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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-third annual Conference on Research in Undergraduate Mathematics Education in Boston, Massachusetts from February 27 - February 29, 2020.

The program included plenary addresses by Dr. Gregory Larnell, Dr. Eric Knuth, and Dr. Elise Lockwood, and the presentation of 148 contributed, preliminary, and theoretical research reports and 74 posters.

The conference was organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in the field of undergraduate mathematics education, each of which underwent a rigorous review by two or more reviewers:

- Contributed Research Reports describe completed research studies

- Preliminary Research Reports describe ongoing research projects in early stages of analysis

- Theoretical Research Reports describe new theoretical perspectives for research

- Posters are 1-page summaries of work that was presented in poster format

The conference was hosted by the Wheelock College at Boston University.

Many members of the RUME community volunteered to review submissions before the conference and during the review of the conference papers. We sincerely appreciate all of their hard work by the 165 reviewers.

We wish to acknowledge the conference program committee for their substantial contributions to RUME and our institutions. Without their support, the conference would not exist.

Finally, we wish to express our deep appreciation for Wheelock College and Boston University. Their support enabled us to have our conference and continually support our community.

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Undergraduate Students' Perspectives on What Makes Problem Contexts Engaging

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Using engaging problem contexts is important in instruction, and the literature contains themes of contexts being realistic, worthwhile, or enjoyable, as well as motivating. Yet, the literature largely lacks detailed student perspectives on what helps problem contexts achieve these characteristics. In this study, eleven calculus students were interviewed to identify features of problems that made them engaging. This led to a new top-level characteristic "variety," and the identification of features that helped contexts have the characteristics described in the literature. In particular, problems that were realistic/motivating contained features including: (a) expansion of awareness, (b) need for math, and/or (c) explicit purpose. Contexts that were enjoyable/motivating contained features including: (a) insertion into problem, (b) teacher's personal story, or (c) absurd story. At the end, we show the usefulness of these results by critiquing problems from the literature in terms of how engaging they might be to students.

Keywords: problem context, student engagement, realistic, enjoyable, variety

Two major reasons given for why students may develop negative feelings towards mathematics are that students cannot see a relationship between classroom mathematics and real life (Boaler, 2015; Boaler & Selling, 2017), or that students do not see mathematics as interesting or enjoyable (Brown, Brown, & Bibby, 2008; Nardi & Steward, 2003). The usage of more engaging problem contexts to introduce and develop mathematical content can help address these issues (Boaler, 2015; Crespo & Sinclair, 2008; van den Heuvel-Panhuizen, 2003). However, the literature generally lacks detailed accounts of students' perspectives on the matter, making it more difficult to ascertain if certain problem contexts actually *are* engaging to students, and what aspects of those contexts helped them be engaging (or not). This study seeks to build on the current literature through the question: What do students perceive as making problem contexts more engaging? We then show this study's potential usefulness by critiquing example problem contexts in the literature in terms of how engaging they might be to students.

Before continuing, however, we need to make a clarification between two closely related aspects of mathematical problems. First, the initial *problem context* is that used to enter students into an exploration. For example, an initial problem context might be, "Given a fixed amount of fencing, what dimensions of a rectangular field maximize the area inside?" The context is about fencing and interior area. Next, within this initial context, there may arise a "problematic situation… unsolvable by [the student's] current knowledge" (Harel, 2013, p. 122). In the example, it might be the need for justifying when certain dimensions actually *do* maximize the area. This problematic situation leads to the mathematics to be learned, in terms of techniques for identifying and guaranteeing extrema. Note that in this study, we are focused *only* on the "initial problem context" in terms of attempting to engage students' interest. We are *not* focused on the "problematic mathematical situation" that might be encountered *within* that context.

Literature Review and Framework on Engaging Context Characteristics

To begin, we reviewed the literature to identify characteristics of problem contexts associated with student engagement. First, much research, including research in Realistic Mathematics Education and in mathematical modeling, discusses *realistic* contexts (e.g., Blum, 1993; Boaler, 2015; Gravemeijer & Doorman, 1999; Kümmerer, 2001). However, Boaler (2015), cautioned that many so-called realistic contexts are "pseudocontexts," which van den Heuvel-Panhuizen and Drijvers (2014) would say is because they are not "experientially real" to students. Thus, we define a *realistic* context as one rooted in the real world containing a problem that a student could see somebody (possibly themselves) having an actual desire to solve.

As a subset of *realistic*, while a student could see someone wanting to solve a real-world problem, there is a difference in whether that problem is seen as important or worthwhile (see National Council of Teachers of Mathematics, 2000). Thus, as another characteristic, we define a *worthwhile* context as one involving a question whose answer is seen by the student as having the potential to make an impact, whether in the narrow world of the student or in the world at large. Note that a context could be realistic without necessarily being worthwhile.

Next, in contrast to the first two characteristics, some studies have mentioned "fun" or "enjoyable" as a characteristic of engaging problem contexts (Brown et al., 2008; Nardi & Steward, 2003; Schukajlow & Krug, 2014). Puzzles, games, or fun situations could potentially create engagement for students (Kebritchi, Hirumi, & Bai, 2010; Kubinova, Novotna, & Littler, 1998). However, definitions for "enjoyable" tend to be vague in the literature, so for our working definition we chose to use the broad idea of personal feelings. That is, we defined a context as *enjoyable* if it provokes a positive emotional response from the student.

Overall, the literature speaks of two main "engaging" characteristics: realistic or enjoyable (with worthwhile as a subset of realistic). We based ourselves in the literature by adopting these two characteristics as those that can make a problem engaging. Of course, we left ourselves open to the identification of additional characteristics in this study. However, within these two main characteristics, it appears that a problem that is technically "realistic" or "enjoyable" still may not been seen as *motivating* to students (see Boaler, 2015; Schukajlow & Krug, 2014). Thus, within these two characteristics, we define a secondary characteristic, *motivating*, as a context that stimulates a student's interest to continue to investigate the initial problem. In this way, while an engaging problem is either realistic or enjoyable, we claim that it must also be *motivating* to truly be engaging. We then define an *engaging problem context* as one that resides in the realistic/motivating intersection, or the enjoyable/motivating intersection, or both.

In summary, past research suggests two broad types of contexts: *realistic* or *enjoyable* (or the intersection of both). Either type must be *motivating* to be engaging. A realistic context does not necessarily need to be worthwhile to be engaging, but being worthwhile might strengthen how engaging it is. Figure 1 shows the interrelationships of these four characteristics, where an engaging context resides in the intersection of realistic/motivating, enjoyable/motivating or both.

Figure 1. Engaging contexts live in the intersection of realistic/motivating, or enjoyable/motivating, or both

Methods

For our study, we designed a two-part interview that we conducted with eleven first-semester calculus students. We chose to involve students in their early undergraduate career because: (a) they have a lot of exposure to problem contexts through their years of schooling, and (b) as older students they might better reflect on and articulate their thoughts about those contexts. In the first part of the interview, the students were shown six problems and were asked to open-endedly explain which ones they found more engaging than others. For this part, the students were not aware of the characteristics in our framework, so that their reasons were more likely to be genuinely their own. To select problems for this part, we narrowed to a specific topic to keep the mathematics generally the same. We chose to use "optimization" problems because they are seen in different mathematics courses meaning it is not *too* narrow in scope, and because optimization lends itself well to the study of context due to the variety of contexts involving maximizing or minimizing. We chose three problems shared among commonly used calculus textbooks (e.g., Stewart, 2015; Thomas, Weir, & Hass, 2014). Then, to help students be specific about what might make certain problems engaging or not, we created three problems with essentially identical mathematical solutions to the three textbook problems but with different contexts. Abbreviated versions of these problems are shown in Table 1.

Table 1. Textbook problem contexts and mathematically equivalent problems with different contexts.

Abbreviated textbook problem	Abbreviated equivalent problem with different context
A farmer has 2,400 feet of fencing and he wants to make a rectangular field that borders a straight river… What are the dimensions of the field that will give it the greatest area?	You're adding on to your home, and you're going to knock out a wall in the back to add an open room … You have enough material for 60 feet of new wall… What dimensions would give the greatest area for this new room?
You are going to launch a boat from point A on the bank of a straight river. You want to reach point B which is downstream on the opposite bank of the river… Where should you land on the opposite bank in order to reach point B as soon as possible.	You are working with a humanitarian aid group to bring safe water supply to villages. One village is located at the edge of a mountain range and its nearest water supply is up in the mountains… What path should the pipe follow in order to minimize the cost of laying the pipe?
Consider the graph of $y^2 = 2x$ and the point (1,4). Every point on the parabola is a certain distance from the point (1,4). Find the point on the parabola that is closest to the point (1,4).	You're working for NASA and a comet is making a close pass by Earth… The comet's path is modeled by the graph of $y = \sqrt{x+5}$ [Earth at origin]. To tell the public which day would be the best day to view the comet… find the point along the graph that is closest to the origin.

In the second part of the interview the students were shown the four characteristics (realistic, worthwhile, enjoyable, motivating) one at a time and a description of each characteristic was given. The students were asked to think of an example problem context they had seen that they felt exemplified that characteristic, and to discuss why. They were also asked to discuss problems they felt *failed* to have that characteristic. While worthwhile is a subset of realistic, and motivating exists only in addition to realistic or enjoyable, we still asked the students about them separately simply to permit additional opportunities for the students to discuss features that might make problem contexts more engaging. While the first part of the interview was somewhat limited due to using a specific topic area, the second part of the interview allowed the students to pull problems they had seen from a range of different classes and different topics, broadening the data set greatly. To conclude the second part of the interview, the students were also asked to identify and discuss what they considered to be the most engaging mathematics problem, overall, that they had experienced in a class.

The analysis consisted of three phases, with the "reasons" given by the students as the unit of analysis. First, we coded the students' reasons in the first part of the interview as *realistic*, *worthwhile*, and/or *enjoyable* according to their definitions, and we also used the category *other* to capture possible reasons beyond our framework. We also used the antonym codes *not realistic*, *not worthwhile*, and *not enjoyable* if a reason was given for why a context failed to have that characteristic. Each reason could additionally be coded as *motivating* if the explanations surrounding that reason suggested that the student's interest was activated because of that reason. Or, it could be coded as *not motivating* if that reason appeared to be one that turned the student off to that context. In the second phase, we looked at the instances within "other" to see if there were any top-level characteristics that the data suggested should be added to the framework (one such emerged). In the third phase, we took all student reasons from both parts of the interview that were coded as (a) realistic and motivating (or not realistic and not motivating), as well as (b) enjoyable and motivating (or not enjoyable and not motivating), to look for themes suggesting features of contexts that helped them live in the "engaging" intersections (or not). This led to the identification of three features of contexts that helped them be realistic/motivating and three features of contexts that helped them be enjoyable/motivating.

Results

The analysis of the data revealed one new main characteristic in addition to the four in our framework, as well as several problem features that helped contexts be realistic/motivating or enjoyable/motivating. Note that due to space constraints, we do not examine the first and second part of the interview separately in this report, but rather describe the resulting themes in aggregate from the first and second parts of the interview all together.

Identification of a New Characteristic to Add to the Framework: Variety

During the analysis, a strong theme emerged in the "other" category that did not fit under any of the characteristics in our framework. To introduce the theme, consider one student who was explaining how some problems that were meant to feel realistic did not seem realistic to her. She stated, "The apple problems were repetitive… When I was a little kid, apples were fine. But as you get older you want to see it in different ways. So, instead of seeing it repetitively, with the same problem over and over, you want to see different applications." Another student echoed a similar sentiment when she explained, "We were always doing rocket problems… It is more engaging when it is something I have never thought about." A third student summarized, "Have a variety of questions… Spice it up with other things, put a new spin on something. It is more enjoyable when we incorporate other things like science, literature." In fact, without being prompted, ten of the eleven students regularly expressed the need for variety in problem contexts. We thus claim *variety* to be a new fifth characteristic to add to the framework, and define it as a context that is new or different from previous contexts in the *students' experience*. As such, variety must be judged relative to commonly encountered previous contexts, suggesting the need for future research to document what those common contexts might be.

Features of Problem Contexts that Can Help them be Realistic/Motivating

When examining student reasons that fell within realistic/motivating (including possibly worthwhile) or not realistic/not motivating, three themes emerged that could help contexts be engaging. First, while the students certainly did agree with past research that problems are motivating when related to their everyday lives (e.g., Bonotto, 2005), the students in our study often expressed the desire for problem contexts to expand past their everyday lives to new views on how math can be used in real-world contexts beyond themselves. One student's example of the most realistic problem she had seen was a case of an oil spill. "The oil slick one was realistic because someone could do that. If I know how big the spill is, I can clean it up. Derivatives were used to find how big the spill was." Another student gave an example of a problem where normal distributions were used to "fine tune a medicine." He explained, "You are saving lives right there." A third student lamented, "The classrooms had these posters hanging up that said, 'Where do you use this math?'… But that did not translate into math problems we did." He wished his classes examined the contexts on the posters since he did not know how math related to them. Another student summarized that contexts can be quite realistic, "even if it is not you." These reasons suggest contexts can be realistic/motivating *without* dealing directly with the students' everyday lives, but that rather let them see *new* possibilities for mathematics. Taken together, we call this feature *expansion of awareness*, and define it as a context that permits students to see mathematics being used in a way they did not previously know it could be used.

As a caveat to the previous theme, however, there was a caution for when problems felt unrealistic. Consider an example excerpt from a student who explained, "No one does this. Sports problems, analyzing the arc of your shot, the quickness of movement. When I go to do these activities, I don't do math." Even if an expert might see mathematics as inherent to such activities, the point is that the student did not perceive mathematics as necessary in, for example, having or developing a good arch. Another student gave a similar explanation. He described a short problem where he was given the force and acceleration of an object and was asked to calculate its mass. He stated, "I cannot visualize that scenario. It would be so easy to get the mass. That is annoying because it is not realistic." He did not believe that one would know the force and the acceleration without already knowing the mass. He summarized, "Problems try too hard to feel realistic… When they are trying to pull math out of something that does not feel like it needs math to solve it." Again, regardless of an expert's perspective on the presence and necessity of mathematics in such situations, it was clear that the *students* did not see math as necessary in some of these situations. This theme suggests a context would benefit from having the feature *need for math*, which we define as a context in which a student can perceive the mathematics as being a necessary tool that a person would actually use in such a situation.

The third theme relates to helping a realistic problem also be worthwhile. Some students made comments about feeling in the dark regarding why a potential realistic problem was important. One student's discussion of the humanitarian aid problem from our study included the explanation, "The key is that it should be obvious how I am helping someone. Make that clear in the problem. I don't care nearly as much about saving cost on laying a pipe… if it is not humanitarian aid. I could abstract the problem being humanitarian aid if it is not there, but… I should not have to abstract that from the problem." The point for this student was that a problem felt much more worthwhile when the context itself gave a reason for why one should care about the problem, rather than him trying to guess from the problem what potential worthwhile uses there might be. Similarly, a student criticized the river problem as failing to be realistic because, "I just can't imagine a situation in which I care why… I cannot think of a single reason where I would actually calculate that." This theme suggests a context would benefit from having an *explicit purpose* attached to it, which we define as a specific reason given as to why the answer to such a problem would be useful or beneficial. For example, if the previously mentioned "arch of shot" context were enhanced by adding a specific reason for why knowing that would be useful, that student might have been more engaged by what was otherwise a "realistic" problem.

Features of Problem Contexts that Can Help them be Enjoyable/Motivating

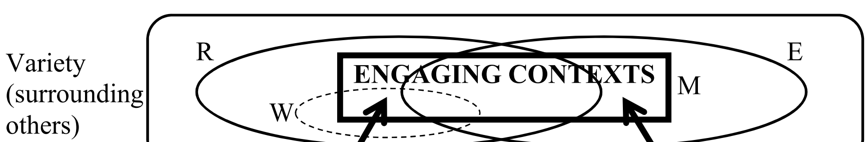
Our data also spoke to three features that problem contexts might have to help them be enjoyable and motivating. First, consider one student's explanation as to why he enjoyed the comet problem: "I can look at it at that time and get a great view." Another student explained why they enjoyed the humanitarian aid problem: "I felt like I was solving an issue." A third student put it well when she said enjoyable contexts might be those that "Give the students an opportunity to put themselves in the shoes of a person within the problem." This theme suggests one possible way to make a problem enjoyable and motivating is if the student can imagine themselves *inside* the problem context. We call this *insertion into problem* and define it as a feature of a context that lends itself well to imagining oneself inside that context.

Second, most students brought up the idea of “stories” being enjoyable and motivating. Two types of stories were discussed by students. One student explained, “My calculus teacher gave us this problem about his son who was trying to make the lantern from [the movie] Tangled. He needed to figure out how to maximize the surface area… Even though I don’t like calculus, I thought, ‘this is a really interesting problem. I kind of enjoy this.’” Another student brought up a situation where his teacher described a “potato gun” his family had and how someone thought it would shoot higher than it really could and ended up making a mess of a gazebo in a park. He stated, “Whenever they give examples from their life it usually ends up being funny and relatable.” Thus, problem contexts that contain a *teacher’s personal story* appeared to be enjoyable and motivating for students.

For the other type of story, a few students explained that sometimes a problem is so “way out there” [student quote] that it is enjoyable. This student explained, “In eighth grade we had this problem about finding ratios and bike tires. It was an absurd problem. The diameter of the tire was massive. The problem became silly and that made it motivating for me.” However, he did add the caution, “It should not be overused. Variety is important. I need some realistic problems. But occasionally putting in ridiculous problems is funny.” Another student described an enjoyable problem where a criminal used calculus to identify the darkest place along a street at night in order to commit a crime there. These contexts contained an *absurd story*, which helped them be enjoyable and motivating. The trick here may be to not overuse this type of context.

Connecting the Results to the Framework

In this study, we identified one new top-level characteristic, *variety*, for problems to be engaging. Note that variety is not contained just within realistic or enjoyable, since the student excerpts on variety included both realistic and enjoyable contexts (e.g., “science” and “literature”). We use this to build on our definition of an engaging context, by defining it as before but with the additional constraint that the context is not seen as repetitive compared to previously encountered contexts. Our study also provides several features that can help a context reside in the intersection of realistic/motivating or enjoyable/motivating. Of course, we in no way claim that these are an exhaustive list of features that could help make problems more engaging. Figure 2 shows how our results have now embellished the original framework.



Inside realistic/motivating:
(a) expansion of awareness
(b) need for math
(c) explicit purpose

Inside enjoyable/motivating:
(a) insertion into problem
(b) teacher’s personal story
(c) absurd story

Figure 2. New characteristic “variety” and (non-exhaustive) features that can help a problem context be engaging

Discussion

Our study adds to the existing literature on engaging problem contexts. For example, some research-based instructional practices within mathematics education are based heavily on presenting students with realistic problems (Blum, 1993; Gravemeijer & Doorman, 1999; Kümmerer, 2001). It is known that these problems should be experientially real to the students, but it is quite possible that an instructor could intend to create a realistic problem that ends up that missing the mark for students. The results of our study can aid instructors in identifying contexts that would engage their students by providing a useful set of features to compare against problems to see whether they would likely be perceived as engaging by students. For example, consider a problem reported on in the work of Schukajlow et al. (2012), who claimed that context ended up having no impact on how engaged students were while doing mathematical tasks. One of the "real-world, modelling contexts" they used is shown in Figure 3. In comparing our study's resulting features to this problem, we can see it fails to be engaging in several ways. We believe it fails to have an *explicit purpose* because there is no specific reason given for why the children in the scenario want to know the cable's length. We believe it fails have *need for math* because even if the children were interested in knowing that, they would likely use other methods to find the length of the cable, such as using a tape measure. While the student audience of this problem might relate to playgrounds and slides, we do not believe the context lends itself to *insertion into the problem*, because the students would not imagine themselves performing such an exploration. While it does provide a story, it seems neither personal nor absurd (nor fun in other ways). With problems like this, we are not surprised that Schukajlow et al's (2012) study produced the results it did. If they had used more a problem context with features that our results suggest are better related to student engagement, the results may have been quite different.

Playground

The Traudt family is vacationing on a farm near Schwandorf in the Bavarian Forest. There is a playground there for the children, Lina and Maria. The playground's greatest attraction is the rope slide. The steel cable of the rope slide is stretched horizontally between two posts 10 meter apart. When Lina reached the middle, the steel cable stretched so much because of her weight that it dropped by 45 cm.

How long is the steel cable now?

Figure 3. A "real-world, modelling context" from Schukajlow et al. (2012, p. 224)

Our study also helps flesh out what makes a problem context enjoyable. As we stated earlier, research promoting "enjoyable" or "fun" contexts have not clearly defined what they mean by these terms (Brown et al., 2008; Nardi & Steward, 2003). Our initial framework provided a definition that is useful by explicitly factoring in the students' emotions. We were then able to elaborate on what helps problem contexts have this characteristic by finding specific features of contexts that seemed to create a positive feeling in the student. Students enjoyed *stories*, and they enjoyed seeing themselves within a problem context. This can aid an instructor in creating problem contexts that stimulate positive emotion by trying to incorporate some of these features.

Finally, our study is important in highlighting just how critical *variety* was for the students in our study. It suggests the need for research work in engaging contexts to pay attention to what contexts are oft-used along the curriculum trajectory from lower grades to higher grades. Future research work could examine this to create a knowledge base of what contexts might be seen as too repetitive, or what contexts might be more unique and new to the students.

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Undergraduates' Perceptions of the Benefits of Working Tasks Focused on Analyzing Student Thinking as an Application for Teaching in Abstract Algebra

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The Mathematical Education of Teachers as an Application of Undergraduate Mathematics project provides lessons integrated into various mathematics major courses that incorporate mathematics teaching connections as a legitimate application area of undergraduate mathematics. One feature of the lessons involves posing tasks that require undergraduates to interpret or analyze the work of another student. This paper reports on thematic analysis of hour-long interviews for eight participants enrolled in an undergraduate abstract algebra course from two different implementation sites. We focus on student work and reactions to these interpreting or analyzing student thinking (AST) applications as they relate to their perceptions regarding the use of AST applications as a mechanism to both deepen their content knowledge and improve their skills for communicating mathematics. Several participants identify positive benefits, but more research is needed to determine the how to incorporate AST applications to accommodate some participants' reluctance to engage in new mathematical contexts.

Keywords: Abstract Algebra, Preservice Secondary Mathematics Teachers, Mathematical Knowledge for Teaching

The Mathematical Education of Teachers II (MET II) Report of the Conference Board of the Mathematical Sciences (CBMS) recommended that preservice secondary mathematics teachers make explicit connections between the undergraduate mathematics content they are learning as part of their continuing education and the primary or secondary school mathematics content they will eventually teach (CBMS, 2012). Furthermore, the MET II report recommends these connections be made in classes throughout their entire degree program and not simply summarized during a culminating capstone course.

An immediate consequence of this recommendation is that mathematics knowledge for teaching must not simply be cursorily included in general undergraduate mathematics courses but in fact emphasized as an application to teaching. This paper reports on the efforts of the *Mathematical Education of Teachers as an Application of Undergraduate Mathematics* (META Math) project to study the effectiveness of lessons which include tasks or applications which focus on interpreting or analyzing student thinking (AST) in an attempt to meet this goal. In particular, we explore the following research question: How do undergraduate mathematics students perceive the effectiveness of AST applications as a mechanism to both deepen their content knowledge and improve their skills for communicating mathematics?

Background and Theoretical Perspective

Wasserman (2018) claims that "despite the strong arguments for how and why studying advanced mathematics might benefit secondary teachers, much of the research has found the opposite to be true: teachers and their students appear to gain little from a teacher's study of advanced mathematics" (p. 4). The MET II Report also asserts that "the mathematics courses [preservice secondary teachers] take emphasize preparation for graduate study or careers in business rather than advanced perspectives on the mathematics that is taught in high school" (CBMS, 2012, p. 5). Similarly, Speer, King, and Howell (2014) observed that "prospective high school mathematics teachers, who earn a mathematics major or its equivalent, do not have sufficiently deep understanding of the mathematics of the high school curriculum" (p. 107). Furthermore, prospective secondary teachers perceive their undergraduate mathematical preparation as unconnected or not useful to their teaching (e.g., Goulding, Hatch, & Rodd, 2003; Wasserman, Weber, Villanova, & Mejia-Ramos, 2018).

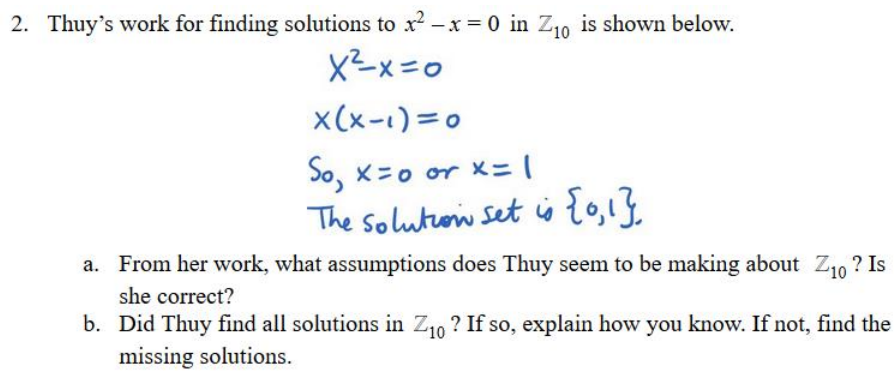
The MET II Report addresses some of these shortcomings by giving examples of several important connections between high school and undergraduate mathematics that they recommend serve as cornerstones of preservice teacher education. These connections are presented as mathematical observances, such as the fact that using inner products to extend the notions of length and angle is "extremely useful background for high school teaching" (CMBS, 2012, p. 57).

While such observations are relevant, the MET II Report does not explain how these connections are to be made explicit in practice. The restructuring of an existing curriculum to include connections for preservice teachers is nontrivially time consuming and potentially difficult for faculty who may not have experience in developing such materials (Lai, 2016; Álvarez & Burroughs, 2018; Álvarez & White, 2018). As such, the development of these materials will necessarily overlap with the study of Mathematical Knowledge for Teaching (MKT), the theory of what mathematics understanding teachers need in order to be successful educators. Originally proposed by Shulman (1986), the concept of MKT is important to preservice teacher education because it posits that traditional content knowledge is not the only mathematical prerequisite required for effective teaching. One particular aspect of this additional knowledge is the ability to "be able to hear and interpret students' emerging and incomplete thinking" (Ball, Thames, & Phelps, 2008, p. 401). Competency in these areas is often directly correlated with not only an instructor's volume of mathematical knowledge but also their appreciation of the structure of and underlying principles connecting mathematical ideas. As such, MKT requires both content knowledge and knowledge applied to teaching where the content knowledge at the secondary level might include mathematics from an advanced perspective such as that addressed in courses such as abstract algebra.

Methodology

To address MKT and the recommendations in the MET II report, the META Math Project has developed inquiry-focused lessons, which can be integrated into various mathematics major courses, that incorporate mathematics teaching connections as a legitimate application area of undergraduate mathematics. The connections, especially targeting relevance for preservice secondary mathematics teachers (PSMTs), aim to link the undergraduate content PSMTs encounter in their major courses and the pre-college content they will eventually teach. One feature of the lessons involves posing undergraduate mathematics students with tasks that require them to interpret or analyze the work of another student (see Figure 1).

This paper will focus on student perceptions of AST applications in two lessons written for an undergraduate Abstract (Modern) Algebra 1 class, "Groups of Transformations" and "Solving Equations in $\mathbb{Z}_n$", which explore how transformations can be examined via group structure and how using traditional high school algorithms for solving equations is affected by working outside of $\mathbb{R}$, respectively.


2. Thuy's work for finding solutions to $x^2 - x = 0$ in $\mathbb{Z}_{10}$ is shown below.

$$x^2 - x = 0$$
$$x(x-1) = 0$$
So, $x = 0$ or $x = 1$

The solution set is $\{0, 1\}$.

a. From her work, what assumptions does Thuy seem to be making about $\mathbb{Z}_{10}$? Is she correct?
b. Did Thuy find all solutions in $\mathbb{Z}_{10}$? If so, explain how you know. If not, find the missing solutions.

Figure 1. An assessment item from the lesson "Solving Equations in $\mathbb{Z}_n$"

To guide the design of the lessons, META Math uses the six categories of MKT identified in Ball, et. al (2008) to develop five *connections for teaching* between college-level mathematics and knowledge for teaching school mathematics: Content Knowledge, Explaining Mathematical Content, Looking Back/ Looking Forward, K-12 Student Thinking, and Guiding K-12 Students' Understanding (Arnold, Burroughs, Fulton, & Álvarez, 2020). In this paper we focus on the category called *K-12 Student Thinking*, in which undergraduates are asked to evaluate the mathematics behind a student's work and explain what that student may or may not understand. These five connections are incorporated into both the "Groups of Transformations" and "Solving Equations in $\mathbb{Z}_n$" lessons by designing an activity-based lesson (separated into pre- and class-activities), homework questions, and assessment items. Instructors were provided with an extensive annotated lesson plan to help them implement the lesson effectively.

Setting and Participants

The two Abstract (Modern) Algebra lessons were implemented by two instructors at two different universities in an upper division mathematics course in Spring 2019. One university is a small public institution and the other is a mid-sized public institution. Both are classified as Hispanic-serving Institutions in the United States (i.e. Hispanic student enrollment comprises 25% or more of total enrollment). As part of their regular coursework, all undergraduates in these sections completed a pre-activity, class activity, homework assignment, and assortment of assessment items for each lesson. For students consenting to participate in the study, their work was examined for mathematical understanding and appreciation of connections for teaching. We also invited consenting undergraduates to participate in an hour-long semi-structured interview at the end of the semester. Four participants from each institution agreed to participate in interviews. To begin the interview, we asked for the student's major and whether they intended to teach in their future (see Table 1).

Table 1: Interview participant information

Pseudonym	Major(s)	Interested in Teaching
Adam	Mathematics	No
Bonnie	Mathematics	Yes (at university level)
Christie	Mathematics	No

Diane	Mathematics	Yes (at university level)
Ella	Applied Mathematics	Possibly
Fleur	Applied Mathematics & Mechanical Engineering	Yes (at university level)
Grace	Mathematics	Yes (at university level)
Helen	Mathematics & Finance	Yes (at university level)

Data Collection and Analysis

The interviews lasted between 45-60 minutes and were audio-recorded and transcribed. During an interview, students were prompted to re-examine their work on the assessment items from each interview (e.g. Figure 1). While reconsidering their work, they were asked to provide an explanation of their thought processes where appropriate, consider alternative approaches, and discuss the potential connections to previous math content. These questions were often posed through the lens of connections for teachers, but interviewees were encouraged to discuss their own perceptions of the assessment items regardless of their intent to formally teach in a classroom environment.

Because the assessment items for each lesson necessarily included a student thinking question, much attention was paid during the interviews to this method of framing connections for teachers. While considering student thinking questions in particular, interviewees were encouraged to also consider how they might use guiding questions to motivate the hypothetical student to correct their work. This line of questioning included a discussion of why undergraduates felt that their sequence of guiding questions would be appropriate.

Finally, interviews concluded by giving the undergraduates the opportunity to discuss how the connections for teachers emphasized in these lessons were beneficial to them even if they did not foresee themselves teaching in the future. Many students took this opportunity to describe how the approach taken to presenting this material, and even mathematics in general, prepared them for their anticipated career.

We used thematic analysis (e.g., Braun & Clarke, 2006; Nowell, Norris, White, & Moules, 2017) to qualitatively analyze the interview transcriptions. Each interview was preliminarily coded for the five connections for teaching. These codes were then expanded inductively with any emergent thematic ideas. These additional codes tended to relate to teaching, the implementation of the lesson, or the format of the activities. Once each lesson was coded independently, we compared our codes until we were all in agreement. Less pervasive codes were eliminated or integrated into broader categories.

Results

Effectiveness of Student Thinking Questions

Out of the eight interview participants, three expressed their unilateral appreciation of student thinking questions, three found them to be conditionally helpful in their own learning, and one thought that student thinking questions did not improve her understanding of the underlying mathematics. Also, one student did not comment directly on her opinion of these types of questions.

Interview participants who broadly approved of student thinking questions often made note of the fact that examining another student's work forced them to consider the problem from a different perspective, thereby complexifying their understanding of the mathematical content. Bonnie verbalized this sentiment by comparing student thinking questions to questions which only require the production of a correct answer:

> Let's say you can memorize the answer. You just know the simplest path to the right answer. You'd be like, OK, I can get it. You can just find a simple answer or something that you remember your teacher told you, right? If you're presented with a wrong answer, then in fact you actually have to justify why this is wrong. You'd actually have to go back to like the main thing. Find a definition or find some reasoning as to why that's wrong.

In a similar manner, Grace said that "if I'm able to identify why it is incorrect, then I'm adding more support to my answer." Both these students emphasized the way in which student thinking questions required them to generate supporting statements or justification for mathematical work. The ability for students to "justify their conclusions… and respond to the arguments of others" is one of the Common Core State Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010); allowing preservice teachers to practice this skill in the context of connections between undergraduate and K-12 mathematical content both improves their mathematical understanding and models successful mathematical thinking skills that they will be more capable of passing on to their students.

The final student who approved of student thinking questions in all circumstances, Ella, expressed that "with math there's like so many different ways to solve one problem. I always think it's nice to know how other people think because they don't think like I do." This was a recurring theme amongst all interviewees, even those with less positive opinions of student thinking questions. For example, Diane said that she would only benefit from doing problems featuring student thinking "when it's correct," adding that "if I look at somebody's mistakes and I don't know what they're doing, it's very hard to understand what is correct and what is not." On the other hand, if student work which is correct but different from her own work is a valuable learning tool since it provides "a different perspective," and for preservice teachers can add "more tools in her toolbox to teach."

Other students with mixed opinions on student thinking questions were Adam and Helen, whose opinions mirrored those of Diane. They valued student thinking questions when the hypothetical student was doing correct but different work, while they found student thinking questions with errors misleading. Helen was particularly vehement about her discomfort with incorrect work, stating that "it would probably make me pick up a bad habit if it's wrong." She implied that not only would the student's error be confusing and cause her to second-guess her original answer, but that attempting to learn new material by examining an incorrect approach might actually cause her to internalize and reproduce the error in her subsequent work. Both students were still in agreement with Ella, however, in expressing the value of learning multiple correct approaches. Adam said that:

> What I love about math is that there's three ways to solve things. You know you can use a lot of different properties, like whatever you want to use to solve something. So if someone took a second route to something I could be like, oh, so I can solve it like that too.

Students who were ambivalent about student thinking questions, interestingly, were often simultaneously proponents of structured group work in class. Diane said that "I do tend to work with another classmate a lot. We'll bounce ideas back and forth and think that's the only way to survive."

The only interviewed student who did not, in some circumstance, feel like her understanding of mathematical content benefitted from student thinking questions was Christie. Instead, she commented on the benefits of student thinking questions from a pedagogical standpoint.

Interviewer: So how does looking at another student's work like this help you understand the material better?

Christie: I don't think it really does. But like if we talk about teaching, then just teaching like- what to put emphasis on.

This student went on to express several other ways that the lesson, in general, benefitted preservice teachers by providing valuable insight into mathematical connections between undergraduate and K-12 content, but she never commented further on student thinking questions specifically.

Benefits for Career Paths Other than Teaching

Seven of the eight interviewed students also described how student thinking questions, the lessons in general, or mathematics as a subject were able to impart a beneficial skill that they found relevant to their futures. These skills extended to fields outside of K-12 teaching.

Two students, Adam and Diane, both noted the advantages of a mathematics education to their future. Diane said that the META Math lessons "[spark] up thinking" by providing a "different perspective." She elaborates on the benefits for non-preservice teachers by adding that "even if they don't teach… they can see things differently in any kind of area." She attributes this shift in perspective to the subject as a whole, adding that "math teaches us to think critically." Adam, who anticipates working in the I.T. field, echoes this sentiment when he observes that he "can critically think now." He goes on to explain how he can apply experiences from his mathematics education to his career choice: "I approach computer problems the same way I approach a Dr. David proof. I've got to assess what the issue is, figure out what the question is asking, and [go] step by step by step by step." Christie, who also wants to work in the I.T. field, explained that the group-oriented structure of the lessons helped her prepare for experience in the workforce when she noted that "even in I.T. you work in groups." While these students did not make explicit reference to ways in which connections for preservice teachers benefited those who might not find themselves working in a K-12 classroom, they were optimistic about the positive effects of their education on their abilities.

Ella noted that it was important for experienced actuaries to be able to explain concepts and procedures to their associates. Other interviewed students explicitly referenced the ways in which the student thinking questions contained in the lessons benefitted them despite the fact that they did not anticipate careers in education. When asked to explain, their answers were similar to Ella's deconstruction of the working relationship between actuaries in that they were centered around the need to provide clear explanations of difficult topics. For example, Bonnie said that

> Even just the test questions where it's kind of like, spot what they did wrong and be able to explain to them why they did it wrong. It forces you to actually be able to do that. For instance, if that would happen in a real life scenario, you're actually be able to do that instead of just going, well it's wrong. I don't know why, but this is my right answer.

Grace mirrored this sentiment when she said that it was important for even non-teachers to be comfortable

> Not only knowing your words and what you're intending to say and different ways to say them but also seeing how another person's approaching the problem and being able to

> think the way they're thinking and see their solution. So I definitely think that even if you're not gonna be a teacher, learning like this is, I think, very beneficial.

These students noted that thinking in a way that is directed at another person's understanding is a skill that transcends the profession of teaching. Interpreting the awareness and possible misconceptions of coworkers and clients is pivotal in establishing good working rapport and a professional environment conducive to learning.

Finally, Fleur noted that, in some sense, the ability to clearly explain a difficult topic transcends the importance of simply possessing knowledge of that topic. She says,

> You might know a lot of things, you know, like math and complex stuff, but if you're not able to share that information, to communicate it well, going from the simple and going to the complex... That might not only be applicable to the teachers but just if you want to explain anything, right?

Implicit in this sentiment is the need for explanations of both simple and complex concepts to be connected and coherent when taken as a whole—a simple explanation should not preclude a teacher from extending it to a more complex explanation. Fleur's response highlights the need for general mathematics courses to emphasize connections between all types of mathematics so that teachers and other professionals who require mathematics content knowledge can provide cogent and consistent justification of all types of ideas.

Conclusion and Implications

Based on the participant interviews described, there is some evidence of the effectiveness of student thinking questions in the context of presenting connections between undergraduate and K-12 content. A majority of participants interviewed felt that these types of questions could be useful to expand their mathematical understanding of the content, either by requiring them to justify their procedure or by providing an example of an alternative or novel approach. Students were also able to recognize that the skills imparted by engaging in deep thought about another person's thinking were transferable not just to teaching but also to other fields that require the dissemination of technical knowledge.

Some students were apprehensive about student work in mathematical contexts in which they were not themselves confident. To accommodate such students, it may be advantageous to position student thinking questions at such a point in a lesson that the new material has already been explored through traditional means; this may mean including student thinking questions at the end of a class activity or on a homework assignment. Such structure would allow students more time to familiarize themselves with new mathematical ideas before using student thinking questions to make a connection to high school curriculum or explore different approaches to justifying a particular methodology.

Operationalizing the connections for teachers as outlined in the MET II Report through the use of student thinking questions appears to support the mathematical learning not only of preservice teachers but of broader categories of students as well. As our participant Fleur notes, "this is a really good skill to have—to be able to share knowledge."

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Assessing the Uptake of Research Based Instructional Strategies by Postsecondary Mathematics Instructors

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For decades, the mathematics education research community has strived to identify, document, and disseminate instructional strategies that support student success in mathematics courses. We now have many research-based instructional strategies (RBIS) which have evidence to support claims that they can be used to help students succeed in mathematics. However, the use of such practices has not become widespread at the postsecondary level in mathematics, nor in the other STEM disciplines. This work, focused on mathematics instructors, is part of a larger project hoping to uncover more about why and how particular instructional practices spread in undergraduate mathematics, chemistry, and physics. Here, we report that single-variable calculus instructors have awareness of, and use, RBIS in their classrooms; however much of class time is spent in a lecture format. Implications of these and future results are discussed.

Keywords: instructional practice, calculus, STEM, national survey

Research-Based Instructional Strategies (RBIS) refer to the host of tools, resources, and modes of instruction that have shown promise in increasing students' engagement in learning and in leading to measurable, and lasting, learning gains. This proposal comes from a larger project tracking the implementation of RBIS in introductory postsecondary STEM courses across the country. In particular, we report on the awareness and usage of particular RBIS in single-variable calculus courses at the undergraduate level, and briefly touch on how instruction in first year mathematics courses compares to instruction in first year chemistry and physics courses.

RBIS are thus called because there is evidence that they can significantly improve learning and retention. This includes many variations of active learning (Freeman et al., 2014), flipped classrooms (Bergmann & Sams, 2012; Lage, Platt, & Treglia, 2000), peer instruction (Crouch & Mazur, 2001; Fagen, Crouch, & Mazur, 2002), formalized small-group work (D. W. Johnson & Johnson, 2009; King, 1993; Kramarski & Mevarech, 2003), and more. Implementing these practices is not easy, and doing so in ways that preserve fidelity of implementation and mirror positive outcomes is not a straightforward task (Andrews, Leonard, Colgrove, & Kalinowski, 2011; Bagley, 2015; E. Johnson et al., in press; Laursen & Rasmussen, 2019).

Our focus on introductory postsecondary STEM courses comes from their outsize impact on students' ability and interest in completing undergraduate STEM degrees (PCAST, 2012; Seymour, 2006; Seymour & Hewitt, 1997). Single-variable calculus is one of several "gateway courses," defined as a foundational course for future studies which is high-risk, in that it carries a relatively high DFW rate, and high(er)-enrollment (Koch, 2017). Shifting the instructional status quo away from straight lecture in these gateway courses has the potential to have a major positive impact on students' undergraduate education. Thus far, however, lecture continues to be the norm (Eagan, 2016; Gibbons, Villafañe, Stains, Murphy, & Raker, 2018; Henderson & Dancy, 2009; Rasmussen et al., 2016; Stains et al., 2018). To measure the changing instructional landscape, and to accelerate it, in this paper we address the following two research questions, and provide some minimal comparisons to chemistry and physics for comparison.

1. How much in-class time is being spent on lecture vs. non-lecture activities in undergraduate single-variable calculus courses?
2. What are the levels of awareness and usage of twelve RBIS among instructors of undergraduate single-variable calculus courses?

Methods

Survey Overview

Data for this project was collected via a web-based survey developed by the research team, drawing on previous experience and relevant research literature. Items and format were informed by previous large studies of chemistry (Gibbons et al., 2018; Lund & Stains, 2015; Stains et al., 2018), mathematics (Apkarian et al., 2019; E. Johnson, Keller, & Fukawa-Connelly, 2018), and physics education (Henderson & Dancy, 2009; Walter, Henderson, Beach, & Williams, 2016). The survey had a few major sections, each of which included items from previous research and validated instruments. The instructional practice section draws from the work of Landrum et al. (2017) and the Postsecondary Instructional Practices Survey (PIPS) (Walter et al., 2016). Subsequent sections of the survey contain questions about departmental climate and culture (Walter, Beach, Henderson, & Williams, 2014); instructor's beliefs about teaching, learning, and students (Aragón, Eddy, & Graham, 2018; Chan & Elliott, 2004; Dweck, Chiu, & Hong, 1995; E. Johnson et al., 2018; Meyers, Livingston Lansu, Hundal, Lekkos, & Prieto, 2006; Tollerud, 1990); and demographic questions related to both professional and personal identities and roles. Each of the items was drawn directly or indirectly from existing research and conjectures about factors affecting instructional practice in undergraduate STEM courses. This proposal draws on data from the first survey section focused on instructional practice, and from a few items in particular (see Figures 1 and 2 in section *Key Survey Items*).

Survey Distribution & Response

The American Institute of Physics Statistical Research Center converted the survey into an online version, developed a roster, and handled the distribution and collection of data. The initial roster was compiled by identifying 1012 postsecondary institutions (545 two-year colleges; 467 four-year colleges and universities) to include in our sample, and then identifying all recent instructors of introductory chemistry, mathematics, and physics courses at that institution. These instructors were then emailed an invitation to participate in the study, which they completed on their own time, within the 8-week period that the survey was open. Reminders were sent approximately every 10 days to increase response rates.

We received 3769 usable responses from instructors at 851 institutions (397 TYC; 454 4YoA). By usable, we mean that the participant consented to participate, had taught one of the target courses in the past two years, and answered at least one item on the survey. Not every participant answered every question, so the sample size varies somewhat from item to item. These responses break down among disciplines and institution types as shown in Table 1. This analysis will primarily focus on the 1349 responses from calculus instructors, however there will be times when we present tables and graphs that contextualize calculus relative to first-year chemistry and physics. The mathematics respondents were two-thirds men; four-fifths White; two-thirds hold tenured or tenure-track positions at their institution; one-third participate in

SoTL and/or RUME. We recognize the existence of response bias within our data, and that this sample is not truly representative of postsecondary calculus instructors.

Table 1. Number of respondents by discipline and institution type.

Institution Type	Chemistry	Mathematics	Physics	Total
Two-year	422	416	261	1099
Four-year	822	933	915	2670
Total	1244	1349	1176	3769

Key Survey Items

The item shown in Figure 1 required participants to report the percentage of class time spent on four activities, (for a total of 100%). We received 3641 responses which totaled 100%, 128 which totaled 0%, and 26 which totaled something else; those which did not total 100% were removed from the analysis.

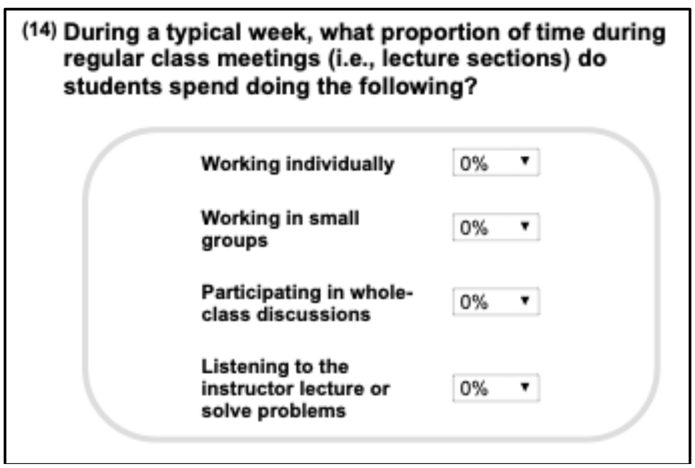
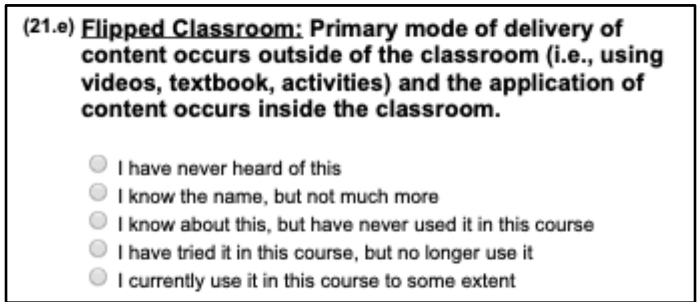


Figure 1. Survey question asking respondents to report percentage of class time.

Participants were also shown a series of items about specific RBIS and asked to rate their awareness/usage of that RBIS. Each RBIS was named, defined, and participants selected one of five options ranging from “I’ve never heard of this” to “I currently use it in this course.” (See the example in Figure 2.) The format of these questions mirrors that of work in physics and chemistry education, which allows us to compare our results with those from previous work in various contexts (Henderson & Dancy, 2009; Lund & Stains, 2015). Nine RBIS were displayed to all participants, with an additional 3-6 more discipline-specific RBIS displayed to people from each discipline. In total, twelve were shown to those teaching mathematics. We drew on prior work in all three disciplines, our own experiences, and the input of other experts in our three fields to select and define our set of RBIS to ensure face validity and the ability to compare with prior work in similar areas.


(21.e) Flipped Classroom: Primary mode of delivery of content occurs outside of the classroom (i.e., using videos, textbook, activities) and the application of content occurs inside the classroom.

- I have never heard of this
- I know the name, but not much more
- I know about this, but have never used it in this course
- I have tried it in this course, but no longer use it
- I currently use it in this course to some extent

Figure 2. Example of a survey item asking about familiarity with, and use of, an RBIS

Results

Class Time

On average, calculus instructors reported that 56% of regular class time is spent with students listening to the instructor lecture or solve problems (*lecture*); the remaining 44% of class time is shared between three *non-lecture* activities. Similar averages were obtained from chemistry and physics instructors (see Table 2).

Table 2. Mean (standard deviation) of reported percentage class time for instructional practices, by discipline.

Instructional Practice	Mathematics (n=1269)	Chemistry (n=1217)	Physics (n=1141)
Students listening to the instructor lecture or solve problems	56.3 (25.3)	58.5 (25.2)	53.9 (25.6)
Students participating in whole-class discussions	15 (14.6)	10.8 (11.8)	13 (11.6)
Students working on problems in small groups	17.1 (17.9)	18.2 (19.4)	22.5 (20.5)
Students working on problems individually	11.3 (11.8)	11.6 (11.6)	10.6 (11.7)

In Figure 3 below we can see how the median report of amount of class time spent lecture, and quartiles, compare for the three STEM disciplines. The medians for each of lecture and non-lecture are quite consistent across the three groups. Physics has the largest IQR, suggesting more variance in instructional practice, and the medians of lecture and non-lecture time are the closest of the three disciplines, suggesting that lecture is less dominant in introductory physics courses than in general chemistry or single-variable calculus.

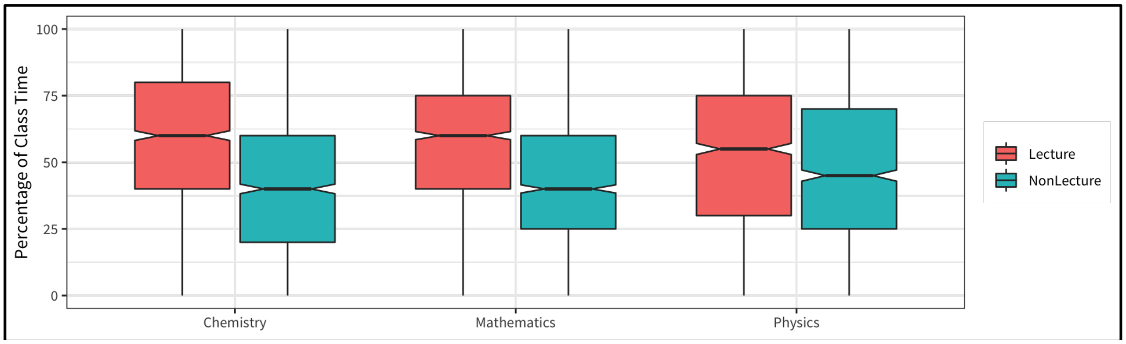


Figure 3. Notched box plots showing the median, confidence interval of the median, and IQR for reports of lecture and non-lecture in-class activities across chemistry, mathematics, and physics instructors.

Of course, averages only give one perspective on these results. Figure 4 show the distribution of responses from mathematics faculty. This distribution suggests first that respondents are more likely to select a decade than a 5% increment, hence the up-and-down pattern in the chart. The data also have negative skew, with a longer tail toward the lower percentages of class time. Thus while the median amount of calculus class time spent in lecture is around 55%, the majority of *instructors* are reporting 50-80% of class time is spent in lecture.

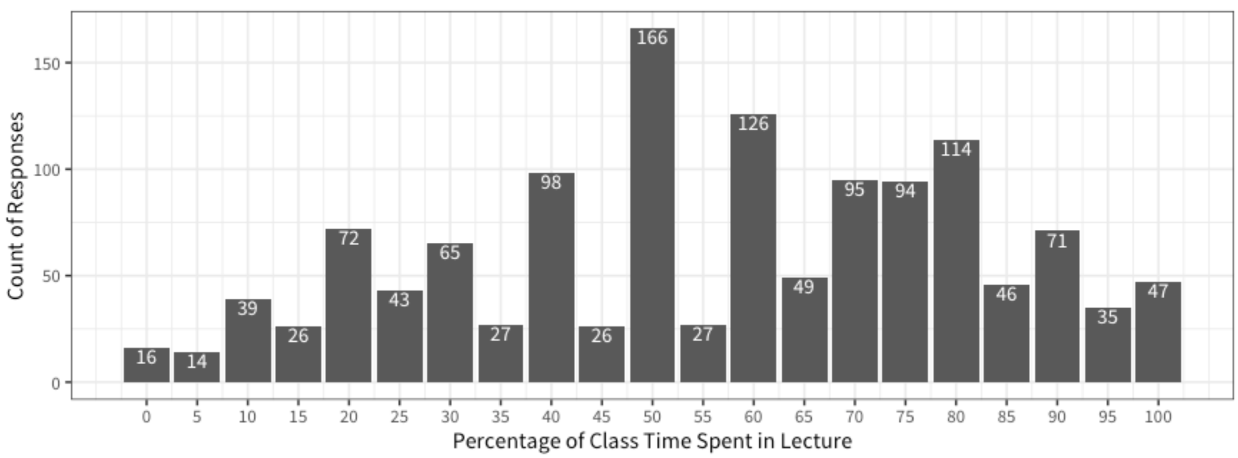


Figure 4. Distribution of responses from calculus instructors about the percentage of in-class time spent with the instructor lecturing or solving problems (n=1296).

Awareness & Usage of RBIS

For this analysis, responses are clustered into current users of an RBIS, knowledgeable non-users, and those with little-to-no knowledge. Current users are those who select “I currently use it in this class to some extent;” knowledgeable non-users are those who select “I have tried it in this course, but no longer use it,” or “I know about this, but have never used it in this course;” those with little-to-no knowledge select “I know the name, but not much more” or “I have never heard of this.” This is similar to the categorizations used in prior work in physics (Henderson & Dancy, 2009). Figure 5 shows the proportion of mathematics respondents who fit into each category for each RBIS. Note that the most well-known RBIS are not necessarily being used the most. For example, small-group work and flipped classrooms are each known by 91% of respondents, but while small-group work is also the most used RBIS (51% of respondents are current users), flipped classes are the 6th (18% current users).

As shown in Figure 5, the RBIS used by the most mathematics instructors are small-group work (51%), teaching with computer simulations and animations (45%) and think-pair-share (37%). These three RBIS are the only ones in use by more than one third of respondents. The most well-known (current users plus knowledgeable non-users) are small-group work and flipped classrooms (91% each); other RBIS known by two-thirds or more of participants are: teaching with computer simulations and animations (81%), IBL (79%), and think-pair-share (76%). The RBIS with the fewest users are: concept maps (10%), reform-oriented textbooks (12%), and POGIL (13%). Two RBIS are unknown to more than half of the respondents: concept inventories (52%) and POGIL (51%); these are followed closely by concept maps (45%), reform-oriented textbooks (42%), and peer-led team-learning (41%).

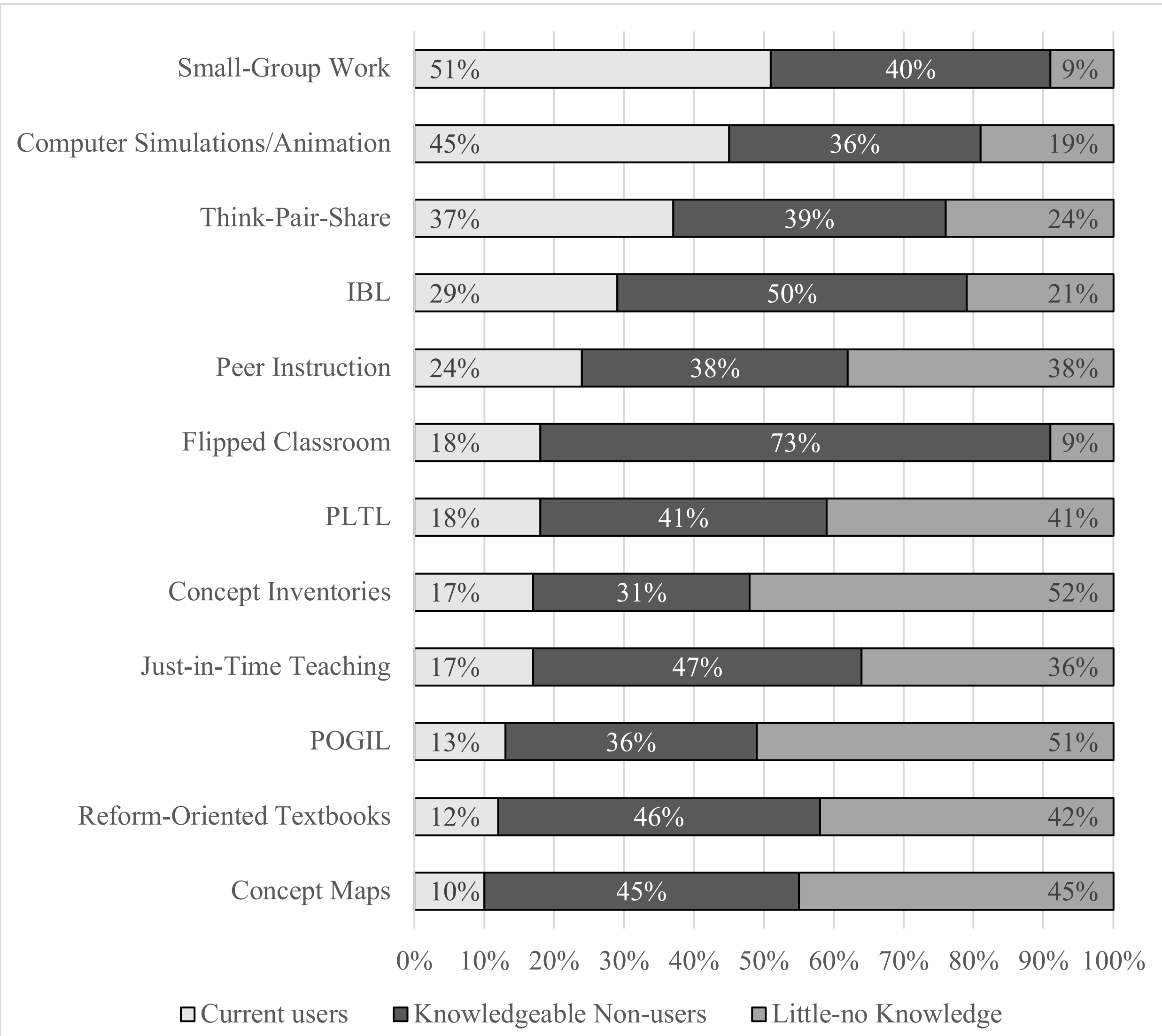


Figure 5. Responses of Mathematics Participants to RBIS questions, grouped into current users, knowledgeable non-users, and those with little to no knowledge of the RBIS. Top-to-bottom, these are ordered by most-least in use.

Further investigation into these responses indicate that knowledge and use is spread out among instructors, not concentrated in a few. Of the 1192 who responded, only 13 (1%) reported no knowledge of *any* of the RBIS on this list; a total of 75 (6%) reported knowledge of zero, one, or two RBIS from this list. In terms of usage, 207 (17%) of participants report that they are not current users of *any* RBIS; 361 (30%) are currently “low users,” using 1-2 different RBIS in their introductory calculus course; 624 (52%) are using 3+ RBIS from this list.

Discussion

Calculus instructors who responded to our survey are aware of RBIS, though we can make no claims about how they understand these RBIS or their familiarity with the research surrounding them. Despite this general awareness, most RBIS are not being used in calculus classrooms. Only three RBIS are being used by more than a third of our participants: formal small-group work, teaching with computer simulations and animations, and think-pair-share. We think it is notable that these are among the least well-defined of the RBIS asked about in the survey, in that they are implemented in a great many ways and do not have particularly concrete definitions or branding. These general RBIS stand in comparison to IBL and Peer Instruction, which are more well-defined. Additionally, in the case of IBL, there has been significant outreach efforts to build awareness and professional development efforts to support adoption (Haberler, Laursen, & Hayward, 2018). We believe these efforts are reflected in our data, with 29% of calculus instructors reporting that they are currently using IBL, and another 50% reporting that they have knowledge of it.

While the majority of our survey participants report that they have knowledge of multiple RBIS, and report using at least one in their course, on average, participants are reporting that the majority of class time is spent with “students listening to the instructor lecture or solve problems.” This indicates that lecture is still the dominant feature of many introductory STEM courses, even when (some) RBIS are being used. More detailed analysis of this data will investigate patterns about which RBIS are being used in majority-lecture courses, as well as fidelity of implementation using data from other elements of the survey. These investigations will be carried out for the chemistry and physics data sets as well, to gauge similarities and differences among instructional practices within the STEM umbrella. The ultimate goals of this project include the identification of relationships and levers between contextual factors (e.g., culture and climate of the department) and individual factors (e.g., beliefs about students and learning) and instructional practice. That information can be used to navigate change initiatives and work to increase the usage of research-based instructional strategies; we are also contributing to baseline usage information which can be used to gauge change in usage from now to the future.

Acknowledgement

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University Calculus Students' Use and Understanding of Slope Conceptualizations

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The purpose of this study is to investigate the visual and nonvisual approaches and conceptualizations of slope used by introductory calculus students when presented with a variety of tasks.

Keywords: slope, university, calculus students

The derivative concept is one of the main topics in introductory calculus. A study of experts' views of central concepts and skills in first-year calculus found that the derivative was unanimously viewed as fundamental to calculus, with three sub-concepts identified: derivative as rate of change, graphical representation of the derivative, and derivative computations (Sofronas et al., 2011). Since the calculus concept of derivative builds upon and extends students' notions of slope (Zandieh, 2000; Zandieh & Knapp, 2006), it is important to know what conceptualizations of slope students bring with them to introductory calculus and how instructors might effectively build on that foundation (Nagle, Moore-Russo, Viglietti & Martin, 2013). Furthermore, since both visual and nonvisual approaches can be used to introduce derivatives in introductory calculus the importance of knowing which approaches to slope students come equipped to build upon in calculus is vital. This study investigates the following questions based on introductory calculus students' responses to a variety of slope tasks:

1. Which conceptualizations of and approaches (*visual* or *nonvisual*) to slope are most common in students' responses?
2. Do students consistently use the same conceptualizations of and approaches to slope or do these vary based on specific tasks or questions?

Theoretical Framing

Meaning is personal and dynamic; it is the way that an individual thinks of a concept and is impacted by the individual's experiences (Vinner, 1992). This is related to Tall and Vinner's (1981) "concept image" which "consists of all the cognitive structure in an individual's mind that is associated with a given concept" (p. 151). If students are able to represent a concept in multiple ways, they have multiple meanings associated with the concept (Moore, Paoletti, & Musgrave, 2013). Since there are often multiple ways that a single concept can be conceptualized, communicated and represented (Duval, 2008), it is desirable that students develop rich, coherent systems of meanings for foundational mathematical concepts. Thompson (2013) goes into detail about the term "meaning" and emphasizes that when instructors are interacting with students what matters most are the students' "current meanings that constitute the framework within which they operate" (p. 62). Instructors should consider students' current, personal meanings by attending to how students think of, link, and organize isolated or discrepant thoughts (Dewey, 1910).

Byerley and Thompson (2017) propose that meaning is context dependent. Different contexts may trigger different meanings; and using particular approaches or representations may influence by the task, its context (Thompson, 2016) or on the extent of the connectedness of the student's ways of thinking about the concept.

The Concept of Slope

Researchers have reported numerous conceptualizations, or ways of thinking, about slope (Stump, 2001a, 2001b; ; Moore-Russo, Conner & Rugg, 2011; Nagle, Martinez-Planell & Moore-Russo, 2019), with some ways of thinking about slope being identified as more useful or productive than others (Byerley & Thompson, 2017). Stump first identified different categories of slope (1999, 2001a, 2001b), which have been revised and termed "conceptualizations" by Moore-Russo and colleagues (Moore-Russo, Conner & Rugg, 2011; Nagle & Moore-Russo, 2013b). For this paper we restructured these into five conceptualizations, each of which was separated into *visual* and *nonvisual* approaches. For details, see Table 1.

Table 1. Slope Conceptualizations and Approaches

Category	Approach	Explanation
Slope as a *Ratio*	Visual	• rise/run or vertical change divided by the horizontal change • includes cases where run equals 1 or where both values are integers
	Nonvisual	• $(y_2 - y_1)/(x_2 - x_1)$ or change in y over change in x
Slope as a *Behavior Indicator* of a Line	Visual	• line increases, decreases, is horizontal, is vertical (i.e., looks like /, \, -, |) for positive, negative, zero, undefined slope respectively
	Nonvisual	• line increases, decreases, is constant, or is not a function in other words (i.e., $y_2 > y_1$ for $x_2 > x_1$) for positive slope, (i.e., $y_2 < y_1$ for $x_2 > x_1$) for negative slope, (i.e., $y_2 = y_1$ for $x_2 > x_1$) for zero slope, or (i.e., $x_2 = x_1$ for $y_2 > y_1$) for undefined slope respectively
Slope as Denoting *Steepness* of Line's Angle of Inclination with respect to Horizontal	Visual	• relates to how inclined, tilted, slanted, or pitched a line is <u>seen</u> as being • greater value of |slope|, line is more steep (e.g., closer to vertical) • closer to zero value of |slope|, line is less steep (e.g., closer to horizontal a line is) • since horizontal lines have no tilt, they have zero slope
	Nonvisual	• relates to how extreme a line is <u>calculated</u> as being • the greater the value of |slope|, the more steep the line over an interval (e.g., the closer to infinity the value of $\|y_2 - y_1\|$ is) • the closer to zero the value of |slope|, the less steep the line over an interval (e.g., the closer to zero the value of $\|y_2 - y_1\|$ is) • since horizontal lines have $\|y_2 - y_1\|= 0$ for all y values, they have zero slope
Slope as a *Constant Parameter*	Visual	• emphasis on the uniform "straightness" of the line's entire graph (i.e., no curvature) • no matter which segment of the line is considered, the slope remains the same between any two points
	Nonvisual	• emphasis that a single constant holds a property for the line's equation/table (not dependent on input) • the slope remains the same between any two points on a line
Slope as *Determining Relationships* between lines	Visual	• two unique lines have the same slope if and only if they never intersect in two-dimensions (i.e., are parallel) • two unique lines have different slopes if and only if they intersect at a common point • two unique, nonvertical lines have negative reciprocal slopes if and only if their intersection is at a right angle (i.e., they are perpendicular)
	Nonvisual	• two unique lines have the same slope if and only if a system of these two lines has no solution (i.e., system is inconsistent since lines are parallel) • two unique lines have different slopes if and only if the system of these two lines has one solution (i.e., system is consistent; there is a shared ordered pair) • two unique, nonvertical lines have negative reciprocal slopes if and only the product of their slopes is -1 (i.e., they are perpendicular)

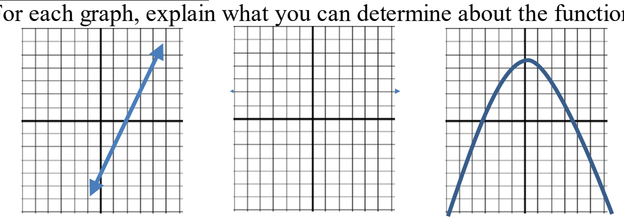
Research has shown that students' meanings for slope are often not connected but rather isolated and fragmented (Nagle & Moore-Russo, 2017; Hattikudur, Prather, Asquith, Knuth, Nathan & Alibali, 2011; Planinic, Milin-Sipus, Kati, Susac & Ivanjek, 2012; Lobato & Siebert, 2002), leading to particular ways of thinking about slope being triggered only in certain contexts (Byerley & Thompson, 2017). For instance, even though the mnemonic device "rise over run" is often associated with slope (Azcarate, 1992; Stump, 1999), this way of thinking about slope is not necessarily connected to thinking about slope as an angle of inclination (Nagle & Moore-Russo, 2013a; Stump, 1999).

Methods

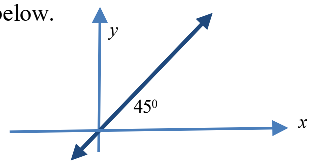
Data were collected from a convenience sample of 231 students in 7 introductory university calculus classes. Of the 231 students, 64 were in 2 business calculus classes, 142 were in 4 STEM calculus classes, and 25 were in an honors STEM calculus class at four small universities in the northeastern United States. All four universities are located within a 3-hour drive of each other and serve students in this geographic region, all of whom are approximately the same academic level upon entry.

The data were collected using a 19-item instrument that was administered on either the first or second day of classes, prior to instruction. Students were given 40 minutes to complete the instrument with only paper and pencil. The instrument, displayed in Table 2, was designed to trigger the use of multiple slope conceptualizations.

Table 2. Tasks and Questions on Data Collection Instrument

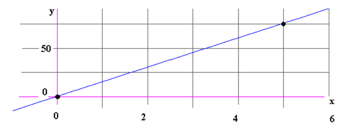
<table>
<tr><th>Item</th><th>Task/Question</th></tr>
<tr><td>1</td><td>What is slope?</td></tr>
<tr><td>2</td><td>How is slope presented?</td></tr>
<tr><td>3</td><td>How is slope calculated?</td></tr>
<tr><td>4</td><td>When is it appropriate to use slope?</td></tr>
<tr><td>5</td><td>The slope of a linear equation is 1/3. What can be determined about the line's graph based on its slope? Explain.</td></tr>
<tr><td>6</td><td>The relationship between time (x, measured in hours) and the amount of water (y, measured in gallons) in a swimming pool is given by $y = -50x+12000$. What does the slope of this equation tell you about the relationship between time and the amount of water in the pool?</td></tr>
<tr><td>7</td><td>Find the slope between the points (3,2) and (5,-8).</td></tr>
<tr><td>8</td><td>Identify the slope for the equation $6x + 3y = 6$. Complete the table for a line that would be parallel to it.
<table><tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td></tr><tr><td>y</td><td></td><td></td><td></td><td></td></tr></table></td></tr>
<tr><td>9</td><td>For each graph, explain what you can determine about the function's slope.
</td></tr>
<tr><td>10</td><td>Complete the table on the right with values so that the linear functions $f(x)$ and $g(x)$ have the same slope.
<table><tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td></tr><tr><td>$f(x)$</td><td>6</td><td>4</td><td>2</td><td>0</td></tr></table><table><tr><td>x</td><td>-4</td><td>-2</td><td>0</td><td>2</td></tr><tr><td>$g(x)$</td><td></td><td></td><td>-1</td><td></td></tr></table></td></tr>
<tr><td>11</td><td>Which is true for two values of x (where $x_1 < x_2$) on the function $F(x)=mx+b$, where m= -1?
a) $F(x_1) < F(x_2)$
b) $F(x_1) > F(x_2)$
c) $F(x_1) = - F(x_2)$
d) not enough information given, need equation of function or point values
e) none of the above</td></tr>
</table>

12 Find the equation of the line below.

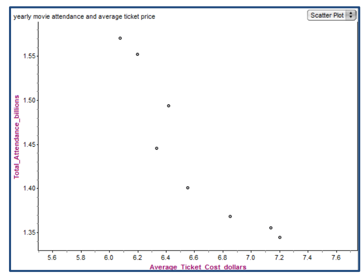


13 There are two waterslides. The first drops 12 feet for every 4 feet you move forward. The second drops 10 feet for every 4 feet you move forward. If both are the same length, which would be a faster ride? Explain your answer.

14 The following graph represents the amount Freda is paid in terms of the hours she works. Determine how many hours Freda would have to work to earn $180. Show your work.



15 The data in the scatterplot shows the relationship between the price per ticket at a movie theater and the number of people who attended the movie.

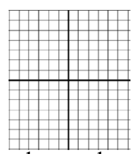


a) Use the data to predict the total attendance for each average ticket cost:
i. $6.30 ii. $6.70

b) Sketch a line on the graph above that you think represents the data well.

c) What can you say about the relationship between average ticket cost and total attendance?

16 Sketch the graph of a function with a negative slope.



17 The teachers at Riverdale High are gathering data related to students and how many pets they have. They are talking to students in groups of 10 to gather the data. The table below shows their findings.

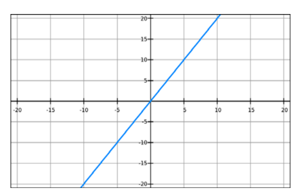
Number of Students	10	20	30	40	50	60	70	80
Total Number of Pets the Students Have	31	58	92	121	148	179	210	241

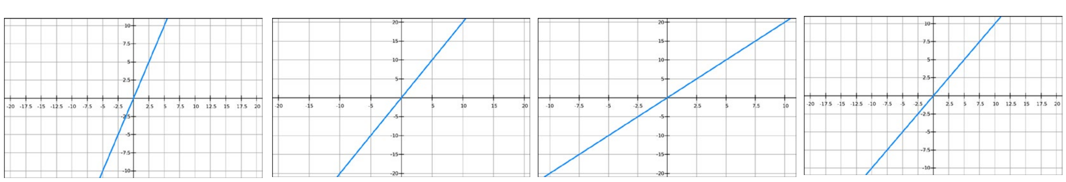
a) Would it be reasonable to use a linear function with slope of 3 to represent the following data set? Explain your answer.

b) What would the slope represent for this data set, in terms of the real life situation?

18 Given a function $y = -2x + b$, find the coordinates of a point (x, y) on the graph of this function given the information that $y = 1$ when $x = 1$.

19 Consider the graph to the right. Determine if each of the graphs below represents the same function or a different function than the graph to the right. Explain each answer.





A student response to a single item was the unit of analysis. With 231 students responding to 19 items, there should have been 4389 item responses. However, 255 of the responses were blank, which left a total of 4134 responses that were analyzed. All student responses were coded as pairs using the coding scheme displayed in Table 1 for a) presence of one of ten slope conceptualizations and b) a corresponding *visual* or *nonvisual* approach. A student's response to an item may have been coded as more than one conceptualization (i.e., *ratio*, *behavior indicator*, *steepness*, *constant parameter*, *determining relationships*) and one or both approaches (i.e., *visual*, *nonvisual*).

Findings

To address research question 1, we consider the conceptualizations of and approaches to slope most common in students' responses across all items. Table 3 below displays the conceptualization-approach counts for the 4134 responses to all 19 items. Note that only 3066 of the responses were noted as showing evidence of any slope conceptualization. The totals in the last row of Table 3 illustrate that *ratio* was the most common conceptualization and was used in approximately one-half of all responses (n=1503). *Ratio* was followed in frequency of use by *behavior indicator* (n=797), with *steepness* (n=319), *constant parameter* (n=271), and *determining relationships* (n=170) used much less frequently.

Table 3. Slope Conceptualization and Approach Use by Item and Overall

Item	*Ratio*			*Behavior Indicator*			*Steepness*			*Constant Parameter*			*Determining Property*			*Total*
	*V**	*N*	*B*	*V*	*N*	*B*	*V*	*N*	*B*	*V*	*N*	*B*	*V*	*N*	*B*	
1	44	35	13	34	7	0	59	6	0	12	5	0	1	0	0	216
2	22	41	8	11	2	0	4	2	0	12	0	0	1	0	0	103
3	29	135	16	0	0	0	2	1	0	3	0	1	0	0	0	187
4	6	7	0	8	1	0	5	0	0	36	25	4	2	3	0	97
5	107	15	4	82	12	0	33	5	0	15	4	1	3	0	0	281
6	26	38	4	11	18	0	0	0	0	3	6	0	0	0	0	106
7	16	185	5	0	0	0	0	0	0	0	0	0	0	0	0	206
8	9	16	0	7	1	0	0	0	0	1	10	0	6	53	0	103
9	101	11	6	118	1	0	7	0	0	35	0	0	4	1	0	284
10	10	100	2	2	1	0	0	0	0	1	13	0	5	88	0	222
11	0	3	0	4	52	0	0	1	0	1	0	0	0	1	0	62
12	18	3	0	5	1	0	79	4	0	5	0	0	0	0	0	115
13	41	82	11	4	2	0	66	38	1	8	1	0	0	0	0	254
14	65	32	0	2	0	0	1	0	0	1	0	0	0	0	0	101
15	23	17	0	179	0	0	0	0	0	12	0	0	0	0	0	231
16	6	3	0	198	0	0	0	0	0	23	0	0	0	0	0	230
17	3	103	1	5	25	0	0	1	0	1	29	0	0	0	0	168
18	3	4	0	0	1	0	0	1	0	2	1	0	0	0	0	12
19	60	13	1	3	0	0	2	1	0	0	0	6	1	1	0	88
Sum	589	843	71	673	124	0	258	60	1	171	94	12	23	147	0	3066
Total		1503			797			319			271			170		

*V = *visual* only, N = *nonvisual* only, B = both

Of the 3066 coded responses, 1714 were coded with a *visual* approach only, 1268 were coded with a *nonvisual* approach only, and 84 were coded with both. Although the *visual* approach was the most common overall, the distribution across conceptualizations varied. For the *ratio* and *determining property* conceptualizations, *nonvisual* approaches were more common. *Behavior indicator* use was heavily dominated by the *visual* approach, and *steepness* and *constant parameter* were also more often used with a *visual* approach. Only a small number (n=84) of the 3060 coded responses used both *visual* and *nonvisual* approaches, and the vast majority of these (n=71) occurred within the *ratio* conceptualization

To address research question 2, we consider if students consistently use the same conceptualizations of and approaches to slope or if they vary their uses based on specific tasks or questions. Since certain tasks might elicit particular ways of thinking about slope, we first consider only the first four open-ended items of the slope instrument, which were intended to trigger student use of multiple slope conceptualizations and which numbered 897 (231 students by 4 items less 27 blank responses). All five conceptualizations were found in these four items. Examining only the first four open-ended items, the order of prevalence of conceptualization was not the same as with the entire instrument. *Ratio* was still the most prominent (n=356), followed by *constant parameter* (n=98), *steepness* (n=79), *behavior indicator* (n=63), and *determining relationships* (n=7). For these open-ended items, the approaches were almost evenly split between *visual* (n=291) and *nonvisual* (n=270). A relatively large number of the open-ended items (n=42) included both *visual* and *nonvisual* approaches, such that exactly half of the responses across all items that incorporated both approaches occurred on these four open-ended items.

Table 3 provides additional information about the conceptualization-approach pairs (henceforth simply called "pair codes") found in student responses for each individual item. No item was coded with all ten pair codes; however, items 1, 2, 4, 5, 9, 11, and 19 were coded with all five slope conceptualizations. Items 5 and 9 generated the most pair codes (n=281 and n=284, respectively) and were the items for which the conceptualization code usages was noted as the most balanced.

Some items triggered specific conceptualizations. For example, responses to item 7 were only coded as the *ratio* conceptualization, with 206 pair codes involving *ratio*. However, when considering items that triggered at least 100 pair codes in a specific conceptualization category, the majority received pair codes involving the *ratio* conceptualization (i.e., n=180 for item 3, n=126 for item 5, n=206 for item 7, n=112 for item 10, and n=107 for item 17). Two items 15 and 16 received 179 and 198, respectively, coded pairs for *behavior indicator*. Two items merited over 100 pair codes for two conceptualization categories (i.e., item 9 had 118 *ratio* coded pairs and 119 *behavior indicator* coded pairs and item 13 had 134 *ratio* coded pairs and 105 *steepness* coded pairs). Item 18 didn't trigger much slope conceptualization use with only 12 codes noted in responses. This might be expected since, despite involving a linear function, a student did not necessarily need to think about slope to generate the point's coordinates.

When considered by item, 12 items (items 1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 17, 19) had at least one response that triggered both *visual* and *nonvisual* approaches within a conceptualization. However, even the most frequent occurrence of this was only 8 times for item 1. In fact, only 56 of the 231 students used both *visual* and *nonvisual* approaches within any of the five conceptualizations in their responses, and 53 of those 56 used both *visual* and *nonvisual* approaches when using the *ratio* conceptualization.

Discussion

The results suggest that *ratio* was the most common conceptualization used. This is not entirely surprising, as students often rely on the mnemonic device "slope is rise over run" and graphical interpretations (Walter & Gerson, 2007), and frequently recall slope as a number to be found using $m = \frac{y_2 - y_1}{x_2 - x_1}$ (Nagle, Martinez-Planell, & Moore-Russo, 2019). *Ratio* also appears to be a versatile conceptualization for students. Not only did it appear within responses students' responses on every item, it also was well distributed among *visual*, *nonvisual*, and both approaches. Also, it was by far the most common of the five conceptualizations for students used both *visual* and *nonvisual* approaches. The patterns in the student responses support the notion that particular items trigger certain conceptualizations for slope, and that items trigger certain conceptualization-approach pairs.

Although both *visual* and *nonvisual* approaches were used, the results showed the distribution was not particularly balanced. Approaches to *ratio* were relatively balanced, with both *visual* and *nonvisual* approaches being used quite often. *Determining property* was used mostly with a nonvisual approach where the remaining three conceptualizations (*behavior indicator, steepness,* and *constant parameter*) all tended to be approached *visually* more often than *nonvisually*. However, perhaps the most interesting result related to the approaches was that relative infrequency of students' using both *visual* and *nonvisual* approaches within a single *conceptualization*. Neither *behavior indicator* nor *determining property* involved both *visual* and *nonvisual approaches* for a single response on any item. Furthermore, *steepness* was only used both *visually* and *nonvisually* on one only response across all the items, and *constant parameter* only twelve times. These results are important because they suggest that although students might approach slope in both *visual* and *nonvisual* ways using all five of the slope conceptualizations, they may not connect those approaches and see them as complementary ways of thinking about a situation or task. In the mind of the student, the *visual* and *nonvisual* approaches may not just be seen as different approaches to the same slope conceptualization, but rather as completely distinct conceptualizations of slope that are appropriate to use in different settings and contexts.

Students' isolation of *visual* and *nonvisual* approaches to slope conceptualizations is also critical as they begin studying derivatives in calculus. For instance, the isolation of *visual* and *nonvisual* approaches to the *behavior indicator* conceptualization of slope is concerning in regards to the importance of students working fluently with this idea when making sense of the first derivative test and the relationships between a function and its derivative both analytically and graphically in a calculus setting.

Implications, Limitations, Future Work

This study expands upon previous work related to students' conceptualizations of slope by looking at the prevalence of their *visual* and *nonvisual* approaches to the various slope conceptualizations. Further, we used four of the items from a previous study, but with a larger sample size (Nagle, Moore-Russo, Viglietti & Martin, 2013). One of the limitations of the study stems from the design as a written assessment where follow-up questions could not be asked. In particular, it would be helpful to further explore whether students would combine their *visual* and *nonvisual* approaches to problems if prompted to do so, or whether these ideas remain isolated for the students.

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A Preservice Teacher's Emerging Concept Image of Function: The Case of Sofia

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For over thirty years, the mathematics education research community has investigated individuals' function conceptions and advocated the importance of a strong function conception. This is particularly important in the case of preservice secondary mathematics teachers (PSMTs) as their knowledge of function will influence their future teaching of function. This study explores how one PSMTs', Sofia's, function concept image changes when she engages with research-based tasks and explorations designed to elicit cognitive conflicts related to function conceptions. Thematic analysis methods were applied to open-ended, tasked-based pre- and post-interviews designed to evoke aspects of individuals' function concept image. Between the pre- and post-interviews, Sofia participated in a course that implemented 11 explorations on functions. Comparing the themes identified in her pre- and post-interviews, researchers identified four changes in Sofia's function concept image. These changes relate to beliefs about functions and equations, non-numerical functions, functions and graphs, and the vertical line test.

Keywords: function, concept image, preservice secondary mathematics teachers

While the study of many mathematical topics rests on the function concept (Dubinsky & Harel, 1992), it remains a difficult topic for many undergraduate students including preservice secondary mathematics teachers (PSMTs) (e.g. Carlson, 1998; Dubinsky & Wilson, 2013; Ohertman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989; Zandieh, Ellis, & Rasmussen, 2017). This difficulty with the function concept is particularly important in the case of PSMTs as a teacher's subject matter knowledge affects both the content taught and how a teacher teaches the content (Even, 1993; Stein, Baxter, & Leinhardt, 1990). In fact, Sten et al. (1990) found that a middle school teacher possessing a rule-based conception of function depicted function to his students as merely an arithmetic operation. Watson and Harel (2013), however, found that when teachers experience studying functions at high level they are able to teach it in a way that lays the foundation for more advanced understanding. It is then essential that PSMTs gain experiences that provide the opportunity to deepen their understanding of functions.

One potential way to provide these opportunities is through a content course consisting of research-based activities designed to elicit cognitive conflicts associated with the function concept. However, limited research is available on how a content course with such activities might influence PSMTs' conceptions of function. In this study, we investigate the following research question:

> How does one PSMT's concept image of function change when she engages with research-based tasks and explorations designed to elicit cognitive conflicts related to function conceptions?

Theoretical Framework

This study draws on Tall and Vinner's (1981) theory of *concept definition* and *concept image*. A concept definition consists of the set of words an individual uses to articulate a concept, and this may differ from the formal concept definition – the definition accepted by the mathematical community (Tall & Vinner, 1981). Although many PSMTs hear and see the formal concept definition of function throughout their secondary studies, they may not actually apply it within a problem context. Tall and Vinner (1989) instead suggest that individuals employ their concept image. A concept image consists of all the memories an individual associates with a particular concept name including examples, nonexamples, relationships, visual representations, impressions, and experiences. Since an individual's concept image is formed around the associated memories, the resulting concept image may not always be coherent, may contain potentially conflicting components, and different contextual situations may evoke different memories or aspects of one's concept image (Tall & Vinner, 1981).

A concept's formation consists of an ongoing interplay between an individual's personal concept definition and concept image. In this interplay, the form of words an individual uses to describe the concept may be affected by associated examples and nonexamples and vice versa. However, once an individual forms a concept image the concept definition may be dispensed (Vinner, 1991). Understanding of a concept then centers on an individual's concept image, and it is essential PSMTs gain experience with a wide variety of examples and nonexamples of function to help form their concept images. In this study, we investigate one PSMT's function concept image as she engages with function-related tasks and explorations.

Research Literature

Research on function conceptions of secondary students, university students, and teachers can generally be divided into two categories: conceptions about function as an entity and characteristics individuals associate with functions. We will discuss each of these categories in the following sections.

Conceptions of Function as an Entity

While investigating conceptions of functions and linear transformations, Zandieh, Ellis, and Rasmussen (2017) identified five clusters of metaphorical expressions their linear algebra students used when reasoning about functions and linear transformations: *input/output*, *traveling*, *morphing*, *mapping*, and *machine*. These metaphorical expressions reveal the types of entities students associate with function. Considering the *input/output* cluster, the associated student responses portray function as an entity that accepts inputs and returns an output. The *traveling* cluster contains statements that depict function as an entity that sends or moves one object to another location. *Morphing* is characterized by references to function as an entity that transforms one object into another. Describing function as a relationship or correspondence entity between two objects is defined as the *mapping* cluster, and the *machine* cluster refers to depictions of function as machine-like-entity that causes one object to change into another.

Vinner and Dreyfus (1989) also identified six types of entities that college students and junior high school teachers associate with the function concept: *correspondence*, *dependence relation*, *rule*, *operation*, *formula*, and *representation*. Similar to Zandieh et al.'s (2017) *mapping* cluster, the *correspondence* categorization refers to descriptions of function as a correspondence, potentially arbitrary, between two sets. Students in this study also described function as a *dependence relation* entity in which there exists some sort of dependence or connection between two variables. *Rule* encompasses descriptions of function as a rule entity with some sort of regularity. The *operation* classification consists of statements that describe function as acting on a number by algebraic operations to produce an image. Vinner and Dreyfus (1989) also found that students describe function as *formula* entity which includes references to function as a formula, algebraic expression, or equation. Finally, *representation* includes descriptions of function as graphs or entities with function notation.

Conceived Characteristics of Function

In addition to conceptions about the types of entity associated with function, students and teachers alike also develop conceptions about the characteristics of function based on their experiences. These conceived characteristics include beliefs about arbitrariness, univalence, and functional notation. Particularly, Even (1993), Carlson (1998), and Vinner and Dreyfus (1989) report that college students and PSMTs believe function should be smooth, continuous, and not "too weird". These beliefs demonstrate a limited understanding of the arbitrary nature of function including potential assumption that functions are graphable with some regularity. Sierpinska (1992), Carlson (1998) and Breidenbach, Dubinsky, Hawks, and Nichols (1992) report an additional belief about arbitrariness: secondary and undergraduate students believe a function should be definable by a single algebraic formula. This conceived characteristic indicates an assumption that all functions are defined on numerical sets and by known rules.

Regarding beliefs about univalence, Even (1993) found that PSMTs understand that univalence distinguishes functions from other types of relations. However, Norman (1992) noticed that secondary teachers frequently apply the vertical line test to determine if a relation is a function even in non-Cartesian coordinate situations. PSMTs also tend to provide students with the vertical line test as a rule to follow perhaps because they do not understand why univalence in important (Even, 1993). Another conception about univalence is related to the one-to-one property. Breidenbach et al. (1992) and Dubinsky and Wilson (2013) report that students believe functions must one-to-one. This may lead to function concept images that do not include constant functions.

Students may also develop conceptions about functional notation. For instance, Vinner and Dreyfus (1989) found that college students are unfamiliar with the relationship between function notation and conceptual aspects of function. Specifically, Sajka (2003) found that a secondary student perceives 'f' as a label for function that does not carry any meaning itself; interprets the symbol '$f(x)$' and corresponding algebraic expression as the formula of the function; and confuses functional notation, $f(x+y)$, with the distributive property, $fx+fy$.

Methodology

This study took place at a large, public, urban university in the southwestern United States with an enrollment of over 42,000 undergraduate and graduate students. Due to the large enrollment of Hispanic students (greater than 30% of the student body), the university carries a U.S. Department of Education Hispanic Serving Institution designation. The university is also described as one of the most diverse national universities in the United States.

The information presented here is part of a larger study consisting of 27 PSMTs enrolled in a mathematics course with a second-semester calculus prerequisite. Materials used in this course, *Explorations on Functions and Equations* (*EFE*), were developed as a part of the *Enhancing Explorations in Functions for Preservice Secondary Mathematics Teachers* Project which was partially funded by the United States National Science Foundation. The *EFE* materials include 11 research-based tasks and explorations for use in mathematics courses for PSMTs as well as instructor materials to assist mathematicians and instructors in implementing the materials in an inquiry-based, active learning environment. Specifically, objectives of the *EFE* include deepening and broadening function-related mathematical content knowledge from school algebra to calculus by exploring relevant topics in an inquiry-based learning situation; and making connections between college mathematics and secondary school mathematics (Álvarez, Jorgensen, & Rhoads, 2019).

Implementation of the *EFE* in the course spanned the first 10 weeks of the 15-week semester. As a part of the larger study, a copy of all participants' written work was collected including written explorations, daily journals, homework, labs, exams, and a midterm project. Each class using the *EFE* materials was also video-recorded using four cameras that were each focused on a different student group.

Of the 27 students who consented to participate in the larger study, we extended invitations to 12 students to participate in video recorded, individual, pre- and post-interviews. Ten students accepted the invitation and completed a pre-interview. All 10 students who participated in the pre-interview remained in the course and were invited to a post-interview; however, only seven students chose to return for a post-interview. The pre-interviews took place within the first three weeks of the semester, and the post-interview occurred between the eleventh and fourteenth weeks of the semester. Questions in the pre- and post-interviews consisted of the same open-ended, task-based questions designed to evoke students' concept definitions and concept images of function as well as draw out their understanding of key ideas associated with function. Examples of questions include (a) how would you define function; (b) list all the distinguishing characteristics of function that you can think of; and (c) do the mathematical terms function and equation mean the same thing. For each of the questions, follow-up questions were used to encourage students to explain and clarify their answers. The post-interview also included questions related to students' perceptions of the course.

Following the completion of the pre- and post-interviews, the video recordings were linked to a pseudonym and transcribed. For the seven students who completed both a pre- and post-interview, we applied inductive thematic analysis methods to the task-based portion of the interviews in order to identify, analyze and report themes of students' function concept images. Inductive thematic analysis seeks to generate codes for data "without trying to fit it into a pre-existing coding frame (Braun & Clarke, 2006, p. 83). Using Nvivo qualitative analysis software, we generated initial codes by analyzing each participant's pre- and post-interview separately for instances that revealed some aspect of the concept image. We then reviewed the instances of each code across all students and interviews and re-coded instances as needed. The remaining codes were then organized into concept image themes, defined and named, and then used to compare aspects of each student's concept image in the pre- and post-interview.

In this study, we use the pre- and post-interview of one participant, Sofia, to report any changes in her concept image of function after interacting with the 11 research-based tasks and explorations in the *EFE*. Sofia self-identifies as a Latina whose native language is English. Prior to the start of this course, the only mathematics courses she had completed at the university level consisted of three-semester calculus sequence. At the time of the study, Sofia was concurrently enrolled in two other mathematics courses, Introduction to Proofs and Introduction to Matrices and Linear Algebra.

Results

Through the coding process described above, we identified over 20 themes, or aspects of Sofia's function concept image, between her pre- and post-interviews. While all of these themes are interesting and reveal important aspects of her concept image as a whole, for this report we will focus on eight concept image themes that, when compared between her pre- and post-interview, depict a change in her concept image: *All Functions can be Equations* (*AFE*), *Functions are Not Equations* (N_{AFE}), *Defined on Numerical Sets* (*DNS*), *Defined on Non-Numerical Sets* (N_{DNS}), *Graphs or Graphable* (*GG*), *Not all Graphable* (N_{GG}), *Vertical Line Test*

(VLT), and *Not Vertical Line Test* (N_{VLT}). Figure 1 depicts the themes that appeared in Sofia's pre-interview and those indentified in her post-interview.

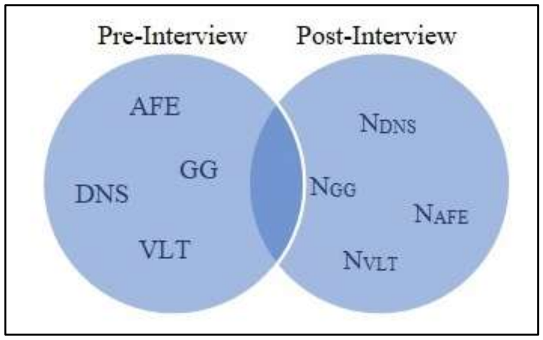

Figure 1. Aspects of Sofia's concept image in her pre- and post-interviews.

The theme *AFE* encompasses statements made by participants indicating their concept image includes the idea that all functions are or can be represented as equations. For instance when discussing whether or not function and equation mean the same thing, Sofia states in her pre-interview, "So like all functions can be equations, but all equations can't be functions." This explanation suggests that Sofia's concept image of function during the pre-interview includes a belief that all functions are equations or can be written as equations. The *AFE* theme was not, however, identified in any of her post-interview statements; rather, the N_{AFE} theme emerged in her post-interview. The N_{AFE} theme represents statements illustrating a concept image aspect in which functions are not equations. Specifically, Sofia recalls once believing that functions are equations, "but then [the students] said that was wrong in class, but [she] never really got why other than the two variable thing." Although she does not fully understand why a function is not an equation, Sofia's experience in the course has partially altered her concept image of function.

DNS represents statements that imply a concept image where functions must be defined on numerical sets, and *GG* includes comments suggesting a concept image in which all functions either are graphs or can be graphed on a coordinate plane. During her pre-interview, the researcher asked Sofia if it would be possible to have a function whose domain was not defined on a numerical set. Sofia's initial reaction is that this would not be possible. She then tries to think of an example for such a function:

> Well, I'm thinking of like graphs that I see or that I know, and it's just like the graph like, the domain is just all the x-values. But if you have it on a graph, that'd have to be like real numbers… There'd have to be real number in order to graph it. And so it can't. I don't think there's any.

This excerpt indicates that Sofia's concept image at the time of her pre-interview does not allow for any functions that cannot be graphed on a coordinate plane or those that are defined on non-numerical sets.

Another theme from Sofia's pre-interview is *VLT* which includes interview responses that suggest a concept image where all functions pass the vertical line test. In both interviews, the interviewer presented Sofia with five statements and asked which statements represented valid mathematical definitions of function. One of these statements defined function as a graph that passes the vertical line test. In her pre-interview, Sofia accepted this statement as a valid mathematical definition of function and says, "That's one of the things that we learn in, when we were taught functions, right, is that it has to like pass that line test." The acceptance of this statement reveals that Sofia's concept image in the pre-interview may only includes functions that pass the vertical line test.

While *DNS, GG,* and *VLT* all appeared in Sofia's pre-interview, none were identified in the post interview. In fact, three quite different aspects of her concept image emerged: N_{DNS}, N_{GG}, and N_{VLT}. Sofia indicated in the pre-interview that functions are only defined on numerical sets. When posed the same question about if a function could be defined on a non-numerical set in the post-interview, Sofia immediately recalls, "[The instructor] used one in class with like people and eye color… I guess people to eye color is a function." This interaction is an example of a statement coded as N_{DNS} – responses that depict that functions defined on non-numerical sets are included in an individual's the concept image.

The N_{GG} and N_{VLT} themes also appear only in Sofia's post-interview. N_{VLT} is characterized by indications that within the concept image some functions do not pass the vertical line test, and N_{GG} represents statements implying that the concept image includes functions that cannot be graphed on a coordinate plane. For instance, Sofia explains why a particular example she provided is indeed a function:

Sofia: …like it only has one output for each input and it passes that vertical test.

Interviewer: Ok. Does it have to do both? Does it have to pass the test and…

Sofia: No, because we said that not all functions are graphable. So I think as long as an input only has one output then you're good because like if it's graphable then it should pass the test. But if it's not, then it should at least have only one output for each input.

Interviewer: Could you give me an example of something like that?

Sofia: I don't know anything that's not graphable. Like off the top of my head I can't…

This conversation suggests that, at the time of the post-interview, Sofia's function concept image no longer requires all relations to pass the vertical line test in order to be considered a function. It appears this is a result of the fact that her concept image now allows for functions to not be graphable on a coordinate plane.

