed (orienting, product, sum, limit, or function), and which of Oehrtman's approximation questions was being discussed. We believe utilizing a framework originally designed to investigate student understanding to analyze instructors' practice will allow us a unique perspective on whether and how these summary lessons align with what we know about student learning of Riemann sums.

Preliminary Results

After watching and summarizing videos from all five instructors, it was clear that all instructors continued to use the *Gorilla, Water,* and *Spring Problems* throughout the next several days of class. The notes provided by the coordinator recommended that each instructor spend one day summarizing the three activities and pointing out the common mathematical structure in all three activities. Even though the context was different (distance covered, force of water on a dam, and energy to stretch a spring), the underlying mathematics all involved adding up pieces of quantities that were defined by a multiplicative relationship. The way in which the instructors covered this varied, and the depth that the instructors provided varied, but each instructor did highlight the sum of products structure. For example, we found it interesting that one instructor used only half of one chalkboard to write notes for the students, while another instructor used six boards (two boards, erased and reused three times), indicating that even with common notes from the coordinator, each instructor enacted their lessons in different ways.

The second day of instruction was devoted to understanding the summation notation in a Riemann sum and/or definite integral. Again, all five instructors continued to use the *Gorilla, Water,* and *Spring Problems* on this day of class, working with the students to express the quantities approximated in each activity as a Riemann sum and/or definite integral. At least one instructor needed to finish this part of instruction on the following day of class, which is why we continue some of our analysis into the third day of instruction after the groupwork activities.

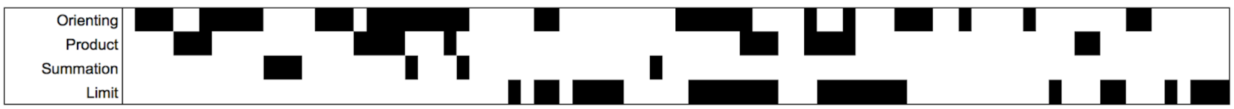
Table 1
Summary of RIF Code Frequency Data

RIF Layer	Example of Coded Transcript	Frequency
Orienting	"Okay guys what were we approximating for the gorilla problem?"	36
Product	"Energy. What was the formula? Force times distance."	17
Summation	"So, we had some small things and we're adding them."	6
Limit	"So, you have to look at the beginning point when the gorilla steps off the roof, and the moment he lands on the ground and split this time interval, from zero to five, into smaller intervals."	28

One way we decided to investigate the data on a deeper level was to look at the total frequencies for each layer of the Riemann Integral Framework (RIF). It should be noted that this approximates relative time spent during the lesson since the coded blocks were not uniform in size. The transcript was broken up using natural breaks in the conversation or when there was a large shift in topic. Table 1 contains a summary of this frequency data for one of the instructors. We see that the most common layer coded for was the Orienting layer with 36 instances, followed closely by the Limit layer with 28 instances, then the Product layer with 17 instances. The Summation layer was the least frequently coded layer with only six instances. A few aspects of these results stand out to us. First, the overwhelming presence of the Orienting layer was surprising given this lecture served as a summary for activities the students had already completed. We had not expected there to be as much attention paid to the context of the tasks throughout the lecture when we set out to code the data. Further investigation of the frequency and content of instances coded as Orienting warrant further investigation. This particular instructor spent a large amount of time discussing the fifth question in Oehrtman's (2008) approximation framework, which asks how many intervals would be needed to obtain an approximation with a predetermined error bound. As such, the Limit layer was quite prevalent in this day of instruction.

Another lens we utilized on the data was a visualization of the progression of the lecture in terms of the RIF. In Figure 1, we have represented the entire coded lecture period horizontally with the RIF layers represented vertically as distinct rows. This perspective on the data allows us to see when each layer was utilized and in what order. We see that the instructor began in the Orienting layer then shifted to the Product layer and back again to the Orienting layer. Next, we see a jump to the Summation layer followed by a return to the Orienting layer. This pattern showcases how there is a regular return to the Orienting layer throughout the lecture period. Additionally, we are able to see a late shift to the Limit layer starting approximately one-third of the way through the lecture. Again, this particular instructor spent a large amount of time on the fifth question of Oehrtman's framework. Interestingly, there is only discussion concerning the Summation layer in the first half of the lecture and it is rather sparse compared to the other layers. While we have not yet done this same analysis for the second day of class, we expect more time to be spent on the Summation layer on the day devoted to covering summation notation. The function layer is omitted from Figure 1 as this lesson did not attend to this layer of the RIF and was not observed in the data.

Figure 1. Visualization of Coded Lecture



We feel the data presented herein allow us to begin exploring the answer to the question: How do the topics of instruction align with Sealey's Riemann Integral Framework? We have seen some differences in how the topics of instruction during these summary lectures compare with how the students worked through the task from Sealey's original study. Where students entered the RIF through the summation layer, the summary lecture began with the product layer. We are encouraged by the high frequency of the product layer code in these data given its importance to student learning for Riemann sums (Jones, 2015; Sealey, 2014). We are interested in exploring whether these trends persist throughout the data for the other instructors, specifically if the other instructors also begin by discussing the product layer instead of with the summation layer like the students.

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Inquiry without Equity: A Case Study of Two Undergraduate Math Classes

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Compelling evidence supports the benefits of active learning environments in undergraduate mathematics. Research shows that such environments can benefit all students, and especially benefit students who have been traditionally underrepresented in mathematics. To move beyond the general idea "inquiry supports equity," we provide an analysis of two inquiry-oriented classrooms to highlight the ways in which equitable participation may or may not be present, particularly in terms of gender. We found some evidence of equitable participation in one of the classrooms, while the other was dominated by men in the class. These early findings suggest that more research is required to uncover the ways in which inquiry-oriented environments may or may not be equitable.

Keywords: Equity, Gender, Inquiry-oriented instruction, Instructional measure, Observational research

Inquiry-oriented pedagogy consists of a teacher exploring students' reasoning and engaging them in authentic mathematical activity (Rasmussen & Kwon, 2007). This gives students space to reconstruct mathematics through critical thinking and mathematical discussion. Laursen, Hassi, Kogan, and Weston (2014) discovered in their multi-institutional study on inquiry-based learning (IBL) that IBL can promote a more equitable learning environment in terms of gender equity when compared to non-IBL approaches. They found that women and men in inquiry-based courses reported statistically equivalent cognitive and affective gains. On the other hand, women in non-inquiry-based courses reported lower gains when compared to the men. Laursen et al. (2014) suggested that "IBL approaches leveled the playing field by offering learning experiences of equal benefit to men and women" (p. 412).

Although the courses in this study helped students achieve more equitable outcomes, what about classroom-level participation? Research shows that talk-based participation plays an important role in learning (Bransford, Brown, & Cocking, 2000). Thus, even though women may have improved outcomes, it is possible that they could be marginalized at the level of classroom participation. This has implications for identity development and belonging, and if women were truly participating less in such classrooms, it would highlight an area for improvement in the use of inquiry-oriented pedagogies for mathematics instructors. Therefore, in this study we attempt to answer the following question: Are the opportunities for talk-based participation in inquiry-oriented classrooms necessarily equitable? Given the length of this brief report we focus only on participation, recognizing there are other consequential aspects to classroom equity.

Background

Inquiry-Oriented Instructional Measure

To measure whether or not a classroom is truly inquiry-oriented, Kuster, Johnson, Rupnow, and Wilhelm (2018) developed the Inquiry-Oriented Instructional Measure (IOIM) as a tool to clearly outline seven practices that coincide with the four inquiry-oriented instruction principles as defined by Kuster, Johnson, Keene, and Andrews-Larson (2017). These principles include: (a) generating student ways of reasoning, (b) building on student contributions, (c) developing a shared understanding, and (d) connecting to standard mathematical language and notation. The classroom enactments of the seven practices of IOIM, which have been described extensively in Kuster et al. (2018), are scored along a 5-point Likert scale from low (1) to high (5).

Equity Quantified in Participation

To measure patterns in student participation, we use Reinholz and Shah's (2018) classroom observation tool EQUIP (Equity Quantified in Participation). EQUIP uses equality as a necessary but insufficient baseline towards equity, recognizing that students who are underrepresented in mathematics typically received less than a proportional share of participation opportunities. This is measured by an equity ratio of actual participation to expected participation based on the demographics of the class (Reinholz & Shah, 2018). For example, if 35% of the students in a class were women then it would be expected that those students would participate in 35% of the classroom discourse to ensure equal representation. However, if the women actually contributed to 70% of the classroom discourse, their equity ratio would be 0.7/0.35 = 2, indicating that they participated more than expected.

To be clear, our argument is not that all students should receive an equity ratio of 1, indicating proportional representation. Rather, we can use an equity ratio of 1 as a point of comparison, recognizing that if students from underrepresented groups are receiving a ratio of less than 1, it would likely indicate a problem. As outside observers, it is beyond us to say what is equitable in a classroom, especially without interviewing students for their perspectives.

Methods

For this study, three coders analyzed lessons from 42 teachers in a broader project focused on inquiry-oriented instruction; 20% of the videos were double coded and Krippendorf's alpha > 0.8 was achieved, indicating sufficient interrater reliability. Prior to this study, each of these classes were analyzed and scored using the IOIM rubric (Rupnow, LaCroix, & Mullins, 2018). We then aggregated the scores across the seven dimensions of the rubric, which ranged from 15.5 (low-level implementation of inquiry-oriented instruction) to 35 (high-level implementation of inquiry-oriented instruction). Of the 42 classes, we chose two classes that received high IOIM scores (Class A scored 33 and Class B scored 34) and had roughly the same number of students. Class A consisted of 5 men and 2 women while Class B consisted of 5 men and 3 women. The gender composition of the classrooms was determined by the coders, where visual and audio cues were used as determinants; we acknowledge that this is a limitation of our study.

We coded participation sequences in the whole class discussions along several EQUIP dimensions including Student Talk. A participation sequence refers to a chain of utterances from a student where a new sequence begins once a new student enters the discussion (Reinholz & Shah, 2018). We provide a brief description of the relevant codes in Table 1. Each participation sequence is coded at the highest level of contribution (e.g., if a student gives both an "other" response and a “how” response within a sequence, the sequence is coded as “how” for Type of Talk). The codes for the sub-dimensions of Student Talk in Table 1 are listed from low to high.

Table 1. Descriptions of Student Talk from EQUIP.

Dimension	Codes & Description
Type of Talk	Other - Student asks a question or does not say a mathematical idea.
	What - Student reads our part of a problem, recalls a fact, or gives a numerical/verbal answer without justification.
	How - Student reports on steps taken to solve a problem.
	Why - Student explains the mathematics behind an answer.
Length of Talk	1-4 words – short single-worded responses.
	5-20 words – a short response consisting of a sentence.
	21+ words – a long response consisting of several sentences.

Findings

We found that even though these classes successfully adopted inquiry-oriented practices in their classrooms as evidenced by their IOIM scores, opportunities to participate were not evenly distributed. With equity ratios below one, we see from Figure 1 that women were underrepresented in the classroom discourse in both classes.

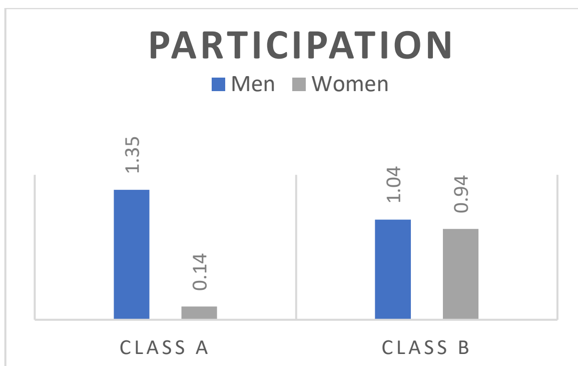

Figure 1. Gender equity ratios by class.

Out of the 51 participation sequences, the women in Class A only contributed to 2 of them. The first sequence played out as follows:

Teacher:	*So, what can we do with that? What conclusion did you draw?*
Michelle:	*When you multiply the (inaudible) a subset with itself you won't get it back.*
Teacher:	*So okay, so you won't get it back …*

The second participation sequence played out in a similar fashion.

Teacher:	*Is there another one that does work?*
Suzy:	*The only one we haven't checked is IFR^3.*
Teacher:	*IFR^3?*
Suzy:	*Assuming that the identity has to be in the identity.*
Teacher:	*Okay. So, let's see we don't have that much time left but maybe we could try to verify if I and FR^3 works.*

The student responses in each of the sequences were short (5-20 words) “what”-type contributions. Both students contributed to the mathematical development in the class by simply stating facts. According to the hierarchy in the EQUIP framework, these are low-level contributions. An example of a high-level “why” contribution from another participation sequence from Class A between the instructor and one of the men progressed as follows:

Anthony: *Um if it's the identity then it has to be abelian. Because that's on identity element, right?*

Teacher: *Um so let's see. So where did abelian come? So you're saying it has to be commutative, where's the rationale behind that?*

Anthony: *Because if you did the identity, if purple's the identity… If you did purple then yellow, you'd get yellow. And yellow on purple should also be yellow. The vertical column would be the same as the horizontal column…*

This student speaks at length (21+ words) about his reasoning (“why”) behind his initial conclusion.

In Class B, the women only contributed to 12 out of the 34 participation sequences. These contributions were either “what” or “why” responses. Although they were overrepresented in overall participation, Figure 2 shows that the men in Class B were underrepresented (equity ratio less than 1) for both “what” and “why” talk. This indicates that in this inquiry-oriented classroom, women were providing the majority of the mathematical explanations.

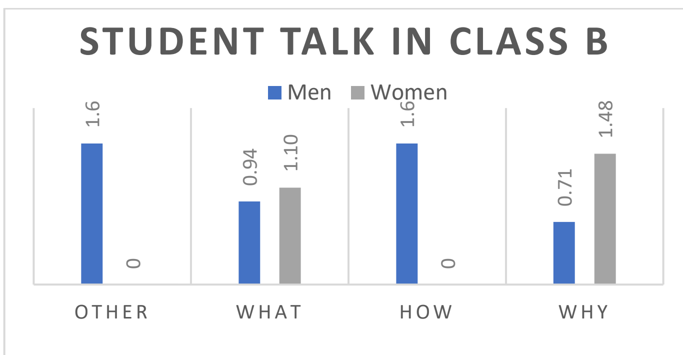

Figure 2. Gender equity ratios for Student Talk in Class B.

According to Table 2, we see that even though most of the participation sequences were more than 1-4 words, the men spoke longer more frequently than the women. In terms of equity ratios, men were overrepresented for the “5-20 words”-length of talk (1.1 to .83) and “21+ words”-length of talk (1.07 to .89).

Table 2. Frequencies for Length of Talk by Gender in Class B.

Length of Talk	Men	Women
1-4 words	1	2
5-20 words	11	5
21+ words	10	5

Discussion and Conclusion

Our findings show that high-inquiry does not imply equitable access to classroom discourse. Both high-inquiry courses provided more opportunities overall for men to participate in the mathematical discourse. The women in Class A contributed to the mathematical development in only 2 out of the 51 participation sequences, all of which were low-level contributions. Likewise, the men in Class B also participated more than we would expect based on demographic representation, but there was a much better gender balance. In addition, the women in Class B were overrepresented in high-level talk ("why" responses), which indicates that when they did participate, they did so in mathematically meaningful ways.

Though the IOIM does measure the degree to which the four principles of inquiry-oriented instruction are implemented (Kuster et al., 2018), it does not highlight which students are driving the discourse. The results from this study demonstrate to us that two courses with a high degree of inquiry can potentially provide a very different learning environment for the students. While inquiry-oriented instruction allows students to interact with mathematics in a meaningful way, it may also amplify inequitable environments by allowing more dominant personalities to overwhelm the classroom discourse and restricting access to other groups of students.

Based on their IOIM score, the teachers of both classes demonstrated that they elicit student reasoning and contributions at a high level. From our findings, we now understand that their implementation was inequitably distributed. This shows us that we must not only be vigilant about increasing student engagement but also conscientious about the ways in which we engage different students. A future direction would be to consider how race factors into student engagement in inquiry-oriented classes. In addition, we would like to further study how equity in opportunities to participate may or may not relate to equity in student outcomes. Here are questions for audience consideration:

1. What is required for inquiry-oriented classes to be equitable?
2. Beyond participation, what other ways should we conceptualize equity in these classrooms?

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Beliefs About Learning Attributed to Recognized Instructors of Collegiate Mathematics

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Six collegiate mathematics instructors, who had all previously won teaching awards, were interviewed about their beliefs on learning. Differences between the beliefs of PhD and non-PhD mathematicians were evident, perhaps connected to the student population with which each worked. Furthermore, the four PhD mathematicians all held very different beliefs about learning and modelled their teaching accordingly. Additionally, each of the four had created at least one teaching analogy for himself (climbing instructor/spark, showman/coach, Sherpa, facilitator) that spoke to the role he saw himself in within the classroom.

Keywords: Beliefs, Learning, Teaching

Research on teachers' belief systems suggests that there are strong ties between teachers' beliefs and their instructional practices (e.g., Thompson, 1992). Ernest (1989), in particular, identified teachers' beliefs about mathematics and the teaching and learning thereof as key. Since the late 1980s, the work on teachers' belief systems has grown to encompass work on college instructors' beliefs systems (e.g., Bruce & Gerber, 1995, Warkentin, Bates, & Rea, 1993) and now includes a noticeable subset focused on mathematics instructors' belief systems, such as LaBerge, Zollman, and Sons' (1997) interviews with 26 mathematicians, Weber's (2004) documentation of a mathematics professor's teaching style and beliefs, and Speer's (2008) documentation of a doctoral student's beliefs about the learning of mathematics.

This study focuses on the beliefs about learning attributed to mathematics instructors—regardless of whether they held a PhD in mathematics—who had received institutional teaching awards. Although we recognize that receiving a teaching award may not imply teaching excellence, teaching award winners represent teaching role models sanctioned by mathematics departments: They represent what is currently valued by mathematics departments. Thus, we believe that understanding their belief systems, particularly about learning mathematics, is a worthwhile endeavor. To this end, the research question that this study attempts to address is: What consistencies or inconsistencies exist in the attributed beliefs about mathematics learning of award-winning instructors of collegiate mathematics?

The theoretical perspective we espouse aligns with Speer's (2005) view that all beliefs are attributed to teachers by researchers, for we agree that differentiating between professed and attributed beliefs ignores the role of the researcher. Furthermore, we follow Speer in viewing interviews as insufficient for gaining a complete picture of an instructor's beliefs. Thus, we see this study as the opening act to a larger study that collects classroom data and allows for fine-grained levels of investigation.

Method

Due to the small pool of recent teaching award winners, we opted to conduct semi-structured interviews with each of the participants. These interviews provided us with the opportunity to ask a number of core questions regarding the instructors' beliefs, while also allowing the participants to expound on anything they brought up as relevant to their teaching. Lastly, careful reading of the transcripts by both authors allowed for the creation of a coding scheme that enabled us to differentiate and categorize different beliefs about learning, teaching, and mathematics in general.

Participants

Six mathematicians at a large Midwestern university were asked and agreed to take part in a study on beliefs. Each of these mathematicians had received at least one institutional teaching award. Of four possible teaching awards, three encourage nominations from faculty, staff, and students and are decided on by committees at the departmental or collegiate level. These committees are composed of faculty members and sometimes students. The last of the four teaching awards requires nominations from chairs and directors, encourages support letters from fellow colleagues, and is decided upon by a collegiate-level committee. Of the six mathematicians, five received at least one of the three former awards, and one received the latter.

Four of the participants held PhDs in mathematics while the other two did not. One of the non-PhD mathematicians was female, and the other five participants were male. Although unintended, this gender ratio approximately mirrored the gender ratio of their mathematics department at the time. Henceforth, we refer to the two non-PhD mathematicians as Aleph and Beth, and the others as Gimel, Dalet, Waw, and Zayn. (All names are pseudonyms.)

Data Collection

Each mathematician was asked to participate in about an hour-long semi-structured interview on their beliefs about learning mathematics. The interviews were conducted by the first author in the first half of 2018, audio-recorded, and transcribed. All participants were asked several core questions about their teaching, what learning (the action) and having learned (the state) meant to them, the roles of student and teacher in the learning process, differences—should they see any—between learning mathematics and other subjects, the goal of learning, and their own learning.

1. *Learning*:
    a. *Abstract*: Abstract discussion of learning (e.g., learning means to …)
    b. *Self*: Discussion of participant's learning (e.g., I learned by …)
    c. *Student*: Discussion of students' thoughts on learning (e.g., students think that learning is …)
    d. *Learner*: Discussion of properties of a learner (e.g., a good learner has property X …)
2. *Teaching*:
    a. *Abstract*: Abstract discussion of teaching (e.g., teaching means to …)
    b. *Self*: Discussion of participant's teaching (e.g., when I teach, I like to …)
    c. *Student*: Discussion of students' thoughts on teaching (e.g., students believe that teachers are supposed to …)
    d. *Teacher*: Discussion of properties of a teacher (e.g., a bad teacher is someone who …)
3. *Mathematics*:
    a. *Abstract*: Abstract discussion of mathematics (e.g., mathematics is a set of rules …)
    b. *Self*: Discussion of participant's relation to mathematics (e.g., I have always loved mathematics …)
    c. *Student*: Discussion of students' thoughts on mathematics (e.g., students think that mathematics is just a bunch of rules …)
    d. *Mathematician*: Discussion of properties of a mathematician (e.g., a mathematician is someone who …)
4. *Miscellaneous*:
    a. *Context*: Discussion of anything that adds context to the interview, typically factual (e.g., Calculus I used to be taught by X in format Y …)
    b. *Other:* Discussion that does not directly match any of the previous codes but might still be considered relevant (e.g., employers look for people who can solve problems)

Figure 1. The coding scheme used to code the interview transcripts. This scheme illustrates both the top-level codes as well as the subcodes.

Data Analysis

As the interviews were spaced out over several months, we were able to reflect on the interviews before beginning the coding process. We realized that although the focus was on beliefs about learning, the conversations in the interviews also turned toward teaching and the nature of mathematics. Furthermore, participants would tell anecdotes or relate factual statements that fell in neither of these three domains. Thus, we agreed on four top-level codes: *Learning*, *Teaching*, *Mathematics*, and *Miscellaneous*. Furthermore, after doing a trial-run of coding on the first interview, we discerned that there was more nuance to the interviews that was not captured by our four top-level codes, and so we decided to add subcodes to each of them. Figure 1 provides a list and explanation of the codes. The examples in Figure 1 are made up by the authors, as the actual coded segments from the interviews would be too long to include. All transcripts were coded with the qualitative data analysis software MAXQDA.

To stay true to the spirit and flow of the interviews, as well as to make codes coherent, codes typically span multiple lines and include surrounding interactions between the interviewer and interviewee to present the complete context of each coded segment. Furthermore, declarative statements explicitly indicating our participants' beliefs as well as participants' succinct summaries of their own responses were separately highlighted.

Results

We shall discuss three results: (a) Depending on their student population, there was a large difference between the ways in which the instructors spoke about learning and teaching; (b) beliefs about learning mathematics were quite varied among the PhD mathematicians; and (c) teaching beliefs appeared to be tied to the PhD mathematicians' learning beliefs.

Student Population May Matter

In listening to the participants, it became clear that there was a big difference in the way PhD and non-PhD instructors spoke about students, students' learning, and teaching. This was possibly due to the student populations they respectively worked with: The non-PhD instructors typically did not teach proof-based classes and did not exclusively teach mathematics majors—if at all. Furthermore, their courses were often large classes of freshman and sophomores.

In the interviews with the two non-PhD instructors (Aleph and Beth), efficiency and students' motivation were of much larger relevance than in the other four interviews. For instance, Aleph noted that students' "motivation is primarily this piece of paper, primarily getting this grade." Beth echoes these thoughts sharing that a lot of her students lack curiosity and that "If you're gonna come to a university like this, you need to understand the point of it, and I doubt, they don't. They think of it as a stepping stone to a job." Both Aleph and Beth work with students who they perceive as possessing neither motivation nor curiosity for mathematics and who are instead driven by the prospect of a degree and its impact on their career paths.

With the lack of intrinsic motivation being such a concern for Aleph and Beth, it is, perhaps, not very surprising that they embrace efficiency. Aleph clearly stated that efficiency is his "theme", and both speak of the need to save time in lectures. This is achieved by preparing course materials containing a lot of text that students traditionally would have had to copy down. Thus, the perceived lack of student motivation as well as the fixed amount of material that needs to be covered result in a push for efficiency. Aleph summarized this teaching predicament by comparing the teacher-student dynamic to an optimization game in which instructors attempt to maximize students' exposure to content to achieve learning, whereas students seek to minimize their exposure to content to the minimum level required to obtain their desired grade.

Beliefs About Learning Vary Among the PhD Mathematicians

Rather remarkable was the extreme variation of learning beliefs among PhD-holders (Gimel, Dalet, Waw, and Zayn). Gimel stated that learning happens mostly when students are by themselves and solve exercises. Although one might pick up a high-level concept from a group or get a hint from others, "mathematics, it really is a [pause] ultimately a solitary activity." Thus, classes are merely an introduction to the exercises, where the "real learning" happens.

Unlike Gimel, Dalet believed that "Certainly, there is times where mathematics is a solitary activity and, uh [pause] but there is also times when mathematics is a very social activity." This more balanced approach is based on Dalet's belief that one should spend some time figuring things out for oneself, but that there is also much to learn from communication with others—not only by listening, but also by explaining.

Almost antithetical to Gimel's were Waw's views. He declared that "[collaborative] learning is a, in some ways the most effective way of learning." Waw later added: "What's essential is that the student must attempt to formulate their own arguments and in addition they need to, uh, be willing to examine other people's arguments with a critical eye." The examination of other people's arguments and the consequent discussion is an aspect of learning that sets Waw's beliefs apart from Gimel and Dalet's. Thus, these three instructors form a spectrum that ranges from learning is a solitary activity (Gimel) to learning is a collaborative activity (Waw), via a blend of these two (Dalet). Interestingly, these views lined up with the ways in which the three instructors themselves had learned and continued to learn.

Zayn added another layer to the solitary-to-social spectrum by pointing out that learning proof-based mathematics involves students overcoming a hurdle consisting of the details and rigor required in proof-writing. How do they overcome this hurdle? "I think the point is they overcome it, there's as many ways of overcoming it as there are students. And the point is that [pause] if you don't try to, like, force them to do it your way, but you just create an environment where they can do lots of trial-and-error …" He is forthright about having a learning style which is uniquely his own and which he seeks not to impose upon his students. Although Zayn is alone in clearly distinguishing his learning from his students' learning, it should be noted that Dalet, in the middle of the solitary-to-social spectrum, made some remarks in a similar vein: In speaking about overcoming mathematical struggles and getting unstuck, Dalet stated that he does not know how to tell students how to go through that process. "I don't even know if we all do it the same. You know, I assume we don't, you know."

Consequently, each of the four PhD mathematicians had a set of beliefs about learning mathematics that clearly set him apart from the others. These beliefs can be said to vary along two axes: first, from learning mathematics is a solitary activity to learning mathematics is a collaborative activity and, second, from not differentiating between one's own and students' learning to making that distinction.

Connections Between Learning and Teaching Beliefs of the PhD Mathematicians

A particularly interesting theme in the interviews is how closely the PhD mathematicians' beliefs about teaching aligned with their beliefs about learning. All of them had, to different extents, even developed analogies of their roles in the classroom.

Gimel, who saw learning as a solitary activity best achieved through exercises, described himself both as a climbing instructor and a spark. It was his goal to point students towards the exercises he carefully crafted, but it was the students' responsibility to do them and learn from them: "The teacher's job is to lead the student to a convenient rock face that he can climb, and then the student has to climb it." Gimel could be the spark, but the students needed to be the fuel.

Dalet, believing that a balance of solitary and collaborative activity might be ideal, saw himself as a showman and coach. He described his classes as a performance in which he tells jokes and jolts people awake: "I get pretty pumped up, I feel the adrenaline before going to class and I think it comes out, you know, I act pretty excited about what I'm doing." In addition to providing this showman-like extrinsic motivation, he also tries to foster students' intrinsic motivation by taking on the role of coach and providing encouragement. Although his classes are lecture-based, he incorporates his beliefs about collaborative learning by seeking to make his classes very interactive. He encourages a back and forth with his students and does not bring notes to class as he is prepared to change his plans on the spot. Furthermore, he may sometimes hold a problem class giving exercises, circulating the room, and letting students work together.

Waw, as a proponent of collaborative learning, preferred a flipped environment. He viewed himself as a Sherpa, a tour guide of sorts. He made clear that he's "not a tour guide who says, 'OK, look at this, look at that, that, that, that.'" Waw is willing to make recommendations when asked for them. He's not the agent in the tour guide–tourist relationship; his students need to approach him with their interests and questions about the mathematical realm they are touring. In response, he does not provide answers: He provides suggestions.

Lastly, Zayn sees himself as a facilitator: "I am simply there to facilitate with as *little help* as possible, but also giving as much help as needed …" Assuming that everyone learns differently, it is his goal to create an environment that allows him not to teach the students, but to facilitate their learning. This is done in a setting with minimal lecturing and a focus on group work. He also recognizes that this will not be beneficial for all students. Yet, he believes that he can help the greatest number of people with the environment he creates in his courses.

Thus, we see that Gimel, Dalet, and Waw's beliefs about their teaching align neatly with their beliefs about learning. Interestingly, Zayn realized during the interview that his own learning filtered through his belief that all students learn differently and affected his teaching: "I guess I made that whole teaching thing sort of. I guess it *is* made in my image, now that I think about it. I went through all that trouble of saying I don't want them to learn the way I learned, but now that you're making me say it …" Consequently, it appears as though all our PhD participants' beliefs about their own learning translated to how they spoke about their teaching.

Implications

We see at least three implications of these interviews. First, "undergraduate mathematics instructors" as a group is perhaps too broad a set of participants for a beliefs study as the undergraduate population one teaches appears to provide important context for one's beliefs. Second, the interviews demonstrate that as different as the PhD mathematicians are when it comes to their learning beliefs, most of them are very similar in not distinguishing between their own and others' learning. Third, all PhD mathematicians' beliefs about teaching mirrored their own learning, regardless of whether they distinguished between their own and others' learning. Thus, awareness of the differences between one's own and others' learning does not necessarily translate into awareness of the frequent—as we discovered—similarities between one's own learning experiences and one's teaching beliefs.

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Exploring the Impact of Instructor Questions in Community College Algebra Classrooms

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We describe a process to characterize the questions asked by instructors and students in community college algebra courses. The goal is to measure the quality of mathematical questions that can speak to the level of student cognitive engagement with mathematics and to connect that quality with student outcomes in the course. As a first step, we explore the relation between frequency of different types of questions and other variables collected in the project. We seek to engage the audience in discussing the affordances and limitations of this work for assessing quality of instruction in connection to students' performance.

Keywords: Community Colleges, Algebra, Question Quality, Student Outcomes

Questions are a form of discourse that have the potential to open up a conversation (Martin & White, 2005). Questioning in mathematics classrooms can play a significant role in advancing student engagement with mathematical content. In community college classrooms, in which the predominant mode of instruction is lecture, instructors say they use questions as a tool to keep students engaged with the content (Burn & Mesa, 2017). While most research has documented that questioning is an important classroom practice, it is unclear how questioning is correlated with student outcomes. Most of the literature on questions documents what instructors and students do when questioning takes place; describing for example the frequency of certain types of questions (Paoletti et al., 2018) or the types of reasoning that they may elicit (Temple & Doerr, 2012). However, assessing the impact of different types of questions has not been pursued, mainly because the work of analyzing classroom discourse is time-consuming, and is typically done on a small-scale basis with few instructors and lessons.

We focus on community college algebra courses because their high failure rate is seen as a reason for students abandoning their plans to complete a degree (Bahr, 2010). As part of a large-scale study of algebra instruction at community colleges, we sought to establish whether and how, the quality of questions relates to various student outcomes in the course. In this preliminary report, we focus on the process of developing a system to code the quality of questions asked by community college instructors teaching algebra courses, and a preliminary analysis that seeks to link the types of questions instructors ask in the classroom with student outcomes in those courses. Our focus was to accurately and reliably code questions asked during instruction and use frequencies of those codes to explore relationships to student performance. Because questions play a predominant role in community college mathematics classrooms, if there is a connection between the quality of questions and student performance, then improving how questions are used in the classroom may lead to more opportunities for student learning. If questioning practices do indeed have an impact on student outcomes in these courses, one could envision a way to use questioning as leverage for improving instruction in ways that can have a real impact on students.

Supporting Literature

Cognitive theory provides strong support for engaging learners in activities that encourage them to draw on their knowledge (factual, procedural, conceptual, metacognitive) using an array of cognitive processes (e.g., remember, apply, evaluate, etc., Anderson et al., 2001). In an environment in which lecture dominates, questions can open a space for cognitive engagement (Mesa & Chang, 2010). The literature suggests that community college mathematics instructors ask a large number of questions as they teach (Mesa, 2010; Mesa & Lande, 2014). These, and other studies of lecturers and faculty, also suggests that that instructors tend to ask questions that for the most part require students to recall information they already know; questions that demand higher level reasoning are asked less frequently (Larson & Lovelace, 2013; Mesa, Celis, & Lande, 2014; Paoletti et al., 2018). However, the quality of questions that instructors ask can encourage students' critical thinking (Boerst, Sleep, Ball, & Bass, 2011). Questions that require students to go beyond what the instructor has presented in the lesson may compel students to bring in information or make connections beyond what is known in the class. Such questions have higher levels of cognitive demand than questions that ask students to recall facts they are expected to know. We may then expect that students in courses whose instructors ask them questions that challenge their thinking or that demand high cognitive work will have better performance.

Methods

As part of the larger project, in fall 2017, 40 different instructors were video recorded teaching at least two lessons in intermediate and college algebra classrooms on one of three topics: linear, rational, or exponential equations and functions. The instructors, who taught at six different community colleges in three different states, volunteered to take part in the study. They filled out questionnaires on beliefs, personal information, and a test of their Mathematical Knowledge for Teaching Algebra (MKT-A). Their students also filled out various questionnaires, including a test measuring covariational reasoning, which was administered twice in the semester: two weeks after the beginning of the term and two weeks before the end of the term. For developing ways to capture the quality of instructor questioning we coded 37 lessons from 15 instructors selected randomly from the high, average, and low scores in the MKT-A questionnaire. Instructor and student scores were not shared with the coders.

To code the questions in a lesson we adapted the taxonomy proposed in Mesa and Lande (2014), which includes two major categories of questions, mathematical and non-mathematical. Mathematical questions were coded as realized if the students provided an answer or the instructor waited a sufficient amount of time after each question (5 seconds or more), and unrealized if there was no answer and little or no wait time. Mathematical questions were placed into three categories: authentic, quasi-authentic, and inauthentic. In this paper, we focus only on mathematical questions. Table 1 presents the categories of questions of the coding process, their definitions, and examples.

Table 1: Coding system for mathematical questions in videos of community college algebra lessons.

Realized Authentic (RA) Questions that, if answered, would require students to use information beyond what they have learned in class. These are often open-ended questions.	T: So, did you guys come up with an example of a situation where the input, right, how I evaluate the output, changes based off the input? What are you coming up with? S: We decided that money could be {... }

Realized Quasi-authentic (RQ) Questions that require some knowledge and have limited possible answers. Questions that can be answered with material to be recalled that is new in that lesson.	T: Anybody see how you could clear all these denominators with a valid mathematical process? S: Inaudible response
Realized Inauthentic (RI) Questions that require using information that is known, expected to be known	T: But remember, $x + x$ is actually what? S: $2x$
Unrealized (UA, UQ, UI) Questions that are not answered by students and the instructor waits less than five seconds for a response	T: And speaking of defining terms, what the heck is a rational function? (no pause) Well, it's {answers question}

Procedure

Seven coders were recruited to assist in the coding of lessons. One difficulty of working in a large group of people, when decisions need to be reached, is the threat of groupthink. Groupthink is "a concurrence-seeking tendency that can impede collective decision-making processes and lead to poor decisions" (Choi & Kim, 1999).The most effective ways to combat groupthink is with the encouragement of dissenters. In our group, we created an environment where all members felt comfortable voicing their opinion. Disagreement among coders was not seen as a hindrance to the study or an obstacle to be overcome, but rather as a key aspect of the objectivity and reliability of our process. By emphasizing each individual's viewpoint, we ensured that the final coding was never reached solely through a member's concession and that every code given had strong collective support. To do the coding, each video file was given to one coder who transcribed and coded the questions. Then, the codes were hidden, and the video file and transcription were given to a second coder, who watched the video again to find any questions the first coder may have missed and to code the questions. The coders resolved any discrepancies by consensus; we held weekly meetings to refine the coding process using understandings from the resolution of discrepancies. To measure inter-rater agreement, we calculated either Cohen's κ (2 coders) or Fleiss' κ (3 coders or more) prior to resolving the discrepancies. Initially κ values were low (around 0.3) but they improved as the protocol was refined after discussions (~0.6-0.8).

There were two areas of difficulty: (1) differentiating between rhetorical questions and unrealized authentic questions, and (2) deciding the authenticity based on the knowledge available to the class. For (1), when introducing a new topic an instructor may say: "What the heck is a rational function?" If students did not respond to the question, and the instructor did not leave time for students to respond, some coders code this question rhetorical and some as authentic unrealized. The final decision was to code these questions as unrealized mathematical acknowledges that they could prompt discussion of new topics and therefore help students create new connections, were the instructor interested in giving students opportunity to engage with the questions. For (2), because instructors in different community colleges introduce topics in different orders, material that may be completely new in one class might have been covered in a previous lesson in another. This made it difficult to know whether a question related to new material or students' prior knowledge. In order to manage this issue, we used (a) the course syllabi to resolve disagreements about what constitutes new material and prior knowledge, (b) relied on coders with experience teaching the course to decide, or (c), when neither (a) or (b) was conclusive, used the "generous" coding approach and assigning the higher level. These decisions helped improve the inter-rater agreement.

Preliminary Findings

We coded 8,323 instructor questions of which about 20% were either authentic or quasi-authentic (n = 1,600) and 40% (n = 3,332) were inauthentic. The average wait time across all 37 lessons after a question was asked was about one second (0.99); only 4% of the questions had a wait time of 5 seconds or more. Across all lessons, instructors asked, on average, five questions every two minutes (2.49 questions per minute). These averages mask variations by lessons. Because the lessons have different lengths, we calculated the rate of questions (# of questions per lesson divided by lesson duration) to facilitate cross-comparison.[1] Table 2 presents descriptive information for several variables that were created.

Table 2: Mean and Standard Deviation for Variables in the Study. (N=37)

	Mean	SD
Rate per minute of Realized[a] Authentic and Quasi Authentic Questions[b]	.116	.069
Rate per minute of Unrealized Authentic and Quasi Authentic Questions[b]	.083	.049
Rate per minute of all Authentic and Quasi-authentic Questions[b]	.498	.402
Rate per minute of Inauthentic Questions[b]	.982	.528
Proportion of Authentic and Quasi-authentic Questions[c]	.198	.091
Average Wait Time (in seconds)	.986	.809
Proportion of Questions for which there is a 5s or more Pause	.048	.047
Normalized Gain in Scores on Test of Knowledge[d]	.102	.064
Final Grade[e]	.589	.153
Proportion of Students Passing the Course	.755	.192
MKT-A[f]	27.6	2.983

Notes: a. A question is realized when there is a student response or a pause of five seconds or more after a question has been posed. b. Estimated as sum of the two categories of questions and divided by the duration of the lesson. c. Estimated as the sum of the two categories and divided by the total number of questions asked. d. Difference between end of term and beginning of term scores divided by the number of questions not answered in the beginning of term test. e. Average final grade on a scale of 100, divided by 100. f. Instructor score in the MKT-A; maximum points in the test was 34.

In these lessons, the instructors asked about one inauthentic question per minute, and about one authentic or quasi-authentic question every two minutes. On average the gains in the student test of knowledge were small: about one and a half more questions responded correctly and about 10% of gain at the end of the term. The average grade in the courses was 59%, and about 76% of students passed their course. To explore possible relations between the quality of questions and the student outcomes, we tested correlations between these variables, using a non-parametric test (Kendall's τ) as the distributions of these variables were not normal.

[1] We also averaged these rates to obtain a single value for each instructor, under the assumption that while there might be differences in lessons that may result in more or less use of different types of questions, the way in which instructors use questioning is a feature of their instruction and not as dependent on the content at stake. In this proposal, we only present analyses at the lesson level.

We found positive and statistically significant correlations between the rate of realized authentic and quasi-authentic questions and the MKT-A score ($\tau = .207, p < .05$) and the percentage of questions for which there is a pause of 5s or more ($\tau = .231, p < .05$). We also found negative and statistically significant correlations between the rate of inauthentic questions and the proportion of students who pass the course ($\tau = -.209, p < .05$) and the MKT-A score ($\tau = -.205, p < .05$). Finally, there was a positive and statistically significant correlation between the proportion of authentic and quasi-authentic questions and the MKT-A score. This suggests that instructors with higher MKT-A scores tended to ask (1) a higher rate per minute of authentic and quasi-authentic questions for which they wanted an answer (i.e., questions that were realized) and (2) a lower rate of inauthentic questions per minute. The rate of questions per minute for authentic and/or quasi-authentic questions was positively correlated with the percentage of questions for which there was a pause. We anticipated this result. And as a consequence of the coding, we believe this suggests that when instructors paused it was likely that they did so for an authentic or quasi-authentic question. We found a positive and borderline statistically significant correlation between the average length of the pauses after the question and the percent grade ($\tau = .192, p = .053$) and the proportion of students passing the course ($\tau = .19, p = .056$). Thus, as the duration of the wait time increased, the pass rate in the course increased (or in courses in which more students passed, there was more wait time after questions posed).

Discussion

These are preliminary analyses, but they suggest, first, that it might be detrimental to ask too many inauthentic questions per minute, and second, that instructors with more knowledge of algebra for teaching (as measured with the MKT-A test) will pose more authentic and quasi-authentic questions with longer pauses. These findings also suggest that when instructors ask authentic and quasi-authentic questions without giving students opportunity to respond or time to think about the questions, such decision might be detrimental for student outcomes. While using rhetorical questions to introduce a topic can be a strategy to capture students' interest and attention in the end they might not be as effective. The connection between scores in the MKT-A and use of types of questions is promising, as it suggests concurrent validity between the two measures. The lack of relationship between the types of questions and student performance on the test of knowledge might be related to the generic character of the coding system.

Questions for Discussion

1. We coded mathematical questions using six categories, based on the level of cognitive demand required to answer it and whether or not the question was realized. What are other possible ways of measuring the quality of questions?
2. Interpreting the anticipated cognitive demand of a question is challenging. What measures should be included to increase the reliability of this type of coding?

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Finding Free Variables as a Conceptual Tool in Linear Algebra.

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This preliminary report examines students' interpretations of free variables in linear algebra. In linear algebra, students build understandings of concepts, such as a (in)consistent system of equations, a linearly independent set of vectors, and a subspace. All these concepts will be the foundation for students' future learning in various fields. Therefore, it is crucial to investigate the notion of free variables as it is one of the constructs underlying work with each of these concepts. Here, I analyzed 110 linear algebra students' written assessments from three different classes using grounded theory (Strauss & Corbin, 1994). The analysis shows that students use free variables as a conceptual tool to answer questions given in different problem settings. This paper reports categories of students' interpretations of free variables and explores what the free variables mean to students in each category.

Keywords: Free Variable, Column Space, Linear (In)dependence, Consistent System

Literature Review and Theoretical Framework

In linear algebra, students reason and compute with a set of vectors in a matrix form. Thus, how to interpret a set of vectors has a large effect on students' learning in linear algebra. Larson and Zandieh (2013) found students have three interpretations to the matrix equation of **Ax=b**, where A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, x=$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, b=$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$; (a) Linear combination interpretation of $x_1a_1+x_2a_2=b$ giving weights x_1 and x_2 to the column vectors of a_1 and a_2 being equal to the resultant vector b. (b) System of equations interpretation $a_{11}x_1+a_{12}x_2=b_1$ and $a_{21}x_1+a_{22}x_2=b_2$ viewing the entries of A as coefficients to the linear equations and entries of x as a solution set to the same system of equations. (c) Transformation interpretation T: x$\rightarrow$b, where T(x)= Ax, reaching the vector b by multiplying A to the input vector x. Larson and Zandieh (2013) offer evidence about how students interpret the matrix multiplication of Ax=b and how students view the matrix A as well. Students may interpret the matrix A as a collection of column vectors or a collection of row vectors or neither of them. Also, Larson (2010) found that students have two different computational strategies for performing matrix multiplication; linear combination column vectors and row-focused computation. In this sense, I adopt Larson and Zandieh (2013) and Larson (2010)'s perspective for interpreting the matrix equation as my theoretical framework.

Possani (2010) pointed out three types of students' difficulties interpreting with the matrix equation Rx=b′ shown in Figure 1, where R is a row reduced matrix obtained by applying elementary row operations to the matrix equation Ax=b. The first type of students is not able to unfold the form Rx=b′ into the corresponding system of equation. The second type of students plugs a few numbers into the free variables but does not know what to do with them. The third type of students finds just a particular solution by substituting numbers for free variables. The three types of students' difficulties raise the issue of how students treat free variables when they appear in a row reduced matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -200 \\ 100 \\ 200 \\ 200 \\ -100 \\ 0 \end{pmatrix}.$$

Figure 1. Rx=b′ obtained from Ax=b (Possani, 2010)

Harel (2017) mentions how limited in-service teachers' conceptions of free variables can be. This issue came out of the discussion about the relation between a solution, $\mathbf{x}=\boldsymbol{\alpha}+t\boldsymbol{\beta}$, of a non-homogeneous system S_1 and the solution, $t\boldsymbol{\beta}$, of its associated homogeneous system S_2, where t is a free variable, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are column vectors. Harel urged the in-service teachers to substitute 0 or 1 into the free variable so that they could find out the relation that the solution for a homogeneous system is nothing but a special case of the solution for the non-homogeneous system. However, the in-service teachers were reluctant to substitute 0 or 1 for the free variables $t_1,\ldots t_k$, wherein the solution for the non-homogeneous is given in a vector form of $\mathbf{x}=\boldsymbol{\alpha}+t_1\boldsymbol{\beta}_1+\ldots+t_k\boldsymbol{\beta}_k$ because they conceived that putting specific numbers in free variables is mathematically illegitimate. The in-service teachers did not allow the free variables to range freely across many values, leading me to question, "what in nature do free variables mean to students?"

Dogan (2018) found that students connect the absence of identity matrix in RREF (Row Reduced Echelon Form) with the existence of a non-trivial type of solutions. In other words, students conceive that the existence of free variables guarantees the existence of non-trivial solutions to the system. Wawro (2014) investigated students' quotes on free variables in her study of students' reasoning about the invertible matrix theorem. A student mentioned that "If RREF is linearly dependent, you're going to have a free variable, then there has to be more than one input to get the same output" This student conceives that linear dependence guarantees a free variable(s) and the existence of free variables allows for multiple inputs to reach a certain value. The studies of Wawro (2014) and Dogan (2017) provide evidence of students' understanding that the existence of free variables is closely related to concepts in linear algebra, such as linear dependence and one-to-one transformation. Additionally, Hannah et al. (2016) pointed out that the number of free variables is also related to the dimension of the column space and row space in students' conceptions.

Many researchers in linear algebra have investigated students' thinking and understanding of various concepts such as linear (in)dependence, span, and eigen theory; however, students' conceptions of the notion of free variable itself have not been investigated much. Even though students regularly use them, free variables have never been a focus of study. Therefore there is a need to examine students' foundational understanding of free variables and how this understanding affects their further reasoning processes. This study investigates what free variables mean to students by asking the following specific questions:

(1) How do students determine if there is free variable?
(2) What are the roles of free variables when students are solving problems in linear algebra?

Methodology

The population of this study is one-semester course linear algebra students enrolled in a large research university in the United States. The data comes from three different classes' written assessments; 34 students with Exam A, 37 students with Exam B, 39 students with Exam C. The exams cover linear algebra concepts, such as solving linear systems, matrices, determinants, vector spaces, bases, linear transformations, eigenvectors, and decompositions. The exams consist of pairs of the multiple-choice question and its follow up open-ended question asking why it is chosen, T/F question and its follow up open-ended question justifying the answer, and independent open-ended questions. Students are required to show their work for each question to receive full credit. All the students' work was digitally scanned before getting graded and documented in the alphabetical order by last names and then shared in Dropbox folder.

This study analyzed the data using the technique of grounded theory (Strauss & Corbin, 1994). At the first stage, what students mentioned about free variables was explored in the context of the problem provided by the full version of the written assessments. As performing the initial open coding based on constant comparative analysis, the first level of categories emerged; (a) row-centered free variable interpretation **R** and (b) column-centered free variable interpretation **C**. The way students view the location of a pivot in RREF determines whether it is **R** or **C**. Focusing on the two interpretations of **R** and **C**, the second level of categories emerged; (a) linear independence **LI** and linear dependence **LD**, (b) column space **CSP**, and (c) consistent system of equations **CS** and inconsistent system of equations **IS**. These three categories disclose how students conceive free variables in relation to other concepts, such as linear independence, column space, and consistency. Students' answers on every one of the concepts with free variables were coded accordingly.

Result

In this section, I report findings on the notion of free variables represented in students' work by the categories that emerged during the two phases of data analysis.

How to determine if there is a free variable(s)

1. Row-centered interpretation "R"; this category connects the existence/lack of pivot in rows with the existence of a free variable. Students in **R** affirm the existence of free variables when there is a lack of the pivot in any "row". Once any row of zeros is found, students recognize the lack of pivot for the row, taking a free variable from that row. Figure 2 illustrates students' work that focuses on the relationship between the pivot row in a row reduced matrix and the existence of the free variable. The students marked boxes of ones representing them as pivots and interpreted that there exists a free variable x_4 due to the lack of the pivot in the last row since the last row consists of all zeros. These are classified into **R** in that students obtain the free variable from the row-centered interpretation. This is consistent with Larson and Zandieh (2013) and Larson (2010)'s perspective in that students' notion of free variables varies with row-centered views on the vectors that make up a matrix.

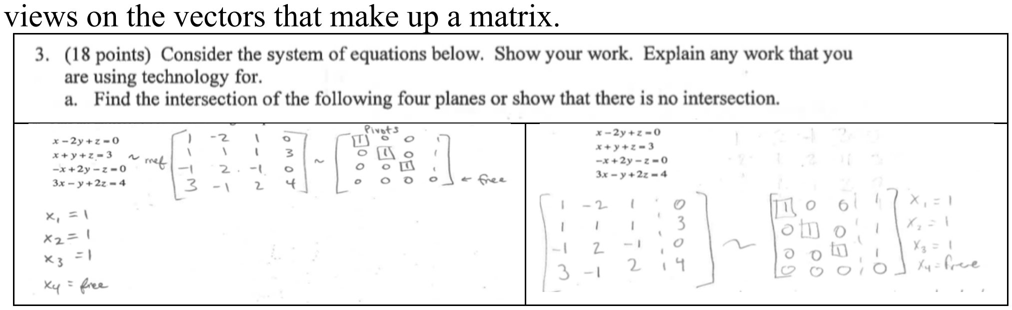


*Figure 2. #3(a) of the assessment and students' answer (Exam A); "**R**"*

2. Column-centered interpretation "C"; this category connects the existence/lack of pivot in columns with the existence of a free variable. Students in C affirm the existence of the free variable when there is a lack of pivot in any "column". Once the lack of pivot for any column is found, students take a free variable(s) from that column. Figure 3 illustrates students' answers that focus on the relationship between the pivot column in a row reduced matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ obtained from $\begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$ and the existence of free variable(s). These are classified into C in that students identify free variables from the column-centered interpretation. This is consistent with Larson and Zandieh (2013) and Larson (2010)'s perspective in that students' notion of free variables varies with column-centered views on the vectors that make up a matrix.

#5. Answer the following questions about the matrix $M=\begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$ and its associated transformation.
(d) If linear transformation T=M**x** is one-to-one, explain how you can tell. If T is not one-to-one, state two vectors **x** that get mapped to the same vector **b.**

it is one one because when Row Reducing the original matrix there are no free variables meaning for every column there is a pivot. “ it is one to one because row reducing the original matrix there are no free variables meaning for every ‘column’ there is a pivot.”	No free variables, pivots in each column. “ No free variables, pivots in each ‘column’ ”

Figure 3. #5(d) of the assessment and student’s answers (Exam B); ***C***

What to do with the free variables

1. Linear independence “LI” vs. linear dependence “LD”; this category shows students’ use of the existence of free variables with the concepts of linear independence/dependence. Figure 4 illustrates a student’s solution that focuses on the relationship between linearly dependence and the existence of free variables. The student finds free variables as a sufficient way to confirm linear dependence. Finding free variables is a tool to determine whether a set of vectors is linearly independent or dependent.

2. Consider the set of vectors $\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\3\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\3\end{bmatrix}\right\}$.

a. (3 points) The set of vectors is (circle one): linearly independent / linearly dependent

b. (4 points) Explain what the answer you circled in a. says about the *Juice Store scenario*.

$RREF = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ there are two free variables. which means that twos of the mixes can be any amount

“ there are two free variables. which means that twos of the mixes can be any amount ”

Figure 4. #2(a), (b) of the assessment and a student’s answer (Exam C); ***LD***

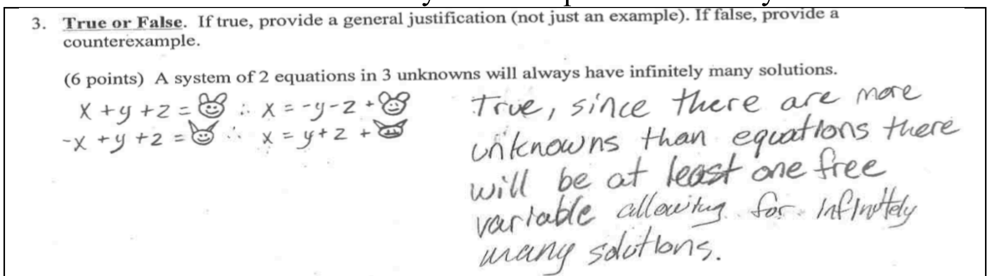
2. Column Space “CSP”; this category shows students’ use of the existence of free variables to find bases of column space. Figure 5 illustrates students’ solutions focusing on the relationship between basis vectors in column space and the existence of free variables. The students interpret that finding columns with a pivot(s) is sufficient for the columns to be the basis vectors in the column space. Students selected the first two columns to be the vectors in the column space since those columns have pivots within it. Finding free variables is a tool to determine whether a column vector could be a basis for the column space.

2. (18 points) Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \end{bmatrix}$. Then C transpose, $C^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$.

(b) List 3 vectors in the column space of C^T, i.e., Col C^T. $C^T \to rref = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Col $C^T \left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$	(b) List 3 vectors in the column space of C^T, i.e., Col C^T. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = -x_3$, $x_2 = -x_3$, $x_3 =$ free Col $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

Figure 5. #2(b) of the assessment and students’ answers (Exam B): ***CSP***

3. Consistent System of equations "CS"; this category shows students' use of the existence of free variables to determine whether it is a consistent system of equations. Figure 6 illustrates a student's answer focusing on the relationship between the number of solutions of a system of equations and the existence of free variables. The student identifies the existence of free variable(s) from more unknowns than the number of equations and concludes that the free variable allows the system of equations to have infinitely many solutions. Finding free variables is used as a tool to determine whether the system of equations is always consistent or not.


3. True or False. If true, provide a general justification (not just an example). If false, provide a counterexample.

(6 points) A system of 2 equations in 3 unknowns will always have infinitely many solutions.

True, since there are more unknowns than equations there will be at least one free variable allowing for infinitely many solutions.

"~ more unknowns than equations… at least one free variable allowing for infinitely many solutions"

*Figure 6. #3 of the assessment and student's answer (Exam C); **CS***

In the second level of categories emerged as LI/LD, CSP, and CS along with the existence of free variables, students have different perspectives to view a matrix; as a collection of columns or as a collection of rows (Larson & Zandieh, 2013; Larson, 2010).

Discussion

I came up with two levels of categories on free variables found in students' written assessments analysis. Figure 7 shows the categories disclosed throughout the different problem settings.

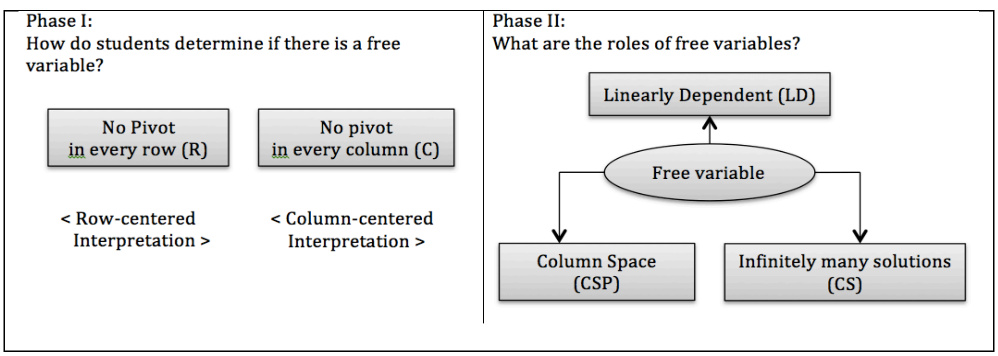


Figure 7. Categories with two phases of students' meanings of free variables

Students mention the term 'pivot' frequently with the term 'free variables', however, I could not determine how students actually define 'pivot' from this written assessment analysis. Nevertheless, I could say that students note the existence of free variables from where the lack of a pivot in RREF appears. In addition to finding free variables, students link free variables to other concepts. In other words, students utilize free variables as a conceptual tool since free variables are used to answer questions related to the concepts, such as linear independence/dependence, basis vectors in column space, and consistent/inconsistent system of equations. Due to the nature of analyzing the comprehensive written assessment, it was not quite easy to discern how students justify the connections between the existence of free variables and other concepts. Despite the limitations, this study discloses many issues in students' learning on a set of vectors of a matrix in linear algebra.

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How Do Mathematicians Describe Mathematical Maturity?

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The concept of mathematical maturity is one that, for some, elicits clear meanings and perhaps illustrations of ideal mathematical students. Mathematicians have been reported to use this term in various ways, yet there is no clear or empirically based description of mathematical maturity at this time. This proposal explores existing descriptions of mathematical maturity as well as descriptions of the related concepts of mathematical intuition and mathematical beliefs. This proposal reports preliminary findings from interviews with mathematicians investigating their understandings of mathematical maturity. Preliminary results include three main components of mathematical maturity: ways of thinking about mathematics, mathematical intuition, and comfort with and competence in mathematics.

Keywords: mathematical maturity, mathematicians, advanced mathematics courses

"Mathematical maturity" is a term often used by many mathematicians to describe some collection of desirable features in their advanced undergraduate students. In some cases, mathematical maturity can even be listed as a prerequisite requirement for advanced mathematics classes or as a learning objective in undergraduate course descriptions and syllabi. The concept is ubiquitous enough within practice to warrant a Wikipedia page which provides the following description:

> Mathematical maturity is an informal term used by mathematicians to refer to a mixture of mathematical experience and insight that cannot be directly taught. Instead, it comes from repeated exposure to mathematical concepts. It is a gauge of mathematics student's erudition in mathematical structures and methods.

Meanwhile, this description is worrisome for multiple reasons. First, describing mathematical maturity as an informal term suggest that it is not used in an official capacity. Second, describing mathematical maturity as "a mixture of mathematical experience and insight" is remarkably vague. Third, the perspective that mathematical maturity cannot be taught is suggestive of a perspective that a learner can inherently succeed at mathematics or they cannot.

This proposal explores the concept of mathematical maturity from the perspective of mathematicians discussing their understanding of mathematical maturity in the context of their undergraduate students. The literature review and theoretical perspective ground the discussion in published opinions of expert mathematicians, mathematical philosophy, and existing work on related topics. Preliminary results from interviews with five pure mathematicians highlight three main components of mathematical maturity and ways these mathematicians report to foster mathematical maturity in their students.

This study investigates the following research questions: How do mathematicians describe mathematical maturity? Is there a difference between how pure mathematicians and applied mathematicians describe mathematical maturity?

Literature Review and Theoretical Perspective

While mathematical maturity may be viewed as a goal of university mathematics and a characteristic of an ideal advanced undergraduate student, the treatment of mathematical maturity in the literature does not reflect this importance. Steen (1983) described mathematical maturity as impossible to define, but suggested that there are "several marks of maturity that most mathematicians will instantly recognize" (p. 99). These marks include the ability to abstract and the ability to synthesize. Steen continues to identify additional "criteria of maturity" all listed as various abilities (for instance, the abilities to use and interpret mathematical notation, to generalize, to perceive patterns) she believed mathematically mature students must possess. Steen's essay is based on the author's and her colleagues' professional opinions, and very little has been published on the specific topic of mathematical maturity since.

Meanwhile, a number of famous mathematicians have published their opinions around their understanding of mathematical thinking, its development, and the mathematical education of an individual. For instance, Tao (2007) argues that mathematical education can be roughly divided into three stages: a pre-rigorous stage, the rigorous stage, and a post-rigorous stage. He suggested that mathematics is first taught informally, then students are taught to be more precise and formal, before they are ultimately to return to the informal—using their intuition which is now supported by their comfort with the "rigorous foundations". Whereas, Thurston (1998) suggested that human language, visual/spatial sense, logic/deduction, intuition/association/metaphor, stimulus-response, and process/time are important for mathematical thinking.

These various existing descriptions of mathematical maturity and important characteristics of successful mathematical thinking and learning do not offer empirically tested frameworks of mathematical maturity, but do provide an impression of what might be important or necessary aspects of mathematical maturity. In particular, these descriptions suggest the significance of a learner's mathematical beliefs and intuitions, which are discussed below.

Mathematical Beliefs

The existing literature on mathematical beliefs is expansive and diverse in its interpretation of the concept. Muis's (2004) review of 33 studies on students' epistemological beliefs highlighted this variety and focused on "beliefs about the nature of mathematical knowledge and mathematical learning" (p. 324). In this review, Muis identified that much of the research surrounding students' mathematical beliefs found the students to possess beliefs that are nonavailing—or those that either do not influence or negative influence learning outcomes. For instance, the literature reviewed identified students as believing that 1) mathematical knowledge is unchanging, 2) the goal in mathematics is to find the correct answer, 3) knowledge is delivered by an authority, 4) mathematical ability is innate, 5) components of mathematical knowledge are unrelated, and 6) students are incapable of constructing knowledge and solving problems on their own. Muis went on to explain the negative effects of students' nonavailing beliefs on their strategies for learning and motivational orientations, calling for future research considering the impact of teachers' epistemological beliefs and their instructional styles on students' beliefs.

Indeed, research has shown that students' experiences within their mathematics classrooms are highly influential in shaping students' beliefs about mathematics. For example, Schoenfeld's (1989) survey of 230 high school students' mathematical beliefs highlighted the separation in students' minds between the procedural, rote mathematics they were accustomed to seeing in their schools from the interpretive and creative nature of mathematics.

Moreover, the mathematics education literature has further shown that students' mathematical beliefs affect their understanding and study of mathematical content. For instance, Szydlik's (2000) study of 27 calculus students indicated that of the students interviewed, those with "internal sources of conviction provided more static definitions [...] and fewer incoherent definitions than students with external sources of conviction" (p. 272). As such, students with nonavailing beliefs, such as believing that the goal of mathematics is to successfully receive knowledge from an authority to directly use the knowledge to achieve the correct answer, may have more difficulties understanding mathematical content than those with availing beliefs.

Mathematical Insight and Intuition

The above descriptions of mathematical maturity given by Wikipedia and Steen (1983), as well as the descriptions of the learning and thinking of mathematics by Tao (2007) and Thurston (1998) all suggest the necessity of mathematical insight and mathematical intuition.

Meanwhile, definitions of both mathematical insight and mathematical intuition seem to have largely evaded the literature. For instance, Keijzer and Terwel (2003) claim that "mathematical insight is widely recognized as an important goal of education", despite failing to provide a definition of the term. Hartmann (1937) offered the generic definition of insight as the "process of making an organism aware of the conditions governing the phenomena to which it is reacting" (p. 19), but does not extend this discussion to explain precisely what is mathematical insight. Griffiths (1971) even suggested that it would be impossible to define mathematical insight and instead offered examples and anecdotes of theoretical students who lacked mathematical insight. Griffiths does continue to suggest that mathematical insight and mathematical intuition are not the same concept; however, others disagree and use them interchangeably.

Mathematical intuition has been more explicitly discussed in the literatures of mathematical philosophy and mathematics education. For instance, Feferman (2000) describes intuition as the "insight or illumination on the road to the solution of a problem" (p. 317) and a "mathematicians' hunches as to what problems it would profitable to attach, what results are expected, and what methods are likely to work" (p. 318). Similarly, Fischbein (1982) described intuition as the unconscious ability to "organize information, to synthesize previously acquired experiences, to select efficient attitudes, to generalize verified reactions, to guess, by extrapolation, beyond the facts at hand" (p. 12). Tall (1980) described intuition as "the global amalgam of local processes from the current cognitive structure selectively stimulated by a novel situation" (p. 2). Thus, we do see an acknowledgement of an unconscious or semi-conscious aspect of intuition as well as the problem-solving aspect of intuition in each of these descriptions above. It is further noteworthy that in Burton's (1999) study involving interviews with 70 mathematicians, most of the mathematicians viewed intuition as a "necessary component for developing knowing" (p. 31). Burton continued that while these mathematicians had this opinion, none of them offered comments on how one might develop mathematical intuition.

Methods

This study took place at a large doctoral-granting research institution in the United States. Participants were recruited via email. Nine mathematicians (five pure mathematicians and four applied mathematicians) volunteered to participate and were interviewed by the author. In the interviews, participants were asked if they had ever used the term mathematical maturity. If the participant indicated that they had, they were then asked several probing questions about the nature of the term. If they participant indicated that they had not ever used the term mathematical maturity, they were asked if they were familiar with the term. If a mathematician was not familiar with mathematical maturity, the interview was terminated.

For the mathematicians familiar with the term mathematical maturity, the interviewer asked 1) in what context they had used the term mathematical maturity, 2) if mathematical maturity is a feature of a person or a mathematical artifact, 3) how they would identify if a student is mathematically mature, 4) what features of a student (or their work) would help them to identify the student as mathematically mature, and 5) if any of the features are clear or key signs of mathematical maturity. Finally, the interviewer asked the mathematician if mathematical maturity is something they aim to foster in their students and if so, in which classes, why those classes, and how they attempt to foster mathematical maturity in their students.

Mathematicians were not compensated for their participation. Interviews ranged in length from 2 minutes to 50 minutes. Mathematicians varied in years of experience teaching advanced mathematics courses and areas of study.

Analysis

Each of the interviews in which the mathematician indicated familiarity with the term mathematical maturity was transcribed and analyzed. The data was analyzed using open coding in the style of Strauss and Corbin (1990). Each interview was individually coded for descriptions of mathematical maturity and various aspects or indicators of mathematical maturity offered by the participant. Categories of these indicators and descriptions were then tentatively identified per interview. The various themes and indicators of mathematical maturity of each of the interviews were then synthesized to identify categories of codes discussed by multiple mathematicians. Transcripts were then reviewed for any additional occurrences of the codes not identified in the earlier pass through the data.

Preliminary Results

Pure Mathematicians and Applied Mathematicians

One striking finding from this study concerns the interviews with (self-reported) applied mathematicians. As noted in the methods, each interview began with the question, “Have you ever used the term mathematical maturity?” Surprisingly, none of the four applied mathematicians has used the term. Moreover, when probed further, none of the four applied mathematicians even appeared to be familiar with the term. At best, two of the four conjectured that the concept was related to a student’s competence in mathematical activities, but each of the applied mathematicians did not feel comfortable continuing the conversation around mathematical maturity. As such, the remainder of the results focus solely on the interview data from the pure mathematician participants.

Aspects of Mathematical Maturity

Below is a table representing each of the codes, and the categories they were sorted into, that resulted from the open coding analysis described above. As seen in Table 1, each code included was present in at least two different interviews and the codes were sorted into three categories: Ways of thinking about mathematics, Mathematical intuition, and Comfort with and competence in mathematics. Due to constraints on the length of this proposal, I note that the Ways of thinking and Mathematical Intuition categories are closely tied to the literature on mathematical beliefs and mathematical intuition (respectively) and provide only a brief description and selected quotes to highlight some of the codes included in the Comfort with and Competence in Mathematics category.

Table 1. Categories and codes for mathematical maturity mentioned by the pure mathematicians.

	M2	M3	M4	M5	M6

<u>Ways of Thinking about Mathematics</u>					
Having a holistic view of mathematics	**X**		**X**	**X**	
Being accepting multiple representations and changing definitions		**X**	**X**	**X**	**X**
Having autonomy or agency over their own learning	**X**		**X**		
<u>Mathematical Intuition</u>					
Knowing what to do with problems		**X**			**X**
Recognizing the crux of an argument		**X**			**X**
Knowing if a solution makes sense	**X**				**X**
<u>Comfort with and Competence in Mathematics</u>					
Having the ability to absorb and use definitions and theorems		**X**		**X**	
Having the ability to effectively communicate mathematics		**X**	**X**	**X**	**X**
Having the ability to abstract and make connections across topics			**X**	**X**	**X**
Having the ability to self-assess, validate, and reconstruct arguments	**X**				**X**

Comfort with and competence in mathematics. This category largely focuses on necessary skills described as indicators of mathematical maturity by the pure mathematicians in the study. For instance, M5 believed the ability to effectively communicate mathematics was essential to one's mathematical maturity. When asked what a specific indicator of mathematical maturity might be, M5 said, "Being able to take an intuitive idea and express it using a sensible notation, and yeah, putting it into words in a sense". M5 continued to explain that being able to express one's ideas, as well as understanding the ideas of others', regardless of the other person's level of comfort with mathematical notation is essential.

When asked if he had used the term mathematical maturity, M3 immediately agreed and continued to explain that mathematical maturity is the ability to abstract in a useful way. As an example, M3 said "if we show our students a proof that there are infinitely many primes, what's the point? It's not really that there are infinitely many primes." Later he explained that by abstraction, he meant "the ability to read the story and understand the moral rather than just seeing that the tortoise beats the hare". Throughout his interview M3 focused on students' abilities to view (or abstract) the bigger picture and motivation behind a proof as indicative of mathematical maturity.

Conclusion

As a preliminary study, these findings are still being interpreted. The findings of this study will also have various limitations due to the small sample size and exploratory nature.

However, mathematical maturity is a concept that has largely been deemed ineffable yet continues to be used in mathematical practice. Moreover, we see that not all mathematicians (notably the applied mathematicians in the study) are familiar with the term. Meanwhile, as mathematical maturity, mathematical beliefs, and mathematical intuition are intrinsically tied, not only to each other, but also to student success, this study aims to provide an empirical first step toward understanding mathematical maturity. Future research considering these topics could lead to future strides for mathematics education research. Such a clearer conception of mathematical maturity can afford future research on fostering and developing mathematical maturity over time and measuring a learner's mathematical maturity.

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Re-Humanizing Assessments in University Calculus II Courses

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Answering the call of Francis Su (Su, 2017) that "math is for human flourishing" and a challenge by Rochelle Gutiérrez to rehumanize math (Gutiérrez, 2018), I changed assessments in two university calculus II courses. The traditional way to change assessments is to change the questions, either by type or by content. Instead, I focused on changing/rehumanizing the structure of exams to include small group discussions between students for part of the exams. This assessment change, along with a consistently enacted classroom mission statement, produced higher exam scores and improved student engagement. Through surveys, focus groups and interview data, students also reported feeling they had a deeper understanding of concepts, as well as a humane and positive math experience in a math class they thought was very difficult.

Keywords: calculus, assessment, rehumanizing, engagement

Objectives/Purpose and Theoretical Framework

Mathemaphobia was first coined by Sister Mary Fides (Gough, 1954), who taught high school mathematics for years and was concerned about students who experienced anxiety in learning mathematics, with a pronounced increase in anxiety during exams, that then impacted their attendance and interest in mathematics and school. This phenomenon was later coined simply as math anxiety (Tobias, 1978), with studies showing correlation between math anxiety and test anxiety (Hembree, 1990). More than 50 years after the idea of mathemaphobia was introduced, we continue to grapple with this problem (Perry, 2004). It is still true that in most math classes exams tend to be high-stakes, since they substantially impact the grade a student receives and grades tend to be how many students (and society) measure success. Additionally, even for college STEM majors who do not have a general fear and loathing of mathematics, as Marilyn Burns called it (Burns, 1998), some of them may experience anxiety on math exams (Perry, 2004).

Knowing that math anxiety is a serious issue, a variety of solutions have been proposed, including both behavioral-related methods and cognitive treatment (Hembree 1990; Perry, 2004). A recent direction comes from ongoing discussions about math education and equity. Within the last year, Rochelle Gutiérrez has coined the term "rehumanize mathematics" (Gutiérrez, 2018), issuing a call to action to bring a more humane element into the classroom by focusing on practices and curriculum that better serve typically underrepresented groups of students. Similarly, another call for a focus on rehumanizing mathematics comes from a mathematician who, in his closing talk as he stepped down from being MAA President, called on mathematicians and math educators to remember that mathematics is "for human flourishing" (Su, 2017).

While there are many ways to implement such goals of rehumanizing math, one potentially impactful leverage point is to focus on assessments as a known stressful and high-stakes part of a math course. Theses changes were implemented in a college calculus II course where I was the instructor of record and thus this is an example of teacher-researcher research. (Schoenfeld & Minstrell & van Zee, 1999; Zimmerman & Nelson, 2000). My goal in enacting these changes with my exams was to answer the call to rehumanize mathematics by playing the game called mathematics and also changing the game (Gutiérrez, 2009). It was to answer the call to "find a struggling student, love them, be their advocate" (Su, 2017). It was to increase students' sense of belonging in their mathematical pursuits.

One important research question I'm exploring is: how does a rehumanizing-math approach to assessment in a large calculus 2 class impact exam scores and student learning?

Methods

I drew on my previous knowledge and experience regarding mathemaphobia and exam anxiety; then combined that knowledge with the current call for change. Thus, I decided to change the structure of the exam and not the content of the exam. I attempted to rehumanize mathematics in my calculus II classrooms by decreasing test anxiety and increasing student interaction, to build a community rather than a competitive setting. Research has shown the importance of discussion for increased comprehension and a positive impact on learning (Roschelle, 1992; Engle & Conant, 2010). Thus, I wanted the students to have access to mathematical discussion during exams to help showcase their knowledge. This change was made in two sections of a high-enrollment calculus II course at a large public research institution. To explore what impact this change in assessment structure had on my calculus II students, I looked at their exam grades over the semester and their final grades, as well as qualitative data from surveys, focus groups and interviews.

In my efforts to rehumanize the classroom and create a cohesive community of learners, every day in my classes, we have the class mission statement written on the board. "This is a kind, inclusive, brave, failure-tolerant classroom." The goal is to consistently remain in the conversation of kindness and to enact this statement every day in class, holding all students, TAs and instructor accountable for this work.

The three midterm exams were each split into two sections, one was a group portion of the exam (worth 32-36% of overall test score), taken during one class, and the other was a solo portion of the exam (worth the other 64-68% of the overall test score), taken on the next class day. The groups were created semi-randomly about 1-2 weeks before the exam. Each group contained three students (or possibly four). Each group portion of the exam contained some of the statistically hardest questions (based on years' of exam data). This encouraged the students to discuss and defend their answers during the group exams. Each student turned in their own group-portion of the exam.

During the group exam, students had 15 minutes to work on their own (silent-solo) and then the remaining 40 minutes to discuss the problems within their group. This way, each member of the group had time to process their ideas and actually solve most, if not all, of the problems on that part of the exam first, and then in the group discussions, everyone had something meaningful to contribute. For the silent-solo portion of the midterms, it was a usual testing structure where each student worked on their own.

The two-hour final exam was one intact exam, not in separate parts. The first 45 minutes was silent-solo, the next 30 minutes was group discussion time, and the last 45 minutes was silent-solo again. This enabled students to work on most of the exam by themselves and then discuss some of their ideas/solutions within their group, get ideas to help them get unstuck, etc. Then, they had time to finish the exam on their own.

Research data consists of (a) three different surveys students filled out throughout the semester which gave information about their attitudes, as well as feedback about their interpretations regarding rehumanizing the classroom, (b) focus group and interview data, and (c) grade data, all from my spring 2018 calculus II courses. Additionally, I taught a high-enrollment calculus II course in the fall of 2017, and the grade data from that class is being used as control group data, since their exams were the standard exam structure.

Data Analysis and Results

Grade Data

Table 1. This table shows the basic statistics from two semesters of Calculus II courses. Fall, 2017, Calculus II course had no change in assessment. Spring, 2018, Calculus II courses had group-portion of exams implemented.

Fall, 2017	Midterm 1	Midterm 2	Midterm 3	Final Exam	Final Scores
mean	79.6	64.4	73.7	82.5	81.9
median	85	64	77	86.5	84.2
stdev	18.6	21	22.8	16.5	15.8
low	24	8	12	7.5	19.38
high	106	106	107	103.5	104.34
n	169	157	151	144	144

Spring, 2018	Midterm 1	Midterm 2	Midterm 3	Final Exam	Final Scores
mean	80.9	80.4	85.8	91	88
median	84	81	88	93	89.36
stdev	14	14.5	13.3	9.6	8.6
low	47	44	38	58	61.94
high	105	106	106	104	102.85
n	167	166	162	156	156

Table 1 and Figure 2 show the statistics comparing exam scores from fall of 2017 (the control group), before implementation of the group portion of exams, and spring of 2018 (comparison group), after implementation of the revised assessment structure. The results show that, in my aim to give students a more humane mathematical experience than a traditional math classroom provides, the spring 2018 scores on three midterms, final exam and final course scores are all statistically higher than the fall 2017 data. Perhaps more interestingly, the standard deviations went down substantially with the new exam structure. (Note: no other changes to the courses were made between the two semesters.)

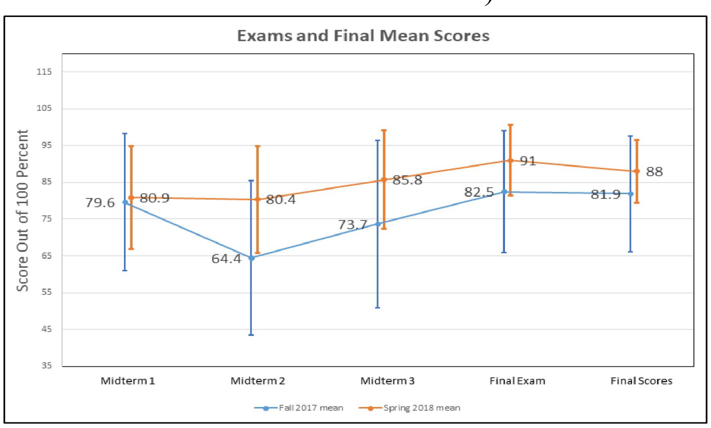

Figure 2. A line plot showing the same data as in Figure 1, with only the means (as points) and standard deviations (as error bars at each point) present in this graph.

The positive impacts of assessment changes were furthermore evident in the lowest exam grades, which were notably higher, suggesting that the changes helped students at the bottom of the class while also supporting students in the mid-performance range. For the top 10% of performers, there wasn't much difference in the statistics. Thus, this new exam structure does appear to either benefit students' grades or have a ceiling effect for the high performers' grades.

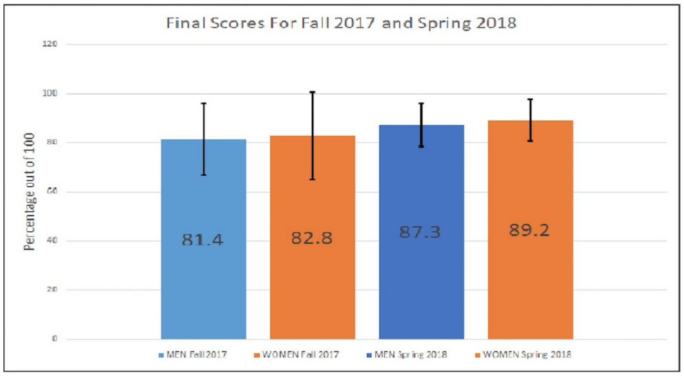

Figure 3: This table shows the final score statistics broken down for each of the two semesters, for men and womxn. (Note: I've chosen to use womxn to denote all students who identify as either female, gender fluid or non-binary.)

In Figure 3, you can see the grade data, for only the final scores, across the two semesters for both men and womxn. Statistically, the grades went up for both groups of students. However, the standard deviation for the men basically stayed the same and the standard deviation for womxn went down surprisingly with the new exam structure.

Survey/Focus Group Data

For the survey data, unique codes were created from within the data and will be revised iteratively. So far, I've gone through the focus group and interview data with one pass, focusing mainly on patterns of comments and have not yet coded it iteratively.

Recurring themes of (a) less anxiety experienced during the exams and (b) a community feeling in the classroom, in both the survey comments and the interview/focus group data, is captured by the following student quotes:

> I feel like being able to start an exam by discussing the concepts and work with other students helps me do better on the solo portion and *eases the nerves a bit*.
>
> I really think the group structure *helps both those who are doing great and otherwise*. Explaining math and understanding it both *require cooperation* in my opinion.
>
> The group portions are excellent, for me I have a tough time starting choosing the methods or test, but *after I discuss with other I gain a better understanding* of which one and why and then I can go forth with my process and usually *complete it on my own*.

The group exam structure not only positively impacted the grades, statistically, for the students, but it also helped build a community inside the classroom where students felt less anxious and more able to pursue mathematics as something to practice and make mistakes, as opposed to something they must be perfect at.

And, finally, when looking through the survey answers to a question about how students interpret or define the idea of re-humanizing mathematics, I see a recurring theme of having a humane classroom environment initiated and enacted by the teacher, as well as the students, as one path to a rehumanized math experience. Here are a couple student quotes that epitomize that theme.

> In my opinion, the idea of 'Re-humanizing mathematics' seems to be the idea of *making math less robotic, cold and conspicuously heartless*. I think the problem is less making math more human, and more *getting people to see that math IS human*. I think teachers could move towards this goal of 're-humanizing' Mathematics by *helping their students see that it is not a 'genius subject'* that only someone with a computer for a brain could do, and that *Mathematics is actually a really fun, cool subject invented by humans to help us with our everyday lives*.
>
> I think teachers/educators can re-humanize math by *making better learning environments* in class, and by *learning how to connect better with people*.
>
> I would like to think this has to do with *framing math practice and math learning as a human endeavor, recognizing how it is connected to other human undertakings, and subject to some of the same biases and flexibilities, instead of being presented as sanitized, rigid, and binary* by necessity. I think that *the way we talk about math in classrooms can influence student receptiveness, and therefore, influence student engagement and comprehension*. If students just feel like they are in class to intake a litany of procedures and formulas, and then to be assessed by reproducing them, mathematical thinking will stay relegated to one portion of their mind, and not interact with other ideas that students may simultaneously processing.

Significance

The group structure of part of the exams more closely mimics how mathematicians actually work, compared to the usual silent-solo style of examination. Mathematicians talk to each other when they're stuck on a problem. Then, they go back to work on the problem by themselves. Allowing students to discuss mathematics on high-stakes exams, defend their solutions to one another and bounce ideas off of someone else when they get stuck allows them to show what they truly know without having test anxiety get the better of them. I argue that this exam structure more authentically assesses student calculus knowledge than silent-solo exams.

Mathematics, as an intellectual pursuit, can frequently feel hostile for students. Many classrooms feel like such a competitive environment that students don't even want to ask questions in class for fear of being told they're wrong or incompetent or unintelligent, and even if we educators don't explicitly say those words, that sentiment is too often portrayed to students (Jackson & Leffingwell, 1999). I argue that this new exam structure, along with the repetition and enactment of the class mission statement, created precisely the type of classroom where students felt a sense of belonging.

I answered the theoretical call to rehumanize the classroom by boots-on-the-ground changes in assessments that produced overwhelmingly positive results. This structure engendered a kind, humane classroom, where students flourished doing mathematics. And, this structure can be replicated to increase the humanity within mathematics classrooms on a broader scale.

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“Let’s See” – Students Play *Vector Unknown*, An Inquiry-Oriented Linear Algebra Digital Game

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Abstract: The results we report are a product of the first iteration of a design-based study that uses a game, Vector Unknown, to support students in learning about vector equations in both algebraic and geometric contexts. While playing the game, students employed various numeric and geometric strategies that reflect differing levels of mathematical sophistication. Additionally, results indicate that students developed connections between the algebraic and geometric contexts during gameplay. The game’s design was a collaborative effort between mathematics educators and computer scientists and was based on a framework that integrates inquiry-oriented instruction and inquiry-based learning (IO/IBL), game-based learning (GBL), and realistic mathematics education (RME).

Keywords: Linear Algebra, Inquiry-Oriented Instruction, Game-Based Learning, Realistic Mathematics Education, Digital Game

Student results in Linear Algebra courses and the extent of students’ struggles in the course are at times surprising to mathematicians and instructors. Making an enthusiastic case for the importance of linear algebra, Tucker (1993) states that Linear Algebra’s “theory is so well structured and comprehensive, yet requires limited mathematical prerequisites” (p. 3). In addition, he states “Linear Algebra is … appealing because it is so powerful yet simple” (p. 4).

The limited number of prerequisites and the simplicity described by Tucker often does not translate into ease for students (Britton, 2009; Dogan, 2017; Dorier & Sierpinska, 2001; Hannah, 2016; Hillel, 2000; Stewart, 2018; Wawro, Sweeney, & Rabin, 2011). Typically, a single course of linear algebra is offered or required in undergraduate education, a situation that presents additional challenges. Tucker acknowledges and describes that “the challenge is to find a middle ground blending vector spaces and matrix methods and at a level that does not scare off the users and yet smooths the transition for mathematics majors to advanced courses” (p. 8).

The inquiry-oriented linear algebra (IOLA) curriculum was created based on principles of RME to guide students through differing levels of activity and reflection and to leverage their intuitive knowledge in the development of more formal mathematics (Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Zandieh, Wawro, & Rasmussen, 2017). Specifically, the curriculum includes a unit known as the Magic Carpet Ride (MCR) sequence that aims to support students in learning the concepts of span and linear independence (Wawro et al., 2012). Expanding IOLA and MCR into the realm of GBL, the promotion of learning by using digital games, may prove to be a productive way to support students’ learning of basic linear algebra concepts. Studies show a clear relation between games and learning (Gee, 2003), especially when thoughtful learning theories are incorporated into the design of games (Gee, 2005; Gresalfi & Barnes, 2015; Williams-Pierce, 2016). The combination of these various perspectives resulted in the development of the game *Vector Unknown*.

Theoretical Framework Utilized for Game Design

The MCR task sequence, which follows RME design principles, aligns well with the structure of game design supported by GBL. Zandieh, Plaxco, Williams-Pierce, and Amresh (2018) developed a framework aligning aspects of GBL with RME and IO/IBL instruction. In considering the three perspectives, Zandieh et al (2018) focused on four aspects of design and implementation: structure of task sequence, nature of task sequence, students' role, and teachers' role. Drawing on specific recommendations from the literature the authors identified similarities along each of these four dimensions for each of the three perspectives. Consider, for instance, the structure and nature of task sequences. Gee (2003) states "Good games operate at the outer and growing edge of a player's competence, remaining challenging, but do-able ... [therefore] they are often also pleasantly frustrating, which is a very motivating state for human beings". Similarly, Rasmussen & Kwon (2007) articulate a perspective for Inquiry-Oriented instruction when they suggest that "challenging tasks, often situated in realistic situations, serve as the starting point for students' mathematical inquiry"; they also assert that students should solve novel problems. Further, Laursen, Hassi, Kogan, & Weston (2014) state that "IBL methods invite students to work out ill-structured but meaningful problems". Our research team has drawn on the design principles of GBL and IO/IBL to convert the first task of the MCR sequence (an RME-based task) to produce the game *Vector Unknown*.

Vector Unknown Gameplay

Gameplay currently consists of five levels and data from Levels 1, 2, and 5 was analyzed. The goal is to guide the rabbit to the basket; a sample screen is displayed in Figure 1. The player moves the rabbit by dragging up to two vectors from the Vector Selection area into the Vector Equation. Adjusting the scalars in front of the vectors in the Vector Equation generates a geometric representation (Predicted Path) of the linear combination. When the player has made selections and presses GO, the rabbit moves along each component vector until it reaches the sum of the rabbit's location and the outcome of the vector equation. The mathematical notation for the move is recorded in the Log.

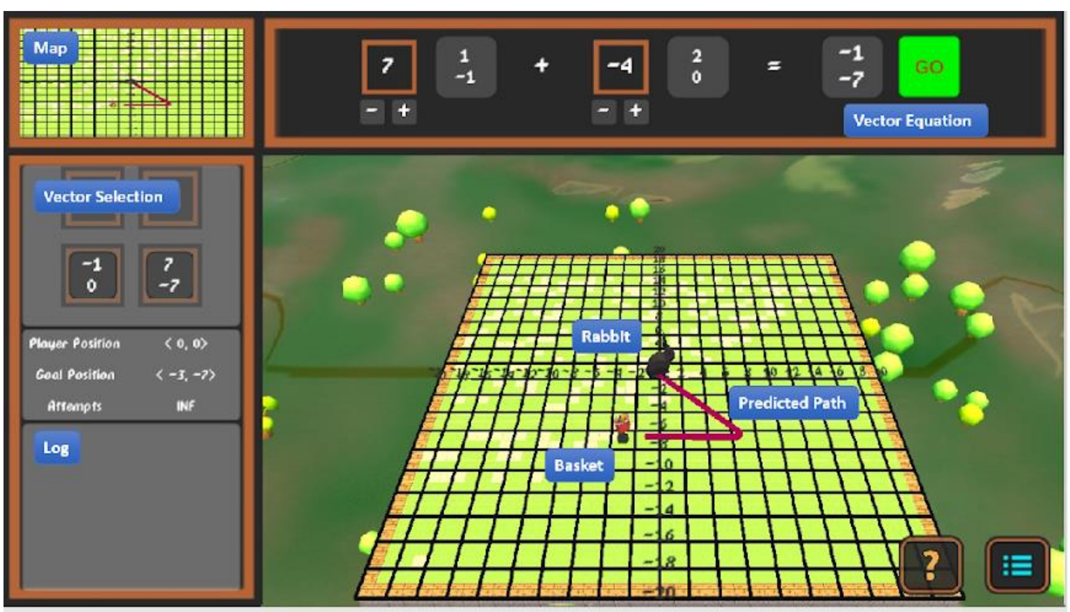

Figure 1. Sample Screen

The game controls reflect common mathematical notation for a vector equation. Scalars were constrained to integers and can be adjusted using plus and minus controls to encourage players 1) to make connections between numerical scalar adjustment and the corresponding change in geometry, and 2) to explore the idea of span. Each level includes a pair of linearly independent vectors along with a scalar multiple of each of the vectors. Level 2 excludes the Predicted Path provided in Level 1, requiring the player to visualize the path on their own or to find the solution using numerical methods. Level 5 includes the Predicted Path from Level 1, but the player must collect one to three keys on the board prior to approaching the basket; this requires the player to consider travel from a point other than the origin.

Research Questions

This report presents some findings from the first iteration of a design-based research study (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) and will focus on answering the following questions:

1. What are students' strategies for completing the game *Vector Unknown*?
2. How do students' strategies vary according to their level of experience with linear algebra?

Methodology and Participants

This project is a collaborative effort of three public institutions: 1) a comprehensive Research I university in the southwestern United States, 2) a multi-purpose regional university in the southeastern United States, and 3) a comprehensive Research I university in the southeastern United States. Eleven clinical interviews were conducted across the three participating universities. Each interview lasted approximately one hour, during which participants were asked to complete three levels of the *Vector Unknown* digital game. As needed, the interviewer provided help on how to navigate the game's screens and use the controls. Interviewers asked scripted questions along with impromptu follow-up questions. Impromptu questions were asked to further clarify and explore the participants' thinking about gameplay as well as any mathematical insights or strategies the participant developed during gameplay.

Participants were diverse with five students identifying as white, five students identifying as black, and one student identifying as Asian; five of the participants were males, and six were female. The students were selected to have a broad range of experience with linear algebra. Participants included Math, Biology, Computer Science, Education, and Engineering majors. The research team reviewed the interviews for strategies used in completing the game, and selected three research subjects to highlight differences in level of expertise in linear algebra. One student had never taken a linear algebra course, one was enrolled in linear algebra, and one had completed a linear algebra course a few months prior.

Preliminary Results

Case Study 1: Gwen - Limited Exposure to Linear Algebra

Gwen has a degree in psychology and will be taking linear algebra in preparation for graduate school. She had no experience with linear algebra prior to playing the game. Her strategy for Level 1 consisted of a trial-and-error approach with vectors and scalars selected at random. Before long Gwen began to realize that the vector equation allows for two vectors to be used simultaneously and attempted to decipher what the scalars did: "I'm trying to figure out what the orange square has to do […] is it 2 times [<0, -2>] to get me 0 over -4?" Gwen completed the Level 1 even though she "had no idea what I just did".

Gwen completed Level 1 again to gain a better understanding of what allowed her to complete the level. On her second attempt, she focused more on the numbers that would get the rabbit to the basket. Despite her numeric approach, Gwen described her strategy as "mindlessly clicking" until the trajectory path showed the correct combination of vectors and scalars. When asked to explain what happened, she responded:

> the little numbers in the orange square are […] multiplying by the numbers given. […] I guess it's what can I multiply in each of these areas to—hold on. […] I'm trying to figure out what I can multiply to get 0 on the x-axis or the numerator while at the same time getting from 0 to 12 on the denominator.

For Level 2 Gwen was more numeric than in her approach to solving the previous level: "I'm not even looking at the position of the rabbit going to the basket. I was just trying to throw in numbers until I got to the position". Level 5 contained one key before the basket unlocked. Gwen immediately selected the vector <-4, -6> from the Vector Selection and scaled it by -1 to reach the key at <4, 6>. Although Gwen had the Predicted Path, she was less dependent on it on Level 5 than on Level 1. In summary, Gwen's guess-and-check numerical strategies evolved during gameplay, and her comments seemed indicative of a growing understanding of the vector equation.

Case Study 2: Andrew – Enrolled in a non-IOLA Linear Algebra Course

Andrew, a senior biology and computation science major who had completed three post-secondary mathematics courses, was enrolled in linear algebra. He focused on making the vector equation yield the goal position. Only after he completed Level 5, where he had to move to the key before moving to the basket, did he begin to direct his attention to the graph. He focused so completely on the equation that he initially noticed no difference in Level 1 and Level 2. However, after he had completed Level 5 and went back to Levels 1 and 2, he noticed that Level 2 does not "show me where it would take me". He mentioned using trial-and-error and intuition and seemed to have strong number sense that allowed him to complete each level quickly.

Andrew's more inquisitive nature came out while talking about scalar multiples as illustrated by the following dialogue; his geometric conceptions seemed to be emerging.

> *Interviewer*: Do you notice anything special about those vectors?
>
> *Andrew*: About the -3 and the 9 and 3? [indicating <-3, -1> and <9, 3>] Well, one of them is both negative and one of them is both positive, and also they are multiples of each other. …
>
> *Interviewer*: So where could you get on the board with just those two vectors?
>
> *Andrew*: Um…can I try and see? [interviewer concurs]
>
> *Andrew*: Alright, let's see! [Andrew moves the scalar multiples to vector equation, scales them up and down, and notes that the bunny was moving along the same line.]
>
> *Andrew*: Alright! [nods and points] Ok, so now I see kinda what it's doing. […] if you add to this one or take away from it [referring to increasing and decreasing one of the scalars], it's still on that same line. Likewise with this one. And since this is the multiple of that one, that means that this is the dependent one on that vector.

Interviewer: You used the word linearly dependent. What does linearly dependent mean to you?

Andrew: So as far as I've learned in my linear algebra class, it means that, basically kinda like what I just said. […] it just means that if you multiply the independent vector by some scalar 1, 2, 3, whatever, -1, -2, you will be able to get that other vector, basically. [Andrew continuously clicks the mouse to change the scalars.]

Ultimately, Andrew began to make connections between the numeric and graphical ideas of linear dependence despite his strong systematic use of numerical strategies.

Case Study 3: Lauren - Completed an IOLA Linear Algebra Course

Lauren was a junior applied mathematics major who had completed six post-secondary mathematics courses, including linear algebra in Fall 2017. Lauren took on the conscious role of game tester and teacher during her interview. She volunteered information about aspects of gameplay that she liked and did not like without being prompted by the interviewer. This perspective precipitated in gameplay that was less focused on reaching the goal during each level and more focused on discovering how adjusting aspects of the vector equation and pressing GO resulted in different movements of the rabbit as illustrated by Figure 2.

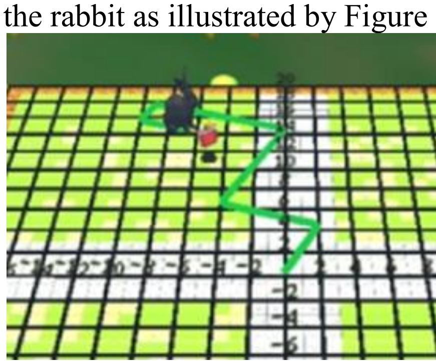

Figure 2. Sample Path for Lauren in Level 2

Lauren's playful nature resulted in an explanation of why two linearly independent vectors span $\mathbb{R}^2$:

> [She chooses two linearly independent vectors.] This diagonal line stretches on forever [points to Vector 1] and this diagonal stretches on forever [points to Vector 2]. However much you multiply that vector, and they start wherever you add them […] And you can start anywhere along this by shifting it [points to Vector 1]. And so you can cover the entire board by starting with this vector [points to Vector 2] anywhere along this vector [Vector 1].

In brief, Lauren used a playful geometric approach and gave indications that she was beginning to conceptualize the idea of span.

Preliminary Conclusion/Questions for Audience

Preliminary analysis of the data reveals that students used a variety of strategies which evolved during gameplay and resulted in mathematical realizations. What are some suggestions for expanding the game to help teach span and linear independence? How could this game be incorporated into a linear algebra course? What instructional sequences in linear algebra could be translated into a level of the game?

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Mathematics Tutors' Perceptions of their Role

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In this study, we investigate the beliefs of undergraduate mathematics tutors. Thirty-three tutors completed surveys and twenty-five participated in interviews to assess their attitudes towards mathematics and their beliefs about the roles of a tutor and instructor. Our analysis provides examples of orientations, goals, and resources that were expressed by tutors in surveys and interviews. Tutors in this study viewed their role as distinct and supplementary to that of a teacher. The orientations, goals, and resources identified in this study provide a foundation for future studies that explain and predict tutor decision making. Although tutors are not content experts, they offer a valuable perspective that is different than that of the instructor.

Keywords: mathematics tutors, beliefs, orientations, goals, resources

Introduction and Literature Review

In a recent study, 97% of universities surveyed offered mathematics tutoring to Calculus students (Bressoud, Mesa, & Rasmussen, 2015), therefore tutoring is a common resource for undergraduate student learning outside of the classroom. Tutoring has been linked to improved pass rates (Cuthbert & MacGillivray, 2007; Patel & Little, 2006) and an increase in grades (Byerley, Campbell & Rickard, 2018; Lee, Harrison, Pell, & Robinson, 2008; Rickard & Mills, 2018), and has been shown to have a positive impact on student attitudes towards mathematics (Bressoud et al., 2015; Croft, Grove, & Bright, 2008; Topping, 1998).

Some studies related to tutoring have emphasized that tutors are not experts in either pedagogy or content (Grasser, D'Mello, & Cade, 2011), but nevertheless, tutoring seems to improve student learning. Grasser et al. (2011) describe several different effective teaching methods and then, by analyzing a corpus of tutoring observation data, they conclude that tutors do not utilize these methods. This deficit perspective of tutoring does not help us to understand what it is that tutors are doing to help students. Also, the context of a one-on-one drop in tutoring session is so different from a classroom setting, it may not be reasonable to expect that an effective tutor would make similar moves to an effective teacher. Defining the differences between a tutor and a teacher can help us to unpack the specialized knowledge that tutors have that may be distinct from teachers' knowledge.

This study will report on undergraduate mathematics tutors' views of mathematics and their perceptions of their role as a tutor, as well as how they perceive their role compares to the role of a mathematics instructor. Our analysis will focus on the orientations, goals, and resources that the tutors mention in their interviews. This foundational understanding of tutors' views of mathematics and their perceived roles is a first step to understanding how these factors influence their practice.

Theoretical Perspective

Schoenfeld (2011) suggests that decision making in an academic setting (and many other settings for that matter) is influenced heavily by a person's orientations, goals, and resources. In this theory, Schoenfeld describes how a person has some set of orientations, goals, and resources which all act on each other during the decision-making process. These factors can be used to construct a model of their decision making process that has explanatory and predictive power.

Through their experiences in education, both as students and tutors helping students, tutors may develop their own perceptions of their role. Because we do not have direct access to a tutor's thoughts and because a tutor may not be aware of them, we cannot say explicitly whether they have specific orientations, goals, or resources. Schoenfeld (2011) skirts this issue by stating that we can attribute goals, orientations, and resources to a person in a manner that explains and predicts their behavior and our model can be adjusted or replaced if the tutor's actions are not in line with the model.

Here, we discuss briefly what is meant by orientations, goals, and resources as given by Schoenfeld (2011). A person's orientations are the set of that person's beliefs, preferences, and values among other things. They are the different meanings and understandings created by the person from their experiences. Orientations play a big role in a person's perceptions of a given situation and can trigger certain behaviors. Therefore, it is important to describe the specific situations as well as the person's orientation.

A goal is "something that an individual wants to achieve, even if simply in the service of other goals" (Schoenfeld, 2011). Goals may be immediate or long-term, and they may have sub-goals. Goals may work together in a given situation or they may work against each other, and the person making the decision may not be consciously aware of their goals at any given time. Goals are prioritized by what the person with those goals believes is more important for the given situation. For this reason orientations play an important role in the prioritization of goals. This set can be adjusted and prioritized multiple times in the process of achieving a goal (or set of goals).

Resources are defined to be everything that is available to use by a specific person. Each individual has their own set of resources, which contain intellectual resources, material resources, and social resources. Some examples of resources include the knowledge somebody has about a certain topic, physical objects or tools that can help a person achieve a goal, or ways of communication that can identify new information.

Putting these three concepts together gives us a way to model a person's decision-making process. Before decisions are made, a person starts with their own resources, goals, and orientations. They will then collect information about the situation. Goals will be either established or reinforced. Then the person will make decisions about how to direct an interaction or a situation that stay consistent with their goals. All of these steps are repeated as many times as necessary as a situation progresses and reevaluation is needed.

Since decision making is influenced by a person's resources, orientations, and goals, we may begin to construct models of tutors' decision making processes to explain or predict their behavior. Thus, we aim to identify what a tutor believes his or her role is in a student's learning process.

Methods

The participants in this study were undergraduate tutors from drop-in mathematics tutoring centers at two institutions: a large research university in the Midwestern United States and a small private university in the Northwestern United States. Thirty-three tutors were given surveys prior to the start of the Fall 2017 semester addressing their beliefs about mathematics, beliefs about mathematics instructors, and beliefs about mathematics tutors. The survey consisted of items that were modified from the CSPCC math attitudes survey (Bressoud, et al., 2015) and the NCTM Teaching and Learning Beliefs Survey (NCTM, 2014).

Twenty-five of the tutors were interviewed to allow them to elaborate on their responses. The interviews were conducted throughout the entire semester as the tutors completed a particular part of their training. At one of the universities, the first semester tutors were not required to complete this portion of the training and thus not all of the tutors who completed surveys participated in the interviews. This study focuses on tutor responses when asked to compare and contrast their perceptions of their roles and instructors' roles. The themes that emerged are presented with representative quotes from tutors.

The surveys gave us some indication that the tutors viewed their role as a tutor slightly differently than that of an instructor. These results gave us a lens through which to look at our interview data. We analyzed the interview data using thematic analysis (Braun & Clarke, 2006). The first phase of our analysis involved transcribing the data and reading through all of the transcripts. Then we generated initial codes and searched for themes among the codes. Lastly we defined and named some orientations, goals, and resources that emerged from the interview data. This theoretical framework was helpful for organizing our analysis of their responses. Note that the tutors were not directly asked to talk about their perceived orientations, goals, and resources. In addition, we should note that the results that we present here are formed from a combination of all of the tutors' responses, and may not reflect the perspective of each individual tutor.

Results

We interviewed the tutors in this study to address any gaps in information given, as well as to elaborate on anything from the surveys. For the purpose of this paper, we only analyzed the portion of each interview in which the tutors expressed how they conceptualize their role as a tutor and how that compares to the role of an instructor.

Since we did not specifically ask the tutors to list their orientations, goals, and resources, the results found are not exhaustive. Instead, we provide an example of each category that the tutors in this study mentioned to help explain what tutors could think or do during a tutoring interaction. Furthermore, a tutor is not limited to only one orientation, goal, or resource.

Orientations

Instructors lay the foundation and tutors fill in the gaps. In the interviews, 82% of the tutors made statements that indicated that the instructor is the one who presents the theoretical material and "lays the foundation" while the tutor's role is to work with individual students on the application of the theory and "fill in the gaps" in their understanding.

> …and I think a tutor is only there to reinforce it, and I think of it like building a bridge, so like the teacher lays the foundation, maybe puts the rough parts of the bridge on there, and the tutor comes around and puts the, you know, the pavement on it. Smooth and easy to travel over. Just to make it just a little bit better. – Anthony

Many of the tutors in this study believe that like instructors, their goal is to enhance students' knowledge of specific mathematics content. However, some tutors expressed that the instructor's role is to present new mathematical ideas while the tutors' role is to reinforce students' understandings and address any gaps in the students' knowledge. The tutors mentioned that instructors are constrained by class size and time, which are not constraints that tutors typically have in the drop-in environment.

Tutors guide students, tutors do not teach students. It is a safe assumption that students who seek help from a tutor are enrolled in a math class. Also, those students usually ask questions about something that was mentioned in class. One belief that these tutors expressed is that the tutors should not be responsible for teaching new material to the students, rather that they should assist the student in understanding new ideas and concepts. Tutors who expressed this belief focused more on what is not part of their role, whereas the tutors who expressed beliefs from the preceding section were focused more on what is part of their role.

Tutors are not experts, but they offer a different perspective than instructors. Although tutors are not mathematics content experts, they do have a different kind of specialized knowledge than instructors have. Because not all tutors are math majors, they can often help students to understand how the mathematics that they are learning can apply in their subsequent engineering or physics courses.

> I'm a physics undergrad, but one of the things that might help them is like more tactile, you know, an example of how stuff works, that's all I run into in my classes. . . I don't try to get into that too much because if I try to start talking modern physics to someone in trig, they immediately glaze over. It's just, it's comforting to them to say, this is real, it's not worthless to learn, there is an application for it. – John

Goals

Help students become independent learners. Part of this goal entails that tutors should help a student towards a correct solution rather than showing them how it is done. Furthermore, a person with this goal in mind may believe that tutors should help students with their problems in such a way that allows the student to succeed at similar tasks on their own.

> So they can kind of go through it themselves and if they get stuck, they can know what to do. Rather than having to go for help every time. – Molly

Determine the needs of the student. Because the tutors in this study work in a drop-in setting, they may not know the needs of the students that come in to get help on any given day. So when a tutor approaches a student who has a question, the tutor has to assess what kind of help the student needs. The tutors in this study talked about four common ways in which they help students: 1. Checking student work for errors, 2. Showing students how to do an example, 3. Leading students through an example by questioning, 4. Helping students understand a specific concept. Tutors do not believe that it is their responsibility to teach a student who has not been coming to class.

Some tutors prefer a more direct approach to figuring out how much a student knows. By probing the student to describe their situation as much as possible, some tutors feel they can better understand what the student needs and then be able to help fill in what a student is missing.

Help students stay on the right path. Tutors can be a valuable resource for students in that they can determine when the student is starting to go in a direction that will not be beneficial for them. It is worth noting here that the phrase "right path" means the path that the tutor believes will lead to a solution to a given problem. In previous research, physics tutors tended to focus on the next step toward a solution on a path dictated by the tutor and did not allow the student to stray too far from the path (VanLehn, Siler, & Murray, 2003). Tutors have different resources at their disposal to address these kind of issues, for example asking questions to direct the students or providing their own explanations of certain problems.

> Ideally, I would just be able to sit there, and they would ask a question, and I would be able to ask a question back that would be able to guide them into the right answer. That would be ideal. – Jonah

> And try to... we can't process it for them. Just try to help them, stay on the right path, I guess. – Ashton

Resources

Tutors have a variety of resources available to them based on their own experiences with mathematics and with tutoring. Since they are undergraduate students, the tutors have taken math courses already. So they have built their own knowledge of problem-solving strategies which may be different from the teachers' methods of problem-solving.

> I mean, the teacher may have one approach that they use in class, and then the tutor may have a different approach that the student can understand better, and that just may not have been mentioned by the professor before. – Veronica

Conclusion

The survey results showed us that undergraduate mathematics tutors view their role as different from an instructor in the sense that they elicit more student thinking and understanding than instructors do, and that they walk students through problems step-by-step more than instructors do. The one-on-one nature of tutoring may account for these differences.

With the interview results, we were able to define a few possible orientations that a tutor might have regarding their role in a tutor-student interaction. From the information provided to us by the tutors that were interviewed, some possible orientations that a tutor may have include the following: 1) Instructors lay the foundation and tutors fill in the gaps. 2) Tutors guide students, they do not teach students. 3) Tutors are not experts, but they offer a different perspective than instructors. The tutors also mentioned some goals they have when helping a student including: 1) Help students become independent learners. 2) Determine the needs of the student. 3) Help students stay on the right path. Lastly, a few resources were named, for example the tutors can use their prior knowledge of mathematics and problem-solving techniques. Some expected resources, such as using the internet and asking a fellow tutor, were not mentioned by any of the tutors that were interviewed. This may be because tutors were not asked directly to name the resources they had or used when they were interviewed.

During this research, tutors self-reported their own thoughts and beliefs about their roles as a tutor. Not having observed the tutors during live tutoring interactions where they may apply their beliefs makes it more difficult to say that a certain tutor's behavior follows a certain orientation or goal. Future research can investigate how the identification of these resources, goals, and orientations work to explain or predict a tutor's in-the-moment decision making while tutoring. Another line of research could investigate how students' resources, goals, and orientations align or are different from the tutor's resources, goals and orientations. Schoenfeld (2011) points out that in an interaction between two individuals, the two parties may have common or conflicting goals. Thus, it may be interesting to see how the student's goals affect the interaction.

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An Exploratory Factor Analysis of EQIPM, a Video Coding Protocol to Assess the Quality of Community College Algebra Instruction

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Evaluating the Quality of Instruction in Post-secondary Mathematics (EQIPM) is a video coding instrument that provides indicators of the quality of instruction in community college algebra lessons. It grew out of two instruments that assess the quality of instruction in K-12 settings—the Mathematical Quality of Instruction (MQI) instrument (Hill, 2014) and the Quality of Instructional Practices in Algebra (QIPA) instrument (Litke, 2015). We present preliminary results of an exploratory factor analysis that suggests that the instrument captures three distinct dimensions of quality of instruction in community college algebra classes.

Keywords: Algebra, Instruction, Video Coding, Community Colleges

Various reports have established an indirect connection between students leaving science, technology, engineering, and mathematics (STEM) majors because of their poor experiences in their STEM classes (Herzig, 2004; Rasmussen & Ellis, 2013). Most of these reports, however, are based on participants' descriptions of their experiences in the classes, rather than on evidence collected from large scale observations of classroom teaching (Seymour & Hewitt, 1997). When such observations have been made, they usually focus on superficial aspects of the interaction (e.g., how many questions instructors ask, how many students participate, or who is called to respond, Mesa, 2010) or their organization (e.g., time devoted to problems on the board, or lecturing, Hora & Ferrare, 2013; Mesa, Celis, & Lande, 2014). Undeniably, these are important aspects of instruction, yet these elements are insufficient to provide a characterization of such a complex activity as instruction.

A key concern in post-secondary mathematics education is the lack of preparation that mathematics instructors receive in their graduate education (Ellis, 2015; Grubb, 1999). We argue that the lack of a reliable and valid method to fully describe how instruction occurs hinders our understanding of the complexity of instructors' work in post-secondary settings and therefore limits the richness of preparation and professional development opportunities focused on the faculty-student-content interactions (Bryk, Gomez, Grunow, & LeMahieu, 2015). As part of a larger project that investigates the connection between the quality of instruction and student learning in community college algebra courses, we have developed an instrument, EQIPM (Evaluating Quality of Instruction in Postsecondary Mathematics), that seeks to characterize

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instruction. In this paper, we present the results of an exploratory factor analysis using ratings generated by coding lessons with EQIPM that suggest three aspects key to instructional quality.

Theoretical Perspective

We assume that teaching and learning are phenomena that occur among people enacting different roles—those of instructor or student—aided by resources of different types (e.g., classroom environment, technology, knowledge) and constrained by specific institutional requirements (e.g., covering preset mathematical content, having periods of 50 minutes, see Chazan, Herbst, & Clark, 2016; Cohen, Raudenbush, & Ball, 2003). We focus on instruction, one of many activities that can be encompassed within teaching (Chazan, et al., 2016), and define instruction as the interactions that occur between instructors and students in concert with the mathematical content (Cohen et al., 2003). Such interactions are influenced by the environment where they happen and change over time. Empirical evidence from K-5 classrooms indicates that ambitious instruction is positively correlated with student performance on standardized tests (Hill, Rowan, & Ball, 2005). Understanding mathematics instruction requires attention to the disciplinary content and the mathematical knowledge for teaching and learning. Therefore, we assume, first, that the experiences of instructors and students while interacting with mathematical content have a significant impact on what students are ultimately able to demonstrate in terms of knowledge and understanding, and second, that it is possible to identify latent constructs that might account for the observed quality of instruction.

Instruction is central to EQIPM. The instrument was designed with the goal of assessing the quality of the interactions defining instruction assessed via three distinct constructs: (1) Quality of Instructor-Student interaction, (2) Quality of Instructor-Content Interaction, and (3) Quality of Student-Content Interaction, supported by the quality of *Mathematical Explanations* and *Mathematical Errors and Imprecisions in Content or Language* that are present in a lesson. Figure 1 illustrates the theorized structure of the coding instrument by showing individual codes within the three constructs. The codes under *Segment features* help characterize the segment (i.e., Mathematics is a focus, Procedure is taught, Modes of instruction, Technology used).

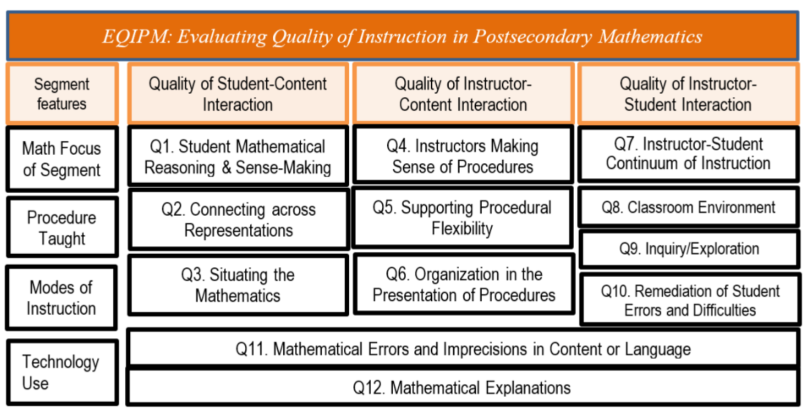

Figure 1. Dimensions and codes for the EQIPM instrument.

Methods

In the Fall 2017 semester we video-recorded 131 lessons in intermediate and college algebra classes from two different community colleges in three different states. The lessons ranged in duration between 45 and 150 minutes, and were taught by 40 different instructors (44 different unique courses video-recorded; 4 instructors taught 2 sections of a course). The lessons covered one of three topics: linear equations/functions, rational equations/functions, or exponential equations/functions. These topics were chosen because they offer us opportunities to observe instruction on key mathematical concepts (e.g., transformations of functions; algebra of functions) and to attend to key ways of thinking about equations and functions (e.g., preservation of solutions after transformations; covariational reasoning), which are foundational algebraic ideas that support more advanced mathematical understanding (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The development of EQIPM was similar to the process used by Hill and colleagues (2008) and by Litke (2015). Their instruments describe and qualify instructional practices from video-recorded lessons by subdividing lessons into 7.5-minute segments and rating all segments within a lesson.

Using version 3a of EQIPM, each 7.5-minute segment within a lesson was coded by one member of a team of 14 researchers using a rubric that described the coding (AI@CC Research Group, 2017). Each code was rated on a 1 to 5 scale and each coder provided a justification for that rating that included evidence linked to a timestamp in the video. The researchers independently coded a maximum of 3 to 4 segments of a lesson to minimize bias due to familiarity with the instructor or the lesson. Ten percent of segments were randomly chosen for double-coding by a pair of researchers. Each pair held calibration meetings to discuss codes with a discrepancy greater than one point between ratings and subsequently reconciled when the researchers scores were more than 1 point apart or when a researcher identified a justification for their code and that justification was "missed" by the second researcher. The reconciled scores of the double-coded lessons were used for the exploratory factor analysis along with single coded segments in the corpus.

We conducted an initial exploratory factor analysis [EFA] using the 12 EQIPM items using Mplus 7.2. We extracted factors using the Mean and Variance Weighted Least Square (WLSMV) estimator with an oblique rotation (geomin). This extraction method is appropriate when using items that do not follow a normal distribution or items measured on a 5-point scale. We preferred an oblique rotation over an orthogonal one because this allows us to freely estimate the correlation between the extracted factors rather than assuming it to be zero. We used a Full Information Maximum Likelihood (FIML) approach to account for missing data. The scree plot for the 12-item EFA suggested that a solution between 1 and 3 factors would be appropriate (see Figure 2). We fit four separate EFAs by progressively adding these possible factors. At the time of this submission, we had coded 17 out of the 88 target lessons (19%) with a dataset large enough to run either a CFA or an EFA (169 segments). We chose to do an EFA because this allows us to make the least number of assumptions about the factor structure of our data. The data set included lessons from all but one of the colleges (8 from College 1, 1 from College 3, 2 from College 4, 2 from College 5, and 4 from College 6), six of which were calibrated. The lessons are from 15 distinct instructors (34% of instructors in sample)

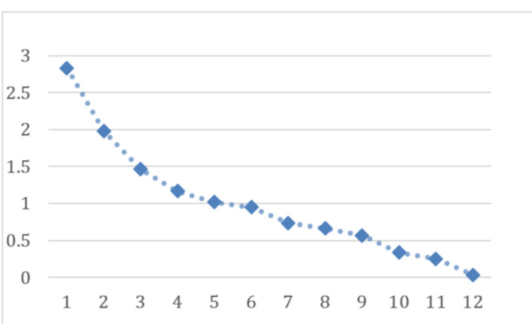

Figure 2. Scree plot

Preliminary Findings

We found that the 3-factor solution was an adequate to good fit to the data: $\chi^2 = 47.466$, $p = 0.049$, RMSEA = 0.051, 90% CI = [0.003, 0.082], CFI = 0.972, TLI = 0.944, SRMR = 0.065 (Hu & Bentler, 1999). This suggested that we did not have to do any model modifications, such as dropping items, in order to reach an acceptable solution.