Discussion

In this study, we discuss four concept image themes that we identified in Sofia's pre-interview. As in the of Even's 1993 study and Vinner and Dreyfus's 1989 study, Sofia's function concept image included an expectation that "functions are (or can always be represented as) equations or formulas" (Even, 1993, p. 111). This is represented in our study by the theme *All Functions are Equations* (*AFE*). The other three themes we discuss from her pre-interview are potentially interdependent: *Defined on Numerical Sets* (*DNS*), *Graphs or Graphable* (*GG*), and *Vertical Line Test* (*VLT*). Specifically, the aspect of Sofia's concept image that suggested all functions are graphable and pass the vertical line test restricted her concept image from including functions defined on non-numerical sets. Carlson (1998) reports that college algebra students believe "all functions must be continuous" which suggests an underlying that belief, similar to Sofia's, that all functions are graphs or graphable (p. 137).

The themes presented in this report do not encompass all the aspects of Sofia's concept image identified within her pre- and post-interviews. However, these themes do depict particular changes in her concept image that occurred while she was enrolled in a course using the *EFE* materials. These changes can be linked to two specific experiences motivated by the *EFE*: an experience considering whether or not a expression of the form $f(x) = x$ should be considered an equation, and an experience considering a relation that maps people to their eye colors.

When considering whether or not function and equation mean the same thing in the post-interview, Sofia remembers deciding in class that functions are not equations, but the only distinction she recalls has to do with equations having “two variables”. At another point in the interview, she explains that her two variable distinction originated from a specific example in class. This example took place within the *EFE* lesson “Functions and Equations” in which each group was asked to share three examples of equation on the board. The instructor also added a few potential examples including “$f(x) = x$”. Students then used the definition of equation (i.e. a mathematical statement that asserts equivalence between two quantities) provided in the lesson to decide in their groups which proposed examples actually constitute as equations. Divided conclusions on $f(x) = x$ resulted in a whole-class discussion. In this discussion, some students asserted that $f(x)$ is not a quantity and therefore $f(x) = x$ cannot be considered an equation.

Sofia remembers this interaction with her classmates and draws on this experience to explain why she believes functions are not equations:

> …then when we started going into more details about equations, and um we had all our examples up and [the instructor] had a paper of all of our examples, and we decided that, I think it was some $f(x)$ function, that it wasn’t an equation because it didn’t have the relationship with two variables.

Although misunderstands the conclusion of why her classmates did not consider $f(x) = x$ to be a function, she does remember the episode, and this experience alters her post-interview function concept image.

Another experience that impacted Sofia’s concept image occurred as a result of an exploration in the *EFE* lesson “What is a function?” At the start of this lesson, student groups generated all the defining characteristics of function they could think of. The groups then shared their defining characteristics on the board and were given the opportunity to agree with or offer a counterexample for the proposed characteristics. As groups examined these statements, they were also asked to consider a few questions including, “Are all functions graphable on the Cartesian coordinate plane?” While these conversations occurred, the instructor added to the board for consideration an example of a relation mapping people to their eye colors.

This specific conversation with her group as well as the example offered by the instructor altered Sofia’s concept image of function. Particularly in the post-interview, she drew on the example of a relation mapping people to eye color to determine that functions can be defined on non-numerical sets. She also remembers discussing with her group whether or not all functions are graphable on a coordinate plane and deciding they were not. Altering this aspect of her concept image then allowed for her to alter the portion of her concept image related to the vertical line test. However, it is interesting, in light of the non-numerical example of function she provided, that she was unable to provide an example of a function that cannot be graphed. This supports the Tall and Vinner’s (1981) suggestion that aspects of an individual’s concept image may be disjoint and different contexts may evoke varying portions of one’s concept image.

Two particular experiences resulted in changes to Sofia’s function concept image. Both of these events were prompted by explorations in the *EFE* materials designed to elicit cognitive conflicts related to function conceptions. These results naturally lead to important questions for mathematics education community about course experiences. Specifically, would inserting more experiences that intend to promote cognitive conflict contribute to improving students’ function concept images? What types of experiences positively impact an individual’s function concept image? How could we interpret whether or not a course experience contributes to or detracts from students’ function concept image? The change we observed in Sofia’s concept image suggests that classroom experiences can positively affect an individual’s concept image. However, further research is needed characterize these types of experiences.

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A Local Instructional Theory for the Guided Reinvention of a Classification Algorithm for Chemically Important Symmetry Groups

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In this presentation I will describe the beginning phases of a local instructional theory that has resulted from a design experiment focused on students work with symmetry groups in the context of chemistry. This local instructional theory will consist of both a generalized instructional sequence intended to support the guided reinvention of a classification algorithm for molecular structures, and the theoretical and empirical rationale for the given sequence. I will also describe the various teaching experiments and how each informed the development of the instructional sequence. The mathematical activity of the participants will be used to describe key aspects of the reinvention process.

Keywords: Realistic Mathematics Education, Abstract Algebra, Group Theory, Symmetry

Group theory, in particular symmetry groups, is an essential tool for chemists when interpreting experimental data and in predicting the macroscopic properties of materials based on molecular structure. The use of symmetry groups to describe molecular structure is often formally introduced in undergraduate inorganic chemistry courses where students are asked to categorize relatively simple molecules by their symmetries in order to find their symmetry group. While students in an inorganic chemistry course are given a pre-existing flowchart to help them determine specific point groups, the focus of the design study reported here was to investigate how students might reinvent this classification system for themselves. Therefore, the primary research question of the study was, *"How can students be supported in reinventing an algorithm for the classification of chemically important symmetry groups?"*

Background and Theoretical Framing

Abstract algebra, the larger field of mathematics in which group theory is taught is notoriously difficult for students (Larsen, 2010; Leron, Hazzan, & Zazkis, 1995; Weber & Larsen, 2008). Many of the recorded student difficulties within abstract algebra can be partially attributed to the abstract nature of the content of the course (Hazzan, 1999, 2001). Investigating the symmetry of shapes has proved to be a productive introduction for students learning abstract algebra. In fact, the investigation of the symmetries of an equilateral triangle has been successfully used as a starting point in the research-based, inquiry oriented abstract algebra course entitled *Inquiry Oriented Abstract Algebra* (*IOLA*) (Larsen, 2016). The results reported here are from a recent effort to engage students in the richness of group theory while simultaneously gaining experience with its applicability to other fields.

Realistic Mathematics Education (RME) serves as the underlying instructional design theory for this study and is built on the theoretical perspective that mathematics is first and foremost a process, a human activity (Gravemeijer & Terwel, 2000). Freudenthal (1971) described this activity as

> an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be matter from reality which has to be organized according to mathematics patterns if problems from reality have to be solved. (p. 413)

Through the activity of *mathematizing* students can be guided in reinventing particular mathematical concepts, as opposed to learning the topic as a ready-made or previously discovered theorem. Therefore, overarching goals of RME include discovering how to provide students an opportunity to reinvent mathematics and also how to support them throughout the activity of mathematizing so that ultimately the mathematics that they develop is experienced as developing common sense (Gravemeijer, 1998).

The theoretical framework of RME has three accompanying design heuristics, which can serve as both guiding principles for instructional design and as a guide for further analysis. These heuristics include the reinvention principle, emergent models, and didactic phenomenology (Gravemeijer, 1998). A local instructional theory, LIT, (Gravemeijer, 1998) describes a generalized roadmap for student reinvention of a particular mathematical concept, in which students feel ownership over the mathematical concepts they investigate. First a context was chosen that offered an opportunity for the students to begin using their own intuitions and experiences to develop informal highly context-specific solution strategies (Gravemeijer & Doorman, 1999) which could later be used in a more formal mathematical reality. Didactical phenomenology is then used to focus on the relationship between a mathematical content and the "phenomenon" it describes and analyses, or, in short, organizes (Gravemeijer, 2004). In this sense the heuristic helps at a global level to inform a good starting point for the reinvention process. Didactical phenomenology was also used at a more local level during the teaching experiments to drive the study by helping to identify ways in which I as the researcher could support the students transform their informal approaches to the molecules into more powerful formal arguments about molecules in general (Larsen, 2018).

Lastly, the design heuristic of emergent models is useful in describing both the character, and the process of evolution of student's formal mathematical knowledge from an initial informal understanding. In Gravemeijer's (2002) description of emergent models he highlights

> ...three interrelated processes. Firstly, there is the overarching model, which first emerges as a model of informal activity, and then gradually develops into a model for more formal mathematical reasoning. Secondly, the model-of/model-for transition involves the constitution of some new mathematical reality – which can be called formal in relation to the original starting points of the students. Thirdly, in the concrete elaboration of the instructional, there is not one model, but the model is actually shaped as a series of symbolizations. (p. 3)

Methods

In order to develop the local instructional theory I conducted a design experiment to produce an initial model of successful innovation (Collective, 2003). Through a series of three teaching experiments, including a pilot study, I was able to test and refine multiple iterations of the LIT. The results shared in this contributed report include findings from all three experiments as they each informed the local instructional theory through the development process. Each of the teaching experiments were conducted with pairs of mathematics students who had no apriori experience with group theory in the context of chemistry. While none of the participants had any formal training in chemistry, the mathematical backgrounds of the participants varied across the three experiments. All students were given pseudonyms and compensated monetarily for their time. Data consisted of video recordings of each episode and all written work was also collected.

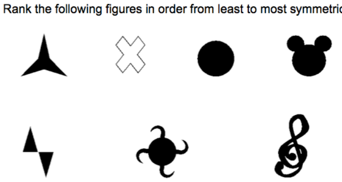
The goal of the pilot study was to explicate *a way* in which a pair of students could successfully classify chemically important point groups and so it served as a kind of an existence proof (for more detail, see Bergman & French, 2019). To better ensure that the students would be successful for the initial attempt, the pilot study (PS) was conducted with a pair of mathematics education graduate students, Emmy and Felix, each of whom had completed a graduate level course focused on classifying groups of finite order. The two main teaching experiments in the study were conducted with pairs of undergraduates. The first teaching experiment (TE1) was conducted with a pair of undergraduate mathematics students, Arthur and Joseph-Louis, who had recently completed a traditional quarter long lecture-based 300-level introductory group theory course. The final teaching experiment (TE2) was conducted with a pair of undergraduate students, Ada an electrical engineering student and Sophie a mathematics student, neither of whom had any experience with group theory. Ada and Sophie had both recently completed a 200-level linear algebra course and were provided two days of selected lessons from the Inquiry Oriented Abstract Algebra (IOAA) curriculum (Larsen, 2016) as an introduction to group theory between step 1 and step 2 of the local instructional theory outlined below.

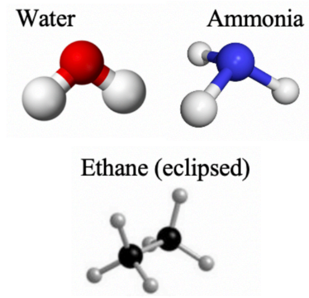
The mathematical activity and inscriptions produced in each of the teaching experiments have been subjected to multiple cycles of iterative analysis. The purpose of this analysis has been to further refine and articulate the local instructional theory informed by the participants approaches.

Results

For the presentation, I will describe the seven steps in the reinvention process and illustrate each with examples from the design experiment. These seven steps, organized into three phases which correspond to the level of student's mathematical activity from situated to formal, constitute the framework for my local instructional theory and represent a generalized instructional sequence by which a classification algorithm for chemically important point groups might be reinvented. For this proposal I will be focusing on the beginning steps of Phase 1. This generalized sequence builds on students' own mathematical activity as they investigate select ball and stick models. The overall local instructional theory is outlined in Table 1 below.

Table 1. A Local Instructional Theory for the guided reinvention of chemically important symmetry groups.

Phase	Step	Rationale/Purpose
1. Find the unique symmetry groups of specific molecules.	1. The Measuring Symmetry Task Rank the following figures in order from least to most symmetric. 	Before investigating symmetries in three-dimensional space students are given a set of planar figures to gain familiarity with identifying various symmetries, and to generate definitions for both symmetry and symmetry equivalence.
	2. Find the symmetry group for each molecule:	Students are given ball and stick representations of three selected molecules to begin their symmetry investigation in three-dimensional space.

		Each of the molecules have different symmetry groups which corresponding to different algebraic structures. The symmetry group for water can be expressed as a direct product of two cyclic groups, $\mathbb{Z}_2 \times \mathbb{Z}_2$, ammonia's symmetry group can be expressed as a semi-direct product of two cyclic groups, D_6, and lastly the symmetry group for ethane can be expressed with three generators as a direct product of a semi-direct product and a cyclic group, $D_6 \times \mathbb{Z}_2$.
2. Develop and describe a system for finding the symmetry group for a given molecule	3. What did we do? What did we use?	Students are asked to reflect on their own activity in Phase 1 so they can begin to articulate a strategy for identifying the symmetry group of any given molecule. As students reflect on their own activity, they begin to inscribe their activity in a sort of list of directions and/or a flow chart. This provides students an opportunity to begin to define particular terms being used, in particular something analogous to principal axis, horizontal and vertical reflections.
	4. Could we describe our method for someone else?	Eventually students are asked to create a model of their approach that is understandable by some unknown future user, preferably a chemist who doesn't want to have to do as much group theory. Students may choose to include an 'instruction manual' or 'glossary' with their approach. This provides students a need to focus on the precision of their terminology and the usability of their model.
3. Develop and describe a procedure for comprehensively and efficiently identifying the symmetry group of	5. Will it work for anything? Determine if your method will work for the following molecules: Ethane (staggered)	The goal of introducing ethane in a staggered configuration is not only to show students that there are more kinds of symmetry elements, but also that there are multiple ways to construct the same symmetry group. While the symmetry group of ethane in a staggered configuration is isomorphic to that of

any given molecule.	C_2H_6	ethane in an eclipsed position, ethane in a staggered position contains an inversion center which is the third and final primitive symmetry in three-dimensional space. The introduction of the inversion center forces students to revise and expand their system.
	6. Did we get everything? Can we prove it work for anything?	The purpose for this step is twofold; first students would need to argue that they have identified all possible flavors of symmetry in 3-dimensions. Next, students would want to prove that they have all possible combinations of symmetries.
	7. Special high order groups	The goal of discussing the special groups is to show the students that there are a few special kinds of groups of very high order. These groups are either linear with an infinite rotational subgroup or have more than one rotational axis of order 3 or higher.

Step 1: The Measuring Symmetry Task

There are two primary purposes for this task, the first is to provide students an opportunity to begin treating symmetries as objects, as opposed to a property of a figure, which is a necessary when combining symmetries as group elements. The second purpose is to have students generate definitions of both symmetry and symmetry equivalence. Chemistry education researchers have found that when students learn about symmetry they often struggle with both visualizing molecules in three dimensions and also determining the relevant symmetry operations (Flint, 2011). I found similar student struggles in the pilot study and TE1 when students began their initial investigations in three-dimensional space. When considering ball and stick models the only symmetry transformation that can be physically performed is a rotation, all others are strictly mental operations. This seemed to contribute to the students in both PS and TE1 creating definitions of symmetry that were heavily focused on rotations, as seen in the excepts below.

Emmy: So in this case, since these are like 3-d symmetry means if I had a shape, if I had it oriented like this (see Figure 1) I wanna do something (she rotates the model 180 degrees) so that it's in the same orientation?

Felix: I think so.

Emmy: Ok.

Figure 1. Emmy considers a 180-degree rotation of a water molecule while describing her definition of symmetry.

> *Arthur:* If I'm looking at this in like a chemistry way in where like at any one point this thing (water) is just randomly been rotated in some configuration here and we're just constantly having some random rotation or something, like this thing (ethane) would seem much more likely to end up in a symmetric configuration than this (water) one. So I would call it more symmetric.

The measuring symmetry task (Larsen & Bartlo, 2009) has been shown to be powerful for promoting student generated definitions of both symmetry and symmetry equivalence. I found similar success in this design experiment. It is important that students have working definitions of symmetry so that they know what to attend to when given the ball and stick models and it necessary to establish a definition of equivalence because there must be an agreement on the set of objects that will constitute group elements. The definition of symmetry equivalence also establishes the combination of two symmetries as a (closed) operation on the set of symmetries.

The measuring symmetry task asks students to rank a set of figures from most to least symmetric; then to count and ultimately define the very characteristics by which students were considering when determining rank. When students were given the measuring symmetry task in both TE1 and TE2 they generated powerful definitions of both symmetry and symmetry equivalence. These definitions, Figures 2 and Figure 3 below, were leveraged frequently when the students progressed to working with three-dimensional figures in Step 2.

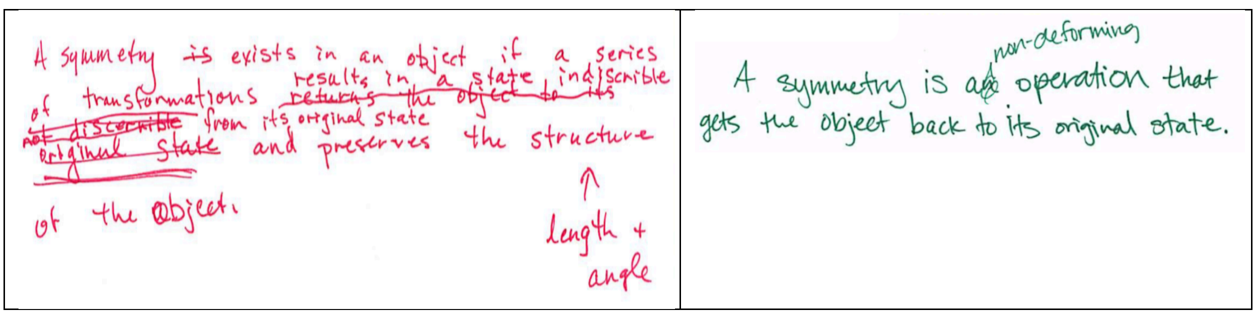


Figure 2. Student generated definitions of symmetry when beginning the Measuring Symmetries task from (a) Teaching Experiment 1 and (b) Teaching Experiment 2.

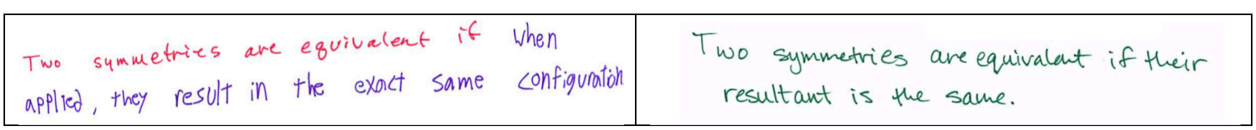


Figure 3. Student generated definitions of symmetry equivalence when beginning the Measuring Symmetries task from (a) Teaching Experiment 1 and (b) Teaching Experiment 2.

Step 2: Find the symmetry group for each of the following molecules; water, ammonia, and ethane.

The purpose for this task is first and foremost to give students experience investigating the symmetry groups of molecules with different group structures. Each of the selected molecules have different combinations of reflections and rotations, symmetries which the students already have some experience identifying from their work in Step 1. The task pushes students to describe the unique collection of symmetries for each molecule and organize them in a manner that helps them to determine what kind of group structure they will produce. I have found that in each teaching experiment the participants began their investigations with water followed by ammonia, and lastly ethane, in an eclipsed configuration.

When considering the symmetry of group for each molecule students start with some kind of inscription system that records the physical effect each symmetry has on the molecule as seen in Figure 4; which was often followed by a Cayley table as seen in Figure 5.

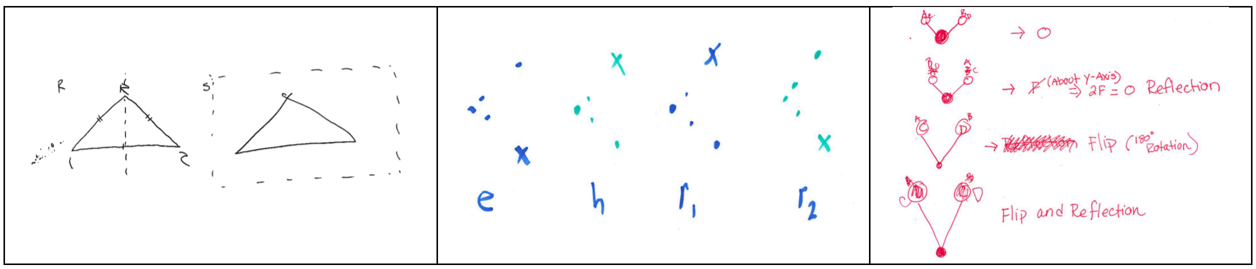


Figure 4. Student inscriptions recording the effects of various symmetries on a water molecule from (a) the Pilot Study, (b) Teaching Experiment 1, and (c) Teaching Experiment 2.

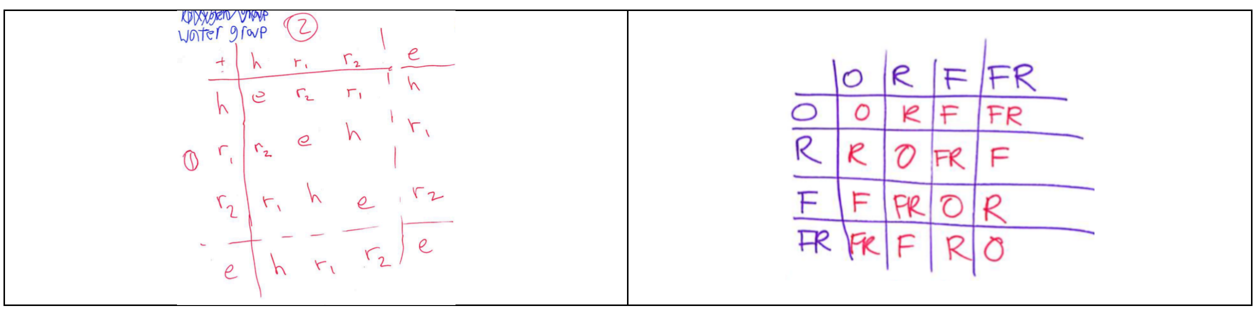


Figure 5. Student Cayley tables for the symmetries of a water molecule from (a) Teaching Experiment 1 and (b) Teaching Experiment 2.

Implications and Significance

The local instructional theory reported here has been tested with students with a wide range of mathematical backgrounds all of whom were able to engage with each task and successfully produce a classification algorithm capturing most chemical structures. Furthermore, all of the students involved expressed truly enjoying the experience and reported a sense of deeper understanding of the concepts of group theory due to their participation. Unfortunately, due to time constraints in this study, I have yet to gather evidence on how students might prove that their classification system is complete, steps 6 and 7 of the current LIT. So while a future study with additional iterations of teaching experiments are still necessary, the LIT reported here already shows potential to be implemented with all kinds of students in possibly a wide range of courses to help students experience the usefulness and applicability of group theory.

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Changing the Script: How Teaching Calculus using Team-Based Learning Misaligns with Students' Views of how Learning Mathematics Occurs

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Active learning has been shown to increase student success and improve student confidence, whereas ambitious teaching potentially leads to decreased student attitudes for learning mathematics. In this study, we examined how 18 Calculus I students' beliefs, attitudes, and expectations were met or challenged across a semester when taught using Team-Based Learning, an ambitious teaching method. Results indicate that aspects of the TBL design, including initial interaction with content outside class, reduced emphasis on instructor lectures, and complexity of the tasks may have contributed to students' negative attitudes. These negative attitudes interplayed with students' beliefs about their competence to learn mathematics on their own and their beliefs about the nature of mathematics and how mathematics should be learned.

Keywords: Calculus, Team-Based Learning, Student beliefs, Student attitudes

The transition of the study of mathematics from the high school to the college level proves challenging for many students. Whether the study of mathematics at the college level demands more rigor than at the high school level or students must seek more support in college (Burrill, 2016), many calculus students display lower confidence (Sonnert & Sadler, 2015) with some opting to switch majors to avoid additional calculus (Ellis, Kelton, & Rasmussen, 2014).

The national MAA study (Bressoud, Mesa, & Rasmussen, 2015) promotes the use of "good teaching" to increase student confidence and attitudes in calculus. In addition, as documented in Freeman et al. (2014), students outperformed their peers when active learning occurred in their STEM courses. At our institution, researchers documented student success in Team-Based Learning (TBL) Calculus I both during the course (Peters et al., 2019), and in subsequent Calculus II and calculus-based physics when compared to their traditionally-taught peers (forthcoming). Based on instructor reports and supported by observations by the second author, TBL Calculus I could be classified as "ambitious teaching" since the course included team projects, unfamiliar problems on homework and exams, written and verbal explanations, and decreased reliance on lecture (Sonnert & Sadler, 2015). As reported by Larsen, Glover, & Melhuish (2015), however, ambitious teaching can lead to decreased student attitudes for learning mathematics. At our institution on their course evaluations, in spite of their success, TBL Calculus students expressed frustration with the TBL process, the workload, and the lack of instruction provided by teachers.

Research Questions

To investigate students' frustrations more closely, we conducted a study to understand students' beliefs, attitudes, and expectations, as they experienced the TBL Calculus course. The questions driving this study included (a) how did the TBL Calculus experience meet or challenge students' beliefs, attitudes, and expectations for learning calculus; and (b) what factors led to changed or unchanged beliefs and attitudes about learning mathematics when in the TBL setting?

Links to the Literature

Seeking to understand how various forms of ambitious teaching impact student beliefs, more should be understood about Team-Based Learning. Developed by Michaelsen, Knight, & Fink (2002), TBL is a flipped, active learning strategy in which students first engage with content outside of class through videos and/or readings and then meet in class with a permanent, heterogeneous team of size five to seven to work on team tasks. Students are held accountable for their work by completing individual Readiness Assessment Tests (iRATs), team Readiness Assessment Tests (tRATs), and application exercises. Administered outside class, the five-question, short-answer or multiple choice quizzes (iRATs) assess students' initial understanding of the main concepts after reading the text or watching the video(s). Students then come to class to take the same five-question quiz with their team. After which, the instructor provides lecture elaborating the big ideas and finer details of the concepts. The application exercises or team projects are rich, problem-based, group-worthy tasks given at a level of difficulty to advance individuals' learning of content and promote team development. The application exercises are designed so each team solves the same significant task, makes a specific choice for an answer, and simultaneously reports an answer. Students complete peer evaluations periodically throughout the course to assess team members.

Knowing that TBL strives to engage learners in both individual and social settings, students' beliefs about the tasks and mathematics influence their level of engagement. Students' beliefs about mathematics and how students acquire mathematical knowledge affect how students approach a course. Schoenfeld (1985) states,

> Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. Beliefs establish the context within which resources, heuristics, and control operate (p. 45).

Students' beliefs in mathematics are individually held though, similar to knowledge, they may be socially constructed (De Corte, Eynde, & Verschaffle, 2002). De Corte et al. (2002) note that beliefs about the class context are determined by the activities, experiences, and interactions in the classroom. Therefore, if students experience classroom practices, such as, listening, watching, and mimicking things the teacher and textbook tell and show them, most students will likely learn and believe that mathematical knowledge is a form of received knowledge, not something constructed (Greeno, 1991). Schoenfeld (1992) identifies additional consequences of traditional mathematics instruction. One, when exposed to curriculum in bite-size pieces, students learn that answers and methods to problems will be provided to them; rarely will they be expected to determine the methods by themselves as they accept their passive role. Second, students come to believe they should have a ready method for the solution of a given problem and the method should produce an answer to the problem in short order.

Attitude is comprised of students' emotions associated with mathematics and their beliefs about themselves, about mathematics, and about their ability to do mathematics (McLeod, 1992). Di Martino & Zan's (2010) Three-dimensional Model for Attitude towards mathematics identifies the deep interplay between beliefs and emotions, especially when students display negative attitudes toward mathematics. The Three-dimensional Model for Attitude links emotional dimension, vision of mathematics, and perceived competence (Di Martino, 2016). Therefore, students may have a negative attitude toward mathematics due to their beliefs about mathematics, their perceived ability to do mathematics, or the emotions related to mathematics (Di Martino & Zan, 2010).

Students' motivational beliefs consider self-efficacy, value, and affect (Kitsantas, et al., 2008, & De Corte et al., 2002). A student's self-efficacy is their belief they can accomplish a task. Value emphasizes why students engage in learning and problem-solving and pursue the learning goal. Affect relates students' emotional reactions to tasks and their performance. Motivation provides the reason students engage in any pursuit, the reason students extend effort toward mathematical activity, and the extent to which students' efforts are viewed as productive (Middleton, Jansen, & Goldin, 2016). When a student engages in mathematical activity, "a combination of intrinsic, extrinsic, social, and individual factors are interrelating" (Middleton et al., 2016, p. 18).

Motivation is a regulatory process as students anticipate, engage, and reflect. When confronted with a mathematical task, students first determine the value of the task, then determine whether or not to engage, next choose a course of action, and finally evaluate their performance (Middleton et al., 2016). As a student's experiences consistently and coherently align with regard to motivational affordances, the person will likely develop a (positive or negative) long-term disposition and identity toward mathematics (Middleton et al., 2016, p. 19).

Theoretical Perspective

Researchers have long observed a link between students' beliefs about mathematics knowledge and the ways in which they engage in learning math (Garofalo, 1989; Schoenfeld, 1992). More recently, Muis and Franco (2009) found that students who viewed knowledge as tentative and complex were more likely to adopt mastery goals, and subsequently more likely to use deep processing strategies. Subsequently, a number of studies provide evidence, though not entirely conclusive, that constructivist learning environments promote more complex beliefs about knowledge (Muis, 2004). However, the relationship between students' beliefs about mathematics and their engagement in active and collaborative learning may be bidirectional. If students' views of knowledge inform their use of learning strategies and self-regulated learning, their epistemic aims (goals related to knowledge/inquiry) and epistemic beliefs (beliefs about knowledge) at the start of the course could influence the quality of their engagement in the course (Chinn, Buckland, & Samarapungavan, 2011). In this study, we hypothesized that epistemic factors might help explain persistence of or change of beliefs about mathematics and the learning of mathematics.

Research methodology

At a large, public midwestern university in Fall 2018, researchers invited all Calculus I students to participate in a study to examine differences in the beliefs, attitudes, and experiences of students. Two sections of Calculus I were taught using TBL. All other sections were traditionally taught in a lecture format. In most cases, students registered for Calculus I without knowing the method of instruction. The academic advisers may have known that students enrolling in a given section may therefore have secured a seat in a traditionally taught or TBL taught section. The information providing instructional method was neither hidden from nor broadly advertised to students.

TBL Calculus implemented the four main principles of TBL as established by Michaelsen et al. (2002). Similar to the description given in Peters et al. (2019), instructors formed heterogeneous, permanent teams, held students accountable for individual and group work, offered assignments promoting learning and team development, and frequently provided timely feedback. The two instructors varied in their teaching experience. One taught calculus for more than 40 years; the other taught calculus for more than ten years. Both taught TBL Calculus I for four semesters prior to the start of this study. The first author taught one of the TBL sections. The second author observed each of the TBL sections three times throughout the semester to identify student engagement and class activities throughout the semester.

In TBL Calculus, students had access to materials through the learning management system (LMS). All throughout the course, students could access the schedule for the iRATs, tRATs, team projects, and homework assignments as well as the videos, textbook, and reading guides. They could review this content and ask the instructors questions about the content at any time. Instructor-made videos accompanied sections of the textbook and ranged in length from 5 to 25 minutes with one to four videos per section. Instructors expected students to watch the videos and/or read the textbook prior to completing an iRAT. Students accessed the iRATs on the LMS during a limited time period. For example, an iRAT covering one or two sections of the textbook opened at 5pm on Monday evening and closed Wednesday at 7:30am, 90 minutes before class started. Once opened, students completed the 5-question multiple-choice or short-answer quiz within 35 minutes. Not informed of their iRAT results, students came to class to complete the team-version of the quiz (tRAT) during a 10-15 minute time period at the start of class. Two instructors and two teaching assistants circulated the classroom to address questions and provide assistance. After teams submitted their answers, the lead instructor lectured the remainder of the class session. Immediately following class, a 5-6 question online homework assignment opened, providing procedural practice of the new content. Students had unlimited attempts prior to the next class session to answer these questions. During the subsequent 1-2 class sessions, students completed team projects designed to engage students in rich conceptually-based problems, applying the new content and involving multiple topics. Additional online homework assignments gave students further practice with concepts and procedures.

Calculus I students enrolled in a more traditionally-taught course could attend three sessions of lecture in large classes (sized 192-360) and one recitation session (sized 32) containing a question/answer session and a quiz. Traditionally taught students completed online homework assignments once each week. All Calculus I students took the same three midterm exams and cumulative final exam.

In the third week of the semester, 606 students responded to the pre-survey asking students about their attitudes, beliefs, and experiences in mathematics. Of the 606, 102 TBL students completed the survey. We invited a subset of the TBL survey participants to complete a series of 40-60 minute semi-structured qualitative interviews about their experiences in the course. Sampling for variation in experiences, we selected students with both instructors and used feedback from instructors to select students in teams with different levels of functioning. We invited several students from each team to participate, with the goal of obtaining at least two participants per team. Finally, although the majority (80.4%) of the students in Calculus I were male, we sampled both male and female students. The result of this procedure was a sample of 18 students (12 male, 6 female) enrolled in TBL sections of Calculus I. Of these, 13 (8 male, 5 female) participated in Interview 2 and 11 (6 male, 5 female) participated in Interview 3. Interview 1 occurred during week 5 and prior to the first exam; Interview 2 took place during weeks 12-13, after the second exam; and Interview 3 occurred the following semester during weeks 7 – 9. Themes for students' statements were identified in the transcriptions using grounded theory and open coding (Glaser & Strauss, 1967; Miles & Huberman, 1994; Strauss & Corbin, 1994) and discussed amongst the three authors. The second author then coded all the transcripts.

Results of the Research

Students' comments during the interviews exposed their beliefs about the acquisition of mathematical knowledge as informed by past mathematical learning experiences. Most of these beliefs opposed the structure and philosophy behind the Team-Based Learning strategy.

The Expert Disseminates the Knowledge

Many students indicated their previous mathematics learning experiences were lecture-based. Expecting students to watch instructor-made videos or read the textbook to gain first exposure to new content challenged students' beliefs related to how knowledge is disseminated.

Will: What I've been accustomed to (in high school) is like a professor just kind of standing up and lecturing, and learning from that. And, then if I have a question afterwards, or a question on a homework problem then go ask.

Students expected, as in their previous mathematics courses, to be taught face-to-face and then be given the opportunity to practice.

Interviewer: What, if anything, seems frustrating or challenging about Team-Based Learning?

Stacy: Just the fact that we're learning with the material instead of ahead of it. We aren't given that foundation. We have to build it ourselves, and it can be very frustrating.

Students did not view the online videos and text as appropriate ways to first provide content.

When asked if they thought they learned more than, the same as, or less than if they were traditionally taught, many TBL students stated they felt they learned less mathematics than if they were in a traditionally taught calculus. Their reasons varied. Some indicated that they depended on the team and thus slacked on their own contributions. Other indicated that with no lectures, all the learning was done outside of class. Time in the TBL class was spent implementing the knowledge on the team projects.

Interviewer: What was the least helpful about your experience last semester?

Eli: Lectures. There were basically no lectures. We barely learned stuff in that class. That class is just a room to do stuff, you know? All you're learning is mainly outside of class applying it to that class. It's practice, rather.

Interviewer: Okay, so you mean you didn't like the learning outside of class?

Eli: You have to teach yourself calc outside of class if you wanna get a good grade. You have to go the extra mile cause the lectures there aren't going to teach you cause there aren't lectures, really. It's more just, 'hey, grab your white board,' and that's about it.

Most students explained during interviews their appreciation to work with other students and gain perspective of how others solved problems. However, very few felt the team work promoted actual learning. They felt their learning occurred individually outside of class, and the work during class reviewed what was self-taught. At least one student, John, recognized the potential for improved learning, when he said the more hands-on approach was "probably better learning in the long run because then you'll remember it more 'cause your brain is the thing that's doing it instead of just being told, 'hey this is how you do it.'"

Asking Questions with First Exposure

In addition to students' expectations for lecture and dissemination of mathematical knowledge during class sessions, students believed asking questions when first engaging with new material contributed to learning. Multiple students addressed frustration at not being able to ask questions when watching the videos and reading the textbook before completing the iRAT.

Amy: I'm definitely more of a person who needs like someone who knows what they're doing to explain it to me. And like, have that lecture time so if I need to ask questions, I can ask questions about like how it's being taught while it's being taught because sometimes if I learn something, but I have questions about it and I don't learn it right, then it's really hard for me to get the right way later on when I ask questions."

At least one student, Francesca, expressed appreciation at the opportunities to ask questions during class when working on team tasks. Because of this, she felt that learning was easier with TBL. "You're able to ask more questions, and be more confident with asking questions along with others on that team."

Many felt they would be able to ask questions in a traditional course when the instructor first discussed the content. However, based on focus-group sessions with traditionally-taught calculus students, questions were rarely asked in the large class sessions as students did not want to interrupt the instructor when they had stated that they had a lot of content to cover.

High Workload and Task Value

Most students interviewed agreed that the workload required too much time. The demands may have resulted in a devaluing of the tasks, thereby decreasing motivation and leading to the use of shallow strategies to complete them. Students felt rushed to watch videos, especially near the end of the semester when multiple videos were 20 or more minutes in length. Students confessed to not watching all the videos or increasing the speed on the videos prior to completing the iRATs.

The frequency of iRATs and online homework assignments frustrated students. Thomas stated, "TBL is a black hole for time for me." Will shared that all the little assignments threw him off. He felt his grades were suffering because he spent his time getting points rather than learning. Miguel said, "Since there's so much work, I was rushing to memorize things or try to understand them as fast as possible and not really get them deep in my brain till I have a deep understanding. I'd have just a basic understanding." John and Eloise stated that they resorted to the use of derivative calculators online to help complete online homework problems quickly, especially if the assistance provided the intermediate steps.

During class, students also expressed feeling rushed to complete the team projects. Eli shared that his team would come to class and encourage each other to simply get the project done as fast as they could so they could leave.

Struggle and Complexity

A few students expressed the struggle the iRATs, team projects, and exams gave them. Eloise felt she had to have team members show her how to do things. "I just feel incompetent sometimes." Stacy commented that the TBL was a struggle. "The information just isn't given to me and I have to go find it."

Part of the struggle students noticed resulted from the differences in complexity in learning mathematics at the high school level than at the college level. John summarizes, "I don't know if it's just in college, but in team-based learning at least, we put a lot of concepts together whereas in high school we'd learn one concept and we wouldn't use it again until the final." Later about a TBL project, he said,

We'd have an in-class word problem like project with our team or whatever and it'd be a concept from the first unit and a concept from the second unit and you'd have to know both of them to be able to do the problem. But it expanded your knowledge because then you learned how to do a problem with both of those concepts.

Another student identified another contrast between his high school experience and the TBL Calculus experience.

Thomas: Our high school calc was more focused like, 'Here's an exact problem, and there's one way to do that problem.' But then here in calc, there's multiple ways to tackle problems….You have three, four weeks of lecture material that you have to remember all of it for a lot of problems. Or you have to remember multiple lectures for one problem.

Interviewer: How is that different than your high school math?

Thomas: You'd have simple problems with just one step. For us, calc exams, they often involve multiple steps.

The complexity of exam problems was addressed by both TBL and traditionally-taught students, and therefore shows one challenge students see in their transition of the study of mathematics from the high school to college.

Overall, based on student-interviews, the TBL Calculus approach challenged and opposed most interviewees' beliefs about mathematics and the learning of mathematics. Students' beliefs persisted since the start of the course in that the expert in a face-to-face environment should establish the foundation of knowledge needed prior to engaging students in tasks. According to students, instructors should highlight the most effective means of solving problems for students.

Applications to and Implications for Teaching Practice

This study shows that many students, similar to those described by Bookman & Friedman (1998) after one semester of reform calculus, reported that the course required too much time outside of class and that "the course taught me very little" (Bookman & Friedman, 1998, p. 118). The negative attitudes portrayed by interviewed TBL Calculus students related to their beliefs about a low competence to learn mathematics outside class. Others expressed negative attitudes due to their belief that mathematics should be disseminated by an expert. Both sets of beliefs misaligned with the foundational structure of the TBL course. As concluded by Bookman & Friedman (1998), students in the TBL Calculus may in another year gradually adopt an attitude consistent with ambitious teaching. For our study, seven or eight weeks into the following semester did not provide sufficient time for attitudes to change. Additional investigation can be done to identify at what point and in what ways attitudes change.

For teaching purposes, students stated they needed to feel they had more control of their learning. In this particular application of TBL Calculus, this likely means a reduction in tasks completed outside of class, including shorter videos to watch and fewer and shorter online homework assignments. Decreasing the time pressures with the iRATs could also reduce anxiety for students. During class, rather than emphasizing the completion of team projects, instructors should emphasize the processing of and reflecting upon some rich questions during whole class discussions.

An additional manner in which students can perceive greater control of their learning is to help them engage in more regulatory behavior. Motivating and training students to anticipate, engage in, and reflect upon tasks could advance students' learning and potentially lend toward changed beliefs about how mathematical knowledge is acquired. This regulation also should decrease the use of shallow strategies including memorization and lend toward deeper learning strategies. Furthermore, students' application of these regulatory skills could ease students' transition in the study of mathematics from the high school to college level.

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Secondary Prospective Teachers' Strategies to Determine Equivalence of Conditional Statements

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Future mathematics teachers must be able to interpret a wide range of mathematical statements, in particular conditional statements. Literature shows that even when students are familiar with conditional statements and equivalence to the contrapositive, identifying other equivalent and non-equivalent forms can be challenging. As a part of a larger grant to enhance and study prospective secondary teachers' (PSTs') mathematical knowledge for teaching proof, we analyzed data from 26 PSTs working on a task that required rewriting a conditional statement in several different forms and then determining those that were equivalent to the original statement. We identified three key strategies used to make sense of the various forms of conditional statements and to identify equivalent and non-equivalent forms: meaning making, comparing truth-values and comparing to known syntactic forms. The PSTs relied both on semantic meaning of the statements and on their formal logical knowledge to make their judgments.

Keywords: Reasoning and Proof, Conditional Statements, Preservice Teacher Education

Mathematics educators generally agree that teaching mathematics in ways that promote reasoning and proof can deepen students' understanding and support retention of knowledge (e.g., Hanna & deVillers, 2012; Harel, 2013). Furthermore, researchers and policy makers support integration of proof and reasoning across all grade levels and mathematical topics (NCTM, 2009; NGA & CCSSO, 2010). In order to bring this vision of mathematics teaching into reality, teachers need to have robust understanding of deductive reasoning, valid modes of inference, proof techniques, and other aspects, which comprise *knowledge of the logical aspects of proof* (Buchbinder & McCrone, 2018). Moreover, teachers must flexibly use this knowledge in the context of school mathematics.

As part of a larger grant to study and enhance prospective secondary teachers' (PSTs') mathematical knowledge for teaching proof, we designed instructional activities aiming to strengthen their knowledge of the logical aspects of proof. The activities were enacted in the capstone course, *Mathematical Reasoning and Proof for Secondary Teachers*, which is a blended content and pedagogy course developed as a part of the grant (Buchbinder & McCrone, 2018).

One set of such activities focused on conditional statements and logical equivalence, or the lack of thereof, between the implication $P \Rightarrow Q$ and various logical forms, such as contrapositive, converse, inverse, but also other forms such as *P if Q* or *P is necessary for Q*. The purpose of the activities was to push the PSTs thinking beyond typical "if-then" statements, in hopes that by identifying ways of moving their own understanding forward, the PSTs would strengthen their knowledge of the logical aspects of proof, which in turn, could inform their future teaching.

As researchers, we sought to understand the process by which the PSTs made sense of various logical forms of conditional statements and to identify strategies by which the PSTs determined logical equivalence. Our work was guided by the research question: How do PSTs make sense of various logical forms of conditional statements and establish equivalence between them?

Background and Theoretical Perspectives

A conditional statement (or a logical implication), denoted as $P \Rightarrow Q$, can be expressed in words as *If P then Q* or *P implies Q*. But many other equivalent forms are possible, for example, *Q if P*, *P only if Q*, *P is sufficient for Q* and *Q is necessary for P*. Formally, two statements are logically equivalent if they have the same truth tables. This definition can be used to determine that a contrapositive $\sim Q \Rightarrow \sim P$ is equivalent to $P \Rightarrow Q$. However, it cannot be applied directly to the logical forms mentioned above. The equivalence of these forms to the original implication is not obvious and many require a few logical steps to show. As an example, consider the form *P only if Q*. Taken by itself, this statement says that *P* is true only under the condition that *Q* is true. In other words, if *Q* is false, it cannot be the case that *P* is true (i.e., *P* is also false). Using such reasoning, one might recognize that *P only if Q* is equivalent to *not Q implies not P* which is the contrapositive of the statement *P implies Q*, and thus is equivalent to *P implies Q* (Chartrand, Polimeni & Zhang, 2018). This type of reasoning can be challenging, as it requires both syntactic understanding of the formal symbolic notation and the semantic understanding of the meaning of each logical form and the related wording (Weber & Alcock, 2004).

Studies have shown that mathematics majors and PSTs often misinterpret mathematical language and experience particular difficulties with conditional statements. These include confusion between an implication and its converse ($Q \Rightarrow P$) (Durand-Guerrier, 2003), interpreting an implication as a biconditional ($P \Leftrightarrow Q$) (Epp, 2003), difficulty understanding the equivalence between an implication and a contrapositive (Dawkins & Hub, 2017; Stylianides, Stylianides & Philippou, 2004) or between an implication and a disjunction ($\sim P \vee Q$) (Hawthorne & Rasmussen, 2015). However, we are not aware of studies that examined how undergraduates, in particular PSTs, make sense of logical equivalence of a broad range of logical forms of conditional statements. We see this as a crucial aspect of teacher preparation, as teachers need flexible knowledge to make sense of and rephrase their students' contributions into more precise mathematical statements in order to determine their validity.

Methods

The Conditional Statements activities took place in the capstone course, *Mathematical Reasoning and Proof for Secondary Teachers*, for which the first author served as the instructor. All PSTs enrolled in the course agreed to participate in the study: 15 PSTs in Fall 2017, and 11 PSTs in Fall 2018. All PSTs were native English speakers. At the time of data collection all PSTs were seniors, thus, they had completed most of their mathematics coursework, including several proof-based classes such as Mathematical Proof, Geometry and Abstract Algebra. Hence, the goal of the Conditional Statements activities was not to introduce PSTs to new material, but to help them refresh and strengthen their content knowledge of conditional statements. The content focus included the meaning and logical notation of the implication $P \Rightarrow Q$, recognition of hypothesis and conclusion in context specific statements worded in different forms, determining truth-value of conditional statements, and recognizing equivalent and non-equivalent logical forms such as a contrapositive $\sim Q \Rightarrow \sim P$ and a converse $Q \Rightarrow P$.

The activity that is the focus of this paper included the following tasks:

a. Working in small groups, determine which of the 11 given logical forms are equivalent to the original implication $P \Rightarrow Q$ and which are not (Fig. 1);
b. Create a poster display of the equivalent and non-equivalent statements and share those posters with other groups;

c. Discuss the answers as a whole class, clarify difficult items, and summarize main points about logical equivalence and its relationship to truth-value.

Each group received a different mathematical statement, but the logical forms were the same across all groups. The mathematical statements of each group were the following:

Group 1: A graph of an odd function passes through the origin (assume f is defined at 0).

Group 2: A number that is divisible by 6 is divisible by 3.

Group 3: Diagonals of a rectangle are congruent to each other.

Group 3

Given a true statement: ***Diagonals of a rectangle are congruent to each other.***

1. Rewrite the statement in an *If P then Q* form. Identify *P* and *Q*.
2. Using P and Q you defined above, write statements in each the forms presented below.

a) P if Q	
b) P only if Q	
c) P is necessary for Q	
d) To infer Q, it is sufficient to know P	
e) Q if P	
f) Q is sufficient for P	
g) If not P then not Q	
h) Not-Q implies not-P	
i) P is sufficient to infer Q	
j) Q is necessary for P	
k) P if and only if Q	

3. Write each statement on a separate index card using different color for true and false statements. Sort the cards in two groups: (i) cards with statements equivalent to the given statement, and (ii) cards with statements not equivalent to the given one. Be prepared to explain why placed the cards the way you did.

Figure 1. A worksheet of Group 3 (Geometry)

The inclusion of a mathematical context and symbolic form in one task aimed to support the PSTs flexible understanding and use of formal logical notation while grounding it in familiar content (Dawkins, 2017; Dubinsky & Yiparaki, 2000).

The tasks were enacted across two class periods and took about two hours to complete. The data were collected in the form of video recordings of each groups' work with a tabletop 360° video camera, a stationary camera to capture the whole class discussion, and PSTs' written work in the form of worksheets and posters. The video recording and their transcripts were the primary data source for this paper. Written artifacts served as a secondary data source.

To analyze the data, we divided the video transcripts into meaningful episodes, and used open coding (Wiersma & Jurs, 2005) and the constant comparative method (Strauss & Corbin, 1994) to identify categories of strategies the PSTs utilized to determine whether certain forms of conditional statements are logically equivalent to the implication $P \Rightarrow Q$.

Results

Our analysis revealed three main strategies in the PSTs approaches to determining logical equivalence: (1) meaning making, (2) comparing truth-values, and (3) comparing to known syntactic forms. These strategies are interrelated and some have further sub-categories, which we elaborate below. The two types of logical forms that appeared to present most challenges to the PSTs were *P only if Q*, and the statements that contained the language of *necessary* and *sufficient* condition. Thus, the data excerpts chosen to illustrate the strategies are taken from these types of statements as they provided the clearest evidence of the PSTs strategies to determine equivalence, although the strategies were observed throughout all types of logical forms. Note, the text in square brackets was added for clarification.

Strategy 1: Meaning Making

This strategy entails rephrasing of the statement (often multiple times) in search of a clearer meaning. This process included three types of sub-strategies: (a) introducing additional or substituting alternative words, (b) attempting to put a statement into an "if-then" form, and (c) relying on counterexamples and "feelings".

Rephrasing by adding or substituting words. When trying to interpret conditional statements that were worded as necessary or sufficient conditions the PSTs initially attempted to substitute open sentences for *P* and *Q*, but quickly discovered that this may result in a nonsensical statement, as shown below:

Excerpt #1. Sam: …how to write that… P is necessary for Q.

Bill: Yeah, I'm trying to think about a word…

Nate: That's a lot of is's. Cause "a quadrilateral is a rectangle is necessary for its diagonals are congruent …it's just awkwardly worded.

Laura: A quadrilateral must be a rectangle for the diagonals to be congruent.

Nate: mmm…nice change.

The PSTs also found it difficult to distinguish between necessary and sufficient conditions. The following excerpt illustrates the PSTs' confusion as they tried to word the statement: If a number is divisible by 6 then it is divisible by 3 in the form *Q is sufficient for P.*

Excerpt #2. Penny: It is sufficient to know that a number is divisible by 3…

Zoe: If it's divisible by 3 then it's satisfies that it's going to be divisible by, …no.

Linda: No, technically not.

Zoe: But that's not what it's saying, sufficient is like if it satisfies… that guarantees… so, if a number is divisible by 3, then it guarantees that it is divisible by 6.

Here, Penny tried to retain the wording of "*is sufficient*" while Zoe struggled a bit but then offered the alternate wording *if it satisfies Q, then it guarantees P*. Not all attempts to rephrase the statement preserved the meaning, but this strategy was highly prevalent and for the most part was efficient in PSTs' attempts to determine equivalence.

Rephrasing in "if-then" form. Much of the PSTs' rephrasing efforts were directed towards putting the statements into an "if-then" form (e.g., Zoe's attempts in Excerpt #2 above), even when this was not entirely appropriate. Consider the excerpt below:

Excerpt #3. Audrey: Umm…and then another one that was really confusing for us was *Q is necessary for P*. So we said, if a number is divisible by 3, it is necessary for it to be divisible by 6, but that's like opposite of what the [original] statement is saying. Cause if a number is divisible 3, it doesn't have to be divisible by 6.

Here, Audrey's group rephrased *Q is necessary for P* incorrectly as *if Q it is necessary for it to be P*, which seems to be further simplified as *if Q then P*. This changed the meaning of the statement indeed transforming it into one not equivalent to the original.

The strategy of rephrasing a statement in an "if-then" form worked for the *P if Q* statement. The PSTs put "if" at the beginning of the sentence: *if Q, P*, and added the words "follow" or "then" to correctly conclude that *P if Q* is equivalent to *if Q then P*, and thus not equivalent to the original implication. However, this strategy failed when interpreting *P only if Q* statement. The PSTs either came to impasse, not knowing how to interpret *only if Q, P* or they simply dropped the word "only," wrongly transforming the statement into *if Q then P* (see Excerpt #5 below)

Counterexamples and "Feelings." The PSTs constructed counterexamples to help them make sense of the statements. For example, Emily used the counterexample of $y = x^2$, a non-odd function passing through the origin to help her make sense of the form *P if Q* and to conclude that it is not equivalent to *P implies Q.* She said: "No, it's not equivalent. [A function] x^2 can pass through the origin. It [*P if Q*] is saying if a graph passes through the origin, then the function is odd. It's flipping it."

In both cohorts, the PSTs came up with the same counterexamples: $y = x^2$ for the statement about functions, an isosceles trapezoid to show that a quadrilateral which is not a rectangle can have congruent diagonals, and numbers 9 and 15, which are divisible by 3 but not divisible by 6. Once someone in the group introduced a counterexample into a public space, the members of the group repeatedly used it to make sense of various forms of conditional statements. However, the PSTs occasionally misused these counterexamples in statements. For instance, Derick attempted to use the function $y = x^2$ to disprove a statement "A graph of the function passes through the origin if the function is odd", which has a form *Q if P*. Derick noticed that the function $y = x^2$ satisfies *Q,* i.e. passes through the origin, but is not an odd function, and wrongly attempted to use it to disprove *Q if P*, without considering that it is equivalent to *P implies Q,* which makes his counterexample non-applicable.

As much as the use of counterexamples was prevalent in the PSTs' strategies, most frequently the PSTs simply relied on their perception to make a claim about the equivalence of statements, without providing any justification, as the next excerpt shows:

Excerpt # 4. Rebecca: Alright, next, *to infer Q is sufficient to know P.*

Logan: Logically, I think it is [equivalent].

Dylan: Really? This is definitely not logically equivalent. Knowing the function is odd is sufficient to infer the graph passes through the origin... Maybe it is.

Grace: Wait, I actually think that's equivalent. It's like exactly the same.

Dylan: Yeah, yeah actually it is equivalent. Oh yeah.

It can be argued that such words as "logically, I think" or "I feel like" (see excerpt #5 below) were used merely as a figure of speech. However, in the absence of any other justifications, we tend to interpret them as discursive mechanisms for convincing oneself or others. We also acknowledge that the PSTs' feelings were probably based on their prior knowledge of both the mathematical content, the formal logical notation, and the counterexamples discussed in their groups. All these could be used to create mental representations of statements, which could then be compared to one another to determine equivalence. However, not surprisingly, the "feelings" were not always reliable, and often led to incorrect conclusions.

Strategy 2: Comparing Truth-Values

A second key strategy the PSTs used to decide if the statements are equivalent or not, was to compare their truth-values. In group discussions, the PSTs quickly agreed that equivalent statements must have the same truth-values and that statements that have different truth-values cannot be equivalent. These conclusions were not obvious to everyone, but the PSTs eventually resolved any initial doubts within their groups, without the instructor's intervention.

A more difficult point to agree upon was whether having the same truth-value made statements equivalent. This point required more discussion and negotiation, especially, since all statements in the worksheets which had the same truth-value to the original implication were also equivalent to it. Thus, the PSTs had to come up with their own examples of statements that have the same truth-value but are not equivalent. This was not always easy, as can be seen in Dylan's comment: "I think, we agreed that if it's equivalent, the truth value has to be the same, but you can have a case that the truth value is the same, but it's not equivalent to the statement. So we were saying we found an if and only if case that was true, it's not going to be equivalent to your original statement, but you can have the same truth value."

If the truth-values could not be used to establish equivalence, the PSTs referred back to relying on meaning making and "feelings", as the next excerpt shows:

Excerpt # 5: Nate: I feel like the *only if* would be the same as saying *if Q then P*. So I think, yeah, the *P only if Q* is like saying *if Q then P*. Which is not necessarily equivalent.

Audrey: I feel like that shouldn't be equivalent.

Nate: It just can be true. Cause I feel like, I feel like…

Audrey: That's true. But doesn't mean that it's equivalent.

While attempting to rephrase *P only if Q* in an "if-then" form, Nate mistakenly transforms it into *Q implies P*, but then correctly concludes that the rephrased statement is not equivalent to *P implies Q*. Both Nate and Audrey appeal to their feelings to seal the conclusion.

Strategy 3: Compare to Known Syntactic Forms

Since all PSTs had previously successfully completed proof-oriented courses, they often drew on their familiarity with logical notation, conditional statements, and other relevant concepts. For instance, the PSTs were aware that an implication is equivalent to a contrapositive, and used it in their work, as Dana's comment shows: "I think I came to the decision that *P only if Q* is not-equivalent [to *P implies Q*]. So *P only if Q* is the same as saying *if Q is true, then P is true*. So then the contrapositive of that would be *not P implies not Q*, and that is not equivalent to *P implies Q* in the first place."

Dana's first step is an incorrect interpretation of *P only if Q* as *If Q then P*. But then she correctly states that the contrapositive of *If Q then P* is *not P implies not Q*, and correctly concludes that it cannot be equivalent to *P implies Q*.

Our PSTs also seem to have a good grasp of non-equivalence of an implication to its converse, as illustrated below.

Excerpt # 6: Angela: 'Cause doesn't the converse like by definition is the opposite truth-value of the original. That's what the converse is? Or no?

Sam: No, you can't determine truth-value by the converse. But the converse could be true in some cases. I think, yeah. It's not necessarily true, but it could be true… Technically the converse is a whole different statement in itself.

Bill: 'Cause we're reversing p and q.

Sam: Yeah, like they're never gonna be logically equivalent.

Here Sam correctly explained that although a converse can have the same truth-value as the implication, it cannot be equivalent to it. Bill justified that by saying that the converse "reverses *P* and *Q*". The other members of the group accepted this explanation. Similar conversations occurred in all other groups, and were resolved correctly without the instructor's intervention. The PSTs also recalled that a biconditional entails the truth of both an implication and its converse, and used it to justify non-equivalence of a biconditional and an implication. Note that in both excerpts above, the PSTs operated solely within a syntactic domain, without invoking semantic meaning of the statements and without appealing to examples.

The Missing Strategy: Negation

Visibly absent from PSTs' strategies was the use of negation. In our data from both cohorts, we recorded only one instance of the use of negation to make sense of *P only if Q*. Except for that instance, the PSTs avoided using negation despite having the relevant prior knowledge. The forms for which the use of negation was critical were *P only if Q*, and the *necessary* condition. Unless prompted by the instructor, the PSTs seemed unable or unwilling to introduce negation to interpret "only if" or "necessary" as "otherwise" or "if not".

Discussion

In this paper, we examined data from two cohorts of prospective secondary teachers' interactions with Conditional Statements activities in the context of a capstone course aimed to enhance their content and pedagogical knowledge of proof. The key feature of the activities was the inclusion of both the formal logical notation and mathematical context from the secondary curriculum: number and operation, geometry and functions. Our study focused on understanding how PSTs make sense of various logical forms of conditional statements and establish equivalence between them. We identified three main strategies the PSTs used to establish equivalence, or the lack thereof, between a broad range of logical forms: (1) meaning making, (2) comparing truth-values, and (3) comparing to known syntactic forms.

Semantic strategies, what we term as *meaning making*, were prevalent in the PSTs' approaches, concurring with the literature (e.g., Dubinsky & Yiparaki, 2000). The PSTs rephrased the statements multiple times to make them more comprehensible either by adding or substituting words or by putting the statements into an "if-then' form. The PSTs used counterexamples to make sense of the statements akin to experts' use of counterexamples (cf. Lockwood, Ellis, & Lynch, 2016). Our data show that in many cases these strategies proved to be helpful in determining equivalence of the various logical forms, but not always. At times, the PSTs would fall back on feelings, relying on their perceptions of the content of statements. This phenomenon reflects the nature of our data, which was captured during the natural discourse among the PSTs. We suspect that the PSTs' perceptions were grounded in prior knowledge, and the PSTs would be able to mathematically justify their thinking, if requested.

The two logical forms that were most challenging for the PSTs to interpret were: *P only if Q* and *P is necessary for Q*. Interpreting these forms requires the use of negation, which did not come naturally to our PSTs, as consistent with the literature (Dawkins, 2017). However contrary to prior studies (e.g., Dawkins & Hub, 2017; Durand-Guerrier, 2003) our PSTs seemed knowledgeable about an implication being equivalent to a contrapositive and non-equivalent to a converse. The PSTs often relied on this formal logical knowledge to determine whether certain logical forms are equivalent or not, without assessing the semantic meaning of these statements.

The studies on knowledge retention in the area of formal logic are scarce, thus, our study adds to this literature by demonstrating knowledge retention of logical aspects of proof, beyond the typical introduction to proof course. Specifically, our study identified particular aspects of this knowledge that were retained and flexibly used by PSTs to make sense of a broad range of logical forms of conditional statements, and determine their logical equivalence. Our data show that the PSTs can apply their knowledge of formal logic to the secondary school context, and use it to interpret a wide range of logical forms of conditional statements, which may come up in classroom discourse. Another contribution to the literature is the design of the Conditional Statements activity that uncovered these strategies and elicited a range of rich discussion among PSTs around the various logical forms of conditional statements.

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Characteristics and Evaluation of Ten Mathematics Tutoring Centers

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Quantitative and qualitative evaluation of math tutoring centers is a critical step to identify characteristics of effective centers. A group of ten math tutoring centers gathered quantitative and qualitative measures of effectiveness as part of an ongoing project to identify characteristics of effective centers. This report summarizes the data collected. We will use this data in a future paper to generate testable hypotheses about characteristics of effective math tutoring centers.

Keywords: mathematics tutoring centers, evaluation, characteristics of effective centers

College-level math support programs are common throughout the United States and globally (Bressoud, Mesa, & Rasmussen, 2015; Matthews, 2013). While these programs have many forms, peer tutoring is a common one, with 89.5% percent of universities surveyed in the US reporting the use of peer tutoring (Johnson & Hanson, 2015). The math tutoring environment has the potential to engage students in a more active way compared to a typical classroom (i.e., Topping, 1996; Topping & Watson, 1996). Tutoring has been shown to impact student self-efficacy (Maxwell, 1994), confidence (Topping, 1996), and have the most impact on at-risk students (Mac an Bhiard, Morgan, & O'Shea, 2009; Rickard & Mills, 2018; Xu, Hartman, Uribe & Mencke, 2001). We study math tutoring centers that serve all students enrolled in eligible mathematics courses. Our long-term goal is to identify characteristics of successful math tutoring centers because there is no research on the characteristics of such centers that contribute to student success (Byerley et. al, 2019).

Literature Review

Although tutoring centers are common, there is little published statistical evidence of the effectiveness of math tutoring centers. One possible reason is the difficulty in measuring the impact of centers on student success (Matthews, 2013). The number of visits to the center is commonly used as a measure of the center's effectiveness, but this measure does not show that visiting the center is correlated with higher grades. In Berkopes and Abshire's (2016) study, the high number of visits by students, particularly by first-generation college students, was evidence of success. In contrast, Marr (2010) interpreted the high number of repeat tutor appointments as a weakness on the part of the tutor to assist the student effectively in their difficulties.

It is hard to imagine a study that would demonstrate that math tutor center visits cause increases in student grades because of self-selection bias (Byerley, Rickard & Campbell, 2018). Students cannot be randomly assigned to a tutoring center treatment. Large numbers of high-achieving students may not seek out tutoring services or, alternatively, more motivated students might be more likely to earn high grades and also more likely to visit the center. Several studies found positive statistical correlations between student visits to tutoring centers and student success (Cuthbert & MacGillivray, 2007; Dowling & Nolan, 2006; Mac an Bhaird, Morgan & O'Shea, 2009). However, these studies do not mitigate the self-selection bias because they do not include control variables in their linear models. In contrast, Cooper (2010) did not find a correlation between tutor center visits and course grades.

Several studies have employed more advanced regression techniques to help account for the biasing self-selection. For example, Xu and colleagues (2001) performed a regression analysis and found that, at Arizona State, visiting the tutoring center predicted higher final exam scores in College Algebra when controlling for well-known predictive variables (gender, SAT, math placement, and high school GPA). Tutoring also had a larger impact on students in the bottom quartile of SAT scores compared to the students in the top quartile (Xu et al., 2001). Byerley, Rickard, and Campbell (2018) found a statistically significant positive correlation between students' tutor center attendance at Colorado State University and Calculus II course grades when controlling for prior student aptitude, same-semester achievement, and motivation (measured by variables such as high school GPA, early test grades, and lecture attendance). Similarly, Rickard and Mills (2018) used a multiple regression model and found that when controlling from prior aptitude, each visit to the tutoring center at Oklahoma State University corresponded with an increase in students' Calculus I final course grades by 0.33%. They found that students with the lowest high school GPAs had the largest benefit from visiting the center.

These studies offer evidence that math tutor center visits can have positive impact on student success. However, these studies were conducted at single institutions and offer few clues about what characteristics of the tutoring center contributed to the effectiveness. These studies do not show if a center was effective compared to other institutions. The tutor centers themselves are likely not representative of tutor centers at a national scale: for example, Rickard and Mills (2018) report a 61% attendance rate at their tutoring center, while the national average is 40% (Johnson & Hanson, 2015). We are interested in more than whether a tutoring center is effective; we are also interested in the attributes that characterize effective tutoring centers. By comparing qualitative and quantitative data from multiple centers we will be able to generate hypotheses about common features of more effective centers.

A group of math tutor center administrators proposed a set of six dimensions that characterize centers (Byerley et al., 2019): (a) specialist vs. generalist math tutor models, (b) strength of relationship between center and math instructors, (c) type and extent of tutor training, (d) types of tutoring services, (e) physical layout and location, and (f) tutoring capacity. The current study extends this work by offering both statistical analysis of the effectiveness of ten different math tutoring centers with qualitative descriptions of the centers in terms of these previously identified dimensions. Due to space limitation, the dimensions are not redefined in this paper; understanding the tables reported in this study depends on being familiar with the six dimensions defined in previous research (Byerley et al., 2019).

Research Questions

This paper answers the following research questions for the ten mathematics tutoring centers that contributed data.

1. What percent of eligible students use the center and how often do they use it?
2. What are the characteristics of each center on each of the six dimensions proposed by Byerley and colleagues (2019)?

3. What is the correlation between visiting a center and students' grades after controlling for students' high school G.P.A. and standardized test scores?

Theoretical Perspective

This paper privileges the questions of practitioners who lead tutoring centers. We understand typical requirements of that job because we oversee or have overseen tutoring centers. Privileging the needs of center leaders often requires conducting research at odds with research guided by constructivist theoretical perspectives. For example, it is well known that passing calculus does not mean a student has constructed productive meanings for calculus. Despite this, we use students' grades in math courses as a measure of success because grades are important to students' academic progression and those in administration who monitor this. For details on the perspective we adopt, refer to the editorials on bringing research closer to practice and reducing isolation between practitioners (Cai, et. al., 2018a, 2018b).

Methods

Data collected from a convenience sample of ten universities includes characteristics of the tutoring centers as well as student academic and visit data. The data on characteristics of the tutoring centers were based on dimensions identified by Byerley and colleagues (2019). They include: the number of students eligible to use the center by virtue of being enrolled in a course served by the center, the percentage of those students who visited the center, the average number of visits to the center per eligible student, tutor hours per student visit, tutor hours per eligible student per week, and the type of tutors the center employs. Tutor hours is the total sum of all hours worked by all tutors in a given semester. In order to standardize analyses across universities, participating center leaders were surveyed to determine what student data were available to them. Initial factors considered were number of student visits to the tutoring center, duration of student visits to the tutoring center, course letter grade, course percentage grade, math placement score, standardized test scores, standardized test math subscores, high school grade point average, high school math grade point average, ethnicity, first generation status, Pell grant status, and number of math course repetitions. Many factors were excluded as not all contributors were able to obtain access to the data needed. The factors that essentially all contributors were able to procure data for were: (a) student visits to the tutoring center, (b) high school grade point average, (c) standardized test scores, and (d) course letter grade converted to grade points. These factors represent only a small portion of factors that might influence student grades in a course. It was nevertheless decided that, in order to ensure similar analyses for each institution, the research team needed to analyze data that all members could access.

Quantitative data from each institution were collected for the fall semester of 2017 and/or 2018 depending on the year data were available. For many tutoring centers, students from any course are allowed to use the tutoring center, including courses the tutoring center may not have tutors for. It was therefore decided that data would only be analyzed for students enrolled in mathematics courses for which the tutoring center purposefully serves. This includes any mathematics course for which the math center specifically provides tutors. Data for all students in a course served by the tutoring center were collected, including those who did not visit the tutoring center. Students with missing data and students who withdrew from the course were removed from the analyses, but these counts are shared in the data tables that follow. Students enrolled in multiple mathematics courses were treated as separate data points, with the number of visits to the tutoring center split equally between the courses taken.

Multiple regression analyses were conducted on the data from each tutoring center using course grades as the dependent variable and student visits to the tutoring center, high school grade point average, and standardized test scores as the independent variables. Due to many smaller enrollment courses, analyses combined all courses within each university.

In addition to collecting data and conducting statistical analysis on it, each tutor center leader wrote a qualitative description of his or her respective math center using the six dimensions from Byerley and colleagues' (2019) framework. The leaders included both positive and negative information about their centers in their descriptions, submitting information to the lead author who blinded it prior to making it available to the rest of the research team. In the future, our team and other experts will analyze blinded qualitative and quantitative features of centers to hypothesize characteristics of effective centers.

Results

While common aspects, such as hiring tutors from pool of students who made high grades in course are not included in Table 1, it describes other less obvious aspects of each center. Some aspects differ widely among center for reasons that are not evident in the table. For example, Bird U has limited physical center space; so, no professors hold office hours in the center, and students enter the center to ask questions but not stay to study. Gorilla U has a number of computers that all students on campus can use, even if not working on mathematics assignments. Dolphin U has high tutor hours per student visit because they offer appointment-based tutoring. Instructors at Dog U offer students tiny amounts of extra credit to correct their tests at the center.

Table 2 displays the statistical results from each center. The R^2 of the model represents the proportion of course grades that can be accounted for by the predictor variables. For five centers the model suggests that a higher number of center visits is predictive of higher grades. The model for Whale U, for example, predicts that if students visit the tutor center 10 times in a semester, their course grade point would be 0.16 higher. Unsurprisingly, high school GPA and standardized test scores are significant predictors of grades at all universities. Dolphin U's model suggests that a higher number of tutoring visits is predictive of lower grades. The data from Dolphin U is from the first semester the center was open, and it was known as a place for struggling students to get help despite the efforts of the director to welcome all students.

Table 3 provides information about tutor training at each center. In addition to considering the number of hours of formal training we also considered if the center had a *generalist* or *specialist* tutor model (Byerley et. al., 2019). *Specialist tutors* must meet some criteria (e.g. being an LA or grader) or complete some event (e.g., training or exam) successfully before they are allowed to tutor a particular course. For example, learning assistants attend the course they tutor to help instructors with group work. Specialists often tutor one course per semester and the tutoring center groups students by course. Cat U and Dog U use specialist tutors. Fish U, Gorilla U, Hamster U and Horse U use a mix of generalist and specialist tutors. A benefit of specialist tutors is that they know the content, the instructors, and the expectations for the course. *Generalist tutors* are allowed to tutor multiple courses without having to meet specific criteria. Dolphin U, Bird U, and Whale U use generalist tutors. Undergraduates usually tutor classes they did well in; however, in some generalist settings undergraduate tutors tutor for classes they have not taken (e.g., Business Calculus). The generalist model allows quick access to a tutor because students do not have to wait for the tutor assigned to their course to be free. However, generalist tutors face difficulties when they tutor more advanced non-coordinated courses that have different concepts, texts, and expectations.

Table 1. This table provides counts and other basic quantitative information about students and tutors at each center.

School	# of eligible students	% of students who visited	Visits/eligible student	Tutor hours/visit	Tutor hours/eligible student/week	Square footage	Drop in or appointment
Bird	1209	9.30%	1.3	0.66	0.04	450	Drop in
Cat	4304	48.40%	6.1	0.32	0.12	4700	Drop in
Dog	1335	67.80%	5.2	0.19	0.07	1738	Drop in
Dolphin	1158	11.50%	0.5	1.00	0.05	1000	Appointment
Goat	1300	16.10%	0.61	0.84	0.03	470	Drop in
Gorilla	4,217	54.40%	7.3	0.20	0.09	8000	Drop in
Fish	8292	8.61%	0.36	1.36	0.03	1607	Drop in
Hamster	6,576	26.20%	1.2	0.29	0.03	2700	Mostly drop in
Horse	2154	28.60%	1.2	1.2	0.10	960	Mostly drop in
Whale	4543	34.30%	2.1	0.48	0.07	1875	Drop in

Table 2. We used linear regression to predict math course letter grade point with number of visits, high school GPA, and SAT or ACT.

School	# of students	R^2	Increase in grade per 1 visit	Increase in grade point per 1 grade point HS GPA	Increase in grade per 1 std deviation SAT/ACT	# of withdraws incompletes
Bird	1096	0.17	0.003	1.00***	0.26***	40
Cat	3270	0.26	0.019***	1.09***	0.59***	540
Dog	1004	0.25	0.035***	0.77***	0.49***	105
Dolphin	1070	0.15	-0.034**	1.63***	Not available	87
Goat	443	0.19	-0.057***	0.57***	0.36***	18
Gorilla‡	2737	0.19	0.015***	0.67***	0.24***	447
Fish	6609	0.09	0.022***	0.71***	0.13***	639
Hamster	5151	0.17	-0.002	1.08***	0.13***	850
Horse	1971	0.12	-0.006	0.20***	0.38***	69
Whale	3453	0.23	0.016***	0.86***	0.26***	360

*** $p < 0.001$, ** $p < 0.01$, * $p < 0.05$ 1; ‡Gorilla U used HS GPA in mathematics courses rather than overall HS GPA.

Table 3. Type of Tutors and Qualifying Criteria/Events For Tutors to Tutor a Course

Institution's types of tutors*	Qualifying Criteria/Events for Undergraduate and Graduate Tutors				
	Training		Previous/Concurrent Course Experience as…		Course-Specific Tasks
	General Tutoring (first semester)	Course-Specific	Assistant	Instructor	
Bird *UGs*	10 hrs	2-3 hrs/semester	none	none	none
Cat *UGs, GTAs*	5 hrs for GTAs before first semester; weekly meetings for new GTAs in fall	10 hrs per tutoring area if no experience for UGs & GTAs	UGs as LAs or graders, GTAs as recitation leaders	GTAs as course instructors	course-specific training exam
Dog *UGs, GTAs*	9 hours for GTAs. 17 hours for UG	weekly coordination meetings	UG and GTAs as LAs in course tutored	GTAs as course instructors	none
Dolphin *UGs*	3 hrs for UGs	none	none	none	none
Gorilla *UGs*	4 hrs for UGs	integrated in general training	none	none	exams for UGs
Goat UGs, GTAs	12 hrs for all	none	none	none	none
Fish *UGs, GTAs*	6 hrs for UGs plus pedagogy course 15 hrs GTAs	6 hrs UGs	UGs as LAs	none	none
Hamster *GTAs*	brief training on software	available but voluntary	drill section leaders	GTAs as course instructors	none
Horse *UGs, GTAs*	limited and voluntary	limited and voluntary	none	none	exams for UGs
Whale *UGs, GTAs*	5 hrs for GTAs; 4 hrs for UGs	none	none	none	none

*Only GTAs (graduate Teaching Assistants), UGs (undergraduate students), and LAs (undergraduate learning assistants) are included in the table even though many institutions reported faculty holding office hours in their centers.

Table 4. Connections between math centers and math department during data collection period.

	Connections to Department
Bird	• Math center director was also calculus coordinator. • Tutors had access to common homework sets for most classes.
Cat	• Math center director and associate director were renewable term faculty members in the math department; associate director was a course coordinator. • Math center director met monthly with all course coordinators. • 3 of 5 course coordinators voluntarily held office hours in math center. • Course coordinators and math department advisor led the course-specific training.
Dog	• Math center co-directors were both math tenure-track faculty. • All instructors held office hours in center and gave extra credit to attend center.

	• Tutors interacted with faculty often because they were LA's in their classrooms.
Dolphin	• Math center director taught calculus for the math department.
Goat	• Math center director was a tenured associate professor of mathematics • Tutors had access to a blackboard site where resources were posted.
Gorilla	• Math center director was a renewable term math faculty member. • Math center director communicated weekly with all course coordinators. • Course coordinators shared instructional materials with the math center.
Fish	• The center is in the mathematics building. • Math center director was a renewable term math faculty member. • Undergraduate learning assistants tutor in center and help with group work in math courses.
Hamster	• Math center co-directors were both renewable term math faculty members. • Since course instructors held office hours in math center, they shared information regarding the center with their students.
Horse	• Math center director taught calculus for the math department. • Tutors visited all classes served by the math center to advertise at start of term.
Whale	• Math center director was a math faculty member. • Math center organized review sessions prior to course exams.

Table 4 includes a summary of the relationship between the math departments and the ten math centers in the study sample. A variety of features indicate a strong relationship between the tutoring centers and the math departments. One feature is having faculty tutoring at the center because this makes it easier for undergraduate tutors to discuss the course they tutor with faculty.

Conclusions, Limitations and Future Work

One limitation of the data set is the use of letter grades instead of percent grades as the outcome variable in the model. Dog U did have access to course percent grades. Students who earn $\leq 60\%$ get an F; yet, a student with a grade of 59% is typically quite different than a student with a grade of 10%. To determine the effect on the model of categorizing all grades as either 0, 1, 2, 3, or 4 rather than using the percentage earned, we ran separate models for Dog U using standardized grade point and standardized percent. The coefficient for the variable "visits to the tutoring center" was the same in each model to two decimal places. In this model the effect of using grade points instead of grade percent is minimal. Note that Dog U had the strongest relationship between visits and course grade. This relationship can be visualized in Figure 1.

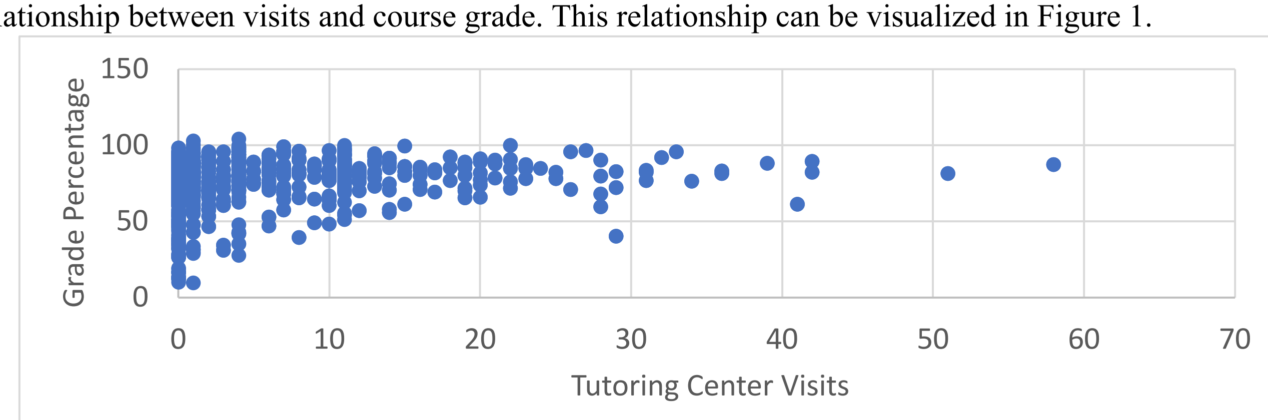

Figure 1. Relationship between grade percentage and tutor center visits at Dog U in Fall 2017.

In the future, our group will use the Delphi method to generate hypotheses about the characteristics of centers that contributed to their effectiveness (Clayton, 1997).

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Exploring the Knowledge Base for College Mathematics Teaching

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In this paper we explore a wide sample of currently available instructional materials intended for college mathematics instructors (textbooks, magazines, teacher editions, lesson plans, teaching articles, classroom notes for flipped classrooms, books, etc.) in order to assess how available materials are building a knowledge base for teaching. We modify a framework from Hiebert & Morris (2009) to look for key categories of knowledge that are fundamental for a knowledge base for teaching mathematics. We found that few articles contained meaningful amounts of multiple categories. We use the categories to describe the nature of current available materials and argue that a new genre of instructional material and scholarly work to create the missing knowledge is needed.

Keywords: College Mathematics Instruction, Knowledge Base for Teaching, Instructional Materials, Lesson Plans

College mathematics suffers from the same lament that Dewey (1929) pointed out about K-12 schooling in the US around a century ago, that teachers take their best ideas with them when they retire, and new teachers are forced to start over when they begin their profession. This is not true of other professions (Hiebert, Gallimore, & Stigler, 2002). A knowledge base for practicing medicine, for example, is continually being built so new doctors can build on the collective knowledge of the doctors that have come before them. In K-12 education, the prime example of sharing professional knowledge are teachers in Japan, particularly the teachers in elementary and junior high school (Hiebert et al., 2002). Researchers have called for a system to develop, store, and share a knowledge base for teaching in grades K-12, mainly through high-quality instructional products, either written or in video (Hiebert, Gallimore, & Stigler, 2002).

How is the college mathematics community doing at creating a knowledge base for teaching? There are several outlets for college mathematics instructors to share instructional knowledge with their peers: books, periodicals, lesson plans, teacher edition of textbooks, magazines, class notes or lesson plans on the web, blogs, etc. However, no one has analyzed the nature or quality of this wide breadth of products, products we refer to as *written instructional products* (WIPs). We analyze a broad sample of WIPs to better understand which WIPs might be best able to capture, or are capturing, a knowledge base for teaching college mathematics, or said another way, which WIPs might provide the best opportunity for college mathematics instructors to learn their craft. Because there has been such little work on analyzing college instructional materials, especially peer-to-peer materials, we draw heavily on work done in K-12 mathematics, where instructional materials have been more heavily studied.

Literature Review

Below we briefly highlight ideas and findings in the K-12 literature on developing a knowledge base for teaching.

Researchers have called for a system to address the lack of instructional improvement: a system for generating, storing, and sharing instructional knowledge. Such a system is called a *knowledge base for teaching* (Hiebert, Gallimore, & Stigler, 2002). The prime example of a knowledge base for teaching is junior high and elementary school teachers in Japan (Hiebert, &

Morris, 2012). With their constant learning of instruction and mathematics through lesson study, teacher math circles, and their wide proliferation of detailed lesson plans and teaching books, teachers have extensive resources to draw on as they strive to become master teachers and develop high-quality lessons.

Even though work to develop a knowledge base for teaching mathematics for US teachers began about two decades ago, few efforts have actually worked on developing specific materials for teachers. Two specific examples have emerged in the literature, both in mathematics teacher education. The University of Delaware (Hiebert & Morris, 2009) and The University of Michigan (Ball et al., 2009) each had groups developing instructional materials for mathematics courses for future elementary school teachers that used annotated lesson plans as a vehicle for storing and sharing a local knowledge base for teaching. The lesson plans took on a nonconventional role for lesson plans in the US, one in which information beyond the instructional plan was provided that would help a teacher understand the background, the content, the purpose, the potential pitfalls, etc., of the instructional choices. This information began to capture some of the thinking and reasoning of expert teacher educators and allowed consumers of the lesson plans to build their knowledge of teaching teachers.

Researchers have found evidence of effective instructional materials that might add insight to building a knowledge base for teaching. In a study by Lewis et al. (2011) researchers found that Japanese teacher's manuals have two features that are found less often in their United States counterparts: anticipation of student thinking and explicit rationale for pedagogical decisions. They found in a randomized study that US elementary school teachers who were given materials that had features identified in Japanese lesson plans had instruction that better addressed students' difficulties and had students that performed better on unit tests. In another study researchers found the highest rated lesson plans from Japan and the US were focused on "specific, varied, and detailed" instances of student mathematical thinking (Corey, Williams, Monroe, & Wagner, in submission).

Hiebert and his colleagues pointed out that storing instructional knowledge is not a trivial issue (Hiebert, Gallimore, & Stigler, 2002). They explained that to build a knowledge base for the teaching profession, knowledge must be shared, examined, revised, and grown in the public sphere of the intended audience. Such involvement appears lacking in the US teaching profession in K-12 mathematics as well as college mathematics, although it may exist in some specific local contexts. Unfortunately, the practice of writing lesson plans that share the knowledge-base of teaching is not common to US K-12 mathematics teachers (Corey et al., in submission) and such lesson plans may be difficult to produce, even with ample time, by top teachers (Ninomiya & Corey, 2016).

Hiebert (2013) argues conversations and resources for changing teaching practice all point towards more specificity. For example, He calls for conversations or resources about teaching to be around specific, lesson-level learning goals where teachers can begin to consider specific actions that are best at achieving those goals in a specific context. He explains that only by focusing on specific details can the cause-effect relationship of teaching and learning come to be understood. Hiebert says:

> In addition to unpacking the details of teaching, studying teaching means seeing the cause-effect relationship between teaching and learning that infuse an ordinary lesson. Many teachers do not appreciate that slight changes in lessons – in the ways they interact with students around content – influence directly what students learn. When teachers see the

effects of the changes they make on what and how well students learn, they can begin to appreciate the powerful impact of studying the details of teaching (p. 53).

Research Question

Our primary research question is: What categories of a knowledge base for teaching college mathematics is captured by currently available WIPs?

Theoretical Framework

We modify the framework of Hiebert & Morris (2012) of a knowledge base for teaching. Their framework includes four important categories of instructional knowledge.

1) An understanding of the learning goals of the lesson and of the rationale for using particular instructional activities to help students achieve these goals.
2) Knowledge of students' thinking [which] is often represented as responses students are likely to give to particular questions and tasks, knowledge gathered from past experiences implementing the lesson.
3) Knowledge of the curriculum as a connected set of ideas and materials.
4) Knowledge of strategies and representations for teaching toward particular learning goals. This is the heart of the lesson plans. . . . [A]ctivities are described and suggested questions and teacher responses are provided. (Hiebert & Morris, 2012)

We have adopted this framework for our current study because it should give us a way to characterize the nature of instructional resources from a broad sample. These four categories capture our view of a knowledge base for teaching. Implicit in this view is that a knowledge base for teaching is shared in specifics. For example knowledge of teaching is really knowledge of teaching specific content (specific learning goals, specific student thinking related to those goals, etc) and in specific curricular contexts, and we would extend these to include knowledge grounded in specific administrative and cultural contexts with specific student populations.

Method

Data

We sampled a variety of sources that might have instructional knowledge for college mathematics instructors (books, journals, magazines, project websites, instructional guides, shared instructional materials online, blogs, etc). Our sampling was based on searching and testing. We searched for any possible published that might qualify as a WIP and analyzed samples of the source until we felt confident we understood the nature of the WIP. Our data consists of 32 publications (books, websites, periodicals, etc). For larger publications (such as books or textbooks) samples of chapters or sections were selected, so each chapter or section was considered a WIP. We ended up with a total of 145 WIPs (articles, chapters, lesson plans, etc) from the 32 publications. We randomly selected (via random number generation) three chapters/sections from each book unless we felt like that was too few to capture the variation across chapters, then we sampled more selected purposely to capture the variation within the publication. For magazines and periodicals, we tried to sample at least five articles that were related to teaching (contain the words *students* or *teaching*) and selected the most recent articles that we had access to. We over sampled on three periodicals because they explicitly create space for sharing instructional knowledge: the *Classroom Capsules* section of *The College Mathematics Journal (CMJ),* the *Classroom Note* section of the *International Journal of Mathematical Education in Science and Technology (IJMEST),* and classroom instruction articles from *Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS)*. For PRIMUS a classroom instruction article was defined as any article 1) focused on sharing instructional knowledge to other college mathematics instructors, and 2) focused on a particular mathematical topic, not an entire course or ideas across courses. We only considered WIPs that pertained to undergraduate mathematics classes, not graduate or mathematics education courses (such as mathematics for elementary school teachers). All sources are listed at the following website: https://tinyurl.com/wipvenues2019.

Coding and Analysis. The unit of analysis was the WIP (article, chapter, section, lesson plan, etc). We began coding each unit to the extent that it contained knowledge in the 4 categories of teaching knowledge explained in the Theoretical Framework section above. We did split the first category into two categories for coding, so there was a code for "understanding the learning goals of the lesson" and another for "the rationale for using particular instructional activities to help students achieve these goals." We added and modified categories during the coding process as elements arose that we thought could add insight to the nature of the WIPs in our sample. One code that we added that we will discuss in the results sections is the extent that the WIP focuses on mathematical content (content-focused). In this short paper, we only share results for the following categories: Content Focused (CF), Knowledge of Strategies and Representations (KSR), Knowledge of Student Thinking (KST), Rationale of Instructional Decisions (RID), and Knowledge of Curriculum (KC).

The extent that a WIP captured knowledge in each category of teaching knowledge was represented on a four-point scale: None, Trace, Minor Focus, and Major Focus. The code "None" represented no knowledge, "Trace" represented one mention (up to one paragraph), "Minor Focus" represented two or three instances (each instance in a separate paragraph), and "Major Focus" represented four or more instances (each instance in a separate paragraph). Only results of Major Focus codes are used to generate results for this paper.

All coding was done by two research assistants, with coding checks done by the head researcher. The research assistants coded the same WIPs until the inter-rater reliability was above 90%, after which only one coder coded each WIP. If a new code was added, the previously coded WIPs units were recoded to include the new code.

After coding was complete, WIPs were analyzed based on the codes. We performed descriptive statistics to look for patterns to understand what categories of a knowledge base for teaching were being emphasized in our sample, and which ones were not.

Results

A strong pattern in our data showed a hierarchy structure to the following codes: CF, KSR, KST, and RID. By hierarchy structure we mean that one code showed up mainly if another code was also present. Few WIPs shared knowledge of strategies and representations (KSR) that were not focused on mathematical content (CF). Few WIPs shared knowledge of student mathematical thinking (KST) that did not include KSR. Similarly with RID and KST codes. The bulk of the WIPs in our sample could be classified, then by how far toward sharing KST and RID they made it. We now share descriptions and examples of the kinds of WIPs in each of these categories: No Content, CF, CF+KSR, CF+KSR+KST, and CF+KSR+KST+RID. The KC (Knowledge of Curriculum) code did not fit well in this same structure and was relatively rare (less than 7% of all WIPs in our sample received a major emphasis on this code).

Categories

No content. There were 10 WIPs that had no mathematical content. Eight of these were from a book on Scholarship of Teaching and Learning (SoTL) for mathematicians (Dewar & Bennett, 2014). This book helps mathematicians understand how to perform SoTL to answer questions that could improve their teaching by performing small educational studies. The book shares many specific SoTL studies by mathematicians, many of which fall into this category. Many of the studies had research questions focused on course structure or administration, or general instructional strategies. An example of the latter is the effectiveness of using a game to review material periodically throughout a course.

Content Focused (CF) only. The majority of the WIPs (116 of 145, or 80%) in our sample had a major emphasis in this code, and about 60% of these did not receive any major emphasis on subsequent categories. Typical to this category were textbooks, some teacher editions of textbooks, some chapters of MAA published books (such as the Notes series), periodical articles (e.g. The Mathematics Magazine, Classroom Capsules from CMJ, some Classroom Notes from IJMEST), and some blogs (e.g. betterexplained.org). Typical examples in this category are periodical articles that show a new proof to an existing theorem that is taught in an undergraduate course, an article explaining an interesting mathematical idea or connecting two seemingly unrelated mathematical ideas, or a chapter in a book or textbook explaining mathematical content that could be taught in a mathematics course.

We tried to sample WIPs that were in some way tied to teaching (in book categories on teaching, mention students or teaching, etc) however, the majority of the WIPs in this category were focused on mathematics and related to teaching in only the slightest sense, if at all. This doesn't mean that these WIPs cannot be brought into the classroom, or are not helpful to teachers. It simply means that explanations of how to teach the content are lacking.

CF+Knowledge of Strategies and Representations (KSR) only. Only about 36% of the WIPS in the previous category (29% of all WIPs) actually provide readers specific tools for teaching the specific mathematical content, not just sharing mathematical content itself. Hiebert & Morris (2012) describe the KSR category as the "heart of the lesson plans" because it provides details for the teacher about what to do during instruction. Typical WIPs in this category were online lesson plans like those from SIMIODE.org (Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations), an online database of modeling activities for differential equation courses. These activities include student handouts to walk them through the modeling activity. However, student responses to the questions on the worksheets were not discussed so they did not get coded as containing knowledge of student thinking. A couple of book chapters of a similar flavor also ended up in this category, explanations of what the teacher should do, but no (or few) supports based on student responses.

CF+KSR+Knowledge of Student Mathematical Thinking (KST) only. Only 13 out of 145 total WIPs were coded a major focus on all three of these codes. These emerged in a variety of sources. Two of them were detailed lesson plans from ProjectMaths.org which include example scripts for teachers (including student responses). Four of them were from periodicals, three were from sections of a Pathways College Algebra (Carlson, 2016) teacher edition, and one was from a section of the MAA Instructional Practices Guide. One example article from IJMEST was about a new technique for helping students evaluate triple integrals without having them sketch potentially complex shapes in order to change the bounds of integration. The article used research that showed how students struggle to sketch the three-dimensional figures required for that technique. The alternative approach uses integration by parts to evaluate the triple integral.

Although we are narrowing in on WIPs matching an increased number of knowledge categories from our framework, most of the WIPs in this category were still not very insightful. The kind of student thinking in many of the 13 WIPs were simply how teachers expected (or perhaps better described as wanted) students to solve a problem or answer a question. There were three that stood out from the others, which we discuss in our next category.

MC+KSR+KST+Rationale of Instructional Decisions (RID). Three WIPs out of the 13 in the previous category also were rated as major on the RID code. More than just adding in explanations about why certain actions were taken, these three WIPs had a different feel to them than the other 10 WIPs. Their explanations and justifications about why certain actions were taken were closely tied to the whole breadth of student mathematical thinking (correct, incorrect, misconceptions, unhelpful, helpful, naïve, advanced, etc.) not just what teachers wanted students to think. These articles tend to report an instructional problem, propose a carefully engineered instructional solution to the instructional problem, provide sufficient background information, describe and draw on student thinking (particularly unhelpful/incorrect student thinking), give specific teacher moves (often justified by how they follow from or impact specific student thinking), and 2 of the 3 give some example teacher-student interactions. We describe one of the three articles to illustrate the nature of these WIPs.

The example we use to illustrate this category is an article published in PRIMUS called *Let's Draw a Picture* (Herrera et al., 2017). In this article the authors help instructors understand how to use number line representations to help students reason about arguments in an analysis class. Herrera, et al. (2017) write that many students believe that pictures or graphs are not helpful for developing a proof because "Many analysis textbooks reinforce [this belief] by refraining from providing or referencing visual representations" (p. 47). One particular difficulty students have when working with a number line is to understand what algebraic expressions might represent (for example (B-A)/2 or (A+B)/2). The authors have specific prompts to help students build these skills. For example, one class activity is to take a number line with two points labeled A and B with A<B, and no other markings on the number line. The teacher then asks students to name a number to the right of B that can be plotted accurately and also to find a number between A and B that can be plotted accurately. The most common initial responses to the first prompt (finding a number to the right of B) are B+A, 2B, and C. Each of these warrants a class discussion about why these numbers might not work or might not be helpful. Students can then return to the problem, thinking more carefully, where some students find that the only distance they can represent is the distance between A and B. Students then work on representing that distance (B-A) and add that distance to B to get a new point on the number line to the right of B. Discussions follow about the ways to represent distances and locations on a number line.

In this short example we have a situation where college mathematics instructors are giving specific instructional moves for them to use in class, with a variety of student responses, so teachers can be prepared beforehand on how to handle the responses and help deepen students understanding and improve their thinking. Moreover, the specific prompts (we have only shared one) are aimed to build an important skill that can be widely used in an analysis class.

Discussion

Our results give a few insights about the nature of current resources for teaching college mathematics. First, although there is a wide variety of resources available the focus of the bulk of the resources is on mathematical content. One way to interpret our findings is that as a community we like to share ideas about *what* to teach, but much less about the nitty-gritty of the actual *practice of teaching* in the way described by Hiebert (2013). Even many of the resources that discuss teaching issues explicitly are focused on general instructional ideas (e.g. MAA Instructional Practices Guide). The combination of specific content with general instructional ideas leaves a lot of work to the instructor. What activities should I do in class? What can I do to make this interesting? To engage students? How will students respond? Where do students struggle with this topic? Why do they struggle? How can my activities help them to change the way they currently think? Etc. Although there will always be some adaptation required by the teacher to meet the needs of his or her individual students and fit his or her context, this situation requires the instructor to do a lot of heavy lifting in creating a high-quality lesson. Furthermore, these materials already seem to assume that instructors know how to do create and/or adapt lessons well, but where are they supposed to learn these skills in the first place?

Second, WIPs focused on teaching specific content often don't take student thinking as important considerations in making instructional considerations. Some materials that did discuss student thinking only used (or assumed) thinking from ideal students – responding as the instructor wanted them to respond, not actually taking on the variety of actual student thinking (correct and incorrect) going on in students heads. Some instructors might not consider actual student thinking as a serious issue, however, we argue that college mathematics instructors' primary medium is student mathematical thinking. The mathematical thinking of students is the raw material instructors work with and shape and refine to create deeper, broader, more robust student thinking as a result of their instructional interactions with students. It seems that WIPs would need to take student mathematical thinking seriously to give instructors powerful tools to use in their teaching. Without a focus on student mathematical thinking, conversations about instructional choices and learning about teaching will not reach the level of "seeing the cause-effect relationship between teaching and learning" explained by Hiebert (2013. p. 53).