Our preferred EFA model extracted three meaningful factors using all 12 codes (see Table 1). We also found that these three factors are weakly and positively correlated: corr Factor 1-Factor 2 = 0.184, corr Factor 1-Factor 3 = 0.245, and corr Factor 2-Factor 3 = 0.391. We noted that Q5 and Q12 have loading patterns that would suggest not to include them in the EFA solution (Worthington & Whittaker, 2006). We decided to retain these items in our model because we plan to confirm the extracted factor structure using a confirmatory factor analysis once the full lesson dataset becomes available. Moreover, the loadings on factor 3 are not high, which may suggest more commonality with the other two factors than what we would like to have. Descriptive information (e.g., item distribution will be made available in the presentation).

Table 1. Factor Loadings for the 3-Factor Solution

	Factor 1	**Factor 2**	**Factor 3**
Q1 – St. Mathematical Reas.	**0.790***	-0.001	-0.028
Q7 – Instructor-Student Cont.	**1.055***	-0.182	0.012
Q9 – Inquiry/Exploration	**0.594***	0.039	-0.016
Q10 – Remediation Std Errors	**0.281***	0.187	-0.057
Q4 – Instr. Making Sense Proc.	-0.179	**0.661***	0.022
Q5 – Supp. Proc. Flexibility	0.051	**0.307***	-0.283*
Q6 – Organization Pres. of Proc.	0.072	**0.454***	0.297
Q11 – Math. Err & Impr. Cont.[a]	0.023	**-0.224***	-0.156
Q12 – Math. Explanations	-0.001	**0.954***	**-0.792***
Q2 – Conn. across Reprs.	0.036	-0.053	**0.300***
Q3 – Situating Math.	-0.05	0.222	**0.325***
Q8 – Class Environment	0.374	0.011	**0.530***

Notes. * Significant loadings at the 95% level.
[a] A high rating in this code implies low quality of instruction.

Discussion

We interpret factor 1 as the **quality of instructor-student interaction**, as it embeds three of the codes under the fourth column of Figure 1 that were meant to address how students and the instructor were working together. This factor also included the *Student mathematical reasoning and sense making* code, which was theorized to be part of the student-content interaction. Being part of the instructor-student interaction factor may suggest that such reasoning occurs through invitations by the instructor. Some corroboration of this conjecture is grounded in the high number of segments in which lecture was the main mode of instruction (94%, 159 of 169 segments coded involved lecture and 90 of those used only lecture). In theory, mathematical reasoning and sense making should be evident without the mediation of the lecture; we anticipate that professional development targeting the importance of this feature of instruction, might yield differences that might align this code under the student-content interaction as theoretically envisioned. As more segments with other modes of instruction appear, we might be able to see if this code continues to be under this factor. We interpret factor 2 as addressing the **quality of instructor-content interaction**; its five codes, three hypothesized under the third column of Figure 1, and the two codes we hypothesized as cross-cutting the three constructs, speak directly about how instructors manage the discussion of the mathematical content. While the cross-cutting code, Mathematical Explanations, allows for students providing explanations that this code loads on the instructor-content interaction suggests that there are few opportunities for students to provide explanations, and could be a consequence of the emphasis on lecturing in these segments. Finally, factor 3, seems to capture the **quality of student-content interaction** by embedding two of the three codes found in the second column of Figure 1 with the *Classroom environment* code which suggests that student engagement in mathematics may be occurring in situations where the classroom environment is supportive of students' interaction with the content. The negative high loading of the code *Mathematical Explanations* in this factor suggests that explanations might not occur when connections across representations, situating the mathematics, and classroom environment are rated highly. This is a puzzling result and merits further investigation.

While these EFA results are encouraging, we recognize that the extracted structure depends on the lessons that were available to perform the exploration, which are not representative of our corpus. We will need to confirm our results using our whole corpus of data. We plan to run a split sample EFA/CFA to explore and validate the instrument's factor structure once we have the full dataset coded. This will allow us to account for the multi-level structure of our data, specifically, segments within lessons.

Being able to identify three distinct factors that can be used to describe the quality of community college algebra instruction is promising for the field: each of the constructs suggest specific areas for supporting the work of instructors in teaching community college algebra. These results also support previous research in K-12 that models learning via assessing the quality of instruction defined as the interactions between teacher, student, and content. The connections between the classroom environment and the quality of student-content interaction rather than quality of instructor-student interaction may highlight the importance of classroom environment on building student engagement with the content. The factors used will be included in the full model of our data to determine links between instructional qualities and student performance. We plan to use the instrument in the design of professional development.

Questions for the Audience

The preliminary factor analysis of EQIPM, version 3a, supports the theoretical dimensions underlying the quality of algebra instruction at community colleges and three possible variables that can be used to assess quality and model student performance in these courses. Given these findings, we have the following questions:

1. Is our interpretation of the EFA results plausible? Are there other links between the codes and the underlying factor structure revealed by the EFA that you believe can be made and be supported by current literature on teaching and learning?
2. EQIPM is based on our conceptualization of instruction, and looks at the quality of interactions between instructors, students, and content. Other than modes of instructions playing a role in the results of the analysis, are there other segment features (e.g., technology) that come to mind that could also impact the loadings? Are there other ideas that come to mind about the EQIPM coding rubric or EFA that could enhance this research approach to describe the quality of instruction in community college settings?

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Student, Teacher, and Institution Effects on Student Achievement and Confidence in College Calculus

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Using the Mathematical Association of America's Characteristics of Successful Programs in College Calculus dataset (CSPCC) of 13,965 students from a variety of institutions nationwide, student characteristics and experiences were analyzed via pre- and post-course survey responses. This research evaluated the effect of student background, student-reported teaching behaviors, and institutional environments on academic achievement and student confidence. The findings of this research could lead to a better understanding of the impact of calculus teaching practices and the implications of retaking calculus for students of all experience levels.

Keywords: post-secondary calculus, pedagogical behaviors, student confidence

Approximately 61% of students taking Calculus I in a postsecondary school have already taken a calculus course in high school (Bressoud, 2015), with students of all demonstrated proficiency levels who took calculus in high school receiving higher grades in post-secondary Calculus I than their counterparts (Sadler & Sonnert, 2018). Existing research has explored the relationships between previous calculus experience and performance, as well as the relationships between different types of instructional strategies in postsecondary calculus classes (Bressoud, Mesa, & Rasmussen, 2015; Ellis, Fosdick, & Rasmussen, 2016; Ellis, Kelton, & Rasmussen, 2014; Mesa, Burn, & White, 2015; Sonnert & Sadler, 2015). However, the interaction of these instructional strategies with students' confidence remains underexplored.

Using the Characteristics of Successful Programs in College Calculus dataset (CSPCC), this study contributes to the existing knowledge of college calculus by examining the impact of previous calculus experience on students' confidence in a post-secondary Calculus I class.

Calculus I in the Postsecondary Setting

The choices made by instructors regarding in-class behaviors and homework have arguable impacts on student confidence and performance. Different categories of teaching behaviors have emerged from the CSPCC dataset. Three factors defined as 'good' teaching, technology, and ambitious teaching were found to have differing impacts on students' attitudes towards math; 'good' teaching had a positive impact, technology had no significant impact, and ambitious pedagogy had a negative impact (Sonnert & Sadler, 2015). These impacts varied further when students were grouped by performance. High performing students responded to progressive behaviors more positively than low performing students (Sonnert & Sadler, 2015). Other research expanded on these three factors and broke 'good teaching' behaviors into three categories: Classroom Interactions that Acknowledge Students, Encouraging and Available Faculty, and Fair Assessments (Mesa et al., 2015). Approximately half of Calculus I homework is submitted on paper, though homework that is graded is most often done so via on online homework system (Bressoud et al., 2015). However, the use of online homework has uncertain effects on student outcomes (Bressoud et al., 2015). This study builds upon the categories previously created by Sonnert and Sadler (2015). and Mesa, et al. (2015) with an exploration of homework categories and textbook choice as teaching behaviors and an additional focus on confidence as a student outcome.

This study contributes to an existing body of literature that has used CSPCC to identify elements of calculus instruction that impact student outcomes such as performance and confidence. These student outcomes are influenced by a student's characteristics and prior experiences as well as the post-secondary Calculus I instruction they receive. In particular, we ask:

- What are the effects of previous math background, classroom interactions, and institutional environments on student outcomes such as student confidence and performance in post-secondary calculus education?

Conceptual framework

This research draws on the framework for instruction as interaction framework (Cohen, Raudenbush, & Ball, 2003), which focuses on the dynamic between students, teachers, and learning environments. Our quantitative study utilizes categorical and numeric variables that fall into different domains of the framework: student, instructor, and institutional levels. In particular, we focus on student and teacher interactions within the institutional environment; analyze the impact of teacher behaviors, we take into consideration other influences on student outcomes at the institution, instructor, and student levels.

Research Methodology and Results

Sample

Our sample draws from the CSPCC dataset, which was comprised of pre- and post-course survey responses from 13,965 students and 496 instructors, at 169 institutions of varying types across the United States. For our analysis, we retained students who had complete data on the variables of interest, resulting in a final sample of 2,831 students. The demographics of the analytic sample closely mirror those of the full sample (Table 1).

Table 1. Full and Analytic Sample Demographics

CSPCC Full Sample			Analytic Sample		
	N	%		N	%
Institutions	169		**Institutions**	131	
Associate's	40	23.67	Associate's	25	19.08
Bachelor's	41	24.26	Bachelor's	30	22.90
Master's	21	12.43	Master's	15	11.45
PhD	67	39.64	PhD	61	46.56
Instructors	496		**Instructors**	333	
Students	13,965		**Students**	2,831	
White	6,947	70.69	White	2125	75.06
Black	456	4.64	Black	67	2.37
Asian	1,340	13.63	Asian	365	12.89
American Indian or Alaska Native	128	1.30	American Indian or Alaska Native	34	1.20
Hispanic	957	9.74		240	8.48
Male	5,688	56.36	Male	1,556	54.96
Prior Calculus Experience	6,837	65.77	Prior Calculus Experience	2,018	71.28

Creation of Composite Variables

The CSPCC dataset has numerous instructional variables and prior work has studied student perception of specific instructor behaviors (Ellis et al., 2014). However, we hope to expand upon this work by considering confidence as a student outcome and by further stratifying types of pedagogical behaviors. To create our instructional composites, we used pre- and post-course survey responses of the students. These variables were created first by combining survey questions that, on a conceptual level, addressed teaching behaviors of distinct types. This was an iterative process that went through conceptual and then statistical testing. The initial instructional categories were: Encouraging students, Fair exams, Interpersonal interaction, Student use of technology (Graphing calculator), Instructor use of technology, Student use of technology (CAS), Cognitively challenging homework, and Valuing students.

Once categorized, we analyzed these collections of questions with a principle component analysis (PCA) to determine the questions with the strongest correlation. After running a PCA on each composite and eliminating survey questions that did not load on the same factor as the others or were not conceptually compatible with other questions, we were left with composites of 3 to 9 items, all with Cronbach alpha scale reliability coefficients greater than .75. After this process was complete, we decided to remove the Student use of technology, Fair exams, and Cognitively challenging homework composites as the factor reports had low scale reliability coefficients. The Valuing students composite was removed because we determined, after additional scrutiny, that it was encompassed by the Encouraging composite.

Analysis

For our analysis, we used hierarchical linear modeling (HLM) with three levels, with students working under instructors who are operating within an institution. We ran a series of HLM models, adding covariates in blocks of related variables. Every model clusters on the three levels. In addition, this model, and all others, are weighted by institution type, as the original data over-sampled universities and under-sampled community colleges (Bressoud, 2015). The sample weight was created using the CBMS (Blair, Kirkman, & Maxwell, 2013) distribution of Calculus I students by institution type. When evaluating student outcomes, performance is measured on a 0-4 scale while student confidence is measured on a 0-5 scale.

Results

Table 2. Course Grade

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Student Variables						
ACT/SAT		2.108***	2.001***	2.315***	2.198***	2.187***
Prior Calculus		0.253***	0.252***	0.333***	0.314***	0.315***
Race						
Black			-0.228	-0.204	-0.182	-0.188
Asian			0.142**	0.159***	0.127**	0.130**
P.I., A.I., or A.N.			-0.04	-0.041	-0.062	-0.075
Hispanic			-0.118	-0.099	-0.093	-0.097
Gender			0.045	0.044	0.026	0.029
Parent Educ. Level						
Some College			-0.071	-0.027	0.003	0.01
College			-0.138*	-0.047	-0.028	-0.023
Graduate School			-0.058	0.02	0.044	0.052
Instructor Variables						
Homework Type						

Physical				0.319*	0.17	0.171
Online				0.256+	0.155	0.154
Student Use of Tech.				0.088	0.103	0.159+
Instructor Use of Tech.				0.148	0.017	-0.04
Class Size[a]						
30-70				0.108+	0.07	0.109+
70+				0.127+	0.110+	0.173*
Textbook						
Hughes Hallett				-0.193**	-0.199***	-0.255***
Thomas				-0.057	0.007	-0.001
Rogawski				-0.155+	-0.023	-0.029
Anton				0.013	-0.037	-0.081
Other				0.007	0.056	0.015
Retaking Ratio				-0.628***	-0.467**	-0.522**
Encouragement					1.496***	1.503***
Interpersonal					-0.111	-0.119
Institution Type						
BA						0.125
MA						-0.049
PhD						0.003
Constant	3.136***	3.016***	3.071***	2.972***	1.994***	1.976***

*+ p<0.10, * p<0.05, **p <0.01, ***p< 0.001*
[a] Class size was estimated from the averages of instructor reports of enrollment.
Note. The baseline student is a white female with an average SAT/ACT score and no previous calculus experience attending a two-year institution. She receives instruction in a small classroom that employs none of the given methods, uses the Stewart text, and does not use homework. Neither of her parents went to college.
Note. Pacific Islander (P.I.), American Indian (A.I.), and Alaskan Native (A.N.) were combined due to sample size.
Note. Models were run using Stata.

Table 3. Student Confidence

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Student Variables						
Prior Confidence		0.569***	0.384***	0.355***	0.354***	0.355***
ACT/SAT			-0.169	0.152	0.152	0.173
Prior Calculus			0.058	0.112**	0.114*	0.115*
Course Grade			0.580***	0.486***	0.490***	0.488***
Race						
Black			-0.033	-0.03	-0.025	-0.022
Asian			-0.191*	-0.212*	-0.203**	-0.195**
P.I., A.I., or A.N.			0.117	0.084	0.068	0.072
Hispanic			-0.098	-0.087	-0.086	-0.077
Gender			0.163***	0.134***	0.132***	0.130***
Parent Educ. Level						
Some College			0.031	0.066	0.055	0.065
College			0.019	0.05	0.046	0.051
Graduate School			-0.035	0.008	0.005	0.018
Instructor Variables						
Retaking Ratio				-0.288*	-0.361**	-0.360*
Encouragement				1.598***	1.566***	1.588***
Interpersonal				0.052	0.146	0.128
Homework Type						
Physical					-0.292+	-0.331+
Online					-0.262	-0.288+
Student Use of Tech.					-0.084	-0.067

Instructor Use of Tech.					-0.105	-0.151
Class Size[a]						
30-70					-0.02	0.006
70+					0.01	0.081
Textbook						
Hughes Hallett					-0.06	-0.06
Thomas					-0.166**	-0.134*
Rogawski					0.007	0.029
Anton					-0.034	-0.077
Other					-0.105+	-0.114+
Institution Type						
BA						0.083
MA						0.138
PhD						-0.022
Constant	1.263***	1.301***	1.471***	0.122	0.604**	-0.233

*+ p<0.10, * p<0.05, **p <0.01, ***p< 0.001*

[a] Class size was estimated from the averages of instructor reports of enrollment.

Note. The baseline student is a white female with an average SAT/ACT score and no previous calculus experience attending a two-year institution. She receives instruction in a small classroom that employs none of the given methods, uses the Stewart text, and does not use homework. Neither of her parents went to college.

Note. Pacific Islander (P.I.), American Indian (A.I.), and Alaskan Native (A.N.) were combined due to sample size.

Note. Models were run using Stata.

Preliminary results indicate that Encouraging students, Homework type, and Student use of technology (graphing calculator) are pedagogical methods that have a significant relationship with student performance. Encouragement and homework also significantly influence student confidence. The overwhelming influence of encouraging behavior paints a clear picture: instructors demonstrating their care for their students can significantly positively impact student confidence and grades. There is an evident need for compassionate Calculus I instruction to boost student morale and achievement.

We also have found students' previous math experiences to be significant in relationship to their confidence; students who have seen calculus before and/or fared well on standardized tests are more confident in Calculus I courses. After observing the influence of previous calculus experience on a student's individual post-secondary calculus experience, we would like to continue our work on the peer effects within Calculus I classrooms and the influence of the ratio of calculus retakers within a classroom as an environmental factor.

The Cognitively challenging homework composite caught our interest despite our inability to include it in our models; we plan to further investigate the influence of homework types, class size, and the student/instructor ratio in future work.

Implications for Teaching Practice and Future Research

This research has implications for the instruction of college calculus courses, specifically encouraging efforts to demonstrate care for students and direct instruction towards students who are taking calculus for the first time. Future work should focus on the effects of homework type and textbook choice as instructor behaviors.

Intended Questions for the Audience

Is there other research that supports this relationship between encouraging behavior and student performance? Any suggestions for a qualitative follow-up study?

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Developing Pedagogical Content Knowledge: Can Tutoring Experiences be Used to Train Future Teachers?

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This preliminary report explores data from a larger study investigating the nature of preservice elementary teachers' content knowledge and pedagogical content knowledge (PCK) in the area of number theory. A prominent theme emergent from the data – a contributing factor in participants' PCK – was the theme of tutoring experiences. Participants explicitly and regularly referenced their tutoring experiences when responding to hypothetical students in PCK tasks. The influential nature of preservice elementary teachers' tutoring experiences on their PCK holds implications for teacher-training, but further investigation is necessary. Questions concerning the design of a future study are proposed for discussion.

Keywords: Pedagogical content knowledge, preservice elementary teachers, tutoring

Research efforts to improve on mathematics education in the United States focus on a variety of contributing factors, key among them is the professional development and education of teachers. The literature has consistently linked student success in mathematics with teacher pedagogical content knowledge or PCK (e.g., Hill, Rowan, & Ball, 2005; Speer & Wagner, 2009), which is an understanding of content that is specific to teaching. A recent study also indicated a link between teachers' mathematical content knowledge and student achievement (Campbell et al., 2014). However, the research suggests that many preservice elementary teachers (undergraduates enrolled in elementary teacher education programs) may lack the mathematical content knowledge and the mathematical PCK necessary to teach mathematics for understanding (e.g., Conference Board of Mathematical Sciences, 2012). This suggests a need for preservice elementary teachers to have additional opportunities to develop their mathematical content and pedagogical content knowledge.

Some researchers have argued that mathematical PCK can only be developed through authentic interactions with students (e.g., Van Driel & Berry, 2010). However, preservice elementary teachers have few, if any, opportunities to engage elementary school students with mathematics prior to their student teaching internships. At many schools, mathematics methods courses for future elementary school teachers emphasize planning and preparation rather than practicum. Practicum can include classroom observations and, occasionally, the teaching of a mini-lesson. Neither of these activities allow for sufficient interactions with students for developing robust mathematical PCK. And during preservice elementary teachers' student teaching experiences, mathematics is never the primary focus; it is only one of many subjects that elementary education majors are required to teach every day.

In this preliminary report, I use data from a larger study exploring preservice elementary teachers' number theory PCK to suggest that tutoring experiences might be used to develop preservice elementary teachers' mathematical PCK. While suggestive, the evidence I present is hardly definitive. I conclude this report with a list of discussion questions concerning the design of a future study with which to further investigate the effects of structured tutoring experiences on preservice elementary teachers' mathematical PCK.

Theoretical Framework

The most prevalent model for mathematical PCK in the U.S. mathematics education literature is Ball and colleagues' (e.g., Hill, Ball, & Schilling, 2008) Mathematical Knowledge for Teaching (MKT). This model distinguishes between types of subject matter knowledge and types of PCK, and it identifies three constructs of mathematical PCK. According to Hill, Schilling, and Ball (2004), one of the three constructs of PCK is knowledge of content and students (KCS), which pertains to "knowledge of students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (p. 17). Another construct, knowledge of content and teaching (KCT), combines knowing about teaching with knowing about mathematics and pertains to instructional decisions as they relate to mathematics. A third construct, knowledge of curriculum, is a knowledge of programs developed for the teaching of a particular subject, concepts covered at a given level, and instructional materials available.

The emergent perspective (Cobb & Yackel, 1996) served as the lens for collecting and analyzing data. I primarily used the psychological lens because the bulk of the data represent individual conceptions. On the other hand, via the social lens I explored the classroom norms, expectations, and experiences that framed participants' perspectives on mathematics teaching and learning. I also drew from Ball and colleagues' MKT (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008) in designing my interview tasks to elicit PCK and again to analyze responses.

Methodology

Data for this report came from an interpretive case study (Merriam, 1998) centered on preservice elementary teachers who were seeking a mathematics concentration and enrolled in a number theory course. Data included classroom observational notes, student coursework for 13 volunteers, as well as responses from two sets of one-on-one task-based interviews with six purposively chosen participants (a subset of the 13 volunteers), which served as the focus of the data analysis. During the interviews, all six interview participants admitted to having had mathematics tutoring experiences, either with grade school students, their peers, or both (see Table 1).

Table 1. *Interview participants' tutoring experiences*

	Brit	Cara	Eden	Gwen	Isla	Lucy
Grade School Tutoring	X		X	X	X	X
Peer Tutoring	X	X		X		X

Many of the interview tasks posed hypothetical student scenarios, designed to elicit number theory PCK. To elicit KCS specifically, I asked participants to identify the hypothetical students' mathematical conceptions and misconceptions. I also asked participants to describe how they might respond to the students in the scenarios in order to elicit KCT. Many of the students scenario task also included a meta-cognitive piece; I asked participants to reflect on why they responded to the hypothetical student in that way.

During the initial stages of my data analysis, I coded data according to the primary constructs detailed in my theoretical framework: KCS, KCT, classroom norms, etc. Within those general, umbrella codes, I conducted open, thematic coding. Finally, I conducted constant-comparative coding (Corbin & Strauss, 2008) until I achieved saturation. Among my efforts to ensure trustworthiness, I used member checking during the interviews and data triangulation afterwards.

Results: The Influence of Tutoring on Participants' Responses

One theme that emerged from the data was the theme of tutoring experiences. Participants frequently referred to past tutoring experiences, sometimes spontaneously, sometimes in response to the meta-cognitive interview questions, in order to justify their proposed responses to the students in the hypothetical scenarios. It was clear from their responses that participants' tutoring experiences contributed to their demonstrated PCK in general, and their KCT specifically. To depict this influence, I detail the results of one such interview task.

During the first round of interviews, I posed the scenario, "Mark suggested that the least common multiple (LCM) of two numbers is equivalent to their product." I asked participants to validate Mark's conjecture, identify his conceptions and misconceptions (KCS), respond to Mark in a way that helped him improve his understanding (KCT), and explain their reasoning for how they responded to Mark. All participants determined that Mark's conjecture was incorrect, and found appropriate counterexamples, but only Brit, Cara, and Lucy correctly determined that Mark's conjecture works for pairs of relatively prime numbers.

When I asked participants why they thought Mark might believe his conjecture to be true, they responded with a variety of insights, which I coded as "KCS" if the statement pertained to "students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004). I also coded the KCS statements as "student reasoning" if the participant referred to *why* a student might believe a statement, claim, or conjecture about number theory is true or false. All six participants had acknowledged at some point during the interview that Mark's conjecture works for some pairs of numbers. Cara, Eden, Gwen, and Isla explicitly cited this as a reason for why Mark may have formed his conjecture. I coded this as "KCS", and "student reasoning", more specifically, because it was a reasonable explanation for why Mark might have believed his conjecture to be true.

After I asked participants why Mark might believe his conjecture to be true, I asked them how they would respond to Mark to help him correct his misconceptions, hoping to elicit KCT. Eden suggested that she would explicitly tell Mark which types of numbers worked. However, Eden's limited understanding of the concepts behind Mark's conjecture led to an inaccurate response. "You could go in and say, 'Yes, this method does work but only for certain types of numbers. And these certain types of numbers would be the prime numbers.'" Here, Eden's content knowledge weakened her KCT.

Gwen suggested that she would discuss a confirmatory example (four and five) with Mark so that he would better understand why it worked, but her explanation was insufficient. The data suggested that Gwen's understanding of the content may have limited any explanation with regards to the role factors play in finding the LCM of two numbers.

In their responses to Mark, Brit and Cara also suggested they would point out that while the product of two whole numbers is a multiple, it is not always the *least* common multiple. This instructional decision did draw attention to the inaccuracy in Mark's conjecture, so I coded it as "KCT." At some point in their responses to Mark, all participants claimed they would present him with a counterexample to explore. I coded these statements as "KCT" as well, because not only were they hypothetical instructional responses, but by strategically picking a specific counterexample the statements pertained to the specific mathematics related to the misconception. Isla suggested the counterexample of two and six, and she went so far as to explain to Mark *why* it was a counterexample.

> *Isla:* If we had the numbers two and six, and if you multiply them together, you get 12. But in the sense of the least common multiple of two and six, it can be six, because six times one is six and two times three is six.

Cara claimed that she would also provide Mark with a counterexample. However, her tack was very different than Isla's. Isla said that she would fully explain the counterexample to Mark, while Cara insisted that Mark explore the counterexample on his own. The other four participants, Brit, Eden, Gwen, and Lucy, suggested that they would give Mark a counterexample to explore using Cuisenaire rods. They all claimed that this would help Mark see, in a tactile and visual way, that his conjecture was not always true. This decision seemed to draw from participants' SCK and experience with Cuisenaire rods from their number theory course. Brit also used this opportunity to draw attention to common factors. "With six and eight, they have that two in common, so they have that stuff to match up before they actually multiply together." Brit went on to say that "we have to look at what they have in common and whether we can match up [the trains] before [the product]." Not only did Brit create an opportunity for Mark to realize that his conjecture was invalid, but she demonstrated KCT by also creating an opportunity for Mark to understand *why* his conjecture does not always work.

Participants offered a multitude of reasons for why they responded to Mark in the ways that they did. I coded all of these responses as "insight to KCT." Brit, Eden, and Isla all cited their tutoring experiences with elementary and middle school students. Brit said that her response to Mark was "just a natural thing" for her because of her years of experience tutoring students.

Eden's tutoring experiences led her to believe that students can be quite adamant that their answers are correct and that it can take a bit of work to convince them of an invalid answer or procedure.

> *Eden:* I tutor some kids in math, and they always think that their method is right, but you kinda show them that, 'if you do it this way I get this answer and it's not the same as yours. How come?' And you kinda slowly take what they're saying and slowly show them why it's wrong. And hopefully they'll connect to it saying, oh yeah, that is wrong.

It is evident from Eden's response that her tutoring experiences contributed to her general strategy for responding to Mark. Eden was one of three students (including Brit and Isla) that had not yet taken a mathematics education course. Later in the interview, she claimed that her tutoring experiences were the only experiences that contributed to her responses to the hypothetical students in the student scenarios. She also suggested that her strategy when working with students mostly consisted of trial and error. She said she would "see what works and what doesn't work."

Isla claimed that her tutoring experiences helped her to recognize the conflicts that arise when teachers tell their students that a "rule" always works when, in fact, it may not. She frequently tutored her younger cousin, a 5th grade student, and she witnessed her cousin attempt to make generalizations about her mathematical understandings from earlier grade levels in order to better understand the current material. Isla claimed that this was problematic. She said, "You're told this rule applies for all, but it really doesn't."

Rarely did participants have experience tutoring the specific content of the interview tasks, but when they did, they would bring it up during the task to justify their responses. In some cases, as we see with Eden's response to Mark, insufficient content knowledge can negatively affect PCK, thus making content knowledge another important contributing factor to preservice teacher PCK.

Discussion

While some might argue that teachers may only demonstrate true PCK (KCT, in particular) in the classroom, others suggest that demonstrations of PCK in a clinical interview may be a sort of pre-knowledge or a subset of the knowledge they could demonstrate in the classroom (Hauk, Jackson, & Noblet, 2010). Even Hill (2010), a contributor of MKT, developed and implemented PCK test items that proposed to elicit KCT. If clinical settings can elicit a subset of a future teacher's mathematical PCK, then more authentic teaching experiences like tutoring sessions are very likely to do so. The data from this preliminary study suggests that not only would tutoring experiences elicit PCK, but they appear to contribute to PCK development. However, to better understand the potential for using tutoring experiences to develop preservice elementary teachers', further inquiry is necessary. Such inquiry should also take into account the effects of content knowledge on PCK. In designing a future study, we might consider the following questions.

Questions for Audience Consideration:

1. How might we structure or supplement mathematics tutoring sessions with primary students in order for the tutors (preservice elementary teachers) to best learn from the experience?
    a. Which design elements will encourage the development of the tutors' KCS? KCT?
    b. How might we ensure that participants' content knowledge is sufficient and being used appropriately?
2. It can be argued that the format of a tutoring session greatly distinguishes tutoring from an authentic field experience. What are some of the limitations (and benefits) of using tutoring sessions in the research design?

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The Physics Inventory of Quantitative Reasoning: Assessing Student Reasoning About Sign

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An increase in general quantitative literacy and discipline-specific Physics Quantitative Literacy (PQL) *is a major course goal of most introductory-level physics sequences—yet there exist no instruments to assess how PQL changes with instruction in these types of courses. To address this need, we are developing the* Physics Inventory of Quantitative Literacy (PIQL), *a multiple-choice inventory to assess students' sense-making about arithmetic and algebra concepts that underpin reasoning in introductory physics courses—proportional reasoning, covariational reasoning and reasoning about sign and signed quantities. The PIQL will be used to not only to assess students' PQL at specific points in time, but also to track changes in and development of PQL that can be attributed to instruction. Data from early versions of the PIQL suggest that students experience difficulty reasoning about sign and signed quantities.*

Key words: Signed Numbers, Negative, Quantity, Physics, Reasoning Inventory

(Physics) Quantitative Literacy

Quantitative literacy (QL) plays a major role in everyday life, affecting how one views general risk, and health and economic choices; quantitative literacy facilitates performance on many tasks. Both everyday sense-making and workplace performance rely on QL, and many K-12 and higher education systems have undertaken systematic attempts to improve student performance, yet progress remains elusive (Madison & Steen, 2003; Steen, 2004). We argue that physics, as perhaps the most fundamental and transparently quantitative science discipline, should play a central role in helping students develop quantitative literacy. We coin *Physics Quantitative Literacy (PQL)* to refer to the rich ways that physics experts blend conceptual and procedural mathematics to formulate and apply quantitative models. *Quantification*, a foundation for PQL, is the use of established mathematics to invent novel quantities to describe natural phenomena (Thompson, 2010; Thompson, Carlson, Byerley, & Hatfield, 2014). Quantification is at the heart of experts' investigation of patterns and relationships, which in turn anchor the quantitative models that are the hallmark of physics. Galileo famously wrestled with the mathematical decision of whether to describe accelerated motion with a ratio of change in velocity to distance traveled, or to elapsed time. Choosing the latter led to the formal concept of acceleration, a foundation for Newtonian synthesis.

Quantification relies on blending physics meaning with a conceptualization of the multiplicative and other mathematical structures of the quantities involved; cognitive blending theory helps to frame this blend (Bing & Redish, 2007; Fauconnier & Turner, 2008). Figure 1 illustrates a double scope quantity reasoning blend, in which two distinct domains of thinking are merged to form a new cognitive space optimally suited for productive work. Findings by Czocher support this view. They observed students enrolled in a differential equations course solving a variety of physics problems, and found that successful students functioned most of the time in a "mathematically structured real-world" in which the students moved back and forth fluidly between physics ideas and mathematical concepts (Czocher, 2016). Fluency within this blended space is a hallmark of PQL. We argue that assessing students' PQL gives us insight into the desired cognitive blend.

Though improvement of PQL is a primary course goal, There is little research to assess how PQL develops throughout a typical introductory physics sequence. To address this need, we are developing the Physics Inventory of Quantitative Literacy (PIQL). The PIQL is an assessment instrument intended to probe students' proportional reasoning, covariational reasoning, and reasoning about sign; these three areas are at the heart of quantification in introductory physics (Sherin, 2001; Thompson, 2010; Thompson et al., 2014).

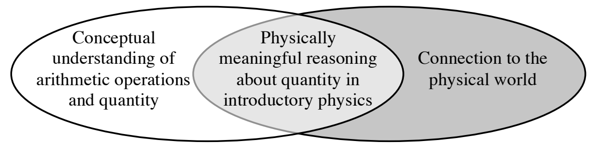

Figure 1: Cognitive blend required for sense-making of physics quantities

In this paper, we discuss recently collected data from a prototypical version of the PIQL (the 'protoPIQL') to preview the types of analyses we hope to achieve using data from more final versions of the PIQL. Our focus in this preliminary report is on instrument items that foreground student reasoning about sign and signed quantities in introductory level physics.

Reasoning About Sign and Signed Quantities

There has been significant research about the different meanings of the negative sign, and student understanding of 'negativity' (Bishop et al., 2014; Vlassis, 2004). Findings indicate that algebraic success is associated with greater 'flexibility' with negativity—that is, students that are able to interpret correctly its use in different contexts show improved performance on tasks such as polynomial reduction (Vlassis, 2004). Flexibility with negativity is analogously important in introductory-level physics, yet no analogous research has been conducted in physics contexts. This paper describes our effort to probe the published natures of negativity (Vlassis, 2004) in a physics context.

Table 1: A map of the different uses of the negative sign in elementary algebra (Vlassis, 2004)

Unary (Struct. signifier)	Symmetrical (Oper. signifier)	Binary (Oper. signifier)
Subtrahend	Taking opposite of, or	Completing
Relative number	inverting the operation	Taking away
Isolated number		Difference between numbers
Formal concept of neg. number		Movement on number line

Table 1, reproduced from Vlassis's 2004 paper, is a map of different algebraic meanings of the negative sign. It served as a guide in our preliminary study of student understanding of the negative sign in introductory-level physics. To begin to probe the effect of introductory-level physics instruction on development of flexibility with negativity, we modified existing signed-quantity questions (Brahmia & Boudreaux, 2017) for use on the protoPIQL. Examples of such questions, and how they fit into the map summarized by Vlassis, are shown in Figure 2.

Figure 2: Items used on protoPIQL representing different algebraic natures of negativity. The acceleration item (left) probes student understanding of the unary (structural signifier, direction of vector component) aspect of negativity, while the work item (center) represents a binary (symmetrical, decrease in system energy) aspect. The position item (right), represents a binary (operational signifier, position relative to origin) nature.

An object moves along a line, represented by the x-direction, and the acceleration is measured to be $_x = -8\ \mathrm{m/s^2}$. Consider the following statements about this situation. Select the statement(s) that **must be true.** ***Choose all that apply.***

a. The object's speed is decreasing.
b. The magnitude of the acceleration is decreasing.
c. The object is doing the opposite of accelerating.
d. The acceleration is in the negative x-direction.

A hand exerts a horizontal force on a block as the block moves along a frictionless, horizontal surface. For a particular interval of the motion, the hand does $= -2.7$ of work on the block. Consider the following statements about this situation. Select the statement(s) that **must be true.** ***Choose all that apply.***

a. The work done by the hand is in the negative direction.
b. The force exerted by the hand is in the negative direction.
c. A component of the force exerted by the hand is in the direction opposite to the block's displacement.
d. Energy was taken away from the hand system.
e. Energy was taken away from the block system.

A person is moving along a line, represented by the x-direction. At a specific instant of time the person is at position $= -7$. Consider the following statements about this situation. Select the statement(s) that **must be true.** ***Choose all that apply.***

a. The person moves in the negative direction.
b. The person is to the negative direction from the origin.
c. The person is facing backwards.
d. The person is moving backwards.

Methods and Analysis

PIQL is designed as a multiple-choice instrument for collecting quantitative data. Quantitative methods are well-suited to our current investigation, as we are not probing students' 'in-the-moment' thinking. Rather, we hope to track changes to and development of PQL over the course of instruction in introductory physics.

The protoPIQL was administered to $N \sim 1000$ students enrolled in each of the three quarters that constitute one academic year of the calculus-based introductory physics sequence at a large, public American university at the beginning of the academic quarter, before significant instruction had occurred. Therefore, for students enrolled in the first quarter of the sequence, the protoPIQL serves as a pretest for the entire introductory physics sequence. For students enrolled in the second and third quarters of the sequence, the protoPIQL acts as a post-test for the previous quarter's course. Thus we are able to determine whether flexibility with negativity in physics changes over the first two quarters of the introductory sequence. In addition, we wished to investigate how flexibility across contexts is correlated with flexibility within a single context, as described below.

The protoPIQL consisted of 18 questions total: 10 on proportional reasoning, 6 on reasoning about negative quantities, and 2 on covariational reasoning. We focus here on the results of the three negativity questions presented in Figure 2. These three questions represent three different natures of negativity in introductory physics. For a_x, the x-component of acceleration, a negative sign indicates the direction of the vector component relative to a coordinate system. Although the position x is also a vector component (position $\vec{r}$ is a vector quantity), it can be considered an 'almost scalar' quantity in this context, as a one-dimensional position measurement along an axis differs from a location on a number line only in its units. Work W on a system is a scalar quantity that is related to changes in the mechanical energy of a system via the work-energy theorem ($W_{net,ext} = \Delta E$); therefore negative net work on a system indicates that the mechanical energy of that system is decreased. In this case, with only one force that does work on the system, negative net work also indicates that the force and the system's displacement have components in opposite directions, as $W = \vec{F} \cdot \Delta\vec{x}$. Thus, a full understanding of negative work requires flexibility within the single context, as multiple interpretations of the negative sign are possible and (in fact) desired.

Changes in Flexibility With Instruction

For our first, preliminary investigation into changes in flexibility with negativity over the introductory physics sequence, flexibility was defined in terms of answers to these three questions. A small percentage of students did not answer the position question correctly; these students were not given a flexibility designation, as we see understanding of the negative sign associated with position as the most basic understanding of a negative quantity (most analogous to a location on a number line). These students, categorized "Nx" were not included in the following analyses. Students answering only the position question correctly were categorized as "Inflexible" (In). Students that answered only one of the acceleration and work items in addition to answering the position item correctly were categorized as "Intermittently flexible." Students answering all three of the mechanics negativity questions correctly were categorized as "Flexible."

Results for students enrolled in a standard introductory physics sequence (labeled Quarter 1, 2, and 3), as well as students in the third quarter of an 'honors' introductory physics sequence (Q3 Honors) and physics graduate students (G) are shown in Figure 3. Although we see an increase in flexibility after a single quarter of instruction (that is, from Q1 students to Q2 students), there is no significant increase in flexibility thereafter.

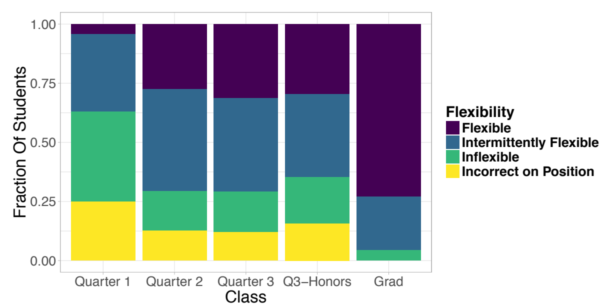

Figure 3: Flexibility for 5 populations of students

Flexibility Within Contexts

To investigate the correlation between flexibility *across contexts* (as above) and flexibility *within a single context*, we consider only students enrolled in the last quarter of the introductory physics sequence (students in Q3 and Q3H, $N = 317$). We also define flexibility differently for this analysis, using a slightly different subset of items: the position and acceleration questions described above, and a third item regarding the meaning of the negative sign associated with a component of an electric field, E_x. Mathematically, the meaning of the negative sign in the electric-field context is similar to that in the acceleration context. We collapse the four categories above into two—students answering zero or one items out of the three were considered to be inflexible, while students answering two or three of these items correctly were considered to flexible. By this metric, approximately 75% of Q3 and Q3H students are flexible (comparable to the sum of Flexible and Intermittently Flexible in Figure 3).

The work item has two correct responses, one that connects the meaning of the negative sign to the relative orientations of the factor vectors $\vec{F}$ and $\Delta\vec{x}$, and one that relates to the system's decrease in mechanical energy. A complete understanding of the negative sign of work requires flexibility within this single context—the negative sign has two correct interpretations. To look at whether inter-context flexibility was associated with intra-context flexibility, we compared performance on the negative work item for students that were rated as inflexible or having emerging flexibility as

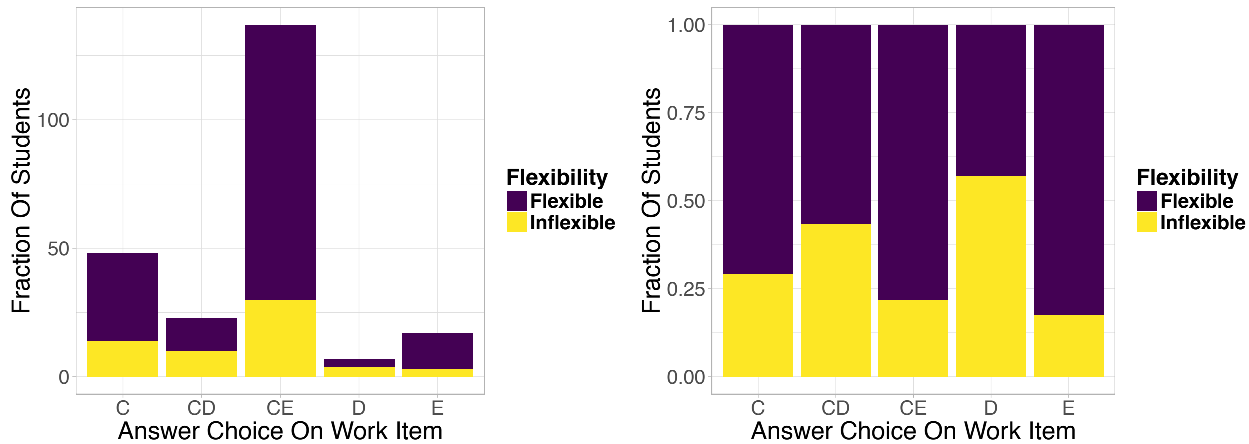

Figure 4: Left: number of students rated as flexible or inflexible based on answer choices. Right: conditional probability of being categorized as flexible or inflexible given answer choice(s).

defined above. (Recall that approximately 75% of these students are flexible by this definition.) The results are shown in Figure 4. Answer choice C compared the relative orientations of the factor vectors of the scalar product, while answer E relates the negative sign to the system's decrease in energy. Answer choice D incorrectly identifies negative work with an *increase* in system energy. X^2 analysis suggests that showing intra-context flexibility by recognizing both possible meanings of the negative sign) is associated with inter-context flexibility ($p = 0.024$). We interpret this result as an indication that flexibility across multiple contexts may help prepare students for the more challenging contexts typical in subsequent physics courses in which there are multiple meanings of signs in a single mathematical statement.

Comments and Future Work

Although the negativity items of the protoPIQL are yielding interesting findings, we believe that our current analysis is limited by the negativity framework developed in the context of algebra. We are developing a negativity framework specific to introductory physics. We find that uncovering the natures of mathematical objects that play multiple roles in physics to be a productive framework for assessment, instruction, and curriculum development. In a related paper in these proceedings, we discuss the Nature of Negativity in Introductory Physics. Such a physics-specific framework will, in turn, necessitate the construction of new items for the PIQL and may inform natures of negativity in the context of quantity used in mathematics education.

Regarding PIQL more broadly, we are creating an analogous map for the natures of covariational reasoning in introductory physics that draws on the extensive work done in the context of mathematics (Carlson, Oehrtman, & Engelke, 2010), and have made progress on a framework for proportional reasoning in the context of physics (Boudreaux, Kanim, & Brahmia, 2015).

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Teacher Candidates' Cognitive styles: Understanding Mathematical Thinking Process used in the Context of Mathematical Modeling Tasks

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In this work we examined modeling routes, mathematical thinking styles and teaching foci of two prospective secondary teachers as they considered two modeling tasks so to consider connections between the candidates' own modeling processes and their approaches to teaching modeling. Two questions guided the study: How might pre-service teachers' preferred mathematical thinking styles impact their modeling routes? How might pre-service teachers' preferred mathematical thinking styles impact their focus while contemplating how they would teach mathematical modeling? Close links were found between teacher candidates' validation methods within the modeling process and their decisions regarding what would be important for school learners to consider. Recognizing teacher candidates' natural approaches to modeling tasks might help teacher educators to be better positioned in developing tasks that motivate reliance on a larger repertoire of representations.

Keywords: Mathematical Modeling, Cognition, Teacher Education

Individuals' mathematical modeling process does not progress linearly (Kaiser, 2013). one's preferences and mathematical competencies influence their choice of modeling routes within the when tackling modeling problems (Blum & Ferri, 2009). While one individual may start the modeling process with a direct connection of a real situation to the mathematical model, another individual might spend time creating a real model first and then move from the real model to a perceived suitable mathematical model (Haines & Crouch, 2010). Ferri (2010) suggests that examining individuals' modeling routes can enable researchers to better understand their mathematical thinking styles.

According to Stenberg (1997), a thinking style is a "preferred way of thinking" or "preference in the use of abilities," something which might be acquired through the social environment. Based on Stenberg's theory, mathematical thinking styles are described as how individuals prefer to understand and learn mathematics (Dreyfus & Eisenberg, 1996; Schoenfeld, 1994). Later empirical studies classified these thinking styles into three categories: visual, analytic, visual, analytic, and integrated thinking (Dreyfus & Eisenberg, 2012).

If we think about the connection between mathematical modeling and mathematical thinking styles, an "individual's preferences" play an important role. Ferri (2010) argued that an one's preferences along their choices of modeling routes can be a strong indication of the individual's thinking styles. It is logical to typologies how teachers' thinking styles affect their behaviors in mathematical modeling implementation: type 1- retrospective formalizer, type 2-realistic validator, and type 3-formalisticrealistic (Ferri, 2018). Although studies which examine the relationship between modeling routes and mathematical thinking styles exist in the literature (e.g., Blum &Ferri, 2009; Ludwig & Xu, 2010; Maltempi & Dalla Vecchia, 2013), these reports have involved work with high school students, teachers, or mathematicians. There is still a need for empirical studies involving teacher candidates who have newly developed knowledge of mathematical modeling and its implementation in the classrooms. In this study, we investigate the relationship between pre-service teachers' mathematical thinking styles, their modeling routes, and their focus while contemplating how they would teach modeling. The following questions guided this study: How might pre-service teachers' preferred mathematical thinking styles impact their modeling routes? How might pre-service teachers' preferred mathematical thinking styles impact their focus while contemplating how they would teach mathematical modeling?

The Framework

We considered three conceptual issues when selecting appropriate mathematical modeling tasks: the participants' ways of mathematical thinking, ways in which their modeling routes proceed in modeling cycles, and their projected teaching behaviors when facilitating mathematical modeling tasks. Hence, three frameworks guided our data analysis process: 1) Blum and Leiβ's (2007) modeling cycle (Figure 1) was used to be able track the participants' modeling routes in their approaches to solving the tasks, 2) mathematical thinking styles (visual, analytic, and integrated) that influenced their modeling process (Ferri, 2010), and 3) Participants' behaviors in teaching modeling (type 1-retrospective formalizer, type 2-realistic validator, and type 3-formalistic-realistic) are considered (Ferri, 2018).

According to Ferri (2010), visual thinkers use mostly illustrative drawings to express real situations/problems while they are working within a mathematical model. They follow the full modeling cycle (Figure 1). Analytic thinkers usually work with symbolic or verbal representations, and they tend to create a mathematical model from a real situation and focus on mathematical results rather than creating a real model in the modeling cycle. Integrated thinkers combine visual and analytical ways of thinking and switch their modeling routes flexibly while creating different representations.

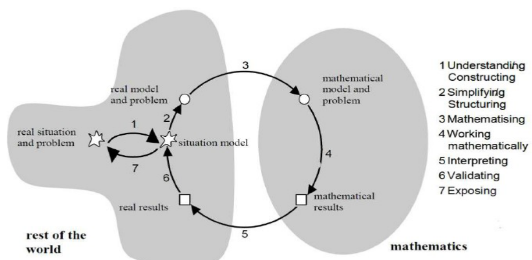

Figure 1. Mathematical Modeling Cycle (Blum & Leiβ, 2007)

Ferri's study (2018) introduces three types of teacher behaviors when implementing mathematical modeling tasks. These are: 1) Type 1- A retrospective formalizer focuses on mathematical solutions and mathematical models, and they use only real facts for the validation of mathematical results; 2) Type 2- A realistic validator focuses on understanding and representing real situations/problems with pictures and graphics, and formalization has a low significance for them; and 3) Type 3- A formalistic realistic focuses on striking a balance between real-world situations and mathematical representations during modeling implementation.

Research Methodology

This research was a qualitative, descriptive account of the modeling routes and mathematical thinking styles of two secondary pre- service teachers. The study was conducted at a public university in a Midwestern state. The participants, Alonzo and Bria (pseudonyms) were enrolled in a methods course on teaching mathematics in secondary schools at the time of data collection. This course is the second of a yearlong sequence of methods courses, in which mathematical modeling was addressed explicitly addressed in three class sessions. The selection of the participants was deliberate, targeting variability among mathematical backgrounds and self- efficacy towards doing and teaching mathematical modeling. Each participant was interviewed individually three times for approximately one hour each.

Interview sessions had a two-part design. In the first part, each candidate worked on a modeling task and were then asked to comment on how they would implement the same task in a classroom. They were asked to comment on challenges they anticipated regarding learners' difficulties, and how these challenges may be addressed in instruction. We used the think aloud technique (Ericsson & Simon, 1980) during the interview process. The three modeling tasks were selected from Three-Act Math Tasks (Meyer, 2011). The common objectives of the three tasks were defining variables, estimation, making assumptions, and modeling with geometry, which required applying concepts of density based on area and volume in modeling situations (e.g., water per cubic foot) (CCSSM, 2010). The tasks access to the mathematics through multiple entry points, and they foster the solving of problems through varied solution strategies. Each of the interviews was videotaped and transcribed, and the written work was digitized. The videos and transcripts were analyzed to capture the relationship between the mathematical thinking styles and modeling behaviors of teacher candidates from tasks to tasks (Auerbach & Silverstein, 2003).

Results of the Research

There is agreement that validation is an essential part of the modeling cycle (Cai et al., 2014). Validation involves comparison of the responses predicted with a mathematical model to the responses in the real world model (Blum & Ferri, 2009). Both participants validated their results at almost every stage of the modeling process. However, their validation methods differed. While Alonzo relied on explaining his reasoning in a formal manner, Bria tended to validate her modeling steps drawing on her experiences, intuitive knowledge, and pictorial representations. In the following section, we show illustrative examples of the participants' work in one of the modeling tasks from the interviews and the modeling routes used in the modeling cycle to show their preferred representational schemes.

<u>Task 1- Part 1: World's Largest Hot Coffee (Meyer, 2011)</u>
The Gourmet Gift Baskets team wants to break the record for the biggest coffee cup. According to the Guinness Book of Records, the World's Largest Cup of Coffee contained 911.5 gallons of coffee in 2007. What should be the size of the coffee cup to break this record? If the rate of filling the cup is 2.1 gallons per minute, approximately how long does it take to fill this cup? Explain your solution.

<u>Task 1- Part 2:</u> What would be your focus if you implement this task in your classroom? How would you assess your students during the modeling process?

Participant Alonzo: Analytic Thinker, Retrospective Formalizer

Alonzo's first solution step was to turn the real problem into a mathematical problem and write down the volume formula $V= \pi r^2h$, where h feet is the unknown height of the coffee cup and r feet is the unknown radius of the coffee cup. Alonzo created a mathematical model based on the volume of a cylinder, then used that formula to proceed to the problem's solution. He brought up another mathematical consideration, which was the conversion from cubic feet to gallons, i.e., 1 cubic foot= 7.48 U.S. liquid gallons (real result). Hence, he wrote "The cup should hold V*7.48 gallons of coffee" (mathematical model) and then revised his mathematical model based on the world record. He suggested that since the biggest cup held 911.5 gallons (real result), the new cup volume must satisfy V*7.48> 911.5 gallons (mathematical model), in which h and r could be approximated based on that calculation (mathematical results). Then, he went back to the 'how long it takes to fill the cup problem, and created another formula, (V*7.48 gallon)/2.1 gallons, which is another mathematical model, this time one to be used to approximate the time it would take to fill the cup. The mathematical calculation and approximations led him to answer "approximately 13 hours are needed to fill the cup." As an analytical thinker, he relied on algorithms and procedures he knew and did not consider further iterations in the modeling cycle's modeling route is presented in Figure 2 below:

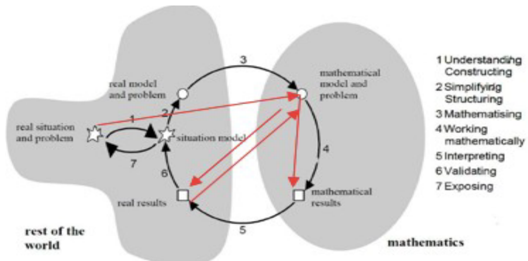

Figure 2. Alonzo's modeling route in Task 1

Alonzo's response to the question "What would be your focus if you implement this task in your classroom?" was: "I believe that students have to think in structures and be able to move within these structures so that they are able to see and to build formulas. I imagine that students would start with simple representations for this problem, and my guidance would be pushing them the mathematical world, such as using more math terms. For me, the most important information in this problem is to be aware of why we use the volume formula." Based on Ferri's (2018) category, Alonzo comments and his methods to solve the problem matched type 1- retrospective formalizer teacher behavior. His formalization of solutions in the form of abstract equations was important, and validating ideas with real-life facts was less important for his instruction.

Participant Bria: Visual Thinker, Realistic Validator

Upon reading the task Bria immediately drew a sketch to create a situation model. She then simplified the resulting situation model to match a real model she believed compatible: the "Coffee cup is similar to a hot tub." Bria mentioned how long it took her to fill up her hot tub. She compared hot tub size (real model) to coffee cup size and estimated that the coffee cup size was 5 x 7 feet. She then drew a 2-gallon water bottle and estimated the size of the bottle as 5 x 10 inches. Bria said: "I imagine myself now, I am using this plastic bottle to pour 2 gallons of coffee per minute to fill up the big coffee cup" (real model/problem). Then, she attempted to mathematize the problem by drawing both the coffee cup and a cylindrical 2-gallon plastic bottle. She planned to divide the big cylinder's volume by the small cylinder' volume to figure out how many minutes it would take to fill up the big cup. She plugged the estimated size values into her formula "V1/V2" (mathematical model) and the first answer was "8.5 hours" (mathematics result)". Bria's answer to the question of "How do you know that your assumption (5 x 10 $foot_2$) on coffee cup size is appropriate for breaking the world record?" was: "I do not know for sure, visually it (coffee cup) seems like little smaller than my hot tub. But, I could figure out the size of the coffee cup by comparing the biggest cup in the world's record." She decided to use 911.5 gallons to be more precise in her coffee cup size answer. She calculated the volume of the coffee cup in ft_3, which was smaller than 911.5 gallons. Then, she revised the formula and wrote down "V1>911.5." she started the modeling cycle once again to determine the size of the coffee cup more precisely. As a visual thinker, her primary method was to draw the coffee cup and a 2-gallon bottle to express the problem situation. Real facts and her life experience guided her in validating her own assumptions. Bria's modeling route is presented in Figure 3 below:

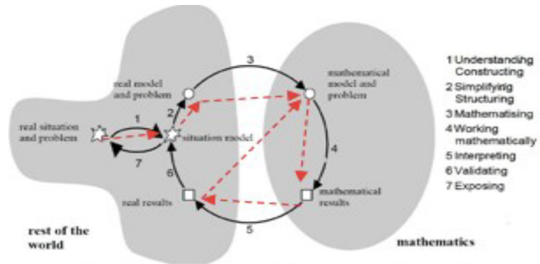

Figure 3. Bria's modeling route in Task 1

Bria's response to the question "What would be your focus if you implement this task in your classroom?" was: "For me, it is not very important that students do everything formally in a correct way and find the mathematical answer of this problem. But they need to understand that mathematics can help them in their way of thinking. My class would focus on encouraging students to start a solution with imagining the real situation. Then, students should be able to explain their mathematical ideas based on what's happening in reality." In contrast to Alonzo, Bria preferred to validate the modeling process with her real-life experiences. She sees visualization and estimation as important tools for interpreting mathematical results in her modeling process. Her focus while she contemplated how she would use this problem in her teaching is more similar to type 2- realistic validator in Ferri's (2018) category.

Implications for Teaching Practice

Similar to the teachers in Ferri's study (2018), teacher candidates' methods for the validation of ideas within the modeling process and making decisions on what to focus on when teaching mathematical modeling depended on their thinking styles. Knowledge about different modeling behaviors based on the teacher candidates' preferred mathematical thinking styles inform both teacher educators and teacher candidates about the articulation of the ways for teaching mathematical modeling and characterizing learners' modelling efforts. This might help teacher candidates to be aware of their own ways of thinking mathematically as well as others' modeling behaviors. Consequently, teacher candidates might be more prepared to anticipate learners' modeling actions, the difficulties learners might face (e.g., building a mathematical model), and to respond their needs in the classroom.

Manouchehri's study (2017) revealed two challenges for teacher educators when engaging teachers in mathematical modeling to include managing teachers' different mathematical backgrounds and creating a collaborative learning environment for mathematical modeling. The current report offers a venue for identifying a baseline knowledge for teacher educators about accessing pre-service teacher's preferred mathematical thinking styles and teaching focuses in the modeling process. Recognizing teacher candidates' natural approaches to modeling tasks might guide teacher educators to be better positioned in developing and using tasks that motivate reliance on a larger repertoire of representations. This approach can not only advance preservice teachers' modeling competencies but also allowing them to gain pedagogical tools for navigating classroom implementations. Based on our findings, we would like to invite the RUME audience to discuss the following questions: How can we assess pre-service teachers for mathematical modeling? What type of tasks would be helpful in developing a research-based teacher learning trajectory specific to mathematical modelling?

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What is Difficult About Proof by Contradiction?

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Although students face many challenges in learning to construct mathematical proofs in general, proof by contradiction is believed to be particularly difficult for them. We investigate whether this is true, and what factors might explain it, using data from an "Introduction to Proof" course. We examined proofs constructed by students in homework and examinations, and conducted stimulated-recall interviews with some students about their thought processes while solving proof problems. Preliminary analysis of our data suggests that students' background knowledge about the typical content domains that appear in indirect proof plays a larger role than the logical structure of the proof technique itself.

Keywords: Indirect Proof, Proof by Contradiction, Teaching Mathematical Proof.

Introduction

The teaching and learning of mathematical proof have received a great deal of attention in the field of mathematics education research, and this emphasis continues to increase. At the college level, the ability to understand and construct proofs is essential for students to transition from computationally oriented calculus sequences to more theoretically oriented upper-division mathematics courses. Many universities have instituted "Transition to Proof" courses to facilitate this. At the K-12 level, the Common Core Standards for Mathematical Practice emphasize the ability to construct and critique mathematical arguments, i.e., proofs.

Indirect proof, also known as proof by contradiction (we will use these terms interchangeably), is an essential form of proof across all mathematical content areas. Instructors' anecdotal experience as well as mathematics education research suggest that students have particular difficulty with this type of proof, where "difficulty" has been variously interpreted as pertaining to comprehending, constructing, deriving conviction from, or simply disliking indirect proofs (Tall 1979, Brown 2018). It is somewhat puzzling why indirect proof would be especially challenging cognitively, given that we use this kind of informal reasoning frequently in everyday life (Reid & Dobbin 1998). The common form of argument, "If that were true, then how do you explain X?" is clearly an informal sketch of a proof by contradiction.