We found it difficult to find WIPs that captured many elements of the knowledge base of teaching. Why are they so rare? There might be many reasons. For example, consider who is tasked or incentivized to produce such work. The work to create this knowledge doesn't seem to fall under traditional scholarly activities of any field, and there is no dedicated outlet for it (so no way for such work to count towards tenure and promotion). The work of generating and documenting knowledge as part of the knowledge base for teaching might look a lot more like engineering (or design) than traditional mathematics or mathematics education research. Our top WIPs were carefully engineered solutions to instructional problems, with documented student thinking as a reason for and in response to instructional decisions (activities, questions, explanations, etc). In this respect, they do not fit any genre of current scholarly work. Even in the broad field of the Scholarship of Teaching and Learning (SoTL), such work is rare (Weimer, 2006), although it still seems to us like the best place to couch such scholarly work. Without dedicated venues the college mathematics instructor community, no one would be generating or consuming WIPs that try to capture detailed instructional knowledge.

We know of college mathematics instructors that put effort into building knowledge about college mathematics teaching as a way to improve their own instruction, perhaps through disciplined methods of refining lessons over time, or by trying small experiments in class to overcome instructional problems. However, they don't do it with an eye towards sharing their knowledge so it is not documented, and of course, there is no venue to have it vetted and shared. Furthermore, the knowledge cannot be shared, examined, revised, and grown in the public sphere of the intended audience which is a necessity argued for by Hiebert, Gallimore, & Stigler (2002). So we are still left with Dewey's problem, that knowledge being taken out of the education system when the professor retires and the new professors having to largely start from scratch.

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Coordinating Two Meanings of Variables in Proofs that Apply Definitions Repeatedly

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In this paper we use data from a recent teaching experiment to characterize a particular challenge in proof comprehension that arises when a proof uses the same definition multiple times in related ways. This difficulty involves coordinating the role and value of a quantified variable in the definition, a distinction we explicate in the paper. We do so by providing two proofs that exemplify the key difficulty and presenting an episode in which two students who were reading a proof together adopted alternate interpretations of a variable (one as value and the other as role), but were for a time unable to coordinate the two meanings. The primary contribution of the paper is to sensitize the research community to this challenge in a way that will help future research on students' reading of mathematical proof.

Keywords: proof comprehension, quantification, variable, universal generalization

As the body of research on undergraduate student learning of mathematical proof expands, researchers have begun to identify particular proof types and frameworks that pose unique challenges to students who read them. Recognizing these nuanced aspects of particular proofs expand our awareness of the many competencies involved in learning to read and write mathematical proofs. In this paper, we identify a specific challenge in interpretation that can arise when a proof text uses the same definition in multiple, related ways. The particular challenge we wish to highlight, which we shall define more precisely in the next section with multiple example proofs, deals with the use and meaning of variables (quantified in the definitions) in such proofs. We explore this phenomenon as it arose in a recent study of novices reading mathematical proofs, drawing implications both for researchers' awareness of the subtle challenges in reading proofs and of the role of understanding of mathematical concepts in comprehending proofs (e.g., Dawkins & Karunakaran, 2016).

Defining and Exemplifying the Phenomenon with Two Sample Proofs

The particular issue we have observed in multiple mathematical contexts can occur when a definition uses the same definition with respect to multiple, closely-related objects (call them A and B). Specifically, the definition must include some quantified variable such that the definition will require (possibly) different values of this variable for both objects (call them x_A and x_B). At this point it is important to note that the label x_A carries two meanings: the particular *value* of the variable and the *role* that variable plays in the definition. It operatively bridges between the conceptual meaning of the definition and the mechanics for how this definition will be used in proof writing. The challenge we wish to examine arises when the value of the variable for object B appears as a function of the parameter x_A. In this case, this new use of the definition distinguishes the value of the variable and its conceptual role in the definition. The example proof in Figure 1 exemplifies this phenomenon.

The variable ε in the definition plays the *role* of an error bound (Oehrtman, Swinyard, & Martin, 2014) as it does throughout analysis. However, it also is understood to stand for an arbitrary, particular *value* that is fixed throughout the text. The use of ε in the first line unifies these two meanings. The next two lines separate these meanings. When the second and third lines of proof bound the error between the sequence terms and the limits they approximate, the value $\frac{\varepsilon}{2}$ now plays the *role* of error bound while ε refers to the same *value* as in the first line.

Definition: A sequence of real numbers a_n **converges** to $L \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - L| < \varepsilon$. In this case, $\lim a_n = L$.

Theorem: Given any two convergent sequences a_n, b_n such that $\lim a_n = L$ and $\lim b_n = M$, the sequence $a_n + b_n$ converges to $L + M$.

Proof: Let $\varepsilon > 0$.
By definition of convergence, $\exists N_a \in \mathbb{N}$ such that $\forall n \geq N_a$, $|a_n - L| < \frac{\varepsilon}{2}$.
By definition of convergence, $\exists N_b \in \mathbb{N}$ such that $\forall n \geq N_b$, $|b_n - M| < \frac{\varepsilon}{2}$.
Define $N = \max\{N_a, N_b\}$.
Let $n \geq N$.
Then
$$\begin{aligned} |a_n + b_n - (L + M)| &= |a_n - L + b_n - M| \\ &\leq |a_n - L| + |b_n - M| \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$
Thus, $\lim(a_n + b_n) = L + M$.

Fig 1. Example proof in which the two meanings of ε become distinguished.

Notice that the proof also coordinates the variable N (an index bound) in multiple uses of the definition, but in this case the notation makes this division of meaning explicit. This is because different variable names are given the different index bounds for each sequence. Each value is named N to notate the shared *role*, but the subscripts distinguish the three *values*. Thus, there is not the same separation of *value* and *role* in a single expression as before. The functional dependence among the index bounds is made explicit in the fourth line. Understanding the use of the parameter N in the proof still requires making sense of this *role/value coordination*. We do not assume that being more explicit is better or will necessarily ease comprehension.

The different uses of the definition can also be distinguished in light of the proof mechanics for quantified claims. The law of universal generalization (Copi, 1954) asserts that a universal claim can be justified by proof about an arbitrary particular, as is done in this proof regarding ε. Because the definition is only being verified for the sum sequence, only its error bound need remain arbitrary. Accordingly, the proof is constructed to verify the inequality in the penultimate line with regard to the *value* ε and not a function of ε[1]. Since the definition is assumed true for the two component sequences, the existence of the index bound N may be asserted for any positive error bound, specifically the function of epsilon $\frac{\varepsilon}{2}$. Since the definition's existential claim about N is not verified using universal generalization, its existence can be satisfied using a function of other *values*, as in line 4. N is also a function of ε, but this remains implicit in the chain of quantities introduced in the text.

The second example proof was presented to students in a recent study from which we take the data presented in this report. Figure 2 displays it as it was presented to the students. This proof displays the same pattern with regard to the variable k, which plays the role of quotient in

[1] There is an alternative tradition of writing such proofs that selects an arbitrary epsilon and allows the final inequality to be proven for any constant multiple thereof. This avoids the need to apply other definitions to (non-obvious) functions of ε, but its application of proof by universal generalization is less immediate. The implications of either system for students' learning of such proofs is worth exploring, though we shall not do so here.

the case that n is divisible by d. In the proof, x is assumed to be a multiple of 6, justifying the existence of the integer k. The definition of x being a multiple of 3 is then verified with $k * 2$ playing the *role* of quotient. This fits our described pattern because the use of k in line 2 unifies the quotient *role* and *value* while the use in lines 3 and 4 distinguishes them.

Definition 1: We say the integer n is a **multiple of d** whenever there exists some integer k such that $n = k * d$. This can also be stated as "n is divisible by d" or that "d divides n."
Notice that in the case that n is a multiple of d meaning $\frac{n}{d} = k$, k is called the **quotient** of dividing n by d.

Theorem to be proven 1: For every integer x, if x is a multiple of 6, then x is a multiple of 3.

Proof 1.1: Let x be any integer that is a multiple of 6.
Then by Definition 1, there exists some k such that $x = k * 6$.
Since $6 = 2 * 3$, $x = k * 6 = (k * 2) * 3$.
Since $k * 2$ is also an integer, according to Definition 1 x is a multiple of 3.

Fig 2. Proof applying the definition of multiple twice.

Relevant Literature

As mathematics education researchers continue to explore the complexities, challenges, and opportunities of the teaching and learning of proof, it is natural that we begin to recognize the diversity among mathematical proofs. Many previous studies have explored the cognitive dimensions of familiar proof types such as contradiction (Antonini, 2003; Antonini & Mariotti, 2008), contraposition (Antonini, 2004; Stylianides, Stylianides, & Phillipou, 2004; Yopp, 2017), or induction (Harel & Brown, 2008; Arnold & Norton, 2017). More recent studies have identified more specific and subtle proof structures that students may encounter such as non-constructive existence proofs (Brown, 2017) or conditional implies conditional proof frames (Zandieh, Roh, & Knapp, 2014). While such nuanced proof techniques may appear less frequently in the curriculum than, say, proof by contradiction, they can be of unique interest to proof research. They should be studied in part because they help reveal a diversity of sense-making competencies that students must develop to read and produce proofs. They also are useful in revealing how students construe proofs.

By breaking from more standard proof frameworks, unique proof structures may afford distinct insights into how students interpret the inferential structure of proofs. As Harel and Sowder (1998) famously described, students may imitate aspects of proof writing as a ritual without unpacking their role in justification. We hope that learners of mathematical proof come to think of proofs as arguments that entail logical necessity. We still lack clear ways to operationalize this central notion. In the past, mathematics educators tried to capture this by asking students if they were absolutely convinced of the truth of claims in the presence of a valid proof (i.e. did not desire further empirical verification; Fischbein, 1982). This is not inappropriate, but appears less reliable in light of Weber, Inglis, and Mejia-Ramos' (2014) findings about how mathematicians gain conviction.

An alternative that Brown (2017) exemplifies is to study how students interpret such proof texts to reveal the inferences they perceive to be at play. She has shown how non-constructive existence proofs are fruitful in this regard because of their subtle inference structure. Such proofs serve as productive boundary objects between the community of mathematicians (of which we mathematics educators and instructors serve as representatives) and the community of novice students who are learning fluency within the genre of proof. We claim that the same can be said of proofs in the category we described above because they help reveal how students make sense of proof by universal generalization (Durand-Guerrier, 1996) and various uses of definitions in a proof text. To our knowledge, there is yet little or no research focused on student learning of the mechanics of proving quantified statements as described in the previous section.

Methods

The data we present here appeared early in a teaching experiment (Steffe & Thompson, 2000) on inference in mathematical proof. It is beyond the scope of this report to detail the experiment in full; rather, we intend only to set the stage for the episode we shall analyze. We recruited two Calculus 3 students at a medium-sized research university in the United States. The teaching experiment consisted of six one-hour sessions held outside of class time. The participants were compensated monetarily for their time. All sessions were video recorded and the participants' written notes were maintained.

The focus of the teaching experiment was to help the students learn about the force of proofs by universal generalization. The primary task we repeated throughout the experiment was to provide students with theorems to be proven and multiple proofs of related claims (the proof in Figure 2 is labelled 1.1 since it was followed by other possible proofs of the given claim). Students were tasked with deciding whether each proof justified the given theorem and, if not, to decide what claim it proved. Of those who volunteered for the study, we selected students who displayed cognitive conflict (Roh & Lee, 2018) on a screening survey. This meant that they either affirmed a mathematical claim while also affirming a proof meant to disprove that claim or they denied a mathematical claim while also affirming a proof meant to verify the claim. The task in Figure 2 was the first one presented to J and Z, who were both engineering students at the time of their participation in the study. Neither student had taken any proof-oriented university courses before participating in the study.

Analysis

The first author served as the teacher/researcher (Cobb & Steffe, 1983) and the second author served as the outside observer (Steffe & Thompson, 2000). Both authors wrote extensive field notes after each session, reviewed video, and met between sessions to form hypotheses about student learning and to plan activities and questions for the next session. Our analysis in this report focuses only on this first session with J and Z, specifically on their work to make sense of the use of definition and use of variables in the proof in Figure 2.

Rationale for this investigation

While we recognize that these students were quite novice with regard to mathematical proof, we find their initial reading activity worthy of study because it teaches us about the reading competencies that students may bring to bear when first learning proof. The proof text we provided is brief, draws upon mathematical properties that should be familiar to Calculus 3 students, and employs direct proof methods. As a result, we expect it mirrors many proof texts that students would see in a Transition to Proof course (David & Zazkis, 2019). By understanding the challenges students face in reading such texts, we may learn about the competencies that Transition to Proof instruction needs to help students develop. As of yet, our research community has relatively little research-based insight into the competencies essential to understanding proofs that might serve as learning objectives for Transition to Proof courses (some insights about competencies like reading strategies notwithstanding; e.g., Weber, 2001, 2015). In that vein, our current analysis seeks to explicate the ways of reasoning that students need to bring to bear to make sense of this proof text, as exemplified in J and Z's case. In this way, we note their difficulties in reading not as deficits in their understanding, but rather as goals for their development as novices. We are particularly interested in the role that students' understanding of the relevant mathematical concepts plays in proof understanding (Dawkins & Karunakaran, 2016), contrary to the assumption we anticipate many may hold that undergraduate students should have an adequate grasp of divisibility/multiple relations.

Results

As mentioned above, their stated task was to determine if Proof 1.1 proved Theorem 1 or something else. The interviewer assured them that the proofs were correct, but they may or not prove the given claim. J and Z took about 50 minutes discussing their initial reading of Proof 1.1, which was their first task in the experiment.

Multiples and Divisors

J began the discussion of the reading by asking about the relationship between k and x. In trying to discern this, Z linked line 1 with the statement of the theorem and explained that it proved that if a number was a multiple of 6, then it could also be a multiple of 3. Z thus saw the parallel between the first and last lines of the proof and the theorem statement. She also linked the equation in line 2 with the given definition, which J said was helpful.

Both students then expressed some difficulty understanding what values k took on. Z expressed two different ways of thinking about the multiple/divisor relation. At first, Z viewed k as the outcome of a division operation, as is suggested in the note below the definition. Alternatively, when she thought about multiples, she engaged in skip counting in which case she did not explicitly attend to a meaning for k. When the interviewer invited her to take an example she read provided (20 is a multiple of 5) and connect it to the written definition, she resorted to dividing again to render k a quotient. A consequence of this was that she began substituting values like 2 or 3 for x and dividing by 6. Her thinking about dividing did not impose the constraint that x be a multiple of 6, though they had already noted this restriction in line 1.

The interviewer pointed out that 2 and 3 were not multiples of 6 because the quotients were not integers. At this point, Z then shifted to thinking k must be a multiple of 6. It appeared that she was trying to recognize how the equation $x = k * 6$ represented the multiple of 6 property. She did not think of the composite unit $k * 6$ as the multiple of 6, so she instead wondered if k were a multiple of 6. The interviewer asked her to consider $x = 12$ to help her recognize that k could take on other values.

Even in this early stage of discussion, we note a few trends. First, J and Z worked to connect the variables to *roles* in the multiple/divisor relation. However, shifting between quotative and multiplicative meanings shifted their attention non-trivially. Z's skip-counting way of thinking about multiples did not give a role to the counter k, but her quotative meaning did not help her maintain the stated constraints on x. Furthermore, the pair initially struggled to overlay those meanings with the equations in the proof text, at least in part because they struggled to think of a product as a single unit.

Two Meanings for k Emerge

In the next part of the session, the pair worked to make sense of the third line of the proof. Z noted that "$6 = 2 * 3$" means that 6 is a multiple of 2 and 3. She describes that the proof is splitting 6 in this way. She then claims that 2 and 3 are now values for the k. One possible explanation is that she is trying to make a structural analogy between the two equations in line 3, rather than viewing the first as a (rather obvious) warrant for the second. However, she has some trouble explaining the meaning of "$(k * 2) * 3$" as it related to the definition of multiple.

J was at this point trying to make sense of these equations in the case where $x = 12$. The pair had seen that in this case $k = 2$. J further noted that 12 divided by 3 was 4, but she had trouble interpreting the expression "$(k * 2) * 3$" as expressing this same relation. While the difficulty of rendering $k * 2$ as a single unit may have played a role, we hypothesize that J was also having trouble because k was no longer unified both value and quotient role as it did in the definition and in line 2.

As the interviewer tried to help J connect the number sentence $12/3 = 4$ and the equation $x = (k * 2) * 3$, Z suggested that k is 4. We interpret this as her way of saying that, with regard to the definition of multiple of 3, the quotient was now 4. While this was numerically inaccurate regarding the equations in the proof, it demonstrated Z's recognition that line 3 is applying the definition in a new way. Accordingly, there is a new quotient when dividing by 3.

For the next few minutes as they worked to resolve this tension, J persisted in seeing k as the value of the variable, which remained 2 throughout, while Z treated k as the quotient, which was 4, relative to the property "multiple of 3." Because neither student was able to coordinate the value and role meanings for k, they both had trouble making sense of the expression $(k * 2) * 3$ in a normative manner. J had trouble seeing $k * 2$ as a unit that could fill a role in the definition. Z did not connect the compound unit $k * 2$ to the quotient of dividing by 3 because k filled that role for her.

Z then began listing various multiples of 6 and computing the values for k in line 2 and the values for k in line 3. She noted that all of the latter values were always even. When the interviewer asked her why this was true, she explained that she noticed it from "doing the math." This provides further evidence that she did not see the expression "$(k * 2)$" as equivalent to the quotient when dividing by 3.

Confirming Evidence

At this point, the interviewer had formed hypotheses about how the two students were interpreting k in different ways and having trouble coordinating the two meanings in the expression "$(k * 2)$." To confirm this, he asked them to interpret the proof when $x = -54$. They computed the quotient with 6 to be -9. Z then substituted this into the equation in line 3 of the proof, yielding -18*3. The interviewer asked the pair what the product would equal. Neither anticipated that this product would still be -54, but rather they proposed computing the product with a calculator. Z went on to affirm that the product was -54. He asked for the quotient of -54 divided by 3. Again, neither student anticipated the answer from the given information. This supported the hypothesis that neither student was currently interpreting the equation "$k * 6 = (k * 2) * 3$" as a way to coordinate divisibility by 6 with divisibility by 3.

The interviewer introduced a new approach by inviting J to divide both sides of the equation $x = (k * 2) * 3$ by 3. J accordingly wrote down the new equation "$\frac{x}{3} = k * 2$." J found this new representation very appealing. Toward the end of the interview, the group read the second theorem to be proven and Proof 2.2 as displayed in Figure 3. Consistent with their stated preference for writing the equations quotatively, the interviewer rewrote the equations in the forms: $\frac{x}{14} = k$, $\frac{x}{7} = k * 2$, and $\frac{x}{2} = k * 7$. He used these equations to further explore how J and Z were coordinating the two meanings of the variable k. He proposed an example in which x was made up of 250 groups of 14, so $x = 250 * 14$. He accompanied his explanation with gestures to represent the groups of 14 lined up next to one another. He then asked how many groups of 7 would be in x. J and Z asked him to restate and clarify the question a few times before J suggested that x would contain twice as many 7s as it does 14s. When asked how many groups of 2 were in x, J said it would be 7 times the number of 14s. While the students were able to apprehend this relationship with guidance, this interchange served to confirm that the students had not previously recognized the equations in Proof 2.2 as stating a multiplicative relationship between the various quotients.

Theorem to be proven 2: For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.

Proof 2.2: Let x be an integer that is a multiple of 14.
Then by definition there is some integer k such that $x = k * 14$.
This means $x = (k * 7) * 2$, so x is a multiple of 2.
This also means $x = (k * 2) * 7$, so x is a multiple of 7.
Thus, x is a multiple of 7 and a multiple of 2.

Fig 3. The final proof discussed in the first session with J and Z.

Discussion

Consistent with our stated goal to identify the particular challenges posed with the proof mechanics that arise when a proof uses a definition repeatedly, we share this episode to demonstrate the significant difficulty it poses to novice students. It was somewhat fortuitous for us in our efforts to model J and Z's reasoning that they adopted distinct meanings for k and maintained them such that we could compare their construal of the text. We had noted a similar issue in previous studies when a real analysis student said of a proof like that in Figure 1, "$\frac{\varepsilon}{2}$ is just another ε." We recognized that this student's first use of the variable referred to the value while the second referred to the role as an error bound. This epiphany that the student was explaining so succinctly demonstrated how he was coordinating the role and value to make sense of the proof technique.

J and Z learned to coordinate the two uses of k in the proof presented in Figures 2, but this seemed to involve constructing both values of k independently (for multiple of 6 and multiple of 3) before relating them multiplicatively. J and Z's pathway to coordination was not through the symbolic expression $(k * 2) * 3$ because they experienced great difficulty assigning the components of the equation to roles in the two definitions. We hypothesize that this relates closely to their meanings for multiple that seemed to exhibit different qualities when they shifted between skip counting strategies and division operations. Because they preferred thinking quotatively, we found that rewriting the equations in terms of division was productive for this pair of students, though it later become unnecessary.

This study contributes to the literature first by sensitizing researchers to the role/value duality among variables in definitions that becomes quite important in proof mechanics. In addition, our analysis should contribute to our shared appreciation of how students' meaning for mathematical concepts play in their interpretation of proof texts. We expect that the literature will benefit from more thorough investigations of how students make sense of the proof mechanics relating to quantified variables in definitions, specifically issues of arbitrariness and independence (Roh & Lee, 2011). It is to these future studies in particular that we hope our characterization of role/value coordination will contribute.

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Student Verification Practices for Combinatorics Problems in a Computational Environment

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Counting problems have been shown to be challenging for students to solve correctly, and one reason is that they can be difficult to verify (e.g., Eizenberg & Zaslavsky, 2004). The field would benefit from further investigations into the nature of students' verification of counting problems. One possible avenue for verification is to have students engage with computational tools and environments. In this paper, we investigate how students verify solutions to combinatorial problems in the context of using Python computer programming. We show that students do not simply use a computed numerical answer when verifying, but they demonstrate a rich understanding between a computer program and a counting process. Further, we exhibit some affordances that using Python to verify solutions provides to the students.

Keywords: Combinatorics, Verification, Computation

Combinatorics is a field of mathematics whose use is becoming more prominent due to its applications in areas such as computer science, statistics, and communications. Despite its growing relevance, students at all ages demonstrate low success rates when solving combinatorics problems (e.g., Batanero et al., 1997; Lockwood & Gibson, 2016). Further, researchers have shown that verification of solutions is particularly difficult in counting (Eizenberg & Zaslavsky, 2004), and many students fail to detect subtle errors in problem solutions. There has been recent activity showing that computational environments can be leveraged to benefit student learning (e.g., Lockwood & De Chenne, in press), and we see an opportunity to investigate how such environments can be used to verify solutions to counting problems. In this paper, we report on a study in which undergraduate computer science students worked within a computational environment as a way to verify solutions to counting problems. We elaborate on ways in which students use relationships between counting processes and computer code to verify solutions, and we discuss affordances of the computational environment.

We seek to accomplish two goals. First, we want to provide an existence proof that students can engage in computational activity in order to verify solutions to combinatorial problems. That is, we hope to provide evidence that students are able to use a computational environment as a method of verification. Second, we seek to illustrate ways in which students leveraged the computational environment as a verification tool. Specifically, we hope to illustrate ways students use a computer program as a tool for verification that are impossible or impractical without a computer. As such, our results focus on examples of verification that were performed with a computer. We seek to answer the following research questions: *Do students engage in computational environments to verify solutions to counting problems? If so, in what ways do the students engage with a computational environment to verify solutions to a counting problem?*

Literature Review and Theoretical Perspectives

Combinatorial and Computational Activity

There has been reported evidence that one way to improve student success in combinatorics is to draw more attention to the relationships between the objects being counted and the way they are counted (Lockwood, 2014). When considering students' combinatorial thinking, we use Lockwood's (2013) model, which frames student's combinatorial thinking into three components: *Formulas/Expressions, Counting Processes,* and *Sets of Outcomes*. *Formulas/Expressions* are the mathematical expressions that yield a numerical value, such as 8*7*6. *Counting Processes* are the imagined or actual enumeration processes in which students engage, such as breaking problems into steps or cases. *Sets of Outcomes* are the elements that are being counted. Often, the cardinality of the set of outcomes determines the answer to the problem. With this framing, we wish to focus student attention to the relationship between sets of outcomes and counting processes. One way to strengthen this relationship is through the systematic listing of outcomes. While systematic listing of outcomes has been shown to be positively correlated with correctly answering combinatorics problems (Lockwood and Gibson, 2016), the sizes of many sets of outcomes can be large enough to deter listing by hand. For example, there are 10,000 possible 4-digit PINs. Partial listing also has drawbacks, as patterns may not extend to all cases and students may fail to detect subtle errors. However, by leveraging the computational speed of a computer, students can reap similar benefits of listing by hand by designing programs and algorithms to enumerate lists.

For our purposes, we define a computational environment as a computer interface capable of receiving and carrying out standard programming instructions, such as for loops, if statements, string concatenation, and integer arithmetic. For example, a Python coding interface such as CoCalc is a computational environment, whereas a four-function calculator is not. When using a computational environment to enumerate sets of outcomes via a listing process, students need to verify that the enumerated list is the desired set of outcomes. Combinatorics is notably difficult to teach and learn in part to there being few readily available solution methods, uncertainty in which methods to employ, and that two equally convincing solutions may yield different numerical values (Eizenberg and Zaslavsky, 2004). These issues persist whether or not a student uses a computational environment, making verification difficult but important.

Verification

We draw on Eizenberg and Zaslavsky (2004) to characterize verification in this study. Eizenberg and Zaslavsky (2004) detail five categories of verification schemes students employ when solving combinatorics problems: *(1) Reworking the solution* (this entails going over and checking all or parts of the solution, but without adding substantial justification)*, (2) Adding justifications to the solution* (this entails justifying components of the solution, or the solution process in a global way)*, (3) Evaluating the reasonability of the answer* (this consists of comparing the computed value to an estimated value, or comparing the computed value to the size of a more global set of outcomes)*, (4) Modifying some components of the solution* (entails either changing the representation of the solution, or using the solution method on a smaller set of outcomes)*,* and *(5) Using a different solution method and comparing answers* (this consists of calculating the answer using an alternative solution method).

In this paper, we take any instance of the above methods as evidence of verification. In particular, we take writing a computer program that lists and calculates the cardinality of a set of outcomes as a different solution method than calculating the cardinality by hand. For example, if a student calculated an answer by hand and also wrote a computer code to determine the answer, we consider this an instance of using a different solution method and comparing answers. As such, we are examining more than just ways in which students verify that a computer program correctly solves a problem; in fact, we will show that students develop rich relationships between writing code to solve a problem and solving it by hand.

Methods

Participants and Data Collection

The data presented here are from a teaching experiment (TE) (Steffe & Thompson, 2000) that consisted of 3-4 hours of contact time with each of four students in 60-90 minute individual interview sessions. In this paper, we focus on two students, Jamie and Allen (pseudonyms), during each of their final hour-long sessions. Both students were Electrical and Computer Engineering majors that were recruited from an introductory computer science course at a large university in the western United States. Both Jamie and Allan were chosen based on their responses to a recruitment questionnaire. The questionnaire indicated that neither student had taken a discrete mathematics class; Jamie had no exposure to computer programming prior to the course from which he was recruited; and Allan had moderate exposure to computer programming prior to the course from which he was recruited (he was self-taught with BASIC, Python, and C languages).

During the TEs, the students sat and worked individually at a desktop computer in the programming environment CoCalc. We gave them paper handouts of the problem statements and provided a brief introductory guide to Python syntax, and the students used CoCalc to write and run Python code. To capture the interviews, we videotaped and audiotaped the interviews, took screen recordings of the desktop computer, and scanned the students' hand-written work.

Tasks

Over the course of the TEs, we gave the students a variety of tasks in enumerative combinatorics. For each student, in the first two sessions we asked them to use a computer to list the set of outcomes of a problem. After completing these tasks, we asked them to find the number of outcomes. In the final sessions, we asked the students only to find the number of outcomes (not explicitly to list outcomes). We told the students that they were welcome to use the computer, although they could choose how to approach the problem. In the final session, after the students appeared to be satisfied with their answers, we asked them how they would verify their solution. In some instances, we asked the students to follow through with their verification scheme. We report on two tasks, the License Plate problem and the Books problem, whose statements are presented in the results section.

Data Analysis

For the results shared in this paper, the first author reviewed videotapes and transcripts of the final interview sessions. In particular, she flagged episodes in which students engaged in verification, creating a document which characterized each episode according to Eizenberg and Zaslavsky's framework. This allowed her to analyze the students' verification practices and create a narrative (Auerbach & Silverstein, 2003) about their reasoning behind their practices. We selected episodes for this report that illustrated trends seen across all of the other students, and they represent particularly insightful episodes that address the research questions.

Results

Due to space, we offer just two instances of students' verification of counting problems within a computational environment. In this section, we report on Jamie's verification techniques on the License Plate problem and Allan's verification techniques on the Books problem. We will see both students use the computational environment to engage in strategies outlined by Eizenberg & Zaslavsky (2004), and we emphasize ways in which the students verified their mathematical solutions by using a relationship between their solution and their code.

Jamie's Work on the License Plate Problem

The License Plate problem states, "A license plate consists of six characters. How many license plates consist of three numbers (from the digits 0 through 9), followed by 3 lowercase letters (from the first 5 letters in the alphabet), where repetition of characters is allowed?" This problem can be solved by reasoning about positions, where an outcome is represented by filling six positions with appropriate characters. Each of the first three positions can be filled with one of ten characters, so there are 10^3 ways to fill the first three positions. Similarly, there are 5^3 ways to fill the last three positions. As each choice of the first three positions can be followed by a choice of the last three positions, there are 10^3*5^3 possible license plates.

Jamie began the license plate problem by splitting it into two parts, first finding the number of arrangements of 0 through 9, and second finding the number of arrangements of ‘a’ through ‘e’. He reasoned that there were 10^3 ways to arrange the digits, and he described starting with 000 and incrementing to 001 and 002 until reaching 999. This logic depicts the process used in three nested for loops, each of which cycle through the digits 0 through 9. Jamie had used similar code in previous interviews. This thought process is also evidenced by the pseudocode in the upper-right corner of Figure 1. Similarly, Jamie reasoned that there were 5^3 ways to arrange the letters, and he describes a similar process of cycling through the letters. After correctly writing the number of arrangements of each part, Jamie wrote his final answer as $10^3 + 5^3$. When the interviewer asked why the operation might be addition, Jamie responded as follows:

> Yeah, I’m not quite sure why it might be plus, because I was thinking about them separately before, and I know like, if I were printing these results out I could concatenate them together and that could give me like an end result. Yeah, but as far as representing them mathematically I’m not sure why it would be plus. I don’t have a good guess for that.

We interpret that Jamie was viewing an outcome of this problem as a concatenation of two separate parts. However, despite being able to correctly count the number of arrangements of each part, he was to unable to justify that addition was the appropriate operation. We note that this issue of not being sure whether multiplication or addition is the correct operation is a perennial one in combinatorics (e.g., Kavousian, 2008; Lockwood & Purdy, 2019). Here, though, we gain insight into how the computational environment gave Jamie an opportunity to verify that his solution (and specifically the operation of addition) was incorrect.

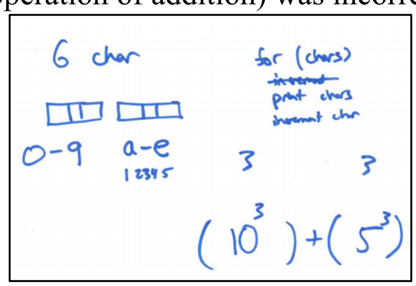

Figure 1: Jamie's initial solution to the License Plate problem.

When asked to verify his solution, Jamie chose to use the computer to implement a program that would list the set of outcomes and give the cardinality of the set. We take this as a method of verification, as it is an instance of Eizenberg and Zaslavsky’s (2004) fifth verification strategy of using a different solution method and comparing answers. After discussing pseudocode that would produce the arrangements of three numbers and pseudocode that would produce the arrangements of three letters, Jamie said the following.

> So I guess I was just kind of like trying to verify like in my head, like how do I wanna combine these together. Because if it was like two nested loops, then I would be, like, multiplying the results together, but if they were not nested it would be like adding them together.

In this quote, we take the “two nested loops” as describing the two pieces of code that would produce the arrangements of three numbers and three letters, respectively. Hence, we infer that Jamie was using the structure of his code—specifically, whether or not one process is nested within the other—to inform himself if the appropriate operation in his mathematical expression should be addition or multiplication. We suggest that this fits within yet another category of verification (such as *Adding justifications to the solution,* Eizenberg & Zaslavsky, 2004), as Jamie was leveraging the structure of the code to add justifications to the mathematical solution. After deciding that the process to generate the letters needs to be nested within the process to generate the digits, Jamie offered the following reflection.

> So maybe, I guess my thought process behind this was like, <u>add the results separately and that's going to like concatenate them together</u>, but like mathematically that wouldn't really make sense because I'm looking for combinations of like one thing on a license plate. So it's not like two separate things, like the first three plus the second three, it's all one.

This quote demonstrates Jamie's understanding of the relationship between his mathematical expression and the code he wrote to verify his solution. The underlined portion highlights how he relates the mathematical operation of addition to the computational operation of concatenation, and thus how his code corresponds to his mathematical expression.

In sum, Jamie used a computational environment to verify his solution by writing a program to print the set of outcomes of the problem. By doing so, he used a different solution method that not only allowed him to compare answers, but also to add justifications to his original solution. In particular, it allowed him to justify that his original solution was incorrect but could be corrected by replacing the operation of addition with multiplication. This is a noted difficulty in Eizenberg and Zaslavsky (2004), who stated that many students who found (or were told) that their solution was incorrect during a stage of verification were unable to correct their error.

Allen's Work on the Books Problem

The Books problem states, "Suppose you have eight books and you want to take three of them with you on vacation. How many ways are there to do this?" Although not explicitly stated in the problem, the order in which the books are chosen does not matter. This problem can be solved by counting the number of ways to arrange three of the eight books, and then dividing by the number of ways to arrange three books. To count the number of ways to arrange three of the eight books, we can reason about positions to find that there are eight ways to select the first book, and the number of ways to pick a book in each subsequent position decreases by 1. Thus, there are 8*7*6 ways to arrange three of the eight books. Here, notice that the 8*7*6 implicitly places an ordering on the three books. For any choice of three books, we can use a similar argument to show there are 3*2*1 ways to arrange them. Therefore, there are (8*7*6)/(3*2*1) ways to take three of the eight books on vacation.

Throughout the previous interviews, Allan had not encountered a problem where order does not matter. His original solution was 8*7*6, which he justified by selecting a first book in 8 ways, a second book in 7 ways, and a third book in 6 ways. While explaining his work, Allan did not address the issue of order; however, later discussion in the interview indicated that he understood the problem statement as meaning unordered selections of books.

When asked how he would verify his solution, he stated that he would use the computer to write a program to count the desired outcomes. We consider this to be verification, as it uses Eizenberg and Zaslavsky's (2004) fifth category of verification. He described the program as three nested loops, each of which cycle through the numbers 1 through 8, where an outcome was printed only if the numbers from each loops were distinct. While the code Allan described overcounts the outcomes of the given problem, it is noteworthy that it would list a set of outcomes with cardinality 8*7*6. That is, Allan correctly described code that would represent his incorrect mathematical expression. However, the following excerpt demonstrates that by writing the code, Allan became aware of the mistake in his solution. During this excerpt, Allen had written pseudocode on paper and, in response to the interviewer's question, was talking through the code and predicting what the first several outcomes of the code would be.

Interviewer: So, when you're listing those, what are, say, the first ten books that you would expect to see in your list?

Allen: The first set would be books 1,2,3, and then assuming this is the last for loop right here [underlines the 3 in 123], we should see 1,2,4 and then 1,2,5, etc., and this would go up to 8. So we'd see, eventually we'd see 1,2,8, and then that would go up to 1,3—and then, oh, I would also have to check for—different—because 1,3,2, that would not be a good combo because it already appeared up here, it's just in a different order.

We take this to be an instance of using a computational environment because Allan was describing the process his code would use, even if he had not yet implemented it. We suggest that in this case, the act of verification by using a computer program informed Allen that his initial solution was incorrect, and, more specifically, that it is incorrect due to overcounting of solutions (such as counting both 1,2,3 and 1,3,2 as distinct solutions, when they should not be).

Allen continued to explore the problem. While Allen was unable to find a mathematical expression that corrected the error in his solution, he was able to adjust his code so that solutions were no longer overcounted. Rather than checking that each character was distinct, Allen's code ensures that each character was in ascending order, as shown in Figure 2. Allen's code in Figure 2 retained the three nested loops structure that is in the previous code; however, each variable cycles through only those numbers that are greater than the preceding variable, ensuring that the printed outcome would be in ascending order. The code produces 56 outcomes, which is the correct answer to the Books problem.

```
In [7]: count = 0

        for i in range(1,9):
            for j in range(i+1,9):
                for k in range(j+1,9):
                    count = count + 1
                    print(str(i)+" "+str(j)+" "+str(k))
        print count
```

Figure 2: Allen's code to solve the Book problem.

After writing the code, Allen verified that it correctly solved the problem by evaluating the reasonability of the answer (the third category of Eizenberg and Zaslavsky, 2004), saying, "I guess it does make sense, because as we go on with each book, the number of possible combinations shrinks, like, rapidly." By this, we interpret that Allen understood that the number of outcomes that start with any digit decreases as the digit increases. This accurately reflects the code, as there are 21 outcomes starting with 1, 15 outcomes starting with 2, 10 outcomes starting with 3, 6 outcomes starting with 4, 3 outcomes starting with 5, and 1 outcome starting with 6. Allan then used the computer to conjecture, find, and verify (empirically through a number of cases) an exact formula for combinations, but those details are outside the scope of this paper.

In sum, Allan used a computational environment to verify that his original mathematical solution was incorrect by showing that the set of outcomes of the corresponding code overcounted the outcomes of the given problem. It is noteworthy that Allan did not attend to the error prior to listing the outcomes of the corresponding code. After verifying that his mathematical solution was incorrect, he was unable to find a mathematical expression to correctly solve the problem. However, Allan then used the computational environment to write code that correctly solved the problem and verified that his code was correct by evaluating the reasonability of the answer. We posit that the computational environment aided Allan in the ability to verify that his solution was incorrect, and that he would not have verified by hand, demonstrating the value of the computational environment in this example. Further, this provides another example of the computational environment being used to find and correct errors during verification, which is a noted difficulty in Eizenberg and Zaslavsky (2004).

Discussion and Conclusion

This paper demonstrates ways in which undergraduate computer science students used a computational environment to verify solutions to combinatorial problems. In particular, the computational environment was used to produce a different solution method that can be compared to a mathematical solution, to add justifications to a mathematical solution, and to show that a mathematical solution was incorrect. Importantly, the students' use of the computational environment during verification changed an incorrect answer to a correct answer in both cases. This addresses a difficulty noted by Eizenberg and Zaslavsky (2004), who indicated that many students who made attempts to verify incorrect answers were unable to detect an error or correct their solution (p. 32), even when the student was aware that the solution was incorrect. While we cannot make claims on the efficiency of using a computational environment when verifying, it does appear to provide students with additional tools that are impractical to use by hand. One such tool is that operations, such as addition and multiplication, relate to how the code is structured, such as when Jamie justified the use of multiplication because loops were nested rather than independent. Another tool is that students can demonstrate that a solution is incorrect by showing that a corresponding code outputs an incorrect set of outcomes, such as Allen did when he verified that his code overcounted outcomes. In sum, the computational environment affords students the opportunity to verify solutions by allowing them to verify more than an imagined counting process.

One limitation of our study is that the students in the TEs were Electrical and Computer Engineering majors who were recruited from a computer programming course. We can make no claim that students with less computer science experience would be comfortable engaging with programming to verify solutions. Further, the students in these TEs appeared to trust their computed solutions more than their by-hand solutions; that is, they used their computed solutions to verify their by-hand solutions. Other students, especially those with little computer science background, may be more comfortable using by-hand solutions to verify computed solutions.

These findings provide evidence of students using a computational environment to verify combinatorial problems, and they offer some methods which the students employed. In future research, it may be useful to evaluate systematically the effectiveness of verification within computational settings, perhaps comparing such activity with traditional methods of verification.

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Teaching Statistics with a Critical Pedagogy

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After the country's focus shifted from K-12 towards higher education, postsecondary schools found themselves under significant public, financial, and political pressure. To close the achievement gap and meet new standards of accountability, higher education institutions began looking for methods to increase student access and success. This quantitative study measured the effectiveness of implementing a critical statistics pedagogy in an undergraduate introductory statistics classroom and its impact on course success, persistence, and mathematical empowerment. Data collected from four classes at a community college found the use of a critical pedagogy had a positive impact on students' overall achievement, increased their awareness of social justice issues, and aided in the development of their critical voice.

Keywords: critical pedagogy, social justice, math empowerment, engagement, agency.

Introduction

Mathematics can act as the gatekeeper from or the gateway to future success, both academically and personally (Lesser & Blake, 2007). In February 2009, eight years after the No Child Left Behind (NCLB) act was signed into law requiring K-12 schools to track the progress and growth of four specific subgroups, the country shifted its focus on education towards higher education, challenging the nation to have the highest proportion of college graduates in the world by 2020. As a result, higher education institutions found themselves under significant political and financial pressure. In North Carolina, General Statute 115D-31.3, which establishes performance funding for community colleges, was updated and reenacted in both 2012 and 2016. The renewed emphasis on persistence and completion, and ties to funding models, enhanced the importance of ensuring student success in mathematics courses, forcing institutions to reevaluate their mathematics curriculums. To better meet the needs of a diverse student body, close the achievement gap, and meet new standards of accountability, some researchers began to explore the use of critical pedagogies (Ukpokodu, 2011).

As community colleges have seen the number of students enrolled in statistics courses triple since 1995, implementing a critical pedagogy in a community college statistics classroom may offer a multi-tiered approach to improving student success and completion (Blair, Kirkman, & Maxwell, 2015). First, numerous studies have shown the use of a critical pedagogy to be an effective way to raise student engagement (Gutstein, 2006; Lesser, 2007). Found to be a predictor of persistence, and completion, student engagement is regarded as an important factor in determining academic success, with students with higher levels of engagement experience greater success (Alvarez-Bell, Wirtz, & Bian, 2017). Second, this approach helps to raise awareness of social justice issues and may expose biases and assumptions, empowering students to create change. Use of a critical pedagogy specifically aims to help students understand how mathematics permeates everyday life and that learning mathematics cannot be removed from understanding and the ability to influence outcomes in the real world (Ernest, 2002; Lesser & Blake, 2007). Lastly, mathematical competency acts as a 'critical filter' in higher education (Ernest, 2002). The math barrier allows mathematics courses to exist as gatekeepers to degree completion and future career opportunities. As mathematics qualifications are often an "admission ticket" to high-paying jobs, a critical pedagogy may become the gateway in moving towards a more balanced economic and social structure (Lesser & Blake, 2007).

Literature Review

Critical pedagogies engage students as critical thinkers, active learners, and challenges them to envision alternative possibilities (Nagada, Gurin, & Lopez, 2003). It achieves its goals through carefully constructed learning experiences that allow students to examine, apply, and reflect upon the content. The term critical pedagogy (CP) was first coined by Giroux (1983), but it was largely founded upon the work of Paulo Freire (Tutak, Bondy, & Adams, 2011). Freire (1970/2005) asserted that learning required the learner to be active and that knowledge is created from a shared process of inquiry, interpretation, and creation. CPs call upon students to develop a sense of agency and begin to speak with their own voices (Giroux, 1983). Freire referred to students learning to understand their lives in new ways and see themselves as transformers of those lives as developing a 'critical consciousness' (Freire, 1970/2005; Tutak et al., 2011). As students discover the source of their beliefs and what structural and ideological factors influence them, they can begin challenge the taken-for-granted ways of their world and create their own history. Freire couples these ideas into praxis which he asserted was "the action and reflection of men and women upon their world in order to transform it" (Freire, 1970/2005, p. 79).

Use of a CP with an emphasis on social justice issues has shown to raise student engagement in mathematics courses through reflective inquiry (Lesser, 2007; Voss & Rickards, 2016; Wright, 2016). Skovsmose and Gutstein are the two well-known scholars of critical mathematics pedagogy. While each offers their own unique perspective, they share a common theme of literacy. As Freire claimed that literacy is more than the ability to read and write, Skovsmose believed that mathematics encompassed much more than the ability to calculate. It is from this notion he developed his idea of *mathemacy* (Skovsmose, 1994). Skovsmose (1994) identifies the three types of knowing on which mathemacy is created from: mathematical knowing, technological knowing, and reflective knowing. One with a well-developed mathemacy has the means to evaluate, or re-evaluate, interpretations of social institutions, traditions, and proposals for political reforms (Skovsmose, 1994). Gutstein's (2006) Teaching Mathematics for Social Justice (TMfSJ) builds upon Freire's definitions of praxis by focusing on helping students to learn to 'read and write the world' with mathematics. 'Writing the world' speaks to the action piece of praxis. It is the writing, or rewriting, of what students read in the world due to a developed sense of agency and seeing oneself as being able to make change. 'Reading the world' is understanding the sociopolitical, cultural-historical conditions of one's life, community, society, and world (Freire, 1970/2005; Gutstein, 2006). To read the world with mathematics means using mathematics to understand relations of power, inequities, and disparate opportunities between different social groups, to understand explicit discrimination, and to evaluate forms of representation of mathematical data and information (Gutstein, 2003).

When students learn to "read the world" they change their orientation with mathematics from being a set of disconnected rules to a powerful tool, causing the belief that mathematics is value-free or neutral to fade (Gutstein, 2006). The non-neutrality of mathematics can easily be seen in the field of statistics. Statistical measurements reflect choices and are not neutral, value-free records of the world; data cannot exist independent from how and why it is used (Weiland, 2017). What is valuable to one group may not be to another, impacting what does and does not get counted. By analyzing data that impacts a student's personal life and community, students become aware that statistics is a powerful tool, analytically and politically.

With a focus on the ability to evaluate and synthesize numerical information, statistics classes continue to gain more students each year (Blair et al., 2015). In the *Guidelines for Assessment and Instruction in Statistics Education College Report* (GAISE), the American Statistical Association (ASA) offer six recommendations and nine learning goals for creating statistically literate students (Carver et al., 2016). Like Skovsmose's mathemacy, ASA's recommendations and learning goals call for statistical, technological, and reflective learning. Five of the goals specifically focus on reflective knowing by requiring students to be critical consumers of statistically based reported results, question the usefulness of investigative processes in statistics, produce and interpret graphical displays and numerical summaries, understand and apply statistical inference, and demonstrate an awareness of ethical issues associated with sound statistical practice (Carver et al., 2016). To achieve this level of statistical literacy, students will need to become 'statistical citizens' and develop a critical understanding of the data in order to act from a more informed position when analyzing ethical issues (Lesser & Blake, 2007).

Largely founded on Gutstein's (2006) TMfSJ and the GAISE recommendations and learning goals, Lesser's (2007) *Teaching Statistics using Social Justice* (TSSJ) is a critical statistical pedagogy that is defined as the teaching of statistics using real world examples related to social justice to enable and empower students to use statistics to 'talk back' to the world. TSSJ "incorporates and facilitates awareness of issues of social justice and prepares students not only to be competitive workers in the economy but also engaged participants in a democracy" (Lesser, 2007, p. 3). Weiland's (2017) critical statistical literacy complements Lesser's (2007) work by specifically defining what it means to be critically statistically literate. The critical statistical literacy (CSL) perspective emphasizes the importance of critically 'reading and writing the world' using statistics to explore sociopolitical contexts (Weiland, 2017). "Reading the world" in CSL includes understanding statistical language and symbols, establishing a statistical way of knowing. Including the need to identify and question data-based arguments of social structures and the evaluation of statistical sources through a social, historical, and political lens adds reflective ways of knowing, for a more critical reading of the world. The same is true for "writing the world" in CSL where one not only communicates the statistical data, but also how one's position impacts their "overall meaning making of the world" and use of statistics to investigate and respond to unjust structures (Weiland, 2017, p. 42).

Framework

As this study was completed in an introductory statistics classroom, the conceptual framework that guided the study was founded upon Gutstein's (2006) TMfSJ, Lesser's (2007) TSSJ, and Weiland's (2017) CSL. Teaching Statistics for Social Justice (TSfSJ) is a critical statistical pedagogy designed for introductory statistics classrooms as it assumes students do not have prior knowledge or experience using statistics, examining and discussing social justice issues, or implementing a critical lens. Class discussion and data sets consisted of real-world, local, nontrivial examples for students to both learn and reflect upon the context of these examples and learn how to use and apply course content (Lesser, 2007). TSfSJ was designed to: (a) increase mathematical empowerment by increasing confidence in their ability to complete statistical procedures that produce results relevant to the surrounding community, (b) instill a sense of social justice to encourage civic engagement, activate critical voice, and increase personal relevance, and (c) increase student engagement through data sets that examine social issues that are relevant to their lives to increase student success. Students were expected to use statistical methods to read and write their world using predetermined data sets examining social justice issues surrounding race, class, gender, and sexual orientation. Through reflective inquiry students discovered the importance and usefulness of statistics, developing a more positive impression of the subject (Voss & Rickards, 2016; Wright, 2016). Reading the world with statistics included "identifying and interrogating social structures and discourses that shape and are reinforced by the data-based arguments" (Weiland, 2017, p. 41) with a particular focus on power, inequities, and disparities between different social groups (Gutstein, 2003). Students then examined these arguments, via statistical analysis and class discussion, to discover what, if any, injustices are being perpetrated, uncover possible inconsistencies, determine whether to accept or reject the arguments, and provide support using statistical investigations to communicate statistical information and arguments (Carver et al., 2016; Gutstein, 2003; Lesser, 2007; Weiland, 2017).

Methods

The purpose of this study was to measure the effectiveness of implementing a critical statistics pedagogy in an undergraduate introductory statistics classroom, and its impact on course success and mathematical empowerment. Intact classroom groups were used as the control and treatment groups, as it was not possible to implement random selection of group members. Two research questions guided the study: (a) does implementing a critical pedagogy increase student success? and (b) does use of a critical pedagogy increase the student's sense of mathematical empowerment? A variety of data was collected via instructor created projects, online homework sets, tests, and surveys to measure the impact of TSfSJ. Course grades, final exam grades, and pre/post CAOS tests were used to examine student success, with success defined as completing the course with a C or higher (a numerical score of 70 or higher). Mathematical Empowerment (MP) refers the student's ability to identify the need for and ability to apply statistics to real world scenarios.

Participants

The quasi-experimental study consisted of four MAT 152 Statistical Methods I courses at Mitchell Community College (MCC). During the Fall 2017 semester, MCC had 1,091 full-time students, 2,093 part-time students, and 20 of unknown status. Of the 3,204 students, 35% of students were matriculated in a transfer degree program and 34% were dual-enrolled students either from one of three Early Colleges affiliated with MCC or from one of three local high schools attending college classes through the state's Career and College Promise program. Excluding missing data from the 20 students with unknown status, students are primarily white (70%), female (61%), and young with 77% of the students being under 25 years of age. They primarily reside in the cities of Statesville and Mooresville, at 41% and 36% respectively. To best serve their students, MCC has two campuses: Statesville and Mooresville. Statesville's 2017 population is noticeably smaller than Mooresville's total population and the demographics differ greatly among the three areas. The population of Mooresville is roughly 87% white, while Statesville is 69% white. The median incomes between the two cities vary more drastically. Statesville, zip code 28677, has a median income of approximately $41,000. Mooresville, zip codes 28115 and 28117, has median incomes of $57,000 and $80,000 respectively (U.S. Census Bureau, 2018). The difference in household income is evident in the high schools. Statesville is served by one public high school, Statesville High. In 2017, Statesville High failed to meet academic growth, incoming student readiness (percent of students considered proficient) was measured at 12.7% and served a population that was 60% economically disadvantaged ("School

Report Card for Statesville High", 2019). Mooresville is served by two public high schools, Lake Norman High and Mooresville High, both of which exceeded academic growth projections in 2017. Lake Norman High had an incoming student readiness of 59.7% and served a population that was 8.9% economically disadvantaged ("School Report Card for Lake Norman High", 2019). Mooresville High's incoming student readiness was 65.9% and its economically disadvantaged population was 30% ("School Report Card for Mooresville High School", 2019).

Data Sources and Collection

During the Fall 2017 and Spring 2018 semesters, one MAT 152 course on each campus was a part of the study. Data was collected from the Fall 2017 Statesville section (10 students) and the Mooresville section (19 students). In Spring 2018, data was collected from 19 students in Statesville and 23 students in Mooresville. Final exam scores and overall course grades were collected from all students who completed the courses. The Fall 2017 sections of MAT 152 served as the control groups and the Spring 2018 sections were the treatment groups. There were two pretests and two posttests administered to all sections included in the study on the first day and second to last day of each course to measure differences between the two groups. One of the tests was the Comprehensive Assessment of Outcomes in a First Statistics (CAOS) course. It consists of 40 multiple choice questions and assesses students' statistical reasoning after any first course in statistics with a focus on statistical literacy and conceptual understanding (Regents of the University of Minnesota, 2006). The second test administered was the Constructivist Learning Environment Survey (CLES). The CLES is a 42-item questionnaire which uses a 5-point Likert scale measuring students' classroom experiences in six categories (CLES, 2017). However, only two categories were of interest in this study: Personal Relevance and Critical Voice. Personal Relevance questions measure a students' perceived relevance of the content learned in the classroom to their out-of-school experiences. These questions examine whether not students are able to read and write the world with mathematics by being able to recognize and apply the curriculum to real world situations (Gutstein, 2006). Critical Voice questions focus on empowerment, ultimately measuring the climate of the classroom and the extent to which students feel able to question or reflect on the teaching and learning process (Lew, 2010; Taylor & Fraser, 1991).

In all four courses, Triola's (2015) *Essentials of Statistics* (5th edition) served as the course text and as a source of course material and resources. MyLabsPlus (MLP) was the courses' online homework platform, with all four courses being assigned the same questions. Final exams and unit tests were created from the textbook's testbank with all unit exams and the final exam remaining the same for each section. There were 10 labs assigned in all sections. All labs were researcher-created, required the use of technology, and were completed in the classroom. Control groups labs consisted of hypothetical scenarios that students examined with the use of online and textbook resources and their calculator. The treatment consisted of embedding an overarching focus on issues of social justice relating to race, class, gender, and sexual orientation by adding examples, questions, and data exploring these topics to the PowerPoints and labs used with the control group. Labs were similar in design and theme, but incorporated real-world, local data to be explored in addition to, or in replace of, the fictional data used in the control.

Results

A total of 12 participants were excluded due to not completing both sets of the pretests and posttests. From Fall 2017, four participants were eliminated from the Statesville control course and five from the Mooresville control course. There were two participants eliminated from the Spring 2018 Statesville treatment course and one participant from the Spring 2018 Mooresville treatment course. The type of analysis used was determined by the level of measurement of data collected.

Research Question One

Course Grades. A Kolmogorov-Smirnov Test (K-S Test) was run on each section's final course grades. For the Statesville sections, both the control ($D(10) = .186$, $p > .05$) and treatment ($D(19) = .152$, $p > .05$) were deemed to be individually normally distributed by the K-S. A Levene's Test for Equality of Variances showed that the two sections had equal variances. An independent *t*-test did not indicate a statistical difference between the treatment ($M = 83.1$, $SE = 2.94$) and control ($M = 85.61$ $SE = 3.53$), $t(27)= 0.523$, $p > .05$. The Mooresville control ($D(19) = .168$, $p > .05$) and treatment ($D(23) = .168$, $p > .05$) sections were also both normally distributed. A Levene's Test for Equality of Variances showed that the two sections did not have equal variances. An independent *t*-test did indicate a statistical difference between the treatment ($M = 91.39$, $SE = 1.69$) and control ($M = 80.65$ $SE = 3.82$), $t(24.922)= -2.573$, $p < .05$.

Final Exam Grades. A K-S Test was run on each section's final exam grades (max score of 40). Both Statesville sections were deemed to be individually normally distributed for the control ($D(10) = .186$, $p > .05$) and for the treatment ($D(19) = .154$, $p > .05$) by the K-S. A Levene's Test for Equality of Variances showed that the two sections had equal variances. The independent *t*-test did not find a statistical difference between the treatment ($M = 31.21$, $SE = 1.37$) and control ($M = 32.8$, $SE = 1.52$), $t(27) = .726$, $p > .05$. The K-S test for the Mooresville control ($D(19) = .204$, $p < .05$) and treatment ($D(23) = .190$, $p < .05$) were not found to be normally distributed. Due to the lack of normality, an independent Mann-Whitney U Test was used to compare the final course grades of the sections and found there was a statistically significant difference between the control ($Mdn = 28$) and treatment ($Mdn = 35$) sections, $U = 358.50$, $z = 3.55$.

CAOS pretest/posttest scores. An independent *t*-test showed that control groups were not statistically different from their treatment groups. Independent *t*-tests of the pretest CAOS scores confirmed that the Statesville control and treatment groups, $t(27) = -0.384$, $p > .05$. , and the Mooresville control and treatment groups, $t(40) = -1.541$, $p > .05$, did not differ statistically

For the Statesville sections, a Shapiro-Wilk (S-W) tests showed both the pretest ($W(10) = .910$, $p > 0.05$) and posttest ($W(10) = .910$, $p > 0.05$) were normally distributed for the control group and the treatment group pretests ($W(19) = .949$, $p > 0.05$) and posttests ($W(19) = .934$, $p > 0.05$). Levene's Test of Equality of Error Variance was not significant ($F(1, 27) = .033$, $p > .05$) showing the variances of the pretests for the control and treatment to be equal. Independent *t*-tests were used to explore the data. Neither the control, $t(9) = -0.9.19$, $p > .05$, nor the treatment, $t(18) = -.109$, $p > .05$, showed a significant difference between the pretest and posttest. In Mooresville, Shapiro-Wilk (S-W) tests showed both the pretest ($W(19) = .929$, $p > 0.05$) and posttest ($W(19) = .965$, $p > 0.05$) were normally distributed for the Mooresville control group. However, S-W tests showed the treatment was not normally distributed for the pretest ($W(23) = .900$ $p < 0.05$) or posttest ($W(23) = .910$, $p < 0.05$). When the outliers were removed from the data set, Shapiro-Wilk reported the treatment as normally distributed, $W(21) = .918$, $p > 0.05$. Levene's Test of Equality of Error Variance was not significant showing the variances of the pretests for the control and treatment to be equal, $F(1, 38) = 0.805$, $p > .05$. The data was retested with outliers remaining in the set. Levene's Test of Equality of Error Variance showed the variances of the two groups to be equal as the test was not found be significant ($F(1, 40) = 3.196$, $p > 0.05$). A paired t-test was used to investigate the control and a Related-Samples Wilcoxon signed rank test was used for the treatment, as the treatment was originally found not to be normally distributed. The paired t-test did not find a statistically significant difference between the pretest and posttest for the control, $t(18) = -0.637$, $p < .05$. However, the difference between the pretest ($Mdn = 16$) and the posttest ($Mdn = 18$) for the treatment was significant, $T = 188.00$, $p < .05$. When outliers were removed from the treatment, a paired t-test also found a statistical difference in the treatment group, $t(20) = -2.274$, $p < .05$.

Persistence. A population proportion test was used to measure the effect of treatment on persistence. Tests were completed via TI-84 SE with z scores rounded to the hundredths. There was found to be no statistically impact of persistence on the Statesville ($z = -0.4$, $p > .05$) or Mooresville ($z = -0.67$, $p > .05$) campuses.

Research Question Two

Two population proportion z-tests were completed on each question comparing pre/post survey data for each of the control and treatment groups with success defined as responding Almost Always or Often on positively worded questions. Except for questions PR1 and PR13 in the control, all questions were found to be significant, $p < .05$. However, for the treatment groups, all questions showed a significant increase in PR and CV, $p < .005$.Two population proportion z-tests were also completed on each negatively worded question comparing pre/post survey data for each of the control and treatment groups with success defined as responding Seldom or Almost Never on negatively worded questions. All questions were found to be significant, $p < .05$.

Conclusion

Despite its limitations, the study provided two promising findings. First, the study showed that the use of a CP in an undergraduate introductory statistics classroom has the possibility of increasing overall course success. As students need a C or better in a course to fulfill prerequisite requirements for other courses or to have the course transfer to another institution, even small increases may result in large, personal gains for the student. By earning transferrable credits, the probability a student completes their degree increases. As “educational attainment is a means of social mobility” this will have an impact on their future lives and career (Trusty & Niles, 2003). Second, the study showed gains in MP in the treatment courses. While the control found the material relevant, students were not able to “read and write” the world with statistics and identify how mathematics touched their own lives. By implementing a CP, students learned “to examine the systems and institutions that are in place and to use mathematics to evaluate and critique these systems and institutions as well as develop individual and social agency” (Leonard et al., 2010, p. 262).

The purpose of higher education has long been to help create well-rounded, forward thinking citizens; people that can gather and analyze information, use that information to address problems, and help improve society. As Stanton (1987) stated “we need a citizenry with a broad understanding of the interdependencies of peoples, social institutions, and communities and an enhanced ability both to draw upon and further develop this knowledge as they confront and solve human problems” (p. 7). By teaching statistics for social justice, students not only learn how to be better employees for the workplace, but how to be better people for all places.

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Influence of Curriculum on College Students' Understanding and Reasoning about Limits

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The Pathways to College Algebra curriculum aims to build concepts that cohere with the big ideas in Calculus, and initial results suggest improved readiness for Calculus by students who use the curriculum. Our study examines similarities and differences of Pathways and non-Pathways students understanding and reasoning about the calculus concept of the limit. We compare students' understanding of limits at the beginning and at the end of the unit. Our findings suggest that (1) students reliance on procedures, combined, or quantitative reasoning was dependent on the calculus instructors' emphasis in the class; (2) students who begin their Calculus class with high covariational reasoning gain a more sophisticated understanding of limits; and (3) when curriculum is coherent students will identify mathematical connections.

Keywords: Pathways College Algebra, Calculus, Limits, Understanding, Reasoning

Mathematics should make sense. Many professional organizations encourage *coherence* across curriculum (NCTM, 2006; NMAP, 2008; Newmann, Smith, Allensworth, & Bryk, 2001; Schmidt, Wang, & McKnight, 2005). Thompson (2008) suggests that coherence is the *ideas* and *meanings* rather than mathematical topics and orderings. Often this coherence is lacking between Calculus and its prerequisite classes, like College Algebra, which frequently focuses on calculations and procedures (see Blitzer, 2014; Sullivan, 2012). While knowing procedures may help students work out calculus problems, often it is hard for students to see how these procedures cohere with the *big ideas* in Calculus of limits, rates of change, and accumulation (see Kaput, 1979; Thompson, 1994).

Levia, Borrowman, Jones, and Teuscher (2019) reported that Pathways students, those who used *Pathways to College Algebra* (Carlson, 2016, hereafter referred to as "Pathways") reasoned about and understood limits differently compared to their non-Pathway peers after the first lesson in their Calculus class. Overall Pathways students relied on quantitative reasoning and used higher levels of covariational reasoning suggesting that the Pathways curriculum may *cohere* with students' initial limit instruction, which is an important aspect of sound curriculum (NCTM, 2006; NMAP, 2008; Newmann et al., 2001; Schmidt et al., 2005; Thompson, 2008). This coherence suggests a possible advantage for Pathways students when encountering difficult concepts such as limits for the first time. The Pathways students were also aware of the ways in which their Pathways curriculum connected to the limit concept in Calculus after just one lesson.

The Pathways curriculum aims to build algebra concepts through quantitative and covariational reasoning and the curriculum was developed to provide foundational knowledge for students, but also to cohere with big ideas in calculus. This study examines one specific area, namely how students' College Algebra experience might influence students' understanding and reasoning about limits after receiving instruction on limits. Our guiding research question is: What differences or similarities are there between calculus students who took non-Pathways algebra versus Pathways algebra, in terms of how they understand and reason about limits at the end of the unit?

Brief Background on Pathways Curriculum

The Pathways curriculum (Carlson, 2016; Carlson, Oehrtman, & Moore, 2017) was developed to provide a coherent and meaningful course for students to understand the foundational aspects of calculus. The Pathways curriculum was informed by research on learning functions (Carlson, 1995, 1998), the processes of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), mathematical discourse (Clark, Moore, & Carlson, 2008), and problem-solving (Carlson & Bloom, 2005). The curriculum contains modules based on research of student learning and conceptual analysis of the cognitive activities conjectured to be necessary to understand and apply the module's central ideas. Mathematics concepts included are rate of change; proportionality; functions: linear, exponential, logarithmic, polynomial, rational, and trigonometric; polar coordinates; vectors; and sequences and series. The curriculum also supports a problem solving approach to mathematics, where students are expected to engage in novel contexts and reasoning to construct mathematics.

Understanding and Reasoning about Limits

In this section, we articulate our perspective on "understanding and reasoning about limits," based on the research literature. Of course, our perspective outlined here does not contain everything that may be involved in understanding or reasoning about limits, because our study deals with understanding after an introductory unit on limits.

We define "understanding" as a student's *concept image* of limit at the end of the unit (Tall & Vinner, 1981), which we would expect to be more complete, yet have possible holes at the end of the unit. Also, because some misconceptions are nearly "unavoidable" (Davis & Vinner, 1986) and take considerable exposure to examples, counterexamples, and contexts to address (Cornu, 1991; Przenioslo, 2004; Swinyard, 2011), we are less interested in documenting student misconceptions. Rather, we are interested in comparing students' understanding with standard *informal* definitions of limit. In our study, we examined cases of both the limit at a point, $\lim_{x \to a} f(x) = L$, and the limit at infinity, $\lim_{x \to \infty} f(x) = L$. While the students in our study participated in class discussions and homework problems throughout the unit, we wanted to know how they understood and reasoned about limits at the end of the unit and in comparison to the beginning of the unit. Our informal definition of limit at a point is that the limit of $f(x)$ is L "if we can make the values of $f(x)$ arbitrarily close to L… by restricting x to be sufficiently close to a… but not equal to a" (Stewart, 2015, p. 83). Our informal definition of limit at infinity is that "the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large" (Stewart, 2015, p. 127).

We define "reasoning" through two aspects of reasoning that the literature claims are important for limits. First, Kaput (1979) stated that "virtually all of basic calculus (the study of change) achieves its primary meaning through an absolutely essential collection of motion metaphors" (p. 289). As such, changing quantities are a part of reasoning. However, standard curricula often focus heavily on algorithms for finding limits (e.g., Stewart, 2015; Thomas, Weir, & Hass, 2014). Nagle (2013) claims that this approach likely leads students to have "independent, unconnected conceptions" of limits that are based on quantities and computation (p. 3). Consequently, the way students use quantitative reasoning versus procedural reasoning is one part of their "reasoning about limits."

Second, some researchers have noted strong relationships between covariational reasoning and understanding limits, due to a limit inherently dealing with two changing quantities (Carlson et al., 2001; Carlson et al., 2002; Nagle, Tracy, Adams, & Scutella, 2017). The informal, "as x approaches a, y approaches L," strongly suggests covariation between x and y. Carlson et al.

(2002) even claimed that, “students' difficulties in learning the limit concept have been linked to impoverished covariational reasoning abilities” (p. 356). Because of the importance of covariational reasoning, we consider how students use covariational reasoning as the other part of their “reasoning about limits.”

Methods

Eight of the 11 Calculus 1 students from a prior study (Levia et al., 2019) at a large private university participated in a second interview for this study. All students had taken College Algebra from various instructors at the university in the previous year; four took Pathways (P) and the other four a non-Pathways Algebra class (N-P). The students’ College Algebra grades and Calculus pre-test scores were similar across the two groups and only one of the N-P students had completed Calculus previously in high school. Students were enrolled in four different instructors’ Calculus courses. Three of these calculus instructors emphasized conceptual understanding and one emphasized procedural knowledge. Students were interviewed a second time about limits the week after they completed their mid-term on limits in Calculus 1. We label the P students as PA, PB, PC, and PD and the N-P students as N-PA, N-PF, N-PC, and N-PD.

The interview contained the same four questions that were given in the initial interview: (1) Explain the meaning of $\lim_{x\to a} f(x) = L$. (2) If you found the limit $\lim_{x\to\infty} \frac{4x^2}{x^2-5x+6}$, , what would you be finding? (3) Select the graph(s) [among six graphs given to the students] that correspond to each limit expression, (a) $\lim_{x\to\infty} f(x) = 1$, (b) $\lim_{x\to 0^-} f(x) = -\infty$, (c) $\lim_{x\to 3^+} f(x) = 0$. (4) The equation $v_{orbit} = \sqrt{\frac{GM}{r}}$ relates a satellite’s required velocity for a stable orbit, v_{orbit}, with its distance from Earth, *r* (where *M* is the Earth’s mass and *G* is a constant). What is $\lim_{r\to\infty} v_{orbit}$? After questions 2 and 4, students were asked to identify any connections they saw from their College Algebra class that might have helped them understand limits.

We analyzed the students’ responses as follows. First, we analyzed how students used quantitative reasoning versus procedural reasoning in their responses. Quantitative reasoning was operationalized as using number sense and relationships between quantities to discuss the limits. Procedural reasoning was operationalized as using an algorithm or memorized set of steps to solve the problem, without explaining why the process worked, regardless of whether the student used the procedure correctly or not. However, if the student explained why the process worked, it was coded as quantitative reasoning, rather than procedural. We applied this analysis only to the questions 2 and 4 where students *could* potentially compute the limit.

Second, we analyzed students’ covariational reasoning behavior from question 3 according to Thompson & Carlson (2017) framework, which includes the reasoning levels: *no coordination, pre-coordination, gross coordination, coordination of values, chunky continuous covariation,* and *smooth continuous covariation*. We only analyzed question 3 because students had to discuss two quantities as they explained their answers. We used the same categories developed from the initial interview analysis (**high**: *coordination of values, chunky continuous covariation, smooth continuous covariation, gross coordination*; **mid**: *coordination of values, gross coordination*; **low**: *no coordination, pre-coordination*) (see Levia et al., 2019). Each student was placed into a category based on their overall demonstrated ability of reasoning. We also attended to whether students were *specific* about the quantities involved in the covariation, or whether they used *ambiguous* language to refer to the quantities (Leatham, Peterson, Merrill, Van Zoest, &

Stockero, 2016). If we were unsure of the quantities the student referenced, we placed the student in a lower category.

The last step of analysis was, at the end of each interview question, to infer students' understanding of limits. We documented whether the students' descriptions were mathematically correct according to our informal definitions. We also noted whether a student's descriptions were consistent across questions, including for *limit at a point* at the end of questions 1 and 3 and for *limit at infinity* at the end of questions 2, 3, and 4.

Results

Procedural versus Quantitative Reasoning

Students tended to reason about limits in one of three ways: reliance on quantitative reasoning, reliance on procedural reasoning, or reliance on combined reasoning. We found that three students (PA, N-PF, N-PA) reasoned as they had during the initial interview at the beginning of the unit. However, the remaining five students moved into different categories of reasoning. Three P students (PB, PC, PD) moved from quantitative reasoning to either combined reasoning or procedural reasoning. One N-P student (N-PD) moved from procedural to combined reasoning, and another (N-PC) moved from combined to quantitative reasoning.

During the second interview P students tended to be more reliant on procedures. For example, in the initial interview, PB reasoned quantitatively, stating: "the idea of this function itself is as x approaches infinity, … this calculation will reach a certain number." Although PB did not actually solve the limit, he discussed what the given equation meant and how he thought about and interpreted his answer. There was no reliance on procedures and he was focused on understanding what was happening with the numbers. Conversely, in the second interview, PB seemed to focus on the new rules he had learned in Calculus. On the same interview question, his response was,

> So something I learned about limits in… [Calculus 1] was that there are certain laws that we can use to kinda move things around… Ok one of the laws says that ... we can take the limit of $4x^2$ as x approaches infinity. We can kind of separate that from this and say that this is… it's the limit of $4x^2$ divided by the limit of … as x goes to infinity. We can say that it's this divided by $x^2 - 5x + 6$ [writes, $\lim_{x\to\infty} 4x^2 \div \lim_{x\to\infty} x^2 - 5x + 6$]. Which is awesome.

Here we see that PB's response is to exemplify the new "laws" (i.e., rules) he learned in Calculus. On multiple occasions, PB mentioned concepts and reasoning behind procedures. However, when given the opportunity, PB, like other P students, discussed memorized rules and procedures from his Calculus class rather than concepts he had come to understand.

On the other hand, two of the N-P students reasoned more quantitatively by the second interview. For example on question 2, N-PC got stuck in his initial interview trying to work through rules and eventually ended up giving up. However, in his second interview, he responded with,

> Whichever x is growing the fastest, that would be the limit you are approaching … So this infinity [points to x^2in the denominator] is being squared so it's going to be much bigger than this infinity [points to $5x$]. Just like 5x. Because a squared number is bigger than a non-squared number. And this number up top [points to $4x^2$] is also squared so it's going to be like the same size of infinity, but it's multiplied by 4 so this is like 4 times as great as this one down here [points to x^2]. So as x approaches infinity, these cancel out like a 1 and you're left with just the 4. So the limit is 4.

In contrast with his initial interview, N-PC provided an answer explaining why the procedure of using the exponents to find the limit worked.

Importantly, during this study we did not control for differences in Calculus instructors. Most of the P students' instructors focused on the procedural process of limits; whereas, most of the N-P students' instructors emphasized conceptual understanding. This could explain the shift in students' procedural and quantitative reasoning. This suggests that students' reliance can change depending on the instructor's orientation in just one month, which was the time between the two interviews. However, more research would need to be done to confirm this.

Covariational Reasoning

After analyzing students' interview responses for instances of covariational reasoning, we identified each student's general level of exhibited covariational reasoning, placing them in a low, mid, or high category. Here, we describe the three categories and the differences between the P and N-P students in the second interview.

High covariation. In the second interview, three P students (PA, PB, PC) exhibited "high" covariational reasoning most often. For example, when PB explained a graph in question 3, he stated:

> We can see that as our value of *x* along this line gets larger and larger [points to each value along the x-axis: 1,2,3, etc], ... this line representing our y-values, gets closer and closer to this constant value of 1.

PB saw the x-axis having intermediary values, thereby exhibiting *smooth continuous* covariation. Students who exhibited this level of covariational ability and were specific during most of their interview were placed in this "high" category.

Mid covariation. Two N-P students (N-PA, N-PF) were placed into the "mid" category. For instance, N-PF explained the correct graph for $\lim_{x\to 3+} f(x) = 0$. She stated, "As the x-values [are] approaching 3 from the right side, you can see that it's getting closer and closer to 0 for the y-value." N-PF is explicit in identifying the quantities she is explaining, but imprecisely describes how the two quantities change together. This kind of reasoning was typical for students placed in the "mid" covariation category.

Low covariation. Three students (PD, N-PC, N-PD) were placed into the "low" category. These students were generally not as specific, or were more easily confused as to which quantity they were discussing in their explanation. An example can be shown by N-PC when explaining how a graph exhibited the limit statement $\lim_{x\to 3^+} f(x) = 0$. He was unspecific and usually called both *x* and *y* "the graph." After being asked a few times to clarify, N-PC describes the graph by saying, "As it's approaching 3... sorry the graph, so as *x* is approaching 3, it's getting closer to 0." When the interviewer asked what is getting closer to 0, N-PC responded: "the graph." This student used the same pronoun for both variables; it is unclear if he was differentiating between the two as they varied together. This confusion may cause the student to see the quantities as separate, distinct objects with little or no link.

Although the students' calculus instructor seemed to influence their reliance on either quantitative or procedural reasoning, this was not the case for students' levels of covariational reasoning. There was no relationship between the instructor's foci and the students' levels of covariational reasoning. In fact, students exhibited little or no change in their covariational reasoning ability between their initial and second interviews. This suggests that students maintained their habit of covariational reasoning and applied it to their new course regardless of how the instructor reasoned. Similarly, if a course is not built with a covariational reasoning focus, students may not improve their covariational ability.

Understandings for Limit

Though no student had a perfect limit meaning, we classified their limit meaning as correct when students showed they were thinking about the input values approaching a certain point or infinity, and that the associated output values approached the limit value. This is exemplified by PC's response to question 1 in his initial interview, "We're not necessarily looking for the value of the function at a specific *x* point. But what *f(x)* is approaching at that certain point, from both sides."

During the second interview we found that some students showed a more sophisticated understanding of a limit. All of the students were asked, when using words like "closer" and "approaches," what they meant by those words. Some students described "narrowing in" on a particular y-value. They indicated that the vertical difference between the expressed y-value and limit value was decreasing. For example, N-PA describes by explaining that:

> First kind of bigger gaps and then they get tighter and tighter … the pattern appears that it gets tighter and tighter around the output of 1. So that the larger the *x*-value gets, the closer and closer the *y*-values get to being 1.

N-PA describes "tighter and tighter" gaps, or vertical distances or differences between the limit value, 1, and the output values on the graph. This suggests that this student's limit meaning is closer to the formal limit definition, and that these vertical distances suggest an understanding of the $L \pm \epsilon$ aspect of a formal limit definition. Of the three students that exhibited this kind of meaning, two of the students were P students and the other was a N-P student who took calculus previously in high school. These three students exhibited either "high" or "mid" covariational reasoning. This suggests that having a high covariational reasoning as these students entered Calculus may have helped their understanding of limits and potentially other content.

Three students demonstrated an improved, but not as sophisticated limit meaning as the previously described group of students. In the initial interviews, four students (one P and three N-P) thought the limit values meant the slope of the function at a specific point. Two of these students (one P and one N-P) continued to confuse the idea of limits and derivatives by identifying a limit as being the slope of a tangent line. This type of thinking happened for both P and N-P students and it was not unique to a specific calculus or algebra instructor. The calculus instructors motivated the need for limits by showing students that they could not calculate the slope of a line tangent to the function at just one point. This may have confused students, and they interpreted it as the limit value is the slope of the function at that point. Even after an entire unit of learning about limits, two students still had misconceptions about the meaning of a limit stemming from that limit introduction.

Connections to Algebra Experience

During the final interviews, as well as the initial interviews, the N-P students generally were explicit in stating that their College Algebra class had few connections to learning limits, and the connections they found were mainly procedures or manipulative skills. The P students expressed that their College Algebra class directly helped them with learning and understanding limits and the connections they made tended to be more about concepts and thought processes.

For example, N-PC said,

> We've used algebra like crazy. So that finally makes sense why we had to learn all that crazy stuff in 110 [College Algebra].... So like finding the domain of a function, finding asymptotes, being able to manipulate like power rules, log rules, natural log, *e*. Being able to factor and simplify and divide. Do all sorts of crazy gymnastics with algebra.

For this student, there were only basic procedures that came from his College Algebra experience and nothing that he felt helped him understand the concepts.

As opposed to the N-P students, PC said,

> We never mentioned the word 'limit' before, so that was a foreign, brand-new concept this last month. But it does relate because it's important to know the general trend of a graph when it's approaching numbers....That relates to limits because that's kind of the definition of a limit.

As PC notes, limits were never taught in the Pathways class, however he is able to recognize that the concepts he learned in College Algebra were directly related to what he was learning about limits.

Discussion

In discussing the trends seen in the results, we caution that our small sample is only suggestive, and cannot imply generalization to the larger Calculus student population. However, within our small sample, we certainly observed differences in trends for how the overall group of P students reasoned about and understood limits compared to their N-P counterparts. Of course, there was overlap in how the students reasoned about limits. For example, students from both groups were seen to reason quantitatively and procedurally. Students from both groups were seen to reason at lower and higher levels of covariational reasoning. Yet, taken in aggregate, P students who had high covariational reasoning to begin the unit, continued to have high covariational reasoning at the end of the unit regardless of their instructor. On the other hand, students' reliance on quantitative, combined, or procedural reasoning tended to be related to what their calculus instructor emphasized. As would be expected after a unit on limits, students' understanding of limits became more sophisticated and were closer to the formal definition of a limit. However, it is interesting that students who had high covariational reasoning at the beginning of the unit, which tended to be P students, tended to have a more sophisticated understanding of limits at the end of the unit.

The Pathways curriculum seems to *cohere* with students understanding of limits, which is an important aspect of sound curriculum (NCTM, 2006; NMAP, 2008; Newmann et al., 2001; Schmidt et al., 2005; Thompson, 2008). This coherence suggests a possible advantage for Pathways students when encountering the difficult limit concept. While it may only be small, if it is combined with a net advantage for other concepts as well, such as the derivative and integral, it begins to build a picture as to why Pathways students might be more successful in calculus (see Carlson, Oehrtman, & Engelke, 2010). In fact, the students themselves seemed aware of the ways in which their Pathways curriculum connected to the limit concept they learned even thinking that limits were taught during their College Algebra class, which was not the case.

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How Do Students Engage with 'Practice Another Version' in Online Homework?

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Homework is one of students' opportunities to learn mathematics, but we know little about what students learn from homework or how they learn it. Online homework platforms contain a variety of resources students might access. This paper explores how students used a 'practice another version' (PAV) feature. Findings indicate students used PAV problems for extra practice, as templates, to see the steps for solving a problem, to make sense of a solution method, to troubleshoot when their own method did not work, to see the form of an answer, to maximize their score, and to check that their method was on the right track. The primary uses were for sensemaking, to see steps, or to use the problem as a template. This work lays groundwork for future work characterizing what students learn from homework and how features such as PAV help (or hinder) their learning.

Keywords: Online homework, Instructional situation, Calculus

Homework, like class time, is one of students' opportunities to learn mathematics. However, we know little about the nature of students' activity while doing homework, or what they learn from homework. Findings from research indicate undergraduate calculus students spend more time doing homework than they do in class (Ellis, Hanson, Nuñez, & Rasmussen, 2015; Krause & Putnam, 2016), but that students' reasoning in homework exercises tends to involve mimicking procedures (Lithner, 2003). Students report that homework and practice exercises are important for their learning, more so than other forms of instruction like lecture (Glass & Sue, 2008). Hence research about student learning from homework has the potential to improve homework as a learning opportunity and increase student learning outcomes in general.

Increasing sophistication in technology has resulted in more and more students having online mathematics homework. Online homework platforms often have a variety of "help" features such as links to relevant textbook sections, opportunities to work through similar examples, and hints. However, we do not know if these features help students learn. Research about student learning from homework has the potential to increase student learning by telling us more about what types of problems are best suited for homework, what sorts of resources aid in student learning, and so on.

To that end, this paper reports on a study of how three differential calculus students used the 'practice another version' feature while completing an online homework assignment about the Fundamental Theorem of Calculus Part II. In a "practice another version" (PAV) feature, the online platform supplies a problem identical to the student's assigned problem but with different numbers. Once the student inputs an answer to the practice problem, the online platform shows the solution and solution method. Krause and Putnam (2016) found students used 'see similar example'-type features more frequently than other resources (class notes, online calculators, their textbook, office hours). Hence knowing more about how students use these features and whether they support or hinder student learning can directly inform the design of better online homework assignments. This paper supports that goal by answering research question *how do students engage with PAV in an online homework assignment?*

Theoretical Framework

Homework is one component of an instructional system (Ellis et al., 2015). In this framing, instruction is the set of "interactions among teachers and students around content, in environments" (Cohen, Raudenbush, & Ball, 2003, p. 122). These interactions take place in a variety of *milieu*, each of which is a counterpart environment in which students have particular goals and resources and in which they obtain feedback on their actions. As students work on a task within a milieu, they have an opportunity to learn the *knowledge at stake* (or content). An online homework assignment is a milieu in which students work on tasks that an instructor has chosen to help them learn the knowledge at stake. Instructors also choose some of the resources students can access while doing homework. The PAV feature is one such resource. This paper focuses on how students engage with this resource. The theoretical framework also informed the work in that I viewed the particular ways in which students engaged with PAV as a consequence of their goals for the homework assignment.

Background Literature

Few studies have explored students' activity while doing homework. Dorko (accepted) found students' answers in online homework may be based on mathematical thinking, guessing, working or mimicking steps of a similar problem, copying an answer from a source (e.g., a webpage), or reasoning based on answers they think they should see (e.g., expecting some answers to be infinity and others to 0). Other research supports the finding that students often work or mimic steps of similar problems (Krause & Putnam, 2019; Lithner, 2003). When finding similar problems, students seem to attend to superficial similarities, such as two sequences with $\cos n\pi$ in the numerator (Lithner, 2003). Relying on these types of "similarities" may hurt students' learning because students may give up when following the steps of the "similar" problem does not work (Lithner, 2003).

In terms of student learning from homework, most research has focused on determining whether homework format (online, paper-and-pencil, or a combination) makes a difference in terms of student achievement. Researchers generally agree that online homework is at least as effective as paper-and-pencil homework in terms of student achievement (Babaali & Gonzalez, 2015; Burch & Kuo, 2010; Hauk et al. 2015; Halcrow & Dunnigan, 2012; Hauk, Powers, & Segalla, 2004; Hauk & Segalla, 2005; Hirsch & Weibel, 2003; LaRose, 2010; Lunsford & Pendergrass, 2016; Lenz, 2010; Weibel & Hirsch, 2002; Zerr, 2007). In terms of a more fine-grained analysis of what students learn from online homework, Dorko (2019) found online homework supported students' learning of procedures and familiarity with sequence notation. This study aims to extend the literature base by documenting how students engage with the PAV feature, laying more groundwork for studying student learning from homework and in particular how "help" features in online homework platforms can support and/or constrain it.

Data Collection and Analysis

The data presented here come from observational sessions and follow-up interviews with each of three students enrolled in differential calculus during the fall semester at a large university in the US. In the observational session, I video-recorded each student while (s)he completed an online homework assignment about the Fundamental Theorem of Calculus Part II (see the tasks for which the PAV feature was enabled in Table 1). The assignment was in a platform called WebAssign and students obtained full credit for a problem if they answered it correctly in 3 tries, partial credit for a correct answer on the 4^{th} or 5^{th} try, and no credit after that. Prior to the session, I requested that students do their homework as they would if I were not in the room, including using internet searches and so on. I did not ask students any questions during the observational session because I did not want to influence their activity. I photocopied students' scratchwork and any class notes they had referenced. Following that session, I made notes about students' observable activity. I first noted in which questions they opened PAV (Table 1). For those problems, I noted what students submitted for answers (if anything) to their own problem before opening PAV, the answers they submitted in PAV, and what they wrote and submitted after seeing the solution to the practice problem. These notes allowed me to ask more informed questions in the follow-up interviews. For example, I observed students submit what seemed like nonsense answers in the PAV problem (e.g., submitting the letter 'r' to each of three parts of a question about an integral) so in the follow-up interview, I asked why they had submitted a particular answer.

In the follow-up interview, the student and I watched the video together and I paused it every time the student opened the PAV feature. I asked questions such as why they had opened PAV at that particular time, why they had inputted a particular answer to the PAV problem, and how the PAV had helped them. I transcribed these interviews and added my notes from the observational session, then used these 'enhanced' transcripts in analysis.

Table 1. Problems for which students opened PAV (A 'Y' indicates that the student opened the PAV feature for that problem; an 'N' indicates that they did not)

		S10	S13	S15
5.5 #6	Let $G(x) = \int_3^x (t^2 - 8)dt$. Calculate G(3), G'(3), and G'(4). Find a formula for G(x).	Y	N	Y
5.5 #7	Find the formula for the function represented by $\int_{\pi/8}^x sec^2(2u)du$. F(x) = ______	Y	N	Y
5.5 #8	Find the formula for the function represented by $\int_x^0 e^{-5t}dt$. F(x) = *"Calc clip" video link*[1]	Y	N	Y[1]
5.5 #9	Find the formula for the function represented by $\int_2^{\sqrt{x}} \left(\frac{1}{3t+2}\right) dt$. F(x) =	Y	Y	Y
5.5 #10	Find the formula for the function represented by $\int_1^{x^2} (t+7)dt$. F(x) =	N	N	N
5.5 #11	Calculate the derivative. $\frac{d}{dx}\int_{\sqrt{x}}^{x^2} \tan(3t)\, dt$	Y	Y	Y
5.5 #12	Calculate the derivative. $\frac{d}{dx}\int_0^x (14t^5 - 16t^3)\, dt$	N	N	N
5.5 #13	Calculate the derivative. $\frac{d}{d\theta}\int_1^\theta$ 7cotu du	Y	Y	N

I analyzed the data using the constant comparative method (Strauss & Corbin, 1994), a systematic approach to identifying themes in data. One of my observations from watching students open PAV was that they frequently seemed to submit random numbers or letters almost immediately upon having opened the PAV screen. This represents part of how students engage

[1] S15 is the only student of the three who watched the Calc Clip, which narrated a step-by-step solution to finding the function represented by $\int_x^\pi \cos(t)dt$.

with PAV, so I analyzed the students' submissions into the practice problem as one way to lend insight into why they had opened PAV as a resource. The data I coded for this were students' submissions into PAV (which I could see in the video of the observational session) and students' answers to the question "why did you submit…" during the interview. The data fell into two categories (Table 2). The categories are not mutually exclusive; because many of the problems had multiple parts, students sometimes submitted an answer to one part based on mathematical reasoning and an answer to a different part with a random letter. The coding based on these categories is shown in Table 3.

Table 2. Themes in answer entered into PAV

Category	Criteria
Based on mathematical reasoning (MR)	The student says they wanted to check that they were thinking along the correct lines; the student submits an answer to PAV similar to work they have been doing on their own problem
Nonsense answer (NS)	The student says they typed a random letter or number in order to see the solution to the PAV; the student enters a number or letter immediately upon opening PAV

Table 3. Coding for what students submitted into PAV. A black box indicates a student did not use PAV on that particular problem. Other designators are given in Table 2.

	S10	S13	S15
5.5 #6	NS		MR
5.5 #7	NS		NS
5.5 #8	NS		NS
5.5 #9	NS	MR	NS
5.5 #10			
5.5 #11	NS	MR	MR, NS
5.5 #12			
5.5 #13	NS	MR	

A second way I sought to answer the research question was by searching for themes in why students had opened PAV and what they did once they saw the solution to the PAV problem. A first pass through all the data resulted in the following themes: sensemaking, template, troubleshooting, see form of answer, maximize score, obtain more practice, see steps, and check that one is on the right track. In order to refine the themes, determine the criteria for a data excerpt to be coded as representing a particular theme, and ensure the themes adequately described the data, I coded all the data using these themes and created a list of every instance I had coded as a particular theme. In coding the data, I coded only work a student had done or a statement specifically related to a particular problem. I did not code hypothetical statements such as "a lot of the times I'd like plug the numbers from my problem that matched up with the numbers on the practice problem" because coding only work/statements on actual problems allowed insight into the relative frequency of students' particular actions.

I used these lists to refine the criteria. I then took blank copies of the transcripts and re-coded all the data using the refined criteria to test that the criteria were specific enough and also that they described the data. The final categories and their descriptions, formed after two iterations of code/revise/code, are shown in Table 4. The results of the final coding are shown in Table 5.

Table 4. Themes in students' use of PAV

Category	Criteria
Sensemaking	Student describes using the answer from PAV to work backward; the student describes trying to understand the PAV solution or a step in it; the student describes that having read the PAV solution gave them insight or understanding into their own problem; and/or the student describes trying to follow the logic of the steps in the PAV solution; the student describes trying to work through the PAV solution steps
Template	Student says they substituted numbers from own problem into PAV steps, and does NOT say anything about having tried to understand those steps
Troubleshoot	Student has answered his/her own problem incorrectly and says (s)he opened PAV to try and diagnose where he/she made a mistake and/or to compare his/her own work to the PAV answer or solution method
See form of answer	Student says (s)he opened PAV to see what the answer would "look like"
Maximize score	Student describes wanting to see an example or do a practice problem because they have a finite number of attempts on their own problem before losing points
More practice	The researcher observes the student entering answers based on mathematical reasoning (see Table 2) or doing work/calculations with the numbers from the PAV problem for at least one part of the PAV problem; and/or the student describes doing the problem to get more practice, to see an example, and/or to check that an idea for a solution method will work. NOTE: reading through the solution is NOT doing the problem – to be coded as this category, the student has to do work with the PAV numbers and/or submit an answer based on mathematical reasoning to the PAV problem.
See steps	Student describes opening PAV to see how to do the problem
Check that one is on the right track	Student describes opening PAV to compare their work/solution (or a solution method they envision) to the PAV solution to see if their work (or idea) is correct.

Table 5. Coding based on themes (black boxes indicate the student did not use PAV on that problem)

	S10	S13	S15
5.5 #6	See steps Troubleshoot		See steps More practice Sensemaking Right track
5.5 #7	See steps Template Sensemaking		More practice See steps Sensemaking Maximize score
5.5 #8	Template See steps		Right track Sensemaking

		[redacted]	
5.5 #9	See form of answer Template See steps Right track	Senesemaking Troubleshoot	Maximize score See steps Sensemaking Template Right track
5.5 #10	[redacted]	[redacted]	[redacted]
5.5 #11	Template	Troubleshoot More practice	Sensemaking More practice
5.5 #12	[redacted]	[redacted]	[redacted]
5.5 #13	Template	Template	[redacted]

Results

The first result corresponds to the frequency with which students used PAV. Table 1 shows the problems for which PAV was available and whether students accessed PAV for that problem. Table 6 indicates the frequency of students' PAV use. While there was variation in the frequency with which students opened PAV, all students opened PAV at least once and the mean was 4.7 times ($n = 3$). The sample size is too small to conclude anything more than the fact that students do use this feature, some students more frequently than others.

Table 6. Frequency of PAV use

	S10	S13	S15
# of problems (of 8) in which student opened PAV	6 (75%)	3 (37.5%)	5 (62.5%)

The second result corresponds to the ways in which students used the PAV feature. Themes in the data indicate students opened PAV to gain extra practice, to use the worked-out solution as a template, to see the steps for solving a problem, to make sense of a solution method, to troubleshoot when their own method did not work, to see the form of an answer, to maximize their score, and to check that their method was on the right track. Of these, they most frequently used the PAV feature for sensemaking, as a template, and to see the steps to solve the problem (Table 7, specifically the rightmost column).

Table 7. Frequency of ways students used PAV

	S10	S13	S15	Row sum
Sensemaking	1	1	5	7
Template	5	1	1	7
Troubleshoot	1	2	0	3
See form of answer	1	0	0	1
Maximize score	0	0	2	2
More practice	0	1	3	4
See steps	4	0	3	7
Check that one is on the right track	1	0	3	4

As an example of how students think about PAV as a resource, S15 said of his use of PAV in problem 11, "I was just trying to understand what was happening…I was looking for similarities between what I thought was supposed to be happening and what actually was happening. And then I was trying to look at their answer and figure out the processes that they used to find that, seeing as they didn't give any of those. And then how, that if there was any, like enough similarities that like if I didn't understand the problem at all if I could plug my numbers into what they did. I try to avoid that but sometimes it comes down to it… I was able to gather what was happening from their answer. Just having something to look at, like look at their problem and look at the right answer, I was able to just, to work out what was occurring."

I coded this particular excerpt as sensemaking because S15 was trying to understand the PAV solution[2]. More broadly, the quote lends insight into why sensemaking, seeing steps, and 'template' were the most common themes. S15's goal was to understand the process and he used the similar example as an aid. However, he knew that if he could not understand it, he thought he could simply substitute his numbers into the PAV steps and obtain a correct answer for his problem. Recall that in the theoretical framing of the project, I followed Cohen, Raudenbush, and Ball (2003) as defining a milieu as a counterpart environment in which students have goals, resources, and obtain feedback on their work as they complete mathematical tasks. S15's quote is an example of how he leveraged resources to achieve two of his goals: understanding and correct completion.

Similarly, students' tendency to submit "nonsense" answers (Table 5) often reflected a goal to see how to do the problem. S10 primarily wanted to see the steps so she could use them as a template (Table 5). She said, "I just like put random things in really fast so I can see how they did it."

Discussion and Conclusions

Some answers to the research question *how do students engage with PAV in an online homework assignment?* are "frequently" (evidence: the three students access PAV a mean of 4.7 times in 8 problems), and "primarily for sensemaking, to see the steps to do the problem, and to use the steps as a template." A limitation of the analysis presented here is that it stops short of describing the details of that sensemaking and what it is students learned from the assignment. A characterization of whether or not the PAV feature helped students learn is ongoing.

Given students' overwhelming tendency to use PAV as a template for their own problem, it might seem that we should disable the PAV feature. However, results from other studies indicate this is probably insufficient for engaging students in the sorts of mathematical thinking and reasoning we would like to see in their homework. Without PAV, students frequently find worked-out solutions in their textbook and mimic those steps to solve their own problem (Dorko, accepted; Lithner, 2006, 2003). On the one hand, a student searching for a similar problem involves a bit more work than opening PAV because identifying a textbook problem as similar to one's own problem can involve some cognitive work. Lithner (2006, 2003) found students often attended to surface features in identifying similarities (e.g., two sequences with $\cos n\pi$ in the numerator) and this hurt their performance the problems had different underlying structures and required different solution methods. In contrast, students may also leverage a worked-out problem to make sense of the underlying idea (Dorko, accepted). I suggest future research investigate student learning from homework when, instead of given access to PAV, they are given hints related to the underlying structure of the problem.

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[2] I did not code this as 'template' or 'see steps' because S15 was speaking hypothetically; the codes in Table 5 reflect codes for activity that I observed occur, or the student said occurred, in a particular problem.

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Investigating the Effects of Culturally Relevant Pedagogy on College Algebra Students' Attitudes towards Mathematics

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College algebra continues to be a barrier for college students (particularly non-STEM majors) to graduate. The rigid teaching methods for developmental mathematics courses continues to exacerbate this issue causing failure rates among students. In an age where colleges are recruiting more students of color, these students are often regulated to these courses and are struggling to successfully complete it in one attempt. With innovate teaching strategies, this study sought to address this problem. The purpose of study was to investigate the effects of a sequence of lessons grounded in the principles of culturally relevant pedagogy on students enrolled in a college algebra course at a historically Black college/university. In particular, the study examined students' views about mathematics and its interaction with culture. Results indicated that students in this course, had positive views about this course, despite their views about mathematics coming in to the course.

Keywords: College Algebra, Equity and Diversity, Student Affect, Curriculum

College algebra continues to be a gatekeeper to graduation for many non-STEM college majors (Van Dyken, 2016). Each year, only 50% of students earn grades of A, B, or C in these courses (Ganter & Barker, 2004). This means that half the students enrolled in this entry mathematics course receive grades of D, F, or withdraw from the course. This is extremely problematic when we couple this with the fact that most college majors require students receive a C or better in this course to make adequate progress toward their degree.

Minoritized students continue to face numerous barriers to achieving in mathematics courses (Saxe & Braddy, 2015). Due to the negative side effects of "gap gazing" (Bol & Berry III, 2017; Gutiérrez, 2008) minoritized students are often regulated and oppressed under the guise of teachers "doing what is best for their students" (Higgins & Moule, 2009). This causes them to take deficit views towards the learning efforts of these students. Deficit perspectives, traditional instruction, and low expectations are factors that help influence whether students take more advanced coursework in high school. When students are not advised to take more advanced courses, they are inadequately prepared for college. This often results in students being placed in college algebra where failure rates are extremely high (Holloway, 2004). There is a clear need to help struggling students in developmental mathematics courses and to improve the way these mathematics courses are taught (e.g. Carnevale, Smith, & Melton, 2011; Fairweather, 2008; Griffith, 2010; Hurtado, Eagan, & Chang, 2010; Pascarella & Terenzini, 2005; PCAST, 2010; Rasmussen & Ellis, 2013). Researchers have explored various ways to help combat negative learning experiences for students of color by teaching for social justice and by using culturally relevant pedagogy (Gutstein, 2003; Ladson-Billings, 1995a, 1995b).

The effect of teaching with culture has been shown to have positive effects on student achievements. Studies showed that students were able to remain focused on tasks during class (Ensign, 2003) and students took a higher interest in mathematics when they were able to create problems related to their own lives (Hubert, 2014). One instructional approach that connects learning to culture is culturally relevant pedagogy. To this end, the research question that motivated this work is: *How are student's views about mathematics affected in a course centered around culturally relevant pedagogy?*

Literature Review

Gloria Ladson-Billings created a model and theoretical framework of teaching, culturally relevant pedagogy, that places cultural respect and appreciation at the center (Ladson-Billings, 1995a, 1995b, 2009). Ladson-Billings (1995b) states that culturally relevant pedagogy is "a theoretical model that not only addresses student achievement but also helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" (p. 469). Culturally relevant pedagogy is also empowering and allows students to use cultural resources they bring to school from home in order to connect with what they are learning to society, politics, and their emotions (Ladson-Billings, 1995a). Culturally relevant pedagogy is built upon three main tenets: (1) academic achievement, (2) cultural competence, and (3) sociopolitical consciousness. Academic achievement concerns itself with student learning, focusing on what "students actually know and are able to do as a result of pedagogical interactions with skilled teachers" (Ladson-Billings, 2006b, p. 34). Cultural competence involves aiding and empowering students to recognize and honor their own cultural beliefs and practices while also acquiring access to wider cultures. Sociopolitical consciousness focuses on students becoming more conscious and aware of sociopolitical issues not only on a national or a global level, but perhaps even more so on a local level. The focus of this study is on the academic achievement tenet of culturally relevant pedagogy, thus additional relevant literature on this tenet follows.

Teachers working with and creating curricula and various curricular activities for their students in order to support student success is possible through "intensive modeling, scaffolding, and clarification of the challenging curriculum" (Morrison et al., 2008, p. 435). For example, teachers used modeling with culturally relevant pedagogy in order for mathematics to be utilized in a classroom with true constructivist teaching practices (Anhalt, Staats, Cortez, & Civil, 2018). In a study by Rubel and Chu (2012), a conceptual model for teaching was used that centered itself around three dimensions (a) teaching mathematics for understanding -- mixing mathematical concepts, procedures, and facts and also engaging students in mathematical sense-making; (b) centering instruction on students' experiences; and (c) developing students' critical consciousness through mathematics (pp. 40-41). When looking at lessons these teachers implemented (planned and enacted), it was found that teachers who had high levels of one of the dimensions also had high levels in the others. Classrooms with strong instructional environments "consistently contained tasks of high level so cognitive demand and offered students the most varied modalities of participation in mathematics" (p. 50). In weaker classroom environments students were disengaged and were prone to disruptions. These classes also "included negative intellectual support of students (by the teacher and by the other students) (p. 50).

Implications of findings by Aguirre and del Rosario Zavala (2013) suggest teachers are able to critically analyze (through self-reflection) the lessons they are offering within a specific day or an entire unit by looking at the various dimensions related to culturally relevant pedagogy. In order to maintain culturally relevant pedagogy, it begins with intentionally planning to do so, and looking at lesson plans is an integral part of promoting academic rigor amongst students.

Methods

The data for this paper was obtained from a larger dissertation study (**author citation**) which utilized an embedded mixed methods of data collection (DeCuir-Gunby & Schultz, 2017) to study the effects of culturally relevant pedagogy in one mathematics course compared to a control course that did not. In this larger study, two College Algebra and Trigonometry I courses at Bernard St. Stephen State University [pseudonym], a large HBCU in the southeast United States, were studied during the Fall 2018 semester. The two sections were taught by the researcher with one section serving as a control group (the course not receiving lessons based on culturally relevant pedagogy; yet designed lessons with effective mathematical practices (Smith & Stein, 1998; Stein, Smith, Henningsen, & Silver, 2009) and the other the experimental (intervention) group (the course where students received mathematical lessons based on the principles of culturally relevant pedagogy).

For this study, the data methods include eight (representative) student interviews conducted in the experimental group at the end of the semester after having received four lessons grounded in culturally relevant pedagogy (Matthews, Jones, & Parker, 2013) during the last half of the semester (spanning three of the five units). This data source was also triangulated with mathematics reflection journals that students completed after each experimental lesson.

Intervention

In accordance to the three paradigms of culturally relevant pedagogy (Ladson-Billings, 1995b, 2014), four lessons were developed that incorporated these dimensions. The course was divided into five units: (1) Numbers and Their Properties, (2) Linear and Quadratic Equations, (3) Graphs of Functions, (4) Higher Degree Polynomials, and (5) Exponential and Logarithmic Functions. The four lessons were incorporated into the last three units. These lessons provided students with a problem statement in which they had multiple opportunities to explore and create visual representations and vocal arguments of proposed support or denial of the statements using a data sets, mathematical manipulations, and information gleaned from websites containing information given to them by the instructor. Students also had the option and were encouraged to utilize other sources of data in order to fulfill the requirements of these assignments, frequently using their prior knowledge and past experiences of various topics in issues in society.

These lessons were intentionally developed with culturally relevant pedagogy and the population of the students in this course during this specific semester. These lessons were rated using Matthews, Jones, and Parker’s (2013) Culturally Relevant Cognitively Demanding (2011) task rubric. An example of one of the lessons was lesson three, which had the guiding questions of: What does incarceration look like in this county and the United States? Does race play a role in incarceration? In this lesson, students learned the foundations of graphing linear equations, a skill needed throughout the rest of the semester. They learn this using real incarceration data of the county where their institution was located and is also related to the race of the students in the classroom. Some of the students have had experiences with people who are incarcerated and/or are knowledgeable about the criminal justice system. Students were asked to think about the guiding questions and to use the data and their past experiences to think critically about the issue at hand which led to critical discussions of the institution of prions and their role in society and Black and African-American communities, specifically.

Results

The qualitative data presented in this section addresses the following research question related to academic achievement: *How are student views about mathematics affected in a course centered around culturally relevant pedagogy?* The reflection journals and interviews were analyzed and several themes and subthemes were identified and will be discussed below. For the purposes of this paper, the results shown will solely center around students' attitudes.

Several student attitude profiles emerged from this data, which are presented below. When describing each profile, I begin by describing who each student is in the form of a brief background to dives into who the students are, not only as learners of mathematics, but also as people. These descriptions offer more insight into who the students are beyond the information provided in the Methods section. Two students embodied each of these profiles; however, due to space, I will include only comments for one of the students.

Positive Mathematics Attitudes: Liked Mathematics Prior to this Course

This profile concerns itself with students who had positive dispositions towards the subject of mathematics and who also had positive dispositions towards the culturally relevant-based course.

Tiffany is an African-American female, first-year nursing major student who received the grade of an A in the course. She stated that mathematics is her favorite subject and she simply just "likes numbers." She attributes this partly to her mother who also likes to "do math" and does so in her career. She describes herself as a hands-on/kinesthetic learner who needs to see the mathematics that she is learning in action before she can truly feel like she understands it.

When asked about how participating in this course changed her thoughts about mathematics she enthusiastically stated:

> *Tiffany*: This course made me love math even more than I actually did. Being that I like learning stuff, I was like: oh, this is easy. This is easy. I can then relate it to the real-world stuff and everything. It was great. Yeah.

She then goes on to talk about how the course also helped reaffirm her love and abilities to do mathematics. She spends a good amount of time commenting on how this course increased her confidence in mathematics and how she was, at one point, thinking about switching to a more quantitative intensive major because of it.

> *Tiffany*: Yeah [this course increased my confidence to learn math] because now after this class, I'm like… I want to know more! Like, what else can you do with it? Yeah. I honestly, I had an opinion, like, maybe I should move my major from nursing to something dealing with math, but then it's like: not so fast! [Laughter] I think I want to stick with nursing because nursing still includes some math.

She and another student felt as if they learned a lot of mathematics throughout the course and that they enjoyed their time in the course. Both came into the course having had positive views towards mathematics and left the class feeling the same way due to the application that the mathematics afforded.

Positive Math Attitude: Just the Numbers

Students who held this disposition loved the subject of mathematics, however, that only extended to the numbers and algebraic manipulations of them devoid of the context. Asia-La'Rae, and African-American first-year student, reiterated her wish to not be in a class where culturally relevant pedagogy is done (literally) every class period. She would prefer the majority of class time be devoted to numerical manipulations and algorithmic learning of procedures. This was the preconceived notions of mathematics that she was used to and, subsequently, prefers. Students often come into mathematics courses with views of how things in a mathematics course should operate (Bennett, 2015; Cobb, Yackel, & Wood, 1992; Kuhn, 2005; Mercer, 2008) and this course conflicted with those preconceived ideas about mathematics. Challenging and pushing to change these norms can be difficult.

Asia-La'Rae elaborates on her experience in this course in the following dialogue:

Interviewer: Do you think other math classes should include lessons like these?

Asia-La'Rae: Yeah, but I will say I wouldn't do too much of it. Yeah, I guess to switch something up or do this like once every unit or every few units or something like that to kind of just get a different… A change in the class and stuff, then yes. But that's not something I would do all the time in math. No.

Interviewer: So, the only ones we did were these four. Do you think that four was too much? Too little? Just the right amount? What do you think?

Asia-La'Rae: Ummm… I don't think it was too much. I think it was a good amount and it's not like they were back to back to back. So, it was pretty good amount for what we were doing and the time we had.

Interviewer: Yeah. Time was a huge factor. So, what would you think if we would've did like one a week?

Asia-La'Rae: TOO MUCH! [Laughter]. I don't want to do that.

Asia-La'Rae's justification for why she thought lessons like these in a mathematics class should be limited came earlier in the interview when she was asked what her least favorite part of the course was.

Asia-La'Rae: Mines was actually those days where you did the comparing, you know, to stuff going on today. Sometimes, I mean, I understood it and it most definitely worked with whatever math we were doing, but sometimes I just rather just; oh yea, know the formula or know how to do it and just do the math instead of just actual like comparing it to outside world sometimes, even though I understand it but sometime I'd just rather not do that – just get straight to the point. Do the math.

Interviewer: Okay. Talk more about these lessons.

Asia-La'Rae: Yeah… [the lessons] were all relevant, I'm just saying. They most def are important to know.

Interviewer: But you'd rather just…

Asia-La'Rae: I don't know, sometimes I just, I don't feel like doing that. Like say for instance, sometimes, I don't want to answer it like, what does this graph mean, you know? I rather just... "oh yeah these are the notes for the graph Figure out those numbers and what they mean and then go about that it like… full-blown questions as if it's like reading or something of that nature because it's math and sometimes you literally want to do math numbers. That's all.

Interviewer: So, you'd rather just have the numbers – to just work with numbers?

Asia-La'Rae: Yeah.

When pressed further about her thinking on these lessons, it was revealed that she preferred mathematics devoid of context. She prefers "just the numbers" as opposed to situations where she could actually see the mathematics being used in a real-world situation. Although she still liked the course and felt like she learned, Asia-La'Rae still prefers "just the numbers" devoid of and context or applications. While the majority of students interviewed did enjoy and like (to some extent) this "change of pace" in a mathematics course, not everyone found it as helpful, and that was okay for them.

Neutral to Positive: Confidence Boost

Participants who displayed the characteristics of this profile were students who came into the course believing mathematics was simply, "ok" or they did not particularly like or love the subject, but they did not hate it either. After experiencing this course, their confidence in mathematics seemed to get a positive boost.

Matthew, an African-American male first-year student, was asked his thoughts about the subject of mathematics. His response, not surprisingly, was predicated on earlier experiences with mathematics. He stated that mathematics used to be his favorite subject, but when he got older, that changed due to bad experiences when attempting to take Calculus in high school. He had to withdraw from the course due to "how hard it was" and he attributed that to not having the right teacher who was capable of teaching him in the way that he preferred or the way that he could learn. He also stated that *"Overall, I don't have a problem with math. I think I'm actually good at it if I have a good teacher, but overall, it's okay, but it's not a big fan of mine."*

Similar to another student who exhibited the traits form this profile, Matthew, when answering a question concerning his confidence after taking this course, he stated the following:

> *Matthew*: Yeah. It boosted my confidence a lot. It really started making sense. Like I said before, it made me think I was actually good at math. It made it easier, the more math that I did… It made my confidence go to an all-time high.

Both students also expressed how they thought that the subject was "important" more so than any of the other participants, which I thought was interesting.

> *Matthew*: Yes, I do [think other math classes should include lessons like these] 'cus I feel like math is a very important subject you should have after school in the real world, but I also think it should be used in real life situations because it will open your mind up, expanding your vision, especially with real-life situation like this that's going on in the real world today.

Responses like this were surprising. Students of this profile described and acknowledged/agreed that mathematics is important, despite of their past negative experiences in mathematics, but positive experiences in this class and seeing that the mathematics they were learning being connected with the real-world and being able to see its application.

Enjoyed the Course After Entering with Negative Attitudes Towards Mathematics

Some students came into the course disliking mathematics. With this negative attitude towards mathematics, none of the student stated that they absolutely loathed the subject; however, they were very candid in their responses with regards to their feelings about the subject: their strong disliking of it. They also left the class still not particularly liking the subject; however, they did enjoy their time in the class and sees the importance and relevance of the subject, yet they still "strongly dislike" the subject.

Kanisha is an African-American female student; however, unlike the other participants interviewed, she is a non-traditional student. She is a 38-year-old woman, recently retired from serving in the military, with a 16-year-old son. She had the following to say:

> *Kanisha*: I. Don't. Like. Math. [Laughter]. I don't like math. I used to like math like um in high school. I used to get As. Algebra – As. Then I went to geometry and dropped geometry and I went to accounting and I passed that. I took accounting and college and community college and I also had to take algebra too and that's where I was just like kind of lost because there was such a big gap and I was kind of disappointed in myself because I remember getting all As. So, for graduating in 1998, I started my college in 2007 –

that's a big gap. So, it's kind of like went out the door and even; I think I took a couple college algebra courses, so even coming back to this; like my memory is really bad. Really, really bad. So, some things was kind of familiar but a lot of things like I would still have to go back like okay yeah, I might have learned this but I know it looks familiar like I remember, you know, but I don't know how to do it. So, I had to relearn it. So, yeah…. and I'm just not good with math. I'm not a big fan of math. I don't hate it 'cus I don't, but I'm not a big fan of math. Like if I had to choose, math would not be my number one.

Kanisha attributed her dislike somewhat to the fact of her past success in the subject in earlier grade levels, yet when reaching higher levels of mathematics, her success diminished as well as her disposition towards the subject. When asked if the course that she just finished taking changed her thoughts about mathematics at all, she had the following to say:

Kanisha: I would say no, but the course was helpful. Like it wasn't bad, I'm just not into math. The course was very helpful though, like very informative. I did learn, I learned some stuff. I think I'll be able to do more stuff; you know. I also think I'll be ready for the next math course I take.

Kanisha thought the class was informative and helpful. When asked if this course increased her confidence in her ability to do math, she stated:

Kanisha: Well I know I can do math. I can. It's just my comprehension is kind of slow sometimes. Yeah, I know I can do it. It's nothing that I can't do it, it just might take me a longer time. It might take some more research, some more learning but eventually I can do it. I don't believe in 'can't.'

This statement shows that she has no issues with her abilities to do the subject, only she needs more time to process information. She just needs a different set of equitable foundations in place for her to succeed in the subject. Perhaps culturally relevant pedagogy aided her in this.

Discussion and Conclusion

When the materials students engage with utilize pedagogy that was in line with students' cultures, a deeper understanding and appreciation of mathematical concepts was had by students, which was true of this present study and one also seen in by Rubel and Chu (2011). By incorporating rich mathematical problems for students to work through and think through, students were provided the opportunity to learn mathematics in a way that was meaningful and relevant for them (Anhalt et al., 2018).

Through the interviews, all students stated they enjoyed the cultural aspects of the course. Since they were able to see the connections with what they were learning and how it related to their own lives, their interest and confidence in their abilities increased. Not all students improved their attitudes towards the subject of mathematics. In particular, Kanisha and Ahshante were not completely void of negative attitudes towards the subject; however, they appreciated the change in the way the content was delivered in comparison to their previous mathematics courses.

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Measuring the Effectiveness of Social Justice Pedagogy on K-8 Preservice Teachers

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This study examines ways in which preservice teachers use mathematics in a social justice context. This study examines preservice teachers' conceptions of teaching mathematics using a social justice lens. Situated at a large, public, predominantly white institution in the southeast United States, where preservice teachers are not required to take a course on teaching diverse populations, participants were asked to respond to questions surrounding their experience with a mathematical social justice activity adapted from Gutstein (2005). Using the preservice teachers' responses from pre- and post-surveys, researchers were able to compare initial conceptions of teaching for social justice to understandings after an activity involving social justice topics of world wealth and population disparity. Preliminary results show that preservice teachers' attitudes shifted from naive/general notions of teaching mathematics for social justice to somewhat more concrete ideas.

Keywords: Preservice Teachers, Equity and Diversity, Instructional activities and practices

The purpose of this research is to see what, if any, knowledge preservice teachers have about topics within social justice and to see if they see the importance of using social justice in their classrooms. This is important because previous research has shown that students who are from backgrounds that are not within the dominant culture benefit from activities where they can relate content with the real world. If preservice teachers have no notions or background knowledge about social justice or whose cultural competence is low (as seen at predominantly white institutions attempting to serve underrepresented populations), then they are recklessly entering a classroom of people who do not look like them and cannot connect with the students that are found within these schools (Baily, Stribling, & McGowan, 2014). Preservice teachers should be more cognizant of teaching to a diverse population of students through social justice techniques.

The goal of this study is to examine preservice teachers' perceptions of using social justice content in mathematics classes. We want to target preservice teachers and the following research questions served as guideposts for this study: (1) *What are preservice teachers' initial conceptions of teaching for social justice?* (2) *How do preservice teachers' view the concept of social justice and its role in mathematics education?* (3) *After the intervention, what are changes in the initial conceptions of preservice teachers on teaching for social justice in mathematics?*

Theoretical Framework

Teaching mathematics for social justice has its roots grounded in critical race theory (DeCuir & Dixson, 2004; Delgado & Stefancic, 2017; Ladson-Billings & Tate, 1995). We must move towards critical mathematical literacy which emphasizes the development of the knowledge, practices, and discourses for transformative purposes (Tan, Barton, Turner, & Gutiérrez, 2012). These transformative purposes come from Freire (2000), which tells us that education should provide opportunities to understand, challenge, and recreate preconceived understandings of the self and of the world. Using Tan et al.'s (2012) framework (modified for mathematics teaching and learning only) as the source in which this research is situated (see Figure 1), in order for the teaching and learning of mathematics for social justice to become a reality, "the terms of both equity and empowerment need to be enacted" (p. 46). This study aims to illuminate how teaching mathematics for social justice can empower and provide truly equitable mathematics classroom that will strengthen students beyond school.

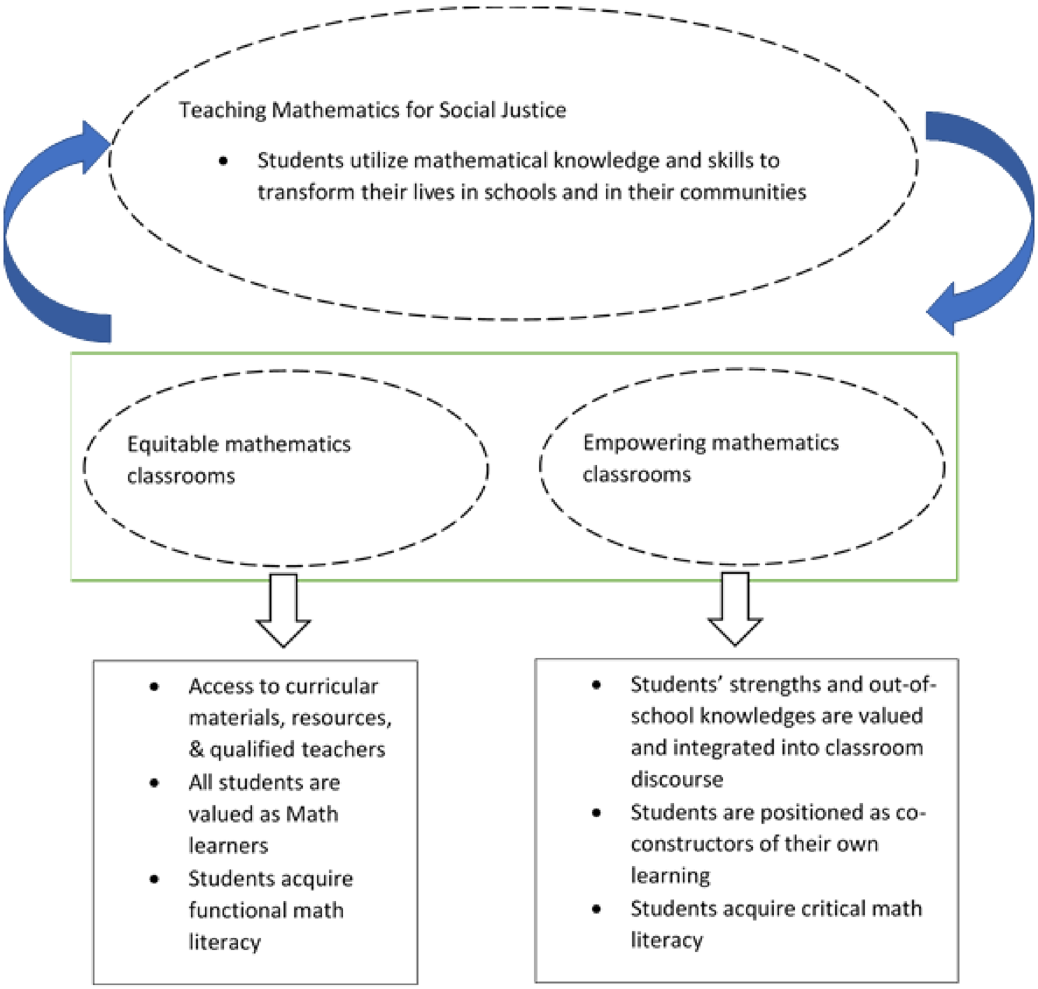


Figure 1. Modified conceptual framework on how teaching mathematics for social justice depends and supports equitable and empowering mathematics classrooms.

Literature Review

Social Justice

While there is a growing body of literature that theorizes the importance of social justice in education (Ayers, 2009; Brion-Meisels, 2009; Michie, 2009; Reynolds, 2009; Torres, 2009), there has been limited understanding of how such theory translates into practice. Researchers have tried to address how teachers engage with social justice curriculum in various forums and with various types of students (Dover, 2009; Hackman, 2005). Facilitating opportunities for university students to reach their full potential requires exposing them to a wide range of experiences, both in and out of the university environment. Students need to be challenged intellectually, as well as creatively, innovatively, politically in areas such as social justice so that they can go out into the world with their eyes, hearts and minds wide open to make it a better place (Hackman, 2005).

Equity/Diversity and Preservice Teachers

While diversity increases among the student population in public schools, the population of preservice teachers remains homogeneous – predominantly White, female, and middle class (Swartz, 2003; Barnes, 2006). One of the challenges for teacher education preparation programs is preparing preservice teachers to teach diverse student populations. An attitude of *naive egalitarianism* is prevalent among preservice teachers. This means that "preservice teachers believe each person is created equal, should have access to equal resources, and should be treated equally" (p. 34). Preservice teachers with these beliefs may lack an understanding of multicultural issues, as well as disregard effects of past and present discrimination (Causey et al., 1999; Finney & Orr, 1995).

In attempts to prepare preservice teachers for diverse student populations, teacher education programs develop multicultural education or diversity courses to address broadly defined issues such as race, class, ethnicity, gender, culture, disability, sexual orientation, and so forth (Garmon, 2004). Prior literature states some of the challenges teacher education programs experience preparing preservice teachers to teach diverse student populations (Melnick & Zeichner, 1998; Barnes, 2006; Causey et al., 1999); however, there is limited research on teacher education programs using social justice pedagogy to prepare preservice teachers to teach diverse student populations.

Teaching Mathematics for Social Justice

Prior literature tells us that if we truly wish to engage in teaching mathematics for social justice, we must attend to the "intersectional nature of justice itself" (Larnell, Bullock, & Jett, 2016). This is the idea that justice in one area (policing, housing, wealth, etc.) is linked to other issues. With that, teaching mathematics for social justice cannot be scripted nor can it be packaged as something that will work for all students in all locales (Leonard, Brooks, Barnes-Johnson, & Berry, 2010). The effort to better equip the next generation of teachers begins with these preservice teachers becoming more "self-aware, reflective, and understanding of the future populations of children that they will be held accountable for in the very near future" (Sanders, Haselden, & Moss, 2014, p. 184).

As students' mathematical ability increases, their understandings of how society works will increase as well (Gutstein, 2003). Of the research that has been conducted, very little of it has commented on the challenges that preservice teachers potentially face with such pedagogy and their possible reactions to it. This study seeks to fill these holes in the literature by providing a thorough investigation on the aforementioned holes.

Methods

Research Design and Context

This mixed methods research was conducted using a convergent mixed methodological design (Creswell & Plano Clark, 2018). This approach is the most appropriate methodological design for this study because it allows for the combination of qualitative and quantitative data in explaining and exploring phenomena. This design allows for the ability to answer the research questions using a variety of approaches in data collection.

This research was carried out at a large public university located in the southeast area of the United States. During the Spring 2017, Fall 2018, and Spring 2019 semesters, students were enrolled in either an introductory mathematics education course or one of two advanced mathematics education methods courses. Both courses were each taught by one of the researchers in this study. All survey data was blinded by the non-teaching researcher in this work.

Out of a total of 40 students that took either course, 29 completed both the pre- and post-surveys. Of these students, 22 were in the introductory, while 7 were in the advanced course. Participants included 12 males (41%) and 17 females (59%) comprised of 3 African Americans (11%), 3 Asian (11%), 1 Hispanic (4%), and 21 White (74%). All students were majoring in Mathematics Education.

Data Collection

Data collected in these courses were considered a normal part of course activities where students received a completion grade upon finishing. In the class before the activity, students were instructed to complete an eight-question open-ended (pre-)survey that gauged students' beliefs on what they think it means to teach mathematics within a social justice context. In the survey, three of the questions were demographic. Three of the questions were qualitative, requiring students to explain what they thought about the social justice and social justice mathematics pedagogy. Those questions included: *What do you think it means to teach for Social Justice?* and *How do you think teachers teach for Social Justice in mathematics?* The final qualitative question was a follow up question asking students to explain their selection from a Likert-scale agreement statement that students had to answer: *I can see myself teaching mathematics using Social Justice topics.*

The pre- and post-surveys were identical except the post-survey asked an additional question asking: *"How have your views about teaching mathematics for social justice changed since completing the pre-survey?"* with a request to explain their selection.

Intervention

In the class after the completion of the pre-survey, students participated in a mathematics activity within a social justice context (adapted from Gutstein and Peterson (2005)). During the lesson, students worked together (in small groups) to complete an activity utilizing proportions and ratios to explore the vast differences in wealth between countries around the world by combining mathematics, geography, writing, and social studies. The activity gave students an opportunity to describe and to formulate conjectures for current power and wealth disparities. During the activity, students received a blank map of the world where they had to guess how many people were in each continent/region. They then created proportions using the actual number of people in each continent/region with the number of students in the classroom to make a mathematical representation on their maps. Students then went through a similar process, but this time for the amount of wealth in the world. They represented these proportions with the number of edible treats provided (again, equal to the number of students in the class). Throughout this activity, students had to work individually and in small groups to compute the ratios and proportions and explore the vast differences between where the majority of people live versus where the majority of the wealth resides. Ultimately, through small and large group discussions, the activity connected students' feelings to the data on world wealth and whether our current world structure was "fair," what the definition of "fair" was, and who gets to decide what "fair" is. After this "intervention class," students completed a post-survey mirroring the pre-survey within three days so that the students would have ample time to reflect on the lesson and the activity.

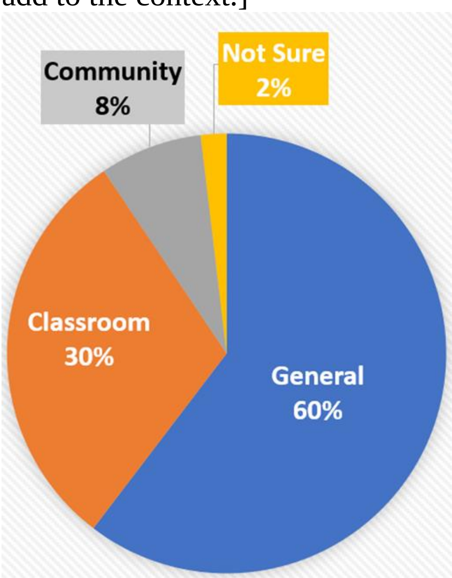
Data Analysis

Data from the pre- and post-surveys were analyzed by the authors. Analysis of the survey data began with the use of open coding in order to conceptualize what has been said by respondents (Strauss & Corbin, 1990). Codes were captured through the use of *in vivo* codes (Creswell, 2013). In order to gain reliability, two members of the analysis team independently coded and the third person mediated to address the differences and the team came to an agreement on the codes before the team progressed. Codes were then refined and a codebook was created. Analysis of the quantitative Likert question occurred using nonparametric hypothesis testing via statistical software.

Results

Research Questions 1 and 2

Preservice teacher responses were placed into three categories: General, Classroom, and Community (Figure 2). Initially, students were not confident or knowledgeable of the question: *What do you think it means to teach for Social Justice?* In the presurvey, 60 percent of the students described general statements about what it means to teach for social justice related to issues of equity and equality of individuals (General). The remaining responses, 30 percent of the preservice teachers talked about how social justice would look specifically in a classroom environment (Classroom), and then 8 percent explained how teaching social justice would look with the community in mind (Community). [Some of the responses are included in Figure 2 to add to the context.]



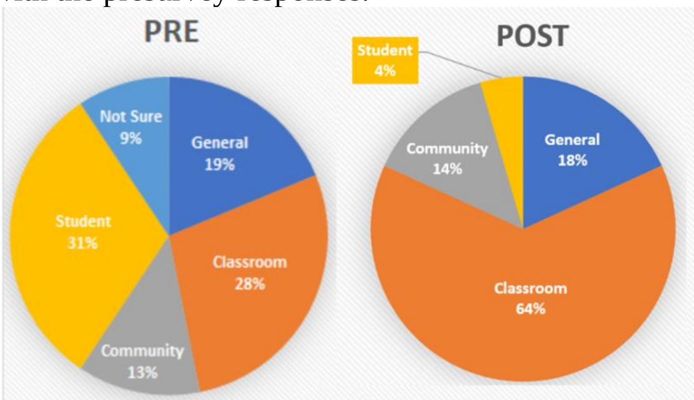
General	PRE
Equity and Equality	9
Help and Support	2
Chance to Learn	7
Influence Admirable Justice System	1
Fair Treatment	7
All Students are Included	3
See Through the Eyes of Others	1
Giving Back to the Community	2

Classroom	PRE
Social Justice Issues	8
Critical Thinking	2
Ending Cycles of Opporession	1
Everyday Classroom	1
Not Discriminate	1
Not Lower Expectations	1
Equity	1

Community	PRE
Social Issues Around the World	4

Figure 2. Responses from questions 4: What do you think it means to teach for Social Justice?

For the next question: *How do you think teachers teach for Social Justice in mathematics?* Student responses were placed into the same categories as previously mentioned and a Student category was added (Figure 3). The Student category included responses where participants referred to teacher-student or student-student interactions as teaching for social justice in the mathematics classroom. For the presurvey, 31 percent of the responses were placed in the student category, followed by 28 percent of responses in the classroom category. Additionally, 19 percent were general statements again related to equity and equality, while 13 percent highlighted interacting with the community as a way to teach for social justice in mathematics. The post-survey results for this same question shows a shift in the preservice teachers' responses. The participants' responses were 64 percent related to classroom interactions, while only 4 percent related to the student. The community and general categories were still consistent with the presurvey responses.



General	PRE	POST
Equity and Equality	4	-
Social justice issues	2	-
Inequality	1	-
Treat others with respect/be kind	-	3
Not related to standards	-	1
Hard to do but through taxes	-	1
Context using social justice as subject matter	-	3

Community	PRE	POST
Project designed to find social justice solutions	2	-
Discussion of social justice issues where students can examine the math of it	2	-
Awareness of social issues	2	-
Project designed to find social justice solutions	1	-
Creating their own social justice activity within the community	-	1
Social issues	-	2

Classroom	PRE	POST
Using math to model Social Justice Issues	6	4
Statistics Classroom	3	-
Equitable Environment	3	-
Critical Thinking/Problem Solving	3	-
Tangible data	-	1
Experiment	-	5
Social Justice/Controversial Issues	-	3
World awareness	-	3
Mix with Social Studies	-	1
Representation of Social Demographics	-	2

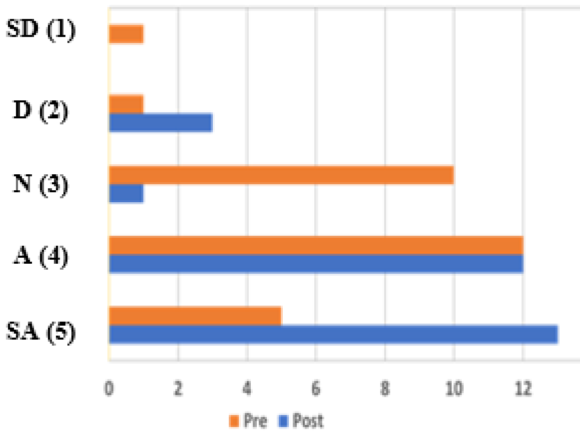
Student	PRE	POST
Non-discrimination of students	4	-
Grading students fairly	2	-
Providing students equal opportunity	5	-
Being patient with students	2	-
Making content relevant to students	2	-
Inclusive to all students	-	7

Figure 3. Responses from questions 5: How do you think teachers teach for social justice in mathematics?

Research Question 3

Question 6 was *I can see myself teaching mathematics using social justice topics*. This quantitative question was addressing research question three (left side of Figure 4). After participating in the intervention, overall, students were more positive and open to integrating social justice topics in the mathematics classroom. Most student responses on the post survey moved from general categories to classroom, teacher, and/or student related. What was more interesting are the actual movement students exhibited from pre to post (right side of Figure 4). Upward movement included seven students who shifted up from Neutral to Agree. Four students went from Neutral to Strongly Agree. One from Neutral to Strongly Agree and a Strongly Disagree to Disagree. There were two downward movements from Agree to Disagree and Neutral to Disagree.

To see if the course, as a whole showed statistically significant movement based on the interaction with social justice less, a Wilcoxon Signed Rank Test was conducted. The results of the test indicated that results from the post-survey responses was statistically significantly higher than the results from the survey responses ($Z = -2.898$, $p < 0.002$).



Student Changes	Strongly Disagree to Disagree	1
	Disagree to Agree	1
	Neutral to Disagree	1
	Neutral to Agree	7
	Agree to Disagree	1
	Agree to Strongly Agree	4
	Neutral to Strongly Agree	1
	Stayed the same	13

Figure 4. Changes pre- to post-survey for question 6: I can see myself teaching mathematics using Social Justice topics.

Discussion and Implications

Further research is needed to identify what connections there are between the social justice activities and the changes in the preservice teachers' perceptions of teaching social justice in mathematics classroom. However, these early results indicate that there may be ways to implement these strategies in more teacher education courses to prepare preservice teachers to teach diverse student populations. This research showcases the importance of preservice teachers needing exposure to teaching mathematics with a social justice lens. If they are not exposed to this style of teaching, they are less likely to either know how to do this effectively or not be privy to the benefits that their future students could experience. Due to the homogeneity of education, it becomes challenging for preservice and in-service teachers to truly find a way to debate the difficult issues that lie at the heart of social justice. It becomes safer to focus on strategies, content, and discipline in an effort to keep from having teachers and teacher educators encounter their own understandings of power, privilege and injustice. This narrow conceptualization of teaching limits the opportunities for teachers to be "relevant, purposeful, collaborative, democratic, and oriented toward social justice and equity" (Wiedeman, 2002, p. 205).

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Departmental Change in Reaction to the Threat of Losing Calculus: Three Cases

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This paper presents narratives of change at three university mathematics departments in response to administrative insistence on improvements in the face of pressure from engineering colleges. Specifically, these three departments were faced with the prospect of losing calculus teaching to another unit on campus and reacted to prevent that outcome. These three departments initiated change efforts including the recruitment of newly appointed leaders to oversee improvements, and all have successfully maintained control of calculus because of perceived program improvements. Drawing on sociological perspectives, we discuss the ways in which change was navigated and negotiated at each site. We found common characterizations of change leadership limited these descriptions of complex change processes, suggesting a need to refine such characterizations in ways that capture more nuanced situations.

Keywords: Institutional change, Calculus, Leadership

Across the United States, concerns about student outcomes in calculus are driving reform efforts. Initiatives to use more student-centered pedagogy have not yet made a significant impact on teaching practices, causing leaders to consider alternatives. One such alternative is to have postsecondary calculus taught outside the mathematics department (NRC, 2013). This specter is of great concern to university mathematics departments, which traditionally provide instructional services to large proportions of undergraduates, particularly those intending to pursue STEM degrees. Removing calculus (or other) teaching from the purview of mathematics departments may decrease financial resources, including faculty and graduate student funding lines. When this specter looms, department actors face pressure to react in ways that protect resources while soothing tensions and addressing underlying concerns about student outcomes.

This paper reports on three similar cases in which the weight of concern, particularly from engineering departments, pushed university administrators to discuss the removal of calculus from the mathematics department. In these three cases, some resolution was reached through negotiations and the introduction of an actor in a new role. These three models of change in response to similar threats suggest that, with effort and collaboration, it is possible to reform calculus programs even at the eleventh hour. Though there are some similarities across these three narratives, we do not suggest that there is one “right” way to change. However, in each narrative, actors in new roles brought outside perspectives while navigating organizational politics at the department, college, and university levels to take steps in the right direction. In each case, these actors restored harmony and improved student success rates in calculus. We focus particularly on the position and roles of these actors rather than the structural changes they supervised, engaging in questions of agency and leadership that can be critical to institutional and departmental change.

Literature Highlights

Universities across the country are seeking to engage in change processes to address persistent concerns about gateway courses (Kezar, 2014; Koch, 2017). Mathematics and other STEM disciplines are contending with low passing rates in foundational courses taken by students in a variety of majors, which have an outsized impact on students' ability and likelihood to complete their degrees and pursue STEM careers (e.g., Seymour & Hewitt, 1997). In response, departmental actors are developing new structural supports and implementing research-based pedagogies. These intentional and planned programmatic changes are designed to accomplish specific goals and outcomes, which separates them from ongoing, unintended, and/or evolutionary changes (Kezar, 2014).

Efforts to enact change in higher education have been deployed at a variety of scales, ranging from individuals to entire institutions (Kezar, 2014). Here we chose to focus primarily within departments because of their relative coherence in terms of policies and norms. Individual departments are generally responsible for designing their own courses. They also represent semi-coherent cultural systems at the intersection of the institution (university) and discipline (mathematics), thus providing an avenue for students to connect to the broader (STEM) landscape (AAC&U, 2014; J. J. Lee, 2007; V. S. Lee, Hyman, & Luginbuhl, 2007). However, efforts to change instructional practice one department at a time have not shifted the status quo away from lecture (e.g., Stains et al., 2018).

Theoretical Perspective

Assessments of change initiatives that have "failed," either to initiate the desired changes or to sustain them long-term, have been attributed to many factors, including the use of implicit, underspecified, and/or narrow theories about change that do not attend to local culture and community factors (Henderson et al., 2011; Kezar, 2014; Reinholz & Apkarian, 2018; Schein, 2010). In this work, we leverage sociological and organization theory to describe and assess the change activities and their impact.

We take a sociologically oriented theoretical perspective to understanding departments and institutional change. At the macro-level, we draw on sociological institutionalism, which is part of the new institutionalism movement within political science (Hall & Taylor, 1996; Ingram & Clay, 2000), and communities of practice (COP) as operationalized by Wenger (1998) and extended by Brown and Duguid (2000). These approaches are consistent in their attention to norms and expectations that inform and constrain the choices available to institutional actors and the consequences of those choices. Both perspectives suggest that individual actors, their behaviors, and beliefs cannot be properly understood outside of the local contexts and communities that constrain not only what people think and do, but what is within the realm of perceived possibilities for thinking and doing.

To frame our results, we leverage Kezar's (2014) four-part framework for understanding change in higher education: (1) identifying the type of change; (2) unpacking the context of change; (3) considering agency and leadership throughout the change process; and (4) combining these to identify an overarching approach to change (Kezar, 2014). For our analyses, we first employed Kezar's (2014) framework to identify the type of change that occurred at each site (i.e., content, focus and sources) as well as the context of the change (e.g., institutional culture, stakeholders). Our analysis foregrounds the third frame, highlighting the roles of particular change agents as they become more central members of their new communities and make choices impacted by the context and culture of their institution (Hall & Taylor, 1996; Ingram & Clay, 2000; Kezar & Gehrke, 2015; Wenger, 1998).

Methods

This work is part of Phase 2 of the Progress through Calculus research project (PtC; NSF DUE No. 1430540; maa.org/ptc). Phase 1 of PtC involved a national survey of university mathematics departments about their Precalculus to Calculus 2 (P2C2) courses and programs; Phase 2 consists of case studies of twelve university mathematics programs. These case studies involve both qualitative and quantitative data collection. Each site was visited by research team members multiple times across two academic years, during which semi-structured interviews were conducted with at least 20 instructors, faculty members, teaching assistants, students, staff, and administrators at each site. Surveys were also distributed to students and instructors each term to gather data about instruction and students' experiences in the P2C2 courses. The design of our research questions and interview protocols draws from previous studies of college calculus (Bressoud, Mesa, & Rasmussen, 2015), and the project evolved to focus on four main themes: (1) course variations; (2) diversity, equity, and inclusivity; (3) course coordination; and (4) change. Here we draw on interview data specifically related to the theme of change.

After completing the data collection visits, project personnel reflected on the totality of the site and drafted thick descriptions. The analysis team read these thick descriptions and conducted an initial analysis of the narratives. Then, using the first two stages of Kezar's (2014) framework, we identified three sites for further analysis since many features of the type and context of change appeared to be similar. Additionally, at each site, our analysis suggested that multiple actors agreed the departmental context had improved as a result of the change process. We then began to cycle through the data and relevant research literature. These iterative phases allowed us to reflexively develop change narratives within the first three pieces of our framework, informed by research on institutionalism, institutional change, and communities of practice.

The three universities being compared in this proposal are referred to as River Rock University (RRU), Rolling Hill University (RHU), and Desert Bloom University (DBU). All three are large public universities classified as R1 institutions (doctoral institutions with very high research activity) and offer both master's and Ph.D. degree programs in mathematics (IUCPR, n.d.; NCES, 2018).

At the end of the reflexive phase, we settled on a framework for understanding change that highlighted six aspects of the process: the reason for the change, who instigated the change, the mathematics department's reaction to calls for change, the role of change agents in the mathematics department, the changes made, and continued monitoring evaluation of change. With these features in mind, the analysis team read relevant interviews from the site visit (e.g., deans, department chairs, client disciplines) in order to understand and describe nuanced features of the change process.

Results

The stories of change at RRU, DBU, and RHU all started similarly: engineering increased pressure on the math department to make changes; upper administrators got involved to force a change; and a new hire entered the scene to drive those changes. We first review the common aspects of the types of change and context for change, highlighting similarities in what led up to the change initiatives. This is followed by separate reports of who was brought to the university and in what capacity, the types of change initiated, and some consideration of leadership and agency within that initiative.

Shared Context of Change

The three institutions experienced a similar problem: high DFW rates that were affecting undergraduates majoring in engineering, the largest client discipline of the math department. While concerns had been building for some time, each institution experienced a "last straw" event. At RRU, a new placement system had recently been enforced, resulting in engineering students being placed further back in the sequence, increasing their time to graduate. At DBU, after years of dissatisfaction with the math department, a semester with particularly high DFW rates was all it took. At RHU, the calculus situation was brought to the attention of a large influential donor who demanded change and threatened to pull his donations until "they did something about the math department" (RHU, S18).

Along with high DFW rates, each engineering division also had the leverage necessary to involve upper administration, which acted as a major catalyst for addressing concerns. Through different means, the problems were brought to the attention of the Provost, who issued a similar threat to each math department: fix the problem or lose calculus (and other courses). Facing this threat, department actors negotiated with administrators and engineering faculty members to begin forging a plan for change. At all three sites, negotiations resulted in the hiring of at least one person to join the department and take on a central role to improve the calculus program.

Differentiated Responses from the Departments

Although each site hired at least one person to take the lead, newcomers' relationships to the university and department varied significantly, as did the plans for change that they created and oversaw. In this section we present narratives of each institution's change story including the department's response and position of the new hire, changes made by the new hire, and current status and continued monitoring.

River Rock University. Determined to show that they were serious about keeping their calculus courses, representatives of the mathematics department met regularly with stakeholders in engineering for several months to negotiate a plan. At this time, the engineering department was under pressure to graduate more engineers and clearly outlined their needs during early negotiations: get students through calculus faster, "fix" placement, and include more applications relevant to engineers. Leading these negotiations was the math chair at the time. His strategy was to cooperate with the engineering department while making sure he had the resources necessary to support their demands. As he described, "[T]he approach we took was to be very accommodating to their needs with the understanding that well, what they were asking was probably gonna take a little bit more in the way of resources" (RRU, Sp19). In response, the committee agreed to hire someone external to the department with the appropriate expertise to design and coordinate a new engineering calculus sequence, who we refer to as James. James was hired in a non-tenure track role, making him officially a part of the mathematics department, but in a role often viewed as peripheral by research faculty at R1 universities.

With funding in place, the new coordinator was hired to design the new engineering courses and met with a committee of content experts to create a pilot. What resulted was the curriculum was trimmed and rearranged to streamline three semesters of calculus into two semesters, and lab sections were created where students would work in small groups on applied problems. These changes also impacted graduate students' experiences with instruction at RRU. First-year graduate students went from teaching their own classes to leading lab sections for engineering calculus, which resulted in more mentoring opportunities.

Overall, the changes are perceived to be successful, as one influential department member stated, "I know the dean of the College of Engineering is very happy with the sequence. They feel the students are being better trained, better educated" (RRU, Fa17). Part of James's role is to act as the liaison between math and engineering, positioning him to act as a broker between communities. He hosts annual meetings with the engineering committee to share local data and solicit feedback. These meetings ensure that math standards are kept high as perceived by the math department while still meeting the needs of the engineering department. These meetings also serve to keep the stakeholders in communication, to ward off future issues by recognizing and addressing them in a timely manner.

Rolling Hill University. Initially, the math department at RHU was not given the chance to negotiate a solution to the problem. For some time, the math department had been perceived as having a bad attitude with an unwillingness to change. With the threat of the loss of an important donor, the Provost declared that math would become a small research department and be almost entirely stripped of their undergraduate teaching responsibilities. In reaction, the mathematics department reached out to an influential member (Don), who was at the time a research fellow outside the U.S. and unaware of the building crisis. He immediately made calls to friends in higher administration, used his status/connections to contact the Provost, managed even to contact the donor, and convinced them that he could solve communication problems between math and client disciplines and address the problems with calculus:

> [B]y me knowing these people when I called them, [...] and they said 'well nobody talks to us in the math department' and that was a big problem over time: the leadership of the math department would not listen to complaints [...] so I said that you know that they can talk to me, that I'll talk to them about it and we [will] try to do something (RHU, Sp18).

Following these conversations, it was decided that Don would return to the math department at RHU as department chair. That he was capable of contacting people across the university indicated that he arrived in this new position already having some legitimacy in multiple communities, with the potential to broker across disjoint communities with dissimilar values.

Don began immediate work to change the "face" of the department. One of his first moves was to appoint a new associate chair, Fred. Together they set out to: (1) change the impression that the department is unfriendly and uncaring toward students, and (2) improve DFW rates by supporting student learning in introductory courses. The first goal involved primarily public relations, and Don instituted a new department policy of "don't be a jerk," asking members to cooperate and work *with* others on campus. To the second, a committee was convened which outlined a three-year plan and the budget required to carry out that plan; following some negotiation this plan and the budget were approved by administrators. Both members of this new leadership team were well-known and respected faculty members in RHU, and that centrality facilitated acceptance of the changes that they introduced:

> [T]his was not going to work with folks who were not respected by their faculty at all and that they had the right... dynamic, engaging, listening personalities to do that outward outreach across campus and really listen and hear and do that gracefully, that's hard. I think the stars aligned for us that we had two super well-respected faculty within the department with the personality and leadership skills to pull this off (RHU, Sp18).

As Don and Fred became central members of the department, the community opened up to cooperating with other stakeholders and changing their departmental practices. The committee's plan included hiring math education-focused faculty, renovating the placement system, instituting a Supplemental Instruction program, improving the learning center, and developing a new co-requisite calculus course. These have all been put into place, adjusted along the way to accommodate evolving state legislature constraints.

During the first year of Don's return to RHU to make changes, the Provost who issued the initial threat was replaced but those involved continued to work for improvement. Now the status of the department seems to be stable, with improved DFW rates and, as the (new) Provost noted, "[T]he reputation that they have on campus and off I mean they- they've become you know state and national leaders in how to implement placement and co-requisite and how to be nice" (RHU, Sp18). The new attitude of department members toward collaboration across departments has resulted in maintained communication about the new program across stakeholders.

Desert Bloom University. At DBU, the Provost's threat to take calculus courses away from the math department was presented as a last resort, with the hope that mathematics, engineering, and administration could negotiate a solution which kept the calculus courses under the purview of the mathematics department. This process started with a series of meetings where each party pitted the blame on the other: "[T]here was three meetings of us yelling at each other before people calmed down and said okay what are we gonna do?" (DBU, Fa17). Eventually, it was decided to make an external hire, and an interdepartmental committee was formed to find someone to be the lead on improving the calculus courses. The committee unanimously agreed on the best candidate during the search process and hired a man we refer to as Kevork.

To avoid tension between tenure-track demands and calculus directorship, a 5-year staff appointment through the Provost's office was created for the new hire, but who would not be filling the position of a mathematics researcher nor faced with the hurdle of achieving tenure on top of managing the calculus improvement efforts. Though funding for the new hire came from the Provost's office, the math department was threatened with a hiring freeze until the position was filled, which increased pressure to find someone quickly. Kevork's position lives in between several groups within a university that had been at odds with each other, coupled with his staff appointment, put him in a peripheral position with respect to multiple communities while the expectation was for him to broker between groups and lead a new system - an inauspicious start.

Although funded and reviewed through the Provost's office, the new hire is functionally part of the math department in terms of his teaching and service requirements. He came in with big ideas for change, but in order to ensure everyone's needs were met, he spent his first year working with others to develop relationships and design a more detailed plan that addressed the needs of multiple stakeholders:

> [H]e met with the engineering departments, and he met with the chemistry department, and the physics department and all of the different, you know, areas, economics, and said 'what do your students need to know to be able to do their work successfully?' and then he met with his colleagues in the math department (DBU, Fa17).

Kevork had the idea to divide calculus into half-term modules at the time of his initial interview, which was refined and adapted to meet the constraints of the university during that first year. Following discussions, each semester of calculus was divided into two half-semester courses. This allows for finer-grained placement into the calculus sequence, more consistent waypoints for students, faster catch-up opportunities for those who fail a course, and a full seven weeks focused on infinite series. The distribution of content among these courses was done in consultation with the mathematics and other stakeholder departments to ensure alignment of objectives. Similar to RRU, the new courses presented mentoring opportunities for graduate students teaching calculus as the new calculus director worked more closely with them.

Passing rates in calculus and first-year retention rates have soared, and the math department remain satisfied with the rigor of the current courses. The interdisciplinary committee formed to find a solution continues to meet twice a semester, with the new hire also in attendance to report on the current status and preemptively address any new concerns; they currently spend a lot of time celebrating successes. The committee is charged with monitoring and evaluating student success in the calculus courses, including careful analysis of available student data.

Discussion and Conclusion

We have presented brief overviews of the processes of change in three mathematics departments when faced with the prospect of losing calculus due to the negative impact of course outcomes on large engineering schools. All three are examples of success, and all three involved the creation of new roles which connected stakeholders and concentrated responsibility in a new actor. The structural changes varied, but we focus on the variation in agency and leadership within the varied institutional contexts. The new actors were assigned different status based on their formal appointments: Don (RHU) was appointed as chair of the mathematics department, James (RRU) was hired as a non-tenure-track faculty member of the mathematics department, and Kevork (DBU) was hired as a staff member under the Provost's auspices. All three have membership in multiple communities at their respective institutions, including: the mathematics department (all), administration (Kevork, Don), and engineering (Kevork, James). Interviews with these three, and others, suggest that they are seen as legitimate participants in the communities which they have membership in, and thus are positioned to act as brokers. Deeper analysis will explore the extent to which this brokering opportunity was taken up, but it is clear that each acted as a broker in some ways (Brown & Duguid, 2000; Wenger, 1998).

Descriptions of change leadership are often collapsed into dualistic concepts like top-down vs. bottom-up and shared vs. collective approaches (Kezar, 2014). While evocative and straightforward, these phrases can obscure the complex ways in which power is distributed through formal hierarchies and social factors (Hall & Taylor, 1996; Schein, 2010; Wenger, 1998). In framing the study, we found that these categorizations were insufficient for making sense of what happened. We see many layers of change agent within each story, with varying motivations and power. For example, while at each site there was intense pressure from "the top" that catalyzed change, the strategies were designed from "the bottom" by members of the mathematics department, some of who were quite peripheral participants in that community. Our future goals include a more in-depth analysis of power, agency, and leadership factors as contextualized by the institutional contexts to provide more nuanced understandings of what constitutes a change agent in postsecondary departmental change initiatives.

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Empirical Re-Conceptualization: Bridging from Empirical Patterns to Insight and Understanding

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Identifying patterns is an important part of mathematical investigation, but many students struggle to explain or justify their pattern-based generalizations or conjectures. These findings have led some researchers to argue for a de-emphasis on pattern-based activities, but others argue that empirical investigation can support the discovery of insight into a problem's structure. We introduce a phenomenon we call empirical re-conceptualization, in which learners identify a conjecture based on an empirical pattern, and then re-interpret that conjecture from a structural perspective. We elaborate this construct by drawing on interview data from undergraduate calculus students and research mathematicians, providing a representative example of empirical re-conceptualization from each participant group. Our findings indicate that developing empirical results can foster subsequent insights, which can in turn lead to justification and proof.

Keywords: Generalization, Patterns, Conjecturing, Justification

Introduction and Motivation

Identifying patterns is a fundamental aspect of mathematical activity, with curricular materials and instructional techniques geared towards supporting students' abilities to leverage empirically-based generalizations and conjectures. However, forming pattern-based conjectures is not sufficient; it is also important for students to understand and justify the patterns they develop. A robust body of research reveals a common phenomenon whereby students are able to leverage patterns in order to develop conjectures, but then struggle to understand, explain, or justify their results (e.g., Čadež & Kolar, 2014; Mason, 1996; Pytlak, 2015; Zazkis & Liljedah, 2002). Indeed, these findings have led some researchers to argue for a de-emphasis on pattern-based activity, dismissing it as unsophisticated (e.g., Carraher et al., 2008; Mhlolo, 2016).

In contrast, we have observed a phenomenon we call *empirical re-conceptualization*, in which participants identify a pattern based on empirical evidence alone to form a conjecture, and then re-interpret their conjecture from a structural perspective. In this paper, we address the following research questions: (a) What characterizes students' and mathematicians' abilities to leverage empirical patterns to develop mathematical insights? (b) What are the potential affordances of engaging in empirical patterning activity? We describe and elaborate this construct across two participant populations, mathematicians and undergraduate students. In this manner, we highlight empirical re-conceptualization as a phenomenon that marks productive mathematical activity at multiple levels and populations. Our findings indicate that developing results from empirical patterns, even those that are poorly understood or unjustified in the moment, can serve as a launching point for subsequent insights, including verification, justification, and proof.

Literature Review and Theoretical Perspectives

There is ample evidence that students are adept at identifying and developing mathematical patterns (e.g., Blanton & Kaput, 2002; Pytlak, 2015; Rivera & Becker, 2008). However, the patterns students identify may not always be those that are mathematically useful. As Carraher et al. (2008) noted, a pattern is not a well-defined concept in mathematics, and there is little agreement on what constitutes a pattern, much less its properties and operations. Students who do identify patterns can then experience difficulties in shifting to algebraic thinking (e.g., Čadež & Kolar, 2015, Moss, Beatty, McNab, & Eisenband, 2006; Mason, 1996). Further, both secondary and undergraduate students who work with patterns struggle to justify them (Hargreaves et al., 1998; Zazkis & Liljedah, 2002). An emphasis on empirical patterning without meaning can promote the learning of routine procedures without understanding (Fou-Lai Lin et al., 2004; MacGregor & Stacey, 1995), or the generalization of a relationship divorced from the structure that produced it (Küchemann, 2010). Further, students' challenges with extending pattern generalization to meaningful learning has been shown to contribute to difficulty in multiple domains, including functions (Ellis & Grinstead, 2008; Zaslavsky, 1997), geometric relationships (Vlahović-Štetić, Pavlin-Bernardić, & Rajter, 2010), and combinatorics (Kavousian, 2008; Lockwood & Reed, 2016), among others.

Despite these drawbacks, researchers also point out the affordances of empirical investigation and pattern development. The activity of developing empirically-based conjectures can support the discovery of insight into a problem's underlying structure, which can, in turn, foster proof construction (Tall et al., 2011; de Villiers, 2010). The degree to which pattern generalization is an effective mode of proof development is an unresolved question, but there is evidence that students can engage in a dynamic interplay between empirical patterning and deductive argumentation (e.g., Schoenfeld, 1986). Similarly, research mathematicians regularly engage in experimentation and deduction as complementary activities (de Villiers, 2010). It may be that students become stuck in a focus on empirical relationships divorced from structure because they lack sufficient experience with this way of thinking. Küchemann (2010) found that with practice, students could learn to glean structure from patterns. Similarly, Tall et al. (2011) argued that attention to the similarities and differences in empirical patterns could support the development of mathematical thinking and proof. These works offer a precedent for positioning empirical patterning as a bridge to insight and deduction.

Structural reasoning. Harel and Soto (2017) introduced five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category, pattern generalization, further distinguishes between two types of generalizing: *Result pattern generalization*, and *process pattern generalization* (Harel, 2001). Result pattern generalization (RPG) is a way of thinking in which one attends solely to regularities in the result. The example Harel gives is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2 + \sqrt{2}}$, $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$, … because the value checks for the first several terms. RPG is typically the type of pattern generalization observed in studies in which students then struggle to shift from recursive to explicit relationships or justify their patterns (e.g., Čadež & Kolar, 2015; Schliemann, Carraher, & Brizuela, 2007). When we refer to the identification of a pattern based on empirical evidence, we are referring to RPG. In contrast, process pattern generalization (PPG) entails attending to regularity in the process, even if that attention may first be initiated by noticing a regularity in the result (Harel, 2001). To extend the above example, Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We define empirical re-conceptualization as the process of re-interpreting a generalization or conjecture from a pattern (identified by RPG) from a structural perspective. By structural perspective, we mean the five major categories of structural reasoning, with the exception of the RPG sub-category of pattern generalization. Thus, a student could reason about the regularity in process from one term to the next of a sequence (PPG). One could also reduce unfamiliar structures into familiar ones, either by constructing new structures or by forming conceptual entities. One may also carry out structural operations in thought without performing calculations, reason in terms of general structures (either by reasoning with conceptual entities or reasoning with operations on conceptual entities), or engage in epistemological justification. In short, re-interpreting a generalization or conjecture from a structural perspective entails the ability to recognize, explore, and reason with general structures.

Figurative and operative thought. The other construct we draw upon to characterize the phenomenon of empirical re-conceptualization addresses a distinction in mental activity (Piaget, 1976, 2001; Steffe, 1991; Thompson, 1985). When engaged in figurative activity, one attends to similarity in perceptual or sensorimotor characteristics. In contrast, operative mental activity entails attending to similarity in structure or function through the coordination and transformation of mental operations. For instance, a student could associate the sine curve with circular motion through conceiving both as representing an invariant relationship of co-varying quantities (an operative association), or through conceiving both as smooth because the motion is perceived as continuous (a figurative association) (Moore et al., 2019). A shift from RPG to PPG is often accompanied by a shift from figurative to operative mental activity, and we consider operative activity to be a hallmark of the ability to reason structurally.

Methods

We drew on interview data from two participant sources, mathematicians and undergraduate students, both stemming from larger projects investigating participants' use of examples to generalize, conjecture, and prove.

Mathematician data. The mathematician data consisted of two sets of hour-long interviews. Thirteen mathematicians participated in Interview 1, and 10 continued for Interview 2. The participants included 7 professors, 3 postdoctoral researchers, and 3 lecturers. Twelve participants hold a Ph.D. in mathematics, and one holds a Ph.D. in computer science. There were 8 men and 5 women. Each interview presented two novel mathematics tasks, which were chosen to be accessible (i.e., they did not require specialized content knowledge) but not trivial (i.e., a solution was not immediately available). For the purposes of this report, we focus on the *Interesting Numbers Task* (Andreescu, Andrica, & Feng, 2007), which we phrased as follows: "Most positive integers can be expressed with the sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9, and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all the interesting numbers?" Below we highlight an exemplar from one mathematician's work on the interesting numbers task, the work of Dr. Fisher.

Undergraduate data. The undergraduate data consisted of a set of hour-long individual interviews with 10 undergraduate male calculus students. The students solved a set of tasks designed to engender generalizing activity (and ultimately to generalize the binomial theorem). For this paper, we report on the *Passwords Task*, which asked students for the number of passwords of length 3, 4, 5, and eventually length n, which consisted of the characters A or B, where repetition was allowed. We asked the students to create tables to organize passwords with a certain number of As, and we also had them reason about the total number of passwords of a given length. Below we focus on one student's work, Raoul, and his reasoning about the total number of length n AB passwords.

Analysis. All interviews were videoed and transcribed, and were also recorded with a Livescribe pen, which yields both an audio record of the interview and a pdf document of the participant's written work. We used gender-preserving pseudonyms for all participants. Using the constant-comparative method (Strauss & Corbin, 1990), we analyzed the interview data in order to identify the participants' generalizations and conjectures and to characterize the mental activity that fostered them. For the first round of analysis we drew on Ellis et al.'s (2017) *Relating-Forming-Extending* Framework for generalizing, which yielded the emergent category of empirical reconceptualization. In subsequent rounds we further examined the participants' talk, gestures, and task responses to further refine the characteristics of empirical reconceptualization.

Results

In order to characterize the participants' abilities to leverage empirical patterns to develop mathematical insights, we present two exemplar cases of empirical re-conceptualization, one from each participant group. In both cases, the participants began with an empirically-based conjecture that they did not understand and could not justify. That initial conjecture, however, then served as a launch point to engage in empirical reconceptualization.

Mathematician Case: Dr. Fisher

Dr. Fisher is a mathematics professor at a large public university. She initially approached the Interesting Numbers Task algebraically by expressing the non-interesting numbers as $m + (m+1) + \cdots + (m+k-1)$. After trying to simplify that expression, she switched to a different approach, in which she listed all of the interesting numbers by ruling out those that were non-interesting. She checked the first numbers 1 to 7 in her head, noting that 3 = 1+2, 5 = 2+3, 6 = 1+2+3, and 7 = 3+4, so only 1, 2, and 4 were interesting. Then she realized she could list all of the non-interesting numbers as sums in an organized table (Figure 1):

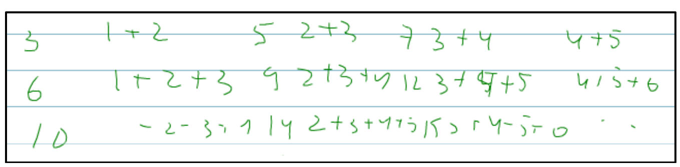

Figure 1. Dr. Fisher's initial table.

Using her table, Dr. Fisher was able to rule out every number up to 24 other than 1, 2, 4, 8, 16, 19, and 23. She recognized the pattern in the first five terms as the powers of 2 and (correctly) conjectured that the interesting numbers were the powers of 2. She made this conjecture even though she had not yet been able to rule out 19 and 23, noting that, "19 and 23 are sort of bothering me from the power of 2 that is showing up in the pattern." Dr. Fisher's conjecture emerged from RPG and figurative activity, in which she noticed a familiar pattern in the (incomplete) data. However, at this time she had no sense of why it should continue.

To determine why the powers of 2 were interesting, Dr. Fisher rewrote her table (Figure 2). She then suddenly noticed a structural pattern in the sums, observing that the sums in the second column could be obtained from the sums in the first column moving diagonally by removing the "+1". Similarly, she saw that the sums in the third column could be obtained from the sums in the second column moving diagonally by removing the "+2". In other words, when moving diagonally up the table, 1+2+3+4 becomes 2+3+4, which becomes 3+4.

1	2	3
1+2	2+3	3+4
1+2+3	2+3+4	3+4+5
1+2+3+4	2+3+4+5	3+4+5+6
1+2+3+4+5	2+3+4+5+6	3+4+5+6+7

Figure 2. Dr. Fisher's refined table attending to sums.

Dr. Fisher: Oh. I see. I see what's happening. […] So, what I see here is that they differ by 1, right? And then they differ by 2. Right? So, I have these numbers [in the first column] and then automatically get these minus 1. […] And then I automatically get those numbers minus 1 and then I get them minus 2 also. And then I get the minus 3 et cetera. So, this actually should tell me most of them by just subtracting.

This insight led Dr. Fisher to formalize the non-interesting numbers as $\binom{n}{2} - 1 - 2 - \cdots - k$, which she simplified to $\binom{n}{2} - \binom{k}{2}$. However, she still did not understand why the powers of 2 did not have that form. To try to answer that question, she returned again to the table and rewrote a third version, this time listing the values of the sums rather than the sums themselves (Figure 3). She evaluated the first row of sums (1+2, 2+3, 3+4, …), then filled in the rest of the table using the diagonal relationship she observed earlier.

3	5	7	9
6	9	12	15
10	14	18	22
15	20	25	30

Figure 3: Dr. Fisher's third refined table.

During this activity, Dr. Fisher noticed a pattern in the second row, "Now what I'm getting here is that these are multiples of 3. Yes, I got it!" To prove this observation, she represented an arbitrary element of the first row by $2k+1$, then used the diagonal relationship to justify that the next element down the diagonal would be $(2k-1) + (k-2) = 3(k-1)$. She gave a similar proof for the multiples of 5 appearing in the 4th row, then realized that a generalized version of the argument would prove that every multiple of $2k+1$ (above a certain point) will appear in the $2k$ row. Her argument proved that every multiple of an odd number is non-interesting, so every interesting number must have no odd factors (i.e. be a power of 2).

Dr. Fisher initially engaged in RPG, generating a conjecture based solely on the numerical pattern she observed in her list of interesting numbers. Once she had the conjecture, however, she shifted from figurative to operative activity, attending to how the sums changed as the number of summands and starting value changed within her table. Through her analysis of the structural relationships in the data, Dr. Fisher was able to engage in multiple rounds of PPG, forming generalizations that led to a partial proof of her conjecture.

Undergraduate Case: Raoul

Raoul was an undergraduate calculus student at a large public university. In his work on the Passwords Task, Raoul leveraged an empirical pattern to develop a conjecture that for an n-

character password made of As and Bs, there would be 2^n total passwords. In the excerpt below, Raoul explained that he saw the pattern based on his prior work determining 3-character, 4-character, and 5-character passwords. However, he did not understand why the total number of passwords was 2^n:

Raoul: Over here I get, for 3 characters, I get 8 numbers. Four characters, 8 times 2, 16. Five characters, 16 times 2. Oh! 2 power *n*.
Interviewer: Okay. Why did you think that?
Raoul: Well I guess, I just started seeing the pattern. I mean 8 is the 2 cubed. I knew that, and I knew that 5 is…no sorry, 32 is 2 power 5. I knew that too, and here is the same, 2 power 4.
Interviewer: Okay.
Raoul: So, I guess, 2 power *n*.
Interviewer: Cool and can you explain, so, why does that make sense? Does it? Why do you think that's true?
Raoul: Um, it doesn't make sense to me why it has to be 2 power *n*. Two power *n*. I have no idea.

Raoul recognized the familiar numbers 8, 16, and 32 as powers of 2, a recognition based in figurative activity. He then engaged in RPG, conjecturing that an *n*-character password would have length 2^n, but he could not justify it combinatorially. The interviewer then asked Raoul why the number of passwords doubled from 4 to 5 characters:

Interviewer: Why would it make sense, let's say from 4 to 5 it doubles, times 2. Why would there be twice as many possibilities here as there were in the 4 case?
Raoul: That's what I'm trying to think. If I can figure that out, I would be able to find why it makes sense to be 2 power n.
Interviewer: Okay, what would you guess for a 2-character password?
Raoul: Two-character passwords? Four.
Interviewer: Okay, how about for 1-character password?
Raoul: There'd be only 2. A and B, so if it's 2, AB, AA, BA, BB. Hmm. Oh, hold up.

Reflecting on a 1-character and 2-character password, Raoul experienced an insight. He began to relate the doubling phenomenon to the combinatorial context of adding another character: "So, I noticed that this is the pattern that I got with 2 characters, so what I find that, *when I increase it to 3 characters* there will be, another character will be adding up, and that could either be A or B, so the number of passwords would be doubled." Important in Raoul's explanation is his shift of attention to what occurred when he moved from the 2-character case to the 3-character case. This marked a shift to operative activity, in that he was now coordinating the mental operation of imagining what happens when increasing the password length by one character. This also fostered PPG, as Raoul could now attend to a regularity in the process and understand why the number of passwords would double each time the length grew by 1. He was then able to explain his reasoning with the case of moving from a 3-character password to a 4-character password: "One character has to be added up, and that character can either be A or B. So, for this one pattern, for 3 characters, there's going to be 2 [options], there's going to be one more pattern if I make it a 4 character."

Raoul began by making an empirically-based generalization that the total number of passwords would be 2^n, but he could not justify his general statement. However, the development of his initial generalization was still important for Raoul's progress. Before he wrote 2^n, the doubling aspect of the relationship was not foregrounded. Once Raoul had produced a generalization, the interviewer could then ask about doubling, which provided an opportunity for Raoul to shift to operative activity, PPG, and ultimately produce a combinatorial justification for why the number of passwords would double when adding a character.

Discussion and Implications

In this paper we have introduced a new phenomenon, empirical re-conceptualization, in which learners develop an initial conjecture based on empirical evidence or RPG, and then are able to re-conceptualize that generalization or conjecture from a structural perspective. Dr. Fisher began with an empirically-based conjecture, but that conjecture enabled her to then begin an in-depth investigation that supported an attention to how the sums changed as the summands and starting values change. She was able to reduce unfamiliar structures into familiar ones and reason in terms of general structures, both elements of structural reasoning. Raoul was similarly able to leverage his empirically-based generalization, 2^n, by shifting from RPG to PPG, and he did so by carrying out structural operations in thought, imagining what would occur when shifting from a 2-character password to a 3-character password, and again from a 3-character password to a 4-character password. Our findings indicate that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and proof. This confirms de Villiers' (2010) claim that "experimental investigation can also sometimes contribute to the discovery of a hidden clue or underlying structure of a problem, leading eventually to the construction or invention of a proof" (p. 215).

Certainly, students frequently identify patterns that they do not understand or cannot justify; this remains a common problem. A danger remains that students will engage in empirical investigation but then not seek to re-conceive their resulting generalizations or conjectures structurally. Our interest is in understanding why some participants in our study were able to engage in empirical re-conceptualization, while others were not. We note that both Raoul and Dr. Fisher had mechanisms by which they could shift their attention towards structural relationships. At times this ability was spontaneous (in the case of Dr. Fisher) and at other times, it required direction from the interviewer (in the case of Raoul). This suggests that directing students towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. In addition, it suggests that when students engage in empirical patterning activities, it is preferable to have them do so within a particular, concrete context. Ultimately, our findings indicate that the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

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Every Mathematics Class is Online: Students' Use of Internet Resources for Self-Directed Learning

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Undergraduate students make extensive use of online resources in lower-division mathematics classes. This explanatory mixed method study (Creswell & Plano Clark, 2011) explores how and why these students are using online resources for their self-directed learning in mathematics courses. Using a survey of 108 students from 13 2-year and 4-year colleges and 26 follow-up interviews, we identify the most commonly used online resources, self-reported description of how these resources are used, and students' perceptions of best practices as well as their concerns about maladaptive practices. We use an Activity Theory framework (Engeström, 1999) that directs our attention to the role that tools play in students' development over time and how these tools are leveraged to achieve learning goals. Our findings reveal undergraduate students as thoughtful and deliberate learners who make extensive use of online resources to help navigate the challenges of higher education.

Keywords: Information literacy, Multi-method, Information seeking behavior

The undergraduate student population is making increasing use of online resources in all areas of studies, but they have become a particular prominent part of mathematical study at the college level. This is attested to by the proliferation of resources which in some cases, such as Khan Academy, have become household names. Many other creators of math lecture videos routinely receive millions of views on YouTube. Students are also making use of answer engines that provide them with step-by-step answers to any math problem they choose and question answering services that host user-submitted solutions.

We employ an explanatory mixed methods study to better understand which resources students are using the most and how they are making use of these resources in order to fulfill their learning goals. We draw on Activity Theory (Engeström, 2001) to develop a model of college mathematics instruction that gives these online resources an appropriate role as part of the learning environment. By looking on institutionalized mathematics instruction as an activity system, we foreground the roles of tools and a social milieu as crucial elements of mathematical learning. This suggests that an accurate accounting of mathematics learning needs to incorporate information about how students are making use of the internet. This concern could be dismissed if the internet were not an important part of students' learning experiences but, as shall be seen below, one of the findings of this work is that a majority of undergraduate mathematics students are making extensive use of the internet and perceive that support as crucial for their success in the classroom.

Literature Review

The Internet and Student Learning

The growing body of literature on the role of the internet on student learning most often focuses on internet use that is structured by the instructor or the curriculum in some way (Lee, Cheung, & Chen, 2005; Precel, Eshet-Alkalai, & Alberton, 2009; Chen, Lambert, & Guidry, 2010), but it is much more difficult to find work that focuses on online learning specific to mathematics.

The internet and mathematics learning. One strand of research on the internet and mathematical learning focuses on uses of the internet that are built into instruction whether this takes the form of a novel online intervention created by the researchers (Rosa & Lerman, 2011; Biehler, Ben-zvi, Bakker, & Makar, 2012) or a course that is based partly, if not completely, on the internet (Timmerman, 2004; Engelbrecht & Harding, 2005; Foster, Anthony, Clements, Sarama, & Williams, 2016). An alternative approach explores how students make use of online resources that are not provided or pointed to by the instructor – we will henceforth refer to such use as self-directed online learning.

Self-directed use of online resources for mathematics classes. There are very few studies that directly address student's self-directed use of online resources in the context of mathematics instruction. A study by Anastasakis, Robinson, and Lerman (2017) surveyed 201 engineering students at a single institution. These researchers asked students to describe the instruments that they adopted and made a distinction between those resources provided by the university and those that students found on their own. They found that their participants were primarily using institutional resources and that they were generally motivated by a desire to do well on exams when they made use of external resources. Notably, this particular university had a robust learning management system which was extensively used by its students. Another study (Muir, 2014) surveyed students in grades 5 through 9, and focused primarily on students' use of Khan Academy as a supplemental resource to classroom instruction. In a couple other cases, researchers examined self-directed online learning by investigating the tracings of such exploration on the websites rather than interviewing students themselves. For example, Van de Sande (2011) studied student interactions in a free online help forum. Similarly, Puustinen, Volckaert-Legrier, Coquin, and Bernicot (2009) looked at submissions to a help forum in order to better understand how French middle school students sought out help with the mathematics that they were learning in school. By focusing on specific resources, none of these studies are able to provide any information about the prevalence of self-directed online learning nor were they able to determine which approaches were most commonly used by students.

Theoretical framework

We make use of Cultural-Historical Activity Theory (Roth & Lee, 2007) as a theory of development that acknowledges the role of the learner's environment both in terms of the people and tools that are incorporated into the activity system.

Activity Theory

Activity Theory was initially developed in order to provide an account of human development that acknowledges the role of tools and the social milieu (Vygotsky, 1997). In this study, we draw on Engeström's (2001) version of activity theory in which the activity system (see Figure 1) serves as the basic unit of analysis for understanding development over time. This perspective acknowledges that any individual/subject is part of a larger community with interactions that are influenced by rules and a division of labor. Tools, in particular, are recognized as both a means to accomplishing goals and as influences on the horizon of possibilities perceived by the subject. This highlights the importance of understanding the tools, specifically online resources, that students are employing as they progress through their classes.

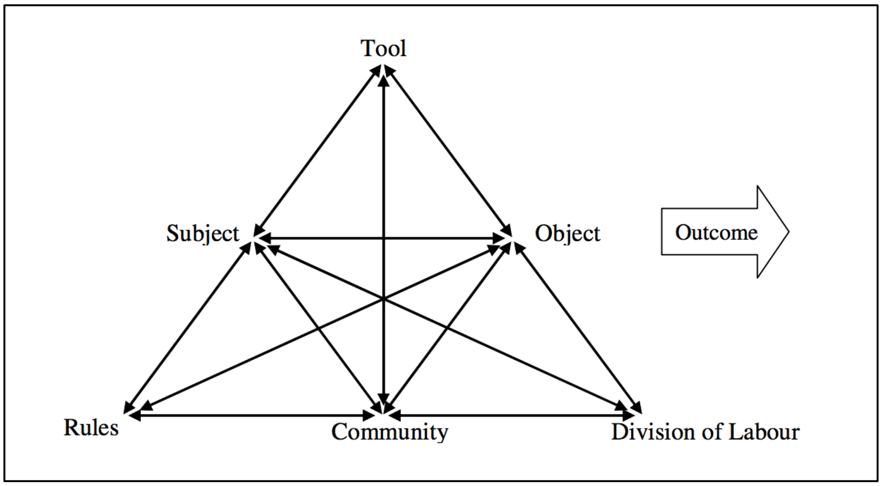

Figure 1. The activity system (Engeström, 2001)

Scanlon and Issroff (2005) made use of activity theory to examine students' evaluation of learning technologies in higher education settings. They argued that in order to reconcile seemingly contradictory responses, the entire activity system needs to be taken into account. For example, they identify seeming contradictions between how students and tutors perceived learning resources since students found a benefit in the serendipitous discoveries allowed by more open-ended resources whereas tutors felt that students would benefit more from a curated set of resources.

Accordingly, this study addresses the following research questions:

1) Which online resources are most frequently used by students as part of their self-directed mathematics study?
2) How do students make use of those online resources and for what purposes?

Methodology

In the section below, I describe the data collection procedures along with our analytical approach.

The Explanatory Sequential Mixed Methods Design

In order to provide a portrait of how students are making use of online resources in general, we distributed a survey across different institutional contexts. These surveys were supplemented by semi-structured follow-up interviews that gave participants the opportunity to elaborate on their initial survey responses. This explanatory mixed methods approach (Creswell & Clark, 2017) allows us to collect large-scale data while deepening our understanding of the quantitative findings. For example, one of the most widely-used resources across our sample was Google, but knowing that students use Google extensively does not help us understand how they are making use of the resource. The follow-up interviews allowed us to create a portrait of the various ways that Google is used to study mathematics, as will be detailed in the findings below.

The Data Sources

We surveyed 108 undergraduate students from 13 different 2-year and 4-year colleges and conducted 26 follow-up interviews with a random subset of those participants. See Table 1 for general demographic information about the participants in the survey. The demographics of the surveyed students is largely representative of the United States undergraduate population although black students and male students are under-represented in this sample. Notably, almost 42% of the surveyed students stated that they are first-generation according to the Department of Education definition[1]. About a quarter of the participants were taking a precalculus or algebra course at the time of the survey while the remainder of the students were either taking a course in the calculus series or a statistics course.

Table 1: *Demographics and Enrollment of the Student Sample (N = 108)*

Characteristic	Frequency	Percentages
Sex		
Female	62	57.4%
Male	40	37.0%
Non-binary / Third Gender	2	1.9%
Prefer to Self-Identify	3	2.8%
Prefer Not to Answer	1	0.9%
Race and Ethnicity		
Native American or Alaska Native	1	0.9%
Asian	11	10.2%
Black or African American	4	3.7%
Native Hawaiian or Other Pacific Islander	0	0%
White, Non-Hispanic or Latinx	64	59.3%
Hispanic or Latinx, Any Race	16	14.8%
Two or More Races	6	5.5%
Other	0	0%
Prefer Not to Answer	6	5.5%
First Generation Status		
Yes	45	41.7%
No	57	52.8%
Prefer Not to Answer	6	5.5%
Current Mathematics Class		
Precalculus or Earlier	28	25.9%
Calculus or Later	39	36.1%
Statistics	40	37.0%
Prefer Not to Answer	1	0.9%

Data Analysis

We report below on the quantitative results of the surveys in order to provide an overview of the resources that undergraduates are using to study mathematics. In the follow-up interviews, each participant was asked about every resource that they said that they used in the survey. They were asked how, when, and why they made use of each resource. We coded the resulting transcripts according to the resources that were referenced by the participant. We then read across every excerpt related to a given resource in order to identify themes that recurred across multiple participant responses. We report on the emergent themes below.

[1] An individual who prior to the age of 18 did not regularly reside with or receive support from a parent or guardian with a baccalaureate degree.

Results

The Pervasiveness of Self-Directed Internet Use

The survey results (see Table 2) suggest that most students in mathematics classes are making frequent use of online resources in ways that are not directed by their instructors. Over a third of surveyed students use resources in this way every day and almost 80% use them at least a few times a week. Furthermore, almost half of the students are using such resources for math courses more often than courses in other subject areas.

Table 2: *Self-Directed Use of Online Resources by Students in Mathematics Courses (n=108)*

How Often	Frequency	Percentages
Every Day	37	34.3%
A Few Times a Week	48	44.4%
About Once a Week	9	8.3%
Several Times a Semester/Quarter	5	4.6%
One or Two Times a Semester/Quarter	7	6.5%
Not at All	2	1.9%
Compared to Other Courses		
More Often	49	45.4%
About the Same	36	33.3%
Less Often	23	21.3%

Table 3: *Percentage of Students Using Selected Online Resources for Summer Mathematics Courses*

Online Resource	At Least Once a Week	Several Times a Semester/Quarter	Once or Twice a Semester/Quarter	Not at all	Do Not Recognize
Google	75.9%	9.3%	8.3%	5.5%	0.9%
YouTube	50.0%	15.7%	25%	9.3%	0%
Khan Academy	25.0%	24.1%	14.8%	27.8%	8.3%
Desmos	22.2%	17.6%	11.1%	13.9%	35.2%
Wolfram Alpha	13.9%	9.3%	11.1%	19.4%	46.3%
Symbolab	13.0%	2.8%	1.9%	8.3%	74.1%
Wikipedia	6.5%	10.1%	13.9 %	69.4%	0.9%

The most commonly used resources (see Table 3) were Google and YouTube. Notably these are tools for accessing content and not the object of students' searching in and of themselves. A majority of student made use of Khan Academy and Desmos at least once per semester/quarter with Wolfram Alpha and Symbolab being used at least once by about 35% and 18% of the surveyed students respectively. Wikipedia was an interesting outlier in that it was one of the least used of the named resources while enjoying a recognition comparable to Google and YouTube.

Patterns of Student Use of Online Resources

Our follow-up semi-structured interviews provided us with additional insight into how students were making use of these tools. Our description begins with Google and YouTube because these resources were the most used and because they occupy a unique position as search engines that bring students in contact with other resources. We will then discuss themes related to the most commonly used resources specific to mathematics instruction: Khan Academy, Desmos, Wolfram Alpha, and Symbolab.

Google. As a search engine, Google is often, though not always, the entry point for students as they seek out resources to aid in their studies. Accordingly, 24 of the 26 interviewees spoke about using Google as part of their self-directed online study. The most often stated reason for the use of Google was convenience, they could enter natural language queries and reliably receive a list of links that responded to their question. The interviewees described a few distinct types of searches using Google: (a) general topics, (b) specific types of problems, and (c) specific problems. There is an interesting overlap between (b) and (c) that has more to do with the goal of the search than with the search terms. When students are looking for a type of problem they may enter in an example problem, but they are not trying to locate a solution to that exact problem, rather they are only looking for a problem that has a family resemblance so that they can get a sense of how the solution method works – this is often expressed as a "problem with different numbers in it". On the other hand, some students are literally looking for the solution to a specific problem and Google will oftentimes provide what they need thanks to websites that host user-submitted solutions to problems such as Chegg, Yahoo Answers, or Mathway.

YouTube. All but one of the interviewees had something to say about YouTube as a source of information. It was described, first and foremost, as a supplement to in-class instruction: students spoke about teachers speaking too quickly and how YouTube would allow them to "catch up and fill in the blanks" or provide them with the opportunity to review material from earlier classes. Students had two distinct ways of interacting with the service. On the one hand, many students relied on specific providers hosted by the platform. Although, rather than advocating for the superiority of any specific speakers, students would generally advocate that others should explore until they found someone that fit their learning style. On the other hand, many interviewees described working with a variety of resources dictated by whatever came up when they searched rather than seeking out specific providers over others. Tactics used to choose specific videos included focusing on the number of likes, the view-count, or the date a video was posted. Finally, interviewees often spoke of the importance of video resources for providing a verbal and/or visual perspective that might not be otherwise available. This visualization component sometimes literally referred to problems that required graphing but students also spoke about the importance of seeing a problem worked out in front of them with an accompanying commentary.

Khan Academy. This resource was often encountered by students before they had entered college through word-of-mouth whether via parents, teachers, or peers. Another reason for the popularity of Khan Academy was its frequent appearance as a search result when a student looked up a topic or question on Google or YouTube. As noted above, many students would default to picking one of the initial search results and so it follows that Khan Academy's prominent placement would contribute to its popularity as a resource. Beyond early adoption and prominent placement in search results, students consistently attributed their use of Khan Academy to reliably clear explanations, availability of worked-out examples and a large quantity of practice problems. They also praised Khan Academy for having videos that correspond to almost any imaginable mathematics topic but this was tempered by a lack of material related to the type of mathematics encountered in upper-division classes. Mathematics majors who had previously made extensive use of Khan Academy would describe a move away from the resource as they realized that it was no longer helpful as they moved into proof-based classes.

Desmos. In contrast with other resources, students were most often introduced to Desmos by an instructor. They emphasized the ease if using the online graphing calculator, comparing it favorably to the actual graphing calculators that they were often required to purchase for college math classes. Furthermore, Desmos was described as a tool that facilitated a deeper understanding of mathematical concepts through their ability to easily zoom in and out of functions and to quickly identify points of intersection.

Wolfram Alpha and Symbolab. Both of these applications are answer engines that can provide step-by-step solutions for a wide variety of math problems although Wolfram Alpha currently requires that users pay a subscription fee in order to get access to all the steps. Students emphasized that they were making use of those solutions in order to better understand how to solve particular types of problems and not to be given answers. These answer engines were particularly attractive to students because of their intuitive interfaces but there was another theme of warning spoken by students who said that it was easy to get overly-dependent on these applications. A student spoke of developing test anxiety because they were too used to being able to look up solutions and others said that they made intentional moves to use them less when they began to get concerned that the answer engines were hindering rather than helping their learning. Notably, these were the only online resources that elicited such warnings.

Discussion and Conclusion

Our survey results provide evidence that undergraduate students extensively use online resources in self-directed ways for mathematics courses. Further, we identified the most commonly-used resources used by these students: Google, YouTube, Khan Academy, Desmos, Wolfram Alpha, and Symbolab. These tools are used by students to supplement in-class instruction with additional lectures, develop a better understanding of how to solve different types of problems, seek out help with specific problems, and provide visual representations that would not be easily available through other means. As is always the case when newly created tools are incorporated into an activity system, it would not necessarily be helpful to ask whether the system previously lacked something that these tools are now supplying. Rather, it may be more accurate to say that the existence of the tools changes the way that the activity is carried out, at least by a sizable subset of students. They can now fast-forward and rewind lectures by different teachers and so now that is a standard way to come to better understand mathematical concepts. They can now cross-reference their work with completely worked out problems whenever they feel less than confident and so that becomes a part of how they study. This research project suggests that, for good or for ill, these online resources are now an important part of the phenomenon of learning mathematics at the college level and should continue to be the subject of study. Accordingly, the next step in our research will be the observation and interviewing of students as they make use of online resources in ways that we found to commonly recur in the present study. This data will allow us to better understand how students are interacting with these resources in a moment-by-moment way and support an analysis of the benefit and drawbacks of these interactions.

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Investigating Combinatorial Provers' Cognitive Models of Multiplication

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Abstract: Combinatorial proof is an important topic both for combinatorics education and proof education researchers, but relatively little has been studied about the teaching and learning of combinatorial proof. In this paper, we focus on one specific phenomenon that emerged during interviews with mathematicians and students who were experienced provers. In particular, participants used a wide variety of cognitive models to interpret multiplication by a constant when reasoning about binomial identities, some of which seemed to be more (or less) effective in helping produce a combinatorial proof. We present these cognitive models and describe episodes that illustrate implications of these cognitive models for our participants' work. Our findings both inform research on combinatorial proof and highlight the importance of understanding subtleties of the familiar operation of multiplication.

Keywords: Combinatorial proof, Multiplication, Counting problems

Introduction

Combinatorics is an increasingly important branch of mathematics, and one class of combinatorics problems, combinatorial proof of binomial identities[1], comes up in discrete mathematics, statistics, number theory, and a variety of other contexts. From our experience working with students in the classroom and research settings (e.g. Lockwood, Reed, & Erickson, 2019), we know that these problems can be tricky even for accomplished counters, and yet this topic has received relatively little attention from the mathematics education research community. A binomial identity is an equation involving one or more binomial coefficients, such as: $\binom{n}{k}k = n\binom{n-1}{k-1}$. In this paper, we take *combinatorial proof* to mean any proof that establishes the veracity of a binomial identity by arguing that each side enumerates the same (finite) set[2]. The validity of these arguments is rooted in the fact that a set can have only one cardinality.

For example, to prove the above binomial identity, one could argue that each side counts the number of committees of size *k* with a chairperson that can be formed from a group of *n* people. In this case, the right-hand side counts this set because there are *n* possible people who could be the chairperson, and then out of the remaining *n-1* people left, there are $\binom{n-1}{k-1}$ ways of selecting the remaining *k-1* people on the committee. As a lead-in to the rest of this paper, we offer the following two questions to provoke thought (we present our research questions at the end of the Multiplication and Counting section). First, w*hy does* $\binom{n}{k}k$ *also count the number of committees of size* k *with a chairperson that can be formed from* n *people?* Second*, how are you thinking of the multiplication of the binomial coefficient by k, and what cognitive model of multiplication are you using?*

[1] Combinatorial proof is a proof technique that can be applied to other types of theorems as well, but we focus on binomial identities in this paper as that was the focus of our research endeavors.

[2] We acknowledge that authors such as Lockwood et al. (2019) and Rosen (2012) have articulated two types of combinatorial proof where one type is that described above, and the second type involves arguing that each side of a binomial identity counts a different set and creates a bijection between the two sets. We do not focus on bijective proofs in this paper.

Literature Review and Theoretical Perspective

Previous Work on Combinatorial Proof

We identified only two prior studies that focused on undergraduate students' thinking about and engagement in combinatorial proof. Engelke Infante and CadwalladerOlsker (2010, 2011) looked at students' solutions to exam questions asking for a combinatorial proof of two binomial identities. They examined the solutions to see what difficulties arose for the students and found that the students appeared to struggle with (a) language mimicking, (b) inflexibility of context, (c) misunderstanding of combinatorial functions, and (d) failure to count the same set (p. 95-96). Engelke Infante and CadwalladerOlsker's (2011) also argued that the students may have engaged in *pseudo-semantic proof production*, which is based on the distinction between semantic and syntactic proof production (Weber and Alcock, 2004). Engelke Infante and CadwalladerOlsker (2011) defined pseudo-semantic proof production as "the attempt to engage in a semantic proof production process, but relying on the syntax of a previously encountered proof when faced with a term that the student cannot explain" (p. 96). This research provides evidence that combinatorial proof can be difficult for students and that they may try to imitate enumerative arguments they previously encounted if they get stuck. Engelke and CadwalladerOlsker (2010) also found that having students write a specific "How many?" question when engaging in combinatorial proof may help them be more successful.

More recently, we conducted a 15-session teaching experiment (Steffe & Thompson, 2000) that covered a variety of combinatorics topics with two vector-calculus students (Lockwood et al., 2019). The last three sessions of the teaching experiment were centered around combinatorial proof of binomial identities, and we found that the students seemed to benefit from two particular instantiations: (a) focusing on a particular context (e.g. counting passwords or committees), and (b) considering specific values of n or other variables appearing in the identity to be proven. We also found that a potentially useful way to prepare students for combinatorial proof is to give them opportunities to generalize while solving counting problems and ask them to solve counting problems two different ways. We used Lockwood's (2013) model of students' combinatorial thinking to frame our findings, and we argued that even though the model was originally conceived to study student thinking while solving counting problems (in other words, questions asking "How many…?), it is also an effective lens when studying combinatorial proof.

This prior work on combinatorial proof is valuable and may help give instructors more pedagogical ideas when covering combinatorial proof. However, questions regarding student thinking about combinatorial proof still remain unanswered, and addressing these gaps in the literature is one goal of the research discussed in this paper.

Multiplication and Counting

As Lockwood et al. (2019) pointed out, interpreting mathematical operations as part of a counting process (Lockwood, 2013) is a key aspect of combinatorial proof. Binomial identities may contain many different mathematical operations, but in this paper we focus on multiplication. Multiplication is a familiar operation to students, yet as Batanero, Navarro-Pelayo, and Godino (1997) found, students do not always know when to multiply while solving counting problems and can confuse situations requiring multiplication versus addition (see also Kavousian, 2008). Numerous researchers have studied how young children think about multiplication of positive whole numbers (e.g., Greer, 1992; Mulligan & Mitchelmore, 1997; Tillema, 2013), yet little work has been done investigating how undergraduate students might conceptualize multiplication. Mulligan and Mitchelmore (1997) used five problem types involving multiplication in their study on young children's reasoning about multiplication, which they referred to as *semantic structures*. These are summarized in Table 1. Some of these semantic structures will be important for our discussion later in the Results section.

Table 1. Mulligan and Mitchelmore's (1997) semantic structures of multiplication.

Semantic Structures for Multiplication	Example
Equivalent groups	3 baskets, each with 5 kittens
Multiplicative Comparison	3 times as many kittens as puppies
Rectangular Array	6 rows, each with 4 four kittens
Cartesian product	The number of possible kitten-puppy pairs
Rate	If each kitten has 2 balls of yarn, how many balls of yarn would 3 kittens have?

While little work has been done studying (potentially) different ways that undergraduate students conceptualize multiplication, there has been some work looking at student thinking about the Multiplication Principle[3] (MP), a guiding principle describing when to multiply when solving counting problems. Lockwood, Reed, and Caughman (2017) found that textbook statements of the MP vary significantly more than the statements of key definitions and theorems in other domains, and then building off of this textbook analysis, Lockwood and Purdy (2019) studied two undergraduate students' reinvention of the MP. Lockwood and Purdy (2019) found that even students who can successfully solve counting problems involving multiplication may struggle to characterize when to multiply, and they found that the MP contains important subtleties that textbooks do not always explicitly address.

These studies reveal that multiplication has a varied and nuanced nature that may go unappreciated in undergraduate discrete-mathematics classrooms. In this paper, we attempt to address the following research questions: *(a) What cognitive models of scalar multiplication do undergraduate students use when engaging in combinatorial proof? (b) What are the implications of various cognitive models of scalar multiplication for students' engagement with combinatorial proof?* Here, scalar multiplication refers to multiplication by a single positive integer constant *k*, such as $\binom{n}{k}k$. We use the term *cognitive models* to mean someone's personal representation of what a given instance of the operation of multiplication entails. This construct is similar to Mulligan and Mitchelmore's (1997) semantic structures; however, our use of cognitive models is intended to go deeper than a classification of pre-existing problem types and attempts to capture their mental representations of what multiplication is doing in a binomial identity[4]. We narrow our results to scalar multiplication (as opposed to other expressions involving multiplication that may occur in a binomial identity, such as $\binom{n}{k}\binom{k}{m}$) due to space limitations.