If indirect proof is indeed uniquely difficult in the formal mathematical context, what are the reasons for this? We explore this question using data from an "Introduction to Proof" course recently taught by one of us, in the context of students *constructing* indirect proofs for homework assignments and examinations.

Purpose and Theoretical Background

The purpose of this study was to identify difficulties faced by students in constructing indirect proofs as part of their regular coursework in an "Introduction to Proof" class. The problems solved by these students are part of the regular course pedagogy rather than tasks chosen by a researcher, for example in an interview setting. Since this study is somewhat exploratory, we wanted a wide range of "naturalistic" proof samples rather than one or two that might reflect peculiarities of those chosen tasks more than general student issues with indirect proof.

Assuming that the literature is correct that indirect proof is uniquely difficult for students, our intent was to test three hypotheses that might explain why, or to formulate additional ones.

1. *Logical hypothesis*. Students have difficulty recognizing what constitutes a contradiction in the strict logical or mathematical sense. If a step in their proof contradicts a piece of their prior mathematical knowledge (which may never have been rigorously proved itself), is that sufficient? They may also manipulate mathematical statements too formally, assigning them so little *meaning* that contradictions go unrecognized (Sierpinska 2007).
2. *Psychological hypothesis*. Indirect proof requires the temporary acceptance for the sake of argument of assumptions that are actually false, and may already be known to be false. Such *counterfactual reasoning* may be more difficult within the domain of mathematics than in everyday contexts (Antonini & Mariotti 2008). For example, I can fairly easily imagine that Hillary Clinton won the 2016 election, but how can I imagine that 7 is not a prime number? What kinds of reasoning can be trusted in such an "impossible world"?
3. *Structural hypothesis*. In a direct proof task, both the hypothesis and the conclusion are known at the outset. That is, one knows where the proof begins and where it will end, providing a structural framework (Selden & Selden 2009). In contrast, the goal of indirect proof is "a contradiction". The prover does not know in advance what this will be, so cannot structure the proof around it.

Our initial research question was, what evidence do students' proofs from their class assignments provide for or against these hypotheses?

Based on our initial data analysis, however, we have broadened our hypotheses. It may be that indirect proof is difficult not (only) because of its logical *nature*, but because of the typical mathematical *content* in such proofs, for example rational versus irrational numbers. The background knowledge and beliefs that students have about such content may be a source of their difficulties. One would then expect to observe similar difficulties in direct proofs dealing with the same content. We have found it useful to think about students' background or prior content knowledge in terms of the *resource framework* (Hammer et al 2005), or the *knowledge-in-pieces* viewpoint (diSessa 2013). In these perspectives, student knowledge does not form a coherent theory, but rather a collection of pieces or "resources" that may not be mutually consistent and may be individually activated in varying circumstances. From such a perspective one would not ask whether a student "really believes" for example that $(a+b)^2=a^2+b^2$ but rather in what contexts this type of assertion is activated.

Participants and Methods

The participants in this study were students in an "Introduction to Proof" class taught by one of the authors at a large public university in the southwestern United States. The class is normally taken following the two-year calculus sequence and is required for all mathematics majors. Of the 106 enrolled students, 72 agreed to participate in the study. The majority of these were mathematics majors, and the rest were from various other STEM majors. There were roughly equal numbers of male and female students.

All homework and exams were graded using the Gradescope system, which preserved the students' work for our later analysis. There were twelve graded proof by contradiction problems, some from the course textbook and some that we added based on previous research or for pedagogical reasons. During the course no attempt was made to match the assigned direct and indirect proof problems for difficulty or content, but for our analysis we selected a comparison group of eleven direct proof problems that we considered comparable in difficulty and subject matter. The comparison is quite rough, since the direct proof problems assigned tended to involve specific topic areas covered in the course, such as equivalence relations or mathematical induction, which do not overlap greatly with the content areas for the indirect proofs. The proof by contradiction problems were clustered in two consecutive homework assignments near the middle of the course, or on the second midterm or final exam. We also solicited student volunteers to be interviewed, but only obtained six volunteers, all of whom were accepted. Nevertheless there were three male and three female interview subjects, representing a range of achievement levels in the course.

Interviews took place just after the second midterm exam. These were semi-structured "stimulated recall" interviews (Shubert & Meredith 2015). Students were shown their own prior work on certain indirect proof problems, and were asked to identify the contradiction they reached and explain why it was a contradiction, how they searched for and then recognized the contradiction, why they chose a particular approach, and what other approaches they had attempted. Sometimes they were shown the work of another student and asked to locate the contradiction or to compare that solution with their own. After discussing specific proof problems, they were asked some general questions, such as what makes an indirect proof work, how they feel about reasoning on the basis of a counterfactual assumption or "impossible" geometric diagram, and whether they prefer direct or indirect proof for any reason.

Homework and examinations are complementary data sources in some respects. Students are under less time pressure when solving homework problems, so one might expect their reasoning to better reflect their capabilities and knowledge about proof rather than careless errors due to time pressure. On the other hand, students have more opportunities to obtain help from friends, teachers, or online sources, when doing homework. We saw evidence for both effects.

At this stage of data analysis, we have examined all student solutions to six indirect and two direct proof problems. We coded the different approaches taken, both correct and incorrect, and created categories of errors or misconceptions exhibited. Of particular interest were the types of contradictions obtained, whether they were reached in an efficient or a roundabout manner, whether any actual contradictions were written down but overlooked by the student, or conversely whether a student claimed to have reached a contradiction when she had not in fact deduced one. We have transcribed the interviews and begun to code them for students' understanding of how proof by contradiction works, ability to recognize contradictions, comfort level when reasoning from counterfactual hypotheses, and so forth.

Results

As an initial rough indication of whether the indirect proofs were "more difficult" than the comparison group of direct proof problems for our students, we compared their mean scores on the two groups of problems using the Welch two-sample t test. The difference in group means was not significant at the 5% level, suggesting roughly similar levels of difficulty.

One of the homework problems assigned was the Angle Bisector problem studied by Baccaglini-Frank et al (2013): show that the bisectors of two angles in a triangle ABC cannot be perpendicular to one another. This is an easy consequence of the angle sum in a triangle being 180 degrees and can be demonstrated by direct (6 students) or indirect proof (48 students). Significantly, most students included a diagram with their proof and showed no confusion in reasoning from this impossible geometric figure (termed a *pseudo-object* by Baccaglini-Frank et al). Asked how he felt about this potential cognitive conflict, one student explained:

> I think it might just be from experience of knowing that hand-drawn pictures can be inaccurate, and then there are a lot of stuff like optical illusions where some things look perpendicular when they're not…It wasn't necessarily to reassure myself that the statement was true because I knew the statement was true, and it was more so to visualize the relationship of that new angle and how it relates to A,B, and C.

This and similar data do not support our Psychological hypothesis.

Another homework problem asked students to prove that no positive integers m and n satisfy the equation $7/17 = 1/m + 1/n$. There are many ways to show this, but only 8 out of 64 submissions were correct. The most common approach was to write $7/17 = (m+n)/mn$, or the equivalent form $7mn = 17(m+n)$. 24 students concluded incorrectly from this that $m+n=7$ and $mn=17$. We coded this reasoning, which we also observed in other problems, as SFE (Strong Fraction Equivalence, the view that equal fractions must have identical numerators and denominators) or SUF (Strong Unique Factorization, the view that $ab=cd$ implies that $a=c$ or $a=d$). These students then easily showed that the few values for m and n consistent with one of these restrictions do not satisfy the other. SFE might reflect students' uncertainty about when it is legitimate to assume that a fraction is in lowest terms (as is often done "without loss of generality" in indirect proof) but this explanation does not seem to account for SUF.

It is not plausible that large numbers of STEM majors "really believe" SFE or SUF, certainly not as part of a coherent set of background beliefs, but clearly these assertions are commonly activated (in direct proofs as well). This and similar observations of ours make more sense in terms of the resource framework. Neither SFE nor SUF seems connected to the logical structure of indirect proof, but the typical content of this type of proof may provide more opportunities than that of direct proof to activate such resources. These observations are consistent with the Logical hypothesis, in the sense that students are working formally rather than attending to the meaning of their mathematical assertions. However, we think it is more productive to locate their difficulties in the content of the proof (integers, rational numbers, divisibility) than in the logical structure of indirect proof.

SFE and SUF can be viewed as parts of a broader category of errors that we observed in our data and termed AVNP, for Algebraic Visibility versus Numerical Possibility. That is, students attend to what is algebraically visible in an equation rather than what numerical possibilities might be consistent with it. For example, students see that an expression is not an algebraic perfect square, so they assert that it cannot be a square for any specific integer values of the variables. Or, a rational expression is in lowest terms (the numerator and denominator have no algebraic common factor) and students assume that it must be in lowest terms for any specific integer values of the variables. Examples occurred in both direct and indirect proofs. The domains of variables are also not explicitly visible in the expressions that students manipulate, and we observed uncertainty about the properties and relationships of the domains **Z**, **Q**, and **R**. For example, concepts like divisibility that only apply in **Z** were used in **Q** or **R**, as has been previously observed (Barnard & Tall 1997).

Conclusion

Our initial analysis of the data suggests that our students have a rather good understanding of and comfort level with proof by contradiction. They can negate claims, identify contradictions, and explain the logic of indirect proof, and they seem generally unperturbed by counterfactual reasoning. The difficulties they encounter in solving proof problems seem to reflect the subject matter of the proof more than the proof type (direct or indirect). Many of the "misconceptions" they exhibit cannot be understood as genuinely held beliefs, which supports viewing them as resources or pieces of knowledge that are activated in particular contexts. The "difficulty" of proof by contradiction may lie in the types of resources that it tends to activate.

Discussion Questions

1. How can we operationally distinguish proof errors that reflect difficulty with indirect proof as such from those that reflect misconceptions about the subject matter of the proof, for example the rational number system?
2. How can we better understand student "misconceptions" as resources activated in specific contexts?
3. How can we improve the design of this study for future replications or extensions?

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Graphs as Objects: Analysis of the Mathematical Resources Used by Biochemistry Students to Reason About Enzyme Kinetics

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Interpreting graphs and drawing conclusions from data are important skills for students across science, technology, engineering, and mathematics fields. Here we describe a study that seeks to better understand how students reason about graphs in the context of enzyme kinetics, a topic that is underrepresented in the literature. Using semi-structured interviews and a think-aloud protocol, our qualitative study investigated the reasoning of 14 students enrolled in a second-year biochemistry course. During the interviews students were provided a typical enzyme kinetics graph and asked probing questions to make their reasoning more explicit. Findings focus on students' mathematical reasoning, with analysis indicating students tended to focus on surface features when describing related equations and graphs, which limited their understanding of the chemical phenomena being modeled.

Keywords: Graphical Reasoning, Rate, Chemistry

Introduction and Rationale

Enzyme kinetics is an area of study within chemical kinetics, which focuses on modeling the rate of chemical reactions. Looking more broadly at the literature related to students' reasoning about rate-related ideas and the use of calculus to model physical systems, it is apparent that students need more support learning these concepts (Bain & Towns, 2016; Becker, Rupp, Brandriet, 2017; Castillo-Garsow, Johnson, & Moore, 2013; Rassmussen, Marrongelle, & Borba, 2014; White & Mitchelmore, 1996). Biochemistry education research is an interdisciplinary and emerging field and little work has been done that seeks to understand how students reason about biochemistry topics such as enzyme kinetics, indicating the need for more discipline-based education research that can provide insight into how teaching and learning can be optimized (Singer, Nielson, & Schweingruber, 2012). Especially relevant for enzyme kinetics are Michaelis-Menten graphs, which tersely summarize large amounts of data. However, understanding the information a graph communicates (regardless of context) is not trivial (Carpenter & Shah, 1998; Phage, Lemmer, & Hitage, 2017; Planinic, Ivanjeck, Susac, & Millin-Sipus, 2013; Potgieter, Harding, & Engelbrecht, 2007). Nevertheless, even if individuals are not pursuing careers in science, technology, engineering, and mathematics (STEM), in order to have an informed citizenry that can interact with global social issues, individuals should be able to interpret graphs and other forms of data, and have an understanding of how data is collected (along with the associated limitations inherent with data) (Driver et al., 1996; Driver et al., 1994; Glazer, 2011; Mahaffy et al., 2017; Matlin, Mehta, Hopf, & Krief, 2016).

These considerations are encompassed in the Next Generation Science Standards' definition of science practices, which reflect the combination of skill and knowledge used by scientists to approach problems and provide explanations for phenomena, including: asking questions; developing and using models; planning and carrying out investigations; analyzing and interpreting data; using mathematics and computational thinking; constructing explanations; engaging in argument from evidence; obtaining, evaluating, and communicating information (National Research Council, 2012). It is within this context that we investigate student engagement in science practices, such as productively reasoning about models (Michaelis-

Menten model of enzyme kinetics) and drawing conclusions from data (graphs). This work was guided by the following research question: *How do students use mathematical resources to reason about enzyme kinetics?*

Theoretical Perspectives

The design of this study was informed by the resource-based model of cognition, in which knowledge is conceptualized as a dynamic and complex network of interacting cognitive units called resources (Hammer & Elby, 2002; Hammer & Elby, 2003). Within the resources perspective, knowledge is framed as context-dependent, meaning that students' specific resources may not be activated in a particular context, which helps explain fragmented and non-normative reasoning (Hammer, Elby, Scherr, & Redish, 2005). Here we focus primarily on mathematical resources called graphical and symbolic forms, which involve associating (mathematical) ideas to a pattern in a graph or an equation, respectively (Rodriguez, Bain, and Towns, Submitted; Sherin, 2001).

In a forthcoming paper, we provide a more complete overview of graphical and symbolic forms (Rodriguez, Bain, & Towns, Submitted). Tersely stated, graphical forms involve focusing on a region in a graph and assigning ideas; examples include steepness as rate (the relative steepness of regions in a graph provides information about rate), straight means constant (a straight or flat region in a graph indicates a lack of change), and trend from shape directionality (attending to the general tendency of a graph to increase or decrease) (Rodriguez, Bain, & Towns, Submitted; Rodriguez, Bain, Ho, Elmgren, & Towns, Accepted). In the case of symbolic forms, originally developed by Sherin (2001), the pattern under consideration is called the symbol template and the ideas assigned to the symbol template are called the conceptual schema. For example, consider a rate law, which has the following general form: rate = $k[A]^a$. The symbol template for this expression would be $\square = \square\square^{\square}$, where each of the boxes represents a term. The pattern of terms implies mathematical relationships and represents a combination of symbolic forms, such as coefficient (a constant or factor that adjusts the size of an effect), dependence (the magnitude of the value on the left is influenced by changing the values on the right), and scaling exponentially (a term raised to a value scales or tunes the overall magnitude). Generally speaking, graphical and symbolic forms derive their importance from their role in supporting reasoning about processes and phenomena (Becker and Towns, 2012; Kuo, Hull, Gupta, & Elby, 2013; Rodriguez, Bain, and Towns, Submitted; Rodriguez, Satntos-Diaz, Bain, & Towns, Submitted; Rodriguez, Bain, Ho, Elmgren, & Towns, Accepted; Sherin, 2001).

Methods

The participants for this study were sampled from a second-year undergraduate biochemistry course for life science majors in the spring of 2018. Students were given a $20 gift card for their involvement, and all aspects of this project were conducted in accordance with the guidelines of our university's Institutional Review Board. After the participants were tested on enzyme kinetics, we collected our primary source of data, which involved semi-structured interviews using a think-aloud protocol and a Livescribe™ smartpen (Linenberger & Bretz, 2012; Harle & Towns, 2013; Cruz-Ramirez de Arrellano & Towns, 2014). During the interviews the students were given a Michaelis-Menten graph (provided in Figure 1), which they were asked to describe. This prompt was intentionally open-ended in order to provide a general idea of students' reasoning. Students were also asked follow-up questions to make their reasoning more explicit and additional questions were asked to provide insight into resources students used as they reasoned about enzyme kinetics, such as ideas that are more explicitly emphasized in general chemistry (e.g., *What is reaction order? What are rate laws? How is that related to enzyme kinetics?*). Following transcription of the interviews, the data was coded using the graphical and symbolic frameworks, inductive analysis, and a constant comparison methodology (Strauss & Corbin, 1990).

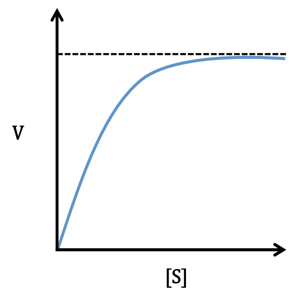


Figure 1. Michaelis-Menten plot provided in the interview prompt.

Preliminary Results

Following analysis we noted student use of mathematical resources was particularly common during (and in some cases isolated to) discussions involving rate laws and reaction order. Generally, students described rate laws in algebraic terms and discussed reaction order in a way that emphasized graphs as objects, affording only a surface-level understanding of the Michaelis-Menten graph provided. However, in some cases, students displayed reasoning that productively integrated mathematical resources and chemistry knowledge, affording a more complete understanding.

Rate Law as Symbol Template

Among the students that discussed rate laws, we observed that the students tended to reason algebraically, which did not productively support their understanding of the Michaelis-Menten model of enzyme kinetics. Looking at the "rate laws" drawn by Tim, Claire, and Alan, we can see there is an attempt to reproduce the rate law by mapping values onto a specific pattern of symbols, which is reminiscent of Sherin's (2001) symbolic forms. In this context, the students were focusing on the symbol template of the rate law and attempting to reproduce some variation of this pattern ($\square = \square\square^{\square}$). This was particularly evident in Tim's discussion where he commented that the rate law for a first-order reaction has two "boxes" (i.e., rate = k[A]), whereas the rate law for zero-order only has one (i.e., rate = k):

> "I think if I remember right, like k and then you can do it to like the first order here and then, there was a, yeah, so there was a rate or something was equal to the k to the first order … If I remember right … [the rate law] had two boxes for here, but I think zero only had one … because there's two, there's two things that are multiplied here, essentially, you have the enzyme and you have the substrate. And so for the rate you have the enzyme, I think if I remember right for first order you had something multiplied by something else … which would leave for me to think it's a first-order, first-order rate reaction."

Following his discussion of rate laws, Tim then stated that the reaction involving the enzyme and substrate must be first-order, because then the two boxes would be filled by the two reactants. Dorko and Speer (2015) observed a similar "box-filling" tendency when they analyzed calculus students' conceptions of measurement in the context of area and volume calculations, noting that students utilized the *measurement* symbolic form ($\square_{\square}$, magnitude and units), often without considering what values filled the boxes (e.g., 144π as adequate to fill both boxes, even though it represents a single magnitude value).

Graphs as Objects

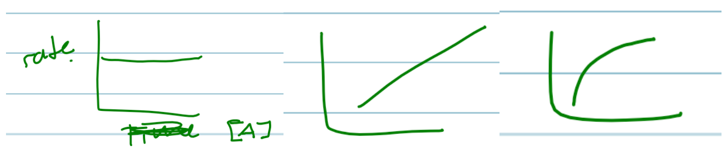
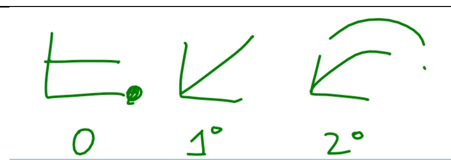
In our dataset the most common conception regarding reaction order involved the association of each order (e.g., zero order, first order, second order) with a particular graph. Eight students in our dataset described reaction order in a way that highlighted the connection between reaction order and graphical shapes, with five of these students explicitly drawing graphs to illustrate this connection. This is analogous to the observed student reasoning about rate laws in the previous section, although in this case the students were focusing on surface-level graphical patterns instead of symbolic patterns. We refer to this type of reasoning as viewing graphs as objects, which is distinct from graphical forms, because in this case the ideas being associated with the graphs are not mathematical in nature.

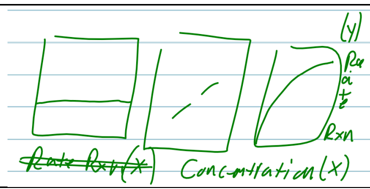
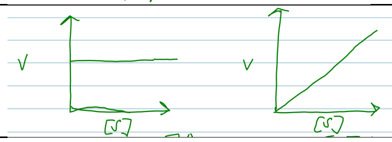
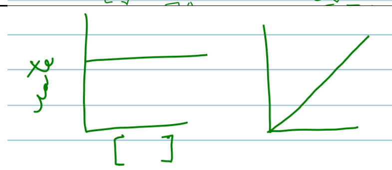
The discussions that accompanied the graphs shown in Table 1 were similar for each of the students, in which they concisely listed the shape associated with each order, focusing on surface features without thinking about the axes (all of the students that drew graphs did not initially draw axes, but some students labeled the axes after prompting by the interviewer, suggesting the salient feature for the students was the shape, and the axes were an afterthought). After discussing the graphical representations of order, the students often attempted to apply shape-centric thinking to reason about the order of the reaction represented in the provided Michaelis-Menten graph, a trend that was observed even for the students that did not draw a graph. For example, in the passage below Amanda discussed the graphs associated with each order, characterizing the Michaelis-Menten graph as a having the second-order "shape":

> "I believe that's first order, second order, and if it's linear then it's first order or something. If it's just a straight line, it's zero order. … I'm gonna take a straight guess and say it's second order [the Michaelis-Menten graph provided in prompt]. … Because it's curved, and it's ... an exponential … maybe it's a log function, something like that, but I just remember it from the picture that it might be a second order one."

Although Amanda did not draw graphs to illustrate her understanding, she verbally traced the shapes using reasoning that is consistent with the other students. Amanda's statement above also provides support for our characterization of students viewing the graphs as objects; in this case the student had an image in mind of the relevant shapes, with which she associated ideas.

Table 1. *Student reasoning about reaction order.*

Student	Written Work
Malcolm	
Karen	

Tim	
Claire	
Carrie	

Conclusions and Questions

The results discussed in this chapter focused on students' ability to connect the Michaelis-Menten model of enzyme kinetics to reaction order and rate laws, which are key tenets of chemical kinetics discussed and assessed in general chemistry (Holme and Murphy, 2012; Holme, Luxford, and Murphy, 2015). As mentioned by Schnoebelen (2018), retention of ideas in general chemistry is higher when concepts are reinforced throughout the undergraduate chemistry curriculum. However, although reaction order and rate laws were discussed in the participants' biochemistry course, they were not the focus of assessment, as is likely the case in other biochemistry courses. Since students study what is assessed, it is not surprising that only a couple of students were able to make the relevant connections, and it should not be assumed that students are making connections between content they are currently studying and content from previous courses (Cooper, 2015). Therefore, we stress the importance of instruction that not only explicitly connects course content (e.g., enzyme kinetics) to relevant concepts previously learned by students (e.g., chemical kinetics), but we also emphasize the role of assessment in student learning, asserting the importance of exams that prompt students to provide evidence they understand these meaningful connections. This work requires further analysis, with the following questions informing our next steps:

(1) What symbolic and graphical forms were productive for reasoning about this context?
(2) How can instruction better support students to make connections between chemistry concepts and mathematical representations?
(3) How do students make connections between the particulate-level mechanism and the graphs/equations used to model enzyme kinetics?

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Undergraduate Mathematics Tutors and Students' Challenges of Knowing-To Act

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Colleges and universities are increasingly providing drop-in tutorial assistance through institutions' learning or resource centers. In this study, we examine one-on-one mathematics tutoring interactions to discover how tutors naturally respond to student requests for assistance with knowing-to act, where a student may be familiar with a procedure, but not know-to use that procedure in the current situation. We contrast three 5-10 minutes episodes; in the first, the tutor appears not to recognize that the student knows-to. In the second, the tutor prevents the student from needing to know-to. In the final episode, a tutor incrementally narrows the vision of her student until the student knows-to.

Keywords: undergraduate mathematics tutoring, types of knowing

Introduction and Review of Literature

While great effort has been and continues to be exerted to study and improve classroom instruction of post-secondary mathematics, comparatively little research has focused on how students study and learn math outside of class. Increasingly, institutions are providing out-of-class assistance for entry level math courses through learning, resource, and tutoring centers (Bressoud, Mesa, & Rasmussen, 2015). These centers have the opportunity to design and implement tutor training, so there is reason to identify tutors' natural tendencies and discover how those tendencies impact student learning.

Studies of human tutoring interactions in disciplines besides mathematics have identified strategies tutors use in effort to assist students, including, but not limited to, direct instruction, error checking, questioning, and hinting (Chi, 1996; Roscoe & Chi, 2008). Examples from these studies show a strong reliance on direct instruction in many cases, which is not surprising considering that most tutors receive little, if any, training, and are not familiar with learning theory (Graesser, Person, & Hu, 2002). When tutors do avoid direct instruction, they tend to use hinting and questioning to guide students toward the tutor's own solution path (Hume, Michael, Rovick, & Evens, 1996; James & Burks, 2018), much like classroom teachers have been shown to use questions in a funneling pattern (Wood, 1998).

This study specifically examines tutoring interactions at a point where the student is needing to focus on the relevant features of a mathematical problem. Students' inattentiveness to particular mathematical features provides a considerable challenge for educators and researchers alike. In multiple areas of mathematics, it has been shown that students can have a solid foundation of specific principles or procedures, yet still be unable to access that knowledge in novel situations (Hoch & Dreyfus, 2005; Schoenfeld, 1980; Selden, Selden, & Mason, 1994). In proof construction, for example, Weber (2001) explains that students who were unable to prove had the necessary syntactic knowledge yet were unable to construct a proof until someone specifically pointed them to the salient facts.

Theoretical Framework

Mason and Spence (1999) refer to this elusive flexibility as knowing-to act. They argue that instruction focuses almost exclusively on knowing-that, knowing-how, and knowing-why, yet success in mathematics relies heavily on students knowing-to act in the moment, which requires an “awareness” or a particular “structure of attention” (p. 138). We use this distinction to identify episodes where tutors are assisting students with knowing-to act and we ask: How do tutors notice and respond to students’ ability to know-to?

Our identification involves assumptions on both the part of the tutor and the researchers. First and foremost, we are selecting episodes where we feel confident assuming the students know-*how* to do the things they may not know know-*to* do without the help of the tutor. For example, in one episode we assume the student is familiar with and has successfully applied the product rule in the past, so the focus of our exploration is his need to recognize the utility of the product rule in his situation, rather than his ability to apply it without error.

In identifying and analyzing episodes, we use a constructivist lens, believing each individual must construct their own meanings (Thompson, 2013). We recognize the tutor and student cannot know what is in the mind of the other. According to Steffe and Thompson (2000), each must create a model of what the other is thinking and react to the other based on that model, rather than what the other is actually thinking. Similarly, we as researchers cannot know what the tutor or student is thinking, so we must formulate models for how we conjecture that both the student and tutor are thinking about the mathematics and their interaction with one another.

We are intentional about viewing constructivism as a learning theory and not a prescriptive teaching method. We believe students can construct meaning for themselves, for example, while listening to a well delivered explanation from a peer tutor. We are not *evaluating* tutors’ responses or methods; rather, we wish to discover what peer tutors believe is helpful to students in a moment of not knowing-to.

Methods

The subjects of this study were eight undergraduate peer tutors at a large public university who work as drop-in tutors for the university’s mathematics department. The tutors had a variety of experience and differing amounts of training. As part of their in-service training protocol, the tutors were required to record a portion of their interactions with students using a Livescribe pen which captured both audio and video of their written work. Each tutor then selected one 5-10 minute episode to transcribe and reflect upon through a written self-evaluation and debrief interview with their supervisor. The interviews between tutors and supervisor were subsequently transcribed by the researchers and pseudonyms were assigned to the tutors.

This study identifies and analyzes tutoring episodes, rather than tutors, because individual tutors often take varying approaches at different times, even within the same short episode (Nardi, Jaworski, & Hegedus, 2005). For this study, we narrow our focus to episodes where the student is asking the tutor for assistance with an issue of knowing-to. To be classified as a knowing-to episode, the tutor must appear to be providing assistance in directing the student’s attention to salient mathematical features of their problem. For example, in one episode, the student must recognize that an expression is two functions multiplied together and decide to use the product rule to differentiate.

We attempt to compare and categorize three episodes by the tutor’s strategies and their intentions as well as the student contributions. While we may describe the tutor strategies and student contributions from the tutoring episode itself, we rely on the tutor’s transcription of the episode, the tutor’s written reflection, and the interview transcript to make conjectures regarding the tutor’s intention for their strategies. We classify the episodes according to the focus or vision of the student and tutor and contrast them based on who was deciding the next move and how the tutor responded to student. We do not claim the three episodes are exhaustive or representative.

Results

Case 1: Divergent Tutor and Student Vision

Case 1 provides evidence that a tutor may not recognize that a student knows-to act if what the student knows-to do is not what the tutor knows to do; that is, if their proposed solution methods differ. In this episode, the student asks for assistance in finding the area of the shaded region, shown below in Figure 1. As the dialogue progresses, we notice that the tutor, Jane, and her student are attending to different aspects of the geometrical figure.

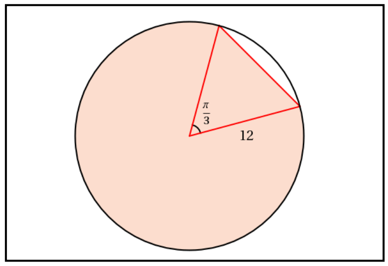

Figure 1: Find the area of the shaded region. (Stewart, Redlin, & Watson, 2016).

Jane: So what do you think are the formulas we're going to be using to go about this are?
Student: We'll use area of a triangle, and then area of whatever this is [area of major sector].
Jane: Yes, so we are on the right track. So the first is area of a triangle. So what's the formula?
Student: ½ ab sine theta.
Jane: Yep, that's correct. Okay, so that will give us the area of this triangle right here, correct? Okay, so we are going to need that. So we are also going to want the area of this [minor] sector because if we get the area of the sector and subtract off the area of the triangle it will give us the area of the non-shaded region. Do you see that?
Student: Yeah
Jane: So it's really area of the sector minus area of the triangle equals area unshaded. And then we have the area of the unshaded we can subtract off the area of the whole region…
Jane: What's our theta?
Student: Um, pi, oh wait, would it be this? [indicating the angle of the major sector]
Jane: …you are right; if we were doing the outer sector then we would use 5 pi over 3, but because we are looking at this sector with the triangle inside it, we're going to do the pi over 3. Does that make sense?

While Jane does not actually ask the student to suggest solution methods, the student's answers to Jane's prompts for formulas and calculations suggest that the student knows-to add the areas of the major sector and triangle. Jane, instead, knows-to find the area of the minor sector and subtract the area of the triangle to get the unshaded region, and then subtract the unshaded region from the circle. Jane does not recognize that the student knows-to because the student's proposed method differs from Jane's. It is not until the interviewer prompted Jane to consider it that Jane acknowledged that the student's path might be viable.

Case 2: Student Vision Unknown

Here we present evidence that tutors may eliminate the opportunity and necessity for students to know-to. In some of these situations, tutors interpret hesitation as not knowing-to. In others, tutors give direction without first giving students an opportunity to demonstrate whether they know-to. In the episode below, the tutor, Abby, asks for student input on the next move, and after a four second pause, explains what to do next.

Abby: So what is the first thing you have to do to take the derivative?
Student: (Ummm…awkward looking at me, indicating they're not sure)
Abby: Okay, so you notice how there's two functions, right? There's x to the fifth and three minus x to the sixth?
Student: Yeah.
Abby: Those are your two, so that means that you use the product rule, right?

During the interview, Abby explains, "…basically they're just like 'umm…'…I interpreted it as 'I don't know where to start on this…I decided to point out, like help them see, like there are actually two functions, like two things multiplied together, and that means that we have to do the product rule."

Later in the same episode, the student is attempting to find the zeros of the derivative. Rather than asking for the student to provide direction this time, Abby provides it herself.

Abby: Awesome. So we have this, and you said earlier that we set it equal to zero. So I'm gonna rewrite it…look good?
Student: Yeah.
Abby: Okay, so it's kind of hard to determine what the zeros are of this function whenever we're, like, adding something in the middle, sooo this is when you want to get things multiplied together.
Student: Okay.
Abby: And you do that by factoring.

Abby explains in her interview that she made this decision to save time, saying "personally, I was like, let's just guide them through this instead of like trying to get them to do it by themselves because that will be quicker." She is not suggesting she knew it would never have occurred to the student to factor, but that it would have taken him longer to realize it than her.

Case 3: Gradually Narrowing Student Vision

At the point where we pick up the episode below, Felicia is helping a student simplify $(1+\cos x)/(\sin x \cos x + \sin x)$. Felicia asks the student to provide direction for the session multiple times. At points where the student appears to be at an impasse, Felicia assists in the knowing-to process by incrementally narrowing the student's focus.

Felicia: …what do you think you're gonna do next?
Student: I kinda was stuck at that point.
Felicia: Okay! So let's look…so we're trying to get some similarity either in the top or the bottom, so that maybe we could cancel something out or just make this guy simpler, right? So, do you see any similarities or any way you could make things look a little bit simpler?
Student: Not really, no.
Felicia: No?
Student: Like the cosine on top and bottom maybe?

Felicia: Um, yeah, there's definitely that. We could try to work with that.... Is there a way you could make that denominator a little bit simpler?
Student: Sine x times cosine x plus sine x…shoot.
Felicia: Well, let's think about it this way. Is there anything that is similar to each of these two terms? (underlines each of the two terms in the denominator)
Student: Both of them have sine.
Felicia: Yeah! So maybe we could?
Student: Take a sine out.

Similar to Abby in the first part of her episode, Felicia first gives her student an opportunity to make the decision for where to go next. However, her response to the student's hesitation is quite different. Felicia seems to be operating under the assumption that the student does know-to. She even challenges the student's response of "Not really, no," when asked if he could make the expression simpler. When the student is stuck or suggests something unexpected, Felicia first suggests some general strategies. ('make it simpler,' 'cancel something out,') When that fails, Felicia narrows the field of vision; she directs the student's attention to *where* he can make it simpler or cancel something out.

Summary

The three episodes illuminate two important variables in tutors' strategies for assisting students with issues of knowing-to. First, tutors decide whether to provide an opportunity for the student to demonstrate whether they know-to. Abby, for example, asked her student how to proceed in the first part of her episode, while in the second part, she stated that factoring was appropriate without asking for input from the student. Although we have some evidence from Abby's episode that time limitations may encourage this strategy, we note that we do not know what motivates these different decisions in different situations and we reiterate that the same tutor can choose different strategies in different scenarios.

The second variable we recognize is how tutors respond to students' indications of their knowing-to or lack thereof. Students can provide solicited or unsolicited information indicating they know-to and tutors can build on or ignore this information. As we see from Jane, tutors may miss a student's knowing-to if it differs from their own. Students can also be silent or verbally claim to not know-to. We see from Abby's episode that a tutor may interpret silence or hesitation as the student not knowing-to. In contrast, as Felicia demonstrated, tutors may not accept a student's claim that they do not know-to. Felicia shows us that tutors are capable of gradually narrowing a student's focus to salient features in a way that still allows them to demonstrate knowing-to at various stages.

Discussion

Future research will provide additional cases and potentially refine those shown above. We seek input regarding the following questions.

- As we analyze more episodes and extend/modify our classifications, should we differentiate based on tutor strategies, tutor intentions, or student responses?
- What existing research, which examines similar phenomena between *instructors* and students, could we build upon in classifying these interactions?
- How do we use this information to design effective tutor training?

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From Friend to Foe to Friend Again:
Eliciting Personification of Pre-Service Teachers' Beliefs of Mathematics

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This paper reports about using eliciting personification (Zazkis, 2015) as a means to study pre-service teachers' (PSTs) beliefs. The method has the PSTs' create a character named Math and describe their relationship with the character. The authors analyzed 68 personifications from college sophomore PSTs' in an elementary math content course. At the end of the semester, the PSTs; revisited the assignment by describing a new character based on the math learned in class and writing a dialogue to themselves. At the beginning of the semester, the PSTs described math as having multiple personalities, out to hurt them, and having a relationship that fell apart throughout the years. The math described at the end was more compassionate, welcoming, and easier to understand.

Keywords: Teacher Beliefs, Narrative Identity, Pre-service Teachers

Introduction

The recently released Association of Mathematics Teacher Educations Standards for Teacher Preparation states that "All teachers, including well-prepared beginners, must hold positive dispositions about mathematics and mathematics learning, such as the notions that mathematics can and must be understood, and that each and every student can develop mathematical proficiency" (AMTE, 2017, p 2.7). A goal for mathematics classes for PSTs should be to try and align the students' beliefs that align with those positive dispositions about mathematics and mathematics learning. Studies (e.g., Philippou & Christou, 1998) have shown that teachers tend to come into these courses with negative dispositions about mathematics.

In this study, we focus on the beliefs of prospective mathematics teachers enrolled in an elementary mathematics content course taught in a mathematics department. The aim of this study is to see if the method of eliciting personification provides insight into PSTs' beliefs that other metrics fail to capture. The results from the assignment were used in several ways throughout the semester to study the PSTs beliefs. We found that the idea proposed by Zazkis (2015) elicited a complex view of beliefs that are not captured by other methods.

Literature Review

When defining beliefs, we are using a definition from Phillips (2007) that treats beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes." (p. 259). Thompson proposed that a teachers' belief about mathematics influences how the decisions they make in the classroom while teaching and they could possess either a conceptual orientation or a calculation orientation towards mathematics (Thompson, Philipp, Thompson, & Boyd, 1994). Teachers who are conceptually oriented focus on engaging students in complex activities with a goal on developing problem-solving strategies or a deeper conceptual understanding while a calculation orientated view may emphasize calculations and procedures more.

Researchers have used several methods such as questionnaires where the participants respond to a series of questions and rate whether they agree with the statement on some scale. An example question could be "if students learn math concepts before they learn the procedures, they are more likely to understand the concept", which the participant can rate their agreement with the statement on a five-point scale from strongly disagree to strongly agree. Di Martino and Zan (2010) criticized this approach, as it uses questions that are entirely positive or negative. They argue that studying beliefs in this way is limiting by ignoring important factors such as the emotions of the individual and proposed a model that incorporated a students' vision of mathematics, emotional disposition, and their perceived competence to mathematics. Other researchers (e.g., Drake, 2006) explored PSTs' beliefs and identity by having them construct mathematical autobiographies. This approach produced a more multi-dimensional view of their beliefs as they described their changing relationship with mathematics by describing influential moments related to their experiences in mathematics.

Zazkis (2015) described a process to study PSTs' relationship with mathematics he called eliciting personification. The process entails having the PSTs' give life like attributes to a character called Math. The participants described this character and wrote a dialogue between themselves and Math. In this paper, we will use eliciting personification to (a) elicit PSTs' beliefs about mathematics, (b) incorporate their stories into the course, (c) have the PSTs construct new narratives about the mathematics they learned during the course, and (d) respond to their original narratives.

Methodology

This study uses the eliciting personification method (Zazkis, 2015) to survey 68 sophomore PSTs enrolled in a semester long elementary mathematics content before they begin any of their teacher education classes.The course is taught out of a mathematics department with a focus on K-5 mathematics, counting, the operations, and rational numbers. Course material focuses on the standards for practice identified by NCTM. PSTs' engaged in open-ended problem solving, analyze the solutions of school-aged children through watching videos (e.g., Cognitive Guided Instruction, the Video Mosaic Collaborative), and look at written work. At the start of the semester, the PSTs responded to a slight modification of the assignment prompt posed by Zazkis (Figure 1).

At the end of the semester the students revisited the personification assignment in two ways (Figure 2). The first part has the PSTs' write a new personification, but about this character called "New Math" based on the mathematics we explored throughout the course. Then the PSTs revisited the assignments they submitted at the start of the semester through the lens of a future mathematics teacher and imagine that a student submitted that assignment to them. They wrote a script for a dialogue they would want to have to this student and what they would like to tell them.

The following assignment is designed to help you explore your own relationship and personal history of mathematics through personification. Be sure to put a good amount of thought into your mathematics character. Being honest and thoughtful with this assignment will be more useful for you with regards to this class and your future as an elementary school teacher.

Who is Math?

Write about who Math is (approximately 300 words). This section should address things such as: How long have you known each other? What does Math look like? What does Math act like? How has your relationship with Math changed over time? These questions are intended to help you get started. They should not constrain what you choose to write about.

Dialogue with Math

Now that you have created a Math-character, write a script for a dialogue that you might have with Math (300-500 words). Here are some examples of questions you might want to think about. As before, these questions are intended to help you get started and should not constrain what you choose to put in the dialogue: Are there things you want to tell Math that you have not said before? Are there things that you think Math wants to tell you? How did you and Math get along in a recent mathematics course that you took? Was the relationship during recent courses different from the relationship you had years ago? What do you suppose that Math turned out the way they did? Do you think Math can ever change? Do you want Math to change? What kind of relationship would you like to have with Math?

Figure 1: Beginning of the semester personification prompt from Zazkis (2015).

7. (20 points) At the beginning of the semester you worked on an assignment that had you give life to a character called Math. For this question I would like you to re-visit the assignment. Your write-up should be approximately 300-500 words. Be sure to put a good amount of thought into your assignment. Being honest and thoughtful with this question will be useful for you with regards to your future as a teacher. Your write-up should address the following:

   (a) In this class I tried to introduce a different way to think about Mathematics. I would like you to write about this "different" Math. What does this Math look like? What does this Math act like? How, if at all, is this Math different than the one you knew before this course?

   (b) Re-read the assignment you submitted at the beginning of the semester (if you are doing this during the exam and not at home - I have your assignment, get it from me). Look at your assignment through the lens of a mathematics teacher. If a student turned in this assignment to you, what would you say to him or her? Write a script for a dialogue that you might have with this student (aka an August 2017 version of you). Are there things you want to tell that student?

Figure 2: End of the semester assignment prompt.

To examine how elementary PSTs' conceptualize mathematics we utilized a narrative identity lens (Kaasila, 2007). Narratives create opportunities to view how the narrator constructs the world around them (Lutovac & Kaasila, 2011). According to Sfard and Prusak (2005), teacher identities are just collections of narratives that can inform future actions. Narratives provide a conduit in exploring ones' own identity and the identities of the personalities created within the narratives. Researchers have also shown self- exploration of created narratives to have positive impacts on mathematics related anxiety (Kaasila, Hannula, & Laine, 2012; Lutovac & Kaasila, 2011; 2014).

The authors analyzed the submitted write-ups by the PSTs' using a thematic analysis (Braun & Clarke, 2006) approach to open-code and develop themes related to beliefs about mathematics through the narrative identity lens. Throughout the process, the authors discussed their codes and themes and talked about any differences they had. In this paper we are presenting common themes across the assignments from the start of the semester and how, if at all, those themes shifted by the end of the course.

Results

Forty-eight of the participants wrote about the first time they met their Math character and of those, 45 described first meeting Math during school, with 42 saying they first met Math in Kindergarten and three students in second grade. Two of the other students mentioned meeting Math as soon as they began counting and one student remarked that "informally, they knew Math all their life, but I was formally introduced to Math when I attended school". A majority of the assignments also mentioned engaging with or avoiding Math in a school setting while others implied Math had not existed outside the school setting/environment.

A second common theme that arose when describing Math was that it had multiple personalities with contrasting personalities. One student described their math character as "kind, but also stern" and another saying "one day Mathius wants to be friendly and cooperate with me and the next minute he wants to cause trouble and make my life difficulty…I have seen these rapid personality changes in Mathius ever since I have known him". Common amongst the described multiple personalities was that one side was friendly towards that and wanted to help while the other personalities were mean and out to intentionally hurt them. A positive attribute assigned to Math was that he was logical, but also that there was only one way to understand his solutions which seemed to give some students comfort as demonstrated in the statement, "....although it may not seem like it, there is always a solution to every problem, I just have to figure it out". Other themes the PSTs' wrote about were growing apart from their character Math and wanting to rebuild their friendship. Algebra, geometry and calculus were described as points where their relationship with Math deteriorated due to the increased complexity in content as many of them claim in their reflections.

In their dialogues, some PSTs' addressed their failing relationship with Math. One student had math telling them that "Your lack of effort is why you didn't do as good as me in Algebra" to which they responded "That's very true. I should have managed myself better to do great in the class". Several students expressed communication issues with Math, such as "In fourth grade Math completely changed on me. No longer did we communicate the same. My teacher decided that Math would no longer speak to me the same way. I was forced to speak to Math in English. Our whole friendship was based on how we had no language barrier in Spanish. We lost the friendship I relied on to help me reach the goals I had placed for myself".

In their second assignment, the "new Math" character described by the students was different than the Math they knew growing up. The student who struggled with being forced to speak to Math in English described this new character as "This Math wasn't trying to confuse me, but help guide me to help others. This Math wasn't out of my grasp this time. Math was almost speaking the same language again…Math was a lot friendlier than I remember and once I gave him a fair shot it wasn't hard to get along". A majority of the participants who described Math as having multiple confusing and conflicting personalities, described the new character as "straight forward", "more approachable", "the old math scared me causing not to speak up and ask a question…This math is different, more patient, understanding and has taught me to look at this subject in a different light". Others described this new Math as having "soft, king eyes that glow with radiance and happiness", "more approachable", "has a lot of very different layers that somehow all come together in a cohesive way", "definitely kinder than the 'mean' one that I portrayed earlier", "lives with me everyday" and it "welcomes me with open arms and wants to help".

When assuming the role of the teacher and responding to their beginning of the semester-self, the students responses included “Always keep trying, never give up on a problem just because you don’t understand it. Ask questions, because you might not be the only student who does not understand what they just learned. I learned that there are multiple ways to solve a problem. You don’t have to solve them a certain way.”, several students suggested that they “don’t be afraid to ask for help”, to “not be discouraged and that a lot of people experience the same problems”, and to focus on “try and understand the reasoning behind how math works rather than trying to just get the right answer”.

Discussion

At the start of the semester the PSTs in the study exhibiting several negative beliefs related to mathematics. PSTs’ that engaged in the task viewed mathematics as an activity that occurred only in schools. They first met math when they entered school and all their described interactions, both positive and negative, occurred in a school setting or by engaging in tasks related to schooling (e.g., homework). They described a volatile relationship where math was alternating between a positive relationship and one which was out to hurt or punish them. They identified mathematics having a logical structure as a positive attribute, but as a result they viewed math as having only one way to approach a problem which was a negative attribute. When they tried to understand why their relationship failed they blamed themselves by saying they did not try hard enough and claimed all they needed to do was try harder indicating that their lack of persistence was the reason for their diminished relationship with Math.

After a class focused on teaching mathematics through a more conceptually based approach, the PSTs’ vision about how the mathematics we talked about in class was a stark contrast to their view of mathematics growing up. Their focus was on a more approachable mathematics that wants to help the students understand. Instead of blaming themselves for not working hard enough, they instead realized that they could ask questions of others to help understand the content and that math was “kind" and wanted the best for them. They also shifted towards more of a conceptual orientation view of mathematics as they focused on understanding and multiple ways to approach a problem instead of rote procedures with an emphasis on finding a solution. They also mentioned the overall utility of the mathematics they were learning and how they would use this “new Math” in their classrooms.

Being able to read their stories gave the authors a unique opportunity to engage with their PSTs’ and provide a way to frame issues that came up throughout the class. The personifications produced painted a complex view of their relationship that is not captured by belief assessments. It also created an opportunity for PSTs’ to re-frame their experiences and produce more productive ways of addressing the issues they identified along with implications for future teaching. Going forward, research is needed to focus on how to leverage activities such as this to help build positive identities and dispositions towards mathematics for future teachers.

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Deaf and Hard of Hearing Students' Perspectives on Undergraduate Mathematics Experience

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Deaf and hard of hearing (D/HH) students face many challenges in the study of undergraduate mathematics. Unfortunately, minimal literature exists in this area, evidencing the need for further research. Through five qualitative survey responses from D/HH students, we identified common themes of concern in addition to a number of specific struggles (and a few successes) encountered by each of the respondents in their own undergraduate mathematics courses. From these students' experience, we can identify further areas of research with the goal of developing new educational tools for mathematics instructors with deaf or hard of hearing students. In doing so, we can help give equal opportunity to mathematics students regardless of their level of hearing.

Keywords: deaf, hard of hearing, student perspective

According to Walter (2010), around 60% of deaf/Deaf/Hard-of-Hearing (D/HH[1]) high school graduates enter some form of higher education, but only around 23% of D/HH ages 25-64 have graduated from college. This statistic makes one wonder about how undergraduate mathematics education for D/HH students is factored, since mathematics is predominantly a general education requirement. We add on to the undergraduate mathematics education literature by exploring D/HH students' experiences in mathematics: What roadblocks and successes happened in their mathematics education?

Background Literature

When analyzing how D/HH students experience mathematics at the undergraduate level, it is important to understand the different K-12 educational backgrounds they come from. Three common educational backgrounds for D/HH students are center schools (schools dedicated to D/HH students), integrated classrooms in main-stream schools (D/HH students in class with hearing peers), and self-contained classrooms (D/HH specific classrooms in mainstream schools) (Kelly, Lang & Pagliaro, 2003). D/HH students entering higher education who take math courses are typically in integrated mainstream classes. This can be a disadvantage to students who did not have the same mathematical experience as their hearing peers before entering their first undergraduate mathematics course. One study has shown that students in self-contained K-12 classrooms and center schools are exposed to fewer discrete mathematical concepts (Pagliaro & Kritzer, 2005); another has shown "teachers of [K-12] deaf students continue to place relatively less emphasis on the development of critical thinking, reasoning, synthesis of information, and other essential skills needed for effective problem solving" (Kelly, Lang & Pagliaro, 2003, p. 116). These teaching differences provide insight into D/HH students' mathematical performance and reasons behind the need of accessibility accommodation in educational settings.

Even when it comes to standardized testing for D/HH students, there is a need for accommodation. The SAT-HI was created with the intention of measuring D/HH students'

[1] It should be noted that the term "deaf" refers to a person who does not hear; "Deaf" refers to the identity of a person who is typically proud to be Deaf and be a part of the Deaf community.

knowledge more accurately than the SAT taken by their hearing peers. These accommodations include translation of testing materials into ASL and screening the students for the correct grade-level test for each section (Qi & Mitchell, 2011). These accommodations are made mainly due to the English barriers D/HH students face. Over 95% percent of deaf children are born to hearing parents (Mitchell & Karchmer, 2004), and many of these children miss vital opportunities as a child for developing language acquisition (Spencer & Harris, 2006). This late language acquisition follows D/HH children through their education and shows itself through testing performance.

A study that analyzed periodic test results from the Stanford Achievement Test for D/HH students from 1969-2003 demonstrated that while D/HH students' standardized testing has shown to be improving over the decades in certain areas, in 2003 the median grade level equivalence for mathematical procedures has decreased to right above a sixth-grade level for students who are 18 years old. (Qi & Mitchell, 2011). This same study showed that the median test scores in 2003 for mathematical problem solving was below a sixth-grade level and reading comprehension was below the fourth-grade level in students at the end of high school. Reasons for lower reading comprehension include the mode of acquisition or the way in which someone learns the meaning of words (Wauters, van Bon, Tellings, & van Leeuwe 2006) and mathematics ability has been shown to stem from "more restricted opportunities for incidental learning" (Kritzer, 2009, p. 418). While many might be testing at a lower grade-level than their hearing peers, there has been an increase of D/HH students entering higher education (Walter, 2010).

Access to higher education for deaf and hard of hearing students has been promoted by the passing of laws such as the Vocational Rehabilitation Act in 1973 and Americans with Disabilities Act of 1990. Both aided in the increase of admission by prohibiting students' rejection by reason of their disability. New admissions have increased the number of D/HH students in postsecondary education, and, because of this, there is a call for a wider range of accommodations. These accommodations tend to be in the form of interpreting services such as a physical interpreter or speech-to-text services. Coming into higher education, D/HH student barriers are addressed by the accessibility resources available to the campus, but perhaps are less informatively addressed by the instructor (Lang, 2002).

Methods

Our methodology for this study was conducting a qualitative online survey to determine some of the first roadblocks or successes D/HH students face in undergraduate mathematics. The reason we chose to conduct a survey for qualitative research was for a few reasons. There is a language barrier between researcher and participant that couldn't be accommodated for due to a lack in financial resources. A two-way in-person ASL interpreter can run for up to $145 an hour. While ASL interpreter over-the-phone rates are cheaper, this would pass the financial obligation onto the participant if they do not qualify for government-funded video phone access. In future research, in-person interviews will be preferred.

The survey was an anonymous online survey consisting of six open ended questions and two multiple choice demographic questions. We sent a survey link via email or in closed Facebook groups to prospective participants that fit our criteria for the survey. The initial criteria for the survey was that the individual identifies as Deaf/deaf or Hard of Hearing and has taken a mathematics course at the undergraduate level. We allowed the survey to be open for all ages over 18; this variety in age of participants can help to establish a roadblock timeline to analyze the evolution of difficulties or successes D/HH students may face in an undergraduate mathematics course. We then used open coding, specifically structural coding, to label using themes (Namey, Guest, Thairu, & Johnson, 2008; Saldaña, 2009).

Results

Table 1. Participants' Background

Participant	Undergraduate Mathematics Courses Taken	Identification	Age Range
Student 1	"Survey of Mathematics 1, Survey of Mathematics 2, Statistics and Probability at [Medium Undergrad University]"	Deaf/deaf	35-44
Student 2	"pre-calculus, calculus I, calculus II, calculus III at gallaudet university calculus I at [Midwest Community College] (audit)"	Deaf/deaf	25-34
Student 3	"I took Data Analysis and College Algebra through [Small Midwest State University]"	Deaf/deaf	18-24
Student 4	"I have taken Elementary Statistics at [South Community College] and College Algebra at [SCC] as well."	"I am culturally Deaf with a classification of Severe on the Audiogram"[2]	18-24
Student 5	"I took college algebra, statistics, and quantitative methods at [Large Midwest State University]"	Hard of Hearing	18-24

In table 1, all five participants and their backgrounds in mathematics education are displayed. When asked about their experience in those courses, four out of five of the students responded with negative descriptions: "a little difficult", "challenging", and "awful". Student 4 went into detail about receiving the letter grade D in College Algebra, and how that grade was out of character: "this is the only D I have received in College, other than that I have no C's, 3 B's, and 23 A's". The student who did not specify a negative experience, Student 3, mentioned their "experience was okay… I made an A in both courses and used CART (Communication Access Real-time Translation) as an accommodation."

Three out of five respondents mentioned struggles related to the instructor writing notes on the board with their back turned. Student 4 stated, "Not only is it hard to see what steps are happening until they stay away [from the board], but I miss almost all their explanation when their mouths are facing the board. I confronted my Physics professor about this & she wish I had told her sooner." Student 1 mentioned there was "not enough of writing problems on the board… [there was] No tutoring support available due to conflict in interpreter schedule and tutoring hours."

[2] An audiogram is a graph that shows hearing test results. A classification of "Severe" is testing 70 to 90 dB higher than normal (NHT Staff, 2014).

In terms of successes, four out of five of the respondents mentioned their grade as a success in their courses: "I had the highest grade," "I made an A in each class," and "…I learned and survived college algebra." Student 4 elaborated on their grade they received in a physics class, and although it is not considered a mathematics course, this participant felt it was important to include in reflection of their undergraduate mathematical experience:

> Earning an A in Physics was greatly credited to my classmates who assisted me & the TA who gave tutor sessions. Nonetheless, I am an extremely visual learner and people who taught me visually helped me advance the most. When people try to explain things with words with no visual aid it's much harder for me to grasp the concept.

Student 1 stated "taking the class with a deaf friend" as an example of a success in the classroom. Two out of five of the students mentioned successes related to online activities. Student 5 recalled having online homework assignments: "we were required to purchase the online homework assignments that explained things with captions and had you practice problems. That is how I learned and survived college algebra." Student 1 mentioned "taking the math course online with ALEKS program" as a success, but then went on to say, "It's not the best. I still struggled…" Finally, Student 5 stated, "Anytime the professor can use one of those projection things that allows them to face the class while working out the problem greatly helped."

Discussion

D/HH Experiences Related to the Course

The main goal of the survey is to determine some baseline difficulties and successes for D/HH students in an undergraduate mathematics environment. These open-ended surveys can assist us in generating more research questions. In this survey, we inquired specifically about the participants' difficulties and successes. First, we included a neutral question about experience to give the respondent a chance to give their initial thoughts about undergraduate mathematics. When asked, 4 out of 5 of the students responded with negative mathematics experiences; this seemed to be common to students regardless of D/HH classification (Betz, 1978).

Three responses specifically mentioned struggles in the course directly related to the instructor facing the board. One of the other responses mentioned under-utilization of the board for presenting materials. Student 5 suggested professors use a projector while teaching; this will still give the students the visualization of the material being learned while keeping the professor turned to the class. If the professor makes a point to face the students while explaining the material, this will help the students who rely on lip-reading as a mode of receiving information. This simple change (where possible) can make a difference in D/HH students learning environment as well as hearing students who learn more efficiently in a visually-engaging classroom.

Another theme that stood out was related to the instructor's speech. Even Student 1, who mentioned having an interpreter for the class, mentioned the professor "spoke through the lesson fairly quickly". For a student that has an interpreter in the classroom, they are relying on not only their own understanding of the material, but the interpreter's relay of information. Receiving information through a secondary perspective will provide its own challenges, and for students who have an instructor that goes over new material quickly, this can heighten that roadblock.

When asked about successes in undergraduate mathematics, four out of five of the respondents mention successes related to their grade in the class. Course grade outcome is unrelated to their deafness, but external motivation and rewards are still considered a large goal with many students. While only one student specified visual aids helped to bring them success in the classroom, online-related responses such as ALEKS add a visual element to the math course. Student 5 mentioned having captions with the online assignments which can help with understanding. Typically, with online courses, students can also dictate the pace at which the materials or assignments are presented, giving them more control over the communication of the course information. Student 3 mentioned using CART as an accommodation, which can help the student avoid missing information when the instructor's back is turned. It also allows more time to read the information on-screen rather than watching an interpreter and taking notes simultaneously.

Representation of D/HH Students

There was only one student, Student 2, who took any type of calculus class in this survey. It should be noted that this respondent took calculus classes at Gallaudet University, the only university designed for teaching D/HH students. Although this is a small sample, this agrees with the previous literature on D/HH higher education statistics (Walter, 2010). Her experience was "challenging as [she] was the only female in most classes." She took all of her non-audit classes at Gallaudet University. The female student population has consistently remained the majority since 1999 according to Gallaudet's enrollment records available online (Gallaudet University Office of Institutional Research, 2018). Since calculus is a required class for most STEM majors, her answer raises questions of D/HH female underrepresentation.

Conclusion

This paper outlines individual D/HH students' initial perspective on their undergraduate mathematical experience. Challenges identified in this study include breaks in communication between instructor and student, speech patterns of the instructor, and possible underrepresentation of D/HH female students. These challenges can be addressed by accessibility resources in higher education institutions and regarded by mathematics instructors seeking information related to teaching these students. Successes identified in this study include visual aids such as online-related elements in the course, usage of the board, and D/HH specific accommodations. Further research should be done to determine potential effects of a math course taught in visually-stimulating environment; developing this environment requires a composition of different visual elements in and out of the classroom similar to the ones mentioned previously. There should also be more exploration into the pedagogical effects of an instructor explaining materials while facing away from the classroom. This study gives direction for future research related to D/HH undergraduate mathematics students whose experiences we believe should be addressed more thoroughly.

Questions for the Reader

1. What pedagogical actions could be taken to create a more visually-stimulating classroom?
2. What pedagogical actions could be taken to lessen communication breaks in the classroom?

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Examined Inquiry-Oriented Instructional Moves with an Eye Toward Gender Equity

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When considering undergraduate mathematics education, gender equity is an ongoing issue and it has been suggested that inquiry-based instruction could make classes more equitable for men and women. In this study, we analyze data from 42 undergraduate instructors and courses and 681 students in the context of inquiry-oriented instruction in either abstract algebra, differential equations, or linear algebra. Specific instructional units were video recorded, watched, and coded to see how teachers distributed opportunities to participate in whole class discussion, how these opportunities were taken up by students, and what teachers did with student ideas. Mathematically substantial opportunities were not distributed equitably between men and women, which was consistent with inequitable student participation observed. Further, instructors tended to leverage women's ideas less than men's ideas when building on formalizing students' mathematical contributions.

Keywords: Inquiry-oriented instruction, Gender, Equity, Whole class discussion

Many teachers use direct instruction that requires rote memorization and, thus, does not support student understanding of mathematical concepts (Quan-Lorey, 2017). This plays a role in engagement and comprehension in undergraduate mathematics courses if students' primary experience with mathematics is through memorization or procedural methods (Chang, 2011). Alternate teaching methods, like inquiry-oriented instruction (IOI), can be useful in prompting students to think critically and immerse themselves in the mathematics they are learning. IOI is a type of inquiry-based learning (IBL), a student-centered method of teaching revolving around "ill-structured but meaningful problems" (Laursen, Hassi, Kogan, & Weston, 2014, p. 407), involving the use of novel, problem-solving tasks that require students to be engaged and active learners (Rasmussen & Kwon, 2007; Kuster, Johnson, Keene, & Andrews-Larson, 2017). These tasks usually involve multiple solution methods, require students to make connections, and call for the use of problem-solving skills. As students inquire into mathematics, teachers inquire into students' reasoning so it can be leveraged in classroom discourse to create shared understandings that can then be formalized mathematically (Rasmussen & Kwon, 2007). Students' work on tasks is leveraged in whole class discussions where students must explain and justify their reasoning whether through their teacher's request or without prompt. IOI has been associated with improved student outcomes (e.g. Rasmussen & Kwon, 2007; Bouhjar, Andrews-Larson, Haider, & Zandieh, 2018). Laursen et al. (2014) found that IBL improved self-reported cognitive, affective, and collaborative gains in all students and leveled significant differences in cognitive and affective gains that existed between women and men in non-IBL courses. However, IOI does not guarantee an equitable distribution of opportunities to participate and engage in mathematical discourse. Our study examines this issue in 42 undergraduate mathematics classes by exploring the following research questions:

1. How did teachers distribute opportunities for students to contribute to whole class discussion, and how did this differ by gender?

2. How were these opportunities to contribute taken up by men and women?
3. In what ways did instructors leverage contributions from women and men?

Theoretical Framework

Laursen et al. (2014) argue that IBL "leveled the playing field by offering learning experiences of equal benefit to men and women" (p. 412). Johnson, Andrews-Larson, Keene, Melhuish, Keller, and Fortune (2018) did not find this to be true, as results in their study showed that men benefit more from IOI as evidenced by significantly different performance of men and women. This difference in findings leaves questions: Does IOI equally benefit men and women? Does it even the playing field? Does it disproportionately advantage men?

We follow Leyva's (2017) argument that gender differences in mathematics are socially constructed and Black's (2004) argument that teacher-student interactions and teacher expectations can shape students' identities and participation in the mathematics classroom. Esmonde (2009) also states that identity development in mathematics in crucial when considering equity. This suggests that a focus on teacher-student interactions will help future research concerning identity development and, thus, equity. In our study, we want to examine interaction patterns in the classroom to better understand gender-based differences in students' experiences in hopes that this will offer insight into differences in outcomes.

Data Sources and Methods of Analysis

Our data comes from a broader NSF-funded study focused on providing undergraduate instructors with support for teaching linear algebra, abstract algebra, and differential equations in inquiry-oriented ways. This analysis focuses on video data of 42 instructors teaching units that varied in length from about 2-4 hours of instructional time. In these videos, a total of 681 students were observed; 452 of these students were identified by coders as men and 229 were identified as women. In this analysis, coders relied on visual and audio cues (e.g. voice, clothing, names or pronouns used) to infer the gender of students. As a result, all claims are based on researchers' binary interpretations of students' gender, a limitation of our study.

Table 1. Codes, subcodes, and subcode definitions

Code	Subcode	Definition
Solicitation Method – (how is speaker selected)	Group	Instructor calls on a group and a particular student speaks
	Individual	Instructor calls on a student by name
	Volunteer	Instructor calls on a student volunteering to talk
	Random	Instructor uses randomization to identify a speaker
	Not Called On	A student interjects without being called on by instructor
Teacher Solicitation (question type)	N/A	Teacher does not ask the student a question
	Other	Teacher asks a general question (e.g., "What did you think?")
	What	Teacher asks a student to read part of a problem, recall a fact, or give a numerical/verbal answer
	How	Teacher asks for a student's solution method
	Why	Teacher asks why something is true/false
Student Talk	Other	Student asks a question or says something nonmathematical
	What	Student reads part of the problem, recalls a fact, or gives a numerical/verbal answer to a problem
	How	Student explains solution method

	Why	Student explains why something is true/false
Teacher Evaluation	N/A	Teacher does not respond to the student's contribution
	Revoice	Teacher repeats student contribution
	Evaluation	Teacher explicitly says the student is correct/incorrect
	Elaborate	Teacher expands on or formalizes the student's idea
	Follow-Up	Teacher asks a new question based on the student's contribution and a new student responds

To examine how teachers distributed opportunities for students to participate in whole class discussion, we used Reinholz and Shah's (2018) observation tool, Equity Quantified in Participation (EQUIP), as a basis for our coding scheme and rules. We refer to our unit of analysis as a *sequence* of talk, where a sequence starts when a new student speaks and ends when another student speaks. With this definition, any length of interaction between the teacher and student is coded as one sequence. On the other hand, if two students are having a conversation, then a new coded sequence begins each time a student speaks so this situation would create many back-to-back lines of code. In this report, we draw on four EQUIP codes (Reinholz & Shah, 2018), given in Table 1. Solicitation Method and Teacher Evaluation were modified for this study, to capture greater nuance in how teachers used student thinking.

Interrater Reliability

There was a total of 104.8 hours of video; and 20% of these videos were double-coded. The coding team consisted of three graduate students. One was the *master coder*, who all of the other students were compared against. Videos were assigned randomly to the three coders, each of whom coded approximately one third of the data. The coders completed double-coding in multiple phases, discussing the results after each phase. Once all videos were double-coded to acceptable reliability (at least 80% agreement on each code), the coders individually completed the remainder of their videos. The coding team met regularly to discuss coding issues that arose to maintain consistency. To compute interrater reliability, we used Krippendorff's alpha (Hayes & Krippendorff, 2007), which is a generalization of Cohen's kappa. An alpha value was calculated for each main level code and each non-master coder; all of these values were over 0.8, which is considered good reliability, the highest category that can be achieved (Carletta, 1996).

Equity Ratios

After all the videos were coded, we used R statistics to aggregate all occurrences of codes and subcodes, and computed equity ratios, which is a ratio of the actual participation of a group to the expected participation of a group based on the demographic composition of the class (Reinholz & Shah, 2018). For instance, if a class was comprised of 40% women, the *expected participation* would be 40% of whole-class talk. An equity ratio less than one means that the observed group is underrepresented (compared to an *equal* classroom), a value greater than one means overrepresentation, and a value equal to one means that the participation of the observed group is proportional to the group's representation in the population (e.g. mathematically equal). While equality is not the same as equity, research shows that underrepresented populations tend to receive less than a proportional share of participation opportunities, so equality can be used as a baseline to move toward equity (Reinholz & Shah, 2018). As outside observers we refrain from describing participation as equitable but can identify participation that is *inequitable.*

Preliminary Findings

For our analysis, we examined how teachers distributed opportunities to participate in whole class discussion by first looking at who teachers called on and then by what kinds of questions they asked, disaggregated by gender. We then considered the nature of student contributions and what teachers did with these contributions, also disaggregated by gender. When organizing and analyzing findings, we look at the speaker selection and the content of interactions teachers have with men and women. We found that when teachers called on students individually or by group, men and women responded at rates comparable to their representation in the population, but this was not the case when teachers called on volunteers or allowed students to speak freely. Overall, teachers asked women less mathematically substantial questions and used women's ideas less when formalizing mathematics. We support our claims by using gender equity ratios to quantify and compare the kinds of questions instructors asked, the kinds of contributions students made, and what teachers did with those contributions.

How Teachers Distribute Opportunities to Contribute to Whole Class Discussion

We organize our findings about teachers' distribution of opportunities to participate to highlight two key aspects of this phenomenon: how they select a speaker and the kind of question they ask. Equity ratios (ERs) for how teachers selected speakers (Solicitation method) and the kinds of questions they asked (Teacher Solicitation) are shown in Table 2.

Table 2. Equity ratios for opportunities for men and women to speak given by teachers

	Solicitation Method: Called on...				Teacher Solicitation: Question Type				
	Group	Individual	Volunteer	Not	N/A	Other	What	How	Why
# Sequences	147	374	372	2545	1295	515	1201	123	303
ER Men	1.02	1.00	1.12	1.07	1.05	1.04	1.09	1.08	1.10
ER Women	.95	1.01	.77	.86	.91	.92	.82	.85	.80

**Note:* Subcodes are organized so that our view of the mathematical rigor of each increases from left to right.

When teachers call on individuals or groups, women participate relatively proportionally to their representation as evidenced by equity ratios of 1.01 and .95, respectively. We interpret this to mean that teachers are treating men and women relatively equally when calling on students by name, and that when teachers call on a group, men and women tend to speak proportionally to their representation in the population. Contrarily, women are much more underrepresented when the teacher asks for a volunteer (ER .77) or in instances where students freely interject (ER .86).

The equity ratios for question type broadly suggest that in interactions with women during whole class discussions, teachers ask mathematically substantive questions (what, how, why) at disproportionally low rates (ERs < 1). We note that women received N/A (a student spoke without the teacher asking a question) and Other category questions (e.g. "What do you think about this) at considerably more equal rates.

How Opportunities Were Taken Up by Students

When women took opportunities to participate in whole class discussion, they were contributing mathematically substantive ideas (What and Why) at underrepresented rates in whole class discussion, as evidenced by the equity ratios shown in Table 3. Interestingly, *how* contributions (which are likely more procedural in nature) are distributed relatively equally between men and women. The link between Student Talk and Teacher Solicitation is also notable as student responses tend to be linked to the teachers' questions.

Table 3. Equity ratios for how students respond to instructors' prompts

	Student Talk			
	Other	What	How	Why
# Sequences	778	2099	209	351
ER Men	.96	1.11	1.02	1.09
ER Women	1.07	.79	.97	.83

**Note: S*ubcodes are organized left to right from least to most mathematically substantive student talk.

What Teachers Did with Student Contributions

Teachers revoiced and elaborated on women's contributions at rates much lower than their representation in the population, as shown by the equity ratios in Table 4. Elaborate often involved the teacher using a student's idea to formalize a mathematical idea and revoice was sometimes used to repeat a student's idea so that the class can hear it or because the teacher is thinking through the student's idea themselves. In either case, teachers leveraged women's ideas in this way at inequitable rates.

Table 4. Equity ratios for how instructors use student contributions

	Teacher Evaluation				
	N/A	Revoice	Evaluation	Elaborate	Follow Up
# Sequences	1397	482	186	846	524
ER Men	1.03	1.17	1.04	1.11	1.03
ER Women	.95	.67	.93	.79	.94

**Note*: Subcodes are arranged left to right from the least to most mathematically substantive use of student contributions.

Discussion

When examining trends in our findings regarding how teachers distribute opportunities to students, we looked at it in two parts: student selection and student-teacher interactions. In analyzing student selection, we found that there was more equitable participation when students were called on individually or by group. Calling on a group could be more equitable because this method creates a smaller pool of students to speak, which creates space for women to share their ideas. Men were more likely to interject or contribute their ideas when asked to volunteer. When examining trends in teacher-student interactions, we notice an interesting link between Teacher Solicitation and Student Talk. Teachers asked women less mathematically substantive questions, suggesting women had fewer opportunities to contribute mathematical ideas in whole class discussion. This might explain why teachers revoiced and elaborated on women's ideas at lower rates, as women were not prompted to give as many mathematically significant contributions. Though teachers likely did not mean for this to happen and are probably unaware of this inequity, the prevalence of these inequities in discussions in mathematics classrooms merits notice and discussion. The fact that the equity ratio for teachers calling on individual students by name was extremely close to 1 suggests that teachers intend for contributions in whole class discussions to be equal between men and women. In the future, we plan to explore the variation of these equity ratios by content area (abstract algebra, differential equations, and linear algebra) as the differences appear to be considerable and this could help explain what gives rise to these phenomena and any links to student outcomes.

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Exploring Experiences of Students of Humanities and Social Sciences in an Undergraduate Mathematics Course and Their Perceptions of its Usefulness

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This hermeneutical phenomenological study explored the experiences of students in the College of Humanities and Social Sciences (CHASS) in an undergraduate mathematics course and their perceptions of its utility. Field observations and semi-structured interviews were conducted. Six themes emerged from the collected data. The phenomenon of being a CHASS student in Topics in Contemporary Mathematics is perceived as enjoyable, but impractical and useless. Moreover, what moves students to be successful are mostly (or only, in some cases) external regulators that do not promote autonomy. A set of implications is provided.

Keywords: Phenomenology, Attitudes Toward Mathematics, Self-Determination Theory

Undergraduate students in the College of Humanities and Social Sciences (CHASS) are required to take what may be the last one or two mathematics courses in their lives as requirements for their degree. Understanding how CHASS students perceive their experiences in an undergraduate mathematics course designed for them, and their perceptions of the course's usefulness are first steps to deciding if the curricula for these courses need revisions to better meet students' interests and needs. Furthermore, caring about their perspectives is crucial for future generations because parents' attitudes toward mathematics significantly predicts their children's attitudes toward mathematics (Mohr-Schroeder et at., 2017). Therefore, the purpose of this study is to explore CHASS students' experiences in an undergraduate mathematics course and their perceptions of its utility. To fulfill this purpose, the following research questions are addressed: (a) How do students feel about being required to take mathematical courses for their degree? (b) What is the perceived relevance and usefulness of the mathematics course they are taking to their life, major, and future career? (c) What motivates them to be successful in the mathematics course? and (d) What are their attitudes toward mathematics?

Literature Review

There is important literature about affect in mathematics and motivation, in general, that can provide meaning and serve as a theoretical framework to this study. This section will focus on attitudes toward mathematics (ATM) and Self-Determination Theory (SDT) as lenses to understand CHASS students' experiences and perspectives of mathematics.

ATM has been extensively studied and this construct has been constantly developing (Middleton, Jansen, & Golding, 2017). It has been shown that ATM is reciprocally correlated with achievement (Ma, 1997; Ma & Kishor, 1997). Additionally, recent studies consider other factors affecting (or affected by) ATM, such as math anxiety, gender, cognition, self-efficacy, problem-solving, cooperative learning, absenteeism, class participation, homework completion, urban and rural differences, socioeconomic factors, teaching materials, teachers' content knowledge and personality, and teaching with real life enriched examples. (Green et al., 2012; Ho et al., 2000; Hopko, Ashcraft, Gute, Ruggiero, & Lewis, 1998; Kolhe, 1983; Mishra, 1978; Saha, 2007; Schoenfeld, 1985). Thus, ATM relates to students' perceptions of mathematics.

Motivation has also been studied in mathematics education, but not as much as ATM. Given that motivation explains a big portion of human behavior in almost every situated context, it is relevant in education. In general, the more autonomous the motivation is, the better students' achievement is (cf. Lee, Bong, & Kim, 2014). Ryan and Deci (2000a) shed light when they introduced different types of extrinsic motivation by presenting them on a continuum of the Organismic Integration Theory starting from less autonomous to more autonomous. When combined with other theories, the seminal Self-Determination Theory is created. Summarized by Ryan and Deci (2007), this theory also explains basic psychological needs for optimal motivation and reward contingencies, which can also serve as lenses to understand students' experiences.

As suggested by Middleton et al. (2017), ATM and motivation should be considered together along with other affect in mathematics and social interactions. Relationships found between ATM and motivation in mathematics are that both are correlated to engagement and achievement (Ma, 1997), both are malleable (Middleton et al., 2017) and tend to change negatively across the school grades (Mata, Monteiro & Peixoto, 2012). Additionally, having positive ATM does not guarantee an autonomous motivation and vice versa (Mata et al., 2012). However, many factors can enhance or diminish positive ATM and an autonomous motivation.

One conclusion that can be extracted from this is that if students are lead into the abstract and demanding mathematics' curricula without providing meaningful activities for them, they could easily lose autonomous motivation and develop negative ATM, both of which relate to mathematics achievement, engagement, enrollment in mathematics courses, cognition in mathematics, and general behaviors in the classroom, among other constructs (Middleton et al., 2017). Now, are we providing students with meaningful and interesting activities related to their majors and future professions so that we promote positive affect and internal motivation?

Methods

This study is a hermeneutical phenomenology study oriented toward the lived experiences of students from CHASS enrolled in *Topics in Contemporary Mathematics*. This course is primarily designed for CHASS students, and it should illustrate contemporary uses of mathematics frequently including topics such as sets and logic, probability, modular arithmetic, and game theory. This study includes six research foci (van Manen, 1990, 2014). Two of them are described as follows: First, the lived experiences in mathematics of whom may be seen as *the others* in STEM, CHASS students, and the research community not paying attention to their voice is an abiding concern to the researcher. Second, this experience is investigated in terms of how the participants live it, and *their reality* will be presented.

The data for this study consists of field observations and semi-structured interviews. During the week of field observations, the researcher was a nonparticipant observer and used natural descriptions (Bernard, 2011). Semi-structured interviews were conducted one week later. Three students enrolled in the course, majoring in History or Political Science, were interviewed. The interviews were audio-recorded and transcribed. The purposes of these interviews were to understand the phenomenon under study from the participants' point of view, to understand the meaning of their experiences, and to uncover their perspectives of the phenomenon (Brinkmann & Kvale, 2015). Therefore, open-ended questions, such as "How do you feel about mathematics being a requirement for your program?", "What has been your experience so far in the course?", and "What moves you or motivates you to study and be successful in that class?", were asked.

Codes were created, defined, and used for significant statements in the transcripts. Then, significant statements were grouped by codes. Afterwards, themes emerged when groups of codes were analyzed together with their significant statements. Finally, textural and structural descriptions were developed to describe the essence of CHASS students' experiences (Creswell & Poth, 2017).

Findings

The six themes that emerged from 63 significant statements will be discussed in this section.

Emerging Themes

Theme 1: Calculus as a turning point. To understand the essence of the students' experiences in the course, it is important to understand their perceptions of mathematics in general. During interviews, two participants discussed their experiences with high school calculus; both Tom and Jake expressed that calculus was very difficult for them and that they did not do well. In fact, Tom expressed, "I liked math before calculus". As Jake mentioned, mathematics was getting "confusingly harder" in high school. Having these experiences, where they were not as successful as they wanted to be, may have created negative ATM and low motivation to learn mathematics. Consequently, this experience and mindset would follow them to undergraduate mathematics courses.

Theme 2: Overcoming math anxiety. CHASS students' previous perception of mathematics was described as: "Confusing. I think math is pretty confusing" and "I just find it harder than anything else". When expressing how they felt upon noticing that their undergraduate program required two mathematics courses, Tom expressed, "I get nervous when I have to take math classes. Um… probably just nervousness. Um, I try to study for math more than anything else." Then, he explained that he felt anxious about taking mathematics, and that he "just wanted to get through it". Additionally, Jake was worried that he was going to have to take "hard math". However, once they were told a description of the course and experienced it, they seemed to overcome math anxiety.

The course's low difficulty level seems to be the key for students to overcome math anxiety. Perhaps knowing that they can do it changes the way they feel about mathematics? Before, they felt worried, nervous, and anxious. Now, Sarah, Tom, and Jake, respectively, expressed: "I don't really mind that class. It is going to be an easy A. I don't have a problem with it anymore", "It's pretty simple and definitely way easier than calc in high school", and "Is pretty simple that is honestly like easier than most of the math courses I took in high school".

Theme 3: Defining a mindset. The type of mindset these students had before and during the experience of being a CHASS student in the mentioned mathematics course was still defining itself or getting more concrete. Statements, such as: "I'm not a math person", "[Math] wasn't my thing" "I am not good at math, like complex math", and "But when you are required to do that, and you don't have a brain that is good understanding math, I think that can really put people off school" evidenced that most of these students had the belief that they do not possess the "ability" to be good at math. Thus, making it clear that they had a fixed mindset (Dweck, 2008). Although all the statements related to their mindset suggested that they did not believe they could master higher level mathematics, the following statement is particularly interesting:

> I've become a different person from high school in college. So, like the person that didn't like math in high school is not around anymore. I think it'll be interesting to go in a high-level class to see if I can perform. Like having an increased in work ethic and trying to have good grades and stuff like that, whereas I didn't care like that in high school.

Thus, suggesting that the experience changes the mindset of some students, but for others it reaffirms their fixed mindset thinking that they are only successful because the course is "easy".

Theme 4: Increasing enjoyment with controlling motives. Students have been enjoying an "easy" math course at college. One student had the following epiphany: "And I honestly kind of miss it, like I like math, and I realize, I think that I realize now that I do like it". Another student expressed that his negative ATM changed because he considers this course to be easy. It seems that, this course, by being “easy”, allows them to enjoy it more.

When asked what moves them to be successful in that course, all replied that grades are what moves them. None mentioned the content of the course or what they may gain from it. If the difficulty of the course matched that of the calculus, they may feel the same negative feelings that they expressed before. Thus, what they seem to enjoy is the low complexity and the “easy A”. For example, Jake mentioned: “But sometimes I just miss like ‘here is a problem and figure that out’ and it’s kind of basic and you can do that”. Therefore, the enjoyment seems not inherent in this case but caused by external factors, such as grades. These external factors are controlling motives and grades in particular are extrinsic motivators with the lowest autonomy in the SDT.

Theme 5: Viewing mathematics as useless. Students showed their perspective that most applications were impractical and that there has not been an evident transfer of knowledge, except for the topic of voting methods. Most topics, according to them, are useless because “one could solve the given problems by flipping a coin” (e.g., deciding where to eat with friends) or simply “cutting the cake however you want”. A student described it as “unnecessary and dumb”. Another student expressed that although it is simple math, he would not know where to apply it. In particular, Sarah mentioned: “I don’t think I’ll use anything to be completely honest. Once I’m out of this class, I’ll keep the notebook and not think about it again.” All expressed an inability to apply what they have learned to real-life situations, except for the topic of voting methods, whose relevance was clear to them and they enjoyed it.

The transfer of knowledge to new contexts is one of the main goals of education (Bransford, Brown, & Cocking, 2000). However, students found themselves unable to make this transfer of knowledge from the mathematics classroom to other contexts. In fact, they do not see the relevance of most topics to them, their careers, or their majors. Thus, referring to SDT, this experience does not seem to promote identified or integrated regulation. Oppositely, the experience lets them stay with low autonomy, low internalization and externally regulated.

Theme 6: Reflecting on its appropriateness. Here, students focused on the characteristics that make or do not make this course appropriate. First, they all explained that the course is appropriate for CHASS students because it is a “simpler math” that is “digestible”. Additionally, they expressed that they have a good instructor and that a “good teacher is what probably makes the difference between really hard or getting through it better”. Then, a dilemma emerges when they talk about the course content:

> Also, it has real world applications which humanities majors usually like. Um, but also, like I said with the apportionment with like the cake problem, it doesn’t seem practical for use in like daily life. Some of the things we learn about, I feel are things I’m never gonna have to use again.

That is, although it has concrete applications, they seem to be impractical, inappropriate or useless for them. Although they clearly enjoy the easiness that characterizes the course, they want more out of that experience. For example, Jake shared: “I wish that like I could have a math course that related to me more”. Sarah added: “So, like, um, not just here you would use it but also why we would need to use it”. Moreover, Jack added: “...so that the course would be more difficult and challenging and also related to what I’m doing, and I would enjoy that”, expressing that more challenging and related topics would be more enjoyable.

According to Ryan and Deci (2007), satisfying students’ basic psychological needs for autonomy, competence, and relatedness leads to optimal motivational function. By contrast, they explained that “whenever the social context thwarts or neglects one of these needs, intrinsic motivation and internalization, as well as positive experience, wither" (p. 7). Students seem to have one or two of these psychological needs satisfied. Having what they consider a good instructor may satisfy the need for relatedness, which makes them feel comfortable and part of a community. On the other hand, it is unclear whether they feel competent in this course. They are doing "good" in this course, but they feel that it is not complex. Perhaps they may feel competent in the course, but not in general. For example, Sarah said: "I also understand that this level of math is easier than the one I had in high school. So, it's good but I also understand why I'm doing good". However, what is clear is that this experience is not supportive of their need for autonomy. For example, when asked about the assignments, a student said: "There just um, basically he'll tell us a formula in class and I will go over one or two examples and it's just a copy of those questions pretty much. The test is the same as the web assigns". Lastly, they have expressed that almost nothing in that course relates to their interests, major, or future careers.

Conclusion and Implications

In general, at the beginning of the experience of being a CHASS student enrolled in *Topics in Contemporary Mathematics*, students felt anxious and had negative ATM. The perceived low difficulty of the course and the positive reinforcement of good grades made them overcome the anxiety and helped them enjoy the course. The instructor also plays a role that, in this case, was positive in helping them enjoy the course. However, except for one topic, their perception is that there is no clear relationship between what they are learning in that course and their interests, major, or future careers. That seems to make transfer of knowledge almost impossible for them. Therefore, the phenomenon of being a CHASS student in Topics in Contemporary Mathematics is perceived as enjoyable, but impractical and useless. Moreover, what moves students to be successful is mostly (or only, in some cases) the grade, which is an external regulator that does not promote autonomy. Nevertheless, they got their desired A.

For CHASS students, what may be the best kind of motivators are those that are perceived as integrated and identified regulators, which are considered as the most internal types of extrinsic motivators in the continuum (Ryan & Deci, 2000b). That is, intrinsic motivation is desired, but we do not want students to do mathematics only because they inherently enjoy doing mathematics. We also want them to understand its importance, why they need it, how they can use it, how they can relate mathematical reasoning in other contexts, etc. We want them to transfer that knowledge in a way that it is useful for them. Based on these students' perceptions, this is not happening in this course, even when the field observations suggested that the professor was teaching for conceptual understanding and also giving applications. The researcher plans to keep interviewing students in the future as it is clear that there is an opportunity to optimize students' experiences in this course. Therefore, this study has the following implications: (a) more activities in which students can feel in control of the learning process, see its relation to them, and, at the same time, feel that it is challenging are needed. Those activities promote students' autonomy, which is required for their cognitive development, transfer of knowledge and optimal motivation; (b) applications given in the classroom should be changed to more practical ones. For example, making a decision matrix to decide where to eat is not practical. Making a decision matrix to decide which graduate schools to apply, where to move, what job to accept, what topic to investigate, and the like, are worthwhile and relevant; (c) topics discussed in the classroom should cater the needs of most students, not only those majoring in political sciences; (d) make instruction more individualized through assignments; (e) collaborate with professors from CHASS to create the mentioned assignments; and (f) more research is needed with this population in mathematics.

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Reasoning Covariationally to Distinguish between Quadratic and Exponential Growth

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In this report, we present preliminary findings from clinical interviews examining inservice teachers' understandings of quadratic growth and exponential growth. The purpose of this pilot study is to investigate how teachers may naturally leverage covariational reasoning to distinguish between the two types of growth. In this report, we first present relevant constructs pertaining to teachers' covariational reasoning and then describe one task we used in clinical interviews. We then present preliminary findings regarding how teachers' leveraged (or did not leverage) covariational reasoning as they addressed this task to differentiate between quadratic and exponential growth. We conclude with preliminary implications and questions regarding how these preliminary findings may have implications for a larger study with pre-service secondary mathematics teachers.

Keywords: Covariational reasoning, quadratic growth, exponential growth

Although several studies have indicated that covariational reasoning can support students develop various mathematical ideas, such as quadratic, exponential, trigonometric, and parametric functions (e.g., Castillo-Garsow, 2012; Ellis & Grinstead, 2008; Johnson, 2012; Moore, 2014; Paoletti & Moore, 2017) the available research suggests that reasoning covariationally is uncommon and minimal among high school mathematics teachers in the U.S. (Strom, 2006; Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017). Particular to quadratic and exponential growth, researchers have indiciated middle school students (e.g., Ellis, 2011a; Ellis, Özgür, Kulow, Williams, & Amidon, 2015) can reason covariationally to construct and reason about quadratic and exponential relationships, but there is limited research examining pre-service and in-service teachers' understandings of these growth patterns. Consequently, the aim of this pilot study was to examine teachers' meanings related to quadratic growth and exponential growth. We present preliminary findings from clinical interviews providing insights we will leverage in implemeting a semester long teaching experiment with undergraduate pre-service teachers. We address the research question: "How might teachers reason covariationally to differentiate between quadratic growth and exponential growth?

Theoretical Perspective

Researchers have articulated varied perspectives of covariational reasoning. Providing a contrast to an emphasis on functions as representing correspondence rules, Confrey and Smith (1994) advocated a covariational approach to function that involves coordinating successive values of one variable (y_m to y_{m+1}) with successive values of another variable (x_m to x_{m+1}). Whereas Confrey and Smith focused on coordinating sequences of numeric values, Saldanha and Thompson (1998) proposed a more continuous perspective on covariational reasoning as "someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). This involves the student imagining both quantities being tracked for some duration and understanding that "if either quantity has different values at different times, it changed from one to another by assuming all intermediate values" (p. 298). For example, as the side length of a square increases continuously from 4 units to 5 units, taking all intermediate values between 4 and 5 units, the area increases continuously from 16 sq. units to 25 sq. units, taking all intermediate values between 16 and 25 sq. units.

Building on these and other researchers' characterizations, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) proposed a framework encompassing five mental actions that students engage in when reasoning covariationally. The mental actions involve identifying *change in two quantities* (MA1), the *direction of change* of one quantity with respect to the second quantity (MA2), *amounts of change in one quantity for equal changes in the second quantity* (MA 3), and the *average and instantaneous rate of change* of one quantity with respect to the second quantity (MA 4-5).

Paoletti & Moore (2017) noted that leveraging these different forms of covariational reasoning can support students in developing more robust quantitative structures and a better understanding of the relationships they are representing. In this report, we focus on inservice teachers' covariational reasoning as they conceived of and described quadratic and exponential growth.

Literature Review: Conceptualizing Quadratic and Exponential Growth

Although there are studies (e.g., Chazan, 2006; Zaslavsky, 1997) pointing to student misconceptions related to quadratic and exponential growth, there are fewer studies providing evidence of students or teachers maintaining productive understandings of these ideas. We briefly describe researchers' characterizations of productive understandings of quadratic and exponential growth using a covariational reasoning lens and synthesize key findings from these studies.

With respect to quadratic growth, researchers (Ellis, 2011b; Lobato, Hohensee, Rhodehamel, & Diamond, 2012) taking a covariational lens compatible with Confrey and Smith's (1995) description have characterized quadratic growth as a student envisioning changing rates of change and identifying that the rate of change of the rate of change is constant. For example, Ellis (2011a, 2011b) showed that middle school students can identify constant second differences to realize the quadratic growth in the area of a growing rectangle.

With respect to exponential growth, researchers have used two of the aforementioned characterizations of covariational reasoning to characterize productive meanings of exponential growth. Confrey and Smith (1994, 1995) leveraged their operationalization of covariational reasoning to describe exponential growth as a juxtaposition of values of one variable changing in arithmetic progression with values of a second variable changing in geometric progression. They reported on students interpreting a table of values by calculating the ratio of successive values of one variable for constant unit changes in the other variable to conceive of exponential growth. Confrey and Smith proposed this conceptualization of a constant multiplicative rate as a foundational idea to approach exponential growth.

In contrast to comparing successive values, Thompson (2008) emphasized, "a defining characteristic of exponential functions is that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument" (p. 39). Drawing on this view of exponential growth, Castillo-Garsow (2012) presented tasks in the context of interest bearing bank accounts to high school students and reported on one student who, consistent with Thompson's description, conceived that the rate of change of the value of the account at a moment was proportional to the value of the account at that moment.

Drawing on both Confrey and Smith's (1994, 1995) and Saldanha and Thompson's (1998) characterizations of covariation, rate, and exponential growth, Ellis and colleagues (2015) examined the activity of three eighth grade students who developed understandings of exponential growth by reasoning about the height of a plant changing over time. The students reasoned covariationally to conceive exponential growth as the coordination of multiplicative growth of height values for constant unit changes of time, through numerical tabular arrangements. Eventually students were able to make these comparisons for non-constant changes in time.

Although the aforementioned researchers noted that students at various ages are capable of reasoning covariationally to develop understandings of exponential growth, such understandings may not arise naturally from school experiences. For instance, Strom (2006) engaged in-service secondary mathematics teachers in a series of tasks she conceived to be related to exponential growth. She noted that a majority of teachers in her study had difficulties coordinating the images of two quantities changing together and concluded that covariational reasoning was minimal in most teachers' responses. Strom's study highlights that although students can develop understandings about exponential and quadratic growth by middle school, experienced teachers do not necessarily have this reasoning immediately available to them.

We note there is dearth of literature examining teachers' (or students') conceptions of quadratic growth and exponential growth in relation to reasoning covariationally. Moreover, there are no investigations we are aware of examining how teachers' may differentiate between the two growth patterns by reasoning covariationally. For instance, even if a teacher can engage in MA3 as described by Carlson et al. (2002) to determine that Quantity A increases at an increasing rate with respect to Quantity B, how might that teacher determine if Quantity A grows quadratically, exponentially, or in some other pattern with respect to Quantity B? Therefore, in addition to adding to the literature on teachers' covariational reasoning and understanding of quadratic and exponential growth, our study aims to better understand how students and teachers can develop more sophisticated understandings of these growth patterns.

Pilot Study Methods and Task Design

The first author conducted four individual task based semi-structured clinical interviews (Clement, 2000) that lasted for 60-90 minutes with in-service high school mathematics teachers. The teachers volunteered to participate from a convenience sample accessible to the researchers. Each teacher had a minimum of ten years teaching experience and had taught a variety of high school math courses.

Carlson et al.'s (2002) framework informed the design of the *Two Quadrilaterals* task which is an adaption of tasks implemented in previous studies that investigated students' reasoning about rate (Johnson, 2012) and quadratic growth (Ellis, 2011b). In this applet, we provided two sliders. Whereas the longer, pink, slider allowed teachers to animate the two quadrilaterals (one in blue and the other in brown) the shorter, red, slider allowed teachers to change the increment the longer slider changed by, thus allowing both seemingly continuous and discrete growth of the two quadrilaterals (see Figure 1 for several screen shots of the task). At the start, the quadrilaterals are congruent. As the pink slider drags to the right, each side length of the blue quadrilateral increases proportionally with respect to the slider's position and thus the area of the blue square can be represented by quadratic growth. As the pink slider drags to the right, the brown quadrilateral doubles in size for each unit change in the slider by first having its width double then its height double, and so on and thus the area grows exponentially. We intended to examine how teachers' may conceptualize and compare the growths of each quadrilateral. The interviewer prompted the teachers to consider how the *areas* of each quadrilateral covaried with the *pink slider*.

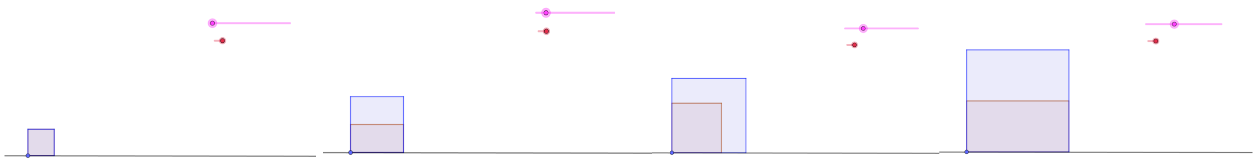


Figure 1. The first four jumps of the Two Quadrilaterals task.

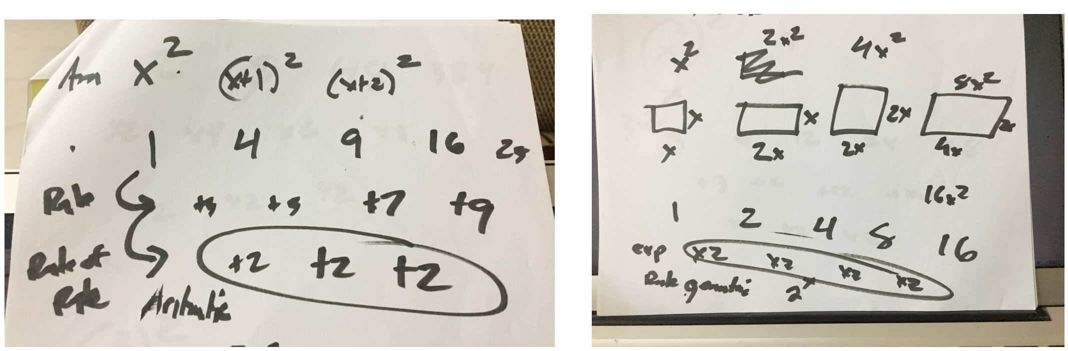
Results

We first present an example of a teacher who explicitly described exponential and quadratic growth when addressing the *Two Quadrilaterals* task. We then briefly synthesize the other teachers' responses to highlight other ways of reasoning the task elicited.

Rick's Ways of Reasoning: Considering Changes to Determine a Relationship

Rick was the only teacher to make statements regarding the type of growth exhibited by the areas of the two quadrilaterals as the slider (or side length) increased. He coordinated how the areas and the slider (or side length) covaried in terms of direction of change (MA2) and the amounts of change (MA3). He then introduced numerical values in order to further analyze and differentiate between the growths of the areas of the two quadrilaterals.

For the blue quadrilateral, Rick noted that both the side lengths increased by one unit if he moved the slider by one unit and claimed "the area is increasing by whatever that side is squared." He considered the initial side length to be 'x' units and described that the areas of the growing square would be x^2, $(x+1)^2$, $(x+2)^2$. Rick next assumed x to take the value of 1, calculated the areas to be 1, 4, 9, 16 and 25, found the differences between these numbers, and also their second differences. Circling the second differences (see Figure 2a), Rick explained "this is the rate of the rate. So the rate of the rate is constant. It tells me it is quadratic." We infer from Rick's explanation that he understood quadratic growth in ways compatible with the characterizations of Ellis (2011a) and Lobato et al. (2012); Rick understood quadratic growth is defined by a relationship such that the rate of change of the rate of change is constant.



(a) (b)

Figure 2. Rick's work describing the growth of area of (a) the blue quadrilateral and (b)the brown quadrilateral.

For the brown quadrilateral, Rick noted that one side length alternately doubles as he moved the slider by one unit. Similar to his aforementioned work (see Figure 2b), Rick described that the area of the brown quadrilateral would be x^2, $2x^2$, $4x^2$, $8x^2$ and $16x^2$ at the first five unit values of the slider, assumed x to take the value 1, and calculated the areas as 1, 2, 4, 8 and 16. He identified that for a unit change in the slider, each consecutive area is double the previous area and stated, “the first rate is constant multiplication which tells us it is an exponential growth. The rate is constant.” We infer from Rick’s activity that he was coordinating the ratio of the area of the quadrilateral with equal changes in the slider to explain exponential growth. His explanation is compatible with Confrey and Smith’s (1994) description of exponential growth as having a constant multiplicative rate.

We note that for each quadrilateral, Rick first engaged in the mental actions described by Carlson et al. (2002) in order to conceive that the area increased at an increasing rate with respect to the incremental changes in the slider. He then introduced numerical values for the side length and the corresponding areas to make the distinction between the two growth patterns.

Other Ways of Reasoning

In contrast to Rick’s activity, two other teachers were able to correctly identify relationships without explicitly examining the underlying growth patterns. Consistent with pre-service teachers’ activity reported on elsewhere (Stevens et al., 2015) these teachers attempted to either derive a formula or recall facts from memory to define the relationships they conceived in the situation. For example, Aman established that the differences in the areas of the blue quadrilateral can be represented by the rule $(2n\text{-}1)\, x^2$ where n is a natural number, but did not describe any pattern in the second differences of this relationship. Similarly, for the brown quadrilateral, he established that the areas can be represented by the formula $2^{(n-1)}\, x^2$. Although Aman successfully determined rules to describe patterns in the growth of the areas, he did not elaborate on what these growth patterns meant with respect to specific function classes.

As another example, David described that the blue quadrilateral remains a square when the slider is dragged and drew the graph of $y = x^2$. However, David did not justify why the graph would be an appropriate representation of the situation. We hypothesize that David recalled from facts that the area of a square can be represented by $y = x^2$ where x represents side length and y represents area. In both this and Aman’s example, we note that, we do not make any claims regarding the teacher’s meanings with respect to quadratic or exponential growth; each teacher may have been able to differentiate between exponential and quadratic growth but did not experience any need to discuss these patterns when addressing the task.

Preliminary Implications and Intended Questions

We note, that Rick first engaged in the mental actions described by Carlson et al. (2002) before providing hypothetical numeric values which supported him in identifying the different growth patterns characterized by previous researchers (Confrey & Smith, 1994; Ellis, 2011b; Lobato et al., 2012). This is consistent with the productive interplay between different ways to reason covariationally as described by Paoletti & Moore (2017). Further, despite being able to animate the quadrilaterals (seemingly) continuously, none of the teachers seemed to naturally leverage continuous reasoning as described by Saldanha and Thompson (1998) when addressing this particular task. Finally, a limitation of our task was that several teachers successfully produced known rules which limited our ability to make inferences regarding their understanding of the underlying growth patterns represented by the two quadrilaterals. This raises several questions for our follow-up study.

Should we expect pre-service teachers to respond differently to similar tasks than in-service teachers? Why or why not? What other task situations would lend themselves to supporting teachers in distinguishing between quadratic and exponential growth? What other constructs (e.g., smooth/chunky reasoning) may be useful to consider when designing tasks and analyzing data?

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First-year Mathematics Students' View of Helpful Teaching Practices

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Research in undergraduate mathematics education has identified various research-based instructional practices to support students' learning. However, little is known about how students experience those practices or how helpful they perceive those practices to be for their learning. As part of a larger national project of first-year mathematics, this study focused on classroom experiences in the Precalculus to Calculus 2 (P2C2) sequence. Using survey data from 4,969 students, we considered how helpful students find various teaching practices and then compared student and instructor reports of how characteristic these practices are of their P2C2 class. Here we report students' ratings of twelve different teaching practices in terms of helpfulness for their learning in and descriptiveness of their P2C2 experience.

Keywords: Precalculus, Calculus, Instructional approaches, Survey

The national study of *Characteristics of Successful Programs in College Calculus* (CSPCC) distinguished between aspects of good and ambitious teaching (Sonnert & Sadler, 2015). In that work, *good teaching* is characterized as instruction that is traditionally accepted as good teaching practices, regardless of pedagogical approach (e.g., being available, grading fairly). *Ambitious teaching* refers to instruction that incorporates more innovative or novel approaches to instruction, including more student-centered strategies (e.g., working with peers in class). The CSPCC study showed that both good and ambitious teaching are beneficial for students' learning in Calculus 1 (Bressoud, Mesa, & Rasmussen, 2015). In this study, we extend that previous work to explore levels of usage of particular good and ambitious teaching strategies and provide insight into students' perception of the helpfulness of those strategies.

Ellis, Kelton, and Rasmussen (2014) laid a foundation for understanding Calculus 1 instruction from both student and instructor perspectives, comparing student and instructor reports of the frequency of specific instructional practices. They found that, on average, students and instructors agree on the frequency of various instructional practices in the classroom, though instructors were more likely to over-report practices identified as part of ambitious teaching. They also noted the presence of more variation among students' responses to items related to ambitious teaching constructs than to items related to good teaching. That is, students in the same course reported a wider range of frequencies for activities like "whole class discussion" than for "lecture." Furthermore, students who reported a lower frequency of ambitious teaching strategies were more likely to switch out of the calculus sequence, suggesting an association between students' perception of instruction and interest in continuing in STEM. This leads us to believe that documenting students' perspectives on not only what happens in class, but how helpful they find it, is important for understanding why students do or do not continue through the calculus sequence.

The undergraduate mathematics and science education research communities have identified many pedagogical practices that have been shown to support student learning. However, there is literature to suggest that students' experiences of those practices are not uniform and their view of these practices is not always aligned with that of an external observer (Ellis, Kelton, & Rasmussen 2014; Rogowsky, Calhoun, & Tallal, 2015; Willingham, Hughes, & Dobolyi, 2015). Additionally, students' perceptions of their learning and the helpfulness of the instructor's actions have an effect on their experience of a course. We suggest that the utility of our work is not to identify what teaching practices *should* be used because students believe they are helpful, but rather to identify where there are discrepancies between students' perceptions and education research literature, either in general or disaggregated based on other student factors. That knowledge may help instructors recognize where to build buy-in, what practices are perceived differently, and address students' concerns about their teaching style in a timely fashion. Implementing ambitious teaching practices in ways that students believe are helpful to their learning should not only improve students' learning of content but also improve their overall experience.

In this study, we identify instructional practices that students find most helpful and identify how characteristic these practices are of their Precalculus to Calculus 2 (P2C2) classes, considering both the instructor and student perspectives. In this report we aim to answer: (1) What teaching practices do students regard as helpful for their learning?, and (2) Do students and teachers describe their class with practices that students deem helpful?

Methods

The data for this study comes from surveys designed for a larger, multiphase national project aimed to examine current P2C2 programs. During the first phase of the project, a large census survey was administered to all universities across the country whose math department offered a graduate degree. Census survey responses were considered to select 12 mathematics departments for in-depth case study sites during the second phase of the project. Specifically, the chosen departments were interesting in regards to the seven features of successful programs identified in the CSPCC study (Bressoud et al., 2015) as well as an eighth characteristic: diversity, equity, and inclusion (Hagman, under review).

The study presented here reports on instructor and student survey data that were administered to all instructors of P2C2 classes and their students. The surveys included 12 parallel items regarding classroom experiences (e.g., 'I guide students through major topics as they listen' and 'I listen as the instructor guides me through major topics'). Instructors and students were asked to indicate whether the statements were descriptive of their P2C2 class. Both surveys used a 5-point Likert scale ranging various levels of descriptiveness of the classroom (5=very descriptive, 4=mostly descriptive, 3=somewhat descriptive, 2=minimally descriptive, 1=does not occur/not at all descriptive). For each item which a student responded with 2 or higher (i.e., indicated the item occurs in their class), they were asked to report how much that aspect of the course helped their learning. The helpfulness item was measured on a 3-point scale (3=very helpful, 2=somewhat helpful, and 1=not helpful).

For this study, we reduced our data to responses from courses with an instructor response and at least five student responses, following the methods of Ellis, Kelton, and Rasmussen (2014). These restrictions resulted in a total of 4,969 student responses from 173 P2C2 classes. More specifically, this includes 1,789 student responses from 55 precalculus classes, 1,806 student responses from 74 Calculus 1 classes, and 1,374 student responses from 44 Calculus 2 classes. We considered descriptive statistics for all of the items as well as conducted a paired samples t-test comparing the student and instructor responses for the classroom experience items. The following section presents sample results to highlight findings regarding how helpful students rate certain teaching practices and students' perspective of how characteristic the teaching practices are of their P2C2 class.

Table 1 offers the student version of the helpful and descriptive items. We coded the items based on Sonnert and Sadler's (2015) factor analysis categories: ambitious and good teaching. Specifically, we coded items that related to traditionally accepted good teaching practices regardless of pedagogical approach as *good* and practices related to student-centered strategies as *ambitious*. Item 1 is reverse coded as ambitious, meaning that more of this practice indicates a less ambitious approach. The items not highlighted as good or ambitious remain uncategorized because there is ambiguity in the terminology which admits many possibilities. Specifically, the nature of feedback, questions, and individual work can support both student-centered and instructor-centered classrooms.

Table 1. Helpfulness and descriptiveness items on student survey.

Ambitious	Good	Other
1. (*reverse association*) I listen as the instructor guides me through major topics	6. The class activities connect course content to my life and future work	10. I receive immediate feedback on my work during class
2. I talk with other students about course topics during class	7. The instructor knows my name	11. I am asked to respond to questions during class time
3. I constructively criticize other students' ideas during class	8. I receive feedback from my instructor on homework, exams, quizzes, etc.	12. I work on problems individually during class time
4. I work with other students in small groups during class	9. My instructor uses strategies to encourage participation from a wide range of students	
5. Class is structured to encourage peer-to-peer support among students		

Sample Results

In what follows, we will focus on the students' perspective and draw on the helpfulness and descriptiveness items on the student survey. Students were asked to rate the items listed in Table 1 regarding how helpful they were for their learning and how descriptive they were of their class. We focus on what students find to be (un)helpful, and then discuss the extent to which the strategies that we categorize as "ambitious" are being implemented.

We will begin by focusing on the helpfulness items. Figure 1 depicts the percent of students that rated each item as not helpful, somewhat helpful, or very helpful. More than half of the students that responded to the first, seventh, eighth, and tenth items indicated that these instructional practices were very helpful for their learning. Specifically, 73.2% of students reported listening to their instructor guide them through topics was very helpful for their learning, 64.6% of students indicated that receiving feedback on assignments was very helpful, 52.7% of students reported that the instructor knowing their name was very helpful, and 51.6% of the students said that receiving immediate feedback in class was very helpful.

At least 90% of the students indicated that the following are either somewhat helpful or very helpful for their learning: listening to the instructor guide them through topics (Item 1), receiving feedback immediately in-class (Item 10) as well as on assignments (Item 8), talking with students during class (Item 2), and working individually on problems during class (Item 12). On the other hand, 16.5% of the students indicated that being asked to respond to questions in class was not helpful for their learning (Item 11) and 23.3% of the students said that constructively criticizing other student's ideas during class was not helpful for their learning (Item 3).

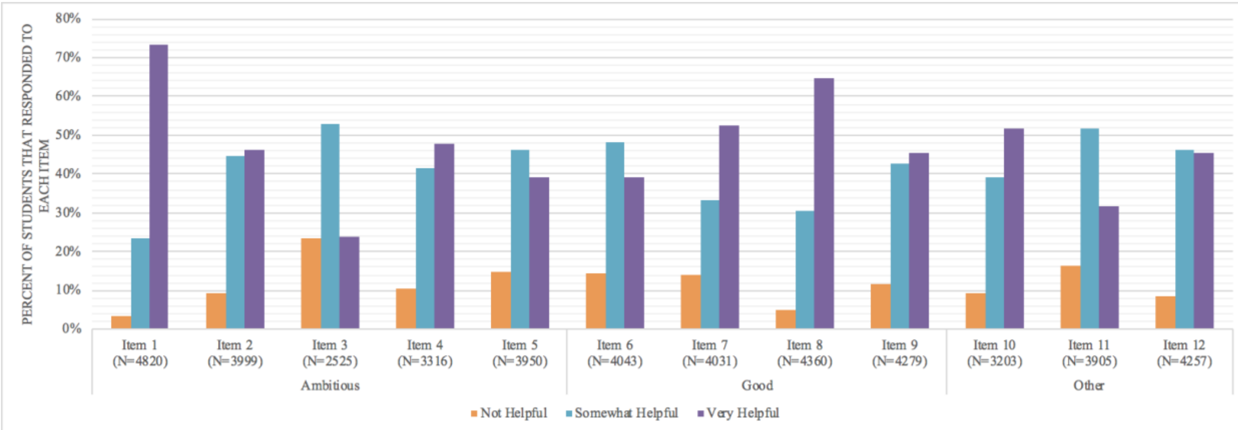

Figure 1. Summary of helpfulness item in percentages.

We further investigated if certain teaching practices were more helpful for students in different courses (Precalculus, Calculus 1, or Calculus 2). Data from several teaching practices were significantly different ($p<0.05$) in terms of helpfulness across courses (i.e., Item 1, Item 5, Item 7, Item 8, Item 9, Item 11, Item 12). For instance, we found that a greater percentage of Calculus 2 students (77.6%) found listening to their instructor guide them through major topics was very helpful compared to Calculus 1 students (70.2%) and Precalculus students (72.8%). Additionally, more Precalculus students (41.5%) than Calculus 1 (36.9%) and Calculus 2 (38.4%) students found class structures that allowed peer-to-peer support very helpful.

Next, we considered students' perspective of how characteristic the helpful teaching practices are of their P2C2 class. Here, we offer the descriptive rating from the students that rated items classified under ambitious as helpful for their learning (either somewhat or very helpful). See Figure 2. We found that 86.3% of the students that indicated listening to their instructor was helpful for their learning (Item 1, N=4655) also said that it was very or mostly descriptive of their class. Additionally, 53.4% of the students that indicated that talking with students during class was helpful (Item 2, N=2431) said that it was mostly or very descriptive of their class. Of the students that found constructively criticizing other student's ideas during class helpful for their learning (Item 3, N=1937), only 11.3% of them reported it was very descriptive of their class, while 35.1% of them said it was minimally descriptive.

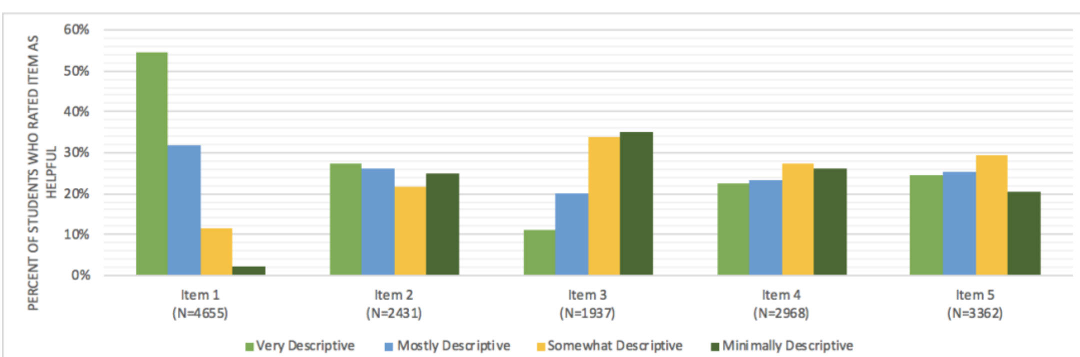

Figure 2. Summary of descriptiveness of ambitious items from students that deemed them helpful.

Alternatively, Figure 3 considers the descriptive rating from the students that reported the ambitious items as *not* helpful for their learning. Comparing Figure 2 to Figure 3, there appears to be a connection between how helpful students report an item and how descriptive it is of their class. Students who deemed Item 2, Item 3, Item 4, and Item 5 as unhelpful more commonly report them as minimally descriptive of their class compared to students who report them as helpful. This suggests that students who experience any amount of these ambitious teaching practices and who think they are helpful for their learning tend to report them more regularly in their class.

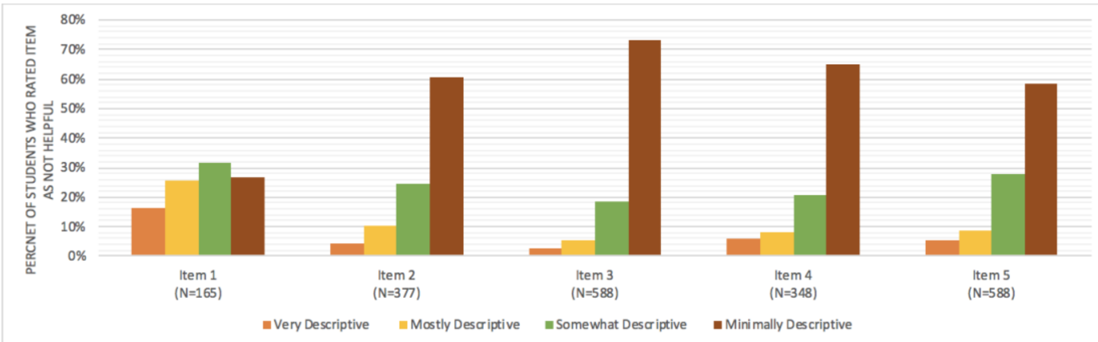

Figure 3. Summary of descriptiveness of ambitious items from students that deemed them not helpful.

Implications

Careful attention needs to be given to the implications of our research, especially for findings that suggest that many students find a particular teaching practice helpful (or unhelpful) for their learning. One might think that the "fix" is to either increase (or decrease) that specific teaching practice; however, the solution may not be so simple. For instance, we found that an overwhelming amount of students think listening to their instructor lecture about major topics is helpful for their learning. Moreover, less than 5% of the students indicated that it was not helpful. Although this finding does not indicate that the same students do not find a more active approach also helpful, it is clear that students value a passive experience. However, it is well documented in the literature that a more active approach is far more productive for students' learning (e.g., Freeman et al., 2014). Thus, it seems more useful to focus on student buy-in for certain practices (ones that are known for being beneficial for students) than to increase the presence of, for example, lecture.

In addition to information similar to what is reported here, in the presentation we will consider the instructors' perspective of how descriptive the items are of their class. We will present subsequent analyses to compare the students' perspective to their instructor's view of how characteristic the teaching practices are of their P2C2 class. The preliminary report and subsequent analyses will contribute to research on how students experience research-based instructional practices in the P2C2 sequence.

Acknowledgments

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Contextualized Instruction as a Motivational Intervention in College Calculus

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Abstract
The purpose of this quasi-experimental study is to measure the impact of a contextualized-instruction intervention on student performance expectations, utility-value, and interest in college calculus courses. Six calculus sections were selected for this study, and three were randomly assigned to take the intervention. Students in the three intervention sections completed calculus tasks that were contextualized to various STEM disciplines, whereas students in the comparison sections did not. Three tasks, one from each in computer science, physics, and engineering were contextualized. The results indicated that the impact of the intervention on student motivation was not statistically significant. However, student motivation significantly changed over time.
Keywords: Motivation Intervention; College Calculus; Design Experiment

Introduction
Calculus students often ask, "why are we learning this?" Students might not see the value or the connections between course material and their lives (Wulf, 2007; Brophy, 1999). If students are not given the opportunity to see this connection, they might lack the motivation (Clarke & Roche, 2017; Harackiewicz, Tibbetts, Canning, & Hyde, 2014). Expectancy-value theory (1983) was the framework used in this study, which posits that students' task values and expectations of success are determinants of students' achievement-related choices and behaviors (Hulleman et al., 2010). Aligned with this theory, interest, utility-value, and performance expectations were investigated: (1) How do the contextualized calculus tasks impact utility-value, interest, and performance expectations in college calculus? (2) How do students' utility-value, interest, and performance expectations change throughout a semester in college calculus?

Methodology
The study followed a quasi-experimental research design. 66 participants from a Southeastern university were selected. Each course section was randomly assigned to intervention and comparison conditions. The intervention was comprised of three contextualized calculus tasks with applications to science, technology, and engineering disciplines. The researcher implemented the tasks in all the sections twice. The data only came from a survey to measure student motivation and *Cronbach's alpha* reliability coefficients were strong (greater than .90).

Results
Linear-mixed effects modeling was used to examine the impact of the intervention on student motivation, and repeated measure analysis was used to investigate change in student motivation across three time points. Results showed that the effect of the intervention on student motivation, although positive in some cases, was not significant. Furthermore, three models—performance expectations, utility-value, and interest were presented for the effects of time and intervention. Results showed that time was a significant factor for change in student performance expectations. One unique aspect of this study was the idea of implementing contextualized calculus tasks that were explicitly developed by potential future instructors of students in computer science, physics, and engineering fields. However, there was not enough evidence to suggest that the intervention had a positive impact on their motivation. More details on the implications and limitations of the study will be discussed.

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Towards Better Mathematics Teaching Assistant Preparation in Graduate Programs

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This work focuses on the preparation of graduate Teaching Assistant in the mathematics department of a large midwestern R1 university. We explore the author's experience with three different versions of the course in three varying capacities: as a student, a mathematics education researcher and a co-facilitator. As we delve into the author's reflections on the evolution of the teaching preparation within this timeframe, we highlight some issues with the course structure and execution. This leads to the development of another, more realistic version of the preparation course.