[3] We provide Tucker's (2002) statement of the MP: "Suppose a procedure can be broken down into *m* successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the *m*th stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes" (pg. 170).

[4] Mulligan and Mitchelmore (1997) studied children's *intuitive models* of multiplication in their research; however, this construct was used to refer to children's multiplicative calculation strategies. We felt it was not relevant to our work studying counters' interpretations of generalized identities involving multiplication.

Methods

We conducted video-recorded, semi-structured, task-based interviews (Hunting, 1997) with five undergraduate students and eight mathematicians. We recruited students from upper-division mathematics courses at a large university in the western United States (desiring students who understood what a mathematical proof is), and we recruited mathematicians from three universities in the western United States. All of the mathematicians in our study had experience conducting mathematics research, and we included both mathematicians who did and did not research combinatorics. Each student participated in five hour-long individual interviews, and each mathematician participated in a single, 90-minute individual interview. The students were asked to solve combinatorics problems, give counting arguments for the veracity of binomial identities, and answer reflection questions about their approach to and reasoning about combinatorial proof. The mathematicians were asked to give combinatorial proofs of various binomial identities and answer reflection questions about their approach to, reasoning about, and pedagogical opinions regarding combinatorial proof. These interviews were part of a larger study aimed at understanding mathematicians' and upper-division undergraduate students' reasoning and beliefs about combinatorial proof.

To analyze these data, the first author re-watched the interviews videos, making note of key episodes related to our research questions and looking for emergent themes and coding according to the *grounded theory* qualitative research methodology (Charmaz & Belgrave, 2007). We coded which cognitive models the mathematicians and students used for multiplication based off of Mulligan and Mitchelmore's (1997) semantic structures, and we also examined the data for other emergent cognitive models that did not align with these semantic structures. Relevant portions of the videos were transcribed. We also followed Lockwood et al. (2019) in using Lockwood's (2013) model of students' combinatorial thinking as an analytical lens for interpreting students' combinatorial proof activity.

Results

Combinatorial Provers Used a Variety of Cognitive Models of Multiplication

We were surprised by the variety of cognitive models of multiplication (summarized in Table 2) that the mathematicians and students used while engaged in combinatorial proof.

Table 2. Cognitive models of scalar multiplication used by students and mathematicians interpreting $\binom{n}{k}k$.

Cognitive Model of Multiplication	Example from the Data Applied to $\binom{n}{k}k$
Equivalent groups	k copies of each k-sized committee
Scaling factor[5]	Each k-sized committee is scaled by a factor of k
Inverse of a probability	The multiplicative inverse of the solution to, "If there is a 1-in-k chance that a committee will form at all, what is the probability that a certain committee will form?"
Element selection	Interpreting k as $\binom{k}{1}$, that is, making a selection of 1 from k people after forming a k-sized committee

As can be seen from Table 2, one of our participants' cognitive models, *equivalent groups*, that aligned with one of Mulligan and Mitchelmore's (1997) semantic structures, while others

[5] The idea of multiplication as applying a scaling factor to some quantity may be related to *linear meanings* of multiplication (Tillema, 2013).

did not. The diversity of cognitive models in our data corroborates previous findings (e.g., Lockwood et al., 2017; Lockwood & Purdy, 2019) that while multiplication is a familiar operation, its representations can vary when used in counting and contains subtleties not always appreciated by students and instructors. Perhaps it is not surprising that the students' conceptions of multiplication in a combinatorial context differed from the multiplicative semantic structures used in studies of K-12 children, but we nevertheless did not expect the variety we encountered.

For example, to illustrate the *inverse of a probability* cognitive model of scalar multiplication, we discuss Riley's work on the following problem, *Write down a counting problem whose answer is* $15 \times \binom{14}{3}$. We will refer to this as the Reverse Counting problem. We acknowledge that the Reverse Counting problem does not ask for a combinatorial proof. However, aligning with Lockwood et al. (2019) we believe that interpreting expressions as having an underlying counting process (Lockwood, 2013) is a critical component of writing a combinatorial proof. In addition, this type of activity is consistent with Engelke and CadwalladerOlsker's (2010) study showing that having students write a specific "How many?" question can facilitate more successful combinatorial proof activity. Thus, we consider Riley's work interpreting $15 \times \binom{14}{3}$ as the solution to a counting problem to be relevant to our research questions about combinatorial proof.

Riley quickly realized that they could use the context of forming congressional committees of size 3 from 14 different senators to interpret $\binom{14}{3}$. However, they struggled to incorporate the multiplication by 15. They first thought that a solution might be, "There are 15 congressional committees, each should be size 3, and there are 14 candidates for each council, and membership in one council does preclude membership in another council." Riley soon realized though that the solution to this counting problem would be $\binom{14}{3}^{15}$, not $15 \times \binom{14}{3}$. After spending some more time thinking, they gave the following solution:

> There is a 1 over 15 chance that a congressional committee will be formed. Given that probability and the fact that there are 14 candidates for the council and three positions, what is the multiplicative inverse of the probability that a given council will be selected? Because you have to overcome the probability that it won't happen at all.

This was a fascinating interpretation of multiplication by 15 that we had neither expected nor encountered previously in our experience teaching combinatorics or conducting combinatorics education research. It suggests that some undergraduate students can more easily interpret multiplication in a probabilistic context than a combinatorial context, which corroborates Lockwood & Purdy's (2019) finding that even successful counters can struggle to articulate when to multiply while counting. However, while Riley's solution to the Reverse Counting Problem was certainly interesting, it was not correct since their solution was not a counting problem. To come up with a correct solution, Riley had to shift to the *element selection* cognitive model of scalar multiplication, which we will discuss in the next section.

A Particularly Useful Cognitive Model of Multiplication for Combinatorial Provers

Revisiting the research questions, we found there was surprising variety in the cognitive models of scalar multiplication that students and experts used when trying to write a combinatorial proof (or give a combinatorial interpretation of an expression). In this section we will discuss which cognitive models of multiplication may be easier or more difficult for students to use when constructing a combinatorial proof. Of the cognitive models outlined in Table 2, only two of them—equivalent groups and element selection—were utilized to come up with a correct combinatorial proof. However, as we will see, these two cognitive models may not be equally useful for students. In fact, only one participant (one of the mathematicians) was able to use equivalent groups successfully, while all five of the undergraduate student participants eventually used element selection effectively.

A combinatorial proof utilizing equivalent groups. One of the eight expert mathematicians interviewed, Emery, successfully proved $\binom{n}{k}k = n\binom{n-1}{k-1}$ using equivalent groups. We do not include all of the details of their work, but they were able to conceive of $\binom{n}{k}k$ as counting the number of ways to make *k* copies of all possible *k*-sized subsets that can be formed from *n* distinct objects. They argued that $n\binom{n-1}{k-1}$ counts the same multiset, because for every fixed object there are $\binom{n-1}{k-1}$ ways to select the remaining *k-1* objects to make a subset of size *k*. This process will also generate *k* copies of each subset (one copy for each of the *k* objects that are fixed), and so the left- and right-hand sides of the identity count the same multiset.

While this combinatorial proof is correct, we hypothesize that constructing an argument that enumerates a multiset containing duplicate objects may be challenging for students. It is not trivial to see that $n\binom{n-1}{k-1}$ counts a collection of size-*k* subsets each with *k* copies, and combination problems that allow for repetition (i.e. copies of some elements) are generally more difficult for students than those that do not allow repetition. Indeed, none of the students used this cognitive model of multiplication to prove any of the binomial identities we gave them.

Element selection was productive for student combinatorial provers. As we saw previously, some cognitive models of multiplication did not facilitate successful combinatorial proof activity for students. However, we found that a particular instantiation seemed to help all five student participants conceptualize "$\times\ k$" as a stage in the MP with *k* options, which we refer to as using the *element selection* cognitive model of multiplication. This cognitive model was so helpful that it occurred in every successful combinatorial proof of a binomial identity involving multiplication by a student participant in our study. The instantiation that helped students use this cognitive model was to represent *k* as $\binom{k}{1}$ in binomial identities involving multiplication by *k*. To illustrate how this instantiation and cognitive model seemed productive for students, we turn to Adrien's work.

Like Riley, Adrien struggled with the Reverse Counting Problem, and after they entertained some ideas that were unproductive, we suggested that they recall that $15 = \binom{15}{1}$. The moment we drew their attention to this fact, they immediately articulated that $15 \times \binom{14}{3}$ could count the number of ways to elect a club president and then a 3-person committee from a set of 15 club members. We asked them if writing 15 as $\binom{15}{1}$ was helpful, and Adrien responded:

> So when I was reading it like this, the way I was reading it was as taking the combination—you have 14 you choose 3. I was thinking of, like, scaling that number somehow. So, I was still thinking of it in terms of that number. But then, when you write it like this, it's like, oh, so you started with 15, took 1 specifically, so now it seems like a 2-step process—like, two separate parts of the problem, rather than one part of the problem and, like, oh, how do I scale that?

We interpret this to mean that Adrien initially used the *scaling factor* cognitive model of multiplication, which was unproductive for them. Once they represented 15 as $\binom{15}{1}$, however, Adrien was able to use *element selection* to correctly argue that $15 \times \binom{14}{3}$ could count committees with a club president. Interpreting operations (such as multiplication) in this manner is a critical step in constructing a combinatorial proof, as argued by Lockwood et al. (2019).

Progressing through subsequent problems in our interviews, Adrien proved several more binomial identities using element selection and the $k = \binom{k}{1}$ instantiation. These include one of the more challenging identities we gave students in the interviews, $n2^{n-1} = \sum_{i=1}^{n} \binom{n}{i} i$. When we first gave Adrien this identity, they initially worked some without making progress, and then we suggested that they write down the identity making the substitution *n=5*. Once Adrien did this, they then re-wrote the summation replacing *i* with $\binom{i}{1}$ and were able to recognize the terms in the summation as choosing *i* from a set of *n* distinct things, and then selecting one of those *i* chosen things. (See Figure 1.) Adrien then gave a nice combinatorial proof of the identity in the context of selecting a finalist from *n* people after two selection rounds:

Interviewer: What would you say both sides are counting?

Adrien: So, you have a group of *n* people, and you're trying to select one of them in two stages specifically. So, you have your first stage, where you just select some group of people—it doesn't matter how large it is—and then out of those candidates you then select the final one. And this [left-hand side] sort of does that in the opposite direction. It's, like, it counts, okay, who was the final one? And then who made it to the second round?

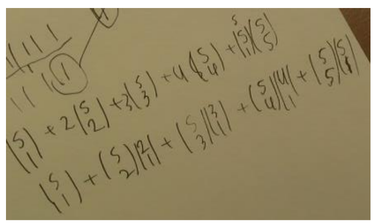

Figure 1. Adrien's work proving $n2^{n-1} = \sum_{i=1}^{n} \binom{n}{i} i$.

We see here again that using the element selection cognitive model to represent scalar multiplication was productive in Adrien's combinatorial proof activity.

Conclusions and Implications

Multiplication is a familiar operation for undergraduate students, and yet we have seen that there are implications for the particular ways in which they reason about it. Combinatorial proof and similar types of problems are a context where the subtleties of multiplication emerge, and we see that it is not a trivial topic, even for upper-division mathematics students. While much work has been done examining the ways that K-12 students reason about multiplication (e.g., Greer, 1992; Mulligan & Mitchelmore, 1997; Tillema, 2013), our study indicates that examining undergraduate students' conceptions of multiplication in combinatorial contexts may be fruitful. While some work has begun to investigate this topic (e.g. Lockwood & Purdy, 2019), how undergraduate students think about and handle the subtleties and variety of cognitive models of multiplication in combinatorial contexts remains largely unknown. Finally, our work suggests that from a pedagogical perspective, some cognitive models of multiplication seem to be more productive for students engaging in combinatorial proof activity than others. Encouraging students to think of multiplication by *k* as $\binom{k}{1}$ if they are stuck may help them more easily see the binomial identity they are working with as corresponding to an underlying counting process that uses the MP (Lockwood, 2013; Lockwood et al., 2019).

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A Model for Assessing ITP Students' Ability to Validate Mathematical Arguments

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In this paper I discuss the process of creating a closed-form multiple-choice assessment of students' ability to validate mathematical arguments at the introduction to proof (ITP) level. This process involved: (1) creating and validating a framework of common validity issues (CVI) in proof writing as a basis for assessment creation through a mathematician survey $(n = 228)$ *and two focus groups (*$n = 4$ *&* $n = 7$*); (2) creating and piloting an open version of the assessment as a means to create distractors for the closed assessment; (3) creating, piloting* $(n = 187)$ *and analyzing the results from the closed form assessment; and (4) conducting interviews with student participants after the pilot to determine the process that students took during the pilot. The results of the processes offer an assessment that, with some refinement, can measure students' ability to validate mathematical arguments from several perspectives in the ITP setting.*

Keywords: Proof, Proof Validation, Assessment, Introduction to Proof

Introduction

Students in advanced mathematics courses spend significant time both constructing and reading proofs. Ostensibly, the rationale for focusing heavily on these two activities is to learn how to construct valid arguments, and to learn the mathematic domain through gaining understanding and conviction in the truth of the mathematical statements (e.g., theorems, lemmas, corollaries, etc.) which define the domain. The former reasoning represents the need to develop the skill of proof constructing, while in the latter, the proofs represent the justifications that each statement is in fact true, further solidifying the primacy of proofs and skills related to proving in the undergraduate settings. One possible hurdle in these curricular efforts is the students' ability to recognize whether an argument is in fact a valid proof. This ability may have an effect on students' conviction with regards to proofs, as well as their ability to understand and construct proofs. Researchers have shown that the relationship between conviction and validity is tenuous for students (Segal, 1999; Weber, 2010) but they have argued that this relationship is important for students to strengthen during their undergraduate course work in mathematics (Selden & Selden, 1995, 2003). It has also been suggested that comprehending and constructing proofs are in some way related to the ability to validate (see Mejía-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Selden & Selden, 1995, 2003), but to date no research has formalized how these relationships are exhibited in practice.

As so, many of these constructs seem to relate to the students' ability to validate mathematical arguments. It becomes necessary to have a systematic means of measuring students' validating ability. By having such a tool, future research could clarify the relationships between validating ability and conviction, proof comprehension and proof construction. The aim of this research was to create a research instrument for measuring students' ability to validate mathematical arguments in the introduction to proof (ITP) setting. The main product of this treatise is a discussion on the methodology undertaken in creating the Assessment of Student Argument Validating Ability (ASAVA). While analysis has begun in earnest on the ASAVA itself, the final product is still in development and requires a final phase of data collection and analysis.

Literature and Theoretical Perspective

In this section, I present an overview of what the literature conveys about undergraduate mathematics majors' ability to validate mathematical arguments as well as a brief presentation of the Common Validity Issue (CVI) framing, the theoretical framing used to guide the creation of the ASAVA. For consistency, I use the phrase *mathematical argument* or just *argument* in place of *proof* and *purported proof*, as in many cases the arguments from this and other studies relevant to validity are often not valid proofs due to the fact they in some way fail to actually prove.

Foundational studies performed by Selden and Selden (1995, 2003) suggest that unaided ITP students are no better than chance at determining the validity of arguments, but with guided intervention can be led to be more reliable in their validation judgments. Alcock and Weber (2005) observed that by the time students are in a real analysis course there are specific aspects of mathematical arguments students have learned to focus on to aid in determining the validity of an argument. Moreover, they noted that these students tend to focus on the veracity of statements rather than the tenability of the statements themselves, meaning students were more concerned with what was or was not true rather than what was or was not supported by the argumentative process employed in a particular argument. Inglis and Alcock (2012) compared the approach that ITP students took in validating arguments to those taken by expert mathematicians. They found that these students tended to fixate on the correctness of mathematical computations, echoing the finds published by Knuth (2002) a decade earlier. They also found that mathematicians were far more active in their *zooming in* – an act described by Weber and Mejía-Ramos' (2011) as focusing on individual portions of an argument rather than on the overall structure – than the students. Additionally, Inglis and Alcock (2012) results supported Selden and Selden's (2003) earlier findings that these students are unreliable in their validating of mathematical arguments. This body of research suggests that ITP students have an underdeveloped sense of validity in the context of mathematical arguments and exhibit behaviors which are dissimilar from their expert mathematics counterparts.

Validity and the Objective/Subjective Dilemma

Proof validation can be thought of as, "The reading of, and reflection on proofs to determine their correctness" (Selden & Selden, 2003, p. 5). Here the literature, as well as my theoretical orientation, recognizes validating as a subjective endeavor dependent on both context and the individual who is validating. This subjectivity at first may appear to make the act of judging and declare an argument wholly valid or invalid as disingenuous. However, I claim that there are validity issues which are so egregious that mathematicians will accept them as universal within the ITP context, thus limiting or removing altogether the effect of the subjective nature of validity. As such, a consensus process was needed to identify arguments that are universally deemed valid or invalid to construct an assessment which meaningfully asks students in the ITP setting to make binary validity judgements despite the normally non-binary nature of validity in mathematics generally.

CVI Framing

One major area not previously explored by current research is a codification of the aspects of proof which ITP students confound when they validate arguments. To better understand this, the CVI framing (Fagan, 2019) was developed as a tool to give specificity to the character of validity issues commonly found in student proofs at the ITP level. This framing consists of six categorizations (see Table 1) extracted from the literature (Alcock & Weber, 2005; Hazzan &

Leron, 1996; Selden & Selden, 1987, 2003; Weber & Alcock, 2005; Weber, 2001) and certified by mathematicians as authentic at the ITP level. In this study, the CVI framework is used as a means of qualifying the validity issues within the arguments of the ASAVA.

Table 1. The Common Validity Issue framework.

Issue (Abbr.)	**Definition**
Assuming the Conclusion (AC)	An argument assumes the consequent (conclusion) of the proposition it is claiming to prove and attempts to show that the antecedent is a direct consequence.
Circular Reasoning (CR)	An argument assumes the consequent (or antecedent) of the statement it is claiming to prove and comes to a trivial conclusion, namely the consequent (or antecedent) once again. Within an argument a claim is made and used to argue to trivial ends, the claim itself. ($P \rightarrow Q \rightarrow \cdots \rightarrow P$). An argument uses the proposition it is trying to prove.
Logical Gap (LG)	An argument omits a portion of reasoning; the argument has a hole. This could be thought of as a lack of a data → warranting → claim turn, or any individual portion of one where such would seem prudent.
Misuse of Notation (MN)	Within an argument, proper notation or variable naming conventions are not adhered to, or notation and variable naming conventions are used inconsistently.
Warranting (W)	Within an argument, an error in justification is made either explicitly or implicitly. This can take the form of an incorrect explicit warrant, or an incorrect implicit warrant which may emerge as an arithmetic or computational error.
Weakening the Theorem (WT)	An argument proves less than what is implied by the statement being proven or begins by assuming more than is permissible.

Though not an explicit part of the CVI framing, *valid* arguments (i.e., proofs) are an important aspect of assessing students' ability to validate arguments. I define valid arguments throughout the study as arguments which do not include the above CVI framing issues. This is an important consideration because if there are no valid arguments to validate amongst a set of invalid arguments, then only some of students' ability is tested, thus the idea of valid not only needs to be defined, but such items become an important barometer for students' true ability to validate.

Limiting Scope

For the standard ITP course (David & Zazkis, 2017), the norm is to introduce students to a wide variety of proof methods (Fagan & Melhuish, 2018). I focused solely on direct proofs in creating the ASAVA to give a clean connection between a student's score and their ability without the possible confounding effect that multiple proof types might introduce. In the example in Figure 1, the proof framework is indicative of a direct proof, where for a conditional statement, $P \rightarrow Q$, the antecedent, P, is assumed true and the desired conclusion is that the consequent, Q, is also true[1]. While it would be interesting to use this assessment to explore students' ability to validate multiple proof types, for now, to assure the best possible instrument and reduce the number of variables in task creation, proof will be defined in terms of direct proofs only.

Theorem: If M is a compact set, then each infinite subset of M has a limit point.

Proof:
Let M be a compact set and N be any infinite subset of M.
$\vdots$
Therefore, N has a limit point. $\square$

Figure 1. The first level proof framework as defined by Selden and Selden (2003) of a direct proof.

Assessments

Educational assessments – especially those designed specifically for research – are tools for observing, documenting, and quantifying a phenomenon (see, Pellegrino, Chudowsky, & Glaser, 2001), but rarely do assessments measure said phenomenon in a straightforward way. On this matter, Mislevy, Steinberg and Almond (2003) noted,

> In assessment, the data are the particular things students say, do, or create in a handful of particular situations, such as essays, diagrams, marks on answer sheets, oral presentations, and utterances in a conversation. Usually our interest lies not so much in these particulars but in the clues they hold about what students know or can do as cast in more general terms. (p. 9)

The overarching implication is that the data from an assessment often means more than the discrete responses to questions and says more about the individual's understanding or ability than the individual tasks.

Test theory

The set of responses which are gathered to evaluate the effectiveness of an assessment must be analyzed to determine the veracity of the assessments capability to meaningfully make inferences about the test taker on a given subject. Classical test theory (CTT) and item response theory (IRT) are the two main forms of analysis which inform this evaluative effort for assessment creation. In modern computer age, IRT became the more common approach as it is often considered the superior method of assessment analysis (Embretson & Reise, 2000).

Characteristic curves for each dichotomous item embody the item's difficulty and discrimination as part of the IRT analysis of an assessment. Using the probability function in Figure 2, these characteristic curves represent the likelihood that an individual with ability θ will get the correct answer for that item. Baker (2001) defines difficulty as a location index for each item on the ability spectrum, where an item with high or low difficulty is associated with high or low ability respectively, and the probability of getting a correct response is .5 for any one item.

$$P(\theta) = \frac{1}{1 + e^{-a_i(\theta - b_i)}}$$

Figure 2. IRT employs a two-parameter logistic model where for any dichotomous item, i, the probability of a correct response, $P(\theta)$, based upon the ability, θ, defined by the item's discrimination (a_i) and difficulty (b_i).

[1] Note that this proof method easily supports proofs which have multiple cases or are proofs of biconditional statements. One such proof could be accomplished by two direct proofs.

Instrument Design

The process for building a large scale student assessment has been outlined in the development of three distinct assessment (Carlson, Oehrtman, Engelke, 2010; Mejía-Ramos, Lew, de la Torre, & Weber, 2018; Melhuish, 2019), all of which borrow from the work on the Force Concept Inventory and Mechanics Baseline Test (Hestenes, Wells, Swackhammer, 1992; Hestenes, & Wells, 1992). As with the work of Mejía-Ramos et al. (2018), the assessment of students' proof validation ability does not strictly fit the model of being a concept inventory. While concept inventories explore students' understanding of a broad set of concepts, the ASAVA is, as yet, focused only on students' ability to validate a subset of proofs from a singular mathematical context, ITP. Despite this difference, the mapping supplied by these concept inventories is still the most comprehensive and sensible solution.

The process of developing the ASAVA involved two phases of assessment creation. Phase one involved developing and analyzing an open-ended survey to create a semi-closed assessment. Phase two involved piloting and analyzing the resultant semi-closed assessment. I will discuss each phase briefly in the following sections. To clarify the conversation, I use the term testlet to denote a theorem, its argument, and all the open-ended or multiple-choice questions dedicated to that theorem/argument pair. I define the terms stem, key, and distractor as they are the major parts of any multiple-choice question: the stem is the question, the key is the correct choice, and the distractors are the set of plausible but incorrect choices (Haladyna, 2004).

Phase 1 – Open-Ended Survey Development and Analysis

Phase one involved multiple steps, the first, and most important of which, was validating the framework and arguments, which are the focus for each item of the student open-ended survey and semi-closed assessment. This validation process involved creating/collecting arguments and expert (i.e., active mathematician) endorsement of said arguments in terms of validity. The process breaks into two stages; (1) item creation[2] and mathematician survey ($n = 228$); and (2) two mathematician focus groups ($n = 4$ and $n = 7$). Next, I constructed, ran, and analyzed the student open-ended surveys ($n = 68$) based upon the twelve best performing items from the mathematician survey, which process gave me a set of possible distractors for the semi-closed assessment that was to follow.

The arguments for each item on the mathematician and student open-ended surveys were a collection of: (1) altered and unaltered student work collected from a pair of ITP courses offered at a large public university in the United States; (2) work collected from the internet site Mathematics Stack Exchange; or (3) altered and unaltered versions of proofs from the common ITP texts (see David & Zazkis, 2017; Fagan & Melhuish, 2018). Arguments were selected based upon their fit within the greater framing of the CVI framework and altered so each argument at most included a single validity issue.

Phase 2 – Semi-Closed Assessment Pilot

The semi-closed assessment was a collection of the eleven best performing items from the student open-ended survey, where each student saw only eight[3]: at least one from each categorization from the CVI framing, and two valid arguments. Each argument was part of a larger testlet – visualized in Figure 3 – aimed at testing students validating ability from two

[2] All told there were 30 arguments to 22 propositions in Phase 1.
[3] The assessment at this point was an anchored assessment with each student seeing the same five arguments and three randomly selected but semi-evenly distributed arguments.

perspectives: (1) their ability to determine the Boolean validity of an argument, and (2) their ability to identify the specific validity issue of an argument. This was accomplished in each testlet by first asking students to decide if an argument was either wholly valid or invalid, and then asking them their rationale based upon a set of possible responses that included a key and a set of distractors. A total of 187 undergraduate students from universities across the US who were at least enrolled in an ITP course – though many had taken more proof-based courses – took part in the semi-closed assessment pilot.

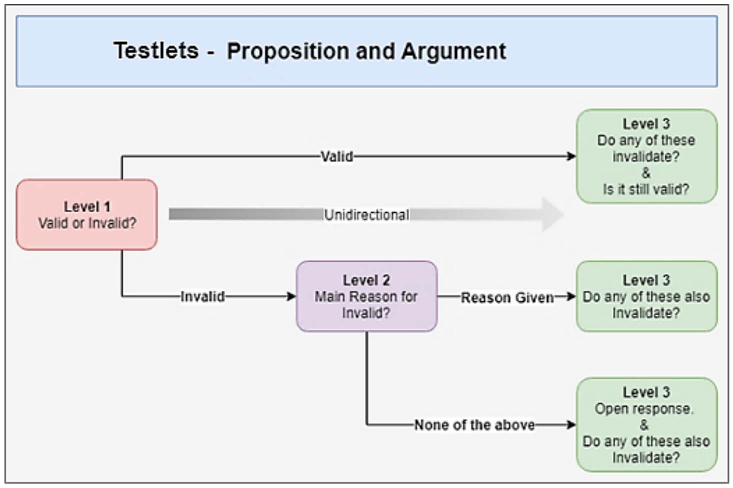

Figure 3. The structure of each testlet was designed to make the validating task non-trivial and to allow students to change their mind about the validity of an argument.

After the pilot, six students were selected to take part in hour long interviews. The participants were selected to account for a wide variety of scores from the semi-closed pilot. The interviews and interview analysis[4] focused on (1) what students were attending to when they were taking the assessment (i.e., were they actually validating or were they doing something else), (2) the quality of the assessment (i.e., were items, distractors, concepts or other ideas worded in problematic ways or incomprehensible due to a lack of explicit explanation or lack of students' prior knowledge), and (3) if students' scores could be accounted for by their ability to validate as opposed to other possible explanations.

Example Testlet

In this section, I present a single testlet from the ASAVA, though the methodology employed is the main contribution of this treatise. I do this to give a deeper understanding of the end product of the process undertaken, not as a means of presenting the overall product as it still needs refinement. This testlet, $T5$ (see Figure 4), was an invalid item with a CVI issue of LG. Each testlet was scored as 1, 0, or -1, with overall scores ranging from negative eight to eight, $[-8,8]$. Consistent with the overall design of the assessment, for $T5$ a score of 1 meant that the student was able to both correctly identify the argument as invalid *and* identify the reason the argument was invalid. A score of 0 meant the student at first identified the argument as valid, but after being asked to decide if any of the key or distractors invalidated the argument, they correctly changed their mind and identified that both the argument was invalid and the correct reason it was invalid. Finally, a score of -1 meant the student was both unable to accomplish the Boolean validation and identify the reason the argument was invalid.

[4] The interviews were analyzed using thematic analysis (Braun & Clarke, 2006).

Proposition: For arbitrary set A, B, and C;

$$(A - C) - (B - C) \subseteq A - B.$$

Argument: Let $x \in (A - C) - (B - C)$. That means that $x \in A$ and $x \notin (B - C)$ which implies that $x \notin B$. Therefore, we can conclude that $(A - C) - (B - C) \subseteq A - B$.

Figure 4.The proposition and argument for testlet $T5$ whose CVI categorization was determined by mathematicians to be LG.

The discrimination for $T5$ was calculated at $d = 15.5592$ with a peak information of 60.52218 at an ability of 0.4590. At first, this made $T5$ appear to be an undesirable outlier as such high discrimination and peak information are not expected in human testing. After further investigation (see Table 2), $T5$ turned out to be a rare case of an item working beyond the norm for human testing. The analysis undertaken suggests that testlet $T5$ performed very well, and that the calculated discrimination was a fair understanding of the testlet's performance in relation to the other testlets from the ASAVA. Overall, students' outcomes on this testlet align in a synchronistic fashion with their overall outcomes, implying testlet $T5$ was not an outlier, but a highly consistent measure.

Table 2. Testlet $T5$ performance against overall performance.

	Total Score																
T5 Score	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
-1	4	3		1	5	4	8	4	5	5	2						
0			1	2		1	5	1	3	1		1	1	2	1		
1											4	2	7	1	14	1	3

Discussion

The main goal of this treatise was to communicate the process by which the research tool, the ASAVA, was developed for measuring ITP students' ability to validate mathematical arguments. The process requires some extension to refine the product of the ASAVA, but the results of the initial process are promising. Items like testlet $T5$ represent a promising representation of what the process outlined in this study can produce. While it is conceivable this study will lead to an assessment focused on validating many types of proofs from a variety of mathematical contexts (e.g., ITP, Algebra, Analysis, and Topology), it is not so at this point. Additionally, while the theoretical framing points to an assessment that measures students' ability to validate both from a Boolean perspective (i.e., valid or invalid) and from an issue identification perspective (i.e., where has the argument failed?), the scoring that was initially put into place does not wholly embrace this sentiment.

Future Efforts

To finalize the ASAVA, a few final refinements must be undertaken. These refinements include (1) generating a more comprehensive scoring system that is both holistic (i.e., total ability) and dual perspective sensitive (i.e., Boolean vs issue identification), (2) removing the anchored structure from the assessment, and (3) collecting and analyzing a second set of student data (i.e., assessment and interviews) to take into account these refinements. Once this process is undertaken, this tool can be used alone or in conjunction with other assessments to answer questions relating to conviction, comprehension, and construction as outlined in the introduction of this study.

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Characterizing Student Engagement in a Post-Secondary Precalculus Class

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This paper reports on several aspects of a larger study aiming to characterize the engagement of students in a Precalculus class at a four-year public university. In line with policy suggestions that advocate for the development of flexibility in mathematics problem solving, we conducted a teaching experiment which utilized a set of "multiple solutions activities" to expose students to alternative solution methods while providing opportunities to examine and critique the reasoning of others. In addition, the instructor and teaching assistant of the course attempted to use these activities to explicitly negotiate productive norms and practices. Nevertheless, several detrimental norms emerged in the class, as the effectiveness of the activities was hindered by several propensity factors, particularly students' prior knowledge and self-regulation.

Keywords: Social Norms. Sociomathematical Norms. Developmental Mathematics. Precalculus.

Purpose

It has long been advocated that post-secondary developmental mathematics courses, such as Precalculus, would benefit from a greater focus on the development of students' argumentation skills, reasoning strategies, and flexible knowledge (Chiaravalloti, 2009; Partanen & Kaasila, 2014; Star & Rittle-Johnson, 2008). Such change of teaching focus could be more conducive of student learning than traditional Precalculus courses, which often tend to emphasize mastering algorithms and reviewing remedial content (Cox, 2015; Grubb, 2013; Mesa et al., 2011). This emphasis usually does not require students to change their mathematical practices and habits, which might have contributed to the need for remediation in the first place (Carlson et al., 2010; Goudas & Boylan, 2013).

Recent policy documents advocate for the importance of students developing *flexibility* in mathematics problem solving, which typically refers to the ability to generate, use, and evaluate multiple solution methods for given problems (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2006; Star et al., 2015). In addition to improving students' conceptual and procedural knowledge, developing flexibility often coincides with providing opportunities for students to practice reasoning skills (Star & Rittle-Johnson, 2009). Developing flexibility may require a form of engagement that students are not accustomed to, which may even contrast with their own preferences. Typically, students with insufficient mathematical skills and backgrounds tend to prefer a dependent learning style that is oriented towards mastering algorithms (Chiaravalloti, 2009; Partanen & Kaasila, 2014). Thus, to develop flexibility in developmental mathematics classes, it is important for instructors to negotiate norms and practices that encourage engagement that focuses on utilizing these skills.

This paper reports on a teaching experiment in a post-secondary Precalculus class at a four-year public university. The course instructor, who is the first author of this paper, utilized a sequence of activities that were designed for students to critique and compare multiple solution methods in small groups, thereby providing opportunities to develop flexible knowledge and reasoning skills. Throughout the course, and specifically when enacting these activities, the instructor and teaching assistant tried to explicitly initiate the negotiation of productive norms and mathematical practices to improve students' mathematical engagement. We analyzed the in-class engagement of two groups of students working on these activities to answer the following research question: what sociomathematical norms and social norms developed in these groups over the semester and what propensity factors mediated the emergence of these norms?

Framework

As our theoretical lens we adopt two frameworks: the interpretive framework by Yackel and Cobb (1996) and the opportunity-propensity framework by Byrnes and Miller (2007), which we detail below. We used the interpretive framework to identify emerging norms, and the opportunity-propensity framework to gauge possible reasons for the emergence of these norms.

The **emergent perspective** characterizes classroom activity from both social and psychological perspectives. From a social perspective, student and teacher interactions in a classroom are regulated by various types of *norms* (Cobb et al., 2001). For example, *social norms* characterize the classroom participation structure and *sociomathematical norms* are those specific to mathematical aspects of students' activity (Yackel & Cobb, 1996). For example, the requirement to justify one's mathematical work is a social norm, whereas what constitutes a mathematical justification is a sociomathematical norm. Students' participation in the negotiation of a norm coincides with each individual's reorganization of their own values and beliefs. Thus, norms and beliefs are said to co-develop, as each of the social constructs is reflexively related to a psychological construct, as shown in the interpretive framework (Figure 1).

Social Perspective		Psychological Perspective
Classroom Social Norms	⟷	Beliefs about one's own role, others' roles, and the general nature of mathematical activity in school
Sooiciomathematical Norms	⟷	Mathematical values and beliefs
Classroom/Communal mathematical practices	⟷	Mathematical reasoning and individual activity

Figure 1. Interpretive Framework (Yackel & Cobb, 1996)

The **opportunity-propensity framework** has previously been used as a theoretical basis for examining the development of flexibility and explaining variance in students' achievement in mathematics (Star et al., 2015; Byrnes & Miller, 2007). The framework outlines two types of factors that mediate students' engagement: opportunity factors and propensity factors. *Opportunity factors* refer to the culturally defined contexts where students are presented with content or skills to learn or practice within a particular domain. These can be formal or informal, and may vary in quantity and quality. Examples of opportunity factors include the types of instructional materials used and a teacher's questioning techniques.

Conversely, *propensity factors* refer to students' willingness or ability to learn content or skills after it has been presented or exposed to them. Byrnes and Miller (2007) report that most propensity factors fit into three categories: ability to take advantage of opportunities, willingness to do so, or factors regarding self-regulation. Self-regulation refers to the means of processing learning material, such as a group's ability to organize and utilize effective learning strategies.

The two frameworks complement each other to help characterize students' engagement. Specifically, the opportunity-propensity framework helps to understand the co-development of psychological and sociological constructs in the classroom, identified through the interpretive framework. Opportunity factors characterize the local social and mathematical contexts in which students engage and in which norms are implicitly or explicitly negotiated. For example, the development of the social norm of *trying to understand others' solutions* may be mediated by providing opportunities to engage in frequent, meaningful discussions. The propensity factors help to understand possible reasons for why certain contexts have been less conducive to student engagement or lack of students' uptake of particular learning opportunities.

Data Sources and Methods

The data were collected in a post-secondary Precalculus class with an instructor and teaching assistant who were Mathematics Education PhD candidates that had each previously taught the course. A majority of the nearly eighty students were retaking the course because they did not earn a sufficient grade the previous semester to meet the pre-requisite for Calculus 1. The class met three days a week for a fifty-minute lecture, during which, the instructor, aided by two learning assistants, engaged students in problem solving in small groups. Two days a week students meet in smaller sections of about 25 students for a lab period, where the instructor and/or the teaching assistant facilitated group-work around solving problems.

A sequence of four *multiple solutions activities* was designed to provide students with opportunities to develop flexible knowledge by evaluating and critiquing different mathematical arguments. These activities were anticipated to improve students' knowledge of strategies, which is critical to the development of flexibility (Star & Rittle-Johnson, 2008). The activities were designed for students to work on in groups of three or four during their lab period. These groups remained constant throughout the semester.

Each activity was composed of three phases. The first phase required students to solve a mathematics problem and to formulate a grading key for it. Second, students were to use their grading key to evaluate three fictitious students' sample solutions to that same problem. By design, the solutions contained intentional mathematical or notational errors and/or incorporated methods or strategies that students had not been previously exposed to in the course. Besides representing a valuable learning opportunity for students, this design would help us identify emerging norms. After evaluating the solutions, students responded to reflection questions that required them to compare and contrast the three solutions. For the last phase of the activity, the instructor and teaching assistant facilitated a class discussion to highlight key aspects of the solutions while also explicitly negotiating particular norms, such as: an acceptable mathematical solution may follow any mathematically valid approach and that solutions must contain explanations. Sustaining such norms in the course would provide opportunities to develop students' flexibility and reasoning skills. These activities served as an essential data source for the study. Six groups of four students were video-recorded using 360-degree cameras during these activities and their written work was collected.

Analysis

The video data was analyzed by classifying utterances and activity within the interpretive framework (Figure 1). Since norms are a social construct, we identified their emergence through interactions between members of the class. Errors and alternative methods included in the sample solutions were intended to represent violations of various conjectured sociomathematical norms, hence students' discussions around these errors and methods were pivotal to our analysis. By coding students' utterances and activity, we hoped to determine whether students deemed these errors and methods as legitimate or not, thus elucidating the development of sociomathematical norms in the groups. For example, both solutions in Figure 2 demonstrate incorrect and inconsistent notation, among other errors. Both groups were quick to condemn the incorrect notation of Tom's final answer, which supported the conjecture that students deemed it necessary for an answer to have the correct notation; this will be further discussed below.

Conjectures of social norms evolved similarly. Our analysis would start with proposed norms. During our coding of utterances and classroom activity, we would try to identify occurrences when students appeared to violate our hypothesized norms. Our proposed norm would garner support if the violation was delegitimized by others. Legitimization on the other hand would require us to revise our proposed norm. Thus, our proposed norms would evolve and gain more support with subsequent iterations of analysis and coding.

Because groups worked separately on the first two phases of the activities, the norms we discuss here are those that were constituted within the students' groups. After identifying norms, we used the opportunity-propensity framework to explain factors affecting their emergence. Consideration of both frameworks led to a more detailed understanding of students' engagement.

In this paper, we focus on two groups of four students each; Group 1: Wes, Ted, Herbert, and Cullen, and Group 2: Molly, Steve, Peter, and Chad (all names are pseudonyms). All eight students were taking the course for the second time. We primarily draw upon parts of our analysis of the first activity of the semester to demonstrate patterns of norms that developed, and were mostly sustained, within the groups over the semester.

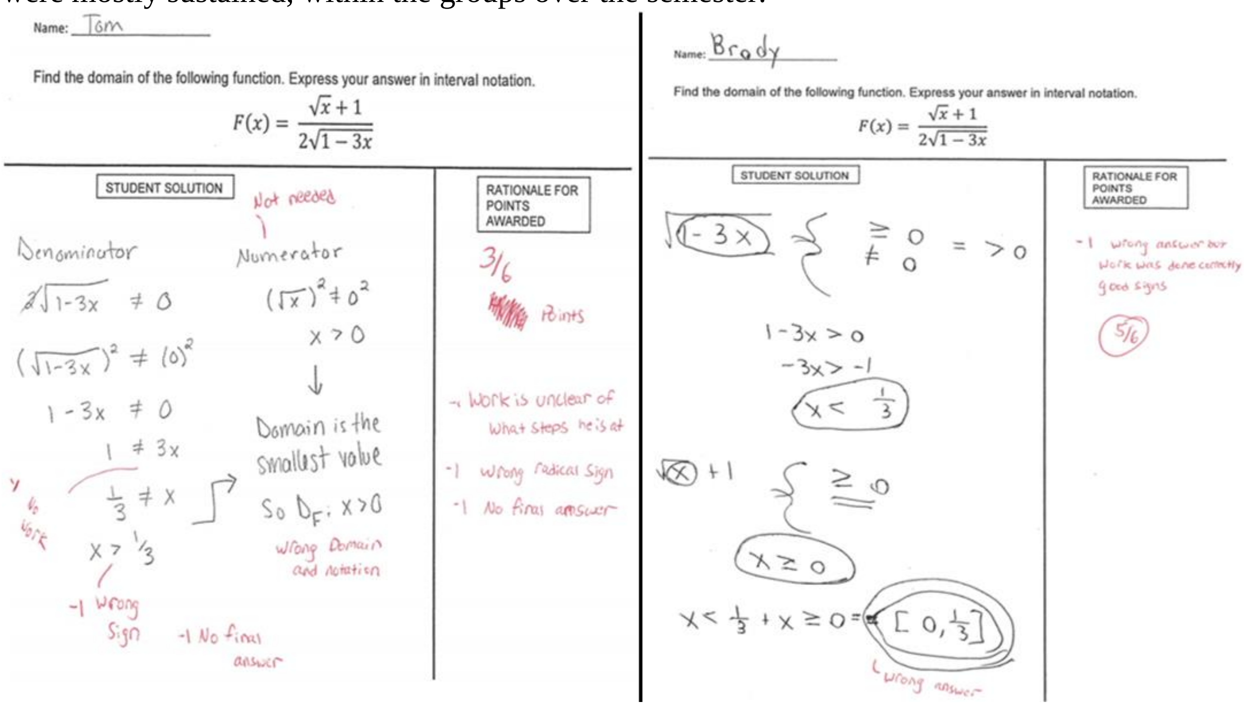

Figure 2. Two solutions from the first multiple solutions activity. The writing in red is that of Wes, from Group 1.

Results

The Role of Notation in What Constitutes an Acceptable Solution

Our analysis revealed that notation played a key role in characterizing the sociomathematical norm of what constitutes an acceptable mathematical solution. Students were keen to be consistent with the instructor and teaching assistant's explicit messages of what notation is improper and what type of notation is beneficial for clearly communicating one's mathematical work. During an activity on finding the inverse of a function f, students were quick to condemn solutions that did not label the inverse function as f^{-1} in the final answer, which was previously stressed by the instructor. But in situations where the instructor and teaching assistant did not previously comment, students' consideration of notation was limited.

The first activity of the semester asked students to find the domain of a function; two of the sample solutions from this activity are included in Figure 2. One of the notational errors in Tom's solution is his incorrect use of a "not-equal" sign, which he eventually incorrectly switched to a "greater-than" inequality. In Group 1, Ted inquired as to whether inequalities were needed for this problem. Wes delegitimized this concern with a shrug, Cullen expressed that he used the same notation as Tom did in his solution, and Herbert suggested that Ted was technically right but that "you don't need to make a big deal about it." The reaction in Group 2 was very similar, as Steve pointed out that, "[Tom] did the whole way it can't equal until the end where he just rewrote it [with the greater than sign]." Molly suggested to Steve that the notation inaccuracies did not matter, as long as the notation was correct at the end, which Steve agreed to.

The notation in Brody's solution did not adhere to standard notation established in class, and included improperly used symbols for informal arguments without clear description (Figure 2). Yet neither of the two groups discussed the poor notation; the only critique of Brody's solution was his incorrect answer. In fact, both groups complimented the work. Group 1 went as far to express that the only thing wrong with the solution was the answer.

The examples above help illustrate one component of the sociomathematical norm of what constitutes an acceptable solution that developed within the groups: *an acceptable solution only needs correct notation in the final answer*. This view was mediated by the propensity factor of deficient prior knowledge. Many of the students had an underdeveloped understanding of mathematical notation. For example, Steve was unable to recognize the interval notation in Brody's solution despite the fact that he had recently taken a quiz on interval notation.

These outcomes illustrate that the instructor's expectations that students will learn proper notation through implicit negotiations that are constituted primarily by modeling in class were overly optimistic. In this case, these opportunities provided to students to learn proper mathematical notation were clearly insufficient. Although the instructor and the teaching assistant were more explicit about the use of notation during the whole class discussions, their efforts were often ineffective for other reasons, which will be discussed later.

Unnecessary to Investigate Alternative Approaches

One social norm that emerged in both groups, and was especially apparent in Group 1, was that it was *unnecessary to interpret alternative approaches*. For example, in the first phase of the activity, when finding the domain of the function themselves, both groups' original answers differed from those of Tom and Brody. The groups did not consider the restriction imposed by the square root in the numerator, but Tom and Brody's solutions did.

As Group 1 attended to the sample solutions, they noticed that Tom and Brody considered the root in the numerator to determine the domain. At first, all group members expressed that it was not needed and some would frequently repeat, "Domain is in the denominator." But eventually the group expressed uncertainty to the role of the numerator. Despite this confusion and doubt, no one in the group attempted to interpret the numerator's role in the other solutions. In general, when the two groups did not understand an alternative approach or strategy, their reaction was often to dismiss, denounce, give up, or some combination of the three.

It was apparent that two propensity factors helped mediate this social norm. The first is students' prior knowledge; their underdeveloped understanding of the concept of domain hindered their ability to productively work on the activity. For much of the semester, it was clear that the students in Group 1 focused on replicating algorithms, of which they did not have a deep understanding, and as result, they implemented the algorithms partially and inconsistently. This preference did not help students effectively develop their understanding of key mathematical concepts, hindering their work on multiple solutions activities. Secondly, the self-regulation within the group was limited. During this activity, the group utilized strategies to confirm the domain restriction caused by the denominator, such as evaluating the denominator with particular inputs. These same strategies could have been effective in understanding the role of the numerator in finding the domain of the function. But despite being familiar with these strategies, the students were unable to determine how to utilize them.

In general, students in both groups were hesitant to learn about multiple approaches and analyze the solutions of others, and expressed their preference for mastering one method instead, as they had done in their prior courses. The students described this difference in a conversation with the instructor:

Steve: That was also like last semester they constantly drilled that in our head that there was only one way to do it.

Molly: Yeah. So that's why I feel like a lot of us, or at least personally why I'm struggling.

Steve: It's a lot different.

Molly: I don't have a set rule to like follow.

Attentiveness in Whole Class Discussions

The class discussion phase represented an opportunity for the instructor and teaching assistant to use the activity as a medium to explicitly negotiate norms that aid the development of flexibility and improve engagement. In relation to the sociomathematical norm discussed earlier, the instructor and teaching assistant stressed that one purpose for writing solutions and using proper notation was to communicate one’s understanding to others; they emphasized the need for using conventional notation lest one’s work would be interpreted differently than intended.

Yet our analysis showed that during this crucial time of the whole class discussion, the students would often disengage, and ignore the instructor and teaching assistant. Students instead would often look out the windows, draw on their papers, or use their phones. This is indicative of a social norm that being attentive during class discussions is not important.

Here, propensity factors relating to students’ unwillingness to take advantage of opportunities to engage in the discussions hindered the development of productive classroom norms. In particular, lack of interest and motivation may have been predominant factors: students were eager to pack up and leave since the discussions occurred near the end of class. The lack of motivation might have been supported by students’ recognition that their participation in the discussions was not graded or the perception that these discussions did not align with traditional assignments and assessments that they expected to receive. Due to this low engagement, unproductive norms were mostly sustained throughout the semester or their change was limited.

Discussion

Our study aimed to characterize students’ engagement in a Precalculus class by examining what social and sociomathematical norms developed among groups of students over the semester, and what propensity factors mediated the emergence of these norms. In particular, we focused on students’ engagement with multiple solutions activities, which aimed to develop their flexible knowledge. The data presented in this paper aims to illustrate some of the norms that emerged in our data, however space constraints do not allow for the inclusion of all of the identified norms.

Our data show that despite the instructors’ attempts to negotiate productive norms while engaging students in mathematical activities that foster flexible solution strategies, the students developed social and sociomathematical norms within their groups that were often detrimental to their own learning. For example, our study revealed several characterizations of the sociomathematical norm of what constitutes an acceptable mathematical solution that developed within these groups, such as (a) it has to follow the algorithm shown in class (Fifty, Buchbinder, & McCrone, 2019), (b) it results in a correct answer, even if the steps are flawed (Fifty, Buchbinder, & McCrone, 2020), and (c) following conventional notation is only important in the final answer, as illustrated above.

Some of the social norms that our study identified were the lack of attentiveness during class discussion and resistance to interpret alternative solutions. Our data show that these social norms acted as barriers to the instructor and teaching assistant's efforts to promote flexible knowledge. For example, because the students were not attentive during the class discussions it should come as no surprise that the groups did not benefit from the instructor and teaching assistant's negotiations. Furthermore, this lack of attentiveness reduces the instructor and teaching assistant’s influence on the development of norms within the groups. These findings represent an instance where social norms may constrain the development of sociomathematical norms.

We attribute some of the students’ reluctance to learn about multiple approaches for solving particular mathematical problems to the effects of detrimental beliefs about the nature of mathematical activity the students came to expect in a Precalulus course. These beliefs could be a consequence of unproductive norms developed in their prior mathematics courses, especially in the previous Precalulus course, as suggested by Steve and Molly’s conversation detailed above. As described by the emergent perspective, norms and beliefs co-develop (Cobb et al., 2001). Even though norms recede with the conclusion of a course, students' beliefs, affected by the development of these norms, may persist and can act as barriers to the negotiation of other norms in subsequent classes. Consequently, our data suggests that instituting productive norms, particularly in developmental courses, can be a gradual process.

To better understand the emergence of these detrimental social and sociomathematical norms, we considered possible opportunity and propensity factors that might have influenced students’ engagement in the multiple solutions activities (Byrnes & Miller, 2007). Some of specific propensity factors that we explored were the roles of students’ prior knowledge, motivation, and utilization of effective learning strategies. Better understanding these propensity factors can help educators reconfigure opportunities to improve students' engagement. For example, given that students seemed unmotivated to engage and participate in the class discussions, one possible way to inspire change would be to more explicitly connect these class discussions to assessment in the course. This could change the social norms that develop in the class, and consequently lead to improved development of students' flexible knowledge.

Despite the wide recognition of the importance of flexible knowledge, exploration of how teachers may provide instruction to facilitate its development in authentic environments remains largely unexplored (Star et al., 2015). Our design and use of multiple solutions activities help fill this gap in the research. The first two phases of the activities presented opportunities for students to discern different methods, while the reflection questions and the class discussion intended to explicate comparisons and clarify nuanced differences between the solutions. This focus on flexibility did not hinder the progression through the curriculum, as each activity was situated within the relevant mathematical context. Additionally, the activities provided opportunities to emphasize general mathematical principles, such as the value of adhering to conventional notation.

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Chavrusa-Style Learning in Mathematics Classrooms: Instructor and Student Perspectives

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MathChavrusa is a novel application of Talmudic study techniques into the context of mathematics education, emphasizing long-term, partnered text study and problem solving. This study implemented MathChavrusa in semester long graduate mathematics courses, collecting data from both instructor and student perspectives on the effectiveness of MathChavrusa in facilitating student understanding of course material and engagement in communicating mathematics. Results indicated that students largely had a positive impression of MathChavrusa, highlighting its impact on student engagement, creating a conducive environment for asking peer-peer and peer-instructor questions, and was a factor in better understanding course materials. This study model has the potential to enhance student engagement in mathematics classrooms and could be adopted more broadly.

Keywords: Peer Learning in Mathematics, Math Chavrusa, Student Centered Learning

Introduction

There is an increasing body of research suggesting that peer learning is beneficial for students in all phases of academic progress (Snyder et al., 2016; Sacerdote, 2011; Tsuei, 2012; Robinson et al., 2005). Though a relatively recent area of interest in educational research, peer learning has been a tradition in Jewish culture, utilized since the first or second century CE in chavrusa Talmudic instruction. Chavrusa, meaning "friendship" or "companionship" in Hebrew, refers to the practice of paired or small group study of the Talmud typically focused around reading, analyzing, and discussing passages in the text. Importantly, the chavrusa method consciously recognizes and embraces the social aspect of learning and its ability to construct new meaning from the materials via debating each individual's perspective (Pace, 1992; Kent, 2010).

MathChavrusa is the application of the Talmudic chavrusa model to the study of mathematics. As practiced in traditional chavrusa learning, MathChavrusa divides the class into one on one peer groups that persist throughout the semester and are either assigned by the instructor or chosen by students. This model is distinct from other peer learning models in a number of respects: (a) limiting the peer size to two creates an atmosphere of intimacy which reduces social anxiety, making it more likely that students will have the courage to ask questions without fear of judgment, (b) the active role each student adopts in a direct dialogue limits external distraction, (c) the discussion model engages students across a variety of levels of academic performance, where advanced students can adopt the role of teacher. Throughout the course, groups are tasked with supporting each other in understanding the material the instructor presents in class. This includes group reading, problem solving, content projects, and homework sets aimed to provide students with multiple methods of interacting with the course material. Emphasis in class is placed on group readings and problem solving, as they are intended to supplement lectures and provide additional background on the subject matter. These activities typically last at least twenty minutes per class to provide groups with ample time to interpret and discuss the implications and connections that supplementary materials bring to bear on the lecture topic. Projects and homework sets remain the students' responsibility outside the classroom: the former are an outlet for students to explore topics beyond the scope of the course, and the latter provide reinforcement of course topics between meetings. Projects and homework sets are then shared with the class as a whole so that all MathChavrusa groups can learn from each other's research.

In this study, we aimed to demonstrate the effectiveness and feasibility of the MathChavrusa model in increasing student engagement, mastery of content, and sense of community within the classroom. Mixed methods of quantitative and qualitative data from three semester-long graduate mathematics courses were collected to assess students' beliefs regarding MathChavrusa's effect on their course experience as a whole, including its influence on each student's course material mastery, peer engagement, and personal involvement. Instructor perspectives came from direct observation, verbal feedback from MathChavrusa pairs during discussion with the instructor immediately after each class session, and reflective essays written by each student at the culmination of each course. Student perspectives came from the student author's observations, from interviews conducted by the student author with a random selection of individual students from each course in which MathChavrusa was implemented, and from formal Likert scale survey results. An instructor-student collaboration was chosen for the format of this paper to overcome instructor bias and to increase the generalizability of these findings to the academic pedagogical system as a whole.

Literature Review

Peer learning is an established practice in mathematics education and comes in myriad forms to suit specific needs and purposes. For instance, groupwork has been used for both short-term and long-term classroom learning arrangements. As an example, a short-term arrangement may take the form of a think-pair-share. Generally, a wide range of activities that contribute towards classroom culture, teaching, review, and assessment can also be done in a brief peer learning context (Udvari-Solner & Kluth, 2018). Such short-term activities are promising, as students have been observed to have an increased understanding of course material and improved interpersonal attitudes after participating in group work (Springer et al., 1999).

Long term forms of peer learning that run over the course of a semester or more have also been implemented, and these are typically characterized by more structured task requirements and grouping criteria (Lindboe, 1998; Quitadamo et al., 2009). Various factors can be controlled in these lengthier groupings, including student ability matching, group objectives, and individual responsibilities. In this context, there are contrasting results as to the usefulness of groupwork in improving mathematical achievement. Meta-analysis of studies performed by Springer et al. (1999) indicated that there was little support that long term groupwork improved mathematical achievement. On the other hand, the same study found that longer grouping periods seemed to improve students' attitudes towards each other and to approaching course material, suggesting that there is still value in such arrangements. Another study by Sofroniou and Poutos (2016) found that by integrating problem-based learning and mixed ability grouping, previously lower achieving students had reduced levels of stress when approaching mathematical problems. Overall, the participants of the study also showed improvements in their content area understanding, measured by the increase in final exam scores in the group participating in peer learning. Lindboe (1998) found similar results when working with lower achieving students. Specifically, when these students worked as peer instructors with each other in undergraduate mathematics courses, their ability to share pedagogical and learning strategies as well as provide social and emotional support for each other improved mathematical achievement. However, despite this body of research demonstrating benefits of group work in higher education, there are situations in which students may be uncomfortable working closely with their peers. In a study looking into attitudes towards groupwork across business administrations, educational settings, and mathematics students, business administration students viewed groupwork more negatively than the comparison groups (Gottschall and Garcia-Bayonas, 2008). The results suggest that though peer learning is shown as beneficial and worth exploring, thoughtfulness should be exercised to ensure that these initiatives resonate with the particular student body.

A subset of the aforementioned long-term peer learning is the Chavrusa method. Research indicates that chavrusa learning is currently seeing increased and expanded use outside its traditional Talmudic setting. For instance, South Korea has seen chavrusa-styled private academies, *hagwons*, emerge in the past decade, supplementing more traditional, mechanical cram schools (Choi, 2017). The model is influencing public education, with the Korean Federation of Teachers' Associations coming to an agreement with the Havruta Cultural Association to train teachers in utilizing chavrusa methods in the classroom (JTA, 2019). Medical education has drawn certain ideas from traditional Talmudic learning, incorporating student-centered, self-directed learning in small group settings. In these models, the role of teachers becomes integrated with group learning as well: rather than solely as experts in their fields, teachers become role models and mentors within a socially interactive framework of group discussion (Notzer et. al., 1998). Similarly, studies within the physical therapy discipline have also confirmed the effectiveness of structured group learning, noting statistically significant improvements between pre and post chavrusa style group-driven academic and motivational interventions (Chung & Lee, 2019).

Methodology

Data was collected from 44 students in three graduate mathematics content courses in the form of reflections, surveys, and post-course interviews. The reflections were collected by the instructor, while the post-course interviews were conducted by the student collaborator. Reflections consisted of written feedback on MathChavrusa and the course as a whole. Likert scale surveys consisted of statements and questions about MathChavrusa specifically. These were collected anonymously, and students were asked to respond to each statement with strongly disagree, disagree, neutral, agree, and strongly agree, with the numbers 1, 2, 3, 4, and 5 representing each of these responses respectively. Respondents were also given the opportunity to write comments about MathChavrusa overall, which were also recorded on the survey handout. Interviews were conducted several weeks after the course completion and were structured around questions designed to compare and contrast with the survey results as well as to identify student background and post course content mastery.

Instructor Perspective on MathChavrusa

By implementing MathChavrusa, the instructor's initial goals were four-fold: to deepen students' understanding of course materials, to strengthen students' communication skills in mathematics, to build a community of relationships and collaboration centered on coursework, and to motivate students to initiate self-driven study from the course "textbook"/readings.

Many of these goals materialized as the course progressed. Students became more adept at reading mathematical texts without direct instruction. Relationships formed between students that expanded beyond the discipline and took on an inherently social and interactive quality. There was a high level of student interaction between groups as well. Students became more proactive and courageous in voicing their questions and deepening clarity on the subject matter.

To evaluate the quality of learning more directly, MathChavrusa pairs were asked to select and teach homework problem sets to the class, and to engage in post-class discussions with the instructor. Within the classroom, student engagement was also assessed by observing how often students turned to their partner or to another group for clarification or instruction. Student engagement was also indirectly assessed by the frequency and the complexity of questions posed by chavrusa partners during office hours. According to the instructor's observations, the more time added to MathChavrusa, the more students actively engaged with the course materials. Students gave attention to their mobile phones less often, were communicating mathematics more explicitly, and were generally more focused on the task at hand. Throughout the courses, students consistently requested more MathChavrusa time. From the onset the instructor observed moments of student excitement while engaging in MathChavrusa reading materials and expressing they "got it". For example, "I can finally see the process of continuous deformation and how the coffee cup is homeomorphic to a doughnut". Students frequently drew on diverse perspectives and were delighted to share how they approached the same problems differently.

As courses moved along, the MathChavrusa structure took on an evolving dynamic. Most of the pairings were of students at mutually similar academic levels. This allowed students to discover insights and work symbiotically. During review sections, and at the students' request, pairs were changed to have "stronger" and "weaker" academic partners. This alternative model worked well for review and summative learning processes. Another organic evolvement was that chavrusa partners filled in the gaps for their own partners, recording lectures, taking notes, and providing verbal reviews for the other in the event of absence. These dynamic changes reflect the importance of consistency, alongside openness to integrating student feedback and needs as the unique structure of each class's pairings present themselves.

An additional benefit that played out as a byproduct of the MathChavrusa model was the drastic transformation of students who entered the course with performance anxiety. While initially fearful and hesitant to present, the format allowed these same students to flourish and overcome their initially hesitancy. By the end of each course, not a single student was held back by fear of presenting and each developed a confident and engaged presentation style.

Many students expressed that struggling through problem sets became more enjoyable and productive, noting that, "having someone to work with really enriched my understanding of the material, and lightened my frustration with solving problems because we struggled together and pieced our ideas together. This was much different from my undergrad where I was isolated, and I had no one to turn to when I struggled." This sentiment was indicative of a theme throughout other students' reflections.

A key performance indicator is test scores. Test score performance was higher in all three iterations of MathChavrusa as compared to courses delivering the same content without MathChavrusa. Only the classes that incorporated MathChavrusa had 100% passing rates, which is unusual for a high-level mathematics content course.

One of the struggles in the MathChavrusa structure is that some pairs seemed to develop a stronger bond than others, which led to different levels of engagement. While all pairs engaged in the course tasks, there was a strong correlation with pair likability and enthusiastic engagement. This did not necessarily translate to varied test scores but did affect perceived students' positive attitudes in the classroom.

Instructor observations are essentially limited by potential subjectivity and bias. To complement this perspective, an instructor-student collaboration was formed to provide increased objectivity and overcome instructor blind spots.

Student Perspective on MathChavrusa

MathChavrusa shares many similarities with models of groupwork prevalent in education today, but also has critical differences that make the experience for the students distinct from schema such as think-pair-share. What follows is a summary of post-course feedback received from students who have experienced MathChavrusa. This feedback was collected in two forms: ten interviews conducted by the student author, as well as analysis from post-course Likert scale surveys completed by students. Four of the interviewees experienced MathChavrusa in two courses. A summary of the statistics from the surveys is shown below in Table 1.

Table 1. MathChavrusa feedback survey summary

Survey Results		
Questions	μ Mean	σ Standard Deviation
1) The MathChavrusa format improved my ability to learn the content for this course.	4.16	0.82
2) The format of MathChavrusa increased my ability to stay engaged.	4.19	0.82
3) The format of MathChavrusa helped me feel more prepared to speak in class.	3.68	1.04
4) The format of MathChavrusa stimulated my interest in reading about course content outside of class.	3.69	0.76
5) The format of MathChavrusa helped me understand key ideas in the course.	4.03	0.81
6) The format of MathChavrusa helped me feel comfortable approaching the instructor with questions or comments.	3.86	1.02
7) I learned more when I worked with my MathChavrusa.	4.19	0.95
8) My MathChavrusa and I worked well together.	4.51	0.91
9) I found it helpful to get feedback from my MathChavrusa on my own performance.	4.24	0.90
10) I needed more instructor guidance during MathChavrusa time.	2.97	1.27
11) My MathChavrusa and I helped each other out during the course.	4.35	0.93
12) I collaborated with my MathChavrusa outside of school.	3.97	1.41
13) The MathChavrusa format contributed to my success.	4.00	1.11
*1, 2, 3, 4, and 5 correspond to strongly disagree, disagree, neutral, agree, and strongly agree		

Benefits of MathChavrusa

Students across all three courses seemed to have positive impressions of MathChavrusa, with all survey questions except one scoring above neutral on the Likert scale. Survey results indicated that students attributed working with their MathChavrusa as a factor in better understanding course materials, with $\mu = 4.16$ and 4.03 for questions 1 and 5 in the survey, respectively. Additionally, student engagement was positively affected by MathChavrusa, with $\mu = 4.19$ for question 2. On an interpersonal level, student responses were even more positive. Question 8, 9, and 11 inquired about peer to peer interactions, with responses of 4.51, 4.24, and 4.35 respectively, indicating a high level of comfort and camaraderie in each MathChavrusa.

These observations were supported and supplemented by students who participated in the interviews. Students stated that MathChavrusa gave them an outlet to easily ask questions to their partner when in need of guidance, and most of the time this allowed them to understand formerly confusing concepts. Beyond clearing up information, MathChavrusa also led to pairs developing organic support systems and specializations, with interviews highlighting various role divisions unique to each MathChavrusa pair. In addition, the act of asking questions within each MathChavrusa led to increased interactions with the class as a whole, with interviewees stating that they often asked MathChavrusa groups around them for advice when internal problem solving was insufficient. This formed bonds between classmates and contributed to newer students finding long-term acquaintances for future classes. In terms of class culture, students remarked that as a whole, the emphasis on peer learning and communication "broke the walls in the classroom," allowing for a comfortable setting to interact with classmates and ask questions freely.

The interviews also produced responses not covered by the survey. An important element of MathChavrusa is joint text study, which some students felt was helpful. Specifically, they appreciated the time given to work on problems directly related to materials presented by the instructor in class, stating that such practice helped make connections and rectify misconceptions in the material. Students also mentioned that they liked the MathChavrusa format as it was a break from traditional lecture-based learning and provided variety within class sessions. Similarly, students appreciated time prescribed for interacting with other MathChavrusa groups, as these interactions resulted in exchanges in knowledge and perspectives that proved valuable in developing each person's depth of understanding.

Considerations for Implementing MathChavrusa

Though generally positive towards MathChavrusa, student interviews illuminated a few important implementation details that need to be carefully considered when utilizing MathChavrusa. First among these is the effectiveness of collaborative reading. While the goal of MathChavrusa is for students to learn new material via textbook reading, students found it challenging and at times confusing. Students from two of the three classes indicated that some readings emphasized concepts and material beyond the scope of the course lectures. Though interesting, these types of texts were found to be confusing as they didn't address fundamental misunderstandings students had about in-class material and assumed competency in recently learned material. Students believed that readings more focused on content directly related to what was being introduced in class would have better improved their grasp of the course material, as well as enable them to then understand the enrichment materials. Related to this, students felt that the collaborative problem-solving portion of MathChavrusa was very useful for improving their understanding, as it gave direct, concrete examples of how material could be presented, as well as opportunities for step-by-step walkthroughs to clarify the logic behind how each step was achieved.

Another salient point students mentioned was the nature of MathChavrusa pairings. Students were given the choice of picking their partner or being randomly assigned, and with class composition and preference for choosing partners, this often resulted in students of similar mathematical backgrounds to be placed together long term. However, students found at times that it was difficult to make sense of concepts when both partners were unable to see the reasoning behind concepts and problems, especially in content dense or unfamiliar areas of the courses. At these points, those interviewed would often attempt to turn to either the instructor or surrounding groups for help, though the latter yielded mixed results due to other groups also being confused and the former only being able to help a limited number of groups during MathChavrusa time. As such, students suggested mixed ability pairings for part of the class, to have at least one person in the group be able to guide the MathChavrusa through difficult content. It should be noted however that students in general enjoyed the freedom they had in selecting their partner, and suggestions for mixed ability pairing only wanted them temporarily to allow for clarification of unclear ideas. Also, though most pairings seemed to be amicable with each other, there were instances of students not getting along with their MathChavrusa partner, which could impact the long term learning outcomes of those students.

Conclusion

MathChavrusa has shown promise as a structure for mathematical learning in postsecondary classrooms. Its emphasis on collaborative text study, problem solving, and long-term peer support enables students to improve their ability to learn mathematics and increases engagement. Moreover, it enables students to discover and implement appropriate strategies necessary to succeed. Students have generally indicated that working closely over a long period of time with their partner developed a positive classroom culture where they felt more comfortable engaging with the instructor and their peers. This in turn led to the development of organic support networks, where students would collaborate on problem sets and in-class materials frequently. Overall, students were positively affected by MathChavrusa and indicated that they learned more. However, feedback also suggests that the implementation can be improved, with important considerations being the interactions between MathChavrusa partners, the significance of readings towards course goals, and ensuring student ability grouping is appropriate for students to grasp course materials.

Future Research

This study provides a foundation analysis of the MathChavrusa model, and suggests the need for further research utilizing objective, performance-based measures of academic performance. Limitations of this study include focusing on the observation of one instructor and one student subject to personal biases. Additionally, further research should isolate the essential components of the MathChavrusa model (social learning, accessibility of instruction, novelty and variety of new academic activities, increased frequency of check-ins with peers and the instructor) for scaling and implementation purposes. It would also be interesting to explore whether student enthusiasm and adaptability vary based on undergraduate versus graduate mathematics courses, as well as the adaptations MathChavrusa needs to make when applied to courses of varying course content areas and academic difficulty.

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What Comes to Mind? A Case Study of Concept Images In Topology

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Prior research has shown that students tend to reason in terms of the informal components of their concept images rather than rely on formal concept definitions when working with topics in mathematics. However, students' concept images may be incomplete or inaccurate, or students' evoked concept images may exclude important features of concepts or classes of examples. This report describes two students' evoked concept images of concepts in general topology as revealed through their completion of proof tasks as well as some of the effects those images had on their success in proving. We also discuss the impact of the students' exploration of their example spaces on their concept images and on their proof writing.

Keywords: topology, proof, concept image, examples

Introduction

Students' concept images can have a substantial impact on their abilities to solve problems in mathematics (Moore, 1994; Rösken & Rolka, 2007; Tall & Vinner, 1981; Vinner, 1983). *Concept image* refers to "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152), while the term *concept definition* describes "a form of words used to specify that concept" (ibid., p. 152), with the formal concept definition being the one presented to students in a mathematics course. As teachers, we may hope that presenting students with a formal concept definition is sufficient for them to solve problems and that students will develop an appropriate concept image to represent the concept. This is, however, not generally the case – in fact, students typically do not consult with the concept definition at all, instead relying entirely on informal aspects of their concept images when responding to a given prompt (Vinner, 1983). Students may retain their concept image of a topic while being unable to produce a definition (Moore, 1994). These factors make the concept image of critical importance for students when solving problems or producing proofs.

A student's consideration of a topic begins with their evoked concept image. Tall and Vinner (1981) defined the *evoked concept image* to be "the portion of the concept image which is activated at a particular time" (p. 152). The totality of an individual's concept image may be sufficient to solve a problem, but the limitation of what is evoked at a given time may act as a barrier to the recognition of key ideas (Raman, 2003). For instance, a student's evoked concept image may be restricted to prototypical examples (Presmeg, 1992) held by the student which, though accurate, may not represent the full breadth of possibilities. This may prove to be especially problematic for the generation of counterexamples, which may require the recognition of highly esoteric cases. Such problems might be mitigated by expansion of the student's example space, i.e., the full collection of mathematical objects the student perceives as examples (Mason & Watson, 2008; Watson & Mason, 2005). Measures to optimize the accessibility (Goldenberg & Mason, 2008) and the generativity (Sinclair, Watson, Zazkis, & Mason, 2011) of the student's example space may improve students' recognition of these cases. Furthermore, a learner's concept image may contain conflicting elements, and the learner may not be aware of this potential conflict until forced to directly grapple with the inconsistency in their concept image (Davis & Vinner, 1986). The inspection of examples may also help students face and overcome these cognitive conflicts.

Our study investigated students' concept images and their effects on students' proof writing in general topology. Studies in topology have explored undergraduates' transitions into axiomatic reasoning (Cheshire, 2017) as well as students' development of schema for reasoning about a topology generated by a basis (Berger & Stewart, 2018, 2019); the latter of these demonstrated that students' schemas for topological concepts may develop very slowly in spite of direct instruction on such topics. Our previous work explored the role that visual representations played in the proof writing process of Stacey, an undergraduate taking a first course in general topology and how those representations helped her to identify the key idea of a proof (Gallagher & Engelke Infante, 2019). The current study addresses the question "How are undergraduates' concept images of topics in general topology structured, and how do their concept images affect their proof writing?"

Limitations in students' concept images can lead to great difficulty in proof construction and in general understanding. A first step toward supporting students' construction of complete, appropriate, and accessible concept images is to gain some insight into students' existing cognitive structures and the nature and limitations thereof. In this paper, we discuss components of the concept images of Tom and Stacey, two undergraduates taking a first course in general topology. We also discuss instances in which limitations or inaccuracies in these students' concept images hindered their understanding or their ability to solve problems, and we describe our informal efforts made toward restructuring their concept images to facilitate knowledge building.

Theoretical Perspective

Moore's (1994) concept-understanding scheme describes the interplay among three features of a student's knowledge about a concept: concept image, concept definition, and concept usage. The last of these "refers to the ways one operates with the concept in generating or using examples or in doing proofs" (ibid., p. 252). According to Moore's scheme, students may have a conceptual understanding of the topics involved in a particular conjecture, but they may still struggle to generate a proof due to a lack of experience or fluency in working with those ideas. Students may not be familiar with notation or language conventions regarding a given concept, or they may not know which of multiple notations might be the most efficient choice for a given application. There are also standard formulations which are used in proofs of certain concepts. For example, in proofs of ε-δ continuity in $\mathbb{R}^2$, the standard procedure is typically to fix an arbitrary $\varepsilon > 0$ and then use algebra to determine the corresponding δ. This formulation follows from the definition of continuity, but a student's ability to repeat this definition does not guarantee the student's knowledge of this proof scheme.

We will also utilize the notions of accessibility (Goldenberg & Mason, 2008) and generativity (Sinclair et al., 2011) of students' example spaces. Students may possess knowledge of examples, yet these examples may not come to mind readily for students in many situations. For instance, when prompted to generate examples of functions, many students may volunteer a number of commonly used functions such as $f(x) = x^2$ or $g(x) = e^x$; it may require quite a bit of probing to get students to consider more esoteric functions such as piecewise-defined functions or functions defined on discrete domains. The students may be aware of these examples, but such examples may be less immediately accessible due to infrequency of use, a perceived lack of importance, or other factors. The generativity of a student's example space refers to that student's ability to create new examples. This ability may depend upon students' abilities to recognize the dimensions of possible variation and range of permissible change within a class of examples (Mason & Watson, 2008); Marton and Tsui (2004) have suggested that students' abilities to generate new examples may not depend only on their exposure to various examples but on their experience identifying dimensions of possible variation.

Methods

Student volunteers were recruited from an introductory undergraduate course in general topology to attend weekly, one-hour Group Study Sessions. The first author acted as Facilitator during all sessions. During each session, volunteers were given either a statement to prove, a statement to disprove, or both if time permitted. Our intent was to create an environment that resembled an authentic studying environment as closely as possible. To that end, volunteers were encouraged to collaborate on all tasks, and they were permitted to use textbooks and class notes as desired. Volunteers were encouraged to speak aloud as they worked, and the Facilitator sometimes asked probing questions to gain more information about the volunteers' comments. The Facilitator sometimes intervened when it seemed that the volunteers would make no further progress on their own. All sessions were video recorded, and videos were transcribed.

Neither of the authors taught the topology course from which the student volunteers were recruited, but the first author attended all class meetings (except for exam days) to make note of topics covered recently by the instructor. Weekly proof tasks were chosen to align with the content recently covered in class. As compensation for the volunteers' participation, the first author offered additional office hours to help them with topology.

Data analysis is ongoing. Our data were initially coded to identify the production and modification of drawings, construction or discussion of examples, and any verbal utterances that might give insight into our students' concept images. Thematic analysis (Braun & Clark, 2006) was applied to these codes to identify prominent elements of Tom's and Stacey's concept images. In this report we discuss the emergent themes of *discrete sets versus continuous sets, connected sets versus disconnected sets,* and *open sets versus closed sets*.

Results

Discrete Sets Versus Continuous Sets

Throughout the course of this study, examples of sets and set imagery were predominantly continuous rather than discrete. For example, in Session 1, Stacey and Tom struggled to make sense of what the notation $f^{-1}(\bigcap_{i\in I} U_i)$ might represent. Recognizing that this notation represented a set, Stacey wrote the set $(2,3]$ on the chalkboard and asked if the "inverse" of this set would "be all of the other elements other than this?" confusing the notation for preimage with the concept of set complement. Stacey, working alone in Session 4, explored the set $[0,1)$ as an example of a set which might be both open and closed.

Session 5 was the first time a discrete set arose in the context of a problem involving topological properties (as opposed to purely set theoretical notions), and this was merely incidental to the students' choice to examine a continuous set. The students' task in Session 5 was to disprove that the boundary of an arbitrary set A must be both open and closed. Stacey quickly surmised (and proved) that the boundary of A must be closed. To disprove that the boundary of A must be open, Tom and Stacey chose to examine the boundary of the set $A = (0,1)$. Using a number line, they identified the boundary as consisting of the points 0 and 1, but when asked to use notation to write the boundary of A, they appeared to struggle. Stacey initially wrote the boundary as $\{0\},\{1\}$. Seemingly unsure, Tom asked if 0 and 1 should be in the same set; Stacey changed her inscription to read $\{0,1\}$, but she continued to stare at it for several seconds in silence as if considering whether this was an improvement over her original pair of singletons.

This led to an unexpectedly frank conversation about the students' preferences for continuous sets over discrete ones. Referring to the set $\{0,1\}$, Tom admitted, "It's difficult to think that a point is a closed set... it's just weird because it doesn't seem like it's a set at all," and shortly after, Stacey added that "It's harder to work with, I feel like, sometimes… like if you're putting a neighborhood in it, it's easier to think you have all these infinite points." We find this result somewhat surprising, as the concept of sets is often introduced using discrete sets rather than intervals and open regions, yet the students in our study seem uncomfortable reasoning about discrete sets. However, the students' topology instructor introduced the course with a thorough discussion of metric spaces, so it is not entirely unreasonable that the students' concept images should be dominated by intervals and open balls.

Connected Sets Versus Disconnected Sets

Although Stacey's and Tom's concept images of sets seemed to include images of disconnected sets, evidence of their evoked concept images suggests that connected sets featured prominently in their minds. In one such instance, Stacey's attachment to connected sets initially prevented her from discovering a counterexample, but she was able to circumvent this difficulty by exploring her example space.

Stacey worked alone in Session 6 on the topic of product spaces. Product spaces were defined in class by means of a basis for the topology; to explore and develop Stacey's understanding of this definition, we asked her to disprove the following statement:

> Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \dots, (X_n, \mathcal{T}_n)$ be topological spaces. Then the set $\prod_{i=1}^{n} X_i$, together with the collection $\mathcal{T}$ of subsets of $\prod_{i=1}^{n} X_i$ of the form $O_1 \times O_2 \times \dots \times O_n$, where each O_i is open in X_i, is a topological space.

Orientation to this problem appeared to be somewhat challenging for Stacey; she expressed unsureness about what individual elements and open sets of such a space might look like in terms of notation. After some discussion with the Facilitator to establish this distinction, Stacey chose to explore $\mathbb{R} \times \mathbb{R}$ as a potential counterexample to the claim, but Stacey showed significant difficulty in understanding this problem, so discussion continued to provide the appropriate scaffolds to allow Stacey to work on the problem.

Working from her knowledge that open intervals are open sets in $\mathbb{R}$, she first generated the set $(-1,1) \times (-2,2)$ and sketched the corresponding open rectangle on a coordinate plane (Figure 1, left). She later generated the set $(1,3) \times (1,3)$ (Figure 1, center), followed by $(1,2) \times (3,4)$ (Figure 1, right) to see if "ones that don't intersect" might violate one of the properties given in the definition of a topology.

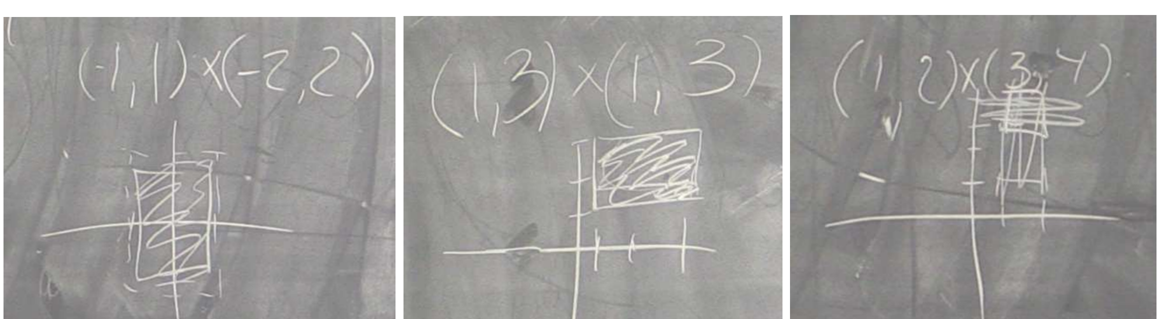

Figure 1:Stacey's examples of open rectangles from Session 6.

At this point, Stacey revealed that she had been viewing sets like $(-1,1) \times (-2,2)$ as the *intersection* of two sets: the tubes formed by $(-1,1) \times \mathbb{R}$ and $\mathbb{R} \times (-2,2)$. The Facilitator established the distinction between the ideas of set product and intersection, prompting Stacey to consider the genuine intersection $(-1,1) \times (-1,1) \cap (-2,2) \times (-2,2)$ (Figure 2).

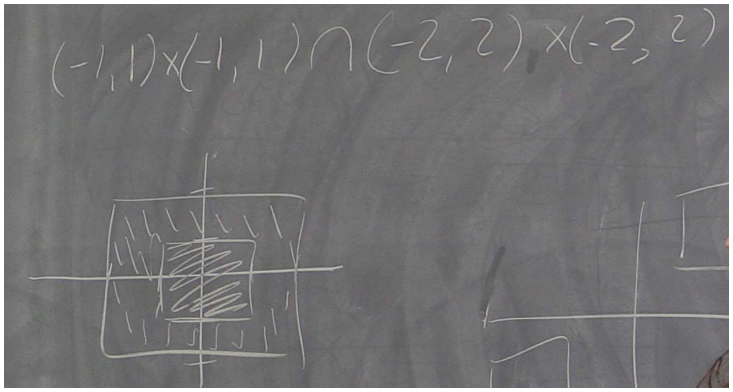

Figure 2: Stacey's example of an intersection of Cartesian products in Session 6.

This seemed to trigger an epiphany (Figure 3): “If you did like, the union of two that were like, say you have like, this open interval over here [drawing], and this one here that are both cross products of things that are, and you did the union of those, you wouldn’t be able to write that as like, an open set cross the open set.” Stacey then went on to produce the specific example $(1,2) \times (1,2) \cup (-1,-2) \times (-1,-2)$ (later recognizing that she had written the interval $(-2,-1)$ with the endpoints in the incorrect order), and labeling her drawing accordingly.

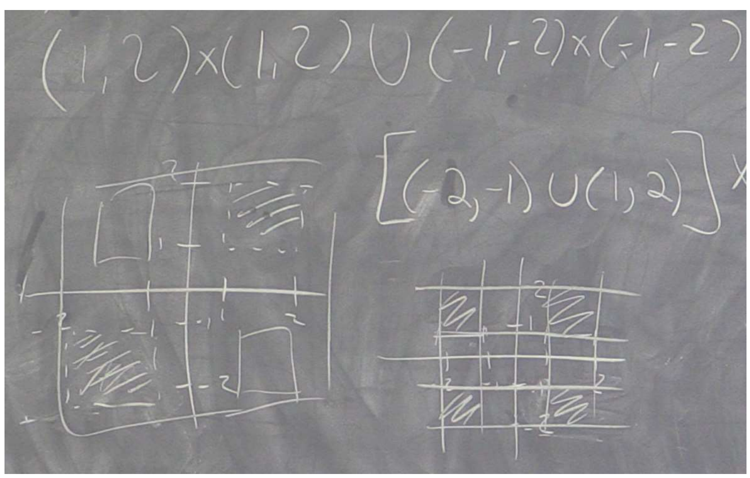

Figure 3:Stacey's counterexample in Session 6.

Open Sets Versus Closed Sets

The idea that the properties of openness and closedness of sets should be mutually exclusive featured prominently in Stacey’s concept image. This misconception was so deeply rooted in Stacey’s mind that it became problematic during Session 4, and Stacey was only able to progress through a proof after directly confronting her misconception. Stacey’s investigation of a concrete example of a set which was neither open nor closed seemed to facilitate her internal restructuring of her concept image. However, even after the error was addressed, it reappeared in Stacey’s reasoning from time to time in future sessions.

The idea that a given set must be exclusively open or closed was most evident and most troublesome in Session 4, and it actually prevented Stacey from applying the definitions of open and closed sets to write a proof. Stacey was asked to prove the following:

> Let $(X, \mathcal{T})$ be a topological space. Prove that $\emptyset, X$ are closed sets, that a finite union of closed sets is a closed set, and an arbitrary intersection of closed sets is a closed set.

She began by reciting that "the definition of a closed set is that its complement is open." She hesitantly identified that the complement of the null set is X, explaining that "for some reason that's tripping me up, because, in X, the complement of the null set … would be X, and we want to say that X is closed, so that can't fit the definition because we need X to be open for the null set to be closed. Or is the null set like a special case with that?" Although the definition of a closed set does not prohibit X from being both open and closed, Stacey suggests here that if X is open, this violates this definition. In order for the null set to be closed, Stacey seemed to require that it be a special case which *violates* the definition. Stacey's cognitive dissonance was so strong that she asked "I *am* trying to actually prove – I'm not trying to disprove this one?"

Stacey addressed her confusion directly in the following exchange:

> *Stacey*: I just can't get out of the habit, $\mathcal{T}$ is open sets. If X is in $\mathcal{T}$, then it has to be an open set, and it can't also be a closed set. But you said there could be open and closed.
> *Facilitator*: Why can't X be closed if it's open?
> *Stacey*: Because [pause] if X isn't closed then its complement [mumbling] open. [pause] Is it, if something is [pause, writing "$[0,1)$"] like is that technically open? Or closed? Or is there like, both?

Following a brief discussion of this example, it was agreed upon that the interval $[0,1)$ is neither open nor closed in the standard topology on the real numbers. This seemed to remove some sort of conceptual barrier for Stacey: after considering this example, she nervously proposed, "If you have an open set, whose complement is still open, then you can consider it a closed set?" She then identified that X was an open set, hence the null set was a closed set, but because the null set is in $\mathcal{T}$, it must also be an open set. She similarly concluded that X was both open and closed.

Despite the revelation that a set might be both open and closed, Stacey's instinctual notion that these properties should be mutually exclusive arose on later occasions. In Session 5, Stacey proposed proving that the boundary of a set A is not an open set by showing that its complement must be open – that is, by showing that the boundary of A is closed. A brief reminder from the Facilitator was sufficient for Stacey to realize her error in reasoning and seek a different strategy.

Conclusions

The limitations in Tom's and Stacey's concept images discussed in our results may each be problematic in their own ways. While a propensity for thinking about continuous sets over discrete sets did not appear to be immediately limiting for Tom and Stacey in the instances described above, consideration of discrete sets is sometimes relevant for the generation of counterexamples in topology. Furthermore, topologies can be defined on discrete sets as well as continuous ones, and students should be able to apply topological ideas to discrete sets. We saw above that Stacey's inclination toward thinking about connected sets over disconnected ones in a product of topological spaces caused a delay in her recognition of the necessary counterexample in Session 6. The perceived mutual exclusivity of openness and closedness of sets acted as a substantial hindrance to Stacey's ability to construct a relatively simple proof in Session 4.

While Stacey initially seemed to be convinced that an open set could not be closed and vice versa, it does not seem to be the case that discrete sets and disconnected sets were not part of Tom's and Stacey's concept images, though perhaps they were not present in their *evoked* concept images. The preference for continuous sets over discrete ones may simply be a difference in accessibility of examples in the example space. We see this in the students' reactions to the discovery that the boundary of the set $(0,1)$ was the set $\{0,1\}$ – they expressed discomfort with the idea, but ultimately were able to accept it. Similarly, despite Stacey's generation of several examples of connected sets in Session 6, she discovered the counterexample of a disconnected set without direct intervention from the Facilitator. This is indicative of the generativity of Stacey's example space, as she is able to manipulate the dimensions of possible variation and their ranges of permissible change in order to generate new kinds of examples.

Tom's and Stacey's abilities to use concepts seemed to be inhibited by the apparent mismatches between their concept images and the concept definitions (Moore, 1994). This was apparent in Stacey's work in Session 4: her inability to adapt to the possibility of a set that is both open and closed, which is allowed by the definitions of these terms but was forbidden by her concept image, significantly hindered her progress in Session 4 and continued to cause problems for her in later sessions. This kind of mismatch created a roadblock in Session 6 as well, as Stacey's concept image of open sets in the product topology as rectangles impeded her recognition of the possibility of taking the union of two such sets. Although this kind of conflict did not seem to hinder Tom and Stacey's success in the context of continuous versus discrete sets, it did give them pause for a moment and introduced doubt, suggesting that it could be problematic in other contexts.

In the case of each dichotomy discussed in our Results, Tom's and Stacey's explorations of their example spaces led to new insights about concepts. In Session 4, even discussion of an inappropriate example (in the sense that Stacey tried and failed to generate a set that was both open and closed) appeared sufficient to help Stacey overcome a pronounced cognitive conflict. It may be the case that even the attempt to produce an example helped to augment or correct Stacey's concept image, or perhaps the awareness that more than two options (open or closed) existed allowed Stacey to accept other possibilities. These results suggest that students' explorations of their example spaces may be beneficial for expanding and correcting students' concept images and for overcoming cognitive conflict by correcting misalignments between the concept image and the formal concept definition.

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An Investigation of an Effective Mathematical Reader and His Interactions and Beliefs About Mathematics and Mathematics Textbooks: The Case of Shawn

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Previous literature has yet to document whether students do not read their mathematics textbooks because they choose not to or because students are unable to read them effectively. This report is a case study investigating the relationship between a student's beliefs about textbooks to learn mathematics and their readings of mathematical text. We find that students may choose not to utilize textbooks as part of their learning despite being effective mathematical readers. Future research should investigate the relationship between what students think it means to understand mathematics and their beliefs about learning through interactions with mathematics textbooks.

Keywords: Reading mathematics, Mathematics textbooks, Effective mathematical reader

Introduction

Mathematics textbooks are an important part of many undergraduate mathematics courses. Instructors of college mathematics regularly assign and expect students to carefully read mathematical text to aid them in developing an understanding of mathematical ideas (Weinberg & Wiesner, 2011). Despite the importance attributed by educators to students reading the textbook as a part of the learning process, literature indicates that students rarely do so, even when instructed (Shepherd, Selden, & Selden, 2009). Cowen emphasized that a goal for instructors should be to teach students to read and understand mathematics in order to provide a path for understanding mathematics, rather than only mastering procedural skills (1991). However, there is no research documenting why students do not read their mathematics textbooks; is it because students choose not to or is it because they are unable to read them effectively, (Shepherd et al., 2009)?

Students' beliefs about learning mathematics has a large impact on how they behave while doing mathematics (Schoenfeld, 1985). Schoenfeld observed that students tend to memorize steps and then practice carrying out these steps, while focusing on whether their answer is correct, rather than why it is reasonable (1985). We believe this behavior is deeply rooted in students' belief that mathematics has pre-existing procedures and results in students learning mathematics passively. Students' beliefs about what it means to learn mathematics is formed through their experiences during their educational careers. As such, their beliefs about what it means to understand mathematics will affect whether students will read mathematics with a goal to develop, or not develop, meanings.

Literature Review and Research Questions

Over the past two decades, research on the use of mathematics textbooks by students has increased (e.g. Berger, 2016, 2019a, 2019b; Lithner, 2003; Österholm, 2006, 2008; Rezat 2006, 2008, 2013; Shepherd, Selden, & Selden, 2009, 2011, 2012; Shepherd & van de Sande, 2014). The researchers of these studies largely focus on how students are reading their textbooks and the students' abilities to accurately solve similar problems to what appears in the reading. Research on why students do or do not read mathematics textbooks as a part of their learning process and the meanings that they construct while reading is limited.

We asked students at a large southwestern university in the United States in two Calculus I courses and one Calculus II in Spring 2018 to choose how strongly they agreed or disagreed with the following statement: I prefer textbooks that focus on showing me what to do and giving me practice doing it. In Calculus I, 77 students (81.9%) somewhat or strongly agreed with the statement, and in Calculus II 30 students (79%) somewhat or strongly agreed with the statement. Weinberg, Wiesner, Benesh, and Boester (2012) reported students claimed that working through example problems in their textbook was to gain understanding of the mathematics. From the same study, 89.4% of the students who responded, reported spending most of their time looking at examples; only 24.7% of the students indicated reading the chapter introductions where the authors develop the concepts and connections to the other concepts in the textbook. Weinberg et al. (2012 reported that what those students meant by "understand mathematics" was to be able to reproduce the steps in the examples to solve a similar problem). Understanding that students' goals for reading and interacting with their textbook are likely different than the goals the instructor or author would have for the student is important in order to help explain how students' beliefs about understanding a mathematical textbook will impact the meanings they create while reading.

There have also been a handful of studies focusing on reading strategies for comprehension related to mathematical texts (Carter & Dean, 2006; Shepherd & van de Sande, 2014), effectiveness of reading strategies based on accuracy of solving problems similar to those in the text (Shepherd et al., 2009), how students read mathematical content-specific texts differently than historical texts (Österholm, 2006), and the ways to categorize students based on their exhibited behavior while reading (Berger, 2019a, Shepherd & van de Sande, 2014). However, even when students exhibited good reading strategies for comprehension, according to the Constructively Responsive Reading (CRR) framework (Pressley & Afflerbach, 1995), there was little evidence to support that the strategies were useful for comprehending mathematical text and its concepts (Shepherd, Selden, & Selden, 2009). Research on how students actually read and use their mathematics textbooks is limited (Shepherd, Selden, & Selden, 2012; Rezat, 2009; Weinberg et al., 2012).

Therefore, research focusing on the relationship between students' beliefs about what it means to learn and understand mathematics, their reasons for utilizing the mathematics textbook, and their beliefs about developing meanings while reading a mathematics textbook is novel and leads us to our questions:

RQ1: For what reasons are students choosing not to utilize the mathematics textbook as a part of their learning process?

RQ2: What relationship is there between the students' beliefs about the role of the textbook in their learning process, what it means to learn and understand mathematics, and the meanings they construct while reading the textbook?

Theoretical Perspective

To characterize students' meanings while reading a mathematical textbook, we must acknowledge that their behaviors and beliefs are individual experiences constructed over time, and therefore we still consider these aspects important when investigating students' meanings while they read a mathematical text.

Previous literature has taken the sociocultural perspective, which supports the categorization of students in the studies based on observable behaviors (reading styles) (Berger, 2019a, Shepherd & van de Sande, 2014). Also notable is reader-oriented theory, in which the reader is viewed as actively constructing meaning from a text though the reading process, but as Weinberg and Wiesner (2011) suggest, many theorists use various perspectives to interpret the views that the textbook is an object and the reader's response to a textbook can also be isolated as an object. Thus, we acknowledge that the frameworks described in previous literature (Berger, 2019a, Shepherd & van de Sande, 2014) offer ways in which to describe students' behaviors while reading mathematical text including *reading for meaning* (Shepherd, 2012). These studies, however, fail to capture the students' beliefs about the role of the textbook in their learning process, what it means to learn and understand mathematics, and the meanings they construct while reading the textbook. We will use the description *effective mathematical reader* to describe someone who consistently constructs meanings in the moment.

Methodology

At the beginning of 2019 spring semester, Calculus 1 students at a large southwestern university in the United States were surveyed to learn about their textbook usage in previous math classes. Of the 154 students enrolled 136 responded. We then scored students' answers focusing on how often students engaged in reading mathematics textbooks in various scenarios and in what ways students use a textbook to prepare for an exam. Using the scores we sorted students into a 3 by 3 table identifying students as low, medium, or high engagement, and as low, medium, or high use for exam prep. We contacted students, via email, categorized in the low engagement and low use for exam preparation (LL), medium engagement and medium use for exam preparation (MM), and high engagement and high use for exam preparation (HH). From the 38 contacted (not all respsoned), we selected 15 students, 6 HH, 5 MM, and 4 LL to interview. Our goal was to have 5 students in each category of LL, MM, and HH, however, only 4 LL responded. All the students agreed to participate in the interviews on a voluntary basis and were compensated monetarily for their time.

We conducted three 90-minute clinical interviews (Clement, 2000) with each students. The first interview took place 4-5 weeks after the beginning of the semester and each interview after was approximately 4-5 weeks after the previous. For each interview, we used the required online textbook, *Calculus: Newton Meets Technology* (Thompson & Ashbrook, 2019). It is important to note that the course and textbook specifically focus on developing a conceptual understanding of calculus. During each interview, students answered questions regarding their current uses of the textbook, and their meanings related to the passages prior to reading passages. Then, students were asked to read several passages from the textbook aloud (as they would normally read the text independently) and to share the meanings they interpreted from reading. Students read the passages displayed on a computer screen and were instructed to use the cursor to point to what they were reading or referencing at any moment. The computer was connected to an iPad to display students' written work on the computer screen, and we captured students' gestures, audio, and work simultaneously using *Camtasia*.