Keywords: Teaching Assistants, Teaching Assistant Preparation, Professional Development

Teaching Assistant (TA) preparation is a relevant topic for graduate programs, including those housed in mathematics departments (Shannon, Twale, & Moore, 1998; Speer, Gutmann, & Murphy, 2005). However, not all schools offer regular, consistent teaching training that help first-time TAs adjust to their new job (McGivney-Burelle, DeFranco, Vinsonhaler, & Santucci, 2001). In this poster we will look into such a preparation program at a large midwestern R1 university and its evolution, as experienced by the author. The math TA preparation at this institution is presented as a semester-long course offered each fall, and mandatory for all incoming first-year graduate students, as well as new math TAs hired from outside the mathematics department.

In this study we encounter reflections on the three versions of the TA preparation course. First is the reflection of the author on their experience in the Version 1 course as a first-year math graduate student in the department. Version 2 is a course model based on the author's research into the extant literature on graduate math TA training. Version 3 is the current Fall 2018 iteration of the TA preparation course, which is co-facilitated by the author of this paper in the role of one of the Teaching Assistant Coordinators.

This poster highlights the variation of the course goals and its hidden curriculum (e.g. Martin, 1976) from year to year, depending on the instructor in charge. For example, the amount of class time spent on discussing "What it means to be a math graduate student?" varied greatly depending on the instructor, as well as attitudes towards teaching. Such variability of goals (e.g., teaching as a primary focus vs. teaching as a secondary focus), assignments (e.g., writing NSF grant proposals vs. writing teaching statements) and the overall tone (e.g., spend as little time on teaching as possible vs. teaching is an important part of your job) of the course has led to highly variable outcomes and student perceptions of the department's TA preparation over the years. Thus, there is a need for more consistency in the course, as well as better practices in its instruction, especially taking into account that this might ultimately be the only teaching preparation that graduate students going into academia would ever get.

As a result of these experiences, the author produces a list of suggestions for Version 4 based on all three previous versions, the extant literature and constraints within the department towards a more realistic and productive course, which could benefit the author's own department, as well as other institutions looking into starting or improving their TA teaching preparation.

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Like it or Love it: Exploring Elements Affecting Student's Mathematical Achievement

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Mathematics achievement, both in high school and early in college, is one of the strongest predictors of college completion. Research conducted within the framework of expectancy-value theory has shown that math interest, utility, engagement, self-efficacy, and identity are related to mathematics achievement. Hence, this study uses structural equation modeling to evaluate Ford's (2017) empirical model linking mathematics beliefs and achievement with a sample of students enrolled in multiple sections of two algebra-focused remedial math courses at a community college near a midsize metropolitan southern city in the United States.

Keywords: remedial mathematics, expectancy value theory, community college, mathematics achievement list

Chen (2016) documents that mathematics achievement early in college is one of the strongest predictors of college completion and community college students complete remedial mathematics courses and graduate at a significantly lower rate than students who start at traditional four-year colleges and universities. Expectancy-value theory (EVT; Wigfield & Eccles, 2000) provides a framework for exploring how students' beliefs and perceptions influence their mathematics achievement. Building on Eccles' EVT model of achievement-related choices (2005) and Middleton's model of mathematics achievement (2013), Ford (2017) proposed an empirical model of mathematics achievement using a nationally representative sample of 9th graders from the High School Longitudinal Study (HSLS: 09; Ingels et al., 2011). This study seeks to evaluate Ford's proposed model linking mathematics beliefs and achievement with a sample of community college students enrolled in sections of algebra-focused remedial math courses near a midsize metropolitan southern city in the US.

Structural equation modeling was implemented using MPlus. Results are shown in Figure 1 with dashed lines indicating non-significant pathways. Similar to Ford (2017), positive pathways linked (1) interest to utility, self-efficacy, and utility, (2) utility to self-efficacy, and (3) self-efficacy to identity; engagement and math achievement were negatively related. In this study, the pathways between (1) efficacy and engagement and (2) identity and achievement were not significant, whereas they were in Ford's study. Additionally, the R^2 for achievement was lower.

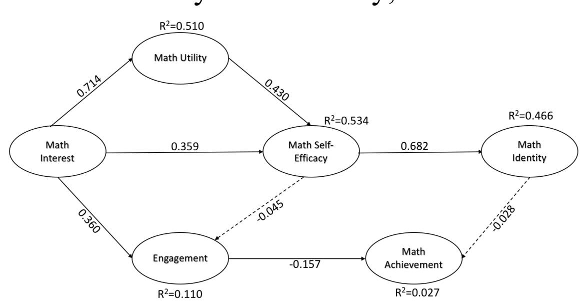

Figure 1. Model with Standardized Estimates

Implications for community college instruction, including ways to increase interest in mathematics, and future research plans with community college students will be discussed.

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Exploring Connections Between Students' Representational Fluency and Functional Thinking

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This study explores the relationship between preservice teachers (PTs) representational fluency and functional thinking. We conducted task-based interviews with five preservice teachers within a growing rectangle context. Although PT create, interpret, and connect representations with covariational and corresponding approach, they identified relationship between area and height an exponential rather than a quadratic. These findings illustrate a challenge of building rich conceptions of functions from emerging representational fluency and functional reasoning.

Key words: Representational Fluency, Covariational and Correspondence approach, Functions

This study is designed to understand relationship between representations and functional thinking by focusing on PTs reasoning with attention to *representational fluency* and *covariational* and *corresponding* approach about functions. Representation fluency (discursive activity in creating, interpreting, and connecting representations) supports meaning-making of mathematical objects (Fonger, 2018; Selling 2016). Students' flexible movement among and created relationships between each of numerical, graphical, tabular and verbal representations strengthened students understanding and sense-making of mathematical objects (Brenner, Mayer, Moseley, Brar, Duran, Reed, & Webb 1997; Yerushalmy, 2006). Covariation (change in one or more columns or compare change among different columns -e.g., change in x and change in y) and correspondence reasoning (a relationship between output values, range, related to input values, domain, and a symbolic equation between dependent and independent values) are important for strengthening sense-making of functions (Confrey, & Smith, 1995; Thompson, & Carlson, 2017). The research question: What is the relationship observed between preservice teachers' representational fluency and functional thinking?

We curated four quadratics tasks set in a rate of change context. We asked five PTs to pick one to solve and reported one results (Emma). We employed a representational fluency framework (Fonger, 2018); each method or approach to a problem was analyzed as a unit of analysis to discern meaningfulness in representational fluency according to four levels (pre-structural, multi-structural, unistructural, and relational). We used interview transcript with artifacts from PTs solutions, then compared and situated it in existing literature (Baxter & Jack, 2008). Despite evidence of PTs use and connection of representations, and covariation and correspondence approaches, PTs were not able to generalize their reasoning about a constant rate of rate of change to symbolize a quadratic function. Emma hypothesized the relationship between dependent and independent variables was exponential rather than quadratics. She said: "So the increases *[change in slope]* are not the same. Which would mean like it's not linear because it doesn't have a consistent slope which would make it exponential..." Our findings suggest that even for PTs, like Emma, who successfully employ covariational and corresponding reasoning with representational fluency on growing rectangle, it is not enough to identify functional relationships. Emma created graphical representations to argue that the function did not have negative domain with inconsistent rate of change, so the function must be exponential (without a symbolic representation to support their claim). This study highlights the importance of networking theories—representational fluency, functional thinking—to understand student reasoning. However, it also points to a need to better support PTs in linking these understandings to understanding and symbolizing functions as relationships of change and dependency relations.

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Faculty and Undergraduate Students' Challenges When Connecting Advanced Undergraduate Mathematics to School Mathematics

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When implementing lessons connecting advanced undergraduate mathematics to school mathematics, challenges arise for faculty and for the undergraduate students. The Mathematical Education of Teachers as an Application of Undergraduate Mathematics (META Math) project has created, piloted, and field-tested lessons for undergraduate mathematics and statistics courses typically part of a mathematics major that leads to secondary mathematics teacher certification. Lessons in calculus, discrete mathematics, algebra, and statistics explicitly link topics in college mathematics with high school mathematics topics prospective teachers will eventually teach. The goal of this poster presentation is to discuss our preliminary observations of the challenges faced by faculty and undergraduate students when implementing or using these lessons. We also wish to gather feedback and suggestions on the study design and potential directions for further research.

Keywords: Pre-service secondary teachers, Undergraduate mathematics, Curriculum modules

Based on recommendations in the Mathematical Education of Teachers II (MET II) report (Conference Board of the Mathematical Sciences, 2012) and the Statistical Education of Teachers report (American Statistical Association, 2015), the *The Mathematical Education of Teachers as an Application of Undergraduate Mathematics* (META Math) project integrates applications of advanced mathematics to high school mathematics and high school mathematics teaching into courses that are part of typical mathematics course sequences taken by mathematics majors intending to teach. The lessons enable faculty to maintain a focus on advanced mathematics topics while simultaneously integrating connections to high school mathematics.

After piloting lessons developed by META Math and monitoring faculty implementation of the lessons, preliminary data analysis and observations suggest that lesson development may need to address more scaffolding for faculty and students to strengthen their understanding of connections highlighted. Discussion with RUME attendees will assist us in identifying additional methodological and design components to strengthen the research team's goal in understanding: (1) How do faculty perceive connections between advanced mathematics and high school mathematics? (2) What connections do undergraduate students make between the advanced mathematics they are studying and the high school mathematics linked in the lessons? (3) How should faculty perceptions and undergraduate students' connections influence lesson design?

Lessons have been piloted at several universities across the United States. We apply a qualitative case study approach, in which each content area is a case. Data consists of in-depth qualitative observations of faculty using modules in their classrooms, preliminary and follow-up interviews with faculty, and cognitive interviews with students. This poster will provide preliminary findings from the field-testing of six lessons.

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Developmental Mathematics Students' Reactions to a Novel Tutoring Program

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Two populations of undergraduate mathematics learners can benefit from extra support: preservice elementary (PsE) teachers and developmental mathematics (DM) students. In this project, PsE students tutored DM students; the former gained experience working with students, the latter received extra, structured support. Results of a survey given to DM students show that although perception of the program was tepid, they were remarkably successful in the course.

Keywords: Pre-Service Teacher Training, Developmental Math, Student Thinking and Learning

Preservice elementary (PsE) teachers train to be generalists and are not required to have significant mathematical training, entering teaching with less mathematical knowledge. They oftentimes also have significant mathematics anxiety (Brady & Bowd, 2005). DM students are unable to place into college-level mathematics classes and often need extra support in other subjects as well (Bahr, 2007), implying that these students need help to develop general study skills. Both populations require low-stakes support to ensure their future mathematical success.

In this study, DM students and PsE mathematics students were partnered in small groups; the PsE students tutored the DM students. The goal was to provide PsE students with a low-stakes opportunity gain experience teaching mathematics and develop their mathematical discourse, while the DM students received extra support in a structured, supportive environment. This study explored the DM students' feedback through two research questions, *what were DM students' perceptions of this tutoring program* and *what were these students' outcomes?*

Population and Methods

The DM algebra students (n=9) were first-year students, retaking a course they failed the previous semester. Many of these students were on academic probation and at risk of being dismissed. They were required to attend one-hour tutoring sessions twice per week with the PsE tutors. Anonymous surveys were given after the final exam asking for feedback on the program.

Results and Discussion

Seven out of nine (78%) DM students passed the class, a remarkable rate compared to similar classes taught by the instructor and found in the literature (Bahr, 2010; Chen & Simone, 2016). Although the program was helpful, the DM students' support for the program was not strong. Four students agreed and five students were neutral to the statement "Tutoring helped me improve my math grade." Four students agreed, four were neutral, and one disagreed with "I am satisfied with the help my tutors gave me." The DM students supported the PsE tutors: six students agreed, two were neutral and one disagreed that "tutors made me feel comfortable and at ease" and eight agreed and one disagreed that "the tutors treated me in a respectful manner".

Open-ended responses show that DM students had a difficult time with scheduling, and were frustrated with tutors' difficulty with the material "they didn't know how to work problems out". As this pilot program was designed to provide confidence and academic support to both groups, future iterations will reset DM students' expectations by having students work cooperatively to solve problems, with PsE teachers more closely resembling experienced problem solving guides rather than mathematics experts. Results from this pilot will be shared to develop students' trust.

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An Inquiry-Oriented, Application-First Approach to Linear Algebra

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The IMAGEMath project combines inquiry based learning with an application-inspired approach. Students first learn about an application, and then, in an inquiry framework they develop the mathematics necessary to investigate the application. A novel feature of this approach is that the applied problem inspires the mathematics, rather than the applied problem being presented after the relevant mathematics has been learned. In this poster, we give an overview of the IMAGEMath modules that use image and data applications (radiography, tomography and heat diffusion) to inspire linear algebra topics. We present results from implementing the modules on a small scale at a few institutions, including student and faculty feedback. We also provide information for faculty interested in using IMAGEMath materials.

Keywords: Linear Algebra, Inquiry-based Learning, Technology, Applications

The IMAGEMath Project (www.imagemath.org), funded by a collaborative NSF IUSE grant (DUE-1503929, DUE-1642095, DUE-1503870, and DUE-1503856), consists of a suite of classroom modules that use data applications to inspire mathematical concepts in upper-division math classes. In this poster, we focus on the two IMAGEMath modules designed for linear algebra. The two data applications (brain scan tomography and diffusion welding) motivate most topics taught in a standard linear algebra course. These modules use inquiry-based learning strategies and group work to guide students to forge mathematical tools.

In brain scan tomography, students' main goal is to reconstruct a 3D view of a brain based on 2D radiographic data. In the process, they discover and work with vector spaces, span, linear independence, linear transformations (and properties such as injectivity and surjectivity), inverse transformations, and pseudo inverse transformations like SVD.

In the diffusion welding setting, students must predict how long it will take for a diffusion-welded rod to cool to a safe temperature. The heat diffuses out the ends of the rod as the weld sites cool, causing the temperature profile to change over time. Along the way, students develop the ideas of eigenvectors, diagonalizability, and long-term behavior.

Active learning and inquiry based learning techniques were studied by Kogan & Laursen (2014), and inquiry methods found to be at least as effective as traditional methods for all students and more effective for some groups of students. Additionally, the incorporation of applications has consistently been recommended as a good practice for linear algebra courses (see (Carlson, Johnson, Lay, & Porter, 1993) and (Zorn, 2015)). While the IMAGEMath project incorporates both of these, the application-inspired approach at its core differs markedly from other application-integrating approaches that illustrate the use of learned tools on real problems. IMAGEMath modules begin by introducing a cutting edge research problem. The solution path inspires the development of mathematical concepts.

We administered pre- and post- content and attitudinal surveys. Content results were positive. Results of the attitudinal surveys were statistically inconclusive. In this poster we also present student and faculty comments. In the future, we hope to study on a larger scale the efficacy of the linear algebra IMAGEMath materials. In addition, should this prove to be a fruitful approach, our vision is to create a community of undergraduate faculty interested in module development using other applications, targeting linear algebra or other courses.

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Observing active learning in mathematics classes: Do we have the right tool?

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Observation protocols allow researchers to document moments of teaching and learning, as well as reveal inequities and opportunities for improvement. In two undergraduate mathematics courses, I used the OPAL protocol to understand if and how active learning strategies created equitable learning environments. In this poster, I share findings from observations and discuss possibilities for adapting observation protocols to align with equitable teaching practices.

Keywords: observation protocol, methodology, active learning, equity

Background and Motivation

Recently, there has been a general effort in STEM departments across the nation to implement active learning (AL) strategies at the university level (CBMS, 2016). While some researchers claim that AL is the best way to help students learn mathematics (Freeman et al., 2014; Prince, 2004), others question whether issues of inequity arise in classrooms where students actively participate and collaborate (Gehrtz, Sampera, & Ellis, 2017). Considering the increased focus on issues of equity in mathematics education research (Adiredja & Andrews-Larson, 2017; Aguirre et al., 2017), I ask how researchers can continue to examine classrooms where instructors and students engage in AL strategies. To this end, this poster illustrates how I used and adapted a well-known observation protocol in order to document qualities of equitable learning environments in mathematics.

This poster will represent research that addresses the following questions:

1. What are examples of appropriate observation tools that explore the qualities of equitable learning environments in active learning mathematics classrooms?
2. How can we use observational data to examine issues of equity in these classrooms?

Methodology and Findings

These preliminary findings report observation data from two undergraduate mathematics instructors who teach entry-level courses at the same large, public university. Although both instructors took a student-centered approach to their teaching, they modified two traditionally lecture-driven courses using various collaborative and technology-based teaching practices.

Over one semester, I observed both instructors multiple times using the Observation Protocol for Active Learning (OPAL) (Frey et al., 2016). OPAL has been validated for undergraduate STEM classes that use an AL approach, and thus was an appropriate observational tool for my study. Codes for this protocol were created by the authors or adapted from the Teaching Dimensions Observation Protocol (TDOP) (Hora, Oleson, & Ferrare, 2013) and the Classroom Observation Protocol for Undergraduate STEM (COPUS) (Smith, Jones, Gilbert, & Wieman, 2013). In addition to the original codes, I developed some of my own after frequent occurrences during observations. For example, I noticed that both instructors frequently called students by name in an attempt to create a comfortable learning community, so I created a code to record these instances. I plan to discuss these new codes with fellow scholars and open the conversation for further adaptations to observation protocols that address equitable teaching.

The poster will provide quantitative and qualitative data obtained from the OPAL tool, as well as comparisons of other observation protocols used in undergraduate STEM courses.

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Predicting Final Grades in Calculus using Pre- and Early-Semester Data

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It is well-known that too many students abandon a STEM career because of their calculus requirement. Therefore, being able to identify early on which students may be at risk of failing is important. Using indicators of mathematical readiness (SAT/ACT and PCA) and attitudes toward mathematics (MAPS), we build models predicting final grades. Our analyses show that all three indicators are significant predictors of success in calculus.

Keywords: PCA, MAPS, calculus, student success

Historically, first-year calculus courses have high D-Fail-Withdraw (DFW) rates and because of these many STEM majors, including STEM education majors, are driven away (Bressoud & Rasmussen, 2015). This situation is highly detrimental for the U.S. as its economy is increasingly reliant on STEM workforce (Olson & Riordan, 2012). In this study, we investigate how to predict success in calculus using pre- and early-semester data. Having such a model would enable instructors to identify early on which students are at risk of failing.

To explore our question, we have combined two approaches used in the literature. First, we have assessed students' mathematical readiness for calculus using the Pre-Calculus Concept Assessment (PCA) developed by Carlson *et al. (2010)*. Second, we have evaluated students' confidence and attitudes toward mathematics using the Mathematics Attitudes and Perceptions Survey (MAPS) (Code, Merchant, Maciejewski, Thomas, & Lo, 2016). We have also controlled for SAT/ACT scores. The population is students in two introductory calculus courses in a large private research university in the Northeast. Data was collected in Fall 2017 and Spring 2018. Students completed the two surveys in the first two weeks of the semester.

Linear models using only one indicator (SAT/ACT, PCA or MAPS scores) indicate that taken individually, these variables are all statistically significant predictors but explain relatively little of the variation of the final exam grades (adjusted R^2 of 0.26, 0.07 and 0.14, respectively). Using a multiple regression model (N=531), we found that SAT/ACT scores (b = 6.21, beta = 0.4007, t(527) = 10.69, $p<0.001$), PCA scores (b = 0.57, beta = 0.206, t(527) = 5.382, $p<0.001$), and MAPS scores (b = 14.42, beta = 0.208, t(527) = 5.823, $p<0.001$) are all significant predictors of final exam scores. Moreover, this model explains a larger variation of the final exam grade (adjusted $R^2 = 0.36$) than linear models using only one of these indicators. Looking at the beta scores, it is interesting to note that the PCA and MAPS have nearly the same effect on the final grade. This supports the idea that both mathematical preparedness and attitude toward mathematics are important for being successful in introductory calculus. The implication for classroom practice is that instructors should not only help students reinforce their mathematical knowledge *but also* support them in developing an expert attitude toward mathematics. A multiple regression model using the MAPS sub-scores (that evaluate different aspects of attitudes and perceptions of math) indicate that Confidence in one's ability to successfully engage in mathematical tasks and Persistence when solving non-routine exercises are the most important of these aspects.

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Mathematical Knowledge for Teaching in Chemistry

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The Mathematical Knowledge for Teaching (MKT) theoretical framework describes effective mathematics teaching in a way that relies on an instructor's subject matter knowledge (SMK) and on their pedagogical content knowledge (PCK). This proposal reports our initial effort to understand MKT within chemistry instruction, namely what MKT could support chemistry instructors' efforts to help students develop a deeper understanding of the mathematics used in general chemistry. Coding of several types of general chemistry problems involving ratios and proportions and covariation are provided as examples.
Keywords: Mathematical Knowledge for Teaching; Chemistry Instruction

Students' challenges with the mathematics used in general chemistry are long-standing (Kotnik, 1974) and persistent (Muzyka, 2018). Efforts to address the deficiencies in mathematics preparation that impact outcomes in general chemistry have largely focused on providing students with more practice of procedures. Simply giving students more practice without building understanding of the underlying mathematics is unlikely to have long-term benefit and will not prepare students to address novel problems.

Suppose that the focus was shifted to building chemistry instructors' ability to anticipate, identify, adapt, and respond to students' difficulties with mathematics in chemistry. We propose that building chemistry instructors' mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008), which combines subject matter knowledge with pedagogical content knowledge (PCK) (Shulman, 1986), would equip them to address the challenges faced by their students and support students in building a deeper understanding of the mathematics used in chemistry.

This poster reports our efforts to characterize mathematical knowledge for teaching applied to the context of chemistry instruction, specifically common content knowledge (CCK) and PCK. We used the *Common Core State Standards for Mathematics* (National Governors Association, 2010) to systematically code the CCK required for general chemistry instruction. In addition, our framework identifies PCK into the categories of known difficulty, pedagogical opportunity, anticipated gaps in prior knowledge, and areas of difference between chemistry applications and mathematics instruction. Our results show the particular importance of chemistry instructors' PCK of ratio and proportional reasoning and covariation, as this content surfaces throughout general chemistry.

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Students' Views of the Relationship Between Integration and Volume When Solving Second-Semester Calculus Volume Problems

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Volume problems are a typical first type of integral application problem that students encounter in second-semester calculus. We will present students' responses to the question, "Why does integration give a volume?" Participants were Calculus 2 and Calculus 3 students enrolled in summer classes at a large, public university. Task-based interviews consisted of students working on and discussing second-semester volume problems. Students had varied and interesting responses that included formula-, units-, and derivative/antiderivative-based reasoning. These results are part of a larger study on how students understand the underlying structure of the definite integral, and how they use pictures and visualizations when solving volume problems.

Key words: Calculus, Definite Integral, Volume, Student Interviews

Second-semester calculus volume problems are a standard first step in the study of applications of integration. Previous research has found that when solving definite integral application problems, students often rely on formulas, patterns, and previously encountered methods for setting up integrals (Yeatts & Hundhausen, 1992; Grundmeier, Hansen, & Sousa, 2006; Huang, 2010). Other studies have shown that students have very little idea of the dissecting, summing, and limiting processes involved in integration (Orton, 1983; Sealey, 2006, 2014; Jones, 2015). The overarching goal of this research is to investigate how students understand the underlying sum-of-products structure of integration when solving volume problems. The focus of this poster will be on student responses to the question, "Why does integration give a volume?"

This research is built on the foundation of the constructivist learning theory (Piaget, 1970), and the framework guiding analysis of student understanding of definite integral concepts is based on Sealey's (2014) Riemann Integral Framework.

Clinical interviews were conducted with 10 students who were enrolled in Calculus 2 or Calculus 3 during the summer 2018 semester. The video-taped, one-on-one interviews involved the participants working through three volume problems and talking aloud about their thought processes and problem-solving strategies. The interviewer asked several questions throughout the interview in order to determine if the students could unpack their methods to explain why they worked. The videos are transcribed and data analysis is ongoing.

Responses to the question, "Why does integration give a volume?" varied from formula-based explanations ("Because the formula does it?") to focusing on units ("…because meter times meter times meter gives you meters-cubed which is a volume") to a derivative/antiderivative connection between volume and area ("Integrating area gives you volume"). As we continue to analyze this data, we will investigate the connection between students' responses to this question and their abilities to solve more complex volume problems.

With this research, we hope to develop activities for second-semester calculus students that emphasize their understanding of the underlying Riemann sum structure of integration and foster a deeper appreciation for how integration can be used in many different situations.

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Exploring Remedial Math through a Number Course for Preservice Teachers

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This proposal describes a pilot study of replacing a remediation mathematics class for undergraduates with a credit-bearing mathematics course designed for elementary education majors. The replacement course focused on number and operations. Results of pass rates, placement test scores (pre and post the course), as well as course feedback will be shared.

Keywords: Undergraduate mathematics, Algebra, Remediation

According to the National Center for Education Statistics report by Chen (2016), "33 percent of students entering public 4-year institutions took a remedial math course" (p. 16). Many of these remedial math courses are targeting algebra, which is a gatekeeper for many students in successfully earning degrees (Moses & Cobb, 2002). Remedial coursework is widespread and impacts both advantaged and disadvantaged populations of students (Chen, 2016). In addition, Complete College America's (2012) report estimates that $3 billion are spent on courses, such as remedial math courses, that are not going to count towards a degree.

In order to address the problem of offering remedial algebra classes at a university in the Northeastern United States, this semester a math course for elementary teachers focused on number and arithmetic was designated as a replacement for the remedial algebra course. This course would allow students to earn three credits towards degree completion. Two sections of the math course for elementary teachers were taught by the same instructor with one section having primarily remediation students (N = 18) and the other section primarily having education students (N = 23).

This pilot study examines the success of the use of the math course for elementary teachers as a replacement for the remedial math course through a few different measures. The analysis focuses on the remediation students (N = 24), compared to the education students (N= 17) and the remedial students in the two sections of the traditional remedial algebra course offered during the same semester (N = 39). One measure of success is examining pass rates of the pilot group compared to pass rates of the other two sections of the remedial math course and the education students. In this regard, success is measured by undergraduate students earning mathematics credits by passing the course as well as the course grade positively impacting the students overall grade point average. Another measure of success will be examining pre-test scores compared to post-test scores on a placement test (ALEKS) before and after the course by both remedial and education students. Information from students around their mindset when taking the pre-test as well as the circumstances of the pre-test will be taken into account as the scores are examined. The pre-test was taken by the students off campus on their own terms. Historically, students have gotten assistance on the pre-test or have not taken the test seriously by taking it on a cell phone or just completing it quickly. Finally, course feedback will be analyzed and shared.

Because the course is being completed now, the analysis has not been completed. The poster, however, will share results of the described analysis. These results will indicate what, if any, measures of success were gained. Future research might investigate changes in beliefs and attitudes about mathematics as well as incorporate qualitative analysis from data sources such as student interviews while implementing a replacement course such as this.

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Students' Proving as a Collaborative Work-in-Progress:
The Case of a Graduate Course in Topology

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We observed recordings of instances from a graduate course in topology where students engaged in proving theorems on the whiteboard in a collaborative environment. We considered the written component on the whiteboard as "the proof", which was aided, in 17 out of 20 instances by some form of verbal explanation. The peculiarity of the class structure allowed each lesson to be followed by an open discussion regarding "the proof". As a result of the discussions, the written component of each proof would undergo improvements. When analyzing the developments of the proofs in this course, we employed the thematic of proof introduced by Mariotti. Stemmed by these proof-presentations, we introduce the idea of proving as a "work-in-progress" activity.

Keywords: Collaborative learning, Proof and proving

As part of a larger project on teaching and learning of topology in a collaborative and discursive classroom setting, we analyzed video recordings of an activity that has been atypical to traditional university courses (Pinto & Karsenty, 2018) – students proving theorems on the whiteboard and discussing their proofs with their peers and the course teacher. Our study was aimed at characterizing such an activity with a special focus on what is said, what is written, what is gestured, and the coordination between the three. In 17 out of 20 instances of the activity that took place during a semester, the written component was treated as "the proof", when the verbal counterpart played an auxiliary role of explaining "the proof". The remaining three instances converged to the written components only as the provers did not accompany their work by verbal speech.

After provers have concluded the described phase, the classroom floor was opened for a discussion, in which the rest of the students raised clarification questions and offered suggestions on how the written component could be improved. This part often resulted in the prover revisiting their writing to account for the received feedback. These developments instantiate what Mariotti describes how all proofs have to undergo through a negotiation that leads to social acceptance. An acceptance, in our context, occurred in a classroom setting. Accordingly, this phase can be associated with Mariotti's theoretical frame positioning the activity of proving "as work-in-progress". Indeed, even after the described improvements, it is plausible to think that students' proofs could be further enhanced.

Several conclusions could be drawn from the presented conceptualization. First, proving as a work in-progress is a social and situational activity that mimics to some extent the activity of professional mathematicians. Second, every proving instance of such a kind is unique since neither the coordination of its written, spoken, and gestured components nor the following discussion may be replicated. This uniqueness challenges a common view of students and teachers, in which a resulting text that emerges from a proving activity is treated as an object with proving powers that are indifferent to time, place, and people who engage with it.

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What Content is Being Taught in Introductory Statistics?: Results of Nationwide Survey

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Introductory Statistics is a course commonly taken by students from a variety of wide-ranging majors, sometimes across departments; however, there is little known about the extent topics are covered generally across courses. Textbooks include more material than can reasonably be covered in a single course, but the non-linear nature of many topics means that from course to course the covered content can diverge greatly. We provide results of a nationwide survey of 148 introductory statistics instructors and assess how often concepts are covered in introductory courses across instructor experience, course audience and course pedagogy.

Keywords: Instructors, Content, Curriculum, Introductory Statistics

Introductory statistics course content varies from school to school, and instructor to instructor. The content that a student is exposed to in an introductory statistics course can vary greatly, particularly when compared to a course like Calculus I. This affects common university interactions such as transfer credits and compromises researchers' ability to generalize introductory statistics course studies to larger populations. Statistics education research papers often dedicate substantial space to articulating what content the course(s) of study covered. Yet, relatively little is known about what content is being covered in introductory statistics courses in the United States. Nationwide surveys that poll the demographics of mathematics content areas have been useful in producing recommendations for best practices in teaching the content and providing insight into areas of future research (Bressoud et al, 2015; Johnson el al, 2018). In the most recent *International Handbook of Research in Statistics Education*, researchers have called for similar efforts (Gould et al, 2018). In response we report on the content taught in Introductory Statistics based on a nationwide survey of instructors. The participants (n=148) were selected through a cluster sample of all possible 2 and 4-year institutions such that every listed instructor of introductory statistics classes from 80 randomly selected institutions for spring 2018 was contacted and asked to respond to a Qualtrics survey (response rate 27.2%). The survey included topics about their course content, instruction decisions and demographic information.

We report results from this survey focusing on course content including the proportion of time content is included in a course. Additionally, we consider how the course content differs over the instructor, audience of the course, and instructional tools employed. For example, 14% of the instructors report covering multiple linear regression; however, of instructors who report having a statistics degree, only 2.8% of them report covering multiple linear regression.

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Assessing Conceptual Learning in Calculus I: Preliminary Results and Future Ideas

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Our initial project focused on assessing conceptual understanding of key topics in Calculus I, specifically measuring changes in the achievement gap between underprepared and prepared students in Active and Traditional classrooms. However, a main hurdle is the lack of instrument for assessing Calculus readiness. In this poster, we present results of student understanding of continuity in Active vs. Traditional settings from 16 sections of Calculus I. We present ideas for refining this study to be able to better assess student growth by creating and validating questions regarding students' initial understanding of Calculus topics: continuity, differentiability, limits, and area. We present our study design and initial findings; we look forward to feedback as we enter the latter half of our project.

Keywords: Calculus, Active Learning, Assessment, Task Design

Calculus I is a crucial benchmark in the path to a STEM education; however, many students rely heavily on memorization and repetition as paths to success in mathematics. These techniques fail when they are asked to explore the abstract concepts *of limits, continuity of functions, differentiability,* and *area*. One pedagogical approach to increasing student understanding and mastery is active learning. Active learning activities provide a setting for students to learn in cooperation with others, thus placing them in an excellent environment to construct complex mental frameworks (Bransford et al., 1999; Vygotsky, 1978). Existing literature supports the idea that active learning techniques can increase student learning outcomes significantly (Freeman et. al, 2014; Bressoud, 2011; Haak et. al, 2011; Boaler & Greeno, 2000). In this project, we study active learning specific to the calculus classroom.

In the initial phases of our project, we targeted the population of students who enter calculus with deficiencies in algebra, trigonometry, and/or pre-calculus. One question we attempted to explore was the following: Does the performance gap between underprepared and calculus-ready students change to a different extent in an active classroom as compared to a traditional classroom? We compared student-learning outcomes in four classrooms employing active techniques to outcomes in four traditional lecture-based classrooms in each of Fall 2017 and Spring 2018. Due to a lack of instrument for assessing calculus readiness, we chose to use the Precalculus Concepts Assessment (PCA) (Carlson, Oehrtman, & Engelke, 2010) to identify students with weak preparation. During both semesters, the active sections discussed each of our target concepts: limits, continuity, differentiability, and area, using a common exploratory activity, discussion, and follow-up assignment. The traditional sections covered the same content, but from a lecture approach. We assessed learning outcomes by scoring performance on in-class exams and again administered the PCA as a post-test. Unfortunately, the PCA was not adequate for distinguishing between prepared and underprepared students or for answering our research question. However, our preliminary analysis of final exam data involving continuity revealed that students in the active sections performed better than their traditional counterparts on the continuity exam questions. Our next plan is to refine our study to be better able to assess student growth by creating and validating questions regarding students' initial understanding of our four target calculus concepts, and we look forward to feedback.

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Connecting Constructs: Coordination of Units and Covariation

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We investigate links between units coordination structures and covariation schemes.

Keywords: *Covariation, Quantitative Reasoning, Units Coordination, Multiplicative Object*

We hypothesize the units coordination structures underlying the construction of multiplicative fraction schemes are also important for covariational reasoning. We used a units coordination diagnostic assessment (Norton et al., 2015) for clinical interviews with college calculus students. We interviewed 25 students about their units coordination structures and fraction and measure schemes. Eleven students agreed to a follow up interview on covariation tasks such as the Bottle Problem (Carlson, et al, 2002). So far we have analyzed eight videos to identify connections between students' units coordination structures and covariation schemes.

Theoretical Perspective. Units coordination refers to the mental operations that begin in childhood with counting schemes that are reorganized in the construction of whole number multiplication and fractions meanings (Steffe & Olive, 2010). Thompson et al., (2017) studied covariational reasoning in 487 teachers and found "one must construct a multiplicative object of quantities' attributes in order to reason about their values covarying smoothly and continuously (p. 128)." Someone with an advanced covariation scheme holds "in mind a sustained image of two quantities' values (magnitudes) simultaneously" and a multiplicative object is formed from the two quantities (Saldanha & Thompson, 1998, p. 299). A person who has constructed a multiplicative object "tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha and Thompson, 1998, p. 299). Frank (2017) explored students' covariation schemes and found that, "imagining little bits of change is essential to construct an image of a quantity's chunky continuous variation (p. 281)." We hypothesize that coordinating two varying quantities and small associated changes in both quantities requires assimilating tasks with three levels of units.

Results. Three of the eight students who have been analyzed consistently assimilated with two levels of units. These students had gross covariation schemes but did not have chunky covariation schemes (Thompson and Carlson, 2017). Further, these students did not talk about associated changes in quantities. Four of the eight students assimilated tasks with three levels of units. Three of these students had constructed smooth or chunky covariation schemes. One of the four only had a gross coordination scheme, suggesting that the construction of three levels of units does not guarantee a student will construct a chunky covariation scheme. The final student sometimes assimilated tasks with two levels and sometimes three levels and she had a fragile chunky covariation scheme. She knew she needed to consider associated changes in quantities but had trouble keeping track of the relationships. Although the sample is small, the empirical evidence is consistent with the hypothesis that constructing three levels of units is important for covariational reasoning. Further research should be done to link research in these two areas.

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Understanding Calculus Students' Thinking about Volume

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We present the methodology and preliminary findings from a pilot study undertaken at three institutions during Spring 2018. Our purpose is to uncover student reasoning around volumes of solids of revolution. Initial findings suggest issues arise in the Product layer of the Riemann Integral Framework (Sealey, 2014).

Keywords: Geometric and Spatial Thinking, Calculus, Post-Secondary Education

Motivation and Research Question

The purpose of this study is to explore student thinking of volume in the context of a second semester calculus course. We are interested in sharing our methodology and in exploring its applicability to a follow-up research project.

The central research question that guides this study is, *how do students think about the Riemann Sum and Integral when approximating and computing volumes of solids of revolution?* We consider the Riemann Integral Framework (Sealey, 2014), and the role of visualization (Giaquinto, 2007; Tall, 1991) to make sense of student thinking evident in our data.

Methods

Student participants in this study were enrolled in Calculus II. We videotaped interviews of three pairs of students from three institutions as they collaborated on mathematical tasks. Students were paired so that we might capture student thinking through discourse. The tasks we created focused on volumes of spheres. Our first problem prompted students to approximate the volume of a rotated semi-circle by slicing into a fixed number of pieces. Follow-up prompts asked students to approximate using an arbitrary number of slices and to compute the exact volume of the resulting sphere. This task also raised questions around the role of dx and Δx in students' computations. In addition to using routine tasks (Brestock and Sealey, 2018), we included Kepler's volume approximation of a sphere (using infinitesimal cones) and asked students to explain this method. Non-routine problems had the potential to reveal student thinking as they moved from *n* subdivisions to the actual volume of a solid.

Results and Discussion

Preliminary results reveal that students had a strong conceptual grasp of the role of dx and Δx in their computations and they understood the exact volume as the limit of a finite sum. However, students had difficulty visualizing appropriate slices of the solid of revolution as cylinders. We partially attribute this difficulty to teaching, where approximating area is given more instructional time than approximating volume. Students also had difficulty distinguishing between varying and constant quantities in setting up a Riemann Sum, e.g. height and radius of a disk. We position this difficulty within the Product layer of the Riemann Integral Framework (Sealey, 2014). As we frame our study based on this pilot project, we hope to revise tasks so that we may gain additional insight into such difficulties. We are also considering how visual thinking arises in the Riemann Integral Framework (Sealey, 2014), as we found this to be essential in constructing volume approximations and deciding on quantities that vary in such problems.

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A Single Case Study of Smartpen-enhanced College Algebra Tutoring

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Students taking courses below calculus are an understudied population in undergraduate mathematics education, as are students with mathematics difficulty or disability. Students seeking additional help are likely to seek YouTube and other outside resources, which may not mesh with in-class instruction. Since secondary education research suggests that targeted tutoring is beneficial to students with mathematics difficulty or disability, this single case study investigated if there was a functional relationship between a Smartpen to create videos for a student with a mathematics disability to listen to during tutoring sessions and her achievement on synthetic division problems over a four week intervention.

Key words: college algebra, disability, single case design,

Students with mathematics difficulty and disability are an understudied population in undergraduate mathematics education. Additionally, while there has been research on college students with disabilities in multicultural and special education (eg. Getzel & Toma, 2008; Wisbey & Kalvodia, 2011), there has never been a study in undergraduate mathematics education focusing on this population of students (Speer & Kung, 2016), and college instructors receive no training in supporting students with disabilities.

Previous research at the secondary level indicated that targeted one on one tutoring is most effective at helping students through algebra, which is similar in content to entry level undergraduate mathematics courses below calculus (Burton, Anderson, Prater & Dyches, 2013). Struggling students are most likely to seek help using online resources such as YouTube (Dibbs, Rios, & Christopher, 2017), but YouTube videos are rarely targeted to the learning objectives or teaching technique of a specific course. Additionally, most YouTube videos intended for undergraduate mathematics students are often 10-20 minutes long and can be difficult for struggling students to follow (Dibbs, Rios, & Christopher, 2017). Although Smartpens have not been studied more in the context of a data collection instrument, group work, or online instruction (Czocher, Baker, Tague, & Roble, 2013; Dibbs, Beach, & Rios, 2018; Fisher & Raines, 2014; Tague & Czocher, 2013).

We conducted a changing criterion single- subject design with one participant enrolled in college algebra diagnosed with a mild learning disability. This design is to evaluate the outcomes of individuals instead of groups and compares the effects of different conditions on individuals. First we collected observation data by watching how the participant performed mathematics. After we have created a stable baseline, we then introduced the Smartpen videos. We used the smart pen during the intervention. We collected data weekly for four weeks. Although the visual inspection of the data revealed a moderate functional relationship between the Smartpen supported tutoring and participant achievement, the maintenance check indicated student retention of learning the material (synthetic division) beyond what the student reported in previous exposure. Furthermore, the social validity check following the conclusion of the intervention indicated that the participant found the Smartpen-supported tutoring to be more effective than the one on one tutoring she had received through both disability support services and the baseline phase of the study.

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Examining the Effectiveness of Culturally Relevant Lessons within the Context of a College Algebra Course

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In an attempt to bring more realistic situations into college mathematics classroom environments, lessons were created that utilized culturally relevant pedagogy for a college algebra course at a large historically black college/university (HBCU) in the south. These lessons were aimed at the growing population of diverse students in an effort to gauge their effectiveness with students, in regards to achievement and self efficacy. This poster will illuminate the conceptual developmental process of four "experimental" lessons and provide some preliminary findings of the course that utilized these lessons in comparison with a control class that did not.

Keywords: College Algebra, Equity and Diversity, Student Affect, Curriculum

While algebra is a gateway course for high school graduation (Moses, Kamii, Swap, & Howard, 1989), college algebra is a gateway to graduation for many non-STEM college majors (Van Dyken, 2016). Each year, only 50% of students are successful enough to earn a grade of A, B, or C in their college algebra courses (Ganter & Barker, 2004). This means that half the students who are enrolled in this entry level mathematics course are receiving grades of D, F, or are withdrawing from the course. This is extremely problematic when we couple this with the fact that most college majors require students receive a C or better in this course to make adequate progress toward their degree. The purpose of this research study is to investigate the following hypothesis: Student outcomes, including self-efficacy, will be improved by participation in a college algebra class at a historically black university, where the instructor uses culturally relevant pedagogy (CRP) in the course. The effect of teaching with culture has been shown to have a substantial increase in self-confidence and self-efficacy; effectively replacing feelings of failure and alienation that is all too common with the subject of mathematics and students of color (Aronson & Laughter, 2016; Dover, 2013; Tate, 1995). CRP is founded upon three principles: academic rigor, cultural competence, and sociopolitical consciousness (Ladson-Billings, 1995a, 1995b). In accordance to the paradigms of CRP, four lessons were developed.

The four lessons were delivered in alignment of three of the units taught in this course: Functions and Graphs, Polynomials and Rational Functions, and Exponential and Logarithmic Functions. The CRP lessons were composed of Matthews, Jones, and Parker's (2013) Culturally Relevant Cognitive Demand Mathematics Task Framework and its corresponding evaluation tool. Each lesson was designed to not only appeal to students' racial backgrounds, but also their cultural backgrounds. The lessons attended to the issues of the urban city the students attend school in, and topics related to college-aged students and experiences that are relevant to them. Specifically, the lessons began with a guiding question that students investigated using mathematics. They were entitled: What does incarceration look like in County X [pseudonym], and the United States?. What are the ramifications of the collegiate 'Cuffing Season'?, What is the true price tag of a college degree? and How is the population of the United States changing?

This poster will contain the conceptual developmental process of these four "experimental" lessons and provide some preliminary findings of the course that utilized these lessons in comparison with a control class that did not.

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Students' Understanding of Trigonometric Functions in an Active-Learning Course

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Minimal research has been conducted surrounding how best students learn trigonometric functions in a precalculus course. Using motivation from a study conducted by Weber (2005), results from this study indicate that students who participated in a college level precalculus course where the unit circle was taught before right triangle trigonometry were better able to utilize a unit circle but struggled to conceptualize some of its properties. This report has implications for mathematics programs looking to determine best practices for the instruction order of precalculus courses.

Keywords: Precalculus, Trigonometric Functions, Student Thinking and Learning

In the mathematics education community of researchers, there is the understanding that the goal of mathematics is to learn mathematics with deep understandings – methods that go beyond memorizing facts and formulas to provide correct solutions on worksheets and examinations (Common Core State Standards Initiative, 2010). Students should be able to use procedures and explain why they are appropriate and justify why concepts in mathematics have the properties they do (Weber, 2005). With this being said, researchers have noted that this does not always apply to the teaching and learning of trigonometric functions in trigonometry or precalculus classrooms (Thompson, Carlson, & Silverman, 2007; Weber, Knott, & Evitts, 2008). They are often geared towards the memorization of mnemonic devices or acronyms.

Weber (2005) states that the form of instruction that students receive will influence how they learn trigonometry. Generally, there are two forms of instruction commonly associated with the learning of trigonometric functions: (1) the method involving special right triangles, where the trigonometric functions are defined as ratios of the lengths of the sides in the right triangles, or (2) the unit circle method where the cosine and sine of an angle is defined to be the x- and y-coordinates of the point that rests at the terminal side of the angle that intersects the unit circle (Kendal & Stacey, 1998). Researchers have tested which instructional strategy will lead to a better understanding of trigonometric functions in order to aid students in overcoming their misconceptions they may have with them (Kendal & Stacey, 1998; Weber, 2005).

This case study is a smaller part of a larger mixed methodological exploratory research study designed to introduce active learning components to study how these new practices are implemented and how they affect student outcomes (Keene, Skrzypek, Downing, & Kott, 2017). Motivated by the dichotomous approaches to learning trigonometry by the work of Weber (2005) and Kendal and Tall (1998), the goal of this study is to see the extent to which students are able to reason through trigonometric concepts after engaging in team activities, in which students worked each other to work through conceptually-based tasks.

The poster will provide the preliminary findings to students' understanding through a semi-structured task-based interview. These results show that students were able to draw upon their experiences with team activities to aid them through tasks. use the unit circle at least as a reference, however, they struggled to conceptualize why it has the properties it does.

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Building Coherence in Circular and Complex Trigonometry with Inquiry-based Modeling

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The trigonometry is first framed on a right triangle, next on a unit circle with a parametrized pair of coordinates (r cos t, r sin t), and then on a complex frame, r (cos t + i sin t) unifying the Cartesian pair (Ekici, 2010). Students often struggle in understanding the connections and the transitions among triangle, circle, and complex trigonometry which serve as a critical mathematical foundation in many STEM fields. It is a challenge for students and teachers to coordinate the multiplicity of these trigonometric frames to develop coherent meanings. To support this transition, dynamic manipulatives using GeoGebra are here developed for student experimentation in modeling with modified circular and complex trigonometric functions. The results show that inquiry-based modeling using these multiple yet interconnected frames facilitate the emergence of coherence observed while validating these trigonometric models.

Keywords: inquiry-based learning, complex trigonometry, circle trigonometry

There is a need for a disciplined inquiry into the problem of teaching trigonometry towards building coherence across Euclidean, Cartesian and Complex frames in the teaching/learning practice with trigonometric functions (Ekici, 2010). Building coherence requires some deliberate focus on the connectedness of alternative mathematical frames in modeling periodic phenomena. Mathematical models can yield multiple solutions depending on the choice of mathematical frame, so the focus less on coming up with a specific answer and more on the validation of the model as framed (Anhalt & Cortez, 2015). Modified circular functions are here introduced here as a composition of sine and cosine functions with different periods. This approach is experimented here as a way to build advanced coherent perspective while modeling periodic functions in rich contexts such as sound modeling using alternative trigonometric frames.

Inquiry-based modeling with multiple mathematical frames is here adopted as a pedagogical strategy (Ekici & Plyley, 2018). GeoGebra applets are designed and offered by the author to help learners experiment and develop their models with dynamic manipulatives. Integrating such technologies for flipped learning provides extended support towards building connections within and between each trigonometric frame with critical reflections and anticipation. The emergence of coherence is observed in modeling with multiple trigonometric frames along a series of IBL lessons connected with a theme across the course. Collaborative action research is adopted to develop and refine an evidence based practice towards building coherence (Stringer, 2014).

The validation of the trigonometric models serves as a critical modeling stage examined across circle trigonometry, modified circular, and complex trigonometry. Through concept maps and reflections, the results of inquiry based modeling demonstrate that interpretation and validation of these multiple yet interconnected trigonometric models facilitate the emergence of coherence. This work informs the trigonometry practices in undergraduate and high school level providing an advanced perspective for teaching/learning trigonometry. The results show that inquiry-based modeling using these multiple yet interconnected frames facilitate the emergence of coherence observed while validating these trigonometric models. Reflections by learners provide evidence of their critical understanding of multiple trigonometric frames by observing how more simplicity is achieved in modeling with Complex frame as opposed to Cartesian.

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Relationship Between Precalculus Concepts and Success in Active Learning Calculus Courses

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As part of an ongoing project to redesign a calculus sequence centered around core calculus concepts through an active learning approach, we aim at understanding the knowledge students need in order to be successful in this setting. In particular, we are interested in exploring what conceptual understandings of precalculus concepts support students in an active learning intensive calculus sequence. We present preliminary results of an analysis carried out to answer the question: What is the relationship between students' precalculus understandings and performance in this newly redesigned calculus sequence?

Keywords: Assessment, Calculus, Student Outcomes, Active Learning

There has been a recent call for an increase in STEM bachelor's degrees (Olson & Riordan, 2012) and it has been suggested that empirically tested and validated teaching practices, like active learning, are critical to attain this goal (Freeman et al., 2014). In response to this call, there is an innovative calculus curriculum currently being developed and implemented by mathematics education faculty and graduate students at Portland State University. This curriculum has been designed with active learning strategies in mind and was adapted from Pat Thompson's DIRACC project which is grounded on research pertaining to students' mathematical thinking and understanding of core calculus concepts (Thompson, Byerley, and Hatfield, 2013).

For this study, we present some preliminary results aimed at answering our research question: *What is the relationship between students' precalculus understandings and their performance in this newly redesigned calculus sequence?* Data for this analysis was collected during the 2018 summer term. The pre-assessment we administered consisted of six items adapted from the Precalculus Concept Assessment (Carlson, Oehrtman, and Engelke, 2010), Aspire MMK assessment (Thompson, 2016), or from tasks created by Hackenberg and Lee (2015) designed to assess students' reasoning with linear equations. One section of each of the redesigned Calculus I (differential calculus) and Calculus II (integral calculus) courses received this assessment. There was a combined total of 64 students in the two courses who took the pre-assessment as well as completed the course.

A Pearson correlation was used to investigate the relationship between Calculus I students' precalculus understandings (M=2.85, SD=1.64) and their final exam scores (M=80.09, SD=16.2). Results suggest a significant positive correlation between precalculus understandings and final exam scores, ($r(30)=0.52$, $p<0.001$, N=32). A similar analysis revealed a moderately significant positive correlation ($r(30)=0.28$, $p=0.058$, N=32) between Calculus II students' precalculus understandings (M=2.93, SD=1.48) and their final exam scores (M=79.3, SD=15.95). Future analyses will investigate correlations between particular final exam items and performance on each of the individual items on the pre-assessment. We hope to use the results of this study to aid in the continual refinement of this calculus curriculum by providing insight into what knowledge our students are coming into calculus with and how that knowledge supports them in the success of this active calculus sequence.

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A Glimpse of Change in GTA PD Programs in U.S. Mathematics Departments

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As part of an ongoing effort to understand how mathematics departments in the U.S. can better support graduate students teaching in precalculus and calculus courses, we are interested in investigating plans (or potential plans) departments are making toward improving their graduate teaching assistant (GTA) professional development (PD) programs. Contributing to a larger national project of first-year mathematics, this study looked at mathematics departments' survey responses to three items regarding changes to GTA PD programs. Out of the 223 departments that responded to the survey 66 of them indicated some level of plans of change to their program. For those schools, we analyzed the open-ended responses elaborating on the current status of the GTA PD program and found several noticeable themes regarding changes or plans to change their programs.

Keywords: GTA, professional development, institutional change, survey

As graduate teaching assistants (GTAs) become more integrated in the teaching of courses in precalculus through calculus two (P2C2) sequences (Vroom, Kirin, & Larsen, 2017), further research is needed to better understand how departments can effectively support GTAs in their teaching (Ellis, 2014). In a recent study, Ellis, Deshler, and Speer (2016) reported that nearly 40% of mathematics departments surveyed said changes to the current graduate teaching preparation program were being carried out or are planned. To better understand what types of changes departments are making, we further investigated departments that indicated changes (or plans to change) by considering their responses to follow-up survey items. Guiding this study is the following research question: *Of the departments that indicated plans or potential plans to change their GTA PD program, what changes did they implement or plan to implement?*

The data for this analysis comes from a census survey designed for a multiphase national project aimed to examine current P2C2 programs. The survey was administered to all universities in the United States granting either a Masters or PhD in mathematics. To answer our research question we looked at responses to three items of the survey regarding changes (or potential changes) to GTA PD programs. Out of the 223 schools that responded to the survey 66 of them indicated some level of plans of change to their program.

For the preliminary results shown here, we used a general qualitative approach to search for patterns in the open-ended responses elaborating on the status of the GTA prep program (e.g., no change, change being implemented, change being discussed). In our initial pass through the data, we recorded some obvious patterns in responses around change: (1) *From pre-semester orientation to more ongoing support* (15%), (2) *change in personnel* (14%), and (3) *creation of a new program* (25%). Note that these results do not add up to 100% since not all schools have been categorized and some responses fell under multiple categories. We hope to continue refining these results by searching for additional descriptive patterns as well as considering connections with previous findings, such as connecting the changes to how departments are evaluating the success of their programs (Ellis, Deshler, & Speer, 2016) and the structure of their programs (Bragdon, Ellis, & Gehrtz, 2017).

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Student Engagement in a Post-Secondary Developmental Mathematics Class

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I report on the first stage of my dissertation project which sought to understand engagement in a Precalculus course at a four-year public university. Breaching instructional activities, student interviews, and classroom recordings were used to study the development of several sociological and psychological constructs to help characterize students' engagement. Despite the instructor's attempts to negotiate productive norms, data analysis shows that some students' detrimental practices and beliefs remained unchanged or were even supported by the course. I examine the roots and consequences of this phenomenon.

Keywords: Social Norms, Sociomathematical Norms, Communal Mathematical Practices

Students of many post-secondary developmental mathematics courses experience long-term mathematics struggles and high attrition rates (Bailey, Jeong, & Cho, 2010; Bahr, 2013). This project explores how one such course, a Precalculus course at a public four-year university, attends to developing students' mathematical capabilities and practices by studying the social norms, sociomathematical norms, and communal mathematical practices emerging in the course. Development of these constructs coincides with development of students' beliefs and practices (Yackel & Cobb 1996), which may improve students' mathematical engagement and support long-term mathematical learning. One way to develop more constructive beliefs and individual practices is to negotiate productive norms and communal practices. The instructor of the studied Precalculus course attempted to negotiate such norms by introducing instructional activities that breach students' mathematical expectations, which allowed for explicit negotiation of productive norms and practices. In addition to 360° video recordings of these activities, data collection included repeated interviews with students which provided information about their beliefs, values, and individual practices, and how these changed or persisted over the semester. The data was then analyzed in conjunction with the interpretive framework of Cobb & Yackel (1996).

I focus on the case of Audrey, who, despite being a diligent and successful student in the Precalculus class, retained her detrimental practices and beliefs. For example, she would enumerate steps to memorize and would focus on repeating algorithms that she did not always understand. Audrey represents a student who is eager to learn, but whose efforts do not allow for extensive advancement of her mathematical capabilities. One critical observation from the study data is that the pedagogical instructional practices and course structure did not require her to change her practices to be successful in the course, and that some of these instructional practices actually supported and perpetuated her own. At the same time, the data shows that the instructor's attempts at negotiating productive norms and practices were often hindered by the need to coordinate his teaching approach and assessments with other instructors of that course.

The results concur with the literature by showing that repeated content exposure will not necessitate changes in students' practices and beliefs (Goudas & Boylan, 2013; Carlson et al., 2010). Although there were some attempts to make changes to the way the course is usually taught, more systemic change is needed. Because of institutional support for reform in Precalculus, the course can be redesigned with a focus on negotiating norms and practices to redirect students' efforts toward more productive mathematical engagement.

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MathChavrusa: A Partnership Learning Model

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In this poster we introduce a new learning modality called MathChavrusa. Inspired by the ancient rabbinic approach to Talmudic study, the chavrusa model pairs students in a partnership of deep text-based analysis, discussion, and debate. Over centuries the model has proved its ability to generate thorough understanding, build skills, develop the courage to question, and demonstrate to students the value of both independent thinking and collaboration. MathChavrusa is a complementary model to other accepted modalities for generating student understanding in mathematics. It is particularly effective when employed after a lecture class. In teaching about the model, we will discuss its origins, how it facilitates deep learning and understanding in mathematics, and techniques for implementation. We have begun to utilize the model in our classes, and are gathering data about its real-world effectiveness. Preliminary data implications will be discussed.

Keywords: Mathematics Identification, Text-based learning, Collaboration, Peer study partnership

Cultivating mathematics identification is critical to engage undergraduate students in mathematics. Peer support and collaboration are critical components for increasing mathematics identification (Walker, 2006). A demonstrably effective educational philosophy exists which moves away from the teacher-centered classroom to student-centered learning environments where learning can happen in a profound way (Freeman et al, 2014). Effective tools for mathematics skill building and mathematics identification are paramount for student success in mathematics. Peer collaboration has been shown to improve students' ability in tasks that require reasoning (Phelps & Damon, 1989). There is evidence that student-centered small group learning alleviates attrition and is beneficial in undergraduate STEM student presence in all demographics (Springer et al, 1999). Students are often too teacher dependent and fear the math textbook. The learning model MathChavrusa is designed to foster independent mathematics learning, peer collaboration, critical thinking, and text-based learning. We introduce and study the practical benefits of implementing this learning model in undergraduate mathematics learning environments.

In summary, MathChavrusa implementation requires students to be paired (they can self-pair or the instructor can do the pairing) and maintain the same partner throughout the duration of the course. Depending on course structure, it is recommended a minimum of twenty minutes per class session to engage in MathChavrusa. Students engage in inside math textbook reading and discussion with their study partner (chavrusa). A study guide and posed questions highlighting the mathematics material can help deepen and increase the benefits of MathChavrusa.

Currently MathChavrusa has been implemented in 6 institutions in the following courses: remedial college algebra, calculus, linear algebra, differential equations, and topology. Currently our data gathering and analysis is focused on qualitative data, evaluating initial benefits such as deeper mathematics understanding, improved mathematics communication, confidence in mathematics assignments and increased aptitude in mathematics text-book learning. We are in the process of analyzing data as to its effectiveness.

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Upgrading the Learning for Teachers in Real Analysis; A Curriculum Project

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Upgrading Learning for Teachers in Real Analysis is a project in which we designed and implemented an innovative real analysis course for pre-service and in-service mathematics teachers (PISTs). More generally, this project provides an alternative model to teaching advanced mathematics to PISTs, a model that more meaningfully connects the teaching of secondary mathematics to the advanced mathematics content. This poster describes the theoretical model, the means of developing connections between real analysis and secondary mathematics content, the 12 modules we designed and how they fit in a standard real analysis curriculum, and presents evidence for their efficacy. The instruction is built from and returns to authentic secondary mathematics classroom situations.

Keywords: Real Analysis, Teacher Education, Mathematical Knowledge for Teaching

We describe an innovative real analysis course that developed for pre-service and in-service secondary mathematics teachers. The course had a multitude of goals: (i) PISTs would learn the real analysis; (ii) PISTs would understand secondary mathematics better; (iii) PISTs would have the pedagogical content knowledge to respond more effectively to pedagogical situations; (iv) PISTs would see the relevance of real analysis to secondary mathematics teaching; and (v) there would be genuine positive changes in PISTs' instructional practice. We developed an instructional model that grounds the study of advanced mathematics in pedagogical situations and asks teachers to revisit those same situations and apply their new knowledge. To accomplish our goals, we developed 12 modules that connect the content and practices of real analysis to the teaching of secondary mathematics. In each module, PISTs are first presented with an authentic classroom situation from high school mathematics in which a teacher needs a deep understanding of mathematics to respond appropriately. From the discussion that ensues, PISTs *build up* from teaching practice to tackle the underlying mathematical issues at play in a real analysis context. After the work in real analysis resolves these mathematical issues, PISTs *step down* to practice and are asked to revisit the original and analogous classroom situations. As such, each module has both mathematical goals (what mathematics are PISTs learning?) and pedagogical goals (what pedagogical practices are PISTs going over in the module?). To design a module we generated pedagogical situations that had three characteristics: (i) the pedagogical situations were authentic (i.e., not contrived but true to situations that arise in teaching), (ii) the prospective teachers were asked to engage in a High Leverage Practices (TeachingWorks, 2013) that are central to the work of a secondary mathematics teacher, and (iii) successfully engaging in these High Leverage Practices required mathematical knowledge that could be informed by or reinforced via real analysis. This poster will show the theoretical model, the means of developing connections between real analysis and secondary mathematics content, illustrate how the 12 modules fit in a standard real analysis curriculum, and present evidence for their efficacy. We will include a QR code that navigates to all of the modules as well as provide printed examples. We argue that the course was effective in many ways, and describe ongoing challenges.

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Humanizing the Coding of College Algebra Students' Attitudes Toward Math

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Through their coding of survey responses, researchers can create spaces to humanize students' attitudes toward math. To account for complexity in students' attitudes beyond positive or negative, we developed three additional codes: mixed, ambiguous, and detached. In our coding methods, we account for a diversity, rather than a binary, of student attitudes.

Keywords: Attitude toward mathematics, College algebra, Humanizing, Research methods

Even before Calculus, College Algebra is a gatekeeping mathematics course, and students' attitudes toward math can impact their persistence in such courses (Bressoud, Carlson, Mesa, & Rasumussen, 2013; Ellis, Fosdick, & Rasmussen, 2016). College Algebra students can express complex attitudes toward math, and we posit that researchers' coding methods should begin to open space to acknowledge the complexities of students' attitudes. Drawing on survey responses as sources of data, researchers have coded students' attitudes toward math as positive, negative, and other/indifferent (Ding, Pepin, & Jones, 2015; Pepin, 2011). In our coding methods, we account for a wider range of students' attitudes, to give more voice to attitudes outside the positive/negative binary. For example, students can express a mixture of positive and negative attitude, ambiguity in their attitude, or a detached attitude toward math.

We administered a fully online attitude survey to College Algebra students at the beginning and end of the Spring and Fall 2018 semesters. We used Pepin's (2011) open-ended question stems, (e.g., "I like/dislike math because…"), because the question stems allowed students to self-narrate a range of attitudes that may not fit into binary categories. Beyond positive and negative, we included three additional codes: mixed, ambiguous, and detached. We coded *mixed* for a response that presented more than one attitude (e.g., positive *and* negative), *ambiguous* for responses that crossed multiple attitudes, and *detached* for a response that separated the person from the mathematics, treating mathematics as something "out there" or not connected to self. Table 1 shows examples of student responses we coded as mixed, ambiguous, or detached.

Table 1. Examples of responses coded as mixed, ambiguous, or detached

Code	Example Student Response
Mixed	I love and enjoy problem solving, but I dislike having to remember a lot of rules.
Ambiguous	I don't care either way.
Detached	Math is the universal language.

Langer-Osuna & Nasir (2016) called for researchers to develop methods that humanize students' experiences. Were we not to have included the additional codes, we would have coded the student responses in Table 1 as "other/indifferent," because they are neither positive nor negative. Yet, the responses presented distinct attitudes, which we valued and wanted to name.

As researchers, our methods are never neutral. In our coding of hundreds of College Algebra students' responses to survey questions, we worked to amplify students' voices to extend possibilities for the kinds of attitudes counted. As a result, we created a richer landscape of possibilities, which requires more than a linear continuum to represent.

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Inquiry-Oriented Differential Equations as a Guided Journey of Learning:
A Case Study in Lebanon

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Integrating innovative pedagogical initiatives within the learning environment at the Lebanese American University in Beirut, Lebanon, has been set as a strategic goal. Active learning, as one medium of instruction, has seen widespread implementation in mathematics classrooms. This study reports on an inquiry oriented differential equations class offered in spring 2018. The focus is on the role of the curriculum in guiding students reinvent successfully key mathematical notions covered in any introductory differential equations class.

Keywords: Inquiry oriented differential equations; curriculum; guided reinvention.

Inquiry Based Learning (IBL), as an active learning medium of instruction, has seen integration in a variety of mathematics classes. An implementation of an inquiry-oriented curriculum is considered successful if it guides learners in reinventing the course key mathematical concepts. *Guided reinvention* (Freudenthal, 1991) allows "learners to come to regard the knowledge they acquire as their own private knowledge" (Gravemeijer & Doorman, 1999, p. 116). True to the nature of an inquiry-oriented learning environment, the Inquiry Oriented Differential Equations (IODE) course was developed by Rasmussen, Keene, Dunmyre, & Fortune (2017). The curriculum drew its inspiration from a dynamical systems approaches to differential equations (e.g. Blanchard, Devaney, and Hall (1998), and Hubbard and West (1991)), representing "a significant departure from conventional treatments of differential equations that emphasize a host of analytic techniques" (Rasmussen and Known, 2007, p. 190).

In spring of 2018 I taught an IODE course in my home institution. The material covered in the course was similar to a traditional course; however, learning was based on the four principles of IODE: Generating students' ways of reasoning, building on student contribution, developing a shared understanding, and connecting to standard mathematical language. The class was divided into seven groups consisting of 3 to 4 students each. Whiteboards, markers, and erasers were distributed to each group at the beginning of every class. To answer the research questions, *To what extent were students successful in reinventing the key concepts of the course and what obstacles were faced in acquiring the desired course outcomes*, I analyzed personal notes taken at the completion of each unit (14 units in all), snapshots of in-class students' work, copies of homework assignments (5 in total), and results of 5 online questionnaires posted on the Discussion Board of Blackboard Learn. Some recent empirical studies on students enrolled in IBL math-track courses have reported "greater learning gains then their non-IBL peers on every measure [such as] cognitive gains in understanding and thinking". (Lauren, S. L., M.L. Hassi, M. Kogan, and T.J. Wetson, 2014). While this study confirms these findings, reinventing knowledge proved to be cognitively demanding and in some cases required the intervention of the instructor to control and guide the discussion. Results also show departing from the conventional treatment of mathematical concepts was the main obstacle students faced.

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Supporting Instructional Change: The Role of Facilitators in Online Working Groups

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Research has shown that faculty benefit from support and collaboration when introducing student centered instruction into their teaching (Henderson, Beach, & Finkelstein, 2011; Speer & Wagner, 2009). The RUME community has some knowledge about how these supports take shape and grow (e.g., Hayward, Kogan, & Laursen, 2015), but work is still needed. A crucial component is researching the facilitation of these supports. In this study, we focus on how the facilitation of online working groups occurs. Our preliminary results indicate that the actions facilitators take play crucial roles in how to use discussions of mathematics to proactively engage in student thinking.

Keywords: Instructional change, online faculty collaboration, facilitators

Faculty are currently making changes to their instruction by introducing different modes of student-centered instruction (Mathematical Association of America [MAA], 2018). Numerous support avenues have become available to these faculty such as faculty collaborations (Nadelson, Shadle, & Hettinger, 2013) and summer workshops (Andrews-Larson, Peterson, & Keller, 2016). In this study we focused on online working groups (OWGs) that supported mathematicians learning to teach inquiry oriented differential equations, abstract algebra, or linear algebra. Previous research has shown the importance of doing mathematics in this process to situate faculty's understanding of these "new" curricula (Andrews-Larson et al., 2016), but facilitating those discussions is largely unexplored. Thus, we aim to answer the research questions: 1) What role do facilitators take within OWGs focused on doing and understanding the mathematical content? 2) How does the topic of conversation shift as a result of the facilitators' actions?

Methods

The current analysis focuses on facilitators who were participants from previous OWGs. Each session occurred via Google Hangouts and was screen recorded and transcribed. The 14 sessions under analysis were chosen to fit the research focus on weeks when the OWG participants were discussing how they solved the mathematical tasks. Two researchers developed a codebook that included a priori codes based on the stated goals of the OWG, and emergent codes from the analysis, and met to discuss and resolve any discrepancies.

Preliminary Results and Discussion

We have found that facilitators regularly use discussions concerning how the OWG participants solved the mathematical task as a springboard for discussions regarding reporting on and student mathematical thinking and more general discussions concerning the pedagogical choices participants made or will make in their classrooms. Our continued analysis will be focused on unpacking the specific ways the facilitators make these transitions and whether/how participants respond to the facilitator's efforts. Implications for this work include showcasing how productive OWGs are facilitated so they can be replicated and have a deeper understanding of how online synchronous professional development programs operate.

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Basic Research on Instructor Practice: What do We Want to Know? …and How?

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There now exist resources (e.g., text- and video-based case activities) for use in the professional development of novice college mathematics instructors. We do not yet know much about the characteristics of effective use of those resources. Even less is known about what facilitators need to know to use the resources successfully. A newly funded project is building meta-materials to help providers use resources. These Provider Packages are a virtual facilitation partner, taking on some of the cognitive load of facilitation (e.g., audio tracks that can be turned on and off, notes from a more experienced peer that are virtual whispers in the ear). The project will examine use of the facilitation support tools in the Provider Packages to identify characteristics of effective facilitation of activities. At the poster, we will seek conversations about research designs that can leverage the opportunities of the Provider Packages.

Keywords: Novice College Mathematics Instructors, Professional Development for Teaching

To create high-quality learning opportunities for undergraduate mathematics students we need to provide opportunities for novice college mathematics instructors to learn knowledge and skills for teaching. Those learning opportunities often occur during teaching seminars led by faculty. Certainly, activities are available for these *Providers* of professional development and some are accompanied by guidance to help Providers use the activities effectively (e.g., Friedberg et al., 2001; Hauk, Speer, Kung, Tsay, & Hsu, 2013). However, from our experience, utilizing Provider guides and facilitating an activity for the first time can be very challenging. Facilitation requires bringing to mind and coordinating several streams of information. We are creating digital *Provider Packages* to serve as a virtual facilitation partner, taking on some of the cognitive load of organization and orchestration (Figure 1). Just-in-time facilitation expertise is available with options to turn on (or off) various scaffolds. For example, with all supports "on" a novice Provider allows the built-in audio of an expert facilitator to lead the session. Or, by selecting only visual supports, a more experienced Provider uses the Package as its lead facilitator. We are developing *Provider Packages* for several publicly-available, case-based activities. The poster will have images from sample *Provider Packages* to illustrate the types of supports provided.

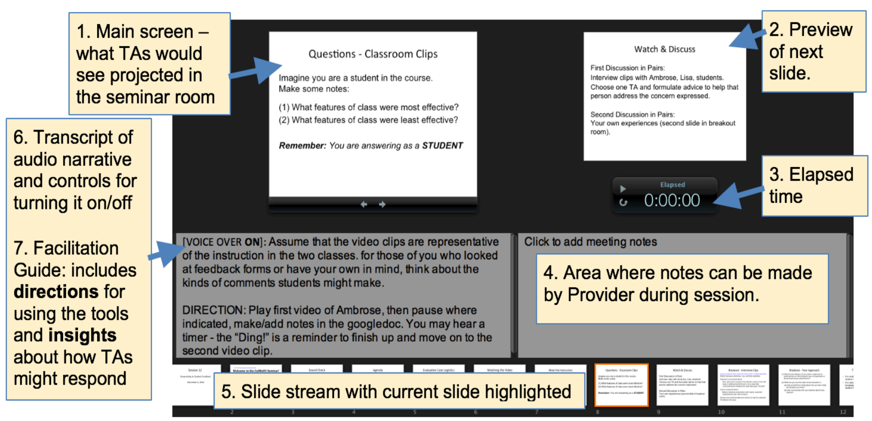

Figure 1. Potential Provider Package view.

As part of this work, we will do research to explore what faculty need to implement such professional development activities. To that end, we seek input from the RUME community about research designs and questions we might use in the context of this project, data that might be most valuable to gather, and ideas for future expansions of the project.

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An Exploration of Math Attitudes and STEM Career Interests for Community College Students

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Survey data for community college algebra students reveals relationships between a student's attitudes towards mathematics and the student's STEM career interests. Results show that while students may not always have a clear understanding of the tasks related to a chosen STEM career area, the student's math interest predicts interest in some STEM careers and not others.

Keywords: STEM, career interests, algebra

In community colleges across America, students are struggling with mathematics. Mathematics has long been a stumbling block for undergraduates, including those pursuing science and engineering degrees (Harackiewicz et al., 2012). The school experiences that create this situation disproportionately affect students from groups underrepresented in science, technology, engineering, and mathematics (STEM) majors (Reardon, 2011), and can drive these students away from STEM fields (Moses & Cobb, 2001). Pass rates in math courses required for STEM careers like College Algebra are low (Howell, 2016). In the present study, we examine whether interest in math is predictive of College Algebra students' interest in STEM careers.

As part of a larger study, students enrolled in College Algebra (n = 367) at a mid-size community college in the southern United States were invited to take a survey regarding their STEM career interests, as well as their interest in algebra and mathematics in general. Male and female students were represented equally, and students were largely 18-24 years old. Respondents were 52% Caucasian, 29% Hispanic, 6% African-American, and 13% other races/ethnicities. The survey (drawn from the Basic Interest Scales; Liao, Armstrong, & Rounds, 2008) asked students to rate their interest in fourteen career areas, first using four questions related to activities one would perform in each area (e.g., "Build a structure to withstand heavy winds"), and then using the name of the career area (e.g., "Engineering"). Additionally, survey items asked students to rate their interest in algebra and mathematics in general using interest survey items from Linnenbrink-Garcia et al. (2010) and Renninger and Schofield (2014).

Analyses were conducted where the career activities aggregates were compared to the students' interest rating in the career area. Results showed these measures were consistently only moderately related, suggesting that students may not be clear on what different STEM careers entail. The relationship was particularly weak for physical science. The career ratings were then compared to student responses related to their level of interest in mathematics and algebra in general. The mathematics interest items were strong predictors of interest in careers in math/statistics and in STEM teaching, and were moderate predictors of interest in careers in engineering, finance, information technology, and mechanics/electronics. Interest in math did not predict interest in a variety of other STEM career areas, including life science, physical science, and social science. Finally, overall differences in math interest between student groups were explored. Results suggest that females in College Algebra have lower math interest than males, and non-Hispanic Caucasian students have lower math interest than other racial/ethnic groups.

The survey results indicate that students in the sample may not have clear ideas of the nature of specific STEM career areas, and their math interests do not align well with their intended career interests in many cases. The larger study hopes to improve learning outcomes for mathematics students by explicitly tying College Algebra course content to STEM careers.

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Students' Responses to Differing Prompts for Reasoning and Proof Tasks

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Students are engaged in various reasoning and proving tasks corresponding to the increased emphasis on reasoning and proving in mathematics education. Students routinely encounter differing language present in prompts for these reasoning and proving tasks. The semantic meaning of the language used in these prompts is not usually explicitly discussed and thus may cause inconsistencies in students' responses to these tasks and in the assessment of their work. The preliminary results imply Calculus I students have various conceptions for prompts such as "prove", "explain", "show", and "convince". This poster will focus on students' various conceptions on the two prompts "prove" and "show."

Keywords: Proving, Reasoning, Prompts

The mathematics and mathematics education community have emphasized the importance of reasoning and proving across the K–16 levels. As a result, curricula and research have asked how students understand reasoning and proving (Harel & Sowder, 1998; Knuth, Choppin, & Bieda, 2009; Weber & Alcock, 2004). Students face differing language within these prompts—such as "prove", "explain", "show", "convince", etc.—both in textbooks and research tasks (Knuth et al., 2009; Otten, Gilbertson, Males, & Clark, 2014). As the semantic meaning of these prompts is not explicitly discussed, researchers have raised questions about the perceived differences between these prompts. For example, a teacher might expect either rigorous proof, or a causal argument when they asks students to "explain" their reasoning (Dreyfus, 1999; Hersh, 1993). Dreyfus (1999) also questioned whether the prompt "show", asks students to generate an actual mathematical proof, or examine some examples. In keeping with such research, we hypothesized that there may be differences in students' responses to each type of prompt. These differences, then, might cause inconsistencies in students' learning, and in the assessment of students' work.

This poster presents preliminary findings based on a study with the following research question: *How differently do Calculus I students perceive and respond to different prompts, such as "prove", "explain", "show", and "convince", for reasoning and proving tasks?* The survey data was collected from 131 students enrolled in a Calculus I course at a large public university in the Midwest United States. The survey consisted of three parts: the students' academic background, questions to choose hypothetical prompts based on given arguments, and Likert–type questions regarding the perceived meanings of these differing prompts.

The preliminary findings indicate the existence of differences in meanings for students for different prompts. Among results, we want to focus on the prompts "prove" and "show" for this poster. Although some students (26.7%) considered "prove" and "show" as synonyms, aligning with mathematicians' understanding of the two prompts as synonyms (Alcock, 2013), the majority of students (55.7%) regarded the prompts "prove" and "show" as different. Some students considered these two prompts to be distinct, and a second group of students thought the meanings of these prompts to have an intersection, but also that each have independent characteristics. Across the data, the students' responses imply that presenting some examples is enough for the prompt "show". This inconsistency in students' responses challenges the notion that students may perceive "prove" and "show" as synonyms and justifies further research on students' perception of the language used for prompts of reasoning and proving tasks.

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Insight into Prospective Elementary Teacher's Beliefs About Mathematics

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This study attempts to answer the question: why do prospective elementary teachers (PTs) have high levels of anxiety learning mathematics and low levels of confidence to teach mathematics? Using a survey of 300 undergraduate PTs and a mixed methods approach, I report the results from the analysis. Identifying and addressing the causes of negative beliefs about mathematics is crucial to ending the negative cycle of beliefs in their future students. A survey of 300 undergraduate PTs was taken and a mixed methods analysis was done.

Keywords: Teacher Beliefs, Affect, Emotion, Beliefs, and Attitudes

The goal of this project was to gain insight into the underlying factors that contribute to beliefs and anxieties about mathematics experienced by prospective elementary school teachers (PTs). Teachers' beliefs about mathematics have a powerful impact on the practice of teaching (Charalambos, Philippou & Kyriakides, 2002). The students of teachers with positive beliefs about mathematics tend to enjoy successful learning experiences that often result in them seeing mathematics as useful and necessary (Karp, 1991). Therefore, it can be argued that teacher beliefs play a major role in their students' achievement and the formation of their beliefs and attitudes toward mathematics. Identifying and addressing the causes of negative beliefs about mathematics held by PTs is crucial for improving their teaching skills and helping them transform the anxieties that they would perpetuate onto their future students.

I administered a survey to over 300 PTs focused on their beliefs about the nature of mathematics, mindset and their prior experiences in learning mathematics. All participants were asked to complete a survey that utilized both quantitative and qualitative items about their beliefs and attitudes on learning mathematics as well as their prior experiences with mathematics. I analyzed the data uwing a mixed method approach to look for correlations between the PTs' beliefs about mathematics and their experiences throughout their mathematics education. Quantitative data was analyzed by using analysis of variance to determine whether differences between sample subgroups were statistically significant. Descriptive statistics were used to analyze qualitative responses and will serve as the basis for conclusions drawn about the nature and etiology of attitudes toward mathematics.

The poster will show the reasoning behind the questions asked and the correlated results of the survey that indicate that there are strong relationships between PTs' previous educational experiences and the level of their anxiety and confidence toward teaching mathematics. PTs' lack of confidence in teaching mathematics was highly correlated with previous experiences learning mathematics that emphasized memorizing procedures and finding correct answers quickly. The findings from this survey provide the first steps in understanding PTs' beliefs about mathematics, and their anxieties and lack of confidence in being able to teach mathematics. Mathematics teacher educators can use these results to address these issues in mathematics education courses so that PTs might have an opportunity to transform their beliefs.

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Using a Scripting Task to Probe Preservice Secondary Mathematics Teachers' Understanding of Function and Equation

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In order to determine preservice secondary mathematics teachers' (PSMTs) conceptual understanding following an inquiry-based lesson on the constructed meanings of the equals sign and the distinctions between the concepts of function and equation, we utilized a scripting task in which the PSMTs individually continued a dialogue between two hypothetical students with opposing viewpoints with respect to an equation arising from a function context. This study is part of the Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers Project which is developing research-based tasks and explorations together with instructor materials to be used in mathematics courses for PSMTs. The goal of this poster presentation is to discuss our implementation of the scripting task to gauge PSMTs' understanding of the nuances between function and equation. We also wish to gather feedback and suggestions on the study design and potential implications of our research.

Keywords: Mathematical Knowledge for Teaching, Preservice Secondary Mathematics Preparation, Functions, Equations

Functions are a foundational component of the mathematics that preservice secondary mathematics teachers (PSMTs) will be expected to teach. However, the research literature identifies ways in which conceptions of functions can be limited for both PSMTs and inservice mathematics teachers (ISMTs). For example, some PSMTs and ISMTs believe that a function can always be represented by an algebraic formula, and others believe that the terms function and equation are interchangeable (Even, 1993; Hitt, 1998). Script writing in the context of a mathematics course for preservice teachers can be a useful tool to investigate and detail nuances in mathematical knowledge and understanding for prospective teachers (Zazkis & Zazkis, 2014). This study aims to detail what script writing revealed about PSMTs understanding of the distinctions between function and equation, particularly following their in-class experience in an inquiry-based lesson.

Data gathered from the scripting task were coded using open and axial coding, then inductive thematic analysis was applied (Braun & Clarke, 2006; Corbin & Strauss, 2008).
Discussion with RUME attendees will assist us in identifying design issues that need to be accounted for in addressing the following research question using scripting tasks: How do PSMTs reconcile their understanding of function and equation with their in-class experiences with equations that arise from functions?

Based upon our initial analysis, though we were utilizing the scripting task to identify obstacles related to PSMTs' capacity to explain distinctions between functions and equations, it seems that the scripting task itself served to improve their understanding by helping PSMTs reflect on the constructed meanings of the equal sign in function and equation contexts.

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Creativity in Problem Solving for non-STEM majors in Calculus Courses

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In this poster we share a qualitative study aimed at investigating creativity in problem solving for non-mathematics tracked students enrolled in a calculus course. Three task-based semi-structured interviews with volunteered participants were analyzed using a modified whole-to-part inductive approach (Erickson, 2006). Our findings suggest that even though students may perceive creativity as a process, this understanding may not necessarily be reflected in their written work.

Keywords: Mathematical creativity, problem solving, creative process

Given the critical role mathematics has played in contemporary innovation, the development of the talent pool in mathematics has great scientific and economic impact. As research studies exploring math majors' creativity in undergraduate math courses commence (e.g., Savic, et al., 2017), there is still a need to explore how such emphasis can be shifted to explore creativity at lower-level math "service" courses such as calculus.

In this poster, we share a qualitative study that aimed to investigate creativity in problem solving for non-mathematics tracked students enrolled in a calculus course. Individual students' problem-solving process and their self-perception of mathematical creativity were documented through interview data. These task-based semi-structured interviews with 3 volunteered participants were analyzed using a modified whole-to-part inductive approach (Erickson, 2006).

Although no explicit description of the creative process in problem solving emerged from the data, each participant was observed to exhibit all four phases of Carlson and Bloom (2005)'s problem-solving framework. Our findings suggest that even though students may perceive creativity as a process, this understanding may not necessarily be reflected in their written work. Teachers therefore need to create opportunities in the classroom to challenge and push students to take risks to develop their mathematical creativity.

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Overview of Evaluating the Uptake of Research-Based Instructional Strategies in Undergraduate Chemistry, Mathematics, and Physics

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Virginia Tech

Research-Based Instructional Strategies have been show to increase learning and retention of students in undergraduate STEM classes but have not been widely implemented in classrooms across this country. While there is research indicating the level of usage of RBIS across the country in gateway chemistry, mathematics, and physics courses, less is known about why instructors choose to use RBIS or not. We report on the design of an ongoing research study to assess the relative impact of individual, departmental, institutional, and disciplinary factors on instructional decisions in key courses for postsecondary STEM-intending students.

Key words: STEM, research-based instructional strategies, instructional practice, survey

There is persistent and mounting evidence that lecturing is *not* the best instructional strategy to support student learning, engagement, and retention which has led to repeated calls for a shift to student-centered instructional practice in undergraduate science, technology, engineering, and mathematics (CBMS, 2016; Freeman et al., 2014; Kogan & Laursen, 2014). Alongside these general calls for more student-centered instruction, researchers have developed many specific instructional strategies referred to as *research-based instructional strategies* (RBIS). Researchers have also demonstrated that RBIS can have a positive impact on student success in terms of learning, retention, persistence, and/or enjoyment of the content. Despite mounting evidence of the impact of using RBIS in classrooms and some student-centered approaches used, lecture (or didactic) approaches to instruction are still the norm in undergraduate STEM classes (Rasmussen et al., in press; Stains et al., 2018). This poster presents the current state of our research project focusing on current knowledge of uses of RBIS and how it lead to Phase 1 of our research study.

Our research project investigates the relative impact of factors which affect instructors' decisions to use RBIS in their classrooms. For this study, we are engaged with an investigation of introductory postsecondary chemistry, physics, and mathematics courses. These three courses are particularly important because they function as *gateway courses* – required of most first-year STEM-intending students, often high-enrollment, foundational for future coursework, and have demonstrably low passing rates (Koch, 2017). By considering instruction and instructors in three disciplines, we hope to learn more about variation across STEM fields. In particular, identify factors which impact across disciplines and which seem relevant in one but not others. This knowledge will support future efforts of change agents by identifying factors that affect the likelihood of using RBIS in classroom and which factors are likely to have the most leverage.

Prior research has identified certain factors related to RBIS usage in these three disciplines, and Phase 1 of our research study involves a national survey querying many of these same factors across all disciplines and all at once. This will allow for partial replication of other studies as well as combining those results across disciplines and factors to build a model of levers for instructional change at scale. It will also provide a data point regarding current levels of RBIS usage which will support further monitoring of the spread of RBIS across the country. Targeted factors include culture and context (e.g., Selinski & Milbourne, 2015), interactions with the education community (e.g., Henderson & Dancy, 2009), growth mindset (e.g., Aragón, Eddy, & Graham, 2018), and instructor attitudes (e.g., Fukawa-Connelly, Johnson, & Keller, 2016).

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Nature of Students' Meanings of Angle Measure and Trigonometric functions in an online interactive forum

Ishtesa Khan

In an online environment that promotes self-learning and online interaction between teacher and students, this poster proposal presents the nature of students' meanings of introductory trigonometry while they interact with each other. Students' communication in the online interactive forum is crucial as their reasoning influence others to make their own meaning of the problem. The poster proposal also presents the difficulties students encounter in developing trigonometric understandings when they work online independently.

Keywords: Online Forum, Interaction, Angle Measure, Trigonometric Functions, Proportional Reasoning

Introduction and Research Questions

Trigonometry has been a difficult mathematical idea for students and using geometric objects only to make sense of angle measure is not helpful to make proper connections between angle measure and trigonometric functions. Moore (2013) discussed the issue of how without having a robust understanding of the process of measuring an angle and how the structure of the unit relates to this process make little sense to students. For this study, I observed students' interactions in an online undergraduate precalculus course that is focused on quantitative, covariational and proportional reasoning. Mathematical ideas here are supported by animations and videos leveraging students' conceptual images. Wallace (2003) addressed the importance of online community as without it an online course is a mere source of information. In the online precalculus course, students were encouraged to use the online forum to discuss their understanding of trigonometric ideas introduced in their online lessons and homework. This proposal investigates the nature of students' meaning making of trigonometry and difficulties they encounter based on their online forum discussion threads. The primary research question driving this study is-

- How do students explain their understanding of angle measure and trigonometric functions to help others who post specific lesson/homework problems addressing that they are having trouble getting it?

Methods and Results

I observed interactions among teacher and 14 students and 4 of them only provided explanations to others but never asked for help. I used grounded theory (axial coding) (Strauss and Corbin, 1990, 1998; Strauss 1987) to categorize and subcategorize students' nature of explanations and difficulties for introductory trigonometry ideas. As findings from this study, a limited number of students were having difficulties to reason quantitatively because of their habit of using Google for help and using SOH CAH TOA for trigonometry was not applicable directly for the problems in this curriculum. A larger number of students who worked through their lessons and watched videos attached in the lessons were successful to provide meaningful explanations that connects angle measure and trigonometric functions and reflect their quantitative and proportional reasoning. Some students struggled to reason proportional relationship among quantities like angle measure and subtended arc.

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Hypothetical Learning Trajectory Leveraging Proportional Reasoning

Ishtesa Khan

This poster presents conceptual analysis and hypothetical learning trajectory for learning proportionality which was previously limited only to figure out missing value using cross-multiplication. Based on a series of clinical interview that investigated students' meaning of proportionality in an online format, I found that students tend to use only cross-multiplication strategy to reason proportionally which did not help them to reason proportionally. That emphasizes how learning proportionality along with the constant rate of change among quantities given in any specific word problem helps to reason proportionality conceptually.

Keywords: Proportional Reasoning, Conceptual Analysis, Hypothetical Learning Trajectory.

Introduction and Theoretical Framework

In proportional reasoning, we are interested in comparing quantities in relation to one another instead of finding the 'missing' number of given situations. The fundamental concept we need for proportional reasoning is the idea of 'ratio'. A ratio is a binary relation which involves ordered pairs of quantities. (Lesh, Post, & Behr, 1988). According to Thompson (1994), a ratio is a result of comparing two quantities multiplicatively. When we discuss proportionality we not only consider one ratio, we compare two ratios with likely quantities. And the rate of change of both ratios remains the same constant in this relationship. By Thompson (1994), a rate is a reflectively abstracted constant ratio. Both definitions of ratio and rate followed by Thompson's 1994 paper are fundamental perspectives to look forward to proportional reasoning.

Conceptual Analysis and Hypothetical Learning Trajectory

A conceptual analysis is a way to describe what students might understand about an idea to reason the way it should be understood (Thompson 2008). To conceptualize and reason proportionality I conjecture that the student will need to achieve seven learning goals I tried to identify in this poster. Simon's (1995) development of hypothetical learning trajectory(HLT) is consist of the goal for the student learning, and hypotheses of the students' learning (Simon, M. & Tzur, R., 2004). Generalizing conceptual analysis (Thompson 2008) and HLT (Simon 1995), this poster is going to present an HLT for proportional reasoning-

1. Students will draw a picture which represents the given situation
2. Students will identify quantities and determine whether they are varying or fixed quantities, and they will always verbalize them with corresponding units.
3. Students will be able to represent the situation graphically with scaled measurements.
4. Students will identify the varying quantities in the given situation and will be able to relate these quantities to the constant rate of change.
5. Students will understand that one quantity is as many times bigger or smaller as the second quantity. If there are more than two quantities in one situation they will be able to understand the relationship among them as well.
6. Students will avoid seeing ratio and proportions only as a tool for performing calculations, applying rules and formulae and manipulating numbers and symbols in proportion equations.

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Investigating Instructional Strategies in Introductory Statistics

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Recommendations for the teaching and learning of introductory statistics at the tertiary level have been set forth by the research community, including recommendations outlining desirable pedagogical strategies, such as the use of student-centered instruction and the integration of technology and resampling methods to support the development of students' conceptual understanding. Yet, surprisingly little is known about how introductory statistics is being taught at colleges and universities across the United States. The research presented here aims to shed light on these aspects of the introductory statistics course by reporting preliminary findings from an instructor survey that was recently completed by 148 instructors nationwide.

Keywords: Instructional Strategies, Introductory Statistics

The importance of teaching for conceptual understanding has been stressed in mathematics and statistics education research at both the K-12 and college level. To support the development of students' ability to think and reason with data, researchers have set forth recommendations for the teaching and learning of statistics, including recommendations that instructors foster active learning and leverage technology as part of their instructional approach (ASA GAISE College Report Revision Committee, 2017). Additionally, emerging empirical evidence suggests that teaching statistics using simulations and resampling methods has the potential to support student learning of statistics (Hildreth, Robison-Cox, & Schmidt, 2018). Since statistics is one of the fastest growing undergraduate degrees of any STEM disciplines (ASA GAISE College Report Revision Committee, 2017), it is important that we understand if the pedagogical strategies used by instructors align with the recommendations made by researchers.

Based on a nationwide cluster sample, instructors from 80 selected universities were surveyed in Spring 2018, resulting in 148 participants (response rate 27.2%). Preliminary findings show that of these 148 instructors, 64% use lecture as their primary instructional format while approximately 28% integrate some form of active learning as part of their instruction (i.e., problem-based learning, inquiry oriented instruction, etc.). Additionally, despite calls for use of statistical software and technology to support student learning, only 46% of instructors report using technology as a fundamental or supplemental part of their course, with many limiting the use of technology to graphing calculators. As part of our poster presentation we will expand on these findings and report on how institutional and instructor characteristics relate to how statistics is being taught in these courses.

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Future Teachers' Identification of Multiplicative Situations

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This study examines 22 future middle school teachers' problem solutions on proportional relationships. Using the appropriateness attribute (Izsák, Jacobson & Lobato, 2011) as a framework, we explored to what extent future middle school teachers were able to appropriately identify multiplication situations (MS), as well as identify and classify as partitive division situation (PDS) or quotitive division situation (QDS). Findings reveal that participants succeeded in identifying multiplicative situations, but more interestingly, with problems requiring division, PDS was recognized four times as often as QDS.

Keywords: Proportional relationship, Appropriateness, Multiplicative situation (MS), Partitive division, and Quotitive division

Multiplicative reasoning contains various operations such as fractions, decimals, ratios, percent, proportions, linear functions and higher level topics (Izsák, Jacobson, and Lobato, 2011). A considerable amount of previous research has shown that future and current teachers struggle to recognize appropriate mathematical operations, especially for multiplication and division situations (e.g., Harel, Behr, Post, & Lesh, 1994; Tirsoh & Graeber, 1990). This present study investigates how middle-grade preservice teachers discriminate multiplicative situations, partitive division situations, and quotitive division situations on a proportional relationship task. According to DTMR, reasoning about fractions entails four attributes including referent unit, partitioning and iterating, appropriateness, and multiplicative comparison (Jacobson & Izsák, 2015). Appropriateness requires identification of an association between the quantities of the given problem and relating this quantitative association with an accurate mathematical operation (Izsák et.al., 2011). Using a uniform definition of multiplication based on equal size groups (MxN=P) that clearly differentiates between the Multiplier "M" (number of groups) and the Multiplicand "N" (# of base units in each group), a quotitive division situation is one where "M" is unknown, and a partitive division situation is one where "N" is unknown.

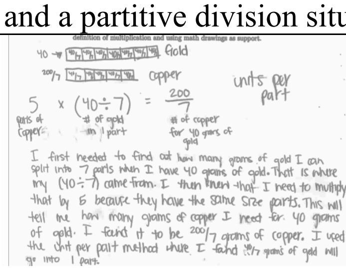
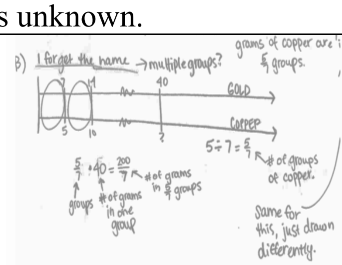
 ➢ Use of MS (i.e., 5x(40÷7)) and division (i.e., 40÷7) ➢ Identification of PDS: "how many grams are in 1 group"	 ➢ Use of MS (i.e., $\frac{5}{7}$x40) and division (i.e., 5÷7) ➢ Identification of QDS: division

Figure 1: Use of MS, PDS, and QDS with indicators for division

Each of the 22 future teachers were asked to solve a task with two different methods, resulting in 44 solutions. Results revealed that all participants could identify MS, however only 15 of the 44 solutions identified division as the appropriate operation. Furthermore, only 3 of these solutions identified QDS, which may signal that it is more challenging. Implications include the importance of focusing instruction on QDS for future teachers.

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Writing Explanations: Provoking Different Knowledge Bases by Context

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Recent research shows the promise of using tasks that situate mathematics in a pedagogical context for secondary teachers, including tasks where teachers are asked to explain a solution to a mathematical task. We use a theory of positionality (Aaron, 2011; Herbst & Chazan, 2003, 2011) to make sense of why explanations might differ when the solver is positioned as a secondary teacher as compared to positioned as a university mathematics student.

Keywords: positionality, mathematical knowledge for teaching

In this poster, we examine the research question: When positioned as a teacher as opposed to positioned as a university mathematics student, what differences in knowledge bases emerge when solving mathematics tasks? To address this question, we interviewed 17 practicing secondary teachers. They wrote explanations to two versions of a task adapted from Biza, Nardi, and Zachariades (2007), first in the context of a university course and second in the context of high school teaching. Using the results of three teachers whose solutions were mathematically valid, we make the argument that positioning as a teacher can elicit the development of mathematical knowledge for teaching (MKT: Ball, Thames, & Phelps, 2008; Silverman & Thompson, 2008) in ways that are not activated when positioned as a university mathematics student. We contribute an illustration of this phenomenon and extend the results of Biza et al. (2007). Figure 1 shows the tasks and positioning as presented to participants.

Positioning as university student	**Positioning as secondary teacher**
Your mathematics professor assigns this problem during a unit on mathematical justification. *Explain why the equation* $|x| + |x + 1| = 0$ *has no solutions.* Write a solution that you would hand in to the professor of this course.	You plan to assign this problem to your high school student during a unit on mathematical justification. *Explain why the equation* $|x| + |x + 1| = 0$ *has no solutions.* Write a solution that you would share with students this course.

Figure 1. Absolute Value Task, adapted from Biza et al. (2007)

The explanations differed based on context, as exemplified by the quotes in Figure 2. In the university context, these participants primarily summarized their deductive reasoning. When positioned as a teacher, they used more representations and attended explicitly to student thinking and instructional moves to guide student thinking. They discussed how to help students generate conviction that the statement is true, but none discussed motivating this idea in the university context. Thus, in the position of secondary teacher, but not university student, participants engaged in all of Silverman and Thompson's (2008) practices for developing MKT.

Positioning as university student	**Positioning as secondary teacher**
This task "requires you to know a lot about what it means to justify it. … that it works for all the cases, is important… understand how absolute values work and how to prove things surrounding them."	"I think [students' knowledge about absolute value is] a good place to start because if they have that knowledge, then they can play with it and make drawings and use number lines and see what's happening and from there convince them self and justify whatever they're saying."

Figure 2. Differences in explanations by context

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Developing the Developers: Lessons Learned from Work to Support Providers of Professional Development for Graduate Teaching Assistants

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Preparing graduate teaching assistants (GTAs) well for their teaching roles is a high-leverage opportunity to improve undergraduate mathematics education. The College Mathematics Instructor Development Source (CoMInDS) seeks to assist people who build and lead teaching-focused GTA professional development (TAPD) at their own institutions. CoMInDS offers direct support to these TAPD providers and seeks to enhance the development and use of research-based TAPD practices. We draw upon project evaluation data and team members' reflections to identify progress, opportunities and challenges in this work.

Keywords: professional development, teaching assistants, instruction, graduate education

As a group, graduate teaching assistants (GTAs) teach mathematics to thousands of undergraduates, particularly in lower-division courses that may serve as students' only college mathematics experience (Ellis, 2014). Yet GTAs often lack good preparation, skills and models for teaching (e.g., Speer, Gutmann & Murphy, 2005; Kung & Speer, 2009). Moreover, many college STEM educators gain their first teaching experience as a GTA (Connolly, Savoy, Lee & Hill, 2016). Preparing GTAs to be effective teachers thus offers a two-fold opportunity to improve undergraduate mathematics instruction, in courses taught by GTAs and in courses taught later by those who go on to careers as college instructors. Indeed, strong, teaching-focused GTA training is linked to good student experiences, retention and success in early college math courses (Rasmussen, Ellis, Zazkis & Bressoud, 2014). But this is not the norm: most GTA training is short in duration and focuses on logistics and uniformity of multi-section courses, rather than seeking to develop GTAs as effective teachers (Ellis, Deshler & Speer, 2016).

CoMInDS supports mathematics TAPD providers, especially newer providers, through intensive and online workshops that model TAPD activities and topics, connections to other providers, and a suite of practical resources—sample syllabi, activities, assessments and program models for TAPD. CoMInDS leaders also work with RUME scholars to help enrich research on TAPD and build research-practice connections (e.g., Deshler, Hauk & Speer, 2015). This poster identifies lessons learned from this work, drawing from evaluation findings and team members' insights to highlight what has worked, what we have learned, and what has challenged us. For example, we find that workshops reduce providers' isolation and foster their sense of TAPD as professional work, but most providers do not (yet) feel strong ties to a wider TAPD community. For many, the workshop is a first exposure to systematic thinking about the goals and design of a TAPD program. Not all providers embrace active learning as a program vision, but most seem open to framing that emphasizes building GTAs' skills in probing and using student ideas.

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Mathematics of Graphic Animations of Solids of Revolution

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The purpose of this study is to look at the covariational reasoning necessary to define surfaces parametrically in 3-space and the mathematics involved in writing statements in Graphing Calculator (GC). This study offers evidence on how developed ideas of covariational reasoning and parametric explanation have an impact on visualizing animations, especially in the case of solids of revolution. Additionally, the analysis of seven calculus textbooks focusing on parametrically defined relationships reveals that the use of parametric explanation is confined to representing familiar graphs parametrically and the trends of describing parametric relationships in the textbooks will be discussed as well. Through the structured way of thinking embedded in the statements in GC, students will be able to understand how the surfaces are formed as the parameters vary.

Keywords: Covariational Reasoning, Parametric Relationship, Solid of Revolution, Technology

Project DIRACC (Developing and Investigating a Rigorous Approach to Conceptual Calculus) utilizes didactic objects to support students' dynamic imagery (Thompson, 2002). In the online calculus textbook 'Newton meets technology' (Thompson & Ashbrook) developed as a part of the project DIRACC, students can use the animations in 3-space to develop productive and meaningful images of solids of revolution.

We analyzed seven textbooks focusing on parametrically defined relationships. The trends of describing parametric relationships are disclosed as follows; 1) Parametric equations are a way of defining a 'curve' in the 'xy' plane. 2) The parametric context is only for finding $\frac{dy}{dx}$ as slope of tangent line. 3) Parametric relationships are just to use the chain rule to find $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

The textbook analysis led us to ponder what would be the most problematic part for students when connecting the parametric representations with the visualization of animations. The mathematical ideas of parametrically defined relationships and covariational reasoning are correlated. In other words, the development of dx and dy as variables throughout the DIRACC textbook is crucial to show the rate at which one quantity changes with respect to the rate of change of another quantity, where these two quantities are related to a common third variable t.

Covariational reasoning and parametric reasoning are implemented in the statements of Graphing Calculator (GC), so that they provide students with the ways of thinking of parameters as varying quantities within finite sized intervals. We created four animations of revolving the graph y=sin(x) around the x-axis and the y-axis, where $x \in [o, \pi]$ using two different perspectives for each revolution; varying height and varying width. All the animations have multiple variables varying simultaneously to create the image. To understand the creation of each surface, a method we found helpful is to consider a particular value of all parameters but one and focus on the changes in the remaining parameter. By focusing on the variation in each parameter individually, we are able to put the variations together more coherently.

Through beginning with this structured way of thinking, students looking at these animations will be able to understand how the surfaces are formed as the parameters vary.

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Students' Consolidation of Knowledge Structures through Problem Posing Activities

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This study explored a pedagogical way to contribute to students' consolidation of their newly formed mathematical knowledge. We applied problem posing activities in complex analysis course. Students posed a problem in a group and then solved this. In the process of these activities, we could observe three modes of consolidation in the students' mathematical knowledge and the epistemic actions emerging in aspects of problem posing. A problem posing activity can act as a practical way to stimulate students' consolidation.

Keywords: Consolidation, Problem posing, Complex analysis

Hershkowitz et al. (2001) introduced Abstraction in Context (AiC). AiC theory explains the process of the construction of mathematical knowledge. Since newly constructed knowledge is fragile, it needs to be consolidated. Consolidation is a process in which abstraction becomes so familiar that it is available to the learner in a flexible manner (Dreyfus, & Tsamir, 2004). Analyzing students' consolidation of knowledge structures can be a way to diagnose their understanding of newly constructed knowledge. However, there is little consideration of pedagogical ways to lead students to the consolidation phase. A problem posing activity can be a way to stimulate such consolidation. This is because problem posing extends students' perception of mathematics, and enriches and strengthens their knowledge of basic concepts (English, 2003). Also, to understand how problem posing can be enacted in classrooms, there is a need for analysis of practice (Cai et al., 2015). So, in this study, we conducted problem posing activities to explore students' consolidation.

The participants were 27 undergraduate students in a course on complex analysis. Two activities were carried out in groups of three. The first activity was posing a problem using the concepts of complex analysis, and the second was solving the posed problems. Two groups were selected for videotaping and post-activity interviews. All their discussions and interviews were transcribed. We analyzed students' individual utterances based on three modes of consolidation: B (Building-with), RfB (Reflecting on Building-with), and Rf (Reflecting) (Dreyfus, & Tsamir, 2004). Also, we traced the students' consolidation by determining their epistemic actions during the problem posing activities in the dimensions of task organization, knowledge base, heuristics and schemes, individual considerations of aptness, group dynamics and interactions, which are repetitive facets in the framework of Kontorovich et al. (2012).

Three modes of consolidation are shown by the students' utterances focusing on concepts and reviewing problems that they had previously solved. The students continued the mathematical discussion based on their own knowledge base and explored the condition of theorems in their group. Also, they discovered relations between their discussion and the concepts they had learned. In these problem posing aspects, we could observe that students consolidated their knowledge, and that problem posing stimulated epistemic actions that initially postponed students' consolidation but induced them to reach the consolidation phase before long. Therefore, problem posing in a group activity contributes to epistemic actions by students that lead to the consolidation phase. This means that this activity can provide an opportunity for students to reflect on their own learning process and enable them to apply mathematical contents in a flexible way.

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Characterizing Transition to Proof Courses: The Case of Liberal Arts Colleges

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Many undergraduate students experience significant difficulty in learning to prove mathematical propositions nationwide. A previous study by David & Zazkis (2017) used document analysis of publicly available syllabi to create a national portrait of approaches to supporting students' transition to proof across a large sample of R1 and R2 universities. Liberal arts colleges (LACs) operate under different sets of institutional constraints and thus offer the possibility of different approaches to this issue. We report results of a preliminary survey study, the goal of which was to enhance previous work on approaches to the transition to proof by specifically focusing on the case of LACs. Analysis of the survey data show that LACs' approaches have distinctive features as compared to R1 and R2 universities. Notably, discrete mathematics courses served as a transition to proof course in almost half of the surveyed institutions.

Keywords: Transition to proof, Liberal Arts Colleges, Instructional Approach

Learning to prove mathematical propositions is a cornerstone of the mathematical discipline (de Villiers, 1990), however, many undergraduate students struggle to learn to prove (Selden, 2012). Mathematics departments have recognized this problem and experimented with different curricular and instructional approaches to supporting students' entry into proof, including courses dedicated to this transition (Smith et al., 2017). A previous analysis by David & Zazkis (2017) showed that numerous departments have developed courses to introduce students to the nature of proof and effective arguments and that these courses have a surprising variability in their form and content. However, David and Zazkis's focus was on R1 and R2 research universities, and the field currently knows little about the range of approaches followed by other kinds of institutions. Our goal for this survey study was to enhance previous work on approaches to the transition to proof by specifically focusing on the case of LACs.

Fifty LACs were selected randomly based on the list of all *liberal arts colleges* by US news. We asked college mathematics faculty involved in the teaching of collegiate transitions to proof courses (or courses that use to facilitate the transition to proof) at those 50 LACs to complete a brief survey that we designed about the approach currently being taken at their colleges. Currently, 15 (30%) LACs have filled out the survey. For those 15 participating colleges, only 2 out of the 15 responded that they do not have any kind of the "transition to proof" courses.

Preliminary results indicate that LACs' approaches to supporting students' challenges through this transition are unique in many ways. The biggest difference is the non-coordinated nature of their "transition to proof" course as evidenced by the following range of approaches: out of the 15 responses, 20% write their own textbooks, 27% said the course varies according to instructor each semester, 20% embed the skills or practices students need in other courses such as number theory or linear algebra. Notably, discrete mathematics courses served as a transition to higher-level math courses in almost half of the surveyed institutions. Despite the differences, there are also similarities with the approaches used by R1 and R2 universities. In particular, while the results of the current study are preliminary, the results allow a broader picture of the range of possible ways to support this difficult transition for students and thus have implications both for LACs college faculty and R1/R2 institutions.

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“Bold Problem Solving” in Postsecondary Mathematics Classes: Validation and Patterns

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This study (a) validates a measure “bold problem solving” for postsecondary students and (b) examines patterns in bold problem solving tendencies within and across various math classes. A confirmatory factor analysis demonstrates the general construct holds for the postsecondary population. Course and gendered differences in bold problem solving tendencies exist.

At the highest levels, math requires inventiveness, experimentation, and risk taking. However, whether early orientations towards mathematical inventiveness and risk-taking have some relationship to those who choose to pursue careers involving advanced math remains unclear. College students who take Calculus tend to be more confident than those in lower level courses (Hall & Ponton, 2005). However, confidence does not suggest actionable *behaviors* that students can develop to help them become oriented toward math. *Bold problem solving* (BPS), a type of mathematical risk taking that involves a preference for solving problems using novel or invented solutions, a preference for working on more open-ended problems, and a desire to work independently, offers one possible avenue to address this. This study:

- Validates the six-item BPS tendencies scale with the postsecondary population.
- Examines relationships between self-reported “boldness” and students’ level of enrollment in mathematics, with a particular focus on gender.

Methods

Data were collected at a large southeastern university during the Fall 2018 semester. Four classes were surveyed: Intermediate Algebra, entry level Finite Mathematics, Calculus I (non-honors), and Discrete Mathematics. These classes loosely capture the standard curriculum of the first few years of the postsecondary mathematics pipeline. The survey asked about students’ demographics, math background, and career plans. The bold problem-solving tendencies scale that was piloted and partially validated with a sample of eighth grade students (Author, in preparation), and several other measures of students’ attitudes towards math, were also included.

Analysis and Results

Using the entire sample, a confirmatory factor analysis (CFA) was conducted which resulted in four of the original six items being retained. Using the four retained items, BPS scores were created for each individual. Independent sample t-tests were run to examine gender differences in scores within classes. Table 1 presents the results from these analyses. Additional results and discussion will be presented on the full poster.

Table 1. Bold Problem Solving Scores and Differences by Gender

	Sample	Female		Male			
Course	BPS	*n*	BPS	*n*	BPS	Difference	Significance
Intermediate algebra	2.937	74	2.838	29	3.19	-0.352	0.016
Finite mathematics	2.874	86	2.89	17	2.794	0.096	0.62
Calculus I	3.223	111	3.079	86	3.41	-0.331	0.001
Discrete mathematics	3.375	26	3.308	48	3.411	-0.103	0.599

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Expert vs. Novice strategy Use During Multiple Integration Problems

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One of the most challenging tasks in multivariate calculus for students is correctly setting up integration problems. The limited research in multivariate calculus suggests that this is due to the need to coordinate algebraic and geometric representations of the problem while maintaining representational flexibility. The purpose of this study was to explore the approaches that both experts and novices take to set up and solve triple integral problems using data collected via Smartpen-recorded task based interviews. The initial analysis of the data suggests that a difference in representational flexibility is one of the main differences between expert and novice approaches to multivariate integration.

Key words: integration, multivariate calculus, representational flexibility

Research examining students' success in introductory mathematics courses consistently shows that students are not learning the intended material (Apkarian & Kirin, 2017). In fact, multiple studies have revealed that students that achieve a high grade in introductory calculus actually have a weak understanding of the course's key concepts. These results put in question whether or not the traditional calculus curriculum is preparing students to use ideas of calculus in future courses (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Ongoing efforts to reform calculus instruction arise from concerns that students are learning calculus as simply a series of algorithms without conceptual understanding (Dawkins & Epperson, 2014). Such algorithmic learning is problematic for students in multivariate calculus, where students need to be able to recognize and convert to appropriate coordinate systems to complete many multivariate integral problems.

The purpose of this basic qualitative research interview study (Merriam, 1998) was to explore how students in multivariable calculus strategize how to solve multiple integrals compared to expert mathematicians. This exploration is significant because there has been no research on integral strategy use beyond introductory calculus (Speer & Kung, 2016). Both the students and experts participated in task based interviews where they will complete multivariable calculus problems using a Smartpen, which allows for an audio recording to be synced with the writing without the use of video recording. The Pencasts were then be analyzed to compare problem trajectories and strategy use with the ultimate goal to compare and contrast expert and novice multivariate calculus users. The research questions for this study were: (1) What are the strategies used by expert and novice multivariate calculus students when solving multiple integration problems? (2) To what extent do experts and novices employ similar strategies?

The expert participants for this study were three tenured faculty members who had all taught multivariate calculus at least five times with the most recent iteration of the course being within two semesters of data collection. The novice participants were six undergraduate sophomores and juniors who had recently completed a multivariate calculus course within one semester of data collection. Expert participants were interviewed alone while the novice participants were interviewed in pairs. All participants completed the same 45-minute task-based interview of representative multiple integration problems using a Smartpen to capture their written work and audio discussion throughout the process. These Pencasts were then open coded for strategy use.

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A Survey of Student Attitudes toward Math in CRAFTY Inspired Classes for Business Students

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Abstract: The author is part of a multi-institution grant that is attempting to implement the recommendations of the MAA Curriculum Foundations Project CRAFTY report on mathematics for partner disciplines. The author's institution is focusing mathematics for business students. Although the CRAFTY report is nearly 15 years old there seems to be little in the literature looking at the effectiveness of implementation of the report's recommendations. This report looks at how implementation changes student's attitudes toward mathematics.

Key Words: Business Mathematics, Spreadsheets, Client Discipline Expectations

Background and Motivation

The MAA's Curriculum Foundation Project and its CRAFTY reports [MAA 2004] looked at the desires for partner disciplines desires for introductory mathematics courses. This was followed by a series of attempts to implement the recommendations in college courses, particularly focused on college algebra [MAA, 2011]. Two RUME reports [May, 2013, Mills, 2015] have looked at the desires of business faculty and confirmed the findings of the CRAFTY report. However there seems to be almost no RUME or SOTL studies on the effectiveness of any implementation or its impact on student attitudes toward mathematics. The work behind this poster is an attempt to start that examination.

Context of the Work

The work is part of a multi-institution grant, NSF-Number, with the author's institution focusing on math for business students. After the CRAFTY report, there were two serious attempts to implement the report recommendations in business calculus projects [Felkel and Richardson 2008, Thompson and Lamoureux, 2002]. All schools using either of those books have gone back to traditional books for there students. The work of [Felkel and Richardson, 2008] served as the inspiration for a online book, WEBSITE that is used at the author's institution and one other school for business calculus. Following the advice of a working group of business and math faculty at the author's institution, the author is adapting CRAFTY inspired materials from another school in the grant to develop a course in college algebra focused on business students.

Research Framework

Anecdotal evidence indicates that the changed focus of the course changes the students attitudes toward mathematics, with the added context, focus on modeling and discipline specific problems, and appropriate use of technology making mathematics more relevant. The students in treatment groups and control groups were given a survey on attitudes toward math and those are compared.

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Alternative Scoring Methods in Collegiate Mathematics Courses

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This poster presentation will highlight the results of a qualitative, multicase study which explored the use of alternative scoring practices in collegiate mathematics classes. Specifically, the researchers explored the use of two different scoring practices: one in an entry-level College Algebra course and one in an upper-level Modern Geometry course. In addition to classroom observations, data collection for each case consisted of two interviews for each course instructor, one interview with each course designer, and interviews with students in each course. This poster presentation will detail themes from cross case analysis which suggest important details for successful implementation of alternative scoring practices in collegiate mathematics courses.

Keywords: Assessment, classroom practices, case study

Grading and scoring practices have been a topic of debate and discussion for more than a hundred years (e.g., Starch & Elliot, 1913). There has been a recent push, however, in collegiate mathematics to implement alternative scoring practices (e.g., MAA, 2018). At a University in the rocky mountain region, there are two different methods used in two different courses. The purpose of this case study was to explore the implementation of each of these methods during the fall semester. In both cases, the scoring practices were implemented by instructors who were mentored in the method by another instructor who designed the course and initially implemented the scoring method. This study sought to describe the nature of alternative scoring practices in collegiate mathematics courses.

Data Collection and Analysis

For each course, the first author conducted pre- and post-interviews with each course instructor. The purpose of these interviews was to better understand their teaching philosophies as well as their use of the alternative scoring practice. She then observed each course for a period of four weeks on days in which the instructor returned scored work. Following classroom observations, she interviewed the course designer as well as three students per course. After data collection, the data was analyzed using an open coding process to determine and explore emergent themes.

Themes and Discussion

The emergent themes from cross case analysis suggest key details for the successful implementation of alternative scoring practices. These themes include instructor buy-in and consistent implementation, communication between instructor and students, and the ability to implement feedback and correct work for an improved score.

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Impact of Historical Mathematical Problems on Student Metaperspectives of Mathematics

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Undergraduate students in a History of Mathematics course engaged with various historical mathematical problems. Reflective journals and interviews were used to analyze their perspectives on meta-issues of mathematics. The results indicate some revision of their metaperspectives and new cultural awareness.

Keywords: Metaperspectives, History of Mathematics, Meta-Issues, Cultural Mathematics

Jankvist (2009, 2011) distinguishes between mathematical *in-issues* and *meta-issues.* In contrast with in-issues, meta-issues are concerned with mathematics as a whole (Jankvist, 2009, 2011), including the nature of mathematics as a discipline and the social and cultural-situatedness of mathematical work (Bishop, 1988, 2002; D'Ambrosio, 1985). Student conceptions of these meta-issues are termed *metaperspectives*, and are important in shaping how they interact with and understand mathematics.

Work of the past few decades has established a number of potential benefits for integrating the history of mathematics into mathematics curriculum (Clark, 2012; Clark, Kjeldsen, Schorcht, Tzanakis, & Wang, 2016; Fauvel, 1991; Swetz, 1995). This project considers undergraduate metaperspectives as students engage with historical problems grounded in primary sources (Barnett, Lodeer, & Pengelley, 2014), investigating the research question: *How do students' meta-perspectives change as they engage with historical mathematical problems?*

Twelve undergraduate STEM majors enrolled in a history of mathematics course completed a series of journal entries reflecting on meta-issues in mathematics and their own experiences encountering historical mathematics. Initial journals included prompts such as "Describe a mathematician", and "Is mathematics invented or discovered?" As the semester progressed, prompts addressed reactions to class work more specifically. All journals were completed online. In addition, five students were interviewed two times each. One interview asked students to expound on passages from their journals, while a follow-up interview at the conclusion of the course prompted reflection on their views of the meta-issues described above. Themes within these journal entries emerged using open coding (Charmaz, 2014).

Results indicate that students initially viewed mathematics as "discovered" - existing independently of any human knowledge of it. Furthermore, an archetypal mathematician was described as an "old, white Greek man". Initial meta-perspectives indicated widespread exposure to a modified Eurocentric perspective on mathematics history (Joseph, 2011), with some awareness of historical mathematical work in Asia. As the semester progressed, students began to describe mathematics as arising from practical needs within a culture and recognize differences in mathematical communication. The proposed poster highlights themes in student metaperspective shifts, particularly new cultural awareness and appreciation of the way mathematics is embedded in cultures that produce it. The results indicate that historical problems prompted students to reflect on mathematical meta-issues and adjust their metaperspectives while not entirely dismissing their existing ones. For example, though more students described mathematics as arising from needs within a culture, they still viewed mathematics as existing separately from any mathematician or culture's conception of it.