Results

We will focus on the survey conducted at the beginning and end of the semester and the three interviews conducted with one of the LL student's, whom we will call Shawn. Shawn received an A for his course grade. We chose to focus on Shawn as a case study for this article for two reasons. First, his interviews indicated that he is an effective mathematical reader who chose not to use textbooks as part of his learning process in previous math courses. Second, Shawn's interviews and survey responses provided insight into (a) his reasons behind his initial decision not to utilize the textbook, (b) a transition in both his beliefs about textbook reading as a part of his learning process, and (c) for his meaning for what it means to understand mathematics. Thus, we believe that Shawn represents an informative case study for discussion of how a student's beliefs about what it means to understand mathematics is related to their reading behavior, and to being an effective mathematical reader.

Previous Experience Survey

Shawn indicated in the first survey that he read assigned sections from the textbook in previous courses but did not read unassigned sections. He did not use the textbook when completing homework, to preview for class, nor after class to understand material covered during class. Shawn also indicated that in prior math courses his only use of the textbook when studying for tests was to find additional problems to practice. The researchers asked if he agreed with the statement: “In previous math classes, I succeeded by making connections among ideas. Shawn indicated that he somewhat disagreed with the statement. When prompted to describe what “making connections among ideas” meant to him, he said “[...] memorizing processes and applying them to other related problems. This is like memorizing basic algebra and being able to apply it to all other forms of math.”

Interviews

Beliefs regarding use of textbook. At the start of each interview students were asked about their textbook usage. During this part of the first interview, Shawn told the interviewer that he didn’t use the textbook in previous math classes because he didn’t need to read in order to pass tests. He found that going to class and completing the homework was sufficient. Shawn mentioned that available time was an influencing factor in his decision to read the textbook or not, and that even when reading the assigned sections he would spend at most 30 minutes with the textbook before class to get what he described as a “general sense of the topic”. He conveyed that he did not read the textbook after class, preferring instead to ask any questions he had during office hours. Shawn mentioned in all three interviews that for the current calculus class he found that reading the textbook before class was helpful, and in the third interview he claimed that his struggle with the current material was, in part, due to the fact that he had not been reading the textbook.

During the first interview, Shawn stated time constraints as a reason to not read or interact with certain features in the textbook. For example, when asked about his skimming of a reflection question during the first interview Shawn said, “I skim it exactly like that. I will look at it and ask myself if I can do this really quick, and if I can’t do it really quick, I will ask if I really want to… and the answer is no.” While reading, Shawn would sometimes skip parts of the text, stating that he did not think that passage was necessary, or that he got the gist of what the passage was talking about. Additionally, there were times when Shawn indicated that he wasn’t sure of the meaning of the passage he just read, but that he would not spend more time reading the passage at that point since he had already learned that topic. Shawn felt he understood the material well enough, so to him, it wasn’t worth the time to try to make sense of that particular passage.

Reading the passages from the textbook. While reading the passages in the text, Shawn demonstrated that he is an advanced reader with effective mathematical reading strategies compatible with descriptions by Berger (2019a) and Shepherd & van de Sande (2014). We offer Figure 1 and Figure 2, with their respective transcripts, to provide evidence that Shawn is an effective mathematics reader.

Excerpt 1.

The Meaning of "Essentially Equal To …"

The idea of a number L being essentially equal to a number represented by a sequence of numbers is that you can make the difference between L and values of all but a finite number of terms in the sequence as small as you please.

Consider the sequence

$$1.9, 1.99, 1.999, \cdots$$

No matter how small a difference from 2 we desire, we can find a term in this sequence so all terms after it are closer to 2 than that difference.

Figure 1. A passage from Shawn's 2nd interview. From Section 4.9: Exact Rate of Change Functions.

Shawn: (Reads Figure 1.) I'm just going to look over this again to try to comprehend what that was. The idea of a number L, that's weird because L is a letter. The number L being essentially equal to, I guess it's a variable, being essentially equal to a number represented by a series of numbers. So L is about equal to a...number represented by a sequence of numbers. So all of this (points to sequence $1.9, 1.99, 1.99$) represents two, I am assuming, and L is essentially equal to two.

Shawn monitored his reading for comprehension by summarizing in his own words what he interpreted the passage to mean. Across his three interviews, he often reread more carefully after he stated he did not know what a passage meant similar to his behavior seen in his reading above.

Excerpt 2.

When we say that $r_f(x_0)$ is the momentary (exact) rate of change of f at x_0, we mean that the value of $f(x_0 + dx)$ is essentially equal to the value of $f(x_0) + r_f(x_0)dx$ as dx varies through a sufficiently small interval containing x_0.

Figure 2. A passage from Shawn's 2nd interview that follows Figure 1 in the textbook. From Section 4.9: Exact Rate of Change Functions.

Shawn: (Reading Figure 2.) When we say that r sub f of x is the momentary exact rate of change of f at x, we mean that the value of f of x plus dx is essentially equal to the value of f of x plus r f of x times dx as x varies through a sufficiently small interval containing that.

Hmmm. I'm going to look over this because that doesn't makes sense. Okay, so the rate of change is this (highlights $r_f(x_0)$) and the momentary exact rate of change of f at..(rereads) is the momentary exact rate of change at f. Okay, so f of x and this (points to $r_f(x_0)$) are equal to each other, I guess. (Rereads) is the moment exact rate of change at f. Yeah, seems that they are equal to each other. We mean that the value of (pauses)…No. This (points to "f at x_0") might be accumulation and that's (points to $r_f(x_0)$) just rate of change. We mean that the value of maybe the accumulation at the initial value plus the change is essentially equal to the accumulation plus…Oh! That makes sense. So, you have the initial value plus the change time the or, the rate time the change within an interval (pointing to $f(x_0) + r_f(x_0)dx$) or something like that. And this (highlighting $f(x_0+dx)$) is just the accumulation at that same value. So it's the initial plus the change is equal to that (pointing to $f(x_0)+r_f(x_0)dx$). That makes sense.

Interviewer: Okay and then when you say the accumulation part…(doesn't finish question).

Shawn: The accumulation, um, well they both are. This (pointing to $f(x_0)+r_f(x_0)dx$) is just say it in a different way. So this (pointing to $f(x_0+dx)$) is the accumulation, at like, we have an initial point (pointing to x_0 in $f(x_0+dx)$) plus uh this change (pointing to dx in $f(x_0+dx)$). So let's say this point (pointing to $x_0$in $f(x_0+dx)$) is like point five and this change (pointing to dx in $f(x_0+dx)$) is point 5, so this (pointing to $f(x_0+dx)$) is the accumulation at one. And this (pointing to $f(x_0)+r_f(x_0)dx$) is the accumulation at point five (pointing to $f(x_0)$) plus the rate (points to $r_f(x_0)$) times point five (points to dx). I am assuming it ($r_f(x_0)dx$) would give you another accumulation and you add that on to the initial accumulation and you would get the same accumulation.

In the second excerpt, Shawn read through in a similar manner as he did in the first excerpt and then dissected the text into pieces. We noticed Shawn read mathematical expressions literally in the first read through, but after stating that what he read did not make sense, he reread the mathematical expressions with meaning. In response to his perceived conflict between what he read and his meanings, Shawn created his own example to clarify what he thought the interviewer might be asking him in the moment (even though the interviewer never finished the question). Additionally, in Shawn's third interview, he created examples to understand what he was reading without any prompting by the interviewer.

Other observations. Due to space limitations we list effective mathematical reading strategies Shawn displayed without supporting evidence. Across the three interviews Shawn demonstrated he is an effective mathematical reader using the following mathematical reading strategies: (a) monitoring for reading comprehension, (b) rereading more carefully when necessary and skimming familiar content, (c) summarizing in his own words passages and big ideas he considers the text to be explaining, (d) verbalizing connections between various portions of the text with other parts of the text, (e) verbalizing connections between various portions of the text with his meanings from class, (f) reading mathematical expressions with meaning, and (g) creating his own examples to compare with various parts of the textbook.

End of the Semester Survey

Concluding the semester, Shawn completed a survey similar to the survey given at the beginning of the semester and was recategorized as a student with medium engagement and low textbook usage for exam preparation. Shawn's reported textbook usage increased in three out of five categories related to engagement and three out of six categories for exam preparation. He reported increased textbook usage in reading unassigned sections, finding examples to help with homework, previewing material before class, reading explanations of concepts, completing the chapter review, and reviewing key terms. The one category regarding engagement that Shawn's usage decreased in was reading assigned sections. The two categories regarding test preparation where his usage decreased was reworking assigned and working unassigned problems in preparation for an exam. His responses to the remaining category in engagement (reading after class) and the remaining category in test preparation (looking for solved problems) were the same.

At the end of the semester, Shawn "strongly agrees" that he succeeded in the current math class by making connections amongst ideas. His meaning for what it means to make connections among ideas changed from "[...]memorizing process and applying them to other related problems." to "[...] understanding concepts and applying them to other concepts".

In response to the question "Has your personal meaning for what it means to understand a mathematical idea changed from the beginning of the course to now?" Shawn responded "Before mathematical ideas were understanding how to use a formula, but now it is understanding concepts. Instead of memorizing formulas, you memorize formulas and concepts."

Conclusions and Discussion

Due to space limitations, we do not argue that Shawn constructed strong or weak meanings while reading because that would require a more in-depth analysis of the passages and the concepts Shawn read about during the interviews. Rather, we claim that Shawn is an example of an effective mathematical reader. We conjecture that Shawn is not unique as a student who is an effective mathematical reader who chooses not to use the textbook often for the purpose of learning mathematics; addressing Shepherd et al.'s call to action (2009) to understand why so few students utilize the textbook as a part of their learning process. His reasons for not utilizing the textbook is due to time and believing that he can learn from other mediums more efficiently.

An extension of this case study should focus on what proportion of students are effective mathematical readers who choose not to read their textbook and why. If a significant portion of students are like Shawn, effective mathematical readers who are choosing not to utilize the textbook, then the reasons why they have made that choice, and reasons for shifts in beliefs as seen in Shawn's survey responses can inform educators on how to address the lack of textbook utilization by students.

Comparing Shawn's responses in the surveys indicated two shifts in Shawn's beliefs. The first shift was in Shawn's belief about utilizing the textbook to understand mathematics. Initially, Shawn believed that reading the textbook was unnecessary to understand mathematics. At the end of the semester he said that pre-reading was an important part of understanding the class lecture.

The second shift was in Shawn's beliefs about what it means to understand mathematics. At the beginning of the semester his meaning for what it means to understand mathematics was computational and focused on the memorization of algebraic processes. However, at the end of the semester he had expanded his meaning beyond computations to also include a focus on mathematical concepts. Additionally, two of the categories where his textbook usage decreased in frequency were when working problems (computational focus) while several of the categories in which his usage increased involved him reading to understand mathematical concepts (conceptual focus). The shift in Shawn's beliefs about what it means to understand mathematics is also supported by his responses to the survey questions. Future research should investigate this relationship more extensively.

Our future research will focus on the relationship between students' meanings, students' beliefs about the role of the textbook in their learning process, and the students' beliefs about what it means to learn and understand mathematics. Moreover, what influence do students' current meanings have on the meanings they construct from reading a textbook?

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Investigating Instructors' Perceptions of IBL: A Systemic Functional Linguistic Approach

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Inquiry-Based Learning (IBL) teaching practices have been hard to characterize in mathematics education research, as mathematics instructors interpret and engage with these practices in different ways. Using systemic functional linguistics (SFL), specifically the appraisal framework (Martin & White, 2005), we analyzed responses to the question "what is IBL" given by 12 mathematics instructors who were teaching with IBL. Our analysis was guided by the following question: What do the language choices of a group of undergraduate mathematics IBL instructors reveal about their perceptions of IBL? We identified how the instructors opened up, closed down, and changed the strength and focus of their messages through their grammatical and lexical choices. We found that instructors softened their language when defining IBL but used sharper and intensified language with less room for negotiation when talking about teacher and student roles in IBL classrooms.

Keywords: Inquiry-Based Learning, Undergraduate, Mathematics, Discourse Analysis, Systemic Functional Linguistics

Inquiry-based learning (IBL) is a form of student-centered instructional practices implemented in undergraduate mathematics classrooms. The IBL literature defines IBL as a form of active learning in which students interact with a carefully scaffolded curriculum—usually a set of problems or a list of definitions and theorems—individually or in groups under the instructor's supervision (Ernst, Hodge, & Yoshinobu, 2017; Yoshinobu & Jones, 2012; 2013). Even though the uptake of reformed teaching practices, such as IBL, has been slow among undergraduate mathematics instructors, instructors are becoming "increasingly aware of the value of inquiry-based instruction" (Laursen & Rasmussen, 2019). Many factors such as class size, students' experience with IBL, instructor's skill level and experience with IBL, institutional resources, and course topic affect implementation of IBL (Ernst et al., 2017). Thus, as IBL grows in popularity among mathematics faculty, how IBL is interpreted and enacted in the classroom has diversified to the point that IBL is currently called the "Big Tent" for all varieties of IBL (Laursen & Rasmussen, 2019). Considering that the IBL literature provides no consistent definition of IBL, we believe it is important to understand how mathematics instructors understand IBL, how their understandings vary, and how these understandings affect their decisions when implementing IBL. In this paper, we aim to explore the meanings that instructors give to IBL. Thus, we started by asking a simple question: How do undergraduate mathematics instructors talk about IBL? To answer this question, we use systemic functional linguistics (SFL, Halliday, 1985) to analyze interview data with 12 undergraduate mathematics IBL instructors. SFL allows us to draw meaning based on the lexicogrammatical choices that instructors make, consciously or unconsciously, as they speak. More specifically, we focus on their appraisal decisions and ask: What do the language choices of a group of undergraduate mathematics IBL instructors reveal about their perceptions of IBL?

Supporting Literature

IBL, as a way of teaching undergraduate mathematics, is traced back to Robert Lee Moore (1882–1974), an American topologist also known for his unconventional graduate teaching methods (Yoshinobu & Jones, 2013). Moore provided his students with a set of axioms, definitions, and theorems and asked his students to prove as many theorems as possible, without any assistance from outside sources or peers (Dancis & Davidson, 1970; Jones, 1977; Mahavier, 1999; Zitarelli, 2004). A few of Moore's students and colleagues taught their classes using various modifications of Moore's method, which seems to be the precursor of IBL as it is used today (see e.g., Dancis & Davidson, 1970; Mahavier, 1999). In contrast to Moore's method, students in IBL classrooms are invited to collaborate with each other, propose ideas to solve the problems, and engage with their peers' mathematical ideas. The instructor orchestrates student discussions and guides them towards the main mathematical goals. Today, IBL is practiced and implemented in many ways (Laursen & Rasmussen, 2019).

As IBL has emerged from college educators' practical work rather than from research (Laursen & Rasmussen, 2019), the mathematics education literature has been conceptualizing IBL either in contrast to lecturing or to instruction guided by core principles (e.g., Ernst et al.'s, 2017, twin pillars; Laursen & Rasmussen's, 2019, four pillars). Lack of a consistent conceptualization of IBL in undergraduate mathematics literature and the numerous ways of implementing IBL leave room for instructors' interpretation of what IBL is and how it is implemented, and may damp efforts to understand its impact on student learning. A clearer conceptualization of IBL needs to start with clarity about the multiple implicit meanings IBL may have.

SFL as a Tool for Analyzing Discourse

SFL, as a model of language (Halliday, 1985) provides various tools for analyzing and drawing implicit meanings from language. By facilitating the "exploration of meaning in context through text-based grammar," SFL allows the researcher to realize the different ways that the writer or the speaker made grammatical choices—not necessarily conscious—to convey meanings (Schleppegrell, 2012, p. 21). Text, "as a social exchange of meanings" (Halliday & Hasan, 1985, p. 11), is used to make three types of meanings: representing an experience, enacting a relationship between the speaker and the listener or the writer and the reader, and connecting the unit of analysis to the other parts of the larger text (Schleppegrell, 2012; Thompson, 2014). SFL has been used for drawing meaning from language, spoken or written, in K-12 mathematics education research. Using SFL, O'Halloran (2000) identified the multisemiotic nature of mathematics discourse in high school trigonometry classrooms; Zolkower and Shreyar (2007) described how a sixth grade mathematics teacher mediated classroom discussions; González and Herbst (2013) investigated the development of oral proofs by ninth graders in a geometry class; and Kosko and Herbst (2012) analyzed how high school geometry teachers used modality (using words such as *may* or *should* to express a measure of indeterminacy) in teacher-to-teacher talk to reveal teachers' perspectives regarding the norms of teaching geometry. In each of these cases, SFL allowed the researchers to systematically investigate meaning from the choices that speakers and writers made when speaking or writing using a variety of SFL frameworks.

We chose SFL because it provides a system for organizing "collection of choices that speakers have when using specific linguistic resources" (González & Herbst, 2013), which is particularly useful, precisely because IBL has emerged from practitioners which suggests that multiple interpretations (and implementations) of IBL exist. Describing what IBL entails has been a goal for IBL scholars; we believe that a well-developed analytical system allows us to detect the nuances of instructors' perceptions of IBL. Compared to thematic analysis that uses open coding of text data, SFL provides established frameworks of grammatical and lexical choices, allowing the researchers to systemically analyze attitudes expressed via language that go undetected using thematic analysis. Thus, SFL minimizes subjectivity and increases the validity of our claims regarding IBL instructor's perceptions of IBL.

Methods

We conducted an appraisal analysis to investigate the evaluative function of text data collected from hour-long semi-structured interviews with 12 undergraduate mathematics IBL instructors in 2012 who were part of a larger study (*N*=70) that investigated how faculty used inquiry-based learning in their courses. The 12 instructors were selected purposefully to be interviewed, in order to have variation in terms of IBL teaching experience (5 participants self-identified as beginner/novice [BN] IBL users and 7 self-identified as advanced/expert [AE] IBL users) and level of the courses in which they used IBL (4 lower division [LD], 6 upper division [UD], and 2 future teachers [FT]).

Coding and Analysis

In order to answer the research question of this study, we analyzed the responses to the first question in the interview: What is inquiry-based learning? This allowed us to ensure that the analysis would yield useful information before engaging in a more systematic analysis of the full interview (e.g., What conditions are present in a successful IBL class? What are helpful aspects of IBL method?). To prepare the data for analysis, we (Saba and Vilma) parsed the responses provided by each instructor into sentences, which became our units of analysis. For each aspect of the framework (discussed below), we coded all the sentences for one instructor at a time and checked our coding with each other before moving to the next instructor. We discussed all the discrepancies until we reached agreement.

The appraisal framework "systematizes a varied set of linguistic resources that speakers and writers use to negotiate evaluations with their addressees and to construct solidarity around shared values" (Thompson, 2014, p. 80). This framework consists of three aspects: attitude, engagement, and graduation. While attitude focuses on how the speaker talks about something or someone (good or bad), engagement and graduation reveal the speaker's choices in negotiating solidarity with the listener and weakening or intensifying the message respectively. In this paper, we report on our results from the analysis of engagement and graduation. Elsewhere (Gerami & Mesa, 2019), we reported our results from the attitude analysis.

Engagement analysis focuses on the ways that writers and speakers open up or close down space for negotiation their evaluation of a phenomenon with the audience. Monogloss engagement refers to statements in which the speaker or the writer leaves no room for negotiation; the engagement is heterogloss when the speaker or the writer opens up space for negotiation. What is evaluated is referred to as the target of engagement. Two main sources—modality and interpersonal projection—are used to invite other perspectives to negotiate the message regarding the target with the audience. Modality points to the extent that a proposition is valid using terms of probability and usuality (e.g., might, may, often, sometimes, always) versus being imposed using terms of obligation (e.g., should, must, have to). For instance, in the sentence "I sometimes have my students work in groups on the problems," the appraisal analysis suggests that the speaker is using *sometimes* to make room for other possibilities for additional activities that can be done in class (at *other* times). Interpersonal projection refers to the use of terms such as *I suppose, I think,* and *to me* by the speaker, which are all considered heteroglossic language. In the two sentences "I think I am doing well in this course" and "I am doing well in this course," the former sentence uses tentative language (*I think*) that is open to the possibility of a rebuttal, that the audience may have a different opinion than what is expressed. The second sentence, in contrast, leaves no room for negotiation.

Graduation analysis focuses on the ways that writers and speakers adjust the strength and focus of their message (Derewianka, 2013). Force is used to increase or decrease the strength of a message regarding a target. For instance, in the sentence "The homework was too hard this time," *too* is used to increase the strength of the difficulty of the homework. Focus is used to sharpen or broaden the options of the target in a message. For instance, in the sentence "You must practice all kind of problem solving," the message regarding problem solving is broadened to all types of problems by using the word *all*. However, in the sentence "You must practice authentic problem solving," the speaker sharpens the focus of the message about the problem to communicate that the problem was of a specific kind, as it was an authentic one.

Results

In terms of engagement, we found that the participants (referred to in here by the letter T and a unique number between 1 and 12) opened up the dialogue when they talked about IBL and closed down the dialogue when they talked about student roles and teacher roles in IBL classrooms. In their responses to "What is IBL?," 11 of the 12 instructors started their responses with heterogloss elements, mainly in form of interpersonal projections:

T4: Um well <u>I think</u> it's learning that's very much driven by um sort of the students their understanding of the subject. [BN, UD]

T8: Well <u>I would say</u> it's about making the students discover the material for themselves … [AE, FT]

T11: Well <u>I guess to me</u> inquiry-based learning is a process by which … [AE, UD]

The instructors' lexical and grammatical choices in these sentences show that they are open for negotiation and alternative interpretation as they talked about IBL by emphasizing that their definition was rather personal than general. Eight of these 11 instructors continued with heterogloss engagement throughout their responses, (e.g., "[IBL] <u>can be</u> in that way a learning process versus being fed") while three of them included some monogloss engagement (e.g., "That [the goal of IBL] <u>is</u> captured and implemented in IBL by careful preparation of mathematical materials"). Only one instructor's response included monogloss engagement throughout the response:

T12: <u>It's</u> basically um <u>it's</u> an approach where the students are um the focus as opposed to the instructor. And um <u>the idea is</u> that students spend their time in various ways but uh <u>the main thing is</u> that they need to be making sense of the mathematics for themselves and the instructor is there more to be a guide or a coach. [AE, UD]

Despite the frequent use of heterogloss engagement when defining IBL, monogloss engagement was more often detected when the instructors talked about the role of the students and instructor in IBL classrooms. This implies that the participants tended to leave less room for negotiation when they talked about student and teacher roles in their definitions of IBL. It is important to note that the phrasing of the interview question was conducive to garner a monogloss response: "What is inquiry-based learning?" as opposed to a phrasing that could be conducive to generate a more heterogloss response such as "How do <u>you</u> define IBL?" or "What is IBL <u>to you</u>?".

Regarding graduation, we found that the instructors softened the focus of their language when they talked about IBL and increased the intensity of their language, and sharpened the focus when they talked about student and teacher roles in IBL classrooms. Force was seldom detected in the instructors' responses with IBL as the target. With student role as the target, force was raised in eight instructors' responses, while it was lowered in one instructor's response. Similarly, with teacher role as the target, force was raised in eight instructors' responses, while force was lowered in one instructor's response.

T12: … the main thing is that they [the students] need to be making sense of the mathematics for themselves and the instructor is there more to be a guide or a coach. (force raised, target: student role) [AE, UD]

T5: I think it is important to uh give them the right problem set the right time uh at the right amount. (force raised, target: teacher role) [BN, UD]

When referring to IBL, the instructors more frequently softened than sharpened the focus and, when referring to teacher and student roles, the focus was more frequently sharpened than softened:

T7: One might think about it [IBL] as sort of a sliding scale in the extent to which the instructor um is uh making things move along. (focus softened, target: IBL) [AE, LD]

T3: I think about it [IBL] as … basically, getting them to generate solutions to problems without you ever having to explicitly show them how to solve these problems. (focus sharpened, target: student role) [BN, LD]

T7 used two sets of terms 'sort of' and 'a sliding scale' to broaden the options when talking about IBL, resulting in softening of the focus. In T3's quote, the instructor's use of *basically* narrows down the roles that students play in IBL classrooms, sharpening the focus of student role. In sum, graduation analysis revealed that the instructors lowered the intensity and softened their language as they talked about IBL, while they used more intensified and sharper lexical and grammatical choices when talking about student role and teacher role in IBL classrooms.

Discussion

Using the appraisal framework, we drew meanings from the implicit ways the instructors' talked about their perceptions of IBL, the parts that they depended on more heavily, and the parts they paid less attention to. Specifically, our findings show that the participants softened their language and opened room for negotiations and alternative interpretations when defining IBL, but used sharper and intensified language with less room for negotiation when talking about teacher and student roles in IBL classrooms.

First, the instructors opening up space when they talked about IBL suggests that they were aware of the various implementations of IBL or that they implemented IBL differently than how it was typically described in 2012, when the interviews were conducted. Second, when the instructors talked about the roles of students and teacher, their language choices did not leave room for negotiation, which suggests that they, consciously or unconsciously, expect their students and themselves to assume very clear and specific roles in IBL classrooms. This is revealed by language choices that suggest they may not be considering alternative implementations of IBL when it came to their *own teaching*. Third, preliminary findings of the second interview question (What conditions are present in a successful IBL class?) that we did not include due to space limitations, indicate that the instructors intensified student roles more often than teacher roles, which express an unbalanced student-instructor power relationship in IBL classrooms, in spite of IBL being declared as a strategy in which the instructor stops being "the sage on the stage." Alternatively, intensifying the student roles may be expressing the instructors' difficulties in reflecting on their own role and practices. This tension may also be the consequence of conceptualization of IBL by Ernst and colleagues (2017) and Yoshinobu and Jones (2013), that were common at the time of the interviews, and that depended heavily on the roles of students to capture the 'twin pillars' of IBL: (i) students engage deeply with mathematical tasks and (ii) opportunities for collaboration with peers are provided. Note that the actors in both of these principles are students; the twin pillars do not describe the role of the instructor: note that in the second statement of the pillar, the actor is not explicitly stated. In the most recent conceptualization of inquiry-based mathematics education Laursen and Rasmussen (2019) added two more pillars: (iii) "instructors inquire into student thinking" and (iv) "instructors foster equity in their design and facilitation choices" (p. 138). These added statements make the role of the instructor quite explicit. We venture that the interviews in our study provide a way for capturing the perceived meaning of IBL at a time in which the role of the instructor was in need of clearer definition.

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An Analysis of Opportunities for Reasoning-and-Proving in a University Precalculus Textbook

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In this study, we provide an analysis of opportunities for undergraduate students to engage in the skill of reasoning-and-proving in a precalculus textbook in a medium-sized university in North Eastern USA. To investigate these opportunities, we focused our analysis on the frequency and nature of reasoning-and-proving opportunities available in both the exercises and narrative section of the book. Findings indicate that there were more reasoning-and-proving opportunities in the narrative section of the book than in the student exercises section. Furthermore, the opportunities in the student exercises section were predominantly of the particular nature while those in the narrative section were general. Implications of these findings are discussed.

Keywords: Textbook Analysis, Reasoning-and-Proving, Precalculus

One of the most important skills in the learning of mathematics at higher levels is the ability to engage in reasoning-and-proving (Stylianides, 2009). Thus, it is important that preparatory mathematics courses provide students with opportunities for engaging in and reflecting on the skill of reasoning-and-proving. An important learning resource that forms an integral part of most mathematics courses at the college level is the textbook. The textbook is not only an important resource that most instructors draw from, but also provides practice problems for the students (Grouws, Smith, & Sztajn, 2004).

A number of researchers (e.g., Hanna & de Bruyn, 1999; Newton & Newton, 2007; Stylianides, 2009) have examined the opportunities available for mathematics students to engage in reasoning-and-proving. Most of these studies, however, focused on school mathematics as well as on instructors' role as curriculum interpreters and implementers. Studies that focus exclusively on undergraduate mathematics are rare. In particular, research on opportunities for reasoning-and-proving available to students in textbooks for preparatory courses such as precalculus are scarce. One of the reasons for this could be the perception that instructors have more influence on student learning than textbooks and other curriculum materials. However, studies (e.g., Porter, 2002; Stylianides, 2009) have shown that mathematics textbooks still remain a major influence on what and how mathematics students learn. Thus, one way of contributing to the ongoing efforts to increase student success in university mathematics courses is to examine the reasoning-and-proving opportunities available for students in preparatory courses. In this study, we sought to investigate the frequency and nature of reasoning-and-proving opportunities available in a precalculus course textbook at a mid-sized private research university in North-Eastern United States. We also investigate the distribution of such opportunities across various sections (such as narrative and exercises) of the book.

Background and Theoretical Framework

Reasoning and proving are among the most important components of the discipline of mathematics (Hanna, 2000; Stylianides, 2009, Stylianou, Blanton & Knuth, 2010). These processes play a central role in both making sense of mathematical statements (Arcavi, 2003; Hanna, 2000; National Council of Teachers of Mathematics [NCTM], 2009; Yackel & Hanna, 2003) and assessing the truth of such statements (Harel & Sowder, 1998). In this sense, therefore, reasoning and proving can serve as powerful tools for exploring mathematical ideas in order to enhance meaningful learning of mathematics. According to Otten, Males, and Gilbertson (2014) engaging in the process of proving is often wide and incorporates aspects of reasoning such as “empirical explorations, conjecturing, justifying, refining, explaining, and so forth” (p. 107). For this reason, it can be argued that the processes of reasoning and proving are intertwined and happen concurrently. In light of this fact, the hyphenated term reasoning-and-proving that highlights the nature of the intertwinement of the two processes has emerged in mathematics education literature (Stylianides, 2009). Consistent with this view, therefore, we use the hyphenated term reasoning-and-proving (RP) for the rest of this paper.

According to Hanna (2000), the process of RP defines the nature of mathematics and is so important that “one cannot be said to have learned mathematics” (p.24) unless they have learned to engage in the process in some way. Despite the importance of RP in doing mathematics, studies have shown that most undergraduate students, including those majoring in mathematics, struggle in mathematics courses that require various forms of RP (Recio & Godina, 2001; Selden & Selden, 2003; Sowder & Harel, 2003; Weber, 2001). In order to address this problem and help students succeed in such courses, many universities offer prerequisite courses that are often meant to prepare students for other courses. One such preparatory course that is common in many universities is precalculus. In many cases, universities give a calculus readiness exam to students and then use those results to recommend that some students take the precalculus course before they start the calculus sequence. However, studies (e.g., Sonnert & Sadler, 2014) have shown that precalculus courses do not necessarily get students ready for taking calculus courses and that several students who take precalculus still fail calculus courses. This calls to question the impact of taking precalculus courses as a way of preparing students to succeed in calculus. As a preparatory course, it is expected that precalculus courses would offer students several opportunities to engage in RP as an important aspect of doing mathematics.

Analytic Framework

Many studies on textbook analysis (e.g., Li, 2000; Senk, Thompson, & Johnson, 2008) have called for the inclusion of both text exposition and student exercise sections in analyzing textbooks for RP opportunities. Text exposition section (sometimes called narrative section) refers to parts of the book that provide information (e.g., theorems, properties, propositions, formulas, etc.) to the reader and to which the reader does not have to take any active role other than reading it. Students exercise sections, on the other hand, are parts of the book that require students to participate in various ways such as completing a task or answering a question.

By drawing from the reasoning-and-proving standard of the NCTM (2000) and other studies that contained aspects of the processes of RP (e.g., Valverde, Bianchi, Wolfe, Schmidt & Houang, 2002), Thompson, Senk, and Johnson (2012) developed an analytic framework for investigating RP opportunities in mathematics textbooks. This framework provides ways of identifying various types of justifications and RP opportunities in textbooks. Moreover, the framework included dimensions for both student exercises and text exposition sections of the textbooks hence heeding the earlier calls by researchers such as Li (2000). Based on the necessity principle (Harel & Tall, 1991) that regards the importance of students being able to see the need for reasoning and proof, Otten, Gilbertson, Males, and Clark (2014) made slight modifications to Thompson et al.’s (2012) framework that included the characterization of the nature of mathematical statements around RP opportunities as *general, particular,* and *general with particular instantiation provided*. A *particular* statement concerns a specific mathematical object or a finite set of objects while a general statement concerns all objects in a given class without exceptions (Otten et al., 2014). A *general with particular instantiation* statement, on the other hand, concerns all objects in a given class but for which one member of the class has been provided for use in reasoning. We found these characterizations of RP opportunities useful since they have a great influence on the kinds of learning opportunities for students.

Method

We analyzed the fifth edition of *Functions Modeling Change* by Connally, Hughes-Hallet, and Gleason (2015). Most chapters in this textbook were structured such that they contained the following parts: chapter sections, chapter summary, strengthen your understanding, and skills refresher. Each chapter section was designed to address particular concepts in the chapter. For example, the chapter named trigonometry and periodic functions had sections such as sinusoidal functions, the tangent function, among others. The chapter summary appeared near the end of each chapter and listed the main ideas (and formulas) in the chapter. The strengthen your understanding section had exercises in form of declarative statements to which students were supposed to supply either a true or false answer. The skill refresher sections provided solved examples and exercises targeted at various key ideas in the chapter. Most student exercises appeared at the end of each chapter section (named section exercises) and the end of the chapter summary (named chapter review exercises). Chapter review exercises tended to cover content from the whole chapter in which they were located. For the purposes of this study, we focused only on the content of the sections and the accompanying section exercises since most of the other sections contained most items that are related to the ones on these sections.

From the preface section of the book, we identified the idea of functions and their properties as the core concept of the textbook. Therefore, we sought to analyze chapters and sections that addressed the idea of functions in a general way as opposed to specific types of functions. We identified two chapters (chapter 2 and 6) as well as parts of chapter one (section 1.1 and section 1.2) as ones that met our criteria. Chapter 2 had a total of six sections (here in called lessons) while chapter 6 had three. Since all identified lessons from the named chapters addressed different ideas about functions in general, we chose to analyze all of them (9 lessons in total).

For each lesson that we sampled, we examined the statements in the examples as well as the expository text and coded them for RP opportunity. Some of the codes related to RP include *make a conjecture, fill in blanks of a conjecture, investigate a conjecture, construct a proof, develop a rationale, outline a proof, fill in blanks in a proof, evaluate or correct an argument or proof,* and *find a counterexample.* We then characterized the nature of these opportunities as either *general, particular,* or *general with particular instantiation provided.* We conducted similar analytic procedures for the student exercises. It is worth noting that some exercises and examples had several sub-items. In such cases, we treated each sub-item as separate items since we noticed that different sub-items could elicit RP opportunities of different kinds.

Results

We present the results of this study in two parts; the first part provides the results for the text exposition section of the textbook while the second part presents the results from the student exercises section. For each part, we investigated both the type of RP opportunities as well as a characterization of the nature of such opportunities.

Reasoning-and-Proving in Text Exposition

Frequency of reasoning-and-proving statements. Table 1 shows the frequency of mathematical statements related to RP in the text exposition section from the three chapters. There was a total of 11 lessons spread across three chapters. Two lessons were from chapter one, six from chapter two, and three from chapter six. The three lessons from chapter six had a total of 19 statements related to RP while the two from chapter 1 had 12. These two chapters, therefore, had the highest average number of RP statements per lesson, that is, 6.33 and 6 respectively. Chapter 2 had the least number of RP statements averaging at 4.5 per lesson.

Table 1. Frequency of Reasoning-and-Proving statements in text exposition.

Chapter	Number of exposition lessons analyzed	Number of RP statements	Average RP statements per lesson
Chapter 1	2	12	6
Chapter 2	6	27	4.5
Chapter 6	3	19	6.33

A notable finding from the textbook exposition section was that most of the RP statements appeared as theorems, postulates, properties, or generalizations following an exploration of a few specific cases. An example of an RP statement from the book was provided when the author was arguing in support of the domain and range of the function $g(x) = \frac{1}{x-2}$:

> "The range is all real numbers that the formula can return as output. It is not possible for $g(x)$ to equal zero since 1 divided by a real number is never zero. All real numbers except zero are possible output values, since all nonzero real numbers have reciprocals". (p. 80)

Since this statement offers an argumentation for a stated claim, it is hoped that readers of the book would provide similar argumentation when solving problems about domain and range. As indicated in the discussion of the framework, providing an argument for why a claim is true or not is one of the RP opportunities. We noticed that although the author offered arguments such as the one above for most examples, the wording of the examples themselves did not explicitly call for the provision of such arguments. Failure to provide such prompts, as will be illustrated later, led us to code some items in the student exercise section as not involving RP when they otherwise would.

Nature of reasoning-and-proving statements in text exposition. We characterized the above RP statements for their nature based on recommendations by Otten et al. (2014). These involved a determination of whether the statements were particular, general, or general with particular instantiation. The results are provided in Table 2.

Table 2 shows that, on overall, general statements were more prevalent in chapters 1 and 2 than either particular or general with particular instantiation statements. There were 9 and 14 general statements in chapter 1 and 2 respectively while there were 2 such statements for each of the chapters. On the contrary, chapter 6 had more particular statements (12) than either general (7) or general with particular instantiation statements (0).

As indicated earlier, most of the general statements in the exposition section appeared as formulas, properties, propositions, generalizations from particular cases, among others. While most of these statements were presented following a logical sequence and deductive reasoning, it is also important to note that readers were not required to take any active role in this process

Table 2. Nature of Reasoning-and-Proving statements in text exposition.

Chapter	No. of general statements	No. of Particular statements	No. of General statements with particular instantiation
Chapter 1	9	2	1
Chapter 2	14	2	11
Chapter 6	7	12	0

other than just reading the text. We see learning in this manner as less cognitively engaging than, say, in cases where readers are asked to answer certain questions as part of the development of these general statements. An example of a general statement that appeared in the exposition section of the book is where the author stated that,

> "In general, we can identify an increasing or decreasing function from its graph as follows: The graph of an increasing function rises when read from left to right. The graph of a decreasing function falls when read from left to right". (p.12)

This general statement was preceded by a few specific cases and then followed by two examples that seemingly aimed at reinforcing the generalization. However, the examples that followed this general statement did not involve any prompt for RP and we coded them as non-RP examples.

Reasoning-and-Proving in Student Exercises Section

Frequency of reasoning-and-proving in student exercises. Each of the lessons from the three chapters had several student exercises at the end. Chapter 2 had the highest number of exercises (419) due to the fact that it had more lessons than the other two chapters combined (5). Chapter 1 and 6 had 115 and 290 student exercises respectively. In total, therefore, we analyzed 824 student exercises.

Table 3. Frequency of Reasoning-and-Proving statements in student exercises.

Chapter	Number of exercises	Number of RP prompts	Number of RP prompts per lesson
Chapter 1	115	8	4
Chapter 2	419	32	5.3
Chapter 6	290	23	7.7

Table 3 shows that, though not a lot, every chapter had some exercises that appeared to involve students in reasoning-and-proving. The percentage of RP prompts out of the total number of exercises was between 7-8% for each of the three chapters. Despite having relatively fewer exercises, chapter 6 had more RP opportunities (7.7 per lesson) than chapter 2 (5.3 per lesson) and chapter 1 (4 per lesson). The increase in the number of opportunities with chapter numbers could be due to the fact that higher chapters often have most definitions and basic concepts already laid out from the introductory chapters. This means that the chapters could then focus more on deeper thinking that involves students in reasoning-and-proving. Indeed, in our analysis, we found more definitions in the text exposition of chapter one than in each of the other two chapters.

It is important to note that, we considered student exercises as RP exercises only if they explicitly indicated so. This means that some of the exercises that we skipped may still elicit RP depending on who is doing them. This happened mostly for questions that did not explicitly ask students to justify or explain their solutions. Although such questions in the exposition section may have been done with complete logical arguments, we assumed that unless students are asked to provide such justifications and arguments, then, they may not do so. For example, an exercise such as “during which time intervals is the function increasing?” (p. 16) was not coded as a RP exercise. However, the exercise “Does the population of the town grow faster between t = 5 and t = 10 years, or between t = 15 and t = 30 years? Explain” (p. 16) was coded as a RP exercise since a non-proof argument is expected from the student. Indeed, several exercises could have been coded as involving reasoning-and-proving if they were modified slightly to involve prompts such as explain, why, how do you know, provide a counter example, among others.

Nature of reasoning-and-proving in student exercises. Table 4 shows the nature of the reasoning-and-proving opportunities in student exercises above. In general, there were more exercises of the particular nature than the other two categories. Exercises of the general with particular instantiation nature were the least prevalent in the exercises.

Table 4. Nature of Reasoning-and-Proving statements in student exercises.

Chapter	Number of general exercises (per lesson)	Number of particular exercises (per lesson)	No. of General with particular instantiation exercises (per lesson)
Chapter 1	1(0.5)	7(3.5)	0(0)
Chapter 2	8 (1.3)	20(3.3)	4(.7)
Chapter 6	9 (3)	5(1.7)	9(3)

Chapter 1 and 2 had the most exercises of the particular nature per lesson (3.5 and 3.3 respectively) while chapter 6 had the least (1.7 per lesson). There were more general exercises per lesson (3) in chapter 6 than there were in chapter 1 and 2 (0.5 and 1.3 respectively). As for general with particular instantiation, chapter 6 had the most (3 per lesson). Chapter 2 had only 0.7 exercises of this nature while chapter 1 had none.

Since particular exercises were the most prevalent, it is worthwhile to provide some detail about such exercises. In reasoning with exercises of a particular nature, students often do not need to generalize their results to all cases. Rather, they do so for just one case or a finite number of cases. Furthermore, it is often easy for students to use the statements provided in the exposition section of the textbook (e.g., theorems, postulates, formulas, among others) in answering exercises of the particular nature. Bearing in mind that students are not required to take any active role in the generation of these statements, one can argue that RP exercises of a particular nature are not often the most fruitful in engaging learners in deeper learning. In reasoning with general or general with particular instantiation exercises, more creativity and a deeper understanding is often required from the students.

Discussion

Before discussing the results of this study, we would like to point out that our analysis focused only on a few chapters of a university precalculus course textbook. Therefore, we are not suggesting that the results are generalizable to the whole textbook or to other situations. However, within the sampled chapters and lessons, we observed trends that could offer valuable insights about the frequency and nature of RP opportunities in the textbook. For example, we discovered that most RP opportunities were in the text exposition section than in the student exercises section. Most RP statements in the textbook exposition were presented in form of theorems, formulas, postulates, properties, or generalizations following consideration of a few specific cases. A similar distribution of RP opportunities was reported by Otten et al. (2014) who investigated high school geometry textbooks. Although the above statements in the text exposition could be designed in a way that requires active participation from students, the way they were presented in the textbook did not require that. It is, therefore, reasonable to assume that while some students could find ways of actively engaging in the generation of such statements, there are still those who would choose to read along. Furthermore, most of the examples following the statements were often followed by particular examples that did not have any prompt for reasoning-and-proving. However, in solving these examples, the author often presented complete and coherent arguments using the statements provided in the text exposition section. Similarly, in the student exercises section, only a few items had such prompts. Given that some exercises and examples had such prompts while others did not, we claim that this situation might be confusing to students in the sense that they may not know when it is necessary to engage in RP actions. In other words, this scenario is against the necessity principle as articulated by Harel and Tall (1991).

In regard to the nature of RP opportunities, student exercises had more opportunities of the particular nature while text exposition had more statements of the general nature. Since statements in the textbook exposition do not crequire active student participation, it is reasonable to conclude that students using the textbook have more opportunity to engage in RP with particular cases than with general ones. Although particular statements are still useful in heling students see the general (Cuoco, Goldenberg, & Mark, 1996; Reid & Knipping, 2010) the insufficiency of opportunities of the general nature, we argue, does not enhance students' ability to reason deductively- a very important skill in most of mathematics. This finding is consistent with other studies such as Harel and Sowder (2007), Herbst and Brach (2006), and Selden and Selden (2003).

A major recommendation that we suggest from this study would be to increase RP opportunities across all sections of the precalculus textbook. There should be a fair distribution of such RP opportunities such that students have sufficient opportunity to engage with statements of *general*, *particular*, and *general with particular instantiation* nature. This is likely to deepen student understanding of precalculus material and hence lead to student success in calculus and, possibly, other mathematics courses offered at the university level.

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An Investigation of a Student's Constructed Meanings for Animations in Construction of a Hypothetical Learning Trajectory

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This study reports results of an investigation into the meanings a precalculus student constructs as she views an animation independently and describes her image of the problem context as it influences her ability to construct meanings in a novel problem context. Using an exploratory teaching interview, the student's initial meanings and image of the problem context shifted once the animation was utilized as a didactic object. An analysis of her meanings and prior research are used to modify a hypothetical learning trajectory for supporting students in their emergence of productive images of novel problem contexts to construct meaningful and appropriate formulas and graphs. The results and the construction of a hypothetical learning trajectory of this study can potentially be useful for a teachers' awareness of students' constructed meanings while designing or selecting mathematical tasks that incorporate applets and animations in her instruction.

Keywords: Hypothetical Learning Trajectories, Quantitative Reasoning, Problem Solving, Applets

Introduction

Prior research has shown students have difficulty with knowing how to approach and productively solve novel application problems (e.g. Carlson & Bloom, 2005; Lesh, 1985; Lester, 1994; Schoenfeld, 2007). Moore and Carlson (2012) illustrated one of the primary issues of students' difficulty in constructing meaningful and appropriate formulas is due to their inability to conceptualize suitable images of the quantities and their relationships within the context of a novel problem. In addition, Carlson & Bloom (2005) expressed that one's initial conceptualization of the quantitative relationships is grounded in how one makes sense of and organizes information from the problem context. The process of writing a formula requires students to represent the covariation between two quantities and to conceptualize variables as a representation of all the values that a quantity can assume (Stalvey & Vidakovic, 2015; Trigueros & Ursini, 2003). Consequently, students have shown difficulty in connecting their reasoning processes for constructing a formula and an appropriate graph (Carlson, Jacob, Coe, Larsen, Hsu, 2002).

As technology improves, instructors are including different mediums of technology in undergraduate mathematics courses that are designed to support students' mathematical learning. Minimal research has shown how applets and animations can be used for improving students' learning, instruction, and performance (Scheiter, Gerjets, & Schuh, 2010; Bérancourt, & Tversky, 2000; Höffler, & Leutner, 2007; Ploetzner, Lippitsch, Galmbacher, Heuer, & Scherrer, 2009). Thompson (2002) stated that the use of diagrams, applets, animations, and other visual aids are most productive for students' learning when utilized as *didactic objects*. For instance, an applet is considered a didactic object once it is used as a tool in which a teacher intends to produce classroom discussions and generate reflective discourse centered around a particular theme or way of thinking. Spector (2016) advocated that the goal of effectively integrating technology, such as an applet, into learning and instruction should maintain a focus on the content and problem-solving activities rather than the technology itself. The idea of effectively integrating and using an applet in the classroom is demonstrated through student's ability to modify or adjust current knowledge to successfully satisfy the requirements of a novel problem (e.g. the quantities and their relationship).

Based on this research, one of the focuses of this study is on the meanings that students develop and how those meanings are formed while viewing an animation. I elaborate on Moore & Carlson's (2012) construct of *image of the problem context* to describe a student's understanding of a problem context as he or she views static images and animations. The driving focus of this study is the awareness of students' constructed meanings that may differ from either that of the teacher's image of what they hoped students to learn or the intended meaning(s) of the designer. Thus, my two research questions were:

RQ1: What meanings do students construct when viewing an animation, independently?

RQ2: How are students' meanings and understandings of a problem affected by watching an animation?

Literature Review and Theoretical Perspective

In order to describe a student's understanding of a problem's context, the theory of *quantitative reasoning* functions as a lens to describe a student's image of the problem context and explain the ways in which students think when engaging in a problem-solving activity. Thompson's (1994, 2011) theory of *quantitative reasoning* describes a *quantity* as a conceived quality of something that is seen as a measurable attribute, either explicitly or implicitly. The notion of conceiving of an appropriate unit of measurement and assigning a numerical value to a quantity is known as the process of *quantification*. A quantity produced by the resultant of conceiving two quantities is called a *quantitative operation*. For instance, the new dimensions of the base of a box after cutting equal sized square cutouts from each corner being additively compared to the original dimensions of a box's base is a notion of performing a quantitative operation. Someone who has conceived three quantities, in which two of the quantities determine the third, is called a *quantitative relationship*. A network of quantitative relationships is referred to as a *quantitative structure.* Essentially, the analysis of a situation into a quantitative structure is better known as *quantitative reasoning*, which describes the initial conceptualizations of quantities and their relationships during the orientation phase. The act of reasoning quantitatively is directly related to the cognitive actions of attempting to construct a mental representation of the problem situation.

The emergence of students' image of the problem's context is related to whether a student can conceive and relate quantities varying. Furthermore, a student's constructed image of the problem's context is known to influence their ability to construct a formula or a graph (Moore & Carlson, 2012). One of the main requirements of creating a meaningful and robust formula or graph is to have a productive image of how two quantities vary simultaneously. The use of *image* is consistent with Carlson et al.'s (2002) notion of an image that consists of the mental pictures, visual representations, meanings, and properties associated with a concept and as the carrier of mental operations. In this study, I conjecture that the emergence of student's image of a problem's context can be supported by questions with a scaffold and the use of an animation as a didactic object. The object only becomes didactic once it is used as a tool in which the teacher can produce classroom discussions and generate reflective discourse around a preferred theme or way of thinking (Cobb, Boufi, McClain, & Whitenack, 1997; Thompson, Philipp, Thompson, & Boyd, 1994; Thompson, 2002).

Prior to selecting an applet and tasks, I considered the student learning goal(s) of the course, the learning activities that can achieve those goals, and the potential thinking and learning that can occur for students throughout the lesson. The process of predicting a learning path that may occur for students' and characterizing the expected tendencies of student learning are actions towards forming a hypothetical learning trajectory (HLT). Simon & Tzur (2004) explains that a teacher's HLT for her students is composed of the teacher's goal for students' learning, the mathematical tasks used to promote student learning, and hypotheses about the process of the students' learning. I hypothesize that act of generating a HLT can be used as a mechanism for selecting and implementing applets that can support students' mathematical learning.

Box Problem Applet & Initial HLT

The Box Problem (BP) activity (Carlson et al., 2018) was created to support students in developing and acquiring the ability to form a mental image of the quantities, their limitations, and their relationship in the situation of constructing a box from a sheet of paper. The activity is based on a real-life situation of cutting 4 equal-sized squares from the corners of an 8.5" by 11" sheet of paper, then folding the sides to form a box. As a student conceives of an image of the sheet of paper forming into various box configurations, he or she must attend to the quantities and develop the ability to imagine the covariation of two or more quantities in the problem situation. For instance, as the length of the size of the cutout increases, then the height of the box increases and the base of the decreases. Based on their conceptualization of the quantities and their relationship, the student must construct a formula and graph that illustrates the relationship between the volume of the box and the length of the equal-sized cutout.

The Box Problem (BP) applet (Figure 1), created in GeoGebra (n.d.) by Grant Sanders and based on the written activity described above, can be a powerful tool for supporting students in reasoning about values of two or more quantities varying simultaneously as one moves the "Cutout Length" slider (i.e. increases the length of cutout) and forms various dimensions of the box can obtain.

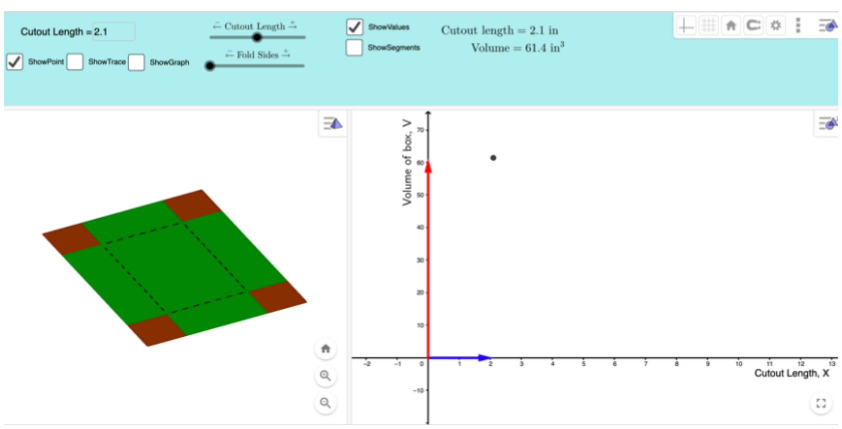

Figure 1. A screenshot of the Box Problem Applet

The following are initial learning goals for a HLT for the Box Problem applet that would support students in developing strategies for making sense of and constructing a productive image of the novel problem context and determine meaningful formulas and graphs:

1. Conceptualize the length of the cutouts, the length and the width of the base, the height and the volume of the box as measurable attributes.
2. Conceptualize the height of the box as equal to the length of the square cutout.

3. Identify the relationship between the variations of the dimensions of the base and the volume of the box with the variation of the length of the cutout.
4. Appropriately conceptualize the dynamic relationship of the dimensions of the box (width, length, height, volume) and the length of the square cutout.
5. Reason about the relationship between the quantities and potentially imagine, then produce a graph that displays the relationship between the volume of the box and the length of the cutout.

Methodology

An undergraduate student was selected from *Pathways Precalculus* (Carlson, Oehrtman, & Moore, 2018) course based on her low grade in the course at the respectable time. The student was videotaped, and all written materials were done on an iPad, then converted to a pdf for analysis. The study was conducted using an exploratory teaching methodology (Castillo-Garsow, 2010), in which the exploratory teaching interview (ETI) consisted of a teaching agent, a single student, and a video camera for recording the single teaching episode. The goal of an ETI is to uncover student's thinking as the student engages with a written mathematical problem or a series of mathematical problems (Castillo-Garsow, 2010).

During each session, I showed the student, Lucie, a static image of the BP applet with four tasks to investigate the meanings she constructed as she viewed the static image and animation independently. I continuously characterized the student's image of the problem in accordance to how productive their image is for successfully achieving the appropriate formula or graph. In correlation with Byerley & Thompson's (2017) definition of "productive mathematical meanings", I define a *productive image* as a mental image that possess a quantitative structure of the problem's context that would be productive for students' long term ability to orient towards and reason through the process of solving mathematical tasks. Towards the end of the interview, I judged the student's image as unproductive when it began to lead them down an unwanted path of reasoning. This action resulted into an intervention where I began to utilize the animation as a didactic object to further support the development of the student's image of the problem's context. The choice to intervene and make suggestions was another way to generate more data about the student's mathematics (Steffe & Thompson, 2000) and enabled the student to potentially complete the tasks easier than before.

Results

The exploratory teaching interview began by displaying a static image of a sheet of paper (Task 1) with the following prompt, "If you start off with a 13″ by 15″ sheet of paper and want to construct a box that is formed by cutting equal-sized squares from each corner of the paper, how do you imagine the box being formed?" Lucie proceeded to describe an image of cutting four square cutouts, then folding the sides to create a box. Then, Lucie viewed the animation and was asked to determine, Does the cutouts of the corner of the sheet of paper have to be square? Or can they be triangles or rectangles?" Lucie describes her initial conception of what quantities comprise the construction of a box to clarify her justification that rectangular, rather than triangular corner cutouts, will produce a box.

Task 2 displayed a static image of various box configurations with the sides of the cutouts labeled *x* with their corresponding values. Lucie was prompted to describe what was shown in the image. Her description entailed her image of the variable x to represent the length of the cutout as a quantity. Lucie stated, "the bigger the cutout length, the smaller the box, but the taller it is”, which indicated she visualized changes in two quantities as the length of the cutout increased. In relation to the height, Lucie stated that the size of the cutouts determined the height of the box. However, in Excerpt 1, her image of constructing the box entailed the variation between the length of the cutout and the base of the box, then the height of the box as a resultant.

Excerpt 1

Interviewer: What do you believe forms the height of the box?

Lucie: Umm.. I believe, I guess I would say the cutouts. The size of the cutouts determines the height of the box.

Interviewer: And, what all do you see is varying in the various images?

Lucie: Um, the cutout to the boxes varying the size, like the area of the box and then the height of the box..

Excerpt 1 displays Lucie’s current image of the box’s quantitative structure did not entail the variable x to represent both quantities, the height of the box and the length of the cutout. After watching the animation that displayed the variation in the different forms of box configurations, Lucie described an image of an input-output table that could be formed from the animation. This description supported her conjecture that the length of the cutout would be the input and the volume as the output, with no mention of the base of the box and the height as determinants of the volume.

In Task 3, the prompt required Lucie to rely on her image of the problem context to construct a formula and graph of the problem description. The third static image and animation of the BP applet displayed colored line segments that represent the original length of the sheet of paper and the variations in the length of the size of the cutout, the height, and base of the box as they vary in tandem. Lucie’s image of the problem continuously changed as she viewed both the static image and animation independently. In tandem, Lucie continued to define (and redefine) her formula that determined the volume of the box given the length of the cutout. Her written work is displayed in Figure 2.

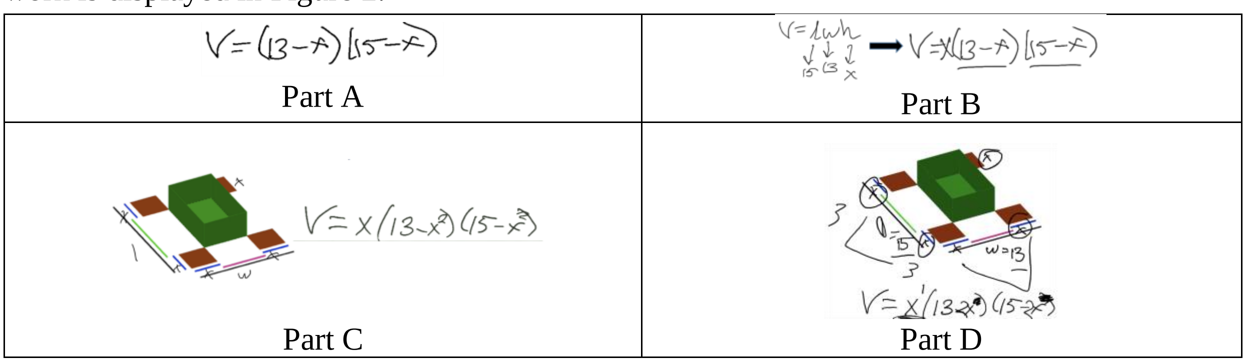

Figure 2. Lucie’s work throughout the interview for defining a formula that represents the relationship between the volume of the box and length of the four equal cutouts.

Part A of the figure shows Lucie’s initial formula for predicting the volume given the length of the side of the cutout. When asked what all quantities determine the volume of the box, Lucie redefined her formula in Part B. From this evidence, Lucie did not conceive of the height as a quantity. Part B further supports the claim; Lucie added on x without referring to x as the height, but as a resultant of folding the sides into a box. After watching the animation, Lucie commented about the four brown squares’ varying lengths that need to be accounted for. She then redefined her formula in (Figure 2, Part C). At that moment Lucie was not advancing her thinking, so I decided that it would be productive to utilize the static images and animations as didactic objects. I then suggested that Lucie determine the volume of the box when $x = 3$. She then focused on how the box configuration changed as the length of the cutout increases. The discussion resulted in Lucie's revised formula (Part D) to accurately represent the quantitative relationships.

Following the construction of a formula, Task 4 prompted her to sketch a graph that relates the volume of the box in terms of the length of the side of the cutout. Lucie stated that she could not draw a graph and she didn't know where to begin. I displayed static images of a graph with various coordinate points as the length of the cutout increases (refer to Figure 1, right panel). Excerpt 2 shows Lucie's response to viewing the static images.

Excerpt 2

Lucie: What I'm trying to figure out is the little black dot on the graph as well. I thought, maybe it was the limit but I'm not sure, I think it's the ...um..(Lucie slides between static images).. I don't know.

This excerpt provides evidence that, at the moment, Lucie had not conceptualized what a coordinate point on a graph represents. However, after watching the animation, that displays the tracing of a graph as the length of the cutout increases, her meaning(s) for what the black dot represented changes. The resulting interactions are provided in Excerpt 3.

Excerpt 3

Lucie: … So, I think the black curved line on the graph represents the whole, like all the possibilities and then maybe the limit at the top, like all the various volumes and the cutout lengths. Then I think that the black dot we've been following up there (slides finger along the graph of the animation) and also with this, (points to the animation) this is where that specific box is, if that makes any sense.

….

Interviewer: So, why do you think it was increasing and then it got to this peak and then it started to decrease? What do you think that happened?

Lucie: So, at first that was like kind of hard to process to relate because that's what I look at is like could be the ball being thrown but kind of describing and talking through like what that little point means and like the limits I think it gets bigger and then smaller I think when the cut out is it looks like about two inches the volume.

Conclusion and Discussion

Several changes occurred in Lucie's image of the problem context and her constructed meanings for the quantities within the situation. Lucie's difficulties in constructing an appropriate formula to determine the volume of the box, given the length of the cutout, were due to her inability to conceptualize how the quantities vary simultaneously. The findings suggest that the use of the animation as a didactic object, in which I created a discussion around the animation, gave Lucie the opportunity to reflect and make connections between her image of the problem context and the symbolic representation in her formula(s). As a result, I conjecture the applet (when used as a didactic object) can be a powerful tool for supporting students in developing the foundational reasoning skills needed for imagining the covariation between two quantities. The results further suggested that the use of the animation can have a positive impact on students' meanings and understanding of a multiplicative object, in which students can conceive graphs as records of simultaneous variation (Thompson, 2002). Lastly, I conjecture that a student's prior knowledge affects their ability to construct productive images of the quantities in an applied context. This can potentially lead to problems when implementing an applet or animation in instruction, as for some students, they are focused more on the technology and how to use it rather than on the quantitative relationships displayed in the applet.

Based on my analysis of the results and prior research, I constructed a HLT, presented in Table 1, that can be used in conjunction of an applet as a didactic object to support students' construction of a productive image of quantitative relationships within the BP activity. The trajectory is comprised of five constructs that are identified as productive images to support students' mathematical reasoning, while engaging in the BP activity and the use of the animation. For each construct, there is a list of learning goals to support students in developing

Constructed Hypothetical Learning Trajectory	
Constructs	Learning Goals
Part 1: Emergence of students' image of a problem's context (Moore & Carlson, 2012)	• Conceiving the length of the cutouts, the length and width of the base, the height and the volume of the box as measurable attributes. • The height of the box possesses a direct relationship with the length of the cutout.
Part 2: Covariational reasoning (Carlson et al.,2002; Thompson & Carlson, 2017)	• As the length of the cutout increases, the height of the box increases; the length and width of the base decreases; and volume of the box initially increases, then decreases at some length of the cutout. • Identified direction of variation in quantities can be used to construct a verbal representation of the process of determining various box configurations and their dimensions.
Part 3: Emergent Symbol Meaning (O'Bryan, 2018)	• Verbal representation can be transformed into symbolic representation by defining variables in correspondence with (1) the verbal phrases and (2) the description of the problem context. • Reflection of evaluating a model for the problem's context can be used for verification of understanding the construction of a meaningful formula.
Part 4: Multiplicative object and multiple representation (Moore & Thompson, 2015)	• A point within a graph is the covariation between two quantities in tandem. • A graph is a collection of quantities' covariation. • The problem context, verbal representation, symbolic representation, evaluation processes, and a point or coordinate point within the graph possess a non-linear connection.

Table 2. The Constructed Hypothetical Learning Trajectory

strategies for making sense of and constructing a productive image of the novel problem context and determine meaningful formulas and graphs. The aim of this study is to contribute to the field of mathematics education's knowledge of students' learning processes and how they can be supported in reasoning quantitatively. A future study involving the implementation of this implicitly based learning trajectory will allow the researcher to investigate a classroom of students' image of a problem context to support students reasoning skills to solve other novel problem contexts.

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Ask Me Once, Ask Me Twice: An Initial Psychometric Analysis of Pre-Service Mathematics Teachers' Responses on a Retrospective Pre-Post Format of the Self-Efficacy to Teach Statistics (SETS-HS) Instrument

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This study reports psychometric evidence for using the high school version of the revised Self-Efficacy to Teach Statistics (SETS-HS) instrument in a retrospective pre-post format with for pre-service mathematics teachers. Analyses at the subscale level indicate adequate internal consistency for "before" and "now" ratings and a high correlation between subscale scores across both ratings. Analyses of individual items' item means, item-subscale total correlations, and the frequency of item mentions in open-ended items are summarized. The retrospective pre-post format yielded SETS-HS scores with some adequate psychometric properties, suggesting a single administration instead of two separate administrations does not degrade the measurements.

Keywords: Pre-Service Mathematics Teachers, Self-Efficacy to Teach Statistics

Within the past decade, the focus on preparation of K-12 mathematics teachers to teach statistics has increased as a result of the inclusion of substantial statistical content in the Common Core State Standards for Mathematics (CCSSM) in grades 6-8 and high school (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010) and other state standards, as well as recommendations for appropriate content and supports for K-12 mathematics teachers provided by the American Statistical Association's *Statistical Education of Teachers* report (Franklin et al., 2015). Relatedly, several findings resulting from a cross-institutional mixed methods study of 200+ pre-service secondary mathematics teachers (PSMTs) (Lovett & Lee, 2017, 2018) demonstrate that current efforts in teacher preparation are not adequate for preparing future teachers to meet the demands of the newer statistics standards. Specifically, approximately 75% of PSMTs did not demonstrate the necessary statistics content knowledge related to K-12 statistics standards, PSMTs felt least confident to teach statistics as compared to other content areas (i.e., algebra, etc.), and PSMTs' self-efficacy to teach statistics was only weakly correlated with their content knowledge.

To meet the demand for high quality materials in order for teacher educators to increase PSMT preparedness and self-efficacy to teach statistics, online modules have been developed by a National Science Foundation-funded project, *Enhancing Statistics Teacher Education with E-Modules (ESTEEM).* The learning opportunities within the *ESTEEM* modules provide critical experiences in pre-service teachers' statistics teaching self-efficacy development. For example, participants engage in statistical investigations, watch video cases of real classrooms, discuss and reflect in forums, and analyze tasks for their potential to promote statistical habits of mind.

Statistics teaching efficacy is an important construct in pre-service mathematics teacher development as teacher beliefs are known to impact their practices in the classroom and, in turn, impact student attitudes, motivation, and achievement (Zee & Koomen, 2016). To date, only one instrument measuring statistics teaching efficacy has been validated with pre-service secondary mathematics teachers in the United States, the high school version of the Self-Efficacy to Teach Statistics (SETS-HS) instrument. However, all validation efforts have focused on a single administration of the instrument, generally towards the end of a teacher preparation program (Harrell-Williams, Sorto, et al., 2014; Harrell-Williams, Lovett, Lee, et al., 2019; Harrell-Williams, Lovett, Lesser, et al., 2019).

However, within the field of program evaluation, the benefits of the retrospective pre-post methodology have been espoused for several decades (Pratt, McGuigan, & Katzev, 2000). Unlike conducting separate pre- and post- measurements at the beginning and conclusion of a program, the retrospective pre-post approach involves a single time-point, simultaneous measurement of both "pre" and "post" data. In addition to the obvious time-saving nature of a single administration, an additional benefit of this approach is that participants have the same frame of reference when answering the items, which has implications for the design of educational intervention studies using the SETS-HS instrument. Hence, this study addresses this gap in the SETS-related literature by reporting psychometric results from the use of the revised SETS-HS in a retrospective pre-post design where pre-service mathematics teachers (PMTs) are asked to rate both their self-efficacy to teach statistics "before" engaging in educational materials focused on statistics pedagogy and "now", after the completion of about two weeks' worth of learning. Specifically, the following research questions were addressed:

1. How well do the subscales perform in terms of internal consistency?
2. How well do the items perform in terms of item means and inter-item correlations?
3. Does the frequency of item number mention in the open-ended items concur with the item means of the Likert-scale items?

Method

Context and Participants

Participants were 187 PMTs from twelve institutions across the United States that offer undergraduate or M.A.T. programs in mathematics teacher preparation covering elementary, middle or secondary grades certification. The number of PMTs per institution ranged from 3 to 26. These institutions are field test sites for the *ESTEEM* project. The sample was mostly female (73.4%), which follows current teacher demographics in the United States (Taie & Goldring, 2017). Demographics regarding ethnicity were not collected.

Participants completed the retrospective SETS survey during 2018-19 after completing the *ESTEEM* foundational module within one of their courses. The module includes 16 to 18 hours of material providing PMTs important experiences with the basics of what it means to teach statistics. The module is organized into two major sections, each one consisting of materials to read and watch, a data investigation using the online tool CODAP, and discussion forums to synthesize and apply participants' learning. The modules are designed to be flexible in that they can be used in a purely online setting but could also be used in a hybrid or face-to-face setting. To maximize the available data, those PMTs who completed items for entire subscales were used, resulting in subscale-specific sample sizes. In addition to 155 participants who completed all three subscales, 23 additional PMTs completed both Level A & B subscales but not Level C, while 9 more completed only the Level A subscale.

Instrument

The revised high school version of the Self-Efficacy to Teach Statistics instrument (SETS-HS) asked the PMTs to indicate their confidence to "teach high school students the skills necessary to successfully complete" 44 statistics tasks using a 6-point Likert scale, with 1 = *not at all confident* and 6 = *completely confident*. Similar to the original SETS-HS, the items were written to mirror the developmental levels (Levels A, B, C) of the *Guidelines for Assessment and Instruction in Statistics Education* (GAISE) *Pre-K-12 Report* (Franklin et al., 2007) and cover most statistical content in the CCSSM for grades 6-8 and the two data analysis strands in high school. Three subscale scores are computed: "Level A - Reading the Data", "Level B - Reading Between the Data", and "Level C - Reading Beyond the Data". Refer to Table 2 for a summary of item content by subscale.

While no factor analysis and reliability results have been published yet on the revised version, the development and validity evidence for the original SETS-HS instrument are documented in Harrell-Williams, Lovett, Lee, et al. (2019) and Harrell-Williams, Lovett, Lesser, et al. (2019). Items in the current version were revised based on (1) psychometric analysis results, (2) feedback from interviews and focus groups consisting of previous/current SETS users, mathematics educators, PSMTs, HS math teachers, and (3) the desire to include more items in the "pose" and "collect" stages in statistical investigation cycle in order to possibly align with stage-based scores on the Levels of Conceptual Understanding in Statistics assessment (LOCUS; Jacobbe, 2015; Jacobbe et al., 2014).

This study used a retrospective pre-post approach for the 44 SETS-HS items. For each item, PMTs reported their confidence to teach each of the statistical concepts, prior to and after their engagement in the *ESTEEM* module. The PMTs also responded to open-ended questions at the end of each subscale to indicate (a) the items they felt most confident to teach and (b) the items they felt least confident to teach, along with their rationale for that ranking.

Analysis

The analyses of the Likert items fall under two categories. At the subscale level, Cronbach's alpha was calculated as reliability estimates for both the "before" and "now" scores and Pearson correlations were estimated between the "before" and "now" scores for each of the three subscales. Item means and item-total correlations were obtained for each item as measures of classical test theory item difficulty and discrimination, respectively. For the analysis of the open-ended items, the item that each student specifically mentioned in their response (by item number or wording) was recorded. Responses that did not explicitly mention a survey item were not included. Frequently, responses mentioned more than one item. The analysis involved creating a frequency distribution for the number of times that each item was mentioned as the hardest item for a PST to endorse with a response of "completely confident", and the number of times that each item was mentioned as the easiest item for a PST to endorse with a response of "completely confident".

Results

Subscale Analyses

The Cronbach's alpha reliability estimates for the subscales for both "before" and "now" ratings were greater than or equal to .93, indicating very little measurement error in the subscale scores (See Table 1). The correlations between a PMT's subscale scores for "before" and "now" ratings were slightly over .60, indicating a strong relationship between the ratings.

Table 1. Subscale-Level Results

Subscale	No. Items	"Before"			"Now"			Corr. Across Ratings
		Alpha	Mean	SD	Alpha	Mean	SD	
Level A	12	.94	36.10	12.14	.93	52.77	9.25	0.61
Level B	15	.96	42.07	16.19	.94	60.62	12.91	0.62
Level C	17	.97	43.80	18.82	.96	62.15	17.96	0.62

Item-Level Analyses

All of the item-total correlations were greater than 0.55, indicating that higher ratings on each item were related to higher scores on the subscales. Item means were generally higher for level A items and lower for level B and C items, with the "now" ratings averaging about a full point higher than "before" ratings. Item standard deviations (SDs) decreased for some items in the "now" ratings, with "before" SDs ranging from 1.2 to 1.5 and "now" SDs ranging from .9 to 1.5.

Table 2. Item Statistics and Frequency of Item Mentions in Open-Ended Responses

Scale	Item & Topic	"Before" Items			"Now" Items			Least Confident Mentions	Most Confident Mentions
		M	SD	*r*	M	SD	*r*		
A	1 Identify data	3.0	1.2	.73	4.4	0.9	.67	2 (2%)	5 (6%)
	2 Collect data	2.9	1.2	.73	4.3	1.0	.69	11(12%)	4 (5%)
	3 Variability sources	2.7	1.2	.67	4.1	1.1	.71	18(20%)	3 (4%)
	4 Choose graphical display	3.0	1.2	.77	4.3	1.0	.71	7 (8%)	3 (4%)
	5 Select statistic	2.7	1.3	.74	4.2	1.0	.74	6 (7%)	2 (2%)
	6 Create graphical display	3.5	1.4	.70	4.8	0.9	.66	3 (3%)	23 (27%)
	7 Describe graphical display	2.8	1.4	.79	4.3	1.1	.72	10(11%)	3 (4%)
	8 Read scatterplot	3.2	1.5	.75	4.5	1.1	.71	7 (8%)	13 (15%)
	9 Variability	3.7	1.4	.70	4.7	1.0	.70	1 (1%)	10 (12%)
	10 Inference limitation	3.0	1.3	.73	4.2	1.0	.69	12(13%)	6 (7%)
	11 Technology to explore	2.9	1.4	.67	4.5	1.1	.69	13(14%)	20 (24%)
	12 Technology to summarize	2.8	1.4	.71	4.4	1.1	.73	18(20%)	17 (20%)
B	13 Question types	2.5	1.2	.68	4.2	1.1	.62	9 (9%)	11 (12%)
	14 Variables & operationalization	2.8	1.2	.77	4.1	1.0	.70	5 (5%)	5 (5%)
	15 Compare statistics	2.7	1.2	.81	4.0	1.1	.73	3 (3%)	1 (1%)
	16 Compare histograms	2.9	1.4	.82	4.1	1.1	.68	8 (8%)	10 (11%)
	17 Compare boxplots	2.8	1.4	.73	4.2	1.2	.64	9 (9%)	23 (25%)
	18 Sampling error	2.7	1.3	.75	4.0	1.1	.68	13(14%)	1 (1%)
	19 Assess linearity	2.7	1.3	.78	3.9	1.2	.71	5 (5%)	7 (8%)
	20 Diff. in distributions	2.9	1.4	.82	4.1	1.1	.69	1 (1%)	9 (10%)
	21 Representative sample	3.2	1.5	.71	4.4	1.1	.63	2 (2%)	14 (15%)
	22 Interpret *r*	2.7	1.4	.75	3.8	1.3	.69	17(18%)	4 (4%)

	23 Study type	2.8	1.5	.80	3.9	1.3	.71	12(13%)	4 (4%)
	24 Corr. vs. causation	2.9	1.4	.78	4.0	1.2	.69	11(12%)	5 (5%)
	25 Sampling variability	2.6	1.2	.84	3.8	1.2	.74	5 (5%)	NM
	26 Random assignment	2.8	1.4	.79	4.0	1.3	.68	10(10%)	8 (9%)
	27 Random selection	3.0	1.5	.78	4.2	1.2	.69	6 (6%)	12 (13%)
	28 Estimate normal prob.	2.4	1.4	.81	3.4	1.4	.72	7(11%)	2 (3%)
	29 Create two-way table	2.3	1.4	.77	3.5	1.5	.74	15(23%)	7 (10%)
	30 Two-way table, relative/marginal dist.	2.1	1.2	.75	3.2	1.5	.79	21(32%)	3 (4%)
	31 Two-way Table, conditional dist.	2.1	1.3	.78	3.3	1.5	.81	16(25%)	2 (3%)
	32 Fit linear model	2.9	1.5	.79	4.1	1.3	.69	2 (3%)	5 (7%)
	33 Fit quad./exp. model	2.6	1.5	.80	3.7	1.4	.76	3 (5%)	2 (3%)
	34 Use residuals for fit	2.4	1.4	.77	3.3	1.5	.74	5 (8%)	1 (1%)
	35 Interpret slope/y-int.	3.3	1.7	.69	4.3	1.3	.57	1 (2%)	20 (29%)
C	36 Evaluate model for simulated data	2.7	1.3	.80	3.7	1.3	.76	3 (5%)	2 (3%)
	37 Randomization	2.9	1.5	.73	4.1	1.2	.66	1 (2%)	10 (15%)
	38 Describe normal dist.	2.8	1.3	.77	4.0	1.2	.70	3 (5%)	10 (15%)
	39 Evaluate conclusions	2.7	1.3	.77	3.8	1.2	.70	2 (3%)	3 (4%)
	40 Point estimate	3.1	1.5	.75	4.1	1.2	.74	NM	5 (7%)
	41 Margin of error	2.3	1.3	.81	3.3	1.5	.82	8(12%)	2 (3%)
	42 Compare treatments	2.5	1.3	.85	3.5	1.3	.81	2 (3%)	1 (1%)
	43 Build physical simulation	2.3	1.3	.82	3.3	1.4	.84	6 (9%)	NM
	44 Technology to run a simulation	2.3	1.4	.74	3.6	1.5	.72	14(22%)	10 (15%)

Note: M = Mean. SD = Standard Deviation. *r* = Item - Subscale Total Correlation. NM = Not mentioned in question responses. Level A: *n* = 187, Level B: *n* =178, Level C: *n* = 155.

Open-Ended Items: Frequency of Mention

The open-ended questions asked the PMTs to think only about the SETS items within each subscale, so the results are presented by subscale. In subscale A, each item was mentioned in both the least and most confident responses. PMTs indicated feeling least confident to teach about sources of variability (#3, 20%) and using technology to explore and summarize data (#11, 14%; #12, 20%). However, two of the technology items were also among the three most frequently mentioned items that PMTs reported feeling the most confident to teach (#11, 24%; #12, 20%), with item 6 (create graphical display) mentioned the most frequently (27%). These results are consistent with the item analysis results, although open-ended results seem to depend on whether PMTs answered based on their initial confidence or after the two-week learning experience. Ratings for item 3 (lowest mean in both the “before” and “now” ratings) and item 6 (second highest mean in the “before” ratings and the highest mean in the “now” ratings) were consistent. However, item 12 moved from the third lowest mean in the “before” ratings to the fifth highest mean in the “now” ratings, and item 11’s mean was relatively low in the “before” ratings but third highest in the “now” ratings. This is likely due to technology use in the learning modules.

In subscale B, all items were mentioned in both the least and most confident responses, except #25 (sampling variability), which was never identified as a most confident topic. PMTs indicated feeling least confident about items 22 (sampling error, 18%) and 18 (correlation, 14%) and most confident about items 17 (boxplots, 25%) and 21 (sample representativeness, 15%). Again, these results are mostly consistent with the item analysis results. The item mean for 22 was third lowest “before” and the lowest “now” rating. Item 18 had a mean that moved from fourth lowest to fifth lowest. Item 21 had the highest mean in both “before” and “now” ratings. Item 17 had a mean around the middle “before” ratings, but increased to the second highest “now” rating.

In subscale C, only 1 of the 17 items was not identified as a least confident topic (#40, point estimates), and only 1 item was not identified as a most confident topic (#43, build physical simulation). In the topics the PMTs identified as being least confident to teach, items about two-way tables (#30, 32%; #31, 25%; #29, 23%) and using technology to run a simulation (#44, 22%) were mentioned the most. PMTs identified item 35 (interpret slope and intercept) as being their most confident topic to teach (29%). In the item analysis, items 30 and 31 had the lowest means in both the “before” and “now” ratings, while item 29 was sixth lowest in the “before” ratings and eighth lowest in the “now” ratings. Item 35 had the highest mean both in the “before” and “now” ratings.

Discussion

The purpose of this paper was to explore initial psychometric evidence for a retrospective pre-post format of the revised SETS-HS instrument. The subscale scores for both the “before” and “now” ratings from retrospective pre-post approach of the SETS-HS exhibited acceptable internal consistency for both the “before” and “now” ratings, suggesting strong inter-item correlations within each subscale. The Cronbach’s alpha estimates (all greater than .90) were similar to those reported in previous studies of the SETS-HS using middle grades and high school PMTs (Harrell-Williams, Sorto, et al., 2014; Harrell-Williams, Lovett, Pierce, et al., 2017; Harrell-Williams, Lovett, Lee, et al., 2019).

At the item level, there was reasonable variability in the item means, both within and across subscales, for both the “before” and “now” ratings. As expected, the item means increased for the “now” ratings as compared to the “before” ratings, but no ceiling effect was observed in “now” ratings. Additionally, item means for subscales B and C were lower than those for A, similar to previous studies using a single-time point measure of “now” ratings (Harrell-Williams, Sorto, et al., 2014; Harrell-Williams, Lovett, Pierce, et al., 2017; Harrell-Williams, Lovett, Lee, et al., 2019).

Based on the results of the analyses conducted here, the scores from responses obtained so far during the *ESTEEM* project have reasonable reliability, difficulty (i.e. item means), and discrimination (i.e. item-score correlation) estimates, indicating that the retrospective pre-post version does not necessarily degrade the psychometric properties of the scores.

Implications

These results have important implications for the use of retrospective forms in research and practice. In ratings prior to exposure to new content (at the start of a course or a new module), PMTs might overestimate their self-efficacy regarding specific topics because they do not anticipate the extent or complexity of the topic. After exposure to strategies for teaching that content, their interpretation of the items might change, possibly leading to indications of no change in efficacy or even a decrease in self-efficacy scores. As previous noted, a benefit of the retrospective pre-post approach is that the PMTs will have the same reference point when responding to the “before” and “now” items. Additionally, this format reduces the amount of time required for collecting baseline PMT data at the start of a class, module, or research project, freeing up valuable instructional time or allowing for other baseline data, such as demographic data or prior knowledge, to be collected in its place without adding to response fatigue.

Acknowledgements

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Students’ Interpretations of the Prompts for Proving Tasks: “Prove” and “Show”

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Proving can be a process for communicating mathematics and people communicating need to be familiar with the language of mathematics. This study focuses on the prompts used for proving tasks not only because the wording of these prompts may impact students’ modes of argumentation, but also because students might bring their preconceptions about these prompts from previous mathematical and non– mathematical experiences. The current study examined how Calculus I students interpret prompts, such as “prove” and “show.” Data was from 131 survey responses and three interviews reveal that these students possibly interpret different prompts in different ways, which can create a gap between students’ interpretations and instructors’ or researchers’ intentions. This suggests that instructors and researchers may need to be more deliberate in choosing such prompts.

Keywords: Proving, Prompts, Language

Proving and proof have been emphasized across K–16 education (Barker et al., 2004; National Council of Teachers of Mathematics, 2000). Among the various roles, some researchers have stressed communication as one important aspect of proof. A proof is a way of communicating mathematical knowledge not only between mathematics researchers but also between teachers and students (de Villiers, 1990; Hanna, 2000). Mathematicians use proof as a way of reporting and distributing mathematical knowledge to society (de Villiers, 1990). Similarly, in certain mathematics classrooms, teachers routinely ask students to prove a mathematical statement, and students respond to the question by attempting to generate a mathematical argument.

For proof to be a way of communication, participants in the dialogue need to be familiar with the discourse surrounding proof (Lew & Mejía–Ramos, 2019). Without knowledge about the specific discourse, it is hard for students to be engaged, and it might affect students’ chance of effective engagement in the mathematics classroom (Lew & Mejía–Ramos, 2019). Hence, students need to learn the discourse, which can be conventions in proving, such as mathematics–specific use of words, grammar, and pattern and formality of argument. There exist, however, few studies focusing in on such conventions in proving. Among the few, Lew and Mejía–Ramos (2019) studied the conventions within written proofs. Based on undergraduates and mathematicians’ responses to the given arguments which had some nonconventional use of mathematics language, the authors found themes of conventions in proof writing.

Although students’ proof responses are important to investigate, we need to examine how we pose questions to students that are intended to elicit mathematical arguments as responses. In this study, we focus on the prompts for proving tasks, which are used in the imperatives of the sentence, such as “prove,” “show,” “explain,” and “convince a classmate.” The following is our research question for the larger study (the complete results of which are not described here): *How do Calculus I students interpret and respond to different prompts, such as “prove,” “show,” “explain,” and “convince a classmate,” for proving tasks?*

Rationale

We had two reasons for choosing to study the prompts for proving. First, the prompts for proving tasks may have an impact on students' modes of argumentation. For example, suppose students are given "show A" where A is a mathematical statement. On the one hand, Alcock (2013) claimed "prove" and "show" are synonymous for mathematicians. On the other hand, Dreyfus (1999) questioned whether the prompt "show" has the implication to produce inductive arguments. Even before students might engage with the mathematical content in a given problem, students need to decide the type of argument or the level of formality of the argument that they need to produce. Hence, if the expectations teachers and students have for a given prompt are different, that inconsistency might negatively impact students' learning of proving (Dawkins & Roh, 2016). Also, it possibly affects the accuracy of student assessments, which is critical for both research and teaching (Morgan, 1998; Schleppegrell, 2007; Weber, 2014).

Our second reason is that students might have some expectations or prebuilt conceptions about each of the prompts. Some of the prompts use words that students commonly use or hear in various non–mathematical contexts. Students might have preconceptions of the words through such experiences, and these preconceptions possibly affect students' understanding of the prompts as part of their mathematics register, which is sometimes referred to as semantic contamination (Pimm, 1987). Mejía–Ramos and Inglis (2011) examined the possible differences in students' responses to "prove" and "proof." The authors presented mathematical arguments and asked questions to students by using either "Is the argument proof of the claim?" or "Does the argument prove the claim?" (Mejía–Ramos & Inglis, 2011, p. 22). Their data revealed that students might consider "proof" to require more formality than "prove." Some students responded that the given argument is "proving" the statement even if the argument is judged not to be a proof. This result aligned with the authors' analysis of using the words "prove" and "proof" from the British National Corpus World Edition, which is a collection of current English language usage. The analysis indicated that "proof" is more used in formal contexts and "prove" is more used in informal contexts. This analysis and the results might mean that students' conception of the prompts might be affected by its use in other contexts as well.

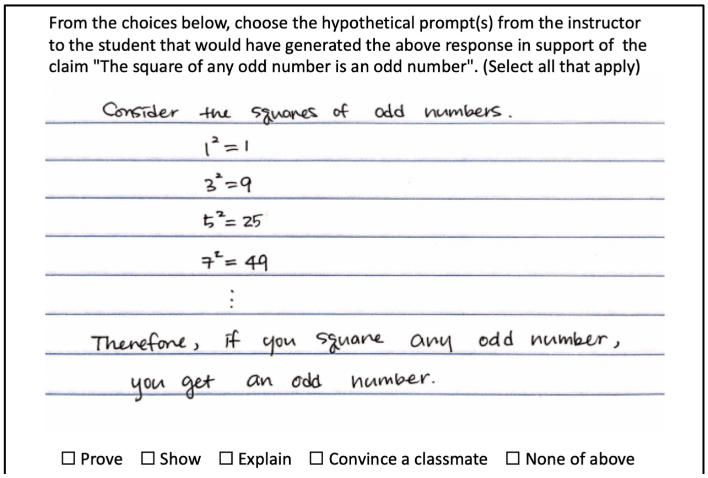
Because students might have come to the proving situation with their previous knowledge about the prompts, and that can possibly affect their learning of proving, there is a need for research about how students interpret and respond to the prompts for proving tasks. In this study, we plan to focus on Calculus I students' interpretations of the prompts because they usually have not experienced a proof–based undergraduate mathematics course. Hence, the population might be appropriate to discuss their previously constructed knowledge from their K–12 mathematics education about the prompts for proving tasks. Therefore, in this study, our objectives were: 1) to describe meanings of each prompt for each participant, 2) to explain connections between prompts for each participant, and 3) to investigate how differently and similarly students interpret the four different prompts.

Our results reveal that students' interpretation of these prompts vary depending on the individual, but these interpretations may not be stable. Taking space constraints into consideration, the current paper reports only on two prompts: "prove" and "show."

Methods

In the reported study, we used surveys and interviews as the main sources of data. The survey consisted of the following: 1) questions asking about participants' mathematical backgrounds; 2) choosing between hypothetical prompts, "prove," "show," "explain," and "convince a classmate," which would have generated given mathematical arguments (see an example in

Figure 1) with optional open–ended asking for the rationale for their choice; and 3) Likert–type questions asking about relationships between the prompts (see Figure 2). We received 265 response from Calculus I students, and we refined our data to 131 responses based on their reliability (e.g., not having the same response to all Likert–type questions). After refining, we analyzed the data by looking for patterns, consistencies and inconsistencies among the "choosing hypothetical prompts" tasks and the Likert–type questions. Based on these patterns, we solicited participants who were found to be exemplars for the type of responses found.


From the choices below, choose the hypothetical prompt(s) from the instructor to the student that would have generated the above response in support of the claim "The square of any odd number is an odd number". (Select all that apply)

Consider the squares of odd numbers.

$1^2=1$

$3^2=9$

$5^2=25$

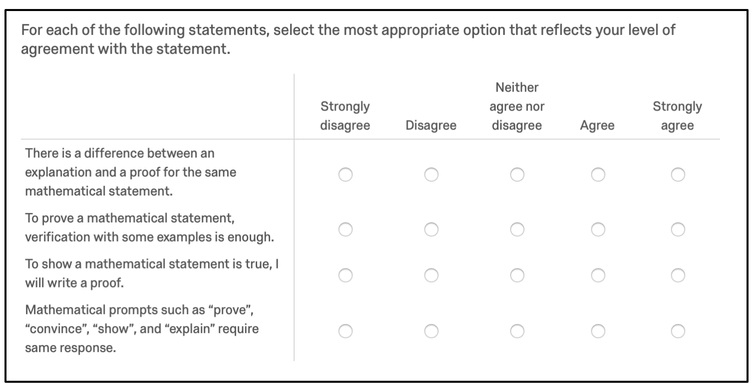
$7^2=49$

$\vdots$

Therefore, if you square any odd number, you get an odd number.

☐ Prove ☐ Show ☐ Explain ☐ Convince a classmate ☐ None of above

Figure 1. An example of a "choosing hypothetical prompts" task.


For each of the following statements, select the most appropriate option that reflects your level of agreement with the statement.

	Strongly disagree	Disagree	Neither agree nor disagree	Agree	Strongly agree
There is a difference between an explanation and a proof for the same mathematical statement.	○	○	○	○	○
To prove a mathematical statement, verification with some examples is enough.	○	○	○	○	○
To show a mathematical statement is true, I will write a proof.	○	○	○	○	○
Mathematical prompts such as "prove", "convince", "show", and "explain" require same response.	○	○	○	○	○

Figure 2. Examples of Likert–type survey items offered to Calculus I students.

We had three participants (Dave, Emma, and Sean; all pseudonyms) that responded to our request for interviews. Before doing interviews, we analyzed their survey responses more deeply and created a narrative for the possible meaning of the prompts and relationships between the prompts for each. During the interview, participants were asked again about their mathematical backgrounds and the "choosing hypothetical prompts" tasks, and in addition we asked about their rationales for choosing and not choosing each prompt. Also, we added a sorting task: participants received six different arguments for the same mathematical statement, and they sorted the arguments to appropriate prompts. Then, for each of the arguments, participants described their rationale for assigning or not assigning the argument to each prompt. The interviews were transcribed and coded by each prompt when participants were mentioning and/or describing their interpretations of the prompts. Based on the coding, we are developing each participant's narrative for each prompt.

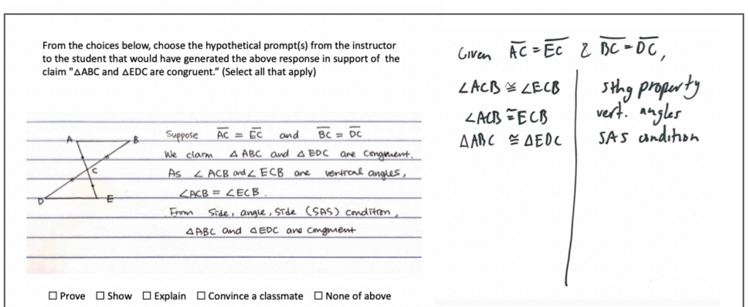
At the beginning of each survey and interview, students were asked to share their mathematical background and major at the university. This was to anticipate their past mathematical proving experiences. With a social constructivist view, students' past experiences offered their learning possibilities of the prompts in the past (Cobb & Yackel, 1996), and researchers investigated possible effects of the context on students' concept of the prompts (Noddings, 1990). For example, Emma, one of the interview participants, referenced her high school mathematics experience when she explained her choice of the prompts during the interview. Hence, we thought it might be helpful to know participants' past mathematics experiences before we started the tasks.

Results

We present the results from the study by focusing on the two prompts: "prove" and "show." Taking the space constraints into consideration, we present a selection of the results about the "prove" prompt, the "show" prompt, and present one possible relationship between the two prompts. For a more complete description of the results of the larger study, please refer to Author (date).

Interpretations of the "Prove" Prompt

Response to "prove" prompt must adhere to a preconceived form. In the survey, only Dave indicated that there should be certain rules that the specific argument generated in response to the "prove" prompt must follow. He wrote, "Prove, one I like because it brings to mind 'proofs', is a very strong indicator to me that I need to write out a response showing that a statement is true by identifying and analyzing theorems." He elaborated on this notion during his interview. In fact, during the interviews, both Dave and Emma claimed that the prompt "prove" requires the generation of an argument that has a series of sequential statements and refers to some mathematical property that satisfies a seemingly arbitrary level of generality. Dave referred such mathematical property as "theory", and it is a necessary part of a response for "prove." On the other hand, Emma's response to the prompt "prove" is more formal and technical. After she rejected S2 as a response to "prove," she suggested a form of two–column proof to describe the kind of argument that she was thinking appropriate for the prompt "prove" (see Figure 3). During the interview, she claimed that "prove" requires formal words such as "given," and "suppose," which she claimed are commonly used by professional mathematicians. Sean also claimed that a response to the prompt "prove" should not include certain less formal words that may detract from its form. For instance, he claimed that the argument prompted by "prove" cannot have the word "you" in it, which implied a more casual situation to him.



S2 from the survey and the interview

Emma's rough response for the prompt "prove" instead of S2

Figure 3. The argument S2 (left–hand side) and Emma's response (right–hand side).

In summary, all three participants seemed to indicate that any response to the "prove" prompt needs to adhere to some preconceived form or rules. We are not completely privy to what the students conceive to be their exact form of or rules for the responses to the "prove" prompt. However, they do provide some glimpses into certain aspects of these forms or rules, and as such, it seems reasonable that the "prove" prompt elicits the deployment of these forms or rules.

Students might not have a consistent notion of proof and the prompt "prove." Our analysis revealed that post–secondary calculus students understandably have an inconsistent notion of proof. As discussed elsewhere, Emma's interpretation of the prompt "prove" seems to indicate that she expects a response that adheres to a strict preconceived form (see Figure 3). However, Sean's response to the prompt "prove" requires multiple arguments from multiple perspectives. For him, it was not enough to use one method to qualify the prompt "prove." In fact, among the nine given arguments, he did not choose one of the arguments as appropriate for the prompt "prove." Further, to construct appropriate arguments for the prompt "prove," he said he would combine three arguments from the provided arguments.

In contrast to the survey results (see Table 1), none of the interview participants agreed on counting an inductive argument as appropriate for the prompt "prove." This result might have suggested that our interview participants have at least some requirements for proof. Dave and Emma claimed that the prompt "prove" requires some level of generality and universality, not just displaying coincidences. Although Sean rejected inductive arguments with few examples as responses to the "prove" prompt, he reported that an argument with "a large enough data set" for the given statement would be appropriate for the prompt "prove." In addition, he rejected the deductive argument S2 (see Figure 3) by claiming that actual measuring of the angles would be needed for the argument if it were to be counted as an appropriate response to the prompt "prove." This indicated that Sean thinks an inductive argument is acceptable as a response to the prompt "prove" in certain cases and not others. This aligns with the survey result that he is one of the 30 students who chose "Neither" for the question "To prove a mathematical statement, verification with some examples is enough" (see Table 1).

Table 1. Survey results for whether an inductive argument is appropriate for "prove" and "show"

		Q14. To prove a mathematical statement, verification with some examples is enough					
		Strongly Disagree (SD)	Disagree (D)	Neither (N)	Agree (A)	Strongly Agree (SA)	Total
Q6. It is not enough to write some example when I asked to show the given statement	SD	1	3	1	2	2	9
	D	4	10	5	20	0	39
	N	0	5	15	12	1	33
	A	4	14	9	16	0	43
	SA	0	2	0	3	2	7
	Total	9	34	30	53	5	131

Interpretations of the "Show" Prompt

"Show" means demonstration. For our interview participants, the prompt "show" might mean demonstration rather than explaining why the given statement is true in detail. Sean claimed an argument prompted by "show" does not come with full sentences or paragraphs, and Emma also claimed the prompt "show" needs less sentences. It seemed the participants' function of having more words in an argument was explanatory in nature, because Emma and Dave's responses implied that the more words an argument has, the less readers of the argument needed to fill the gap between the lines. Emma said that the argument prompted by "show" can leave the most possibility of interpretation to the reader among the four given prompts. She does not expect to write every part of her thinking process in the argument. Similarly, Dave claimed that the prompt "show" asks how the given statement works, but not in a way that describes why nor helps someone to understand the statement. Both Emma and Dave implied that demonstration is a key point of the argument prompted by "show."

"Show" can lead to various form of arguments. Our interview participants' responses to the tasks implied the argument prompted by "show" can include visual arguments and inductive arguments, as well as deductive arguments. They claimed that the argument prompted by "show" could have "straight equations," "calculation," and "technical algebra." In fact, in the sorting task, all three interview participants chose an argument with only algebraic work present without a single word as a response to the "show" prompt. This aligns with the survey results in that some of the survey participants emphasized the importance of number and equations in their written response to the "show" prompt.

In addition, Emma and Dave claimed that a visual argument, which they described as using some visual representations such as graphs or charts, can be counted as a response to "show." Emma claimed that the prompt "show" is the most "visual–slanted" prompt. After she saw an argument that relies on the graphic calculator, she chose that one as appropriate for "show." With that claim, she used analogies from daily life to describe the meaning of the "show" prompt. The following is one such analogy:

> Emma: Okay, I think to me, a graph– It probably makes more sense to say that show is the most visual–slanted of all of these prompts. In everyday language, if someone says 'Show me how you like your shirts to be cut' … I would either bring them an actual shirt and say, 'I want it cut like this,' or I would draw them a diagram and say, 'Okay, here is my measurements' ... Like the architects do, where they have the brackets, and they're like '14 inches across here, 21 across here.'

The above excerpt indicates that Emma thinks a visual argument is an appropriate response for the prompt "show." Dave also chose "show" for the same argument as it is presenting the statement visually. In addition to visual arguments, both Dave and Emma claimed that measuring the angles is appropriate for the prompt "show." Emma did not choose "show" for the deductive argument S2 (see Figure 3). However, she claimed that she would choose "show" if the argument included measuring the angles.

One Possible Interpretation of the relationship between "Prove" and "Show"

Based on that the types of arguments prompted by "show," including deductive arguments, it seemed reasonable to assume that, if one argument satisfies the prompt "prove," then the argument satisfies the prompt "show" as well. Dave claimed that "if you prove something, you've essentially shown it." He claimed that this is because the "prove" prompt requires calculations written in an appropriate order, and to "show" implies that "you just write out your calculations, you have your answer at the bottom and you're done." Emma also mentioned that she cannot think of an example that satisfies "prove" but does not "show." Furthermore, she claimed that "prove" requires the most strict and thorough argument, which in turn also satisfies the ones stemming from the other prompts. However, both Dave and Emma claimed that the other direction is not possible. That is, the arguments generated in response to "show" would not satisfy the requirements to be a response to "prove." Dave stated that the response to the "show" prompt "wouldn't satisfy for a proof". And Emma claimed that "show" and "prove" are "quite close", but "prove" is a bit more demanding of the properties or form required of the response. This implies that a response for "show" may not fulfill the requirements of a response for "prove."

In summary, our data analysis revealed that students have their own interpretation for the "prove" and "show" prompts. For the prompt "prove," participants emphasized certain requirements for the form of the response, although their understanding of "prove" might not be fully developed yet. Students indicated that the prompt "show" can have various interpretations, with seemingly the common goal of demonstration. That is, the argument prompted by "show" can either include inductive reasoning, rely on a diagram, and/or reason deductively. Also, the responses for "prove" seem to automatically satisfy the "show" prompt, but not vice–versa.

Discussion

Our research demonstrates the possibility of different interpretations of different prompts, especially for the prompts "prove" and "show." The results were inconsistent with the commonly accepted notion that these prompts are synonymous for professional mathematicians (Alcock, 2013). Thus, asking students to "prove" and/or "show" without considering their possible differences in meaning for students can have negative impacts on accuracy of assessment for teachers, and on interpretations of research data for researchers since the desired intent for a task and the implementation of the task by students can be different. This is similar to situations that happen when designing classroom tasks or curriculum (Remillard, 2005; Stein, Grover, & Henningsen, 1996). Hence, choosing prompts more deliberately or having more description about the intent of the task might help preciseness of the response from the students. Therefore, we claim that researchers and teachers may need to analyze carefully how research or teaching tasks are implemented and whether they align with their intended interpretation.

There are certainly values to our current study and future directions to pursue, but also some limitations that exist. Some of our participants seemed to be developing their interpretations of the prompts during the interview and their involvement in our study itself might have implied there exists certain differences between the prompts for the participants. In addition, we do not completely know whether students are actually influenced by the prompt. In fact, Emma claimed that she usually does not consider the meaning of prompt, and if so, it depends on how careful she is being in the moment. However, we posit that if the participants developed interpretations of the prompts during our study, it is also possible that they develop such interpretations while interacting with other non–research related tasks. We do not claim that students are always careful about the prompts and have static interpretations of the prompts. Rather, we claim that there exists the possibility that students' interpretations of the tasks can influence their responses.

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Future Teachers' Use of Multiplication and Division to Formulate Linear Equations

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We report results from administering a paper-and-pencil survey to 56 future middle grades and secondary grades mathematics teachers enrolled in teacher preparation programs at two large public universities in the Southeast. We intentionally designed one task on the survey so that it could be used with the variable-parts perspective on proportional relationships. The task asked teachers to show relationships among quantities described in the text by drawing a picture of the situation, writing an equation, and writing a few sentences to explain their equation. Analysis of the responses led to three results: (a) most future teachers represented relationships between parts and wholes appropriately, either by comparing parts to parts or by combining parts to make wholes, (b) future teachers were successful using division as often as multiplication when writing linear equations that combined two different units, and (c) when future teachers made errors using multiplication, reversal errors were the most common.

Keywords: Teacher education, multiplication and division, proportional relationships, linear equations, reversal errors

Background

Strengthening the teaching and learning of algebra remains a central challenge for mathematics education in the United States (e.g., National Mathematics Advisory Panel, 2008; RAND Mathematics Study Panel, 2003; Stevens, Ellis, Blanton, & Brizuela, 2017). Past research has identified several areas of frequent difficulty, including interpretations or meanings for the letter x (e.g., Filloy, Rajano, & Solares, 2010; Kieran, 1992, Küchemann, 1981), interpretations of the equal sign (e.g., Kieran, 1992; Stevens et al, 2017), and inappropriate extrapolation of rules for symbolic manipulation (e.g., Matz, 1982). Results from the present report contribute to research on a further area of difficulty often referred to as the reversal error (e.g., Clement, 1982; Soniera, González-Calero, & Arnau, 2018).

Clement (1982) reported results from administering a written test to 150 college freshman engineering students. One of the test items asked the students to: "Write an equation using the variables S and P to represent the following statement: 'There are six times as many students as professors at this university.'" Only 63% of the students answered with the correct equation $S = 6P$, and the most common incorrect answer was $6S = P$. Clement examined two possible sources for this reversal of the normative location for the letters S and P. For the first source, termed *word order matching*, students simply follow the word order when writing terms left to right. They read the phrase "six times as many students" and write $6S$; and, they continue by reading the phrase "as professors" and completing their equation, $6S = P$. For the second source, termed *static comparison*, students attend to the relative sizes of the two groups and use the equation $6S = P$ to express a correspondence or co-occurrence of the group of students and the group of professors. Clement also presented sample protocols in which students wrote both $6S = P$ and $6P = S$ back-to-back but then settled on the former. These results suggest that challenges expressing relationships among quantities with an equation can be deep seated.

More recently, Soniera et al. (2018) reported two experiments with future primary teachers in Spain in which they examined *word order matching* and *static comparison* as sources for reversal errors. In the first experiment, the researchers found no statistically significant difference in the incidence of reversal errors in two groups, where one group was asked to build equations using names for quantities that referred only to objects (i.e., professors and students) and a second group was asked to build equations using names that referred to numbers of objects (i.e., number of professors and number of students). Thus, more precise meanings for literal symbols, alone, did not result in statistically significant differences in the rate of reversal errors. In the second experiment, the researchers did report statistically significant differences depending on whether the sentence structure supported using *word order matching* to arrive at a correct equation or a reversal error. Similar to the original students-professors problems, all the Spanish-language descriptions used constructions indicating multiplicative comparisons.

When multiplicative comparison language is included in the stem of a task, as was the case with tasks used by Clement (1982) and by Soniera et al. (2018), it is hard to determine whether such language led students to make reversal errors that they might not make otherwise. In the present study, we examined the equations that 56 future middle and secondary teachers in United States wrote in response to a task that described a situation *without* any multiplicative comparison language. The task, described in greater detail below, asked future teachers to make a math drawing of the described situation, write an equation, and provide an explanation for the equation. Our research question was the following: What can we say about the range of ways that future teachers coordinate quantities when writing equations, correct or incorrect, in response to this task? As we report below, reversal errors still dominated the incorrect equations.

The Present Study

The present report is from an on-going study in which we are investigating how a coherent approach to topics related to multiplication can help future teachers see a common structure that connects a range of topics in the multiplicative conceptual field (Vergnaud, 1983, 1988). Such topics include multiplication and division with whole numbers and with fractions, proportional relationships, and linear equations. We express that structure through the annotated multiplication equation shown in Figure 1. The most important ideas are intertwining multiplication with measurement and viewing multiplication as arising when a given quantity is simultaneously measured in two different units, which we refer to as base units and groups.

N	$\bullet$	M	$=$	P
How many base units make one group?		How many equal groups make the product amount?		How many base units make the product amount?

Figure 1. An annotated meaning for multiplication.

In past reports, we have given theoretical presentations of multiplication as a coherent operation that organizes topics in the multiplicative conceptual field and have identified several consequences of systematically analyzing problem situations using a consistent meaning of multiplication (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019). We have also reported empirical results from content courses taught in which future teachers leveraged our structural approach to multiplication to generate and explain multiple, sound methods for dividing by fractions (Izsák, Kulow, Stevenson, Ölmez, & Beckmann, 2019), solving proportions (Beckmann, Izsák, Ölmez, 2015; Izsák et al., 2019), and generating and explaining equations (Beckmann & Kulow, 2017). The present report extends our prior results by examining capacities that future teachers bring, prior to instruction in the measurement meaning of multiplication summarized in Figure 1, for reasoning about proportional relationships and equations that model problem situations.

In the rest of this section, we explain enough about our approach to multiplication and equations to motivate the design of the task at the center of this report. Our discussion also sets up key results that we present later. We begin by pointing out, using the meaning for multiplication summarized in Figure 1, that there are two ways to form a proportional relationship. First, one can fix N and let X and Y co-vary in the equation $N \cdot X = Y$. This perspective on proportional relationships, which we refer to as *multiple batches*, is well-represented in discussions of unit rates (e.g., Kaput & West, 1994) and composed unit reasoning (e.g., Lobato & Ellis, 2010). Second, one can fix M and let X and Y co-vary in the equation $X \cdot M = Y$. This perspective on proportional relationships we refer to as *variable parts*. Elsewhere (Beckmann & Izsák, 2015) we have argued that the variable-parts perspective has been largely overlooked but holds promise for understanding connections between proportional relationships and geometric similarity, including slope of a line. Because we see promise in developing the variable-parts perspective, we include it in the content courses we teach for future teachers.

In the present report, we examine capacities of future teachers to reason about the Sand and Gravel task (see Figure 2 and 3), a task we designed to afford opportunities to reason about proportional relationships and equations from the variable-parts perspective. In particular, we examine teachers' performance before instruction in multiplication or variable parts.

Consider this information about a fleet of trucks carrying sand and gravel:

- There are nine trucks, each carrying the same weight.
- Four of those trucks are filled with sand; Five are filled with gravel.
- All together, the four sand trucks are carrying a total of X tons.
- All together, the five gravel trucks are carrying a total of Y tons.

a. Make a math drawing to show the relationship between the quantities of sand and gravel. Indicate where X and Y are in your drawing.
b. Write a few sentences to explain how to develop an equation relating X and Y, reasoning about the quantities of sand and gravel. Interpret the equal sign in your equation.

Figure 2. Parts (a) and (b) of a variable-parts task for generating equations.

In this situation, each truck is a variable part. The first key feature of the variable-parts perspective is that each truck carries the same weight. The second key feature of the variable-parts perspective is that number of trucks remains fixed, but the weight of each truck varies depending on the total tons of sand and gravel.

We present two different solutions the Sand and Gravel task that use variable-parts reasoning. In the first solution, one can express the number of tons in one truck, $X/4$ tons for one truck of sand and $Y/5$ tons for one truck of gravel. The constraint that each truck carries the same weight justifies the equation $X/4 = Y/5$. In the second solution, one can compare the number of trucks carrying sand to the number of trucks carrying gravel. In particular, 5/4 of the number of trucks carrying sand is the same as the number of trucks carrying gravel. Therefore, 5/4 of the weight of sand is the same as the weight of gravel. This perspective leads to a second equation, $X \cdot 5/4 = Y$, following the meaning of multiplication shown in Figure 1.

We designed parts (a) and (b) of the task (Figure 2) to simulate initial portions of semi-structured cognitive interviews, which we have conducted with smaller numbers of future teachers using similar tasks. Part (a) mimics a common practice we use during interviews and class—draw a picture. A drawing also serves as a check on reading comprehension. Part (b) mimics a typical follow-up question in which we ask future teachers to explain their reasoning.

We designed parts (c) and (d) of the task (Figure 3) so see if future teachers could interpret the constant of proportionality, 5/4, from a variable-parts perspective and then use that interpretation to formulate an equation. Part (c) asks future teachers to interpret the constant of proportionality as the measure of one total quantity (e.g., tons of gravel) in term of another total quantity (e.g., tons of sand). Part (d) asks future teachers to use the measurement 5/4 that relates the total tons of sand and total tons of gravel to formulate an equation like $X \cdot 5/4 = Y$.

c. If 4 sand trucks make one load, how many loads make the total weight of 5 gravel trucks exactly?
d. Using your answer to part (c), write a few sentences to explain how to develop an equation relating *X* (tons of sand all together in 4 trucks) and *Y* (tons of gravel all together in 5 trucks) by reasoning about the weights of sand and gravel. Interpret the equal sign in your equation.

Figure 3. Parts (c) and (d) of a variable-parts task for generating equations.

Methods

We administered a paper-and-pencil survey to 56 future middle and secondary mathematics teachers enrolled in preparation programs at two large public universities in the Southeast. We recruited the future teachers using an approved IRB protocol and paid each participant \$50. The survey was treated as a homework assignment. We distributed copies of the survey in class and collected them 1 week later.

The full survey consisted of six items about proportional relationships and equations, including the Sand and Gravel task shown in Figure 2 and Figure 3. To analyze responses to parts (a) and (b) (Figure 2), for each of the 56 responses, we examined the future teachers' drawings, their equations for the situation, and their written explanations for their equations. We examined these responses to infer the future teachers' interpretations of the described situation and how they made sense of their work, including when it was non-normative. In some cases, we could see from drawings that a future teacher assigned *X* and/or *Y* to be the weight of just one truck, which is at odds with statements in the task. In other cases, future teachers wrote that they were using the equal-sign to indicate the co-occurrence of weights of sand of and gravel.

The first author used a bottom-up approach to coding the future teachers' responses. He used iterative cycles in which he first noted similarities across different responses, generated initial categories, and then refined those categories through additional cycles. The categories were based on expert judgement about which quantities future teachers used, how they related those quantities in equations, and how they explained their equations. The second author then took those categories as the basis for her own, independent coding of the data. The authors resolved discrepancies through discussion and reevaluation of student work.

To analyze each of the 56 participants' responses to parts (c) and (d) (see Figure 3), we examined whether the participant formulated a new equation or stated a new relationship in words in part (d), or explained an equation or relationship differently than in part (b).

Results

Equations and Interpretations Formulated in Parts (a) and (b)

Table 1 summarizes the categories into which we placed responses to parts (a) and (b) of the Sand and Gravel task, provides example equations for each category, and lists counts for each category. We make several points. First, 23 of the 56 teachers (41%) both wrote correct equations where the number and type of unit on the left hand and right hand sides of the equal sign were the same and justified those equations using relationships in the situation. Second, when future teachers wrote correct equations in two units (e.g., trucks and tons), about half the time they did so using division to equate the weight of one truck of sand with the weight of one truck of gravel (i.e., X/4 = Y/5). A about one quarter of the correct equations were based on using 4/5 as a multiplier operating on tons of gravel or 5/4 as a multiplier operating on tons of sand. These are the two solutions explained above. Third, in six cases (11%), future teachers wrote correct equations in just one unit. In equations of the form $X + Y = 9k$, future teachers explained that k was the weight of one truck and so expressed all terms in tons. In the equation 4/9 + 5/9 = 9/9, all the terms were expressed in parts, where one part corresponded to one truck.

Fourth, of the remaining 33 future teachers, six (11%) provided no equation and six (11%) gave explanations for their equations that seemed based on rote translation from text to equation without consideration for particular relationships between trucks and tons. For example, one teacher wrote $4/5 = X/Y$ and explained that he or she was "matching like terms on top and bottom." Another teacher wrote $4/X = 5/Y$ and explained "X and Y with 4 and 5 are proportional." Fifth, the most frequent error (13 cases, 23%) was the reversal error discussed above. Seven future teachers wrote $4X = 5Y$, when $4Y = 5X$ would have been correct, and six incorporated similar reversal errors when adding parts of sand and gravel to make a whole.

Finally, nearly all future teachers represented relationships between parts and parts or parts and wholes appropriately. This held even when they mixed units, as in the equation $X + 1 = Y$, in which a future teacher added X tons of sand and one truck of sand to arrive at Y tons of gravel. The future teachers never equated an expression for just the trucks of sand, or the trucks of gravel, with an expression for the trucks of sand and gravel combined.

These results demonstrate that nearly all the future teachers engaged in reasonable sense making about parts and wholes when coordinating equations with the problem situation, that some were more successful than others when combining the units of tons and trucks, and that combining the units of tons and trucks using multiplication was particularly challenging.

Additional Equations and Interpretations Formulated in Parts (c) and (d)

There were 29 responses to parts (c) and (d) (see Figure 3) in which future teachers produced either an equation or a description of the relationship between X and Y that was different from their response to part (b). Of these responses, 22 were correct and seven were incorrect.

Correct responses. We classified the 22 correct responses into three categories. In first category, six future teachers wrote $5/4\ X = Y$ or $Y = 5/4\ X$. All but one response seemed to draw on the interpretation of 5/4 from part (c) either to develop an equation or to interpret an equation that was derived through algebraic manipulation. Thus, the prompt in part (c) to consider the four trucks of sand as a unit led to interpreting 5/4 as a single multiplier, appropriately.

In the second category, 10 future teachers developed an equation other than $5/4\ X = Y$ or $Y = 5/4\ X$ and oftentimes used 5/4 in a way other than as a single multiplier. One future teacher stated: "If, we look at X as the whole since 4 trucks make 1 load, then Y would be 5/4 loads" and wrote $X/4 \bullet 5 : Y$. Another future teacher seemed to interpret 5/4 as an invariant ratio expressed both in trucks and in tons when writing 5/4 = *Y*/*X* and explaining: “If I’m using 4 trucks to load an equivalent of a 5 truck load and the proportion is 5/4 (gravel to sand), then it’ll equal the same proportion for the amount of sand and gravel.”

In the third category, six future teachers described the relationship between *X* and *Y* in ways that suggested that they did some additional productive thinking after responding to part (b). For example, one future teacher who had previously produced 4/5 *Y* = *X* and 5/4 *X* = *Y* noted the reciprocal relationship: “I am comparing 4 truck loads (of equal wt per truck) and 5 truck loads. They will remain reciprocals of each other.”

Responses with incorrect equations. Seven responses in part (d) included an equation or expressed a relationship that was normatively incorrect. Of these, five included an equation with a reversal error involving the fraction 5/4 or 4/5. As one example, a future teacher wrote the equation 5/4 *Y* = *X* and explained: “In part C, I found that 5 gravel trucks makes 5/4 loads. So since *Y* represents tons of gravel all together in 5 trucks, then I came up with 5/4 *Y*.” Perhaps this future teacher viewed 5/4 *Y* as expressing that 5/4 corresponds with or makes *Y*. Another future teacher, who gave the answer 4/5 in part (c), produced the equation *Y* = 4/5 *X* and explained: “A load of *X* is 4 trucks and *Y* has 5 trucks so a load of *Y* would be 4/5 of *X*.” Because this future teacher was not confused about the numbers of trucks corresponding with *X* and *Y*, perhaps he or she meant that *Y* corresponds with *X*, which consists of 4/5 as much.

Discussion and Conclusion

This study suggests that the field still does not have adequate understandings of difficulties that college students face generating linear equations and, therefore, does not have the knowledge base to support them in coordinating such equations and problem situations more successfully. We would like to understand underlying causes that result in less than half the future teachers in our sample generating and explaining equations for a situation like that presented in the Sand and Gravel task. Previous studies that have reported reversal errors (e.g., Clement, 1982; Soniera et al., 2018) have relied on tasks that included the words “times,” “times as much,” or phrases of the form “for every __ there are __,” which could lead participants towards errors based on word order matching or static comparison. In contrast, the Sand and Gravel task did not use any such language. Because we presented a bulleted list to highlight the different quantities separately, and asked future teachers to make math drawings before generating equations, our results about reversals (that include fractions not mentioned in the problem statement) suggest that word order matching or static comparison are incomplete explanations for such errors. While acknowledging that future teachers sometimes use non-normative but sensible interpretations of notations, we suspect that interactions among understandings of multiplication as a model of problem situations and meanings for the equal sign contributed to the reversal errors we observed.

We designed the Sand and Gravel task to foster variable-parts reasoning as an avenue into developing and explaining linear equations in a standard $y = \mathrm{m}x$ form by interpreting the constant of proportionality, or rate, as the answer to a measurement question that relates total quantities. That only a few of the future teachers did so, even after being prompted with a measurement question in parts (c) and (d), suggests that developing this aspect of the variable-parts perspective will require focused instruction. On an optimistic note, even without instruction, the measurement question in part (c) and the request to use its answer to develop an equation in part (d) did seem to lead to some potentially productive ideas about how to relate the variables *X* and *Y*, even if they did not always lead to correct equations.

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Table 1. Future Teachers' Initial Equations for the Sand and Gravel Task.

Correct Equations in Two Units		
Category	Examples	Freq. (N = 56)
Equating Weight of 1 Truck	$X/4 = Y/5$	9
Equating Weight of all Trucks Other	$4/5 \bullet Y = X$ $5/4 \bullet X = Y$	4
Other	$X/4 \bullet 5 = Y$	4
Correct Equations in One Unit		
Category	Examples	Freq.
Equations in Tons	$X + Y = 9k$	4
Equations in Trucks	$4/9 + 5/9 = 9/9$	2
Reversal Errors		
Category	Examples	Freq.
Comparing Part-to-Part	$4X = 5Y$	7
Adding Parts to a Make Whole	$4X + 5Y = Z$ $4X + 5Y = 9$	6
No Response or Rote Matching		
Category	Examples	Freq.
No Equation	-	6
Rote Matching	$4/5 = X/Y$ $4/X = 5/Y$	6
Other		
Category	Examples	Freq.
Comparing Part-to-Part	$4/9 = X$ tons $5/9 = Y$ tons	2
Adding Parts to Make Whole	$X + 1 = Y$ $X + Y = 9$	2
Misc.	$X = Y$ X/Y $9(X + \mathrm{Y}) = \mathrm{Z}$	4
Totals		56

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Exploring the Genetic Decomposition of Interior and Exterior Angles of Polygons with the Use of Computer Programming and GeoGebra

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Pre-service mathematics teachers need to have an advanced perspective of geometry concepts and be able to extend and generalize the geometry content they will teach. This study documents the continued refinement of a geometry lesson using computer programming and GeoGebra to iteratively teach generalization over the concept of interior and exterior angles in polygons and, ultimately, helps to refine a previously developed genetic decomposition for continued improvement of this and future lessons.

Keywords: Preservice Teachers, Polygons, APOS, Genetic Decomposition

In order for preservice teachers to be successful in their mathematics teaching career, they must master the subjects they will be teaching. Shulman (1987) asserts that "teaching necessarily begins with a teacher's understanding of what is to be learned and how it is to be taught" (p. 7). Teachers must be able to provide multiple representations and alternative explanations of the same concepts. Jones (2000) simply stated that effective geometry teachers must know "a good deal of geometry" (p. 109). However, as preservice teachers advance in their training, the amount of geometry being taught decreases, sometimes disappearing completely at the university level. Consequently, teachers are often expected to teach geometry at both the elementary and secondary levels when, in fact, they have done very little with geometry since they were in secondary school (Jones, 2000). Ward (2004) observed that preservice teachers did not recall important high school concepts before a lesson, but could apply formulas after a lesson on a concept while still holding on to contradictory concept images. Usiskin et al. (2003) hold preservice teachers need an "advanced perspective" on their geometry concepts in order to teach for understanding. They assert teachers need to know how to extend and generalize the content they teach. Dindyal's (2007) study of high school students' conception of the sum of the interior and exterior angles of a polygon found that students could recognize patterns, but encountered conceptual obstacles to generalization.

APOS theory is a framework for investigating how a mathematics learner acquires and uses knowledge about a specific mathematical concept, where a learner must have appropriate mental structures in order to acquire a specific mathematical concept (Dubinsky & McDonald, 2001). The degree to which the learner has constructed the required mental structures needed for the mathematical idea is described by four levels in APOS theory: Action, Process, Object, and Schema (Asiala et al., 1996). Actions are interiorized as Processes, and Processes are encapsulated into mental Objects. APOS practitioners do not believe the progression of a learner is linear through the constructs, rather learners may move back and forth as well as hold positions between levels. A genetic decomposition (GD) is critical in APOS theory to understand how students conceptualize a mathematical idea (Arnon et al., 2014). The GD is a hypothetical model of mental constructions required by the learner for a given mathematical concept, and it details the types of behaviors a student might exhibit at the Action, Process, Object, or Schema levels. This type of detail informs the pedagogy about a concept and provides researchers clarity in determining where students are on the APOS scale over a concept.

APOS theory also states that computer programming activities can act as a catalyst in developing the associated constructions of mathematical concepts (Dubinsky & Tall, 1991). In 2011, a team of researchers (Jenkins et al., 2012; Stenger et al., 2017) developed an instructional model (IM) that uses such computer programming activities to promote mathematical reasoning and explicitly teach generalization and abstraction. The computer programming exercises are designed to stimulate the development of mental constructions that support the transfer of learning in a mathematics context, specifically abstraction and generalization. The approach is modeled on APOS theory and constructivist pedagogy and uses computer programming as a laboratory for the learner to explore the behavior for a specific concept under investigation. This IM exploits iterative thinking to encourage learners to generalize and abstract.

The GD for this project comes from a pilot study (Jackson, Stenger, Jerkins, & Terwilliger, 2019) and was developed for the study of interior and exterior angles of regular polygons. It begins with the behaviors a learner may exhibit at the pre-Action level and then delineates the Action, Process, and Object level indicators. Employing the instructional model developed by Jenkins et al. (2012) as a guide, the researchers then developed this study's instruction using this foundation and utilizing two iterative approaches designed to parallel the construction of this geometry concept. This report details the investigation of the proposed genetic decomposition.

Methodology

We applied an iterative-based IM (Jenkins et al., 2012) to the concept of interior and exterior angles in polygons to a geometry content class for elementary education majors (N=12, all female). The class was randomly split into two groups of six students each. The lesson was identical for each group up until the computer programming stage of the IM. Before enrolling in the course, all of the students had taken a prerequisite mathematics class, either Pre-Calculus Algebra or Finite Math. All participants completed a pre-test before any instruction and a post-test once the instruction was complete. The course final exam also assessed students on these same topics. At various points during their instruction, the students completed response sheets about specific topics or events just covered. These motivating questions were created for participants to answer during their exploration. Special attention was paid to the form of statements in the classroom activities so that general expressions were a natural outcome of the exercise. In the lesson development, conjectures and proofs were created using the general expressions. Learners were guided into making conjectures during the lesson and, using the general expression they discovered, made convincing arguments about their conjectures.

To begin the lesson, a pre-test over the concept was taken by all learners. Questions asked students to calculate the measures of the exterior and interior angles of a regular octagon and to write a convincing argument for their answers. A final question which included an aerial view of the Pentagon (that we have labeled the "Pentagon problem") reads as follows:

> At the Pentagon, a courtyard is enclosed by the interior of the inmost wall. A sidewalk runs along the exterior of this courtyard. When it is cleaned (by power-washing), the machine can cover the entire width of the sidewalk in one pass. However, it must make turns at each angle, and the design of the machine makes turning more expensive. Outside of the normal cost of cleaning the entire sidewalk, an extra "turning expense" is

always added to the final bill. If $10 is charged for each **degree** of turn needed (not each turn, but each **degree** of turn), how much extra can the Department of Defense expect to be added to their bill as a "turning expense"? Would the turning cost be less if the building were shaped like a rectangle instead of a pentagon? Explain your answer.

Next, relevant background and definitions were presented by the facilitator. A brief lecture on the definition and characteristics of a polygon began the instructional process. Examples of polygons were shown and on each, the interior and exterior angles were highlighted and defined. An emphasis was then placed on exterior angles, and each was highlighted on an image of a hexagon projected on a screen. At this time, the groups were divided based on the type of computer programming used for instruction. During the computer programming stage, one group used Python programming and the other used GeoGebra, which is a web-based application for teaching and learning mathematics and science.

The Python group began with a brief introduction to the language and then learners wrote programs to produce tables depicting the measure of exterior and interior angles in polygons, iterating the number of sides. Participants were asked to add instructions to their programs to show what the measure of a regular polygon's exterior angles would be as the number of sides increased. Learners were encouraged to experiment with their computer programs and make observations about any relationships. Once the initial table was constructed, the participants were ushered through a series of program modifications and written responses, including writing a convincing argument (i.e., proof) that as the number of sides of a polygon increases, the measure of each of its exterior angles decreases. Next, the students were instructed to tinker with their programs in order to produce a third column, one that depicted the measure of interior angles in a regular n-gon. Learners wrote programs to show those measures and were then encouraged to experiment with their programs and make observations about any relationships. The IM was designed so that repetition with various program modifications would stimulate the desire to generalize the observed behavior and make conjectures about the mathematical construct.

In the GeoGebra group, the instruction continued with each participant being given a handout of the same image that had been projected and told to cut out each exterior angle and put them together, allowing them to "discover" that they formed a circle. GeoGebra was then used to show this same concept on various polygons. Next, a more traditional type of lecture was employed about curves and polygons in the plane, convex and concave figures, polygonal curves, triangles, quadrilaterals, and regular polygons. The lecturer then returned to GeoGebra to revisit the specific topic of interior and exterior angles of regular polygons, showing iterations of regular polygons with more and more sides. The students eventually acknowledged that the polygon came to look like a circle. A further discussion was then conducted on the measure of interior and exterior angles of these figures as the number of sides approached infinity. A worksheet was distributed after the instruction which emphasized the classification of triangles, polygon identification, measures of individual interior and exterior angles in a polygon, and the sum of the measures of all interior and exterior angles in a regular polygon.

In both groups, motivating questions were presented periodically throughout the instruction by the facilitator, and participants were guided until the necessary general expressions were uncovered. Finally, a conjecture was made by the facilitator and a convincing argument was presented using a discovered general expression. Learners were then asked to make their own conjectures and convincing arguments. Participants wrote responses to questions and reflections on their observations on response sheets, including generalizations of behavior observed in their activities. Observations on the relationship between the number of sides of a polygon and the measure of its exterior angles were solicited, and participants were taught how to denote the general expressions in mathematical language (e.g., $(180 - 360/n)$). At the conclusion of the lesson, participants took a post-test which contained the exact same questions as the pre-test. On the end-of-semester final exam, the instructor incorporated seven questions particular to the instruction from this study that covered various levels in APOS, as well as the Pentagon problem (from the pre-test and post-test). Also added was a new problem, the "Linear Pair" problem shown in Figure 1, which was designed to aid in assessing the Process level of conception.

24. Shown below is a "close-in" view of the interior and exterior angles of two regular polygons, where they meet at one pair of their adjacent sides. Which polygon, A or B, do you think has the most sides? Explain your answer.

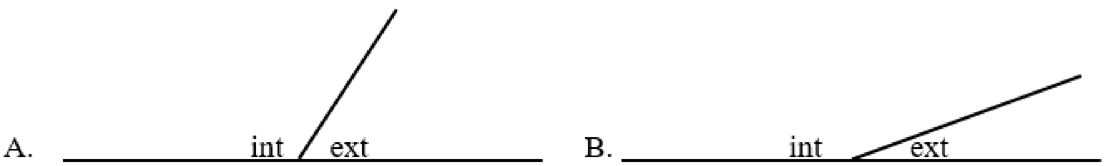

Figure 1: Linear Pair problem

All of the data collected from pre-tests, post-tests, and the final exam was reviewed and scored using APOS theory (Asiala et al., 1996) as a guide. In our study, a ranked set of scores was devised to denote pre-Action (0), Action (1), Process (2), and Object (3) levels based on the genetic decomposition and recorded for each subject's submissions. Three scorers ranked each participant. In the event that authors disagreed, a discussion and further analysis of the data was used to reach consensus. The GD was used as a guide to evaluate responses.

Genetic Decomposition

A genetic decomposition details the types of behaviors a student might exhibit at the Pre-Action, Action, Process, Object, and Schema levels. This type of detail informs the pedagogy of a particular concept. Additionally, it can provide researchers clarity in determining where students are on the APOS scale over a concept. An initial genetic decomposition for interior and exterior angles of polygons was developed in a prior study (Jackson et al., 2019) and does not include Schema. It reads as follows:

Genetic Decomposition for Interior and Exterior Angles of a Polygon

Pre-Action. Students may not know what a polygon is. They may not know what an interior or exterior angle is.

Action. At the action level, the learner has been given the formula for finding the measure of an exterior angle in a regular polygon. Then, the learner can take the number of sides in a polygon, plug it into the formula, and then calculate what that measure is. Additionally, they read or are told that the measure of an interior angle of a regular polygon is (180º - the measure of the exterior angle) and are then able to plug in the previously determined value into this formula to find that measure of this interior angle.

Students, in this case, will not be picturing a triangle in their mind but are only strictly adhering to the given formula. They may not even see the relationship between an interior and exterior angle. Also, they do not notice that as the number of sides of the regular polygon increases, the measure of the exterior angles decreases, and the measure of the interior angles increases. However, if the learner does start to recognize that as the sides of a regular polygon increase in number, the measure of its exterior angles decreases or the measure of its interior angles increases, they are starting to imagine the process in their mind, making a step toward generalizing the relationship between the number of sides in a polygon and the measure of its exterior/interior angles.

Process. At the process level, students begin to imagine in their minds what happens as the number of sides in a regular polygon increases. They start to play that pattern as sort of a video sequence in their mind. They might first imagine an equilateral triangle in their minds and visualize its exterior angle. Next, the students could think of a square and visualize its exterior angle. This process would continue with a regular pentagon, a regular hexagon, a regular heptagon, etc. As they do this, they picture in their minds that the measure of the exterior angle decreases as the number of sides increase. Alternatively, a student might think of the sequence numerically as 360/3, 360/4, 360/5, 360/6, etc., and notice that this numerical calculation of the exterior angle is decreasing each time. In both cases, the student has learned to generalize the behavior of how this relationship of the number of sides in a regular polygon and the measure of its exterior/interior angles is related.

Object. At the object level, they might see the interior/exterior angle relationship as a linear pair, recognizing them as an entity so that one angle does not need a specific measure to know the measure of the other, but that if one angle has a measure of α, then they know that the other has a measure of $(180 - \alpha)$. They might see the process of adding more and more sides as a totality, imagining the sides moving toward an infinite number (and even completing the process), where the polygon is now a circle.

Results

Generally speaking, most of the students made some improvement during the lesson. Eleven of the twelve participants ranked at pre-Action (0) level on the pre-test. Only four participants were still at this level on the post-test. Five students were at Action (1) level, two were at Process (2) level, and one student was at Object (3) level on the post-test.

Minimal Recall of Important Geometry Concepts

The pre-test shows many pre-Action samples and a lack of deep understanding of this concept by most participants. This was seen by many students guessing at measurements of the interior and exterior angles of an octagon. Most students either thought the interior and exterior angles were the same or they reversed the definitions and measures of each.

To illustrate, student LP01 had the same measures and explanations for the interior and exterior angles. “There are 8 sides, and if each exterior angle equals 45, then it will add up to 360.” Regarding the interior angle, she said the exact same thing, only changing the word “exterior” to “interior.” Student LP12 gave an incomplete and incorrect description of both exterior and interior angles. She said that the measure of any exterior angle would be between 90° and 180°, explaining that “all exterior angles of a regular polygon are obtuse angles.” Conversely, she thought that all interior angles had to have a measure between 0° and 90°, but then noted that the interior angle in a regular octagon “will be supplementary to the exterior angle.”

Memorizing Formulas and Calculating Angle Measurements

On the final exam, a question asking for the measure of the interior angle in a regular octagon was answered correctly by *all* participants. On another question asking students to sketch and label the interior and exterior angles of a hexagon, all but one student answered the question correctly. This question reads as follows: "Sketch a regular hexagon, label one set of interior and exterior angles, and determine the measure of an exterior and interior angle. Show your work and explain your answers." One student's (LP06) response is shown in Figure 2.

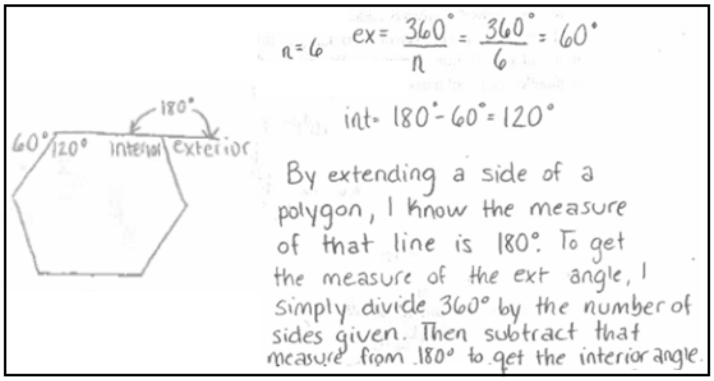

Figure 2: Student response to final exam question

Understanding Linear Pair Relationship

Another question on the final exam asked students to examine the linear pair relationship. Student LP12 stated: "The measure of any interior angle of a regular octagon will be 135º because the exterior and interior angles are a linear pair. Since these angles are a linear pair, you can take 45º and subtract it from 180º; the difference will be the interior angles measurement."

All of the students answered the Linear Pair problem (shown earlier in Figure 1) correctly.

The Pentagon Problem

Only one student, LP04, answered the Pentagon problem correctly on the final exam, even though this was the third time they all had seen the problem. This student was able to apply the concepts from the instruction, particularly the iteration of sides and noting the measure of exterior angles. Her explanation suggests she was able to see the process as a totality, recognizing that the sum was always 360º. She calculated the measure of the interior angles in a pentagon (72º) and a rectangle (90º) and multiplied each by the number of sides to get a total of 360º. She then reaffirmed that "the turning cost would not be less if the building was a rectangle because the sum of all exterior angles is 360º, so the price will be the same *no matter what polygon it is* [emphasis added]."

Revising the Genetic Decomposition

Based on analysis of student responses with respect to the GD, the researchers made the following additions:

Pre-Action. The learners may not know how to calculate the measure of interior or exterior angles. They might simply guess at angle measure based on observation and/or estimation. Students may think that the measure of an exterior angle is always obtuse and that an interior angle is always acute, or vice-versa.

Action. No additions.

Process. No additions.

Object. Students recognize the difference between more sides and longer sides. They should see that as the number of sides increases, the measure of the interior angles increases and that, at the same time, the measure of the exterior angles decreases; but, while the *sum* of the measures of interior angles grows, the *sum* of the measures of exterior angles remains 360º. The learners are aware that even if the side lengths or overall area of the polygon increases, it does not affect the sum of the measures of interior or exterior angles. Students are able to view the process of summing the exterior angles as a totality, recognizing that the sum is always 360º.

Discussion and Conclusion

In this study, student work on pre-tests, post-tests, and final exams was analyzed for three goals: (1) to test and refine a proposed genetic decomposition, (2) to determine pre-service elementary teacher (PSET) readiness to apply, extend, and generalize the content they will teach, and (3) to improve the instructional model designed to teach generalization over this concept. PSETs did not come into the lesson with a deep understanding of the concept, and their improvement during the lesson can be largely attributed to memorizing formulas. Still, progress was made toward all three goals.

After the IM, all PSETs could calculate angle measures using memorized formulas and recite general statements about relationships (e.g., sum is 360). This is good news, but the goal of the IM is, in APOS terms, to push the learner beyond the action level to a higher level of conception. It can be argued that all of the PSETs did acquire a more advanced standpoint on the geometry concept than was evidenced in the pre-test. This was shown by student work on the final exam.

All students answered the Linear Pair problem correctly, a question encountered for the first time on the final exam. Their responses indicated an advanced perspective for the problem and required, according to the GD, a process conception. However, having acquired this advanced perspective on the concept, they were not able to apply their understanding to the Pentagon problem (featured on all three assessments). Situating the content in an applied setting added a layer of complexity that was seemingly insurmountable. In the context of an APOS framework, PSETs could imagine the process of exterior angle measures decreasing as the number of sides increased, as shown on the final exam's Linear Pair problem. Yet, only one participant showed evidence she could encapsulate the process into an object (e.g., finish the motion of sides increasing and exterior angle measure decreasing) and view it as a totality. That is, she saw the sum of the measure of the exterior angles is 360 degrees regardless of the number of sides. When asked, all students could state that the sum of the exterior angles was 360 degrees (Action level), but only a couple demonstrated a Process conception of these angles always summing to 360 even when the number of sides increased and the exterior angle measure decreased. Only one student had a conception robust enough to transfer this knowledge to the new setting, and recognize that the extension or generalization of the concept allowed it to "fit" perfectly or apply. This student was previously assessed at an Object conception on the post-test according to the GD, and she was the only student who solved the Pentagon problem correctly. This student's work suggests that solving the application problem may have been the action that pushed her to encapsulate the process of sides increasing and exterior angle measure decreasing into an Object conception of the sum as 360 degrees.

For future work, a description of Schema should be developed that describes the coordination of the processes described above and incorporates the role of the applied problem. When the GD reflects this new result, the IM should be modified to account for it.

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A Comprehensive Hypothetical Learning Trajectory for the Chain Rule, Implicit Differentiation, and Related Rates: Part II, a Small-Scale Teaching Experiment

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Students learn more deeply when conceptual understanding is at the forefront and connections are made between topics. With this in mind, we have created a hypothetical learning trajectory (HLT) for the chain rule, related rates, and implicit differentiation to teach them in a conceptual, connected way. In a previous paper we outlined the creation of the HLT based on nested multivariation (NM). In this second paper we describe a small-scale teaching experiment done to test the HLT. Our results suggest NM was an appropriate construct to base the HLT on, and we report on these students' developing understandings. Based on the results, we made final adjustments to the HLT in preparation for a full-scale classroom teaching experiment.

Keywords: chain rule, implicit differentiation, related rates, multivariation, covariation

Among the goals of mathematics education are (a) conceptual understanding and (b) connections across topics (Hiebert & Carpenter, 1992; Hiebert et al., 1997; Schoenfeld, 1988). As a key part of undergraduate education for STEM students, calculus education should especially aspire to meeting these goals. Yet there are topics within calculus that are still often treated as separate, and often procedurally, even though they could be taught in a conceptual, connected way. In particular, the topics of the chain rule, implicit differentiation, and related rates contain powerful, interrelated ideas (see Austin, Barry, & Berman, 2000). Although researchers have suggested conceptual connections between them (Clark et al., 1997; Cottrill, 1999; Infante, 2007; Martin, 2000), these topics tend to be examined in isolation of each other, inhibiting our ability to create instruction that tightly connects them. We propose to unify these topics into a coherent whole through the construct of nested multivariation (NM) (Jones, 2018), defined in the next section. In a first paper (Jeppson & Jones, 2020), we outlined the development of a hypothetical learning trajectory (HLT) (Simon, 1995) for teaching this portion of the calculus curriculum in a conceptual, connected way. In this second paper, we describe a small-scale teaching experiment done to test this HLT. In particular, this study is meant to address three questions: (1) How was NM used by students as they progressed through the HLT? (2) What kind of understandings did students develop for these three topics? (3) Where did students struggle in a way that suggested a need to revise the HLT?

Nested Multivariation

Previous research states that covariation is important for these topics (Infante, 2007). Yet, we realized that because multiple quantities are in play it may be more appropriate to conceptualize these topics through *multivariation* (Jones, 2018), which builds on and extends covariation. In brief, multivariation refers to situations in which *more* than two quantities may change in relation to each other. While multivariation refers to the situation, the reasoning used in such situations is called *multivariational reasoning*. Next, Jones (2018) used the term *nested multivariation* specifically for situations with a function composition structure, $f(g(x))$. That is, if x changes, it induces a change in g, which in turns induces a change in f. If one interprets the derivative fundamentally as a rate of change, as we do, then NM is inherent to each of the chain rule, implicit differentiation, and related rates. The chain rule says that for a function composition,

$f(g(x))$, then $df/dx = df/dg \cdot dg/dx$. In other words, the covariational relationship between x and f is uniquely determined by the NM relationship going from changes in x to changes in g to changes in f. For instance, suppose g changes twice as fast as x (i.e., $dg/dx = 2$) and f changes three times as fast as g (i.e. $df/dg = 3$). Then for a change in x, the corresponding change in f is $3{\cdot}2 = 6$ times as large (i.e. $df/dx = 6$). The multiplicative nature of the chain rule comes from scaling the change in f relative to g (df/dg) by how much g changed with respect to x (dg/dx).

Similarly, we see NM inherent in implicit differentiation and related rates. We define an *implicit function* as one in which one variable in an equation is conceptualized as a function of another variable in that same equation, such as thinking of y in $x^2 + y^2 = 1$ as a function of x (by restricting the range). This leads to the function composition $[y(x)]^2$. NM is present in $d[y^2]/dx$ through the nested changes $x \rightarrow y \rightarrow y^2$. For related rates problems, we define a *function of an implicit variable* as a variable in an equation conceptualized as a function of another variable *not* present in the equation. For example, thinking of x and y in $x^2 + y^2 = 1$ as functions of t makes the equation $[x(t)]^2 + [y(t)]^2 = 1$. A related rates problem is finding the rate of change of one variable with respect to the implicit variable, when the rate of change of the other with respect to the implicit variable is known. This again similarly leads to NM for the derivatives $d[x^2]/dt$ and $d[y^2]/dt$ through the nested changes $t \rightarrow x \rightarrow x^2$ and $t \rightarrow y \rightarrow y^2$.

Research on the Chain Rule, Implicit Differentiation, and Related Rates

Before showing our HLT, we briefly acquaint the reader with the research on these topics that helped lead to our HLT. Some background studies on the derivative suggest the importance of quantitative meanings of the derivative as a multiplicative comparison between changes (e.g., Thompson, 1994). We made sure that our HLT opened by developing this meaning for derivatives. In addition, the scant research on the chain rule suggests the importance of basing the chain rule in the students' understanding of function composition (Clark, et al, 1997; Cottril, et al, 1997). With regard to related rates, Infante's (2007) dissertation contained several items that helped create our HLT. First, Infante made sure to build on chain rule understanding while exploring related rates. Second, Infante used a "delta equation" of the form $\Delta x/\Delta t = \Delta x/\Delta y \cdot \Delta y/\Delta t$ to connect related rates problems to that chain rule understanding. We did so, as well, but elected to use a differential "*d*" equation, $dx/dt = dx/dy \cdot dy/dt$, instead. Third, Infante highlighted the importance of making "time" an explicit quantity in related rates. Fourth, Infante noted that students rarely referred back to the diagram or model, so we regularly asked them the meaning of each variable to help make connections back to the diagram or model. Greater details on how this research literature scaffolded our HLT is given in Jeppson & Jones (2020).

Our (Much Abbreviated) Hypothetical Learning Trajectory

Simon (1995) defined HLTs as consisting of learning goals, learning activities, and a hypothesized learning process. In our previous paper (Jeppson & Jones, 2020), we outlined the creation of an HLT. In brief, our HLT contained five stages, meant to be traversed over four 50-minute interview sessions. Each stage had one overarching learning goal, accomplished by meeting several subgoals inside that stage, as shown in Table 1. Table 2 then shows abbreviated versions of the learning activities and key questions meant to accomplish the goals.

Table 1. The Hypothetical Learning Trajectory's Learning Goals

Stage	Description of Goal
Stage 1	**Develop the multiplicative nature of the chain rule.**
1a	Interpret *df/dg* as how many times as large the change in *f* is than the change in *g*.
1b	Interpret *dg/dx* as how many times as large the change in *g* is than the change in *x*.

1c	Interpret *df/dx* as how many times as large the change in *f* is than the change in *x*.
1d	Conceptualize how changes in *x* affect changes in the other two variables simultaneously.
1e	After finding specific values of *dg/dx* and *df/dg*, construct $df/dx = df/dg \cdot dg/dx$ at a point.
Stage 2	**Generalize the chain rule and gain procedural fluency.**
2a	Continue to construct the multiplicative nature of the chain rule.
2b	Generalize the chain rule to any function composition *f(g(x))*.
2c	Gain procedural fluency with the application of the chain rule.
Stage 3	**Develop the idea of variables being functions of the implicit variable time and recognize subsequent existence of compositions of functions.**
3a	If variables change with time, conceptualize them as functions of time and represent them as such
3b	Create compositions of functions structure for functions of implicit variables
3c	Recognize the need for, and correctly use, the chain rule for functions of implicit variables
Stage 4	**Develop the idea of implicit functions and recognize existence of compositions of functions.**
4a	Given an equation with variables *x* and *y*, conceptualize *y* as an implicit function of *x*, or *x* as an implicit function of *y*. Represent these symbolically as *y(x)* or *x(y)*.
4b	Recognize the need for the chain rule in taking the derivative of an implicit function
Stage 5	**Extend ideas to more complicated implicit differentiation and related rates contexts.**
5a	Continue to gain competence in recognizing compositions of functions in these types of problems
5b	Recognize need to use the chain rule for functions of implicit variables or implicit functions
5c	Gain procedural fluency with more complicated related rates and implicit differentiation problems

Table 2. Abbreviated Activities in the HLT and Relation to Subgoals

Abbreviated Activity Contexts	Sample key questions (related to HLT subgoals)
Stage 1: "Let's say you make $9/hr at your job and you're obsessed with chocolate… You can buy 0.15 lbs of chocolate per dollar." [A similar context/questions are used relating temperature, expected attendance at a carnival, and revenue. A third is then used relating time elapsed, distance run, and calories burned.]	1a-b: What is the value and meaning of *dD/dh* (and *dc/dD*) in our context? What are the units of *dD/dh* (and *dc/dD*)? 1c-d: If we could find *dc/dh*, what would that mean in our context? 1e: What is the value of *dc/dh*? How do you know?
Stage 2: [Build on previous *running context*] "Let's say you run at a constant rate of 0.1 miles/minute… Consider the functions $D(t) = 0.1t$ and $c(D) = 20D^2 + 40D$." [This is then repeated for the same context but new functions $D(t) = 2^t$ and $c(D) = D^3$.]	2a: What are *dD/dt* and *dc/dD* at 20 minutes? What is an equation for $c(D(t))$? What is *dc/dt* at 20 minutes? 2b: How can you use what you have found so far to write an equation for *dc/dt* at any time *t*? What patterns do you notice? Given any function for $c(D(t))$, what is *dc/dt*?
Stage 2: Given $g(x) = \sin(f(x))$, calculate *dg/dx*. [Repeated for $h(x) = \sin(x^2)$ and $g(x) = (\sin(x))^2$.]	2c: How can you relate what you have found so far to write an equation that gives *dg/dx* for any *x*?
Stage 3: "The body of a snowman is in the shape of a sphere whose radius is melting at a rate of 0.25 ft/hr. Assuming the body stays spherical, how fast is the volume changing when the radius is equal to 2 ft? Remember that for a sphere, $V = (4/3)\pi r^3$."	3a-b: What are all of the things that are changing with time? How can we write that symbolically? 3c: What are we trying to find? How can you represent it as a derivative? How can you represent 0.25 ft/hr as a derivative? How can we find *dV/dt*?
Stage 4: "A blue square with side length *x* has a corner cut out, also in the shape of a square of side length *y*. The leftover blue area must always equal 1, $x^2 - y^2 = 1$. As *x* changes, by how much does *y* have to change?" [A second problem is then given, which shows the equation $x^2 + [f(x)]^2 = 9$ and asks what *df/dx* is.]	4a: Can you write what we are trying to find as a derivative? How does $[y(x)]^2$ relate to what we have been discussing in our previous sessions? How much does the blue area change as *x* changes (i.e. *d[blue]/dx*)? 4b: What is the difference between dx^2/dx and dy^2/dx? How do our previous discussions relate to finding dy^2/dx?
Stage 5: (1) Related rates problem involving a shuttle launching upward at a rate of 0.2 miles/second and finding the rate at which the angle of observation is increasing. (2) Related rates problem involving an airplane flying horizontally 1 mile in altitude and finding the rate at which the distance to a station on the ground is change. (3) Implicit differentiation problem to find *dy/dx* for equation $\sqrt{x+y} = x^4 + y^4$.	5a-c: Allow students more independence in conceptualizing the problem, the functions, and the derivatives, and in solving the problem. Instructor reminds students to consider ideas already discussed in previous sessions.

Methods

To test our HLT, Author1 conducted small-scale teaching experiments with four students taking first-semester calculus for the first time. The teaching experiment was done with each student individually, over four 50-minute interview sessions, to focus on each student's personal developing understanding. In their class, the students had been introduced to the derivative, but had not yet seen any of the material covered in our HLT. We analyzed the interview data in three major phases. In the first phase, we marked any time nested multivariational reasoning (NMR) was used by the students throughout the sessions. To help identify instances of NMR, we took Carlson et al.'s (2002) and Thompson and Carlson's (2017) frameworks for covariational reasoning and imagined what might be analogous mental actions for NMR (for more details, see Jeppson, 2019). We then looked for themes in terms of how students used NMR as they learned these three topics. In the second phase, we broke each student's data along the five stages of the HLT and identified all of the units of data related to each of the learning subgoals in that stage. We also looked for pieces of data that suggested a student either had an incorrect understanding related to the subgoals or was inhibited in reaching them. For each subgoal, we looked at a given student's data related to that subgoal and made a determination as to whether their understanding related to that subgoal was "complete", "incomplete", or "missing." Complete understanding meant there were multiple units of data suggesting understanding of the subgoal. Incomplete understanding meant there was some evidence of reaching the subgoal but where there was not sufficient evidence to be certain of their understanding, or if there was some evidence of a misunderstanding related to that subgoal. Missing understanding means there was no evidence of understanding the subgoal. Lastly, in the third phase, we identified places in each stage of the HLT where the four students tended to struggle, suggesting a need to revise the HLT.

Results

Use of NMR in the HLT

NMR was most obviously observed in stage 1 as students used it to make sense of the way the variables in function compositions were related to and depended on one another. They also frequently used it as they first thought about the different rates within the function composition, how the quantities changed together, and how two rates could multiply to make a third rate. For example, in the chocolate context in stage 1 (see Table 2), Student A exemplified typical thinking to find the derivative, dc/dh, of the function composition $c(D(h))$. She said she could multiply the two derivatives because, "You know that you get 0.15 lbs per every…dollar. But, you can't go from pounds to hours because there is a medium ground you have to hit first before you can get to that… This [0.15 lbs/dollar] would just symbolize 1/9 of an hour, and then you would multiply it by 9 so it would give you, for the full hour, how much." Such reasoning allowed the students to build intuition for the multiplicative nature of the chain rule, in which two rates multiply to give a third rate. Reasoning like this in the other stage 1 contexts as well led all four students to observe patterns that helped develop and generalize the chain rule structure.

One important result was that, while it took instructional investment to develop this initial understanding of multiplying rates, the students were able to frequently use it later without needing to recreate it. Most instances of NMR after stage 1 were recognizing the existence of a nested relationship and the need for the chain rule in differentiation. However, it was also true that using NMR proved important for the students in reasoning about how the quantities changed together within related rates contexts and implicit function equations, such as interpreting the nested relationship structure between quantities and how that related to function composition.

Students' Developing Understandings as They Progressed Through the HLT

Stage 1. All of the students exhibited evidence for *complete* understanding (as defined in the methods) of the first three subgoals in stage 1 by interpreting different derivatives within function compositions. For example, Student D explained *dc/dD* = 0.15 by first identifying the units "lbs/dollar" and stating, "For every dollar that I have… I would be able to get 0.15 [lbs]." Similar interpretations were made for other derivatives in stage 1. The students also all used NMR to discuss the relationships between the three variables in function compositions, exhibiting *complete* understanding for subgoal 1d. For example, in the chocolate context Student A said, "The chocolate is dependent on the amount of dollars which is dependent on the hours." Lastly, as seen in the example in the previous subsection, all of the students used NMR develop the multiplicative nature of the chain rule at a point, demonstrating *complete* understanding of subgoal 1e. In summary, the students demonstrated *complete* understanding for all subgoals in stage 1, suggesting that there was no need to revise the HLT for stage 1.

Stage 2. In the running context with functions (Table 2), the students found derivative values at specific values for *D* and *c*, and again used the multiplicative nature of the chain rule to then find *dc/dt* for those values. This showed *complete* understanding for subgoal 2a. To progress toward subgoal 2b, the students were then asked to find equations for $c(D(t))$ and *dc/dt*, which was the first time they were asked to make connections between the chain rule and symbolic expressions. In doing so, the students were asked how to differentiate symbolically given a composition of functions. Until this point, they had identified that $dD/dt \cdot dc/dD = dc/dt,$ but with this prompt, they were able to reverse it to $dc/dt = dD/dt \cdot dc/dD.$ For example, Student B said, "If we take the derivative of the inside times the derivative of the entire thing, we can find the derivative of the composition of function, *dc/dt*."

However, Student B encountered a difficulty related to subgoal 2b. After finding a correct symbolic form for the chain rule, he came up with an incorrect conclusion. Using a kinematics example, he stated that if we knew acceleration and position, but not velocity, then, "plugging these, a function inside a function, I can make the relation… So, position has velocity as a derivative, and the derivate of velocity is going to be acceleration… the function itself can be combined… and I can find derivatives." Here Student B was suggesting that any time three variables were related to one another, in apparently *any* way, they could automatically be placed into a composition of functions (e.g., perhaps $a(v(s))$) and the chain rule could be used to find a missing rate). Because of this incorrect conclusion, Student B was the only one to have *incomplete* understanding for subgoal 2b.

When finding the derivative of $h(x) = \sin(x^2)$, three students connected their understanding of rates with physical quantities to abstract symbols. However, Student C struggled in recognizing the composition of functions. He compared it to the running problem and said, "I feel like I had 3 things [in the running problem]. We had *d*, *t*, and *c*… But here, I am not sure if we even have three things. I think sine comes in somewhere and cosine but I am not exactly sure how." While Student C had some understanding, we labelled it *incomplete*. The struggles of Students B and C generalizing the chain rule to symbolic expressions seemed to come from underdeveloped understandings of function composition (cf. Clark, et al, 1997). Thus, even anticipating this difficulty, we still underestimated how big the difficulty might be.

Stage 3. In the snowman problem, the students all gained the ability to recognize that a quantity changing can be considered a function of time. For example, all noted that volume and radius changed with time, and when Author1 suggested that they can be thought of as functions of time, the students wrote them symbolically as $V(t)$ and $r(t)$ and used these representations in the rest of the problem. Given the overall thinking students used, we considered this to demonstrate *complete* understanding of subgoal 3a. The students also understood that these functions corresponded to the quantities in $V = (4/3)\pi r^3$, allowing them to reconceptualize it as $V(t) = (4/3)\pi[r(t)]^3$. By recognizing the function composition structure in $[r(t)]^3$, the students reached subgoal 3b. In moving toward subgoal 3c, Student C was initially unsure of how to get from the formula relating V and r to the rate of change of volume. To get there, he first identifed the given rate, $dr/dt = 0.25$ ft/hr, and the rate that he was trying to find, dV/dt. To find dV/dt, he eventually recognized that he could multiply dr/dt and dV/dt, to create the "d equation" $dV/dt = dr/dt \cdot dV/dr$. In this way, he made a strong connection to the chain rule he had developed in stages 1 and 2. Student D's work was similar to C. Students A and Student B also recognized V and r as functions of the implicit variable time, but without using a d equation. After creating an equation that related volume, radius, and time, both of them recognized immediately the existence of a composition of functions, $[r(t)]^3$, and the need to use the chain rule in taking the derivative. These connections suggest *complete* understanding for 3c.

Student D did struggle in a specific way during stage 3, due to misunderstandings between a derivative value and an average rate of change. Because of this, he did not know how to complete the process of solving for dV/dt. Thus, while he did recognize the need for the chain rule, he was unable to work it out, and we determined he had *incomplete* understanding for 3c. In our analysis, this appeared more to be an issue with how he first learned derivatives than with our HLT. However, it does reinforce the need for derivatives to be taught previously in a conceptual way that can be built on and elaborated on within these three topics.

Stage 4. For the implicit differentiation problem in this stage, all of the students exhibited evidence of conceptualizing y as an implicit function of x. They represented it symbolically as $y(x)$ and substituted that into the equation: $x^2 - [y(x)]^2 = 1$. Due to the relations of these acts to earlier thinking, we considered this as evidence for *complete* understanding for subgoal 4a. Then, as an example, Student B pointed to $[y(x)]^2$ and stated that it was "a function within a function." This helped him see the function composition structure and the nested changes $x \to y \to y^2$. However, when Student B differentiated the equation, he wrote the incorrect result $2x - 2[y(x)] = 0$. Author1 prompted him to look at his previous work in other problems, after which he recognized the need to multiply *both* the rates dy^2/dy and dy/dx together to get $d[y^2]/dx$. This seemed to be an important step in making sure that his previous understanding of the chain rule was connected to his understanding of implicit differentiation. In the end, all students exhibited *complete* understanding of subgoals 4a and 4b by conceptualizing implicit functions and recognizing the need for the chain rule in implicit differentiation.

Stage 5. In this last stage, all of the students became adept at recognizing compositions of functions in these types of problems, satisfying subgoal 5a. However, a major issue was that the students moved through these problems more slowly than anticipated. We had expected that their previous experiences would allow them to do these three problems more independently in one 50-minute session. Student A was the only one to even reach the implicit differentiation problem. This suggests a need to revise this portion of our HLT to better scaffold it for students.

Further, for the implicit differentiation problem, Student A first subtracted $x^4 + y^4$ to make the equation $\sqrt{x+y} - x^4 - y^4 = 0$, with a constant on one side of the equals. While not wrong, it was unnecessary and seemed to stem from a limited understanding of this type of problem. We realized that because both prior implicit differentiation problems had constants on one side of the equals, $x^2 - y^2 = 1$ and $x^2 + [f(x)]^3 = 9$, she may have come to believe that implicit differentiation was only possible in this form. As such, this suggested another needed revision to the HLT.

Similarly, in the two related rates problems, all four students seemed to not realize that they could take the derivative of both sides of the equation with respect to time. For example, the shuttle problem led to the equation $\tan(\theta) = d/2$, which the students correctly conceptualized as $\tan(\theta(t)) = d(t)/2$. Yet, instead of applying the chain rule to $\tan(\theta(t))$, all of them tried solving for θ to get $\theta(t) = \arctan(d(t)/2)$. While not wrong, it drastically complicated the procedures. We realized that all previous cases had the format of $h(x) = f(g(x))$, and that they may have tried to follow that format here, as well. As such, this exposed another needed revision to the HLT.

Discussion and Revisions to the HLT

This study was aimed at testing our HLT in a small-scale experiment. This study suggested NM was a reasonable construct on which to base an HLT for these topics. It also confirmed that there are several important background understandings that must be in place for these deeper topics in calculus to be well understood. For example, Oehrtman, Carlson, and Thompson (2008) explain that it is critical for students to understand functions in order to understand calculus, and Clark et al (1997) speak of needing to understand function composition in particular. This was the case for our students, as seen by the difficulties Student B and C. It also seems critical that students understand derivatives quantitatively as a multiplicative comparison of changes (Jones, 2017; Thompson, 1994). This impacted Student D's ability to progress in stage 3.

Further, our study demonstrated the need for revising the HLT in three ways. First, during stage 2, students may need more time becoming familiar with "inside" versus "outside" components of a function composition. Second, the students' tendencies to solve equations for a single variable (in related rates) or to get a constant on one side of the equals (in implicit differentiation) implies that in stage 4 it may be necessary for students to better understand these equation structures. For example, Mirin and Zazkis (2019) have recently suggested the need for students to conceptualize each "side" of an equation as separate functions that are equal to each other. For example, $x^2 + y^2 = 1$ can be conceptualized as $f(x) = x^2 + y^2$ and $g(x) = 1$. The derivatives of these equal functions can be thought of as equal. Such understanding might have helped the students realize that $\tan(\theta(t)) = d(t)/2$ does not need to be solved for θ, or that $\sqrt{x + y} = x^4 + y^4$ does not need to be set equal to 0. Rather, each "side" in these two equations can be thought of as a separate function for which the chain rule can be used. Third, it became clear that stage 5 was not adequately set up to help the students take what they had recently developed and use it proficiently for those three more complicated problems. Rather than expect students to be fairly autonomous at this point, the same types of questions used in previous stages should be explicitly repeated in stage 5. We believe this may help the students make connections more quickly and be able to solve those problems a little more efficiently. While these changes do require more time earlier on, they may pay off in a more efficient stage 5.

Conclusions

Overall, this small-scale teaching experiment suggested that NM is a reasonable construct on which to base an HLT for the chain rule, related rates, and implicit differentiation. The students used NMR frequently to create the chain rule structure and to see that structure in related rates and implicit differentiation. This study allowed us to see how individual students' understanding developed over the course of our HLT. Some difficulties were detected along the way that have permitted revisions to our original HLT. Through this, we now view our HLT as prepared to transition from a small-scale, interview-style test to a full-scale, whole-classroom teaching experiment. This subsequent experiment would be necessary to see how this HLT might function with all the complicating dynamics of an entire classroom.

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Tutoring Beyond Show and Tell: An Existence Proof

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Drop-in tutoring has been shown to positively impact grades. However, little is known about what occurs in drop-in tutoring. This study sought to fill this gap by answering the following research questions: 1) How do tutors interact with students? 2) What factors influence tutors' decisions regarding how to navigate interactions? 3) How do these factors impact tutors' interactions? This study provides a descriptive case study of one tutor's practices in the context of drop-in undergraduate Calculus I tutoring. Natural tutor session recordings and stimulated recall interviews were collected. Multiple lenses, including teacher decision-making, social and socio-mathematical norms, responsive teaching, and cognitive apprenticeship, were utilized during analysis. Results found tutor-student interaction patterns are more complex than previously reported in the literature. In addition, results suggest tutors are capable of changing their practices within the short time span of a semester and further, tutors are capable of enacting "best practice" teaching strategies.

Keywords: Tutoring Practices, Calculus, Stimulated-Recall Interviews

Introduction

In 2010, the Characteristics of Successful Programs in College Calculus study sought to gain insight into the institutional characteristics of successful calculus programs. Undergraduate mathematics tutoring, or the presence of a learning center, was cited as one of the primary features of institutions with successful calculus programs (Bressoud, Mesa, & Rasmussen, 2015). In addition, 97% of the institutions surveyed offered some form of calculus tutoring with the majority offering tutoring by undergraduates. Further, attending drop-in tutoring has been shown to positively impact calculus performance as measured by final course grades (Rickard & Mills, 2018). This indicates tutoring plays an important role in undergraduate mathematics education, yet little is known about what occurs in drop-in tutoring or the work these tutors do.

What research exists on tutoring is often context independent, examining the general pedagogical practices of tutors, such as tutor questioning (Graesser & Person, 1994). In addition, studies often examine the practices of 'expert' tutors, most of whom have had teaching experience (Arcavi & Schoenfeld, 1992; Schoenfeld, et al., 1992), as well as explore tutor moves in the context working with a student to learn a pre-determined topic in a fixed amount of time and number of sessions (Arcavi & Schoenfeld, 1992; Chi, 1996; Graesser & Person, 1994; Schoenfeld, et al., 1992). Little research has attended to the specific context of undergraduate tutors working in a drop-in tutoring site. In particular, research has yet to examine the practices of the same tutor across multiple sessions and over an extended period of time.

The aim of this study was to provide descriptive case studies on tutoring practices in the specific context of drop-in undergraduate Calculus I tutoring. In particular, it sought to answer the following research questions:

1) How do tutors interact with students?
2) What factors influence tutors' decisions regarding how to navigate interactions?
3) How do these factors impact tutors' interactions with students?

This study establishes baseline data on which to build future studies regarding tutors and tutoring practices.

Literature Review

This work was grounded in two conceptual principles: 1) in response to others, humans make decisions based on what they assume is important, drawing from what they notice in the course of interactions (Herbst & Chazan, 2003) and 2) responses humans make in the course of interactions draw on current understandings of the event, such as its circumstances and content, and resources they have at their disposal (Schoenfeld, 2011). The data was analyzed with this in mind to examine the tutor's references to factors which influence decisions. We elaborate in detail in the following section how these points shaped the analysis of data in this study.

We also drew on literature regarding teaching practices that support student learning. In order to create a learning environment that supports students' conceptual understanding, certain social and socio-mathematical norms are accepted as critical to be negotiated by the instructor and student (Cobb, et al. 1992; Cobb & Yackel, 1996; Yackel & Cobb, 1996). Further, scholars have argued that in order to encourage the student to be an active participant and build on students' thinking, instructors need to develop eliciting skills to engage the student in the interaction and draw out the students' thinking. While the student is talking, the instructor needs to listen to the student in order to build an understanding of what the student is thinking (Teuscher, Moore, & Carlson, 2016). Instructors should not be listening for specific answer but instead engage in listening in order to negotiate meaning with the student they are working with (Davis, 1997). Tesucher et al. describe the instructor's cognitive process in listening to the student, "de-centering". In de-centering, the instructor builds a model, or way of explaining the student's understanding based on what the student communicates. The instructor then uses this mental model to make in-the-moment instructional decisions on how to proceed.

Methods

The case presented was part of a larger study conducted using a multiple case study approach with each case being a Calculus I tutor in their first year of tutoring. This report will focus on one these cases, Jason. Jason was a junior biomedical engineering major with no previous tutoring experience working in the Calculus I room in a drop-in tutoring center.

In the center, students came to the tutoring room either with questions in mind or to work on their homework and ask questions as they arose. Typically, students asked for help on a specific homework task rather than for help understanding a general concept. Tutors generally helped students with their particular task and then moved on to the next student. A typical tutoring session lasted 5-15 minutes. While tutors may have worked with a particular student in the past, it was more likely they had not worked together before.

As part of his training requirements, Jason recorded three tutoring sessions at four points during his first year of tutoring for a total of 12 sessions. A session was defined as an interaction with an individual student, typically on one task. At each of the four points in time, Jason had approximately two weeks to audio record his three sessions and submit the recordings and photographs of the task and written work. Jason collected a set of recordings during his first weeks as a tutor (beginning of September), halfway through his first semester (mid-October), at the end of his first semester (December), and at the beginning of the subsequent semester (January). It was left to Jason to select the sessions he recorded and submitted. These recordings and photographs constituted the tutor session data for the study.

In addition, Jason met with the researcher twice to discuss his recorded tutoring sessions. Each meeting was conducted as a stimulated recall interview, having Jason listen to each of his sessions. At key points while listening to the session, the interviewer stopped to ask Jason to explain what he thought was going on, why he did what he did, or what he was trying to do or get at. This allowed the researcher to better understand Jason's decision making and thought processes rather than making assumptions. In addition, during the interviews, the researcher asked Jason questions regarding his beliefs about tutoring and learning.

Data Analysis

Analysis of the data occurred in waves and was conducted on a group of four tutors. First, each tutor's tutoring sessions from September and October were transcribed and indexed to describe the actions of each talking turn. Then these transcripts were open coded to look for patterns of behavior to begin to formulate hypotheses regarding these behaviors and possible rationales. Session transcripts were then marked for passages where these behavior patterns emerged. It was at these marked points in the tutoring sessions that during the stimulated recall interviews, tutors were asked questions regarding their thoughts and behaviors. In addition, the hypotheses for factors influencing the tutors' behaviors were also addressed through additional interview questions.

After conducting the first interview, the interview transcript was analyzed using a grounded theory approach to look for themes which emerged from the data. This phase of the analysis drew on several bodies of literature. Because this was an exploratory study seeking to find baseline data on tutoring, the exact frameworks which may have been useful in examining tutoring sessions were not known. Employing grounded theory techniques allowed the researcher to look for aspects of the data which come out as key features. However, of particular note was the literature on social norms (Cobb, et al. 1992), decentering (Teuscher, Moore, & Carlson, 2016), and teacher decision making (Schoenfeld, 2011).

During analysis, social and socio-mathematical norms were inferred based on the patterns of interactions. Tutoring sessions were examined for negotiation of norms such as justifying answers, the student writing, or pushing against the idea of the tutor as the authority on mathematics (Cobb, et al., 1992; Yackel & Cobb, 1996). Further, tutors' rationale for their behaviors attempting to negotiate these norms was analyzed. To analyze the data for decentering, Ader and Carlson's (2018) levels of decentering framework was employed. The framework provides observable instructor behaviors as indicators they are working at a particular level of decentering.

In light of the analysis of the first interview, an initial model accounting for Jason's decision-making and factors influencing his work was constructed. Jason's tutoring sessions from December and January were then analyzed and again marked for stimulated recall interview questions. In particular, these sessions and the second interview were examined for data which confirmed or contradicted the existing theory.

Results

Research Question 1: How do tutors interact with students?

In many ways, Jason's tutoring was fairly consistent over time. One of the major changes was his increase in use and variety of questions. Throughout Jason's tutoring, the students appeared inclined to write, talk, and ask questions without being told. Jason maintained these social norms by allowing students to work on their own without taking over. Because Jason felt understanding the student's thinking was important, he allowed students to explain their reasoning with Jason asking questions to better gauge their understanding. Jason listened to his students and, based on what he understood their thinking to be, would respond to students in a way that addressed the student's individual thinking. Jason was consistently working at decentering level three or four (Ader & Carlson, 2018). For example, in a problem taking the limit as x goes to zero of a rational function, the student suggested an incorrect method for solving the task.

Student: =Okay and would I multiply by one over x squared? Or one over x to the fourth?

Tutor: So when you expand this, the highest power is gonna be x to the fourth, so you would do um divide by x to the fourth.

When asked what he thought the student was thinking and why he responded the way he did, Jason said,

> So I think what they were thinking is like when you take the limit as x is approaching infinity, like they're taught to divide by the highest power and then that eliminates all the other terms essentially…I think what I was thinking is if they like continued on the path that they were going and either saw one way or another if it worked or didn't work they would still I guess learn.

Jason indicated he thought the student was thinking about applying a method using for limits at infinity with rational expressions. Because of this, he answered the student's question as if the method were valid in the scenario in order to allow the student to explore the incorrect path, working at decentering level four.

However, the manner in which Jason responded to students changed over time. In Jason's September sessions, his responses often took the form of explanations while in his October session, these responses were information gathering questions (Boaler & Brodie, 2004). In December, Jason asked a range of questions for a variety of purposes.

In addition, Jason applied practices from cognitive apprenticeship (Collins, Brown, & Newman, 1989). Jason listened to his students in order to determine the level of help they needed. He would adjust his level of help based on what he perceived to be the student's understanding. Jason strived to have students do as much on their own as possible but when they were stuck, he would offer scaffolding to assist them. If scaffolding did not work, he would model. Jason wanted to avoid modelling and only used it as a last resort. Jason said he wanted the student to be "in the driver's seat".

Research Question 2: What factors influence tutors' decisions regarding how to navigate interactions?

In working with students, Jason drew on his own mathematical thinking and experience learning calculus to direct him to possible difficulties as well as ways of assisting students past those difficulties. He also drew on his accumulating experience working with students. Jason learned common errors and ways to address those errors. Further, it was important to Jason that he not only helped students solve the task at hand but that he also worked with them in a way that they could solve future tasks on their own.

Research Question 3: How do influencing factors impact tutor interactions with students?

Jason's behaviors were not predictable in the same way as other tutors in the larger study. This was partly because Jason's students would offer their own thoughts unsolicited. This led to Jason's decision-making (Figure 1) being centered-around student behaviors as they were the primary directors of the sessions. The branches in Jason's decision-making tree depended on whether or not students offered thinking, the type of thinking they offered, the correctness of students' mathematics, and the questions students asked. This follows from Jason's priorities. Jason wanted students to do the work on their own and be the "driver" of the session. Jason only wanted to be involved when asked. When students were stuck, Jason preferred to help by giving hints rather than telling.

Jason prioritized determining what students were thinking to give direction to the sessions. All of Jason's sessions started with students expressing what they were thinking. Even when students were not sure how to start the problem, they would generally offer some thoughts. When students were unable to offer an idea of how to start the problem, Jason would ask a general question, such as how to approach problems of that type, consistent with his priority to give students hints rather than answers. However, in his earliest sessions, Jason would at times tell students what he noticed in the problem that indicated to him what to do. In doing this, Jason would also tell the student what to do. Jason admitted that when he first started tutoring, he was guided by a priority to get the answer. Later, however, Jason said he aimed to get students more involved and work with their ways of thinking. Jason would ask students questions to provoke different ways of thinking about what was causing them difficulty.

The way Jason responded to students during the introduction corresponded to broader patterns throughout his sessions. Jason's responses to student impasses were each unique because students were able to express very specific difficulties. For example, throughout Jason's sessions, he was primarily silent unless directly asked a question or the student made a mistake. This allowed students to offer their own knowledge of what to do, how to do it, and facts. This aligned with Jason's priority that students do their own work as much as possible. When Jason did prompt students to continue working, he would do it with a question about what to do next or how to do something.

However, there were two situations in which Jason would directly explain. One was when explicitly asked a question and the other was when he had tried to ask questions but the student was still not able to move forward. In both situations, Jason would not only explain what the student was not understanding, but also why it was the case. As previously stated, occasionally in his earliest sessions, Jason would tell a student what to do or how to do something when he saw they made a misstep. However, for the most part, Jason would address procedural mistakes by asking how they got their answer so he could determine how to address the cause of the error. At other times, Jason would allow students to explore the incorrect path so they could realize the error on their own.

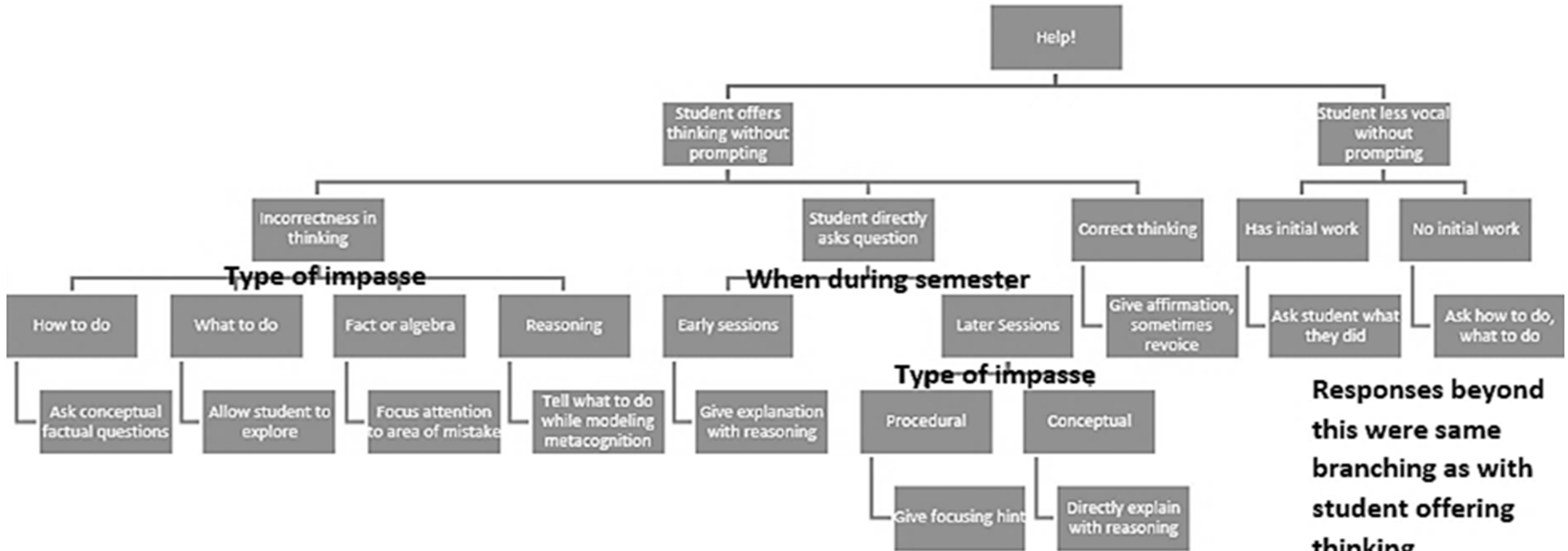

Figure 1. Jason's decision-making tree

Discussion and Future Directions

In contrast to previous research findings (Graesser et al., 1995), the current study indicates that at times tutors do use sophisticated pedagogical strategies. Although the goal of the study was not to confirm or refute presence of specific pedagogical strategies, it became evident that Jason engaged in strategies such as scaffolding, converging on shared meaning, error diagnosing, and addressing motivation. As such all features of Graeser et al.'s sophisticated pedagogical strategies were present in Jason's work. Further, in contrast with Chi's (1996) findings, Jason attempted to understand and build on student thinking, allowing students to dictate the direction of the session. In addition, when Jason sought out student thinking, he tried to address any misunderstandings he uncovered.

Further, although the goal of this work was not to assess the effectiveness of tutor moves, productive tutor moves identified in previous literature were present. For example, Jason allowed students to attempt to work on their own. In addition, Jason used questions to help students determine and explain their steps, providing explanations only if the student was unable to do so (VanLehn et al., 2003). Jason used questions and hints to help students (Lepper & Woolverton, 2002). Further, Jason referenced using various types of knowledge during his work and expressed a desire to be empathetic toward students. Jason used indirect criticism to avoid being harsh but also refrained from overpraising. Jason also described making decisions based on pedagogical knowledge for teaching (Gutstein and Mack, 1999). Findings demonstrated that tutors are capable of interacting with students in ways that align with best practices. While it is not known whether Jason would have demonstrated these best practices in the absence of training, data revealed that even first-year tutors are able to interact with students in ways that moved beyond show and tell. Tutors are capable of trying to understand student thinking and make moves based on responses to questions tutors asked. In addition, Jason either developed or held pedagogical content knowledge (Hill et al., 2008) in ways that allowed him to better assist students. This study indicates tutors are capable of enacting evidence-based pedagogy. Future research can look more closely at tutoring behaviors using frameworks such as the NCTM Mathematical Teaching Practices (2014) to look for direct evidence of the presence or absences of evidence-based best practices.

Further, Jason was able to enhance his practices even within the timespan of a few months. While the goal of this study was not to uncover reasons for these changes, it seems reasonable to consider two factors. First, tutor behaviors may have changed over time simply due to the experience gained working with students. Second, changes may have been due to tutor training. It is plausible to assume that both factors contributed to developing pedagogies. That tutor training can change over a short timespan indicates tutor training is not a futile effort. Although tutors often only work at a center for one to two years, training may make an impact quickly. This also indicates that time and money spent on tutor training may be worthwhile. The current study did not examine how the tutors' ongoing training during data collection influenced their practices and beliefs. It is not clear whether and how the Jason's practices and rationales for their decision-making would have differed had they not undergone training or been subject to alternative training. Future work can examine the impact of the training curriculum on tutors' practices. It is important to understand how different components of the training programs may influence tutors and specific aspects of their work.

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Student Self- and Simulated Peer-Evaluation of Proof Comprehension: Tina

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Research shows that students in mathematics can benefit from participating in self- and peer-evaluation. However, many students lack the self-confidence in their ability to grade their peers' work and often provide feedback that is unhelpful. This study investigates the criteria used by a student to evaluate responses to questions on an assignment pertaining to comprehension of a proof by contradiction. Two interviews were conducted with the student more than one year apart in which she evaluated her own responses and responses written by an instructor to the questions in the assignment. Through the lens of achievement goals, results provided show a shift in the way she used her criteria to provide evaluations of these responses. Further, analysis indicates students may benefit from regular activities and practice in self-/peer- evaluation of proofs without assigning a grade to help their reflection and sense-making abilities.

Key Words: Self-Evaluation, Peer-Evaluation, Proof Comprehension, Indirect Proof

Introduction

Many studies have investigated the effects of self- and peer-assessment/evaluation on student performance and understanding of mathematical concepts (e.g., Beumann & Wegner, 2018; Scott, 2017; Dupeyrat et al., 2011, Brookhart et al., 2004; Stallings & Tascoine, 1996). Beumann and Wegner (2018) explain there are three key dimensions of self-evaluation: use of criteria to self-evaluate learning, interactive dialogue between the teacher and student, and assigning a grade to the work. With regard to student use of criteria in self-evaluation, Stallings and Tascoine (1996) state that students originally did not include the idea of conceptual understanding in their criteria, and tended to weigh effort heavily in their self-evaluations. Research shows that students sometimes underestimate their comprehension, but more frequently overestimate it (Lin, Moore, & Zabrucky, 2001). Beumann and Wegner (2018) call for more research on how students assign the grades associated with their self-assessment and invite others to investigate topics with self-assessment in mathematics.

Researchers state that both peer- and self-assessment methods improve student autonomy and learning and help foster skills associated with metacognition (Brown & Knight, 2012; Elwood & Klenowski, 2002). While much research supports the use of peer-assessment, Wen and Tsai (2006) found many students lacked the self-confidence in their ability to grade their peers' work and were apprehensive about criticism from their peers. Brophy (2005) implies student concern about peer comparisons are likely to distract their focus while attempting to understand a concept. Scott (2014, 2017) suggests *simulated peer-assessment* as an alternative to peer-assessment, in which students are provided with an anonymous example of an incorrect solution to a question and asked to provide feedback after analyzing it. By removing real student work, it may remove issues such as "apprehension about being criticised, provision of deleterious feedback, influence of social relationships, and differing peer abilities," (Scott, 2017, p. 111). In this study, explanatory qualitative case study methodology (Baxter & Jack, 2008) is utilized to consider a student's simulated peer-evaluation and self-evaluation responses to a proof comprehension activity.

We use this type of case study methodology to explain a shift in this student's evaluation criteria over time and present results and discussion on the following research questions: What criteria does an undergraduate mathematics major use in evaluating responses to questions about proof comprehension? In what ways do these criteria change (a) depending on whether the student is partaking in self-evaluation or simulated peer-evaluation and/or (b) over time (after completing other upper level mathematics courses such as Modern Algebra I, Analysis I and II, etc.)? Results of this paper imply practice in student self- and peer- evaluation of a proof comprehension activity without assigning a grade may help students to improve their sense-making abilities. In particular, by better evaluating their own work, students can reflect on their understanding to better develop mathematical concepts and proof skills.

Theoretical Framework

There are four achievement goals described in Elliot (1999) and Pintrich (2000) that will be used in this report to analyze data and describe results: Performance approach, performance avoidance, mastery approach, and mastery avoidance. Performance-oriented goals represent those in which a student may tend to compare themselves to others or focus on a grade, while mastery-oriented goals are those that focus on learning and understanding a mathematical task. We summarize these goal orientations in Table 1 below as they pertain to approach and avoidance states, as adapted from Pintrich (2000).

Table 1. Goal orientations and approach/avoidance states, as adapted from Pintrich (2000).

	Approach State	Avoidance State
Performance Orientation	• Focus on being the smartest or best in comparison to others • Use standards such as being the best performer in the class, receiving the highest grade, etc.	• Focus on avoiding inferiority or looking dumb in comparison to others. • Use standards such as not being the worst in the class, not receiving the lowest grade, etc.
Mastery Orientation	• Focus on mastering learning and understanding • Use standards such as deep understanding, self-improvement, etc.	• Focus on avoiding misunderstanding or avoiding not mastering a task • Use standards such as not being wrong, not doing things incorrectly on a task, etc.

Kroll and Ford (1992) assert that students with poor error detection skills and overrated comprehension were significantly more performance goal oriented. Research shows there are at least three sources of information that students may use for self-assessment: absolute standards (requirements of a task), self-referenced standards (one's own past performance), and social comparative standards (performance of others) (Wayment & Taylor, 1995). Dupeyrat et al. (2011) suggest that further research be done to investigate what type of information students use in their self-assessment process, which may help advance the understanding of biases in self-assessment and the relationships these have to mastery and performance goals. For this study, these goals were used to analyze the biases and criteria used by a student to identify how her goals motivated her evaluation of responses to a mathematical task.

Methodology

Case Study Selection

This research study was conducted at a large, public university located in an urban center in the southeastern United States. The initial phase of data collection took place in October 2017 in an introduction to proofs course. Of the 14 students enrolled at the time of recruitment, two math majors were asked to voluntarily participate in interviews for this study. These two students were targeted because the instructor perceived one to be an "average" student and the other to be a "strong" student based on past grades and their progress in the current course. The idea behind selecting two students with varying mathematical skill sets was to investigate how different students perceive and evaluate understanding in proof comprehension. By considering an "average" and a "strong" student in an introduction to proofs course, we hoped to find possible trends in the thinking of mathematics majors to design activities that would assist in helping students better excel in their future mathematics courses. For space limitations, we only focus on results from the "average" student, Tina. The courses taken and grades earned by Tina prior to the first interview included: Calculus I (B), Calculus II (B), Calculus III (C), Ordinary Differential Equations (B), Linear Algebra I (C), Mathematical Statistics I (A). Tina earned a C in the introduction to proof course in which she was recruited. The courses Tina completed and grades earned after this course, prior to the second interview included: Mathematical Statistics II (B), Analysis I (C), Linear Algebra II (B), Applied Combinatorics (A-), Analysis II (B-), and Introduction to Statistical Methods (A-). Tina consistently attended and was an active participant during the introduction to proof class and attended office hours as necessary.

The Task

As part of the curriculum for the introduction to proofs course from which Tina was recruited, students completed an assignment outside of the classroom pertaining to proof comprehension. This assignment included an indirect proof that $\sqrt{2}$ is irrational (see Table 2). Tina was asked to read the proposition and proof and answer the questions also seen in Table 2. Since this proof is not a typical proof provided to students for this statement, it may allow the opportunity to probe for students' problem-solving skills and sense-making abilities. We note responses to this task when it was assigned is what we refer to as *initial* or *original* data.

Table 2. Proof comprehension assignment given in class (including the proof and questions).

Proposition: $\sqrt{2}$ is irrational. **Proof**: 1. Let $A = \{b \in N: \exists a \in N \ni a^2 = 2b^2\}$. 2. Suppose A is nonempty. Then by WOP, A has a least element b_0. Let $a_0^2 \in N$ be such that $a_0^2 = 2b_0^2$. 3. Then a_0^2 is even which implies a_0 is even. 4. Substituting $a_0 = 2k$, we have $(2k)^2 = 2b_0^2$, and so $b_0^2 = 2k^2$ is even.	Questions: 1. Explain the purpose of statement #1. 2. Why does a_0^2 being even imply a_0 is even? 3. How do we know $\frac{a_0}{2}$ and $\frac{b_0}{2}$ are natural numbers?

5. Thus b_0 is even. 6. Now $\frac{a_0}{2}$ and $\frac{b_0}{2}$ are natural numbers and $\left(\frac{a_0}{2}\right)^2 = \frac{a_0^2}{4} = \frac{2b_0^2}{4} = \frac{b_0^2}{2} = 2\left(\frac{b_0}{2}\right)^2$. 7. Therefore $\frac{b_0}{2} \in A$, which is a contradiction. 8. Thus A is empty and $\sqrt{2}$ is irrational.	4. How do we know $\frac{b_0}{2} \in A$? 5. Why does $\frac{b_0}{2} \in A$ imply we have a contradiction? In particular, be sure to address what is being contradicted and why this completes the proof.

In April 2018, as part of an interview, Tina was given typed versions of her own and instructor responses to these questions, and told they were from two random students. Tina was asked to analyze each student's responses and provide her thoughts about them. We deem this as *simulated peer-evaluation*, similar to simulated peer-assessment described in Scott (2014, 2017), since there was some form of anonymity and no indication that Tina realized one of the responses was her own from the assignment six months prior. Tina was given about 30 minutes to read and write her comments to the two students' responses. This was followed by the semi-structured interview to probe for elaboration on her comments.

Finally, in May 2019, as part of a similar interview, Tina was given the same responses to the questions and asked to comment on them. However, now a self-evaluation task, she was informed that Student 1's responses were her own from 17 months prior (also given in her handwriting now) and the others were from a key, or instructor responses. Tina was given time to read and write her comments about these responses which was followed by a similar interview as before. This change in format between interviews was to help answer the research question pertaining to criteria used in self-evaluation versus simulated peer-evaluation. Both interviews were audio and video recorded and transcribed. Using the achievement goal framework (Elliot,1999; Pintrich 2000) described above, researchers analyzed all of Tina's written and verbal comments from both interviews. Analyses were compiled and discussed to identify trends in the data. Results from this analysis are presented in the following section.

Results

To answer the research questions in this paper, results are presented through the lens of the achievement goals described in Table 1 (Elliot,1999; Pintrich 2000). To illustrate many of the criteria identified as important to Tina, we provide her comments during both interviews pertaining to her own and the instructor responses to the comprehension question, "Why does a_0^2 being even imply a_0 is even?" (Table 2, Question two). Her comments for this question were chosen for this paper as they were representative of her overall achievement goals and criteria that were exhibited during her interviews.

Simulated Peer-Evaluation Interview (April 2018)

We provide the responses Tina was given to evaluate in both the simulated peer-evaluation and self-evaluation tasks during the interviews in Table 3. As seen in the table, Tina's response proved that the square of an even number is also even, erroneously asserting a direct proof of the converse of the statement proves the original statement. Then, we provide Tina's comments about these student responses in the 2018 simulated peer evaluation interview in Table 4.

Table 3. Anonymous, typed responses presented to Tina in first interview.

Question 2	Student 1 *(Tina's response on the original assignment in October 2017)*	Student 2 *(Instructor solution)*
Why does a_0^2 being even imply a_0 is even?	$a^2 = 2b^2 \leftarrow a^2$ here is even ⇨ Implies a is also an even number, if we have an even number => n = 2k If we square it $n^2 = (4k^2) => n^2 = 2(2k^2) => 2k^2 = m$, m is an integer and $n^2 = 2m$, which is an even number.	Suppose a_0 is odd. Then $a_0^2 = (2k+1)^2 = (4k^2+4k)+1$, an odd. Thus, by contraposition we have if a_0^2 is even then a_0 is even.

Table 4. April 2018 Simulated Peer-Evaluation Comments.

Written Comments - Student 1 Response	Written Comments - Student 2 Response
"This response is more accurate and a direct way to explain statement #3. By justifying what an even number is $\boxed{n = 2k} \leftarrow$ and explaining from there, directly, it is a straight forward way to help the proof of #3."	"This response is also true and I would say that the first response explains the statement more accurately."
Verbal Comments - Both Responses	
"This is my personal way of algebra, the best way to explain something like that; This one [pointing to Student 2] is good too but I feel like Student 1 is using the definition of even number, but you didn't' even have to use the definition of odd [pointing to Student 2]. So my preference is Student 1."	

In her comments to each response during the simulated peer-evaluation task, Tina talked about how well the response was *explaining* an idea. For example, in Table 4, she used phrases such as "a direct way to explain statement #3", "explains the statement more accurately", and "the best way to explain". Not included here, in her verbal comments about the responses for question three of the assignment, Tina stated, "Student 1…would be helpful to explain it to someone else, if someone was sitting there…" This is just one example of Tina focusing on the students' abilities to explain concepts to another person. By using phrases such as "a straightforward way to help the proof of #3" and focusing understanding and explaining ideas to another person, Tina indicated holding a mastery orientation with a tendency towards mastery approach versus mastery avoidance. We note that she displayed some signs of mastery avoidance in that she avoided faulting student work, even in situations when it was obvious that some statements did not make sense. This was consistent throughout her interview.

Self-Evaluation Interview (May 2019)

In May 2019, Tina was given the same responses and explicitly told the two sets of responses were her own (in her handwriting) and an instructor's. We note that since the last interview, she had completed several upper level math courses, listed in the *Case Study Selection* section of this paper. In Table 5 below, we present Tina's comments during this self-evaluation task.

Table 5. May 2019 Self-Evaluation Comments.

<table>
<tr><td><u>Written Comments - Her Own Response</u>
“The response is clear, uses aglebra to explain the question also uses previous knowledge of integers/even numbers. And is also broken down.”</td><td><u>Written Comments - Instructor Response</u>
“The response uses previous knowledge of contraposition which tells the grader/student has connected previous knowledge with current knowledge.”</td></tr>
<tr><td colspan="2"><u>Verbal Comments - Both Responses</u>
“For number 2, Student 1 it was clear, it was very broken down, it was understandable, and it was student language, and I felt using algebra to explain was neat…using the knowledge of integers, the previous knowledge and all of that…Here [pointing to the key]…it was using the contraposition…basically, uses previous knowledge. To use that is one level above Student 1 which means that the key answer was connecting the material together which is needed…”</td></tr>
</table>

A large shift in the type of comments made by Tina was noted in the analysis. She focused on how she believed an instructor or grader would read the response rather than how well ideas were explained. Although she continued to employ phrases like, “uses algebra to explain”, there were several instances in which she explicitly referred to a grader, saying things like “uses previous knowledge of contraposition which tells the grader/student has connected previous knowledge with current knowledge.” She often specified that she preferred her own responses to the instructor responses because of how they were “more broken down” and referred to her own work as using “student language,” as in Table 5. She also implied that since the instructor solution used a proof by contraposition, it was “one level above Student 1.” The analysis indicated she was trying to explain that a grader would be more satisfied to see a student using a proof by contraposition instead of a direct proof because she perceived it to be more advanced.

We note that even though Tina was aware Student 1 was her own work, she referred to these responses as “Student 1” during the interview. It may be possible this was to help her feel comfortable evaluating her own work while minimizing the feeling that she was being graded. This would support Scott (2017), since Tina may have been apprehensive about being criticized, which is a reason the author argues for simulated peer-assessment. It is also possible that after completing more upper level coursework, Tina may have begun to rely on self-referenced standards (one’s own past performance) and was conditioned to think about which response she felt would earn more points on an assignment instead of which response made the most sense to her. Often, she implied the more information a response had, the better it was. This, coupled with her focus on grades or being graded during the self-evaluation task, implies she developed a tendency to think a teacher would prefer unnecessarily providing knowledge in a response.