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A Case Study of Student Motivation and Course Structures in Introductory Calculus

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Student success in introductory calculus is imperative to obtaining a degree in STEM. Calculus I is a main gatekeeper course for STEM majors, and many students leave the class with a diminished motivation to pursue further courses related to mathematics. This poster reports a qualitative case study from a larger mixed-methods project aimed at exploring the relationship between course structures (hybrid, traditional, and large active learning) and student motivation in calculus. Using the theoretical framework of self-determination theory (SDT), six students were interviewed to investigate how each course structure was related to students' perceptions of their competence, autonomy, and relatedness. Emerging themes showing differences in student motivation between the three course types will be presented.

Keywords: Calculus Success, Motivation, Active Learning

The Mathematical Association of America (MAA) national study of *Characteristics of Successful Programs in College Calculus* revealed that introductory calculus occupies a gatekeeper role for STEM majors across the country. Even if students persist through Calculus I, they leave the class with a diminished confidence and enjoyment of mathematics and a decreased desire to continue pursuing further mathematics (Bressoud 2015). Thus, the goal of this research study was to provide a better understanding of the relationship between learning environments and student motivation in introductory college calculus. Results of this work will help guide mathematics faculty and administrators to create environments that are most conducive to fostering students' motivation, thus supporting their academic achievement in calculus.

The theoretical framework of self-determination theory (SDT) was used to guide this study. SDT is a macro-theory of motivation and has been widely used to study the social factors of an environment under which people thrive (Ryan & Deci 2000). According to SDT, three basic psychological needs are essential to fostering a student's motivation and engagement: competence, autonomy, and relatedness. Competence refers to students feeling confident and effective in the classroom, autonomy means they have a sense of agency and authority, and relatedness incorporates students' need to feel a sense of belonging in the classroom (Niemiec & Ryan 2009).

This poster will report the qualitative piece of a larger mixed-methods design that investigated the interaction of course structures, students' basic psychological needs satisfaction, and motivation. Three different course types of Calculus I were sampled at a large research university, which included traditional methods, hybrid online, and a large-enrollment active learning classroom. The Basic Psychological Needs Scale (BPNS) and the Situational Motivation Scale (SIMS) were administered to students in the three course types (N=323). Six students were purposefully selected based on their survey responses, and one-on-one interviews were conducted to determine what aspects of each course structure were contributing to students' perceptions of their competence, autonomy, and relatedness. This poster will present emerging themes from the case study analysis (Merriam 1998), and implications for mathematics faculty will be discussed.

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Understanding the Impact of Supports on Adjunct Mathematics Instructor Knowledge

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This proposal describes findings of an ongoing project designed to support adjunct instructors' teaching of undergraduate Precalculus. We are studying the impact of supports on Precalculus instructors' knowledge through interview and assessment data. Using Shulman's (1987) components for a teaching knowledge base, we discuss shifts in instructors' perspectives.

Keywords: Teacher knowledge, Adjunct instructors, Undergraduate mathematics, Precalculus

We know students' persistence in pursuing STEM degrees is heavily influenced by their experiences in undergraduate first year mathematics courses (Pampaka, Williams, Hutcheson, Davis, & Wake, 2012). In this regard, the quality of pedagogy can make a difference in retaining students, as improved instruction may motivate students to learn more mathematics and consider pursuing a STEM degree (Ellis, Kelton & Rasmussen, 2014). Moreover, current trends in higher education are to employ more part-time, non-tenure track faculty to teach introductory courses in science and mathematics (Curtis, 2014). These trends have motivated the field to better understand how institutional policies and practices can improve part-time instructors' professional growth (Kezar & Sam, 2013). This proposal presents findings from a study of Adjunct Mathematics Instructor Resources and Support (AMIRS) to explore the impact on Precalculus adjunct instructor knowledge in an effort to address these issues.

To investigate Precalculus adjunct instructor knowledge, we adapted three components for teaching knowledge base Shulman (1987) argued, allow teachers to develop deeper understanding of their subject: Structures of subject Matter (SOM), Principles of conceptual organization (PCO), and Principles of inquiry (POI). We looked at how supports (e.g. course coordination, summer workshop, PLC meetings) impacted their knowledge through pre- and post-interviews and content assessments aligned with an adopted research-based curriculum. Based on interview data, we found differences in SOM by observing changes in the depth of instructors' content knowledge in terms of thinking about specific structures of precalculus (e.g. tangent being the slope of a curve). Second, although instructors had previous experience teaching mathematics, and therefore prior conceptual webs of precalculus topics (PCO), there is evidence that teachers not only began reorganizing their knowledge but also valued this reorganization as a benefit for their students' understanding. Finally, regarding POI, instructors moved from general to more specific ideas about how students can engage in mathematical inquiry, while also citing opportunities for students to model situations to problem solve, and for students to drive instruction. Currently, we are analyzing results from content assessment to better understand the nature of these changes.

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Student Engagement while Establishing Classroom Mathematical Practices

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This study investigates student engagement while learning through use of an app that collected student engagement reported by participants during a classroom teaching experiment. This paper discusses preliminary results on students' engagement in the process of learning. Though not anticipated, we observed differences between male and female students' engagement while working in mixed-pairs worthy of investigation.

Keywords: Student engagement, Classroom mathematical practices, Preservice teachers

There is a significant connection between student engagement and performance achievement. Klem and Connell write, "student engagement has been found to be one of the most robust predictors of student achievement and behavior in school, a conclusion which holds regardless of whether students come from families that are relatively advantaged or disadvantaged economically or socially" (2004, p. 5). However, student engagement is complex, and currently relationships to outcomes such as mathematical understanding and learning are elusive (Fredricks, Blumenfeld, & Paris, 2004; Middleton, Jansen, & Goldin, 2017). This study investigates student engagement in the process of learning.

Theoretical Framing

From the perspective of flow theory, student engagement is comprised of interest, enjoyment, and concentration (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003), where interest and enjoyment make up emotional and behavioral aspects and concentration accounts for cognitive engagement. The emergent perspective describes learning as social and individual, where classroom mathematical practices comprise collective learning and individuals' ways of participating in such practices reflects individual learning (Cobb & Yackel, 1996). We consider students' affective and cognitive experiences through these theories.

Methods

We conducted a classroom teaching experiment (Cobb, 2000) with 6 preservice teachers (3 female and 3 male) to address the question, *what characterizes relationships between student engagement and learning?* Participants were sent a 5-item survey on engagement at two random times during each one-hour session through a mobile app on their smart phones. One-on-one recall interviews were conducted based on survey responses and mathematical contributions. All sessions and interviews were video recorded. Data were analyzed for participants' engagement and classroom mathematical practices (Stephan & Rasmussen, 2002).

Results and Conclusions

We observed differences between male and female students' engagement while working in mixed-pairs surrounding important mathematical contributions from female partners. Female students described situations in which they perceived of male partners overlooking valuable contributions towards completing tasks, resulting in dips in engagement. With regards to data collection, the app and survey effectively gathered information on student engagement, which was triangulated by students' descriptions in recall interviews.

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TRANSFORMATIVE LEARNING THEORY: A LENS TO LOOK AT MATHEMATICS COURSES FOR PREPARING FUTURE TEACHERS

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We suggest a 4-step cycle for helping prospective teachers transform their mathematical understandings from procedurally-based to more conceptually-based understandings. We use Mezirow's (1991) idea of Transformative Learning Theory (TLT), which is an application of androgogy, or the methods of teaching adults. In this poster, we share our model of a TLT cycle and illustrate it using an example of a proportional reasoning problem for prospective teachers.

Key Words: Learning Theory, Teacher Education-Preservice, Andragogy

Research has shown that prospective teachers (PTs) enter their mathematics content courses with procedural understandings of mathematics (e.g. Thanheiser et al., 2014). However, they will be required to know and understand more than just *how* to solve mathematics problems (AMTE, 2017). We believe that it is our job as mathematics teacher educators to help PTs develop the conceptual understandings and specialized mathematics content knowledge that they will need in their work as teachers. In studying how to help PTs transform their understandings of mathematics, we look at what makes PTs' relearning of mathematics different from children learning mathematics for the first time. We begin by studying the concept of *andragogy*, which involves methods of teaching adults. (This contrasts with pedagogy, which are methods for teaching children.) Malcolm Knowles (1984) researched the concept of andragogy in the 1980's and proposed four assumptions about adult learners. *Self-concept* relates to the idea that adults are more responsible for their own learning than children. The *role of experience* encourages instructors who work with adult learners to take the experiences they bring into account when planning and implementing instruction. Adults are often motivated by how the content they are learning applies to them and their future careers. An adult's *readiness to learn* and *orientation to learning* are tied to their internal appreciation of how the information applies to their lives.

Transformative learning theory (TLT) is an application of andragogy that attempts to establish and clarify a learner's prior assumptions and then transform these assumptions (Mezirow, 1991). The theory claims that only after learners are aware of their assumptions can they develop strategies to transform these assumptions. We present a 4-step implementation cycle based on TLT to help university instructors plan and implement lessons to help their students deepen their mathematical knowledge. In the first step the instructor presents the learner with a *disorienting dilemma* where his/her preconceived ideas are challenged and perturbed, or where the procedures that they already believe they know are not enough to solve the problem. In the second step students are asked to work through the dilemma while *reflecting on their previous assumptions*. The third step focuses on *justifying and explaining* their proposed solutions with peers in order to reach an equilibrium between their prior assumptions and the disorientation presented by the task. The fourth and final step involves *making connections* between the procedural fluency and the conceptual understanding, helping students to see how the procedures they learned (their prior assumptions) are related to their new knowledge (their transformed understanding). We suggest that TLT can be a valuable resource for helping PTs and other undergraduate students to expand and transform their mathematical understandings.

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Conceptual Desires and Procedural Demands: Conflicting Aims in University Mathematics Students' Work on Tasks in Seminar Groups

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Small groups teaching as part of first semester was studied through observations of the seminars, analysis of the tasks, and students' responses in surveys and interviews. The research question posed is how the teaching activities enabled the students to develop their conceptual and procedural knowledge. The results show a desire for a development of conceptual knowledge but the demands on procedural knowledge placed the students with conflicting aims.

Keywords: university mathematics teaching, conceptual knowledge, procedural knowledge

In the area of university mathematics education there is a growing interest in the teaching (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). Small groups teaching has shown reasonable effects on students' conceptual understanding (eg. Jaworski, Robinson, Matthews, & Croft, 2012), but still there is need for research on the use of small groups teaching to understand students' learning in such settings. The poster presents results from a project at a mathematics department in Sweden where small groups teaching was part of the schedule for the first semester mathematics students (Pettersson & Larson, 2018). The aim of the project was to develop the small groups teaching and a study was set up to better understand the situation. The so called seminar groups met once a week, included about 10-15 students in each group and were led by an experienced lecturer. There were also four times a week ordinary lectures given in a lecture hall for all the students (>100 students). In the seminars students were discussing tasks that promoted conceptual knowledge. An example of a task is "What does it mean for a function that the derivative is missing in one point?" The students were before each seminar obligated to hand in written solutions of one or two tasks. These tasks are marked and commented on by the teacher of the seminar group. An example of such a task is "Give the global maximum and minimum for the function $f(x,y) = \ln(1+x^2+y^2) - x$ on the disc given by $x^2+y^2 \leq 4$."

The research question for the study presented here was: How did the small group teaching activities enable the students to develop their conceptual and procedural knowledge? Data collection includes observation of small group sessions, analysis of the given tasks, and students' responses to surveys and interviews. For the data analysis a deductive thematic analysis has been used where procedural and conceptual knowledge are used as defined by Baroody, Feil, and Johnson (2007). Starting from Hiebert (1986), further discussions argued that both procedural and conceptual knowledge could be of different quality (eg. Star, 2005) and that both procedural and conceptual knowledge need to be developed and to be intertwined (Baroody et al., 2007).

The results show that both the students and the teachers desire a development of conceptual knowledge. The discussion tasks were posed in a way to promote this kind development. The students appreciated this and wanted time to even more think about conceptual tasks. However, the tasks that the students are obligated to work with and deliver for grading were mostly engaging the students in development of procedural knowledge. The students used a lot of time to get every symbol right. So, even if the lecturer in the seminar raised conceptual aspects according to the comments made, the students were not able to fully take on that. The demand for procedural knowledge gave rise to a conflict with the desire for conceptual knowledge, and that hindered a development where procedural and conceptual knowledge got intertwined.

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Introducing IOLA-G: The Inquiry Oriented Linear Algebra Game

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This poster introduces early results from IOLA-G, a project exploring the possibilities of supporting inquiry-oriented instruction with digital videogames. During the poster presentation, we will describe our game design process, including the theories we have drawn upon during its development. We will also show small samples of gameplay from a corpus of data we collected with undergraduate students at three different universities. Additionally, we will demonstrate gameplay on laptops, providing poster visitors with first-hand experience playing the game.

Keywords: Linear Algebra, Inquiry Oriented Instruction, Videogames, Game-Based Learning

In recent years, videogames have gained traction as an educational tool, leading to theoretical perspectives to inform game design and a growing body of research into best practices for supporting student learning through gameplay. Within the same period, developments in internet and mobile technologies have led to a ubiquitous market of digital games that are being produced at staggering rates and with rapidly increasing sophistication. However, relatively few of these games have explicit educational goals and a small fraction of those games draw on best practices as identified through research, especially in order to support the learning of undergraduate content. The goal of IOLA-G is to modify our team's research-based Linear Algebra curriculum toward a theoretically oriented videogame for Linear Algebra (Zandieh, Plaxco, Williams-Pierce, and Amresh, 2018). In this endeavor, we leverage best practices from multiple communities, drawing on theory from three different bodies of research – Game-Based Learning (GBL; Gee, 2003; Gee 2005; Gresalfi, 2015), Realistic Mathematics Education (RME; Gravemeijer, 1999), and Inquiry-Oriented (IO) and Inquiry-Based (IB) Instructional practices (Zandieh & Rasmussen, 2010; Zandieh, Wawro, & Rasmussen, 2017; Rasmussen & Kwon, 2007).

In our first year of the project, the IOLA-G team worked with undergraduate capstone Computer Science students to develop an initial version of a videogame called "*Vector Unknown*." Throughout the game' development, our team explicitly drew on the various theoretical framings cited above to make decisions about game design, instructional goals, and player experience. The goal during gameplay in the current version of the game is for the player to help a bunny avatar reach a goal using vectors from a given set. In order to move the bunny, the player must insert vectors into blank boxes on a control panel to create a linear combination. The player changes the scalars of the vectors using "+" and "–" buttons to increase and decrease the scalars of the vectors in integer increments. On some levels, projections of the resulting vectors emanate from the bunny's location before the player clicks a "Go" button. On other levels of the game these guides are removed to help necessitate a change in the player's strategy.

We have collected and are currently analyzing our first round of individual interview data with undergraduates who have varying levels of Linear Algebra experience playing the game. Additionally, we are now working with a second group of programmers to improve the game beyond its current state. With this revision, we intend to draw on our experiences collaborating with the first team as well as the insight we continue to gain through our analysis of the interview data. During our poster presentation, we will describe the gameplay and our development process in greater detail. We will also present visitors with a laptop on which to play the game during the poster session and also with a link to the current version of *Vector Unknown*.

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Adjunct Instructors' Opportunities for Learning Through Implementing
a Research-based Mathematics Curriculum

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This study explored adjunct instructors' opportunities for learning as they faced challenges while implementing a research-based mathematics curriculum. Three case studies explored adjunct instructors' experiences as they implemented a research-based precalculus curriculum for the first time over the course of two semesters. The similarities and differences between the challenges faced by the instructors and the opportunities for their learning were analyzed.

Keywords: Teacher learning, adjunct instructors, research-based curriculum

Teachers play an important role in providing meaningful learning experiences to their students and can influence their decisions to stay in STEM fields (Cohen & Ball, 1999; Ellis, 2014). Research suggests that teachers should be supported as they implement research-based curricula (Cohen & Ball, 1999; Remillard, 2000). Implementing such curricula can provide learning opportunities for teachers in addition to their students (Ball & Cohen, 1996; Remillard & Bryans, 2004; Doerr & Chandler-Olcott, 2009; Drake & Sherin, 2009). In this poster I present findings from a study exploring the opportunities for teachers' learning that arise as a result of their interaction with a research-based precalculus curriculum. My specific research question was:
How does engagement with a research-based Precalculus curriculum provide opportunities for adjunct instructors' learning?

Remillard and Bryans (2004) found that teachers' unique ways of engaging with a curriculum can provide opportunities for student as well as teacher learning. They define *opportunities for learning* as arising from "events or activities that are likely to unsettle or expand teachers' existing ideas and practices by presenting them with new insights or experiences" (p. 12). These opportunities arise as teachers engage with a curriculum while making instructional decisions for effective student learning experiences.

Case study methodology (Yin, 1994) was used to study the opportunities for three adjunct instructors' learning while they implemented a research-based precalculus curriculum over two semesters. The study took place at a Ph.D granting institution in the northeastern United States. Three adjunct instructors were selected as participants for this study with each having over ten years of teaching experience. This was their first semester teaching precalculus using the new research-based curriculum. As part of this implementation the instructors participated in weekly online meetings where they discussed the curriculum and their experiences implementing it.

Data was collected in the form of semi-structured interviews and classroom observations conducted at the beginning and end of Fall 2016 and Spring 2017 semesters. In addition, audio recordings and chat logs from instructors' participation in online meetings were also collected. Qualitative data analysis methods were used to code and sort data (Saldaña, 2009). The key findings from this analysis were as follows. In order to avail the emergent learning opportunities, teachers should: 1) be mindful of their own challenges, 2) be able to explore in depth what those challenges entail, even if it includes analyzing their own teaching practice, and 3) be willing to take the necessary steps for overcoming the challenges. These findings have implications for adjunct instructor professional development.

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Graphical Forms: The Adaptation of Sherin's Symbolic Forms for the Analysis of Graphical Reasoning Across Disciplines

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This work involves a methodological presentation of an analytic framework for characterizing mathematical reasoning, introducing the construct of "graphical forms". Graphical forms build on Sherin's (2001) symbolic forms (i.e., intuitive ideas about equations) by focusing on ideas associated with a pattern in a graph. In addition to providing an overview of the symbolic forms identified in the literature, we describe how we expand the symbolic forms framework to encompass graphical reasoning. Analysis involving the graphical forms framework is illustrated by providing examples of interpretations of graphs across disciplines, using introductory biology, calculus, chemistry, and physics textbooks. Our work suggests the broad applicability of the framework for analyzing graphical reasoning across different contexts.

Keywords: Mathematical Reasoning, Symbolic Forms, Interdisciplinary

The framework we describe in this work is informed by the resource-based model of cognition, which posits that knowledge is composed of fine-grained cognitive units ("resources") that form a network and are activated in response to specific contexts (Hammer et al., 2005). Resources reflect ideas that may be characterized as conceptual, epistemological, or procedural (Becker, Rupp, & Brandriet, 2017). Symbolic forms can be considered "mathematical resources" that describe intuitive ideas associated with patterns in an equation, such as attributing the idea of "balancing" to the pattern "☐ = ☐" (a box represents a term or group of terms) (Sherin, 2001). Sherin's (2001) initial work involved characterizing algebraic operations in physics problem solving, but recent work has utilized the symbolic forms across disciplines, characterizing advanced mathematical reasoning about topics such as differentiation, integration, and vectors (Dreyfus et al., 2017; Dorko and Speer, 2015; Hu and Rubello, 2013; Izsak, 2004; Jones, 2013, 2015a, 2015b; Rodriguez et al., 2018; Schermerhorn and Thompson, 2016; Von Korrff and Rubello, 2014).

It is also worth noting that the symbolic forms framework has focused on students' reasoning, but we assert that experts have access to a similar set of mathematical resources about equations. Furthermore, we describe an analogous type of reasoning about graphs, graphical forms. Using examples from graphs presented in introductory biology, calculus, chemistry, and physics textbooks, we illustrate examples of graphical forms, including "steepness as rate" (the relative steepness of a graph provides information about rate), "straight means constant" (a straight line indicates a lack of change), and "curve means change" (a curve indicates change). Graphical forms, like symbolic forms, have broad utility and applicability for interpreting mathematical reasoning because they are not context-specific. In addition, these mathematical ideas serve as an anchor to attach meaning and describe phenomena. In this work we hope to draw attention to the role of intuitive mathematical ideas in interpreting graphs and provide a potential avenue for future research across disciplines.

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Student Reasoning about Basis and Change of Basis in a Quantum Mechanics Problem

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In this study, we explore how quantum mechanics students understand linear algebra concepts in the context of two spin-½ probability problems, the second of which required a change of basis. In particular, our research question is: what problem solving approaches do students use, what mathematical concepts are involved in that approach, and how do students reason about basis and change of basis as they engage with the problems? Data come from individual, semi-structured interviews with twelve quantum mechanics students from two different universities. Our poster will share preliminary results for all parts of the research question.

Keywords: linear algebra, change of basis, quantum physics, student reasoning, problem solving

Several studies (Adiredja & Zandieh, 2017; Hillel, 2000; Stewart & Thomas, 2010) explore student understanding of basis. For example, Adiredja and Zandieh (2017) explored students' conceptual metaphors for basis related to real-life contexts. Students' verbs related to bases generating or describing a space, and their adjectives described bases as minimal, maximal, essential, representative, different, and non-redundant. However, little is known about student understanding of change of basis, especially in a upper-division physics context.

In our study, semi-structured interviews (Bernard, 1988) were conducted with 12 quantum mechanics students at the end of the semester. Eight were from a junior-level spins-first course at a large public research university in the northwest US, and four were from a senior-level spins-first course at a medium public research university in the northeast US. Interview questions were designed to prompt student reasoning about linear algebra concepts used in quantum mechanics. For this study, we analyzed responses to: "Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$. (a) Calculate the probabilities that the spin component is up or down along the *z*-axis. (b) Calculate the probabilities that the spin component is up or down along the *y*-axis." The follow-up question of interest was: "How do you see this problem relating to basis or change of basis?"

Through a grounded analysis (Strauss & Corbin, 1998), preliminary results indicate that to complete the spin up portion of problem (b), students used two main approaches: changing $|\psi\rangle$ to be written in terms of the *y*-basis, or changing *y*-basis vectors to be written in terms of the basis the given $|\psi\rangle$ was expressed in. The linear algebra concepts involved in at least one of these approaches include linear combinations, inner product properties for orthonormal bases, squared norms of inner products, and systems of equations. We also found some nuance in the ways that the students discussed change of basis. Students used phrases in which the object of focus is either a basis or a vector as they discussed the result of changing the basis. Of the students whose object of focus was a vector, some indicated that change of basis is a way of rewriting the vector, and some indicated that change of basis is a process of transforming the vector in a way that the post-change vector becomes a different vector than the original. Students whose object of focus was a basis referred to switching or changing the basis of a vector, rather than the changing the vector itself. Additionally, we examined the context of the phrase "in a basis" as it appeared in the students' dialogue. This is a common phrase used in many different contexts, sometimes imprecisely. We noticed that students talked about any of the following as being "in a basis": a vector, a procedure, a person, or the problem setting.

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Examining College Precalculus Teachers' Noticing of Mathematics Department Curriculum Materials

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This preliminary report will focus on how college precalculus teachers, mostly graduate teaching assistants, interact with department-provided curriculum materials. We specifically address what collegiate teachers notice in curriculum resources while planning. Comparisons will be drawn between first-time instructors and those with more experience, ultimately informing what and how collegiate teacher educators might incorporate experiences for precalculus teachers to develop curriculum use practices.

Keywords: Curriculum, Graduate Teaching Assistant, Precalculus, Teacher Training

Curriculum materials are at the core of lesson planning, influencing what teachers plan for and enact in their classrooms (Brown & Edelson, 2003) and ultimately influencing what learning opportunities are provided. However, we still know little about how this influence is exerted (Stein, Remillard, & Smith, 2007), and we know even less about undergraduate education.

Collegiate teaching practices are widely underrepresented in the literature (Speer et al., 2010) and specifically, we know very little about how collegiate teachers learn to use curriculum, and further how varying designs influence use. This preliminary report will focus on how college precalculus teachers, mostly GTAs, interact with department-provided curriculum materials, specifically what they attend to within a lesson, and inform what and how teacher educators might incorporate experiences for collegiate teachers to develop curriculum use practices.

The data was collected at a large doctoral-granting university, with half the teachers being first-time instructors of record. Further, the mathematics department, as part of an ongoing effort to support active learning in their classrooms, developed an Open Educational Resource (OER) to be put into use for the first time during the semester. This joined the other curriculum resources already provided - a course packet of worksheets to be used by students in class, online teacher lesson guides, and a set of online homework assignments.

The main research question addressed in this report is *how, and to what extent, do precalculus teachers interact with department-provided curriculum?* Narrowing the scope of this broad question, this study aims to address the following questions:

1. What do teachers *notice* while using department-provided curriculum materials to plan?
2. In what ways does the department-provided curriculum inform teachers' lesson plans?

We use the Curricular Noticing Framework (Dietiker et al., 2018) to describe this interaction. Curricular noticing is a set of skills "that enable teachers to recognize, make sense of, and strategically employ opportunities available within their curriculum materials" and is comprised of three interrelated skills: *Curricular Attending, Curricular Interpreting*, and *Curricular Responding*.

In this preliminary report, we specifically focus on how what is in the curriculum and the format of the curriculum influence teachers' attention when planning a lesson and how this informs the work of teacher trainers and educators. Further analysis of this data will link teacher interactions and use of curriculum to teacher training, pedagogical goals and beliefs, and student factors such as target audience and achievement gaps, thus providing a more complete picture of the role teacher noticing plays in precalculus courses.

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Studying the Relationship Between Students' Perception of the Mean and Their Understanding of Variance

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This poster explores how Introduction to Statistics students think about and compare the mean and variability of four datasets. They explored the datasets through various representations (e.g., balance beam, leveling off) and ranked them from most to least variance. When exploring the mean, the students found value in both the balance beam and leveling off representation, but preferred the balance beam for reasoning about variability. However, when reasoning about the variance using the balance beam representation, the students focused on the wrong properties and made faulty inferences. When reasoning using the leveling off representation, they focused on the correct properties and made sound inferences about the data.

Keywords: Statistics Education, Content Knowledge, Mean, Variance

A key idea in an Introduction to Statistics course is the mean as not only a measure of central tendency but as a measure of variability. Policy documents (e.g., The Gaise Report) stress the importance of students having multiple conceptions of the mean such as a measure of center and as a balance point. In this poster, we intend to explore how a student's conception of the mean influences their thinking about variation by having students view the mean through some of the popular representations and seeing what features of the representations are they attending to when trying to determine the mean and variance of a dataset. This study took place at a large four-year college in the southern part of the United States. The participants (n=7) are students who were taking an Introduction to Statistics course during the time of the interview, which took place at the end of the semester after the course ended. They engaged in an hour long video-taped task-based interview (Maher & Sigley, 2014) with two of the authors where they: (1) describe what they thought the mean and variance are, (2) identify the mean and variance of a series of histograms and then ordered the histograms in terms of least variance to most, (3) engage in a task that had them construct a distribution on a line using Unifix cubes and then move the cubes to show a distribution that would have more and less variance than the one they constructed, (4) use a program developed in Mathematica (White, Straughn, & Guyot, 2016) to dynamically explore different interpretations of the mean, (5) re-rank the initial histograms (from least variance to most) based on the different interpretations, and (6) select one interpretation they had the most trust in being correct. When viewing the mean as a balance point, the students preferred the approach, but focused on the mode and symmetry of the data around the mode when considering variability which led to faulty inferences. When focusing on the leveling off representation, the students focused on the distance of the data points from the mean, which led them to making correct inferences about variation.

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Developing a Reasoning Inventory for Measuring Physics Quantitative Literacy

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In an effort to improve the quality of citizen engagement in workplace, politics, and other domains in which quantitative reasoning plays an important role, Quantitative Literacy (QL) has become the focus of considerable research and development efforts in mathematics education. QL is characterized by sophisticated reasoning with elementary mathematics. In this project, we extend the notions of QL to include the physics domain and call it Physics Quantitative Literacy (PQL). We report on early stage development from a collaboration that focuses on reasoning inventory design and data analysis methodology for measuring the development of PQL across the introductory physics sequence. We have piloted a prototype assessment designed to measure students' PQL in introductory physics: Physics Inventory of Quantitative Literacy (PIQL). This prototype PIQL focuses on two components of PQL: proportional reasoning, and reasoning with signed quantities. We present preliminary results from approximately 1,000 undergraduate and 20 graduate students.

Keywords: Quantitative Literacy, Physics, Assessment, Psychometrics

The development of students' PQL is an important goal in many introductory physics courses, but previous research suggests that students often do not achieve robust learning gains (Brahmia, 2017). We aim to develop a valid and reliable reasoning inventory to measure students' PQL. We present preliminary results from an 18-item reasoning inventory focusing on two constructs as proxies for PQL in general: reasoning using ratios and proportions (Arons, 1983; Boudreax et al., 2015), and about signed quantities (Brahmia & Boudreaux, 2016; Brahmia & Boudreaux, 2017; Bajracharya et al., 2012; Hayes & Wittmann, 2010; Vlassis, 2004). Future iterations will include items involving co-variational reasoning (Carlson et al., 2010).

Data for our primary analyses are comprised of responses from 1,076 undergraduate introductory physics students. We use descriptive statistics and classical test theory (CTT) to analyze our results. Overall, scores are fairly normally distributed (small but negative values of both skewness and kurtosis, -0.3 and -0.2, respectively) with an average (mean, median, and mode) of 11 out of 18 correct, and a standard deviation of 3.0. The internal reliability is below the commonly accepted threshold for making measurements of individuals: Cronbach's 〈 = 0.67 < 0.80 (Doran, 1980). CTT results indicate that some questions may be too easy for our target population, with difficulty > 0.8. In addition, student performance on no single item strongly correlates with the overall score, i.e. CTT discrimination < 0.6 (Wiersma & Jurs, 1990). The test is a work in progress and will continue to be revised based on our analyses.

Results from graduate students show that one multiple-choice-multiple-response item about negative charge is particularly difficult: only 3/22 students answered completely correctly, compared to at least 18/22 for five other items. This highlights the interesting case of the sign of charge being used as a label for a type of charge, which is uncommon for scalar quantities.

Future work will involve interviewing students and faculty to validate the interpretations of inventory items (Adams & Wieman, 2010), as well as item development and refinement.

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Students' Use of Programming as a Problem Solving Strategy in Probability

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In this work we examined the effects of programming as a problem solving heuristic on students' mathematical work on tasks involving probability and expected value. Analysis of student performance on a unit of instruction that focused on students' competence in both programming and calculating probabilities and expected values revealed that students can see programming as a valid problem solving strategy and use it effectively.

Keywords: Computational thinking, Curriculum, Probability, Programming, Simulations

Computational thinking is a fairly recent development in the mathematics education research arena. Its origins are in computer science, but Wing (2006) states that it is a "fundamental skill for everyone, not just computer scientists" (p. 33). It includes practices with data, modeling and simulation, computational problem solving, and systems thinking (Weintrop et al., 2016). Drawing on this perspective, this study focused on the utility of programming as a problem solving strategy when examining probability tasks in one mathematics classroom. One broad question guided data collection and analysis: does the use of programming in addition to virtual/physical simulations and theory enhance student understanding of these topics?

Methodology

A unit of instruction was developed and implemented in a precalculus class consisting of 10 students. The curriculum utilized several strategies, including programming, to teach concepts related to probability and expected value. The unit, which consisted of five lessons, first introduced students to several programming concepts, including conditionals and loops. It then included simulations related to three specific games of chance. Upon completion of the unit, students were asked to evaluate the curriculum and what they seemingly gained from the unit.

Results

Students were engaged throughout the unit and showed persistence in solving tasks, which involved modeling games of chance. Post unit assessment results indicated that a majority of students became competent in writing basic programs to solve probability tasks. Students also identified the potential of programming as a complement to other strategies with which they had familiarity. Several students used the word "shortcut" when evaluating programming as a strategy. It appeared that they questioned the theoretical value of programming for solving problems, even if the method yielded the same result. Students used creative strategies when writing programs, some of which were unanticipated by the teacher/researcher.

Implications

Because the sample group for the curriculum was small and the scope of curriculum fairly limited, more research is needed to further expand students' perceptions of and facility with programming when solving problems. With increased interest in inclusion of computational reasoning skills in the curriculum more systemic research is valuable in defining effective ways to help students not only acquire programming skills but also realize its value for extended mathematical work.

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Knowledge Used in Teaching Undergraduate Courses: Insights from Current Literature on Knowledge for Teaching Across STEM Disciplines

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Research on Mathematical Knowledge for Teaching has helped the education community understand the complex, knowledge-related factors that shape instructors' practices and the learning opportunities they create for students. Much of this work has occurred in the context of K-12 teaching. Although expanding, research on knowledge for teaching undergraduate mathematics is not extensive. A similar situation exists in science education. To help support these research efforts and theory development, we analyzed existing literature on knowledge for teaching undergraduate STEM content. Findings take the form of cross-disciplinary themes and differences that can help inform research efforts in this area. We seek feedback from the RUME community about our representations of knowledge for teaching, ideas about findings from research on Mathematical Knowledge for Teaching that have been especially useful, and/or ideas for research investigations that would be particularly useful to inform curriculum development, professional development for teaching or theory.

Keywords: knowledge for teaching, novice college instructor professional development, STEM disciplines

Evidence-based instructional strategies can improve outcomes for all students and the retention of students from underrepresented groups in undergraduate STEM degrees (Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, 2014; Laursen, Hassi, Kogan, & Weston, 2014). As a result of this potential, there have been repeated high-profile calls for substantial reform in teaching practices in undergraduate STEM. Achieving widespread adoption and effective use of evidence-based teaching strategies demands attention to the role of college instructors, including what instructors know and are able to do as evidence-based teachers. Although work in this area has increased in recent years, undergraduate mathematics instructors' knowledge and teaching practices have not been extensively researched (Speer, Smith III, & Horvath, 2010). Examining the role of teaching knowledge in evidence-based instruction and how to support its development is crucial to progress in reforming undergraduate instruction.

Although also not extensive, research also exists on undergraduate instructor knowledge and practices in science disciplines. In an effort to encourage and support additional research in this area, an inter-disciplinary team of researchers has conducted a review and analysis of literature about studies of knowledge for teaching across STEM disciplines. In this report, we share findings from our review and highlight key challenges and opportunities in this research area. We discuss how major categories of knowledge for teaching that appear in multiple disciplines (e.g., pedagogical content knowledge) are defined in those different disciplines. We also discuss research on types of knowledge that can apply across disciplines (e.g., pedagogical knowledge) and knowledge types that currently appear only in descriptions of knowledge used to teach mathematics (e.g., specialized content knowledge, horizon content knowledge).

We seek feedback on our representations of the knowledge for teaching across STEM content areas and on our suggestions for next steps to advance research on undergraduate instructors' knowledge and practices.

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Developing Freshmen Math without Developmental Math

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Abstract: Most universities and community colleges are struggling with how to prepare incoming students for the rigor of college-level mathematics courses. At Governors State University, developmental courses are not offered, although a significant number of students do not have the required prerequisite mathematics knowledge for college-level courses. This poster has three themes: an analysis of institutional data on freshmen mathematics, a discussion about navigating conflicting goals and ideas from university leadership, and an examination of mathematics interventions, both those that were tried and recommendations for next steps.

Keywords: developmental mathematics, remedial mathematics, freshmen

At Governors State University (GSU), the Board of Trustees mandated that they would not offer or require any developmental mathematics courses, and that the only courses offered were at college level for college credit. At the same time, the university supports a diverse body of non-traditional students, many of whom do not have the prerequisite knowledge to succeed in a college-level mathematics course. After four years, and given the body of research on success rates for students who may need developmental mathematics, it is not surprising that most of these freshmen were not successful in their first-semester mathematics courses (Bailey, 2009).

Instead of required remedial instruction, the university offers a variety of resources for students, including the Academic Resource Center, mathematics success workshops, Supplemental Instruction, Smart Start Mathematics, and other initiatives. What administrators did not do is examine the possible reasons why many freshmen do not succeed in mathematics. Without this critical information, any interventions produced to remedy high rates of D/F/W grades are not likely to be effective (Ashby & Sadera, 2011, Sadler & Sonnert, 2016).

The university now has four years of data that can be mapped to local-area high school curriculum, the placement exam, research on strategies in higher education and mathematics, and possible intervention techniques; both ones that have been tried and others that haven't. For this poster, I will summarize the findings based on data from four years of enrolling freshmen at GSU. Each intervention implemented is correlated with success rates. A review of research and an exploration of interventions offered by other universities will help point to more targeted solutions that meet each student's individual needs (Melguizo, Kosiewicz, Prather, & Bos, 2014, Jaggers & Stacey, 2014, Woodard, 2004).

A second theme for this poster is navigating between colleges, administration, and faculty who all have different ideas on how to approach freshmen mathematics and the specific needs of our students, whether they are accurately understood or not. For example, many university mathematics professors may lack the pedagogical skills required by a high school teacher, and may be resistant to change, particularly if it seems more work will be required (Brownell & Tanner, 2017). Qualitative data is also being collected in order to better understand reasons why a student may not be successful in college-level mathematics.

The final theme of the poster details the changes recommended for this university, based on current data analysis and research. It is the author's expectation to implement some of these changes beginning with the Spring 2019 semester, and so by the next RUME meeting, a full paper with results will be forthcoming.

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Development of Equity Concepts During Professional Learning About Teaching

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Using a stakeholder-centered design, this interactive poster presents a research framework for attending to equity and supporting transformative change for faculty who teach courses for future K-8 teachers. The poster reports on a research and development project that is creating and examining the impact of professional learning modules for these faculty. One aim of the project is intentional awareness development among faculty about their own views of mathematics and opportunities to learn it, those of their undergraduate students, and those of the children their students will one day teach. The particular focus of the poster is our research attempt to identify and capture aspects of equity that factor into instructor decisions in each phase of their professional learning experience (motivation, construction, and organization).

Keywords: Equity, Faculty Professional Development

The NCSM and TODOS position paper, *Mathematics Education Through the Lens of Social Justice: Acknowledgement, Actions, and Accountability* emphasized mathematics education should include "fair and equitable teaching practices." In response, Hauk and D'Silva (2018) proposed a process for attending to equity in college mathematics education research and development through purposeful, stakeholder-centered design (Figure 1). Using such a design, we are creating and investigating a mini-course for faculty who teach future K-8 teachers. It is made up of a series of online learning modules. Modules include explicit attention to the importance of cross-cultural interaction in K-8 and collegiate teaching and learning. Across the mini-course, faculty engage in two cycles, each activating a set of three phases of professional work: motivation, construction, and organization (Kubitskey et al., 2014). In motivation, faculty identify their own problem of practice. In construction, they participate in opportunities to learn (e.g., a within-module activity) and prepare for change in practice. In organization (between-module activity), faculty rely on the knowledge constructed and activate its use in the classroom, with emphasis on monitoring progress (to support reflection and identifying subsequent instructional aims). The poster presents the conceptual framework for attending to equity, using the cycle in Figure 1, for the development and implementation of the modules. Given the model in Figure 1, the visuals about current project research efforts, and goal of equitable teaching practice, the **question driving poster conversation is:** How do we improve the documentation, research, and feedback on equity in faculty decisions for each phase of professional learning?

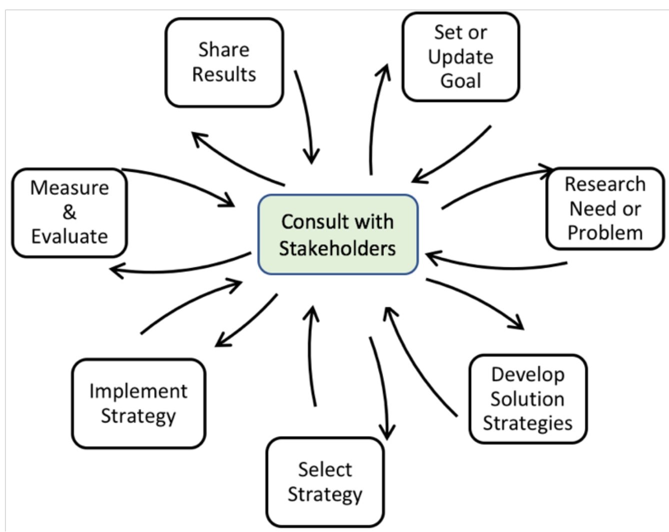

Figure 1. Stakeholder-centered design cycle.

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Using Identity to Frame Mathematics Educational Learning Experiences of Historically Marginalized Students

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The goal of this study was to illustrate how notions of identity could be used as an analytical tool to account for such diverse perspectives along with issues of power in the context of Latinx and Native American students. Interviews and classroom observations revealed an array of perspectives regarding what counts as mathematics within a classroom, yet is reflective of an ongoing assimilationist practices that have negatively impacted Indigenous peoples for centuries. I argue for the need for mathematics educators to identify dehumanizing practices in mathematics by seeking the perspective of Indigenous educators.

Key words: Diversity, equity, identity, sociocultural, sociopolitical, Indigenous, Latinx

The underrepresentation of students in historically marginalized groups are often illustrated and framed in the research as an achievement gap in STEM between students from these underrepresented groups and students from white backgrounds (Gutiérrez, 2008). However, research centered on closing such achievement gaps relies on narrow notions of learning and equity (Gutiérrerz & Dixon-Román, 2010). I seek to answer the research question: to address the research question: How can we use identity to better understand the various forces impacting the mathematical learning experiences of Native American and Latinx students? This study utilizes the notions of *normative identity* and *personal identity* (Cobb, Gresalfi, & Hodge, 2009) as well as Martin's (2000) multilevel framework that seeks to describe the interaction of influences from both inside and outside the classroom on students' mathematical learning experiences by considering the agencies made available to and exercised by students, school level forces, community and family, and sociohistorical influences.

Semi-structured interviews and classroom were collected from Native American and Latinx students and their classroom, their mathematics teacher, the parent of one of the students, and an assistant principal in order to account for the multiple influences on a student's mathematical learning experiences. Analysis of the interviews from the teacher, parent, and assistant principal reflected strong influences from top level government and educational policies and a historical disconnect between the perspectives of students' communities and their schools.

I draw upon the findings of this study to argue for greater integration of perspectives from Indigenous education researchers. In particular, I argue for the need for frameworks in research in undergraduate mathematics education that incorporate notions of identity while accounting for multilevel sociopolitical and sociohistorical forces on mathematical learning experiences to better describe the damage of assimilationist practices in higher education.

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Computational Thinking Mediating Connections Among Representations in Counting

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Abstract. There is increased focus on exploring the role of computation in students' learning of mathematical concepts, and the notion of computational thinking has gained prominence. In this poster, we demonstrate ways in which students make connections among different combinatorial representations, and we argue that computational thinking mediated such connections.

Keywords: Computational Thinking, Combinatorics, Mathematical Representations

The role of computation in students' learning of mathematical concepts has received increased focus in recent decades. As such, the notion of computational thinking (CT) has gained prominence among computer scientists and STEM educators. We define CT as a way of thinking that one uses to formulate a problem in such a way that a computer could effectively carry it out (Wing, 2014). In this study, we interviewed two pairs of undergraduate students, who were novice counters and relatively novice programmers. Each pair was interviewed for 15 total hours, during which they sat together at a computer and used basic Python coding to solve counting problems. In this poster, we share one aspect of this project that focuses on the role of CT in helping students connect mathematical representations as they solve counting problems. The ability to make connections between mathematical representations is highly valued by mathematics educators as a means for students to make sense of mathematics and deepen their understanding (Pape & Tchoshanov, 2001; Stein, Engle, Smith, & Hughes, 2008). Counting problems can be difficult for students to solve (Batanero, Navarro-Pelayo, & Godino, 1997) and facilitating connections between representations may improve student understanding of such problems. We present findings that answer the following research question: *How does CT help students make connections among multiple representations in counting problems?*

We identified five mathematical representations that arose in our interviews: i) computer codes, ii) outputs, iii) lists, iv) tree diagrams, and v) expressions. In the poster, we demonstrate ways in which students make connections among combinatorial representations and discuss how CT mediates these connections. For example, one pair answered the question "*Write a program to list all possible outcomes of flipping a coin 7 times.*" They related output of code (Fig. 1 shows partial output) and a tree diagram (Fig. 2), saying, "So, this [the tree diagram] is what we were talking about with the third column [of the output] as like it's splitting into four." In the poster we offer additional examples and explore why we think CT afforded such connections.

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tails tails heads tails tails tails tails
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tails heads tails tails tails tails tails
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128
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Figure 1: Partial Output

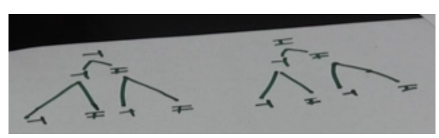

Figure 2: Tree Diagram

In our observations of students working with combinatorial problems, we are beginning to notice not only a unidirectional affordance between CT and mathematics, but a bidirectional effect where computational and mathematical knowledge are co-constructed. Our findings could have broad implications in terms of influencing practices that instructors employ when teaching combinatorial problems in particular, and possibly other mathematics problems in general.

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Exploring Co-Generative Dialogues with Undergraduates to Improve Teacher Feedback Practices in a Probability and Statistics Class

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Providing students with explanation feedback on their homework has been shown to highly and effectively support their learning. However, there is limited research on how students might play an active role, in collaboration with teachers, to tailor feedback, to meet students' needs. This study, carried out as a form of practitioner-inquiry in a teacher's own classroom, explores how a teacher used their own and their students' shared experiences to refine and develop explanation feedback that supports students' mathematics learning in an undergraduate Probability and Statistics class. Data include records of co-generative dialogues between the teacher and her ten students, and the students' homework worksheets. Emergent findings show that co-generative dialogues provide an effective opportunity for teachers to learn from their students how to improve their pedagogical practices, especially in providing effective explanation feedback.

Keywords: Probability & Statistics, Homework, Explanation Feedback, Co-generative Dialogues

There are many potential positive consequences of providing students with feedback on their homework (Fyfe, 2016; Landers & Reinholz, 2015). Explanation feedback is conceptualized as information provided by a teacher regarding students' performance, with details on how to improve. Inviting student voice into the nature of explanation feedback makes it possible for students to agree with teachers on how their work was judged, then use those standards in producing new work (Landers & Reinholz, 2015). Consistent with this finding, this study was guided by a reality pedagogy framework, wherein teaching is guided by a teacher's developing understanding of students' experiences (Emdin, 2011). The aim of this study was to investigate how student voice can be leveraged to refine instructional practices of giving effective explanation feedback on students' homework. We hypothesized that when students have more agency in determining the nature of explanation feedback on their homework, teachers may improve feedback practices, and ultimately support students' learning.

We carried out a practitioner-inquiry study (Samaras & Freese, 2009) in the first author's freshmen Probability and statistics class at a university in the Northeast of the United States. We enacted co-generative dialogues, defined as a form of structured discourse, where teachers and students engage collaboratively to identify and implement positive changes in a classroom teaching and learning (Martin, 2006). Data were collected through 15 minutes of co-generative dialogues after class, with all students, once a week, for each of the six weeks of study. The dialogues were active and reflective, as Wambua and her students discussed the explanation feedback she had provided in the previous assignment, then agreed on ways of refining future feedback. The data were recorded via teacher fieldnotes and students' notes. Students' homework worksheets were also analyzed to note how students implemented the feedback.

Findings indicate that the type of explanation feedback preferred by students vary based on their self-identified needs. For example, while some requested for specific details on why they were wrong, others needed details for correct solutions. Working with students, the teacher learned how to give specific yet probing feedback. This implies that, working closely with their students better positions teachers to provide feedback that support individualized student needs.

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Pre-service teachers' concept image of *numeral*

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David Tall's Three Worlds of Mathematics provides a framework through which to study the development of mathematical thought, the three worlds being the embodied, the symbolic, and the formal. This poster shares initial investigations and findings into how the concept of numeral is situated in the symbolic world, beginning with developing a concept image for numeral that is held by pre-service elementary teachers. An analysis of eight mathematics textbooks for pre-service teachers provided an initial definition. Additionally, students in a college mathematics content course for pre-service elementary teachers completed a survey. The participants in the survey were given several symbols and prompted to respond whether or not they believed the symbol represented a numeral. Results of the textbook analysis and the survey were used to understand the students' concept image of numeral.

Keywords: Pre-service teachers, Numerals, Three Worlds of Mathematics

This study represents the initial stage of a planned study to better understand the development of the concept of *numeral*. David Tall's Three Worlds of Mathematics framework (Tall, 2013) suggests that embodied actions or rote memorization leads to procedures with symbols. The symbols that are used represent both a procedure and an object. This dual nature occurs as a result of a compression of the concept from an action to an object. While the ultimate goal of the project is to understand how people apprehend the dual nature and the compression process of a numeral, this current study attempts to develop a concept image of numeral held by pre-service elementary teachers (PSETs).

Students in two sections of a capstone mathematics content course for PSETs were participants in the study. In order to establish a baseline for their understanding of numeral, a textual analysis of eight textbooks commonly used in mathematics content courses for PSETs examined the book's stated definition for *numeral*. One textbook did not provide a definition. Of the seven other books, six defined a numeral as a symbol for a number with the seventh calling a numeral a name for a number. Instruction in the course regarding numerals was in line with the common textbook definition of numeral as a symbol for a number.

Near the end of the course, sixty-four students completed an online survey that probed their beliefs about numerals. Of the participants, fifty-five provided complete responses to the prompt, "State your definition of numeral." Twenty-seven participants stated that a numeral was a symbol for or a representation of a number, twenty-one participants conflated numeral with number, and seven participants gave a mixed definition. The results indicated that despite numerals being defined and treated as numerals, less than 50% viewed a numeral explicitly as a symbol.

Participants were then shown a series of symbols and asked to indicate whether or not they considered the symbol to be a numeral. For example, 95.3% of participants believed 512.37 to represent a numeral, while 70.3% believed 22/7 represented a numeral. Moreover, participants who defined numerals as symbols identified 70.3% of the symbols as numerals, while participants who defined numerals as numbers identified 62.4% of the symbols as numerals. While analysis of the data is ongoing, the preliminary analysis suggests that students may not fully apprehend the dual nature of a numeral.

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Exploring Preservice Teachers' Views of Students' Mathematics Capabilities Within Mediated Field Experiences

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This research explores preservice teachers (PSTs) views of students' mathematical capabilities (VSMC) within mediated field experiences (MFEs) and the role of beliefs on instructional decisions. In MFEs, teacher educators serve as instructors, coaches and supervisors as PSTs plan, enact, and debrief instruction (McDonald et al., 2014). The research questions were: What is the nature of PSTs VSMC across an MFE cycle? How might PSTs beliefs impact instructional decisions? What role might MFEs play in developing productive VSMC? Findings showed that some teachers believed students were incapable of engaging in rigorous instruction, and consequently, would not always respond to student difficulty in ways that helps students participate in rigorous mathematical environments. Results suggest the need to study how teachers might develop more productive VSMC and better support students who struggle. The analysis also revealed how daily debriefs within MFEs supported PSTs to glean general instruction principles to inform their teaching.

Keywords: Preservice Teacher Education, Teachers' Beliefs, Field Experiences

Studies show teachers' beliefs, specifically, how teachers *frame* student difficulty in mathematics, will determine the type of support teachers give to students, and therefore, ultimately play a role in how they will support students who struggle in mathematics. (Jackson, Gibbons, & Sharpe, 2017). Therefore, it is important to explore ways to support preservice teachers (PSTs) to develop productive view of students' mathematical capabilities (VSMC) within teacher education programs. In the realm of PST education, teacher educators have used mediated field experiences (MFEs), or methods courses held on university campuses and at K-12 schools, as a context for supporting PST learning (Campbell & Dunleavy, 2016). Daily co-planning sessions and lesson debrief discussions are rich sites to discuss critical moments and reframe student difficulties in terms of supports rather than lowered expectations.

The research questions for this study were: What is the nature of PSTs VSMC across an MFE cycle? How might PSTs instructional decisions in classroom enactments relate to their beliefs about students' capabilities to engage in high-quality mathematics instruction? What role might MFEs play in supporting PSTs to develop productive VSMC? This qualitative study involved interviewing, surveying and analyzing the written work of seven PSTs enrolled in an elementary mathematics methods course embedded within an MFE. To explore the nature of PSTs' VSMC throughout the study, I employed the analytic framework, *Views of Students' Mathematical Capabilities* (Jackson, Gibbons & Sharpe, 2017), to analyze whether PSTs framed student difficulties from an asset or deficit perspective (their *diagnostic framing*), and to categorize the nature of supports they feel are appropriate for students who struggle in mathematics (*prognostic framing*). Findings showed that some teachers believed students were incapable of engaging in rigorous instruction, and consequently, would not always respond to student difficulty in ways that help students participate in rigorous mathematical environments. Results suggest the need to study how PSTs might develop more productive VSMC and better support students who struggle. Our analysis also revealed how daily debriefs within MFEs supported PSTs to glean general instruction principles to inform their teaching.

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Students’ Interpretations of Animations Supporting Dynamic Imagery

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The study of calculus focuses on change and thus dictates a need for instruction involving dynamic imagery. DIRACC (Developing and Investigating a Rigorous Approach to Conceptual Calculus) utilizes animations to support students’ dynamic imagery. This poster investigates how students use and understand animations in the DIRACC textbook in connection with associated calculus topics.

Keywords: Calculus, Technology, Animations, Variables

Calculus can essentially be explained as “*using how fast a quantity is varying at every moment to find how much of that quantity there is at every moment, and vice versa*”. Accordingly, it is important that students conceptualize variables as truly varying (dynamically). In a calculus course developed by Thompson and Ashbrook emanating from an NSF-funded study called DIRACC (Developing and Investigating a Rigorous Approach to Conceptual Calculus), the curriculum takes the position that dynamic imagery of smoothly varying quantities is important for learning calculus ideas (Thompson, P. W., Byerley, C., and Hatfield, N., 2013; Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., and Hsu, E., 2002). DIRACC utilizes animations as didactic objects (Thompson, 2002) to support dynamic imagery in an online textbook. This poster investigates how students interpret the animations and discusses how the animations were refined as a result.

This study arose from a survey given to DIRACC students regarding their usage of these online animations where a majority of students indicated difficulties understanding them clearly upon their first watch. This led to our team to investigate the following: “How do students interpret animations in the online textbook?”, “In what ways do animations assist students in developing dynamic imagery of varying quantities?”, “How do the animations help students construct the meanings we intend, and how do they not help?”.

In this study, four Calculus II DIRACC students were clinically interviewed (Clement, 2000), asked to watch three different animations and respond to a set of questions related to their interpretations of the animations. These interviews provided data on how each animation was conveying information, whether intended or unintended, as well as implications on how animations can be used and improved. Across all three animations, students initially responded with a general idea but had trouble articulating details. It was through pausing animations and prompting students with probing questions that elicited answers where students reflected on their understandings. This suggests that animations should deliberately make students pause and reflect on what they saw, and teachers should be trained to use animations as didactic objects in class, including pausing and questioning of students as part of class discussion. Additionally, interviews indicated that students interpreted one of the animations in an unintended way, resulting in a refinement of the animation within the textbook. This refinement demonstrates the iterative aspect of research-based curriculum design in the context of DIRACC.

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A Student's Meanings for the Derivative at a Point

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Abstract: The purpose of this study is to examine the meanings and interpretations a student has about the derivative at a point. The responses given by the student is representative of many Calculus 1 students and their beliefs about derivative.

Key Words: Derivative at a Point, Calculus, Student Thinking

This poster discusses one particular student's reasoning about the following task:

Task 4 – The Approximation Derivative Problem
Given that *P(t)* represents the weight (in ounces) of a fish when it is *t* months old,
a. Interpret the statement $P'(3) = 6$
b. If $P(3) = 15$ (and $P'(3) = 6$) estimate the value of $P(3.05)$ and say what this value represents.

Figure 1: Task 4. Interpreting Derivative at a Point

The purpose of this study is to build models of students' mathematics, termed the mathematics of students (Steffe & Thompson, 2000). In this study I attempt to build a model of a student's understandings of the derivative at a point and the factors that might have contributed to the student's responses. The study of derivatives is fundamentally about change and thus dynamic situations, yet as evidenced by Zandieh (2006) students tend to recall the finished static product and not the dynamics involved. The research questions this poster endeavors to address is "What images do students have of derivative at a point? Is it dynamic or static?".

Due to students' wide range of beliefs about functions (Szydlik, 2000) and students' tendency to recall a finished product Zandieh (2006), I theorize that students' will not consider the derivative at a point as concerning a small interval, but rather a point. This notion is reminiscent of Harel and Kaput's (1991) discussion of pointwise versus uniform operators and bolsters this idea on student thinking about functions. Students are often introduced to function as a correspondence (Sfard, 1992) and see one input being mapped to one output. It should be natural then that as this notion is rarely challenged, student's conception of function as a mathematical object (Thompson & Sfard, 1994) has the property of only being concerning with a singular input value. Despite dynamic teachings of the derivative that involve secant lines converging towards a tangent line, Zandieh (2006) notes that students forget the dynamic motion and recall the finished product of the tangent line. This informs the possibility that students may interpret a statement such as $P'(3) = 6$ as one that is focused solely on one point.

This poster presents a study of one student's meaning for the derivative at a point in a quantitative context. I take the constructivist approach (Glasersfeld, 1995) espousing that it is impossible to know completely a student's knowledge and hence the goal is to model the student's beliefs. In this study, the student's responses indicate a consistency with Zandieh's (2006) assertion that students recall a finished product. His responses either noted completed change, or anticipation of change to come both of which lacked dynamic imagery for the point involved. This student's responses indicate a need for teaching of the derivative to flesh out meanings of the derivative at a point so that students might construct a productive meaning for it.

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Developmental Mathematics Reform: Analyzing Experiences with Corequisite College Algebra at an Urban Community College

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Roughly half of the nearly 44% community college students referred to developmental mathematics never make it into, let alone through, the college-level mathematics courses required for their academic major. The disproportionate number of these students who are from underrepresented groups combined with the low success rates has prompted many community colleges to undertake developmental mathematics reform. The purpose of this study is to provide a multi-perspective account of one community college's program redesign using Gutiérrez's equity framework, Tinto's model of persistence, and activity theory to analyze and interpret the experiences of students, instructors, and administrators.

Key Words: Developmental Mathematics, Corequisite Support, Activity Theory, Equity

Through an institutional equity initiative in 2013, Lowry Community College (LCC, a pseudonym) discovered that their African-American students had a much lower rate of degree completion or transfer within three years of first enrollment than all other groups at the college. Furthermore, African-Americans were disproportionately represented in developmental mathematics, not succeeding in developmental mathematics, and underrepresented in college algebra. The initiative not only led to a targeted goal of increased student success for African-American students but brought to light the struggles of many of LCC's developmental mathematics students similar to the published research (e.g., Bailey, Smith Jaggars, & Jenkins, 2015). The equity initiative provided the impetus for LCC's developmental mathematics reform.

Following the recommendations of Bailey et al. (2015) and Complete College America, LCC implemented a newly-designed college algebra course incorporating a corequisite model of support to replace the previous sequence of developmental mathematics courses leading to calculus. In this model, developmental mathematics content is taught in service to the college algebra content, first during an intensive 5-week on-boarding class at the beginning of the semester, and then as a separate, mandatory corequisite course that meets either directly before or directly after the college algebra class during the remaining 10 weeks of the semester. The course was first implemented in the Fall 2018 semester.

This study adopts the theoretical perspectives of activity theory (Engeström, 1987), Tinto's (1975, 2006) model of persistence, and Gutiérrez's (2009) equity framework to provide an account of the experiences resulting from the course redesign. In activity theory, individual and collective experiences can be characterized through goal-directed activity systems, allowing the interpretation of interactions between and within systems (Engeström, 1987). Students, instructors, and administrators form distinct activity systems with components informed by the constructs of equity and persistence, which also indicate contradictions as drivers of change within and between systems.

Preliminary results of the initial data analysis will be discussed.

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