The summative analysis indicated Tina was focusing on how someone else would be analyzing or grading the responses instead of how someone else would be understanding the response as an explanation. Directly comparing her own responses to another and focusing on which would be assessed higher is evidence she was operating with a performance goal orientation with a tendency towards performance approach instead of performance avoidance. Tina’s lack of ability to find errors within her own responses and often favoring them over the instructor responses is consistent with Dupeyrat et al. (2011) and Kroll and Ford (1992) in that students with poor error detection skills and over-rated comprehension were significantly more performance goal oriented.

Discussion and Concluding Remarks

The purpose of this study was to identify criteria used by a student in simulated peer-evaluation and self-evaluation, how these criteria change over time/with more coursework, and how they change based on whether the student is comparing responses via simulated peer-evaluation or self-evaluation. Tina demonstrated that she felt the use of algebra/symbols, the amount of knowledge shown (focusing on making connections between previous and current knowledge), and how detailed a response was were important criteria in her evaluation. In the simulated peer-evaluation, she focused on how each response explained an idea if someone (typically a peer or fellow student) was reading it. In the self-evaluation task, Tina was more focused on how each response would be assessed if a grader were reading it. Thus, her orientation during the simulated peer-evaluation task was more focused on mastery, while her orientation during the self-evaluation task was more focused on performance.

We hypothesize that this shift in focus occurred for one or both of two reasons: (a) the ways in which she perceived to have been graded throughout her college experience had caused her to focus on how the responses would earn points on an assessment, or (b) since she was made aware one of the responses was hers and one was the instructor response in the self-evaluation task, she personalized the responses and began to reason where she should earn at least partial credit for her responses. The data indicate that in the self-evaluation task, she was conditioned to focus on a grade and was relying on her previous college mathematics performances and assessments to evaluate these responses. This is evidence she began to use her past performance experiences to analyze these responses (e.g., Butler, 1995; Régner, Escribe, & Dupeyrat, 2007). However, analysis indicated a shift in the information Tina used for assessment from absolute standards (requirements of a task) to self-referenced standards (one's own past performance) in a way that may have hindered her learning and proof comprehension, since her focus was no longer on sense-making. Thus, evaluation tasks or activities more focused on sense-making with no grade attached may be beneficial for students' proof comprehension skills. Such activities may help shift this focus from assessment to understanding, from performance to mastery goals.

Future research could further investigate differences in how students peer- and self- evaluate and how these methods change over time. Specifically, investigation into the differences in achievement goals of students in peer- and self- evaluations and how these affect student biases in evaluations/understanding would be useful. Through these types of investigations, researchers can identify obstacles math majors face in introduction to proofs courses that, if overcome, could help them develop a better foundation for evaluating their own work and thus, opportunity to better understand mathematical concepts and develop proof skills. There were several limitations to this study, one of which was this is one case study. Although Tina represented an "average" student in October 2017, this does not mean all average students would have similar responses, as each student constructs knowledge uniquely.

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A Calculus Student's Thinking about the Idea of Constant Rate of Change

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This study focuses on a student's thinking about a constant rate of change in the context of a conceptual approach of calculus (DIRACC). DIRACC calculus defines a constant rate of change as variations measured in two covarying quantities being proportional to each other. This paper also presents a theoretical framework that characterizes students thinking about the constant rate of change in relation to variation and covariation.

Keywords: calculus, constant rate of change, differentials, proportional reasoning, covariational reasoning.

Introduction

The idea of the rate of change and the constant rate of change play critical roles in calculus. Thompson (1994a) investigated some fifth grader's concept of speed, its relationship to concepts of rate and revealed a lack of productive distinction between the concept of ratio or proportionality, and the concept of rate. Another research by Thompson (1994b) about students' images of rate and understanding of Fundamental Theorem of Calculus (FTC) reported unproductive meanings for the concept of rate in calculus students. My primary goal is to study how students think about the idea of a constant rate of change conceptually, and therefore, I refer to a conceptual approach of calculus curriculum DIRACC (Thompson & Ashbrook, 2019). DIRACC calculus treats differentials as variables explicitly. A differential in 'x' is a variable whose value varies through the interval $(0, \Delta x]$ repeatedly (Thompson & Dreyfus, 2016). Therefore, dy is a variable whose value varies through the interval $(0, mdx]$, where m is the constant rate of change when dy is a variation in y as the value y varies in relation to the variable dx. To understand the rate of change function and the constant rate of change function conceptually, envisioning differentials as variables are central in DIRACC. When f represents a functional relationship between the variables x (independent) and y (dependent), and we say x varies smoothly through its domain, the differentials in x as dx are small variations in the variable's value as dx vary within tiny intervals of infinitesimal length (Δx) in x's values. Therefore, a function has a rate of change at the moment if, over a suitably small interval of x containing the moment, f's value varies at essentially a constant rate with respect to x.

In this study, I developed a two-dimensional theoretical framework to explain students thinking about the idea of the constant rate of change with respect to students thinking about variation and covariation. This study focuses on the idea of a constant rate of change using the notion of differentials (as variables) being proportionally related quantities. The research question motivated this investigation is: - How do students think about the idea of the constant rate of change in DIRACC?

Literature Review

Mathematics educators have been interested in students' understanding of proportionality, rate and ratio; conceptualization of rate of change functions, derivatives and differentiation in terms of modeling and graphical approach; and mathematicians' conceptual approaches for differentials and infinitesimals for some time being (Pulos & Tourniaire, 1985; Doerr & O'Neil, 2011; Orton, 1983 & Yoon, Byerley & Thompson, 2015). Results of these studies have provided insights to investigate students' understanding of the idea of a constant rate of change function and differentials as varying quantities and being proportional in variation of their measures. Thompson (2008) discusses that the constant rate of change is foundational to understand linear functions and subordinate ideas as the average rate of change, proportionality, and slope. Studies have discussed the disconnections students, and many teachers have as they see these ideas as a separate set of procedures and as associated with unrelated contexts (Lobato, 2006; Lobato & Siebart, 2002; Lobato & Thanheiser, 2002 & Coe, 2007). According to Thompson (2008), the common meanings students and teachers have for constant rate of change is insufficient to understand the foundation of linear functions. Hackworth (1995) reveals that students' meanings for a constant rate of change rarely includes thinking about two quantities changing together and do not involve all the possible changes in the value of one quantity being proportional to corresponding changes in the value of other.

There have been researches in the field of mathematics education that address the issue of students' images of two varying quantities changing with respect to one another (Carlson, 1998; Carlson, Persson & Smith, 2003; Carlson, Jacobs, Coe, Larsen & Hsu, 2002). Thompson & Carlson (2017), revisited the covariational reasoning framework and clarified that variational reasoning is a person's thinking of variation of a quantity, covariational reasoning is how a person envisions the change in one quantity or variables value is related to change in another quantity or variable's value; and how the person envisions both varying simultaneously with respect to one another. Castillo-Garsow (2013;2012 & 2010) argued that smooth images of change are more potent over chunky images of change. According to Castillo-Garsow (2013), smooth thinking allows students to think about completed change and change in progress within intervals and from one interval to another. To investigate students' thinking about a constant rate of change in this study I will be aware of 'chunky thinking' and 'smooth thinking' to make sense of students' cognitive activities during the tasks. Ely & Ellis (2018) argue that scaling-continuous reasoning is a productive way of thinking about key ideas in calculus like variation of quantities, the instantaneous rate of change, limit, differentiation, and the definite integral. By saying 'scaling-continuous reasoning,' the authors elaborated Leibniz's notion of infinitesimal increments. Their study entails the importance of continuous covariational reasoning (Thompson & Carlson, 2017) epistemologically for students to develop the foundational ideas of calculus. However, scaling continuous thinking does not rely on the image of motion (according to Newton) or an underlying parameter. Having Leibniz in mind, the authors proposed scaling-continuous reasoning as an alternative approach to smooth-continuous covariational reasoning, which aligns with Newton's way of calculus. Newton and Leibniz independently invented Calculus during the last third of the 17th century (Kleiner, 2001). They conceptually approached the concept of derivative differently. For Newton, the fundamental concept of Calculus is to recognize variables vary with time, and they vary at some rate every moment. DIRACC curriculum takes Newton's approach of Calculus as the underlying conceptual framework.

Thompson & Dreyfus (2016) surveyed seventeen textbooks, where they found that most of these do not mention differentials at all for single-variable calculus. According to them, "*a differential in* $'x'$ *is a variable whose value varies through the interval* $(0, \Delta x]$ *repeatedly.*" Therefore, dy is a variable whose value varies through the interval $(0, mdx]$, where m is the constant rate of change when dy is a variation in y as the value y varies in relation to the variable dx. In this approach, the idea of the differential is not based on the idea of derivative; instead, a differential is introduced consistently with a linear variation prior to the concept of derivative. Linear variation essentially represents the proportional relationships between the differentials (two covarying quantities) dy and dx, where m is the proportionality constant.

Theoretical Framework

In this paper, I am presenting a theoretical framework that emerged from a conceptual analysis of the idea constant rate of change in relation to variational and covariational reasoning. I have used some well-developed constructs and explained how they support the data analysis. Therefore, I suggest a two-dimensional theoretical framework- levels of students' thinking about the idea constant rate of change with respect to levels of students thinking about variation and covariation of varying quantities (variables).

Levels of Students' Thinking about the Idea of Constant Rate of Change-

Level 5 (L5CROC)- students think of the constant rate of change as a relationship between two covarying quantities remaining same. If y is a varying quantity that changes with respect to another varying quantity x then the rate of change $dy/dx = m(\Leftrightarrow dy = m.dx)$ always remain constant. In this level, students treat 'dx' and 'dy' as variables, and these variables represent tiny variations among x and y. Students thinking about a constant rate of change in this level entail strong proportional reasoning. Their meanings for the average rate of change for any function (linear/non-linear) in a tiny interval also entail proportional reasoning.

Level 4 (L4CROC)- students think about the constant rate of change as a relationship between two covarying quantities remaining same. If y is a varying quantity that changes with respect to another varying quantity x, then the rate of change $dy/dx = m$ $(\Leftrightarrow dy = m.dx)$ always remain constant. In this level, students treat dx and dy as small changes (not variations) in fixed intervals of changes (Δx and Δy). Students thinking about the average rate of change for tiny intervals in this level can also entail strong proportional reasoning for linear functions. However, their meanings for an average rate of change for any non-linear function in a tiny interval not necessarily entail proportional reasoning.

Level 3 (L3CROC)- students think about the constant rate of change as a relationship between two covarying quantities remaining same. If y is a varying quantity that changes with respect to another varying quantity x, then the rate of change $\Delta y/\Delta x = m$ $(\Leftrightarrow \Delta y = m.\Delta x)$ always remain constant. Students think about the constant rate of change in terms of fixed intervals (perceivable chunks) and fixed arbitrary small amount of change within those intervals in the domain. Students thinking about the average rate of change within those fixed intervals are associated with proportionality but their thinking about the average rate of change smaller changes within those fixed intervals not necessarily entail proportional reasoning for linear and non-linear functions.

Level 2 (L2CROC)- students think about the constant rate of change as a relationship between a varying quantity and a fixed number. For any value of x, they think y as a constant number that can be represented as a horizontal line graphically. In this level, students' thinking about the constant rate of change is not associated with proportionality.

Level 1 (L1CROC)- students' thinking about the constant rate of change as a relationship between two numbers (quantitative reasoning for this level of thinking is less reliable). Students' struggle to imagine the graphical representation of two quantities remain linear in this level, and their meaning for the idea of a constant rate of change lack proportional reasoning.

Levels of Students' Thinking about Variation and Covariation of Quantities-

Level 5 (L5VCO)- students' thinking about a variation of a varying quantity and covariation between two or more covarying quantities consistent with 'smooth-continuous variation/covariation' discussed by Thompson & Carlson (2017). A student thinks 'dx' and 'dy' as varying values of 'x' and 'y' changing by intervals anticipating that within each interval the variable's value varies smoothly and continuously. Also, for y as a variable depending on variable x, a student in this level envisions dy changing simultaneously with respect to dx smoothly and continuously and the ratio of the varying values of y and x remains same.

Level 4 (L4VCO)- students' thinking about variation and covariation is consistent with the construct 'scaling-continuous variation/covariation' discussed by Ely & Ellis (2018). A student using scaling-continuous reasoning does not necessarily treat dy and dx as varying values. However, they consider dy and dx as such tinier changes in such a scale that the smaller and smaller increment in quantities creates an image that can be an alternative to treat dy and dx as variables.

Level 3 (L3VCO)- students' thinking about variation and covariation align with 'chunky continuous variation/covariation'(Thompson & Carlson, 2017). A student thinks about variation of a variable's value or covariation between two covarying quantities as changing by intervals of fixed sizes. In this level, students treat dy and dx as smaller changes within a larger fixed amount of changes as Δy and Δx.

Level 2 (L2VCO)- students' thinking about variation and covariation is consistent with gross variation and coordination of values (Thompson & Carlson, 2017). A student in this level envisions some relationship of increasing or decreasing pattern when the quantities vary, but a student gives little or no attention how their values are changing within the quantities or if there is any multiplicative link between overall changes in two quantities values.

Level 1 (L1VCO)- in this level a student's thinking of variation or covariation can be consistent with any of the following constructs discussed in Thompson & Carlson (2017)- Discrete variation, no variation, variable as symbol, precoordination of values and no coordination.

A student in L5CROC is likely to associate with L5VCO. Thinking about differentials as variables is an outcome of being able to think in L5VCO. A student thinking about a constant rate of change in L4CROC can be associated with L4VCO or L3VCO. Students' sophistication of thinking about the average rate of change for non-linear function in a tiny interval will help to distinguish between L4VCO and L3VCO. As Ely & Ellis (2018) described scaling-continuous reasoning as an alternative of smooth-covariational reasoning and therefore, L4VCO might have some association with L5CROC but not as strong enough as L5VCO. Students having chunky continuous variation and covariation thinking L3VCO can be associated with L4CROC an L3CROC. The difference between association with L4CROC and L3CROC depends on students thinking about changes in terms of 'dx' and 'dy' as smaller changes within 'Δy' and 'Δx' or students thinking about the constant rate of change only in terms of 'Δy' and 'Δx'. The association between L2CROC with L2VCO and L1VCO and L1CROC with L2VCO and L1VCO can be flexible enough as all these levels produce weaker variational and covariational reasoning with unproductive meanings for the idea of a constant rate of change.

Methods

I used Exploratory Teaching Interview (ETI), which is inspired by one of the elements of Teaching Experiment (Steffe & Thompson, 2000). An ETI is a one-on-one interview consist of an interviewer, a single student, and video/audio recording devices. ETI was an appropriate methodology for this study as the spontaneous nature of creating on-moment hypotheses would support the research question to construct a model of students' understanding of the constant rate of change. The subject, Anna (pseudonym) who participated as an interviewee was a Calc 2 student. Anna was already exposed to the idea of the rate of change functions and accumulation function in Calc 1 using DIRACC. The methods of data analysis used in this study are qualitative analysis that is supported by grounded theory (Strauss & Corbin, 1994) and conceptual analysis (von Glaserfeld, 1995; Thompson, 2008) parallelly to construct a model of Anna's thinking.

The first interview question was 'What comes to your mind when you hear the term *constant rate of change*?'. The goal was to learn about Anna's free response to this question and her overall thinking about the idea constant rate of change. There were three other tasks for the ETI-

Task 1. *Jim ran 100 meters at a constant speed. He moved 0.6706 meters from 2.1 seconds after the clock started to 2.2 seconds after the clock started. What was Jim's speed from 6.81 seconds after starting to 6.87 seconds after starting?* The goal for using this task was to investigate whether Anna can reason that while Jim ran 100 meters at a constant speed, the speed remains the same for any fraction amount of time.

Task 2. *A common definition of a constant rate of change is given in section 3.15-*

y varies at a constant rate with respect to x if for a fixed amount of change in x (i.e., Δx) the amount of change in y (i.e., Δy) is constant. And another definition for the constant rate of change mentioned in 4.1 is- constant rate of change always involves two quantities varying together smoothly and continuously. Two quantities vary at a constant rate with respect to each other if, and only if, variations in one are proportional to variations in the other. Suppose that y varies at a constant rate with respect to x. Then dx is a variation (large or tiny) in x as x varies, and dy is a variation in y as the value of y varies in relation to the variation dx.

The relationship between variations is $dy = m \cdot dx$ for some number m.

Q. Do you think these two definitions have the same meanings for the constant rate of change? If yes, then why do you think they are same. If no, explain why they are not. The goal of this task was to investigate Anna's thinking about variation within a variable's value and her thinking about the constant rate of change in terms of differentials.

Task 3. The purpose of this task was to investigate Anna's understanding of the constant rate of change entailing proportional relationship within a large interval and super tiny interval. In this task, I used a GC animation file that showed a function's average rate of change over a tiny interval. However, from a zooming out position, the tangent line appears to be intersecting the graph of the function instead of being tangent over the interval. There is an image of the GC animation.

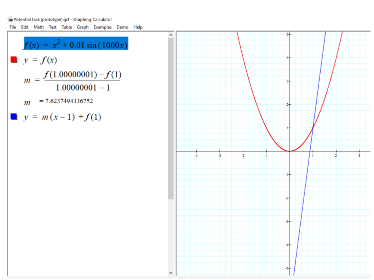


Figure 1. A picture of the GC animation.

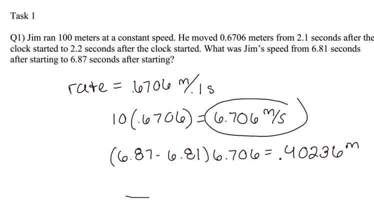
Results

The interviewer initially asked Anna to explain what comes to her mind when she hears the term *a constant rate of change*. Anna responded that she thinks about a rate of something that is changing with time and the rate remains the same relative to the time. According to Anna, her thinking about the rate of change of any quantity relative to time is a result of the kind of examples and instructions she had received from her DIRACC experience. Then the following interaction took place-

Interviewer: Do you think it is necessary that the change of a quantity needs to be with respect to time only or you imagine any two quantities changing with respect to one another?

Anna: Any two quantities; if you have a rate of change of a quantity relative to another quantity as long as they are proportional and related, they have a constant rate of change.

While working on task 1, Anna first figured out Jim's constant speed in meters per second. To her, if Jim moved .6706 meters in 0.1 seconds then in one second Jim has moved 6.706 meters, and his speed is 6.706 m/s. She went continued by calculating Jim's distance traveled from 6.81 seconds to 6.87 seconds but then realized that she had gone beyond the question that was posed. Anna mentioned that Jim's speed remains the same as 6.706 m/s as he is running 100 meters at the same speed. She described, the distance with respect to time remains proportional in this situation. While working on the task, she wrote speed as time and explained that she feels comfortable to think about speed as a rate of distance with respect to time. Here we present her work for the task below-


Task 1

Q1) Jim ran 100 meters at a constant speed. He moved 0.6706 meters from 2.1 seconds after the clock started to 2.2 seconds after the clock started. What was Jim's speed from 6.81 seconds after starting to 6.87 seconds after starting?

rate = .6706 m/.1 s

10(.6706) = 6.706 m/s

(6.87 - 6.81) 6.706 = .40236 m

Figure2. Anna's work for Task 1.

The thinking Anna exhibited when responding to Task 2 revealed her conception of the idea of change and the notion of differential when considering a context involving a constant rate of change. To Anna, initially, both definitions seemed similar. The interviewer asked her to explain why they seemed the same to her; she took some time to think about it and decided they are not the same. She answered by saying that Δx intervals are fixed and dx is a change since x starts varying within any interval. Anna conveyed that, it is possible to make the size of Δx's very, very small, but they would still be fixed. Anna relied on dx to vary the variable x. She mentioned that she was not paying enough attention to the vocabulary of the textbook and used her common language-

Interviewer: In your common language, do you think Δx and dx have the same meanings?

Anna: No. Δx is a fixed interval, and fixed change and dx is a movement of x within that interval.

Interviewer: How would you describe the movement dx?

Anna: the change in x.

Interviewer: As you said, Δx represents a change in x, and dx also represents change in x. Are the changes the same or different?

Anna- Δx is fixed, and dx varies. dx is never fixed; it is always varying.

For task 3, the interviewer explained what appears in GC window and asked how the tangent line for the function f would look like over the tiny interval between 1 and 0.00000001. Anna said there would be a little straight line touching the curve near that interval. The interviewer then graphed it, and an intersecting line appeared. Anna mentioned that it is not what she expected it, but the tangent line would be touching the curve over that tiny segment. Anna further explained that she expected to see a small portion of the original function where the tangent line would look like it touches the graph of the function. The interviewer zoomed into the tiny interval, and Anna mentioned that then it looked like what she expected. The interviewer zoomed enough, and a triangle showed up in GC graph over the interval between 0.00000001 and 1. The interviewer asked how the change in x is related to the change in y over that interval. Anna mentioned that they are changing together at a constant rate. The following interaction occurred-

Interviewer: Can you tell what the constant rate of change is within the interval between 1 and 0.00000001?

Anna: (focusing on the triangle that appears in GC and the average rate of change in GC window) hypotenuse of the triangle is the constant rate of change m for this section.

Interviewer: How would you represent the relationship between two covarying quantities within this interval?

Anna: dy is equal to this m times dx (she writes $dy = (7.6240)dx$).

Interviewer: So, you mean within this interval the variation in dy and the variation in dx....(did not let the interviewer finish)

Anna: the variations in dy and the variation in dx are going to be proportional to each other.

Discussion and Conclusion

The findings suggest that Anna was consistent in thinking about the proportional relationship between two covarying quantities throughout the tasks focused on the idea of a constant rate of change. The tasks presented in this study do not focus explicitly on how students think about variation in a variable's value. However, tasks 2 and 3 provide some insights on students thinking about variation within a variable's value. Anna demonstrated an awareness of the difference between the notion of change and the notion of differential, a critical distinction DIRACC intends for students to make. Her thinking about the idea of a constant rate of change aligns with the L5CROC described in the theoretical framework. Her thinking about the idea of a constant rate of change entails proportional reasoning between two covarying quantities. Anna's thinking about variation and covariation was close to L5VCO. It is difficult to make explicit comment about her variational thinking as none of the tasks mentioned in this study investigates students' thinking about variation in a variable's value in detail. However, Anna showed no signs of chunky thinking about variation in any of the tasks. Anna treated dx, a differential as a varying quantity in both tasks 2 and 3.

Therefore, DIRACC, as a conceptual approach of calculus, has potential to encourage students' thinking about variation in a variable's value and covariation when two quantities are varying together smoothly and continuously. The revised definition of the idea of a constant rate of change in terms of differential in DIRACC entails proportional reasoning when calculus students think about the idea of a constant rate of change. The theoretical framework mentioned in this paper can also be used to explain precalculus and traditional calculus students' thinking about the idea of a constant rate of change in association with variational and covariational thinking.

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The Role of Gestures in Teaching and Learning Proof by Mathematical Induction

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When talking about mathematics, teachers and learners actively use hand gestures to support their speech as well as to describe ideas that are not expressed verbally. In this study, I investigate the gestures that were utilized by an instructor and his students during a teaching episode on proof by mathematical induction. Alibali and Nathan's (2012) typology of gestures are employed to code the observed gestures. The study reveals that the use of gestures plays an integral role in teaching and learning induction. I show that pointing gestures helped to reduce ambiguity in classroom discussion, representational gestures were useful in describing specific subcomponents of induction, and, finally, metaphoric gestures were independently introduced by a teacher and students to describe the nature of proof by mathematical induction.

Keywords: gestures, embodied cognition, proofs, mathematical induction.

Literature Review

A number of scholars argue that gestures convey meaning and should be considered as an important part of communication (Lakoff & Nunes, 2000; Goldin-Meadow, 2003; Radford, 2003). People use gestures not only to support their speech, but also to describe ideas that are not expressed verbally, even without realizing it. Goldin-Meadow (1997) characterized gesture as a "window to the mind". For this reason, gestures have become the focus of attention for many psychologists, neuroscientists and educators; a comprehensive analysis of gestures may help to understand the way of human's thinking.

The investigation of gestures in mathematics takes place within a philosophical perspective that frames cognition as an embodied phenomenon. Wilson (2002) described the nature of embodied cognition by noting that "the mind should be understood in the context of its relationship to a physical body that interacts with the world" (p. 625). As a result, human cognition has deep roots in sensorimotor processing. Alibali and Nathan (2012) argued that the cognition is embodied in two senses – being based in perception and action, and being grounded in physical environment.

Gestures play an important role in the learning and development of children (Piaget, 1959). Research confirms that gesturing facilitates students' learning of mathematics in different contexts, such as learning to count (Alibali & diRusso, 1999), symmetry (Valenzeno, Alibali & Klatzky, 2003), equivalence (Singer & Goldin-Meadow, 2005), ratio and proportion (Abrahamson, 2003), motion and graphing (Nemirovsky, Tierney & Wright, 1998). Students often benefit from their gesturing while engaging in challenging, abstract mathematics since the hand gestures help to convey meaning without requiring an overwhelming amount of cognitive resources (Cook, Duffy & Fenn, 2013; Goldin-Meadow, Nusbaum, Kelly & Wagner, 2001). Gestures may help to represent concepts students find difficult to remember and allow learners to coordinate abstract mathematical relationships when processing multiple tasks (Alibali & Nathan, 2012; Ping & Goldin-Meadow, 2010). Gestures, therefore, are key elements in students' processes for knowledge objectification. Gestures help learners to put together different pieces of information and understand conceptually difficult mathematical objects.

Gestures are also actively used by teachers (Flevares & Perry, 2001; Neill, 1991). Literature suggests that students may benefit from their teachers' gestures. Research confirms that learners can detect conceptual information expressed in gestures, and that information that teachers express in gestures facilitates learning (Kelly & Church, 1998). Many studies have shown improvement in students' performance on a posttest after lessons taught by a teacher actively using gestures, compared to lessons that did not contain gestures (Church, Ayman-Nolley, & Mahootian, 2004).

Understanding the connections between different mathematical objects plays an important role in learning mathematics, and guiding students to make those connections is an important aspect of instruction. Several studies aimed to identify how teachers communicate connections among ideas in instruction in classroom settings (for example, Richland, Zur, & Holyoak, 2007). Their findings confirm that teachers often employ gestures to refer to the ideas being connected.

Proof by mathematical induction is known to be conceptually difficult for undergraduate students (Harel, 2002; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007). This method is used to prove that the statement P(n) holds for any natural number n. To prove P(n) by mathematical induction, one must check two assumptions: (a) the validity of P(1) (the base case), and (b) if the statement P(k) is true for some natural number k, than it is also true for P(k+1) (inductive implication). The purpose of this case study is to investigate the role of gesturing in teaching and learning proof by mathematical induction. More specifically, the study is guided by the following research questions:

- *How does the instructor use gestures in teaching proof by mathematical induction?*
- *How do students use gestures in learning proof by mathematical induction?*

Theoretical Framework

I draw on Alibali and Nathan's (2012) typology of gestures manifesting embodied cognition. *Pointing gestures* reflect the grounding of cognition in the physical environment. This type of gestures is the most commonly used in mathematics (Alibali, Nathan & Fujimori, 2011). Pointing gestures are usually used to indicate objects, location, inscriptions or students.

Representational gestures convey simulations of action and perception and depict semantic content, literally or metaphorically, with the help of handshape or motion trajectory. According to Alibali and Nathan (2012), actions and perceptions are intimately linked: "when humans perceive objects, they automatically activate actions appropriate for manipulating or interacting with those objects" (p. 254). From this perspective, actions and perceptions are similar and it does not really matter what exactly, action or perception, a gesture represents.

Metaphoric gestures are a subset of representational gestures and reflect conceptual metaphors that are grounded in the body. These metaphors transmit understanding and perceptions of the reality. The conceptual metaphors that underlie mathematical ideas are often expressed in the gestures that humans produce when speaking about these ideas. Thus, metaphoric gestures provide evidence for psychological underpinnings of mathematical concepts.

Data Source

The study utilized a 40-minute teaching episode on proof by mathematical induction in a large research university in the southeastern United States. The course is a junior-level mathematics course designed to teach mathematics major students typical mathematical proof techniques. The participants were a white male mathematics professor and 20 students, who agreed to participate in the study. The video-data were part of a larger project studying teacher-students' interactions during instruction on mathematical induction. The videotapes were completely transcribed by the author, and the segments of classroom discourse when the teacher or a student gestured were marked and recorded on the transcripts. These segments were then analyzed qualitatively for the types of gestures that were used and the role the gestures played in facilitating the construction of shared meaning between the teacher and students.

Methods and Procedures

Before documenting any of the teacher's gestures, the author watched the videotapes and read the transcripts a few times. Then, using classroom videotapes, all of the teacher's and students' gestures were time indexed for starting time and duration, and classified in accordance with Alibali and Nathan's (2012) framework. Given that gestures occur very quickly, I re-watched the videotapes to assure that all the gestures and accompanying speech were documented. The gestural episodes that did not refer to mathematical instruction or mathematical conversation were omitted from the analysis.

Results and Discussion

In total, there were 132 instances of teacher gesturing and 15 incidents of student gestures. The huge difference between the number of teacher and student gestures may be explained by the fact that the observed teaching episode was an introduction to proof by mathematical induction, and, therefore, a considerable part of classroom discourse was held by the instructor. I report on the use of gesturing in accordance with Alibali and Nathan's (2012) typology.

Pointing gestures

Alibali and Nathan (2012) argue that pointing gestures reveal indexing of speech to the environment. In this subsection, I present how the instructor and students utilized pointing gestures to index mathematical ideas in proof by induction to the physical world.

The data confirm Alibali, Nathan and Fujimori's (2011) claim that pointing gestures constitute a majority of gesturing: 70% of the teacher's and 54% of the students' gestures may be characterized as pointing gestures. All the instances of pointing gestures were accompanied by an utterance "this," "that," or "it," or by addressing a mathematical symbol written on the board. The use of pointing gestures not only supported verbal communication between teacher and students, but also helped to circumvent the ambiguity of the words "this" and "it" in a classroom discourse.

Figure 1. Example of a pointing gesture.

One of the key aspects of teaching proof by mathematical induction is helping students distinguish between the truth for proposition P(k) versus the implication P(k) → P(k+1). During the observed teaching episode, the instructor purposefully pointed to either P(k) or to the sign of implication written on the board to indicate which one he referred to (Figure 1). However, students' responses show that they did struggle to understand the difference:

Teacher: What integers do we know that the proposition works for? … I know, **it** works for 1. How do I know, **it** works for 2?
Student: I'm saying that there is an integer $k \geq 1$. So, we can say that k is equal to 1. So, we only know that k works for 1 and 2.
Teacher: How do we know that **it** works for 2, that's my question… How do we know that **it** works for 3?
Student: Because the P(k), k is equal to 1.

Here, the student conflated the truth for proposition P(1) with the truth for implication $P(1) \rightarrow P(2)$. In this example, the teacher's word "it" was not supported with the corresponding pointing gesture and, consequently, could lead to confusion among the students.

Representational gestures

As it was noted by Alibali and Nathan (2012), "representational gestures simulate real-world objects that ground or give meaning to mathematical ideas" (p. 264). The present subsection describes how this type of gesture was employed in the classroom.

Any statement that may be proved by mathematical induction contains quantifiers. In order to prove that implication $P(k) \rightarrow P(k+1)$ is valid *for any* k, one should first assume that P(k) is true *for some arbitrary* natural k. The difference between these two quantifiers is subtle but critical. That is why it is crucial for students to understand the role of quantifiers in proof by induction.

The instructor and students used representational gestures in multiple ways. However, a large majority of teacher's gestures were aimed to represent quantifiers "for all", "for any" and "for some". The instructor repeatedly relied on the hand gestures that appeared like a fountain (spreading out and up) when he referred to the goal of the task – to prove the validity of proposition P(k) *for all* positive integers:

Teacher: Ok, let's put your ideas together. We know that P(1) is true. And there is some $k \geq 1$ s.t. the proposition is true for k implies the proposition is true for k+1. Is that enough to know that the proposition is true **for all** (Figure 2) natural numbers?

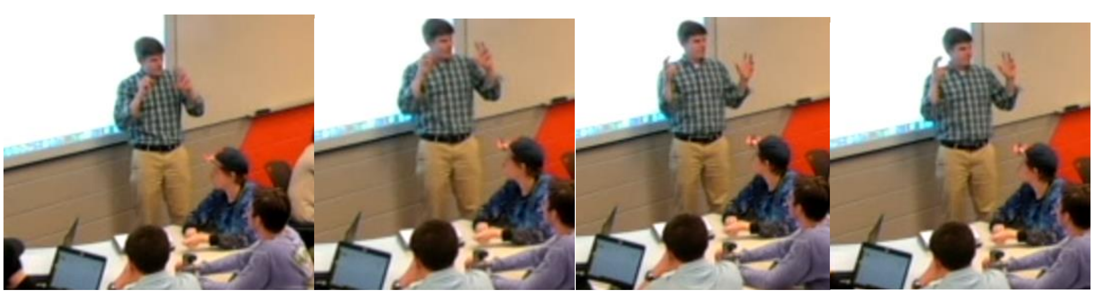

Figure 2. Example of a representational gesture used by a teacher to illustrate the quantifier "for all" ("fountain gesture").

In contrast to the "fountain" illustrating "for all" quantifier, the teacher temporarily fixed his hands motionless in the air to indicated that he fixed "some arbitrary" integer k:

Teacher: The “for all” is really important here as we just saw. So, if for all n, working for n implies working for n+1. We’d be done. Now I’m not gonna get really picky about this, but some people will want you to change the letter here. Because with n we’re talking about general statement that’s supposed to work for all n. And I might switch it to k for the step that [one of the students] was just talking about because I might want to say ‘ok I’m just talking about some arbitrary value (Figure 3). I’m not talking about all natural numbers anymore.’

Figure 3. Example of a representational gesture used by a teacher to illustrate the quantifier “for some arbitrary” (“rigid hands”).

Another purpose of using representational gestures was to attract students’ attention to the logical implication between propositions P(k) and P(k+1). Norton and Arnold (2017) showed that the implication P(k)→P(k+1) may be considered as either an action that transforms P(k) into P(k+1) or as an object representing an invariant relationship between P(k) and P(k+1). Dubinsky (1991) hypothesized that treating implication as a single object is crucial for promoting students’ understanding proof by induction. For this reason, the implication P(k)→P(k+1) deserved considerable attention during the observed teaching episode.

The teacher actively involved hand gesturing to facilitate the discussion on implication between P(k) and P(k+1). Typically, he used his hands in the air to track the trajectory of transition P(k) ↔ P(k+1):

Teacher: I know P(k+1) only in this [motions forward] direction. You can do an induction in both directions. You can try to show that P(k+1) **implies** P(k) (Figure 4).

Figure 4. Example of a representational gesture used by a teacher to illustrate the implication.

The students, in their turn, used similar gestures describing implication.

Teacher: All I see is 1!
Student 1: But it works for all integers.
Student 2: When k is 1 and we know 1 works then P(1+1) works (Figure 5), so 2 works.

Figure 5. Example of a representational gesture used by a student to illustrate the implication.

Metaphoric gestures

Through metaphoric gestures people put an abstract idea into a more literal, concrete form. Metaphoric gestures are similar to representational gestures in that they have a narrative character, but the images produced relate to abstract objects and processes. McNeill (1992) proposed that mathematicians have distinctive gestures for mathematical terms, and that these gestures are "somewhere on the road to a gesture language, but not all the way there" (p. 164). Thus, these are metaphoric gestures, where abstract mathematical objects are situated.

The teacher used metaphor "engine" to informally describe the idea of proof by mathematical induction:

Teacher: … So there's a lot of conjectures y'all been struggling to prove that kinda go on forever. Like that Fibonacci one where every third….umm...element of the Fibonacci sequence is even. And you start going and going and going and you can't go on forever. This [points to inductive implication] goes on forever for you. This is the **engine** that's doing all the work for you. So that's what we want to try to do. Is try to apply this **engine**.

Interestingly, a metaphoric gesture reminiscent of an engine was introduced by one of the students right before the teacher used the metaphor "engine" for the first time:

Student: Then you can plug 2 back in for k and **the logic repeats itself**.

The student revolved his hands around each other to support his idea about the repeating logic. Notably the teacher was not looking at the gesturing student (Figure 6). For this reason, it seems that the teacher *independently* came up with the gesture (Figure 7), which was employed whenever the metaphor "engine" was used.

Figure 6. One of the students is using a revolving hand. Gesturing student is put inside a white circle.

Figure7. The teacher uses hands revolving around each other to address to the metaphor "engine".

Conclusion

In line with prior research (e.g., Alibali & diRusso, 1999; Valenzeno et al., 2003; Singer & Goldin-Meadow, 2005; Abrahamson, 2003; Nemirovsky, Tierney & Wright, 1998)), the teacher and students actively used gestures in classroom discourse, and different types of gestures played different roles in mathematical conversation. This study contributes to prior research by demonstrating how different kinds of gestures played different roles in discourse on mathematical induction. Pointing gestures reduced ambiguity when the teacher was using the words "it," "this, or "that." Using this type of gesture, the instructor attempted to disambiguate the students' conflation between the truth for proposition P(k) versus the implication P(k) → P(k+1). When the ambiguous word "this" was pronounced without gesturing, some of the students immediately became confused. Thus, the study confirms students' difficulty identified in the research literature about distinguishing P(k) and P(k) → P(k+1) (Dubinsky, 1991; Norton & Arnold, 2017), and suggests pointing gestures as an instructional tool for ameliorating this difficulty. Further, the teacher employed numerous representing gestures to draw students' attention to the subcomponents of mathematical induction, such as quantification and logical implication. Some of these gestures were readily adopted by students. Finally, the results confirm the hypothesis that people deliberately employ metaphoric gestures when talking about abstract mathematics (McNeil, 1992; Alibali & Nathan, 2012). In this study, metaphoric gesture of "revolving hands" was independently used by teacher and students when talking about the "engine" metaphor, describing the nature of proof by mathematical induction. Future research may be conducted on comparison other instructors' gestures in teaching proof by mathematical induction.

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One Mathematician's Epistemology of Proof and its Implications for her Comments and Marks on Students' Proofs

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Balacheff (2008) offers the notion of epistemology of proof to account for researchers' views on what a mathematical proof is from a teaching-learning perspective. This study argues that an awareness of mathematicians' epistemologies of proof might provide insight into their rationales and actions when teaching university students how to prove. This argument is illustrated by the case of one mathematician who provided comments and marks on proofs submitted by her graduate students as part of her course instruction of topology. A commognitive framework is mobilized to discern the mathematician's epistemology of proof.

Keywords: commognition, mathematicians' pedagogy, proof and proving, topology, university mathematics education

Introduction and Background

In the last few decades, several scholars have recognized that researchers use the notions of 'proof' and 'proving' in profoundly different ways (e.g., Reid, 2005). Balacheff (2008) maintains that these usages come hand in hand with researchers' *epistemologies of proof* and *rationalities*. The former notion captures one's relationship with truth and validity as it determines one's "view of what a mathematical proof is from a teaching-learning point of view" (p. 502). The latter notion is associated with "the system of the criteria or rules mobilized when one has to make choices or decisions, or to perform judgements" (p. 502). Balacheff illustrates research rationality by the choice of a theoretical framework, questions to explore, and methodology. He also argues that awareness of researchers' epistemologies of proof is instrumental in interpreting their research rationalities and findings.

This study extends Balacheff's (2008) perspective beyond researchers and proposes that an awareness of teachers' epistemologies of proof might be valuable for making sense of their proof-related rationalities and practices, including mathematicians' teaching of mathematics to university students. Then comes the question of how mathematicians' epistemologies of proof can become accessible. In contrast to those researchers who explicate their epistemologies in their papers (see Balacheff, 2008 and Reid, 2005 for analyses and examples), there might be not many mathematicians who voluntarily share their well-defined views (Sfard, 1998). One way around this could be to solicit these views through direct questioning (Alcock, 2010; Harel & Sowder, 2009; Weber, 2012). An alternative approach taken in this study assumes that a discourse – talk and practices – that mathematicians propagate in their classrooms can be productively leveraged to construct their epistemologies of proof; if not in an exhaustive sense, then at least to delineate some of its key features.

Let me illustrate the above assumption with university courses that the literature often names 'transition-to-proof' (e.g., Alcock, 2010; Moore, 2016). Even before dwelling on the pedagogies and curricula of these courses, their title tells us something about the epistemologies of the mathematicians who institutionalized them. Indeed, the word 'transition' summons a discourse where the prior mathematics education of students taking such a course is positioned as not having addressed proofs in the same way that this course does. Also, while most mathematics courses are structured around specific content areas (e.g., calculus, linear algebra, topology), this course posits that it is possible to learn and teach proofs as a single content-general activity. (To appreciate the particularity of this position, consider the last time you encountered a course entitled 'transition-to-problem solving' or 'transition-to-definitions'.) I am acquainted with a few mathematicians who reject these positions under the premise that proving is fundamental to all mathematics disciplines, and thus it should be discussed in all content-specific courses. Some of these mathematicians eschew teaching 'transition-to-proof' courses to keep a 'corrupted' practice at arm's length. Such eschewing showcases how teachers can enact their epistemologies of proof even without entering a classroom.

To illustrate the potential usefulness of the extended notion of Balacheff, I explore the *enactments* and *affordances* of the epistemology of proof of one mathematician. The enactments under scrutiny pertain to the written feedback that she provided on students' homework assignments as part of the course instruction. The focus on feedback-providing has been instigated by recent studies arguing for the instrumentality of this practice for students' proof comprehension, capability to write proofs, and development of the notion of proof (Miller, Infante, & Weber, 2018; Moore, 2016; Moore, Byrne, Hanusch, & Fukawa-Connelly, 2016). Borrowing from Hutchby (2001), I associate 'affordances' with "functional and relational aspects which frame, while not determining, the possibilities for agentic action in relation to an object" (p. 444). In this study, the agentic actions are the mathematician's teaching and the students' learning how to prove. The central agent of this study is a practicing topologist whose actions unfolded in a topology course that she taught to a small group of graduate students.

Theoretical Framework

I draw on Sfard's (2008) communicational framework of *commognition*[1] to capitalize on its epistemological and analytical potential. This framework has been widely acknowledged by the international mathematics education community (e.g., Felton & Nathan, 2009; Sriraman, 2009; Wing, 2011), and specifically, by researchers interested in university teaching and learning (e.g., Nardi, Ryve, Stadler & Viirman, 2014). Commognition focuses on *discourses*, which are defined as "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). Proving, from this standpoint, can be viewed as a constituent of a broader mathematical discourse, where particular routines and narratives are endorsed (i.e. sanctioned as 'proofs') by those who are positioned as a mathematical authority in their communities[2]. This view is in accord with the social perspective on proving as a human endeavor that aims "to persuade the appropriate mathematical community to accept the knowledge as warranted" (Ernest, 1994, p. 34). In a university setting, a communication of this ilk is typical to courses where students are asked to capture their knowledge in texts and submit them for warranting by a mathematician, who is the course representative of a broader mathematics community.

Some scholars expand the original scope of commognition to account for the practices of mathematics teaching (e.g., Nachlieli & Tabach, 2015). Combining this framework with Cultural Historical Activity Theory (Engeström, 1987; Roth & Lee, 2007), Nachlieli and Tabach (2015) define *teaching* as a communicational activity where a pedagogue leads a classroom discourse

[1] The neologism brings to the fore the intimate relations between communication and cognition, which is the cornerstone of the framework.
[2] Sfard (2008) distinguishes between discourses through keywords, visual mediators, routines and narratives. For the purposes of this study, it is sufficient to associate routines and narratives with specific mathematical processes and their outcomes.

with the aim of bringing a change to the individual discourses of the students. Changing one's discourse in a lasting way is a commognitive interpretation for *learning*.

One of the cornerstones of commognition is *objectification*. This term is used to refer to a special way of verbal communication about non-tangibles (e.g., mathematical concepts) as living outside of human discourses, just as their material congeners do. Objectification is achieved through the mechanisms of *reification* and *alienation*. "Reification is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44). Alienation erases the human agency from a narrative, which results in impersonal sentences (an elaborated example is presented in the section on Dr Brownstone's epistemology of proof).

The Study

Dr Brownstone, who is in the focus of this study, actively participates in (at least) two communities of a mathematical discourse: a research community that redefines the edges of topology as a mathematical discipline; and a course community that, at the time of data collection, encompassed students at the beginning of their professional journeys, while possessing a sizable experience in university-level mathematics already. This study delineates key features of Dr Brownstone's epistemology of proof through attending to its enactments in and affordances to discourses that took place in the course. The delineation revolves around written constituents of these discourses: *texts* that students submitted as a response to homework assignments asking them to prove mathematical statements, and *feedback* (i.e. comments and marks) that Dr Brownstone provided on them.

I report on an explorative case study, the data for which were collected from a one-semester long course in topology that was offered in a large English-speaking university in the Southern Hemisphere[3]. The course was concerned with standard topics in point-set topology (e.g., continuity, convergence, compactness) and in algebraic topology (e.g., covering spaces, fundamental groups, homology theory). Overall, six students participated in the course, four of whom were studying towards post-graduate degrees in mathematics, one student who was completing his undergraduate major in mathematics, and another student who was a beginning university lecturer who, in her own words, "was filling educational holes".

Dr Brownstone is a well-regarded topologist with dozens of publications in professional journals. She has more than a decade of teaching experience in school and university mathematics, and a genuine interest in improving her students' learning. Towards the middle and at the end of a twelve-week long semester, Dr Brownstone designed two homework assignments that were intended for individual solution by the course students. The assignments comprised of four and five problems respectively that asked students to prove topological statements (see Figure 1 for example). The students were given nearly two weeks to complete each assignment, each of which contributed 7.5% to their final grade.

The data for this study came from: (i) written comments and marks that Dr Brownstone provided on students' submissions as part of her course teaching, and (ii) audio-recorded semi-structured interviews that I conducted with her a few days before and after she completed (i). In the one-hour-long session that preceded (i), Dr Brownstone shared her general approach to assessing students' assignments and situated it in the broader set of her pedagogical views. The session ended with Dr Brownstone asking, "So how do you want me to go around [assessing] the assignments this time?" and me proposing that she apply her regular assessment practices. My

[3] This general description is aimed at protecting the confidentiality of Dr Brownstone and her students.

response was driven by the acknowledgment of her teaching professionalism and this study's interest in her routinized teaching practices. When asked a posteriori about how her assessment was different from usual, she replied, "I assessed in a way that I would usually do. Since I knew that we were going to talk about it, maybe I was a bit more detailed in my comments to the students."

Cantor's ternary set, C, is defined to be the intersection of the sequence $C_0, C_1, \ldots$, where $C_0 = [0,1]$ and for $i > 0$, C_i is the union of 2^i closed intervals obtained from the 2^{i-1} closed intervals of C_{i-1} by removing from each of these intervals the open middle third of the interval. Thus

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}$$

Prove that C is homeomorphic to $\mathbf{2}^{\mathbb{N}}$ where $\mathbf{2} = \{0,1\}$ and has the discrete topology, and $\mathbf{2}^{\mathbb{N}} = \prod_{\alpha \in \mathbb{N}} X_\alpha$ where $X_\alpha = \mathbf{2}$ for each $\alpha \in \mathbb{N}$.

Figure 1. A task from the course homework assignment

Once the students submitted their assignments, I marked and commented on their solutions to each problem in parallel to Dr Brownstone and without accessing her feedback. When doing so, I attempted to mobilize a social perspective on proof as a preparatory step to analyzing Dr Brownstone's feedback, which I expected to be driven by epistemologically different perspectives. Then, I systematically juxtaposed both feedbacks through the constant comparison technique (Glaser & Strauss, 1967). To be explicit, by engaging in the same practice as Dr Brownstone, I aimed at raising my own sensitivity to the choices of and rationalities behind the actions that she carried out.

My analysis of Dr Brownstone's feedback started with identifying words and phrases that appeared frequently in her comments and continued to identifying patterns in their usage. The patterns emerged from iterated examinations of the relationships between Dr Brownstone's marks, general-level comments that she provided on texts that students submitted as a response to a problem, and line-by-line comments that referred to particular sentences and inferences that students made. Emerging patterns were contrasted within each assignment and between assignments submitted by different students. Some clarification questions and hypotheses emerged at this stage, and they were discussed through a forty-minute long interview with Dr Brownstone. There, she was asked to elaborate on selected comments and clarify her intentions and messages to the students. These elaborations and clarifications led to another analytical cycle that resulted in identifying preliminary features of her epistemology of proof. These features were discussed with her in another half-an-hour-long interview, which led to additional rounds of analysis and modification.

Dr Brownstone's Epistemology of Proof

I start with central features that were identified in regard to Dr Brownstone's epistemology of proof. The space limitations allow me to give only one example of how this epistemology was enacted. Additional examples will be shared in my research presentation.

Proofs as objects requiring detection and representation

When providing feedback on students' texts, Dr Brownstone extensively used the word 'proof'. While commenting on 'proofs' in the assignments that requested students to prove mathematical statements may seem like the most obvious thing to do, commognition is sensitive to instances where an interlocutor refers to discursive creations as self-sustaining entities rather than to the human actions that produced them. In conjugation with such adjectives as ‘correct’, ‘clear’ and ‘thorough’, ‘proofs’ almost start to sound like material objects that are capable of living outside the course discourse that the students and Dr Brownstone maintained through writing[4]. Such objectified talk also delivers the message that the property of proof-ness is ingrained in the texts that the students submitted regardless of time, place, and the reading audience.

The general-level comments that Brownstone provided were saturated with sentences attesting to the students’ mathematizing. For instance, “you articulate your arguments very well”, “you have a very good grasp of how to accurately express terms”, and “a well presented proof”. Indeed, once a ‘proof’ is treated as an extra-discursive “thing-in-the-world”, the students’ role becomes to “articulate”, “express” and “present” it. Metaphorically speaking, such a relation between ‘proofs’ and students resembles the relation between a client and an attorney: the client exists regardless of the attorney’s actions, but these actions are still of significance in the court of law. In our case, the assignment requests “to prove” mathematical statements ensured the existence of ‘proofs’ regardless of whether students detect them or not. Once detected, the students’ role becomes to represent ‘proofs’ properly, i.e. in alignment with the rules of a literate mathematical discourse.

Interestingly, the interviews revealed that in the course discourse, a ‘detection of proofs’ is not always tantamount to ‘students constructing proving texts by themselves’. There were cases in which Dr Brownstone was convinced that the students were using external resources, such as the internet. Sometimes she even shaped their ‘proof-detection’ by intentionally name-dropping mathematical notions in the formulations of the problems in order to lead students towards particular resources (e.g., a product topology and Cantor set in Figure 1). For example, regarding the problem in Figure 1 she explained:

> I am absolutely sure they used external resources through the problem, and in this particular problem I’m happy with it. Because my goal for them is to understand what is going on with product topology and also get the benefit of finding out a bit more about a Cantor set.

Thus, in some cases, course students were expected to realize the metaphor of proof-detection almost literally.

In the interviews, Dr Brownstone drew parallels between her epistemology of proof in the teaching-learning context and practices that are characteristic to her research discourse. In her words,

> In a sense, I think of assignments as reflecting what you would do normally in a research publication. […] Because there are two elements to a proof, loosely speaking: one, is this spark of insight as the how the proof works – what makes it go through. And the other is writing it down in a logical manner, which supports the idea and makes it true mathematically for the editor and reviewers. So they [the course students] have to obviously have the idea of how the proof works, how you get from here to here. And then, they have to write it down in the way that conveys to me that it is logical.

This excerpt illustrates that in her research discourse, Dr Brownstone also distinguished between detecting a ‘proof’ (“the spark of insight”) and representing it (“writing it down”) for an editor and reviewers. Furthermore, she considers the activity of students submitting written texts for her

[4] In this section onwards, I use ‘proof’ to signal to the readers that Dr Brownstone’s epistemology is under discussion and to remind them that alternative discursive approaches could be used.

assessment as a course counterpart of the research practice of submitting a manuscript to a journal.

Example of implication of the above epistemology

One notable characteristic of Dr Brownstone's feedback pertains to what may seem like a mismatch between the high marks that she allocated to students' texts and the seriousness of their flaws that she pointed out in her comments. For instance, in his response to the problem in Figure 1, one of the students stated that "$2^{\mathbb{N}}$ has a discrete topology", which suggests that any set of infinite binary sequences is open. However, the assigned problem stated that the set 2={0,1} is the one having a discrete topology, while $2^{\mathbb{N}}$ is an infinite product with a conventional product topology. In this way, the chain of logical arguments that the student developed on his way to the desired homomorphism contained a faulty assumption. Yet, he received five marks out of six. When asked in the interview to elaborate on her rationale behind this mark, Dr Brownstone explained:

Dr Brownstone: The components of what he is doing are on the right track. Because he misinterpreted the product topology he is going to fail, he is not about to prove it [the homomorphism] because he is not proving the right thing […]. But the basic idea, the format of his proof is correct, he just misinterpreted the topology on $2^{\mathbb{N}}$, which was one of the things that I was hoping he would take out of this question. […] I must've thought that he did everything else very well because I've taken only one mark off.

Igor': What is "everything else" in this case?

Dr Brownstone: Well, the components of the proof are to describe the Cantor set in a way that you can talk about a bijection with the $2^{\mathbb{N}}$, and then you need to show that it is bijection, and then you about to show that it is open and continuous. But setting up this bijection is key. […] So he has understood all of that but he has misinterpreted the topology on $2^{\mathbb{N}}$.

This excerpt shows that Dr Brownstone expected students' 'proofs' to contain three 'components': capturing the Cantor set in a way that allows manipulation, creating a bijection with $2^{\mathbb{N}}$, and showing that the bijection is continuous and open. These components were evident in the text that the student submitted: he presented a general point *x* in the Cantor set as $\sum_{i=1}^{\infty} \frac{x_i}{3^i}$ (xi∈{0 or 2}), and he introduced a bijective function *F* that matches the set to the infinite binary sequence $\left(\frac{x_1}{2},\ \frac{x_2}{2},\ \ldots\right)$. The assumed discreteness yielded that *F* is trivially open, and then the student proceeded to show its continuity. Accordingly, while he did not prove the assigned statement, he engaged with the expected 'components' of the sought 'proof'.

Dr Brownstone's assessment of Diego's text illustrates a broader pattern suggesting that her marks and comments did not always deliver the same message. Nearly all the marks that Dr Brownstone allocated were either full or one less than full, which might have been read by the students as acknowledging their submissions as high quality. However, the line-by-line comments often pointed at multiple issues requiring the students' attention. In the interviews, Dr Brownstone explained that she typically takes marks off for incorrect arguments, omitted cases, and additional instances where the "proof did not go through". She also explained that "to take more points off would not give students an adequate mark for what they have done". Full-mark texts tended to be accompanied by many comments as well, via which Dr Brownstone sought additional explanations, pointed at confusions with terminology and notation, and raised questions about mathematical entities that the students used without explicit defining.

Commognitively speaking, the comments demonstrated Dr Brownstone enacting her role as a teacher who leads students' mathematical discourses to better align with mathematical rules.

However, her marks signaled that abiding by some of these rules is more important than others. These practices resonate with her epistemology of proof: the issues justifying a mark reduction could be ascribed to detection of expected ‘proof-components’, while other issues pertain to ‘proof-representation’. Stretching the metaphor of a court of law, Dr Brownstone can be viewed as an honest judge who does not punish a client who is innocent (i.e. “a proof that goes through”) even when there are ‘holes’ in the client’s representation.

Concluding Remarks

The theoretical contribution of this study is the expansion of the domain of Balacheff’s (2008) notion of epistemology of proof to include mathematicians – key participants in a discourse through which university students learn how to prove. I use ‘expand’ and ‘include’ to highlight that mathematicians’ epistemologies can come into being through disciplined inquiry (Shulman, 1997) and through the encounter with epistemologies of mathematics education researchers (cf. Barad, 2007). My aim of coupling these epistemologies is two-fold. First, it is intended to afford further research on potential gaps between declared and enacted epistemologies of mathematicians (e.g., Davis & Hersh, 1981; Mura, 1983). Second, epistemological incommensurability of ‘typical’ mathematicians and researchers who keep up with contemporary theories in mathematics education is well known (e.g., Sfard, 2008). The proposed extension turns this incommensurability into an asset that can be leveraged to gain deeper insights into mathematicians’ teaching rationalities and actions.

The extended definition proposes that a mathematician’s epistemology of proof is idiosyncratic. For instance, the detection and representation constituents of Dr Brownstone’s epistemology do not allow it to be labelled as an instance of some ‘extreme’ perspective (e.g., purely Platonist or fully social). However, some of its features resonate with the voices of additional mathematicians. Indeed, it is not rare to find mathematicians who use the objectified ‘proof’, and argue that it should ‘stand alone’ based on its ‘components’ (e.g., Rota, 1993). Thurston (1994) elaborates on the importance of communication of proof within the mathematics community. Accordingly, it is plausible that some aspects of Dr Brownstone’s epistemology are more representative of a broader mathematics community than it could be assumed.

Strengths and limits could be discerned in the enactment of any epistemology of proof, and Dr Brownstone’s one is not an exception. Imagine, for instance, how useful her component-based epistemology would be when instructing students on how to prove a homeomorphism. Also, this epistemology summons an organic marking scheme where points are allocated based on the number of ‘proof-components’ that were detected. According to the presented analysis, the issues emerged when Dr Brownstone endorsed the validity of ‘proof-components’, but she was not fully satisfied with their representation. In some cases, this yield a discrepancy between her comments pointing the student towards all the changes that need to be made, and the full marks that she still allocated to the submission. This case invites further research on how mathematicians’ epistemologies of proof afford their teaching and students’ learning.

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Minding the Gaps: Algebra Skills of University Calculus Students

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This study builds on existing research detailing calculus students' algebraic errors ("gaps") by connecting data from a standardized, proprietary placement instrument. Results suggest skills that are rarely versus commonly demonstrated prior to calculus. We also report on success and failure for students in the STEM-track Calculus 1 based on the different skills.

Keywords: calculus, algebra, misconceptions, prior knowledge, mathematics placement

National data indicate that about half of freshman who declare STEM majors as they enter college leave these fields before graduation (Chen, 2013). Since most STEM degrees require the completion of at least one calculus course, calculus serves as the gateway course to success in a STEM field. Bressoud, Mesa, and Rasmussen (2015) assert that requiring a calculus course often creates an insurmountable hurdle or, at the very least, one that discourages students from pursuing STEM degrees. Research supports this assertion and has shown that more than half of students are deterred from pursuing a career in STEM due to their difficulties in completing calculus (Crisp, Nora, & Taggart, 2009; Mervis, 2010).

Most university calculus mathematics instructors are aware that students in their courses face challenges. Some of these challenges are related to students transitioning from high school to university mathematics courses (Clark & Lovric, 2009; Seldon, 2005) whether it be due to the difference in cultures of the two environments (Brinkworth, McCann, Matthews & Nordström, 2009; Moore-Russo, Wilsey, Parthum & Lewis, 2017) or the different perspectives that high school teachers have compared to university instructors (Hong et al., 2009). However, some of the challenges are related specifically to how student think about mathematics material prerequisite to calculus (Nagle, Moore-Russo, Viglietti & Martin, 2013). Most calculus students enter college having taken algebra and precalculus in high school (Bressoud, 2015), and may even have performed well in those courses. Still, they enter their university calculus classes with varied and numerous challenges involving algebra content (Cangelosi, Madrid, Cooper, Olson & Hartter, 2013; Reeder, 2017; Salazar, 2014; Top, 2018). Understanding how to navigate the challenges students will face with algebra in university calculus courses should be of concern to calculus instructors and the mathematics education community.

Many universities, frustrated by underperformance in mathematics, are focusing on student success in early mathematics courses in light of placement (e.g., Pleitz, MacDougall, Terry, Buckley, & Campbell, 2015; Revak, Frickenstein, & Cribb, 2000). Many are relying on various data beyond high school grades and course completion to help place students in mathematics courses that are at the appropriate level for them (Ratterree, 2018). Often placement assessment data come from in-house placement exams developed by the college or university, standardized assessment scores such as the ACT or SAT, proprietary assessments (e.g., Accuplacer, ALEKS-PPL), or the combination of a standardized assessment and other criteria (Heiny, Heiny, & Raymond, 2017). Regardless of the assessment, it is common for universities to utilize some method to determine appropriate placement of students in college level mathematics, including introductory calculus, in order to support student success. The purpose of this study is to examine the use of a common mathematics placement instrument to shed light on which algebra skills and abilities align to student success or failure in calculus, first with respect to common errors and second with respect to course outcomes.

Theoretical Framing

To distinguish between knowing a subject and knowing how to teach it (Ball & Bass 2000), Shulman (1986) introduced the concept of pedagogical content knowledge (PCK) as a category of mathematical content knowledge unique to teaching. PCK should be distinguished from subject matter knowledge, general pedagogical knowledge (e.g., understanding of assessment, classroom management, educational aims), and context knowledge (i.e., understanding of the particular educational institution and students). Among other things, PCK involves "… the conceptions and preconceptions that students … bring with them to the learning" (Shulman, 1986, p. 9). Ball, Thames, and Phelps (2008) elaborated on Shulman's earlier work to define three domains within PCK, one of which was the knowledge of content and students (KCS). KCS includes what instructors know about the mathematics to be taught and about the students in their classrooms. KCS involves instructors using their knowledge of students' previous educational experiences, students' emerging and incomplete thinking, and the content struggles that are common to students in their classes to structure instruction so that lectures, discussion topics, tasks, and examples that are implemented in class are appropriate for students.

Simon (1995) based his idea of *hypothetical learning trajectories* on the works of constructivists, such as Cobb and Yackel (1996), indicating that teachers should understand the typical trajectory for their students understanding of mathematical concepts. Simon argues that teachers need to know what path to take with their instruction so as to build on students' prior knowledge. Based on an assessment of students' difficulties learning the subject matter, teachers need to devise a new path, or paths, until the goal is achieved.

Background

Stewart's (2017) recent book features studies that analyze the occurrence of algebra errors by students in undergraduate mathematics courses. They describe categories of algebra mistakes that they found on exams or other assessment instruments. Also included are studies (e.g., McGowen, 2017) that discuss ways to prevent algebra mistakes in more advanced courses. The general consensus is that algebraic errors in calculus, and even more advanced courses, are challenging and require study.

To employ KCS, calculus instructors need to be aware of the algebraic skills that their students demonstrate. Of course, students are individuals and there is not a one-size-fits-all. Still, the general information on common algebra errors would be valuable for calculus instructors in their planning of their instruction, especially in light of the growing popularity of co-requisite instruction (Vestal, Brandenburger, & Furth, 2015). It would be helpful to instructors if they were able to determine how robust students' facility with algebraic skills is and if there are certain areas in algebra that may be more challenging for students than others. A calculus instructor might be inclined to just send the students back to an algebra or precalculus course. However, if students do actually have facility with most of the algebra content, with just a few gaps, then it seems most prudent for instructors to "mind the gaps." We adopt this expression that is commonly announced in the London Underground Subway system (aka the "Tube") for passengers who are boarding the trains but have to navigate the space between the platform and the train. Instructors should be helping students navigate their algebraic gaps so that they are able to board the train to STEM majors successfully. However, this is only possible if those gaps are identified.

For this reason, the current study involves a data set gathered from the mathematics placement instrument administered to incoming calculus students. These data show, in fine-grained detail, exactly which algebraic skills calculus students demonstrated (i.e., they were able to perform tasks related to this skill) on the placement instrument. We compare this information with the algebraic topics where errors are most likely to occur on calculus exams (Stewart & Reeder, 2017). The specific research questions for this study are now listed.

1. Previous research has put forward areas of observed algebraic difficulties in students' calculus work. Which algebraic skills associated with these difficulties are students able to successfully demonstrate during placement into calculus? Which are they not able to demonstrate?
2. Which of the least commonly demonstrated algebraic skills correspond with students who successfully complete the STEM-track Calculus 1 course? Does a student's prior demonstration of the skills that align with frequent algebra errors in calculus correspond with a difference between success and failure in the STEM-track Calculus 1 course?

Methods

Data were collected in fall 2018 at a comprehensive, doctoral-granting university with more than 30,000 students (49% male, 64% white, 48% receiving some need-based financial aid). This study considers only those students enrolled in a STEM-track Calculus 1 class in fall 2018. These students tend to be primarily engineering and natural science majors, including a few math majors, similar to the calculus composition reported by Bressoud (2015). The study only considers those students described above who were placed into the course via the placement instrument and not those who were placed into the course by taking the prerequisite university precalculus course.

At this university, all incoming and transfer students were placed into their first math course after taking a proctored ALEKS-PPL placement instrument. This placement instrument was developed as a product of Knowledge Space Theory, which is deeply described in Doignon and Falmagne (1999). Falmagne, Cosyn, Doignon, and Thiéry (2006) outline the theoretical and historical development of the instrument. The first detailed implementation analysis was conducted by Reddy and Harper (2013) showing that, over several years and hundreds of students, ALEKS-PPL accurately placed students into math courses through Calculus 1. This placement instrument determines a student's abilities on 314 distinct mathematical skills (their "knowledge state") in the areas of arithmetic, geometry, algebra, functions, and precalculus.

The placement instrument returns a knowledge state for each student, i.e. a vector of 314 binary entries, where a 1 indicates students have demonstrated evidence of the mathematical skill and a 0 indicates they have not. If students in the study had multiple attempts at the placement, their maximum scores for each skill were recorded in the data set. Thus, students who demonstrated a particular skill on at least one of their attempts would have been marked as demonstrating that skill. Any students whose data resulted in mismatches in student identification numbers and ALEKS identifiers were removed prior to de-identification. The result was valid data for N=166 students. Altogether, there were 52,124 units (166 students x 314 skills) that were analyzed for the study.

The research team used the work of Stewart and Reeder (Reeder 2017; Reeder & Stewart, 2017, Stewart, Reeder, Raymond & Troup 2018) to identify categories of algebraic errors that calculus students commonly exhibit. These categories include – *simplifying* (SI), *distributive property* (DP), *variable isolation* (VI), *operations with rational expressions* (RE), and *operations with radicals* (RA). A category that is not explicitly algebraic, but that anecdotally results in many calculus errors, was *composition of functions* (CF). For the current study, all six categories were considered.

> **ALGE062:** **Solving a rational equation that simplifies to quadratic: Binomial denominators and numerators.**
>
> **Solve for** x.
>
> $$\frac{x-3}{x-7}+1=\frac{x-5}{x-6}$$

Figure 1: Sample task from placement instrument for the A062 skill in RE (ALEKS/McGraw Hill, 2016).

Using the categories for algebraic errors, the team then reviewed the skill descriptions for the placement instrument (Higher Education Math Placement, 2019) to determine which skills tested by the placement instrument aligned with the six categories (SI, DP, VI, RE, RA, CF). The research team identified 36 skills that aligned. See Table 1 for the matching of these 36 placement instrument skills to the common algebraic error categories. Note that the item codes have been modified from those used by ALEKS/McGraw Hill. ALGE, the code used to refer to algebra, has been replaced with A, and FUN, the code used to refer to functions, was replaced with F. An example of an A062 skill (ALEKS/McGraw Hill, 2016), which we connect to the RE category, is found in Figure 1.

Table 1. Placement skills aligned with the categories of algebra errors observed in calculus exams.

Simplifying	Distributive Property	Variable Isolation	Operations with Rational Expressions	Operations with Radicals	Composition of Functions
A034	A033	A011	A053	A088	F022
A080	A180	A013	A054	A089	
A084	A604	A179	A056	A090	
A710	A606	A743	A057	A091	
A798	A735	A744	A058		
A640	A835	A810	A060		
A811			A062		
			A205		
			A206		
			A620		
			A622		
			A767		

Table 2. Skills from placement instrument, ranked from most rare to most common. Score is the fraction of students (N=166) who demonstrated the skill on the placement instrument. (Higher Education, 2019).

Code	Category	Score	Description Given by Placement Provider
			Rare Skills
A062	RE	0.20	Solving a rational expression that simplifies to quadratic: Binomial denominators and numerators.
A057	RE	0.23	Adding rational expressions with different denominators: ax, bx.
A622	RE	0.24	Adding rational expressions with different denominators: x+a, x+b.
A091	RA	0.24	Solving a radical equation that simplifies to quadratic: One radical.
A206	RE	0.28	Solving a rational equation that simplifies to linear: Unlike binomial denominators.
A767	RE	0.29	Complex fraction simplifying: GCF and quadratic factoring.
A640	SI	0.35	Simplifying a product of radical expressions: Multivariate.
A084	SI	0.36	Simplifying a sum or difference of radical expressions: Multivariate.
A811	SI	0.36	Simplifying a higher radical expression: Multivariate.
F022	CF	0.43	Composition of two functions: Basic.
A088	RA	0.47	Rationalizing the denominator of radical expression using conjugates.
A034	SI	0.51	Simplifying a ratio of multivariate polynomials.
			Common Skills
A058	RE	0.80	Simplifying complex fraction involving multivariate monomials.
A056	RE	0.82	Adding rational expressions with common denominators and binomial numerators.
A620	RE	0.94	Multiplying rational expressions involving quadratics with leading coefficients of 1.
A080	SI	0.98	Simplifying a radical expression with an even exponent.
A710	SI	0.99	Simplifying a ratio of polynomials: Problem type 1.
A735	DP	0.99	Multiplying univariate polynomial by monomial with positive coeff.
A835	DP	0.99	Multiplying a multivariate polynomial by a monomial.
A179	VI	0.99	Solving a linear equation with several occurrences of the variable: Fractional forms.
A744	VI	0.99	Algebraic symbol manipulation (isolating variable in an equation): Problem type 2.
A053	RE	0.99	Multiplying rational expressions involving multivariate monomials.
A054	RE	0.99	Dividing rational expressions involving multivariate monomials.
A798	SI	1.00	Simplifying a sum or difference of two univariate polynomials.
A033	DP	1.00	Multiplying binomials with leading coefficients of 1.
A180	DP	1.00	Multiplication involving binomials and trinomials in two variables.
A604	DP	1.00	Distributive property: Integer coefficients.
A606	DP	1.00	Distributive property: Whole number coefficients.
A011	VI	1.00	Solving a linear equation with several occurrences of the variable: Variables on the same side and distribution.
A013	VI	1.00	Solving a linear equation with several occurrences of the variable: Variables on both sides and distribution.
A743	VI	1.00	Algebraic symbol manipulation (isolating variable in an equation): Problem type 1.
A810	VI	1.00	Introduction to algebraic symbol manipulation.
A060	RE	1.00	Solving a rational equation that simplifies to linear: Denominator x.
A205	RE	1.00	Solving a rational equation that simplifies to linear: Denominator x+a.
A089	RA	1.00	Solving a radical equation that simplifies to linear: One radical.
A090	RA	1.00	Solving a radical equation that simplifies to linear: Two radicals.

In order to also look at the grades the students received in their Calculus 1 course, a college employee (not on the research team) linked placement instrument results and course grades the students in the study. The de-identified data were then shared with the research team.

Results and Analysis

In Table 2, each skill listed in Table 1 is given with a brief description (*Higher Education Math Placement*, 2019) and the fraction of our sample of students who demonstrated acquisition of that skill on the placement instrument. Since each of the 166 students had placed into Calculus 1 via the placement instrument, all had demonstrated many of the 314 skills, so it is not surprising to see many skills that every student had demonstrated.

The skills are listed in Table 2 by increasing order of demonstration by the students (i.e., from the most difficult of the 36 skills that are listed in Table 1, for students in the sample, to the easiest). Interestingly, they separate into two distinct categories. There are 12 skills with scores that are quite low, ranging from 20%-51% of students. We call these **rare skills** for students to have demonstrated prior to calculus. However, the remaining 24 skills have success rates over 80% (13 were demonstrated by every student), so we denote these **common skills** for students have demonstrated prior to calculus.

The rare skills include many from the *operations with rational expressions* (RE) category; see Figure 1 for an example. We also see students having difficulty with topics from the *operations with radicals* (RA), *simplifying* (SI), and *composition of functions* (CF) categories. A large number of students entering calculus had not demonstrated the following crucial skills for calculus: composing functions, adding rational expressions with different denominators, and solving equations involving radicals. Therefore, instructors should look for ways to revisit and review these topics, possibly in a just-in-time approach, to help assist students in gaining acquisition of these skills.

On the other hand, the population had demonstrated every skill relating to the *distributive property* (DP) and *variable isolation* (VI). In these areas, calculus students tend to make errors even though they demonstrated these skills on the placement instrument prior to the course. This implies that an intervention targeting these skills should not focus on re-teaching algebraic content, but rather on mitigating other factors that could be causing mistakes.

Next, we restrict our analysis to the 12 rare skills and ask if any of these correspond with student success in their introductory calculus course. We divided our population into the group of students who completed their Calculus 1 course with an A, B, or C and those who received a D, F, or W in the course. We looked at the fraction from each group that had each of the 12 rare skills. The results are shown in Table 3.

Table 3. Skills from placement instrument by student course success. Score is the fraction of students (N=138 or N=28) who demonstrated the skill on the placement instrument.

Course Grade for Student	N	Skill											
		F 022	A 034	A 057	A 062	A 084	A 088	A 091	A 206	A 622	A 640	A 767	A 811
ABC	138	.44	.54	.23	.20	.36	.44	.23	.27	.25	.34	.31	.36
DFW	28	.39	.32	.29	.21	.32	.61	.29	.36	.21	.39	.18	.36

The results are mixed. We see that half of the skills (A057, A062, A088, A091, A206, and A640) were more commonly demonstrated by students who were *not* successful in the course than among those who were. There were only two skills (A034 and A767) that were clearly demonstrated more often by successful students, both involving operations on rational functions. Even composition of functions (F022) did not correlate strongly with student success. From this, we might conclude that students who exhibit enough skills to be placed into calculus can be successful even if they are initially lacking some key algebra skills.

Discussion

Stewart (2017, p. vii) states that, "as the complexity of mathematical ideas increases rapidly, the unresolved high school algebra problems mount up progressively and continue to create further distress." Our goal in this paper was to identify algebra skills that students do and do not demonstrate prior to entering an introductory calculus course, so that instructors will better understand how to plan instruction so that students are successful. We also wanted to know if students' incoming algebra skills impacted their grades in calculus.

In light of the first research question, we matched the algebraic areas where calculus students most often make mistakes, identified by Stewart and Reeder (2017), to items on the placement assessment. This allowed us to analyze students' prior demonstrated skills in those areas. The matched skills fell into two distinct categories. Some algebra skills were demonstrated by nearly all students; yet, prior research shows that calculus students still make errors in those areas. Other skills were not widely demonstrated prior to the calculus course, identifying areas where instructors might intervene with supplemental instruction.

For the second research question, with the exception of simplifying rational functions, we did not find any correspondence between the least commonly demonstrated algebraic skills and students who successfully completed the calculus with a grade of A, B or C. In fact, when looking at the students who were not successful in the course, those students had demonstrated half of the rare skills in greater numbers than the successful students prior to the start of the course. While it is possible that the skills not yet demonstrated by the students where not of importance for succeeding in the course, another more likely explanation is that the placement instrument accurately placed students who had sufficient backgrounds such that they were able to, for the most part, acquire these skills during the calculus course.

Implications and Future Work

The results of this study help differentiate between skills that are rarely and commonly demonstrated prior to calculus. This information should be useful to calculus instructors as they plan instruction since knowing what students understand and where they are likely to make mistakes is a large part of the KCS used by course instructors. The findings will help calculus instructors mind the gaps. It is particularly vital information for those who work with calculus co-requisite models, that is, implementing remediation of prerequisite material alongside the topics in a Calculus I course (e.g., Vandenbussche, Ritter, & Scherrer, 2018). Future work in this area should consider the extent to which calculus instructors, even those not teaching co-requisite courses, are aware of the algebraic errors that incoming calculus students make to determine how they take this into consideration in their instruction.

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Construction of a Mathematics Learning Assistant's Fragile Mathematics Identity

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This paper shares results of a discourse analysis of an interview with a female student who was working as an undergraduate mathematics tutor and learning assistant. The student is shown to have developed a mathematics identity made up of a strong identity as a doer of mathematics and a weak sense of belonging within her mathematics community. I use robust mathematics identity to refer to a mathematics identity that includes both a strong identity as a doer of mathematics and a sense of belonging in mathematics. A fragile mathematics identity is one in which one or the other of these constructs is not present. This interpretation of these terms is closely aligned with, but slightly different from, the way these terms are used in other literature in the field. This student is shown to have a fragile, rather than robust, mathematics identity.

Keywords: Mathematics Identity, Sense of Belonging, Learning Assistant, Discourse Analysis

Problem Statement, Background and Context

Women and people of color are persistently underrepresented in the field of mathematics. While women make up approximately half the students in college Calculus 1 courses they are much more likely than their white male peers to drop out of mathematics after the first semester of Calculus (Ellis, Fosdick & Rasmussen, 2016). Students of color often experience high school learning environments that are under resourced and do not provide access to high level mathematics courses. These learning experiences can lead to placement into Precalculus rather than Calculus in their first semester in college, a reality that places an additional barrier to continuing in mathematics or mathematics dependent fields of study. Increased use of active, inquiry and reform based instructional practices in college math courses has been associated with greater rates of persistence in mathematics (Freeman et al., 2014; Laursen, Hassi, Kogan, & Weston, 2014), at least through these early undergraduate courses. However, students who are members of persistently underrepresented groups in mathematics continue to describe their experiences in these courses less positively than their white male peers (Voigt, 2017). In order to support a diverse mathematics population to participate in doing mathematics, it is essential to understand the structures that support positive math experiences and pathways for development of robust mathematics identities among members of marginalized groups in mathematics.

The experience of working and helping other students as a tutor or as a Learning Assistant (LA) more generally, may offer a role of legitimate participation (Wenger, 1998) that could potentially counteract some of the marginalization experienced by women and people of color in mathematics and mathematics classes. More specifically, I am interested in learning whether these roles may support development of more robust mathematics identities and hold potential as a pathway to central participation (Wenger, 1998) in mathematics for the LAs themselves. In the longer term, if the experience of working as an LA proves to be positive, and this experience is shown to support development of robust mathematics identities for the students working in these roles, this could possibly lead to a more diverse group of mathematics instructors and to more positive experiences for women and students of color in mathematics courses more generally.

The research question in this article was: How do experiences as a female working as a tutor in the Mathematics Academic Resource Center impact mathematics identity development in ways that afford and/or undermine access to a role of central participation in mathematics?

Literature Review: Understanding Mathematics Identity

Mathematics education research on mathematics identity includes work related to perceptions and understandings of what it means to do mathematics, some of which use the term "identity as a doer of mathematics" (e.g. Boaler & Greeno, 2000; Langer-Osuna, 2011; Park, Kondrak, Ward, & Streamer, 2018), and also work related to the sense of self in relation to mathematics, particularly with regard to gender, race, ethnicity and/or culture, some of which use the term "sense of belonging in mathematics" (e.g. Good, Rattan, & Dweck, 2012; Radovic, Black, Salas, & Williams, 2017; Solomon, 2012).

Identity as a doer of mathematics refers to students' vision and understanding of themselves and the nature of their participation in what they understand to be the practices of doing mathematics. Articles that attend to students' "identity as a doer of mathematics" (Cobb, Gresalfi, & Hodge, 2009, p. 44) include those in which authors use "mathematics identity" more narrowly in reference to people's vision of their ability to do mathematics successfully (Pietri et al. 2018; Nasir et al. 2008) or their decision regarding whether or not to participate in what they perceive to be the practices of doing mathematics (e.g. Boaler & Greeno, 2000). Much of this research attends primarily to the ways such decisions are made, or the social processes through which such visions or identities are formed (Boaler & Greeno, 2000; Langer-Osuna, 2011; Turner, Dominguez, Maldonado, & Empson, 2013), with the primary concern being the degree to which students come to see themselves as what is sometimes called a "math person."

Sense of belonging in mathematics relates to students' sense of being valued as themselves (their racial, ethnic, gender, cultural and class identities are honored) and for their own unique ways of doing mathematics and connecting mathematics to their lives. The term "sense of belonging" is related to Langer-Osuna & Esmonde's category of "membership identity" (2017) which is based on social membership(s) related to race, gender, language, culture, etc. Good, Rattan & Dweck (2012) claim that sense of belonging "involves one's personal belief that one is an accepted member of an academic community whose presence and contributions are valued" (p. 701). The mathematics education research that attends to matters related to sense of belonging addresses how and to what degree students feel fully accepted in mathematics spaces.

The concept of robust mathematics identity is present in a limited number of mathematics education articles. This term appears in the literature to describe mathematics identities that support students' persistence in mathematics through resilience strategies (Joseph, Hailu, & Boston, 2017), resistance to dominant discourses about who does mathematics (Stinson, 2008) and maintenance of high levels of academic achievement in spite of "marginalization, stereotyping and other forms of racialization" (McGee, 2015, p. 599). McGee (2015) also develops the concept of fragile mathematics identity to describe mathematics identities that develop primarily in response and reaction to racialization and racial stereotypes, and which aligns conceptually with mathematics identities that develop in reaction and response to gendered experiences in mathematics. Gutierrez (2018) states that "beyond being seen as a legitimate participant (a "doer" of mathematics), a student should be able to feel whole as a person – to draw upon all of their cultural and linguistic resources – while participating in school mathematics" (p. 1). When I use the term "robust mathematics identity" I am referring to a mathematics identity that consists of both positive identity as a doer of mathematics and also positive sense of belonging in mathematics. I suggest that experiences in mathematics that support students' development of robust mathematics identities may support students to persist in mathematics because of, rather than in spite, of those experiences.

Theoretical Framework

Students who become participants (Wenger, 1998) within their mathematics learning communities take up a form of learning that supports their ability to envision themselves pursuing further study in mathematics (Boaler & Greeno, 2000; Cobb & Hodge, 2002), and social contexts have been shown to impact the ways that people learn and do mathematics (Nasir, Hand, & Taylor, 2008). Students' development of an identity as a doer of mathematics depends on their vision of themselves that aligns with their perception of what it means to do mathematics (Boaler & Greeno, 2000). In other words, students who experience roles as central participants in mathematics learning communities, and whose experiences of doing mathematics in those communities align with their visions of themselves as people, are more likely to develop identities as doers of mathematics and to persist in mathematics or STEM fields of study.

Boaler & Greeno (2000), citing Holland, Lachicotte, Skinner & Cain (1998) and stating that they are building on the theory that "identities develop in and through social practice," use the term "'positional identity' to refer to the way in which people comprehend and enact their positions in the worlds in which they live" (p. 173). This attention to people's positions in the world points to the importance of drawing from research on the sociopolitical nature of mathematics and mathematics learning experiences (Gutiérrez, 2013; Nasir & McKinney de Royston, 2013; Valero, 2004). Aguirre et. al. (2017) state that "a sociopolitical approach allows us to see the historical legacy of mathematics as a tool of oppression as well as a product of our humanity" (p. 125). Taking a sociopolitical perspective enables me to attend to the ways that "people's positions in the worlds in which they live" (Boaler & Greeno, 2000) are related to their positions in the mathematics worlds in which they take part.

Research has demonstrated that members of certain populations (e.g. students of color and women) experience reduced opportunities to participate in the learning and doing of mathematics due to structures which are systematic and pervasive (e.g. Berry, Ellis, & Hughes, 2014; Langer-Osuna, 2011). This restricted access to roles of central participation is accompanied by limitations on access to agentive participation through which people are able to influence the nature of mathematics and mathematics communities. Varelas, Settlage & Mensah (2015) describe agency as a person's capacity to engage with cultural schemas and mobilize resources in ways that did not exist before, creating new contexts and practices (p. 439). This process is central to the capacity of new participants to participate in and shape the nature of mathematics.

This research has taken up sociocultural learning theory to help contextualize how students develop mathematics identities, the nature of such identities as fluid and socially constructed within mathematics learning environments, and the ways that students themselves enact agency over the development of their own mathematics identities as well as mathematics itself.

Research Methods

The Subject and Context of the Interview

The interview which provided the data for this paper was conducted with a white female undergraduate student named Tara (a pseudonym). Tara held a position as a mathematics learning assistant (LA) and, in association with that position, she was working as a tutor in the Mathematics Resource Center (MRC), a context which I will describe in more detail below. Tara was a sophomore at Large University at the time of the interview. She was majoring in mathematics with a focus on secondary education in preparation for teaching and had taken multiple math courses during both college and high school.

The interview was conducted in the MRC, which is a large, open room that is well-lit with large windows where undergraduate students enrolled in mathematics classes can come for tutoring and support. The tutors sit at tables around the periphery of the room, and they each have a sign indicating the name of the course to which they are primarily assigned and available to provide assistance. Each tutoring space has a chalkboard that the tutors often use when providing explanations, and occasionally tutors and students use the chalkboards together as a space to work collaboratively. The interview was conducted during a quiet time in the MRC, and we began by clarifying that students looking for tutoring would take priority, meaning that we would pause the interview, and I would step aside, if any students came by looking for tutoring.

Data Collection and Preparation

The data analyzed for this paper is comprised of a transcription of data from the first 19 minutes of an original audio recording of a 38-minute interview with Tara. The interview segment chosen for this analysis consists of narrative in which Tara describes her mathematical experiences in ways that provide insight into the discursive construction of her mathematics identity. In one segment Tara talks about her own identity as a doer of mathematics and her views about the type of math classes she imagines she would like to teach one day. In another segment she describes some of her experiences as a tutor in the MRC, including her sense that she is not often asked for help and some of her thoughts about the reasons behind that. She also describes numerous aspects of her experience as a female student in mathematics and in her life outside of mathematics communities.

Methods of Analysis

The interview used for this paper was analyzed with the goal of understanding the construction of Tara's mathematics identity, including how Tara makes sense of her role and position within various contexts inside and outside of mathematics. Particular attention was paid to how Tara's mathematics identity was associated with affordances and/or constraints with regard to her access to a role of central participation in mathematics.

My approach to this analysis incorporated "the social psychological tripartite model of identity (Brewer and Gardner 1996; Sedikides and Brewer 2001) into an interactional sociolinguistic framework for the purposes of discourse analysis" (Cohen, 2010, p. 69). Cohen (2010), citing Auer (1992) and Williams (2008), claims that "good interactional analysis can maintain a dynamic view of identity and yet account for the more stable 'brought along' features" (p. 69). This claim highlights the importance of recognizing identity construction as fluid and responsive to social contexts, while still honoring the ways that an individual maintains agency and ownership of their own identity. The tripartite model of identity attends to personal, relational, and collective aspects of identity construction and provides the ability to understand the role of each of these three layers in the construction of identity. This approach enables me to consider how Tara constructs her positive identity as a doer of mathematics relative to personal, "brought-along" experiences, in relation to a specific teacher, and in the collective context of her mathematics education classes. I am also able to consider how Tara's weak sense of belonging in mathematics is constructed personally with regard to how she sees her identity as a female impacting her work as a tutor in the MRC, relationally in reference to interactions with students, and collectively in contexts where she sees very few other female students. This weak sense of belonging is at the heart of Tara's struggle to access a role as a central member of the mathematics community of practice or to develop a robust mathematics identity.

In this analysis I also attend to Tara's use of other contexts outside of mathematics at Large University in her construction of her identity. This portion of the analysis is informed by Modan and Schuman's (2010) work on invoking place in identity construction in which they state, "likewise, the relationships among narratives can be structured through the relationships between the social meanings of the places where the narratives are located" (Modan & Schuman, 2010, p. 85). "More specific to spatial concerns, scholars such as Hill (1995) have shown that speakers use spatial shifts between deictic centers to set up moral geographies in which alignments between people and places set up reputations for both" (Modan & Schuman, 2010, p. 85). Tara explicitly references her high school experiences in an apparent effort to add credibility to the construction of her positive identity as a doer of mathematics, and she makes reference to her sorority in a way that serves as a contrasting image, perhaps illustrating her full membership in a social context where she can enact her identity as a female without restrictions or repercussions.

Findings

Construction of an Identity as a Doer of Mathematics

Personal: Brought along identity as good at math. During the early portion of my conversation with Tara she establishes her view of herself as someone who both likes and is good at math. Her explanation of why she would like to eventually teach high school honors Precalculus includes the statements "'cause I love math" and also "I wanna teach honors trig precalc, it's my favorite math." The following except from the interview transcript demonstrates Tara's confidence in her mathematical abilities while also indicating that her positive identity as a doer of mathematics is, at least largely, "brought-along" from her high school days. Tara is responding to a question about why she thinks she would like to teach honors Precalculus.

> *Tara*: I was always in honors classes and I can, like, relate to those kids a little bit more… A lotta times we're like not the best students because if you're advanced in high school it's usually not 'cause you've worked hard for it, it's because you were naturally born with that gift and so, like, math just comes easy to you,… like I was in honors trig and, like, I never did my homework and still, like, pulled a 98 percent in that class.

Tara constructs herself as someone who was always good at math in high school. In the following segment, however, Tara describes some struggles that she had with Calculus 2:

> *Tara*: In high school my calc BC teacher, and she actually was my calc 3 teacher too, she was a great teacher and people did really well in her class. I struggled, really like every time she explained things it just like … I'm terrible at Calc 2 anyways. It proved to be the same thing in college, even when I put, like, forth, like, a lot of effort. But the way she explained things just, like, never made sense with my mind, and she was so good at math, and so advanced, that, like, she couldn't, like, dumb it down enough for me to be like "oh now I see it now."

The strength of Tara's brought along identity as someone who loves mathematics and is good at doing mathematics is evidenced by the fact that the challenges she faced in her Calculus 2 class do not appear to have undermined the strength of her identity as a doer of mathematics. In her explanation of why she would like to teach Precalculus someday she states, referencing her college coursework, "I've mastered these skills, like it's my fourth year taking classes like this, so, like, how do I, like, go back to, like, when I remembered that I didn't know how to do it."

Relational: A teacher who made it easy to put in effort. Tara's brought along identity as a doer of mathematics is reinforced through her experiences taking Calculus, both 1 and 2, in spite of her claim that Calculus 2 was difficult for her in both high school and college. This strengthening of her identity, at a stage when it could well have been undermined due to her challenges with the content, took place in relation to her positive experience with a particular instructor. She describes some aspects of that experience in the following interview segment:

Tara: Um, I think the biggest one is my Calc 1 and Calc 2 instructor. … if he would have taught Calc 3 this semester, I, I would have been in Calc 3 this semester. But um, I think the biggest thing is that he just, like, was very available to students… and he, like, explained things very clearly and I also think he was just, like, really excited about the material which made you, like, feel more comfortable…. I've had some math teachers, like at (institution blinded), that I felt like I could never go into their office hours 'cause I didn't like them as a person or, like, I didn't feel comfortable going into their office hours, but with him I'd feel comfortable. I took 8 AM math two semesters in a row 'cause I liked him, so definitely, like, he made it easy to put in a lot of effort.

Collective: Math education track. Tara notes an important difference between her experiences in mathematics education courses and those in her other mathematics classes. She states, "since I'm in secondary education, like, there's more women in those classes." Tara's descriptions of her experiences indicate that the only context that seems to support her effort to maintain her brought along identity as a doer of mathematics is her secondary education courses.

Invoking another place: High School. Tara constructs her identity as a doer of mathematics partly around other aspects and details of her life that extend well beyond her individual relationship with mathematics. She shares that "both my parents were math majors at UCLA, and my dad is currently a math teacher at one of the public high schools by me," referring to where she grew up. Furthermore, Tara's personal experiences that have contributed to her strong positive identity as a doer of mathematics occurred mostly prior to her arrival at Large University. She relies heavily on her high school experiences (e.g. "I was always in honors classes" and "I was in honors trig"), much more than those she's had in college, when she makes claims supporting her identity as someone who is good at mathematics.

Summarizing the evolution of Tara's identity as a doer of mathematics. Tara's brought along identity as a doer of mathematics, which was largely developed during high school, is maintained and reinforced through her experience with a university Calculus instructor she liked and enjoyed learning from, as well as through her experiences in secondary education.

Coping with a Limited Sense of Belonging in Mathematics.

Personal: being young and female. Tara talks extensively about her experiences as a female mathematics student. Her descriptions of these experiences shed light on how stereotypes about women in mathematics have limited her sense of belonging in mathematics.

Tara: my first time in the MRC I helped, like, a lot of people, …and then, like, it's definitely died down. I think just 'cause, like … I mean this could be like stereotypical to say, but like, I'm young looking and I'm also like… a female.

Relational: Students in the MRC. Tara's description of the evolution of her experiences as a mathematics tutor lead with her description of her personal identity as a young female, described in the previous section, and this explanation continues with the following observation in relation to students seeking tutoring in the MRC:

Tara: I think, like, that also is like, people who sitting in the MRC are like, "Oh, she's not the first person I'm 'gonna go to," even in the Precalc they're like "I'm 'gonna go to someone else first before I go to her."

Tara continues with additional reflection on the ways that her youthful appearance seems to affect students' willingness to seek and accept her help with mathematics:

Tara: everyone, they're like no way you're older than me and I'm, like, "yah." Like they're, like that is, they're like "yah, I feel so dumb." It doesn't mean anything… like 'cause I'm younger than you, like, doesn't mean anything, so I think a lot of times people associate age with, like, wisdom and, like, knowledge.

Collective: Math classes. Tara's efforts to make sense of her experiences as a female student doing mathematics outside of her education classes reflect an apparent sense of isolation. She notes that, "I have not had a single female math teacher." She also describes the population of women who are members of her sorority:

Tara: there's only two of us who are math majors, um, and there's probably like a handful of us that are, like, engineering…. Since I'm in secondary education, like, there's more women in those classes. …I have a class in the engineering center and like walking to the engineering center I'm like "wow, I only see men right now." …It's more like visual than like people being like "oh you can't do this because you're, like, a female.

Interviewer: Are there ways that, as a group of females in that class, you support each other?

Tara: …in my number theory class it's definitely, like, there's a math major and … she's like "yah, I think our professor likes us cause we're, like, women in the STEM field." …. She's like "there's not a lot of women in my classes" and there's a lot, like, more women in that class, definitely, but I think we do, like, make comments about it sometimes.

Interviewer: What about in the MRC? Do the women, do the other female tutors, are there people that you're friends with? Or not so much?

Tara: We don't really talk about that, 'cause I don't think it comes up that much, 'cause I don't see them that often. But I bet it's on everyone's mind.

Invoking another place: A sorority. Tara's invocation of her sorority provides a backdrop that reinforces her identity as a female and her commitment to performing her gender identity in ways that feel authentic. At no point in her interview does she indicate that she performs her identity differently within mathematics spaces, but rather that her identity as a female impacts the ways that others view her and also affects how she experiences her mathematics classes.

Conclusion

Analysis of this interview data makes clear that working as an LA and/or tutor may not contribute to the development of a robust mathematics identity or to a central participatory role in mathematics; if it contributes to these constructs it does so incompletely or with qualifications. This story serves as a reminder of the power of the hegemony in mathematics communities and the ways that experiences in mathematics among women and students of color that result in persistence in mathematics frequently do so in spite of, rather than because of, those experiences.

I claim that hegemony works in mathematics to perpetuate the current balance of gender and race groups in mathematics, and the homogeneity of the population of people doing mathematics serves to constrain the nature of interaction and collaboration, the strategies and methods for solving problems, and the applications and uses to which mathematics is applied. This work contributes to our understanding of how some students may experience participation in mathematics communities through an in-depth analysis of one particular students' mathematics identity development. This work increases our knowledge about and understanding of the social problem of mathematics as a persistently white male field.

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Framework for Characterizing University Students' Reorganization of School Mathematics Understandings in Their Collegiate Mathematics Learning

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In an attempt to address the problem of discontinuity between school mathematics and collegiate mathematics, the current study aimed at characterizing how university students' previous knowledge constructed in school mathematics can be reorganized in their learning of collegiate mathematics. To this end, a construct called transformative transition and its accompanying categorical framework were developed and explained with student data. A transformative transition involves a qualitative leap in existing understandings as an individual encounters a new construct and integrates it into his/her cognitive system. Its four categories—extending, deepening, unifying, and strengthening—delineate different ways in which that qualitative leap might take place. Results from the analysis of interviews with six mathematics-intensive majors revealed that the context in which the participants could actively revisit, observe, reflect on, and interrelate their existing understandings at a higher level and from a different angle helped them to reorganize their school mathematics understandings.

Keywords: Double discontinuity, Unique factorization theorem, Transformative transition

As suggested by Klein (1908/1924), the *double discontinuity* refers to the disconnect that prospective teachers experience as they first encounter college mathematics as students coupled with the disconnect that they experience as they re-enter the school mathematics classroom as teachers. As corroborated more recently (e.g., Cofer, 2015; Ticknor, 2012), in both transitions, their school mathematics understandings can remain isolated from and unaffected by their knowledge construction of collegiate mathematics. The goal of this study was to address the question of "how might university students come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics?" and to observe, document, and explain important growth in university students' understandings that builds on connections between school and collegiate mathematics.

Teachers of collegiate mathematics can better support their students when they understand how construction of new knowledge can be accompanied by understanding of how the new piece of knowledge can possibly fit into what a learner has previously learned. The current study aimed at characterizing such knowledge development by investigating the ways in which mathematics-intensive majors—i.e., mathematics majors and secondary mathematics education majors—reorganize their school mathematics understandings within teaching interview [TI] context based on Abstraction-in-Context [AiC] framework (Dreyfus, Hershkowitz, & Schwarz, 2015) and abstract algebra content. An empirical merit of the current study is to move forward our discussion of the double discontinuity problem that was centered on a deficit in students' understandings by characterizing meaningful ways students reorganize their existing understanding of school mathematics in collegiate mathematical learning.

Framework

Fundamental to the problem of double discontinuity is the way that individuals construct new knowledge in their learning of collegiate mathematics. I examined the problem through the lens of knowledge construction theories and the literature on the nature and development of an individual's mathematical understandings from which a construct called transformative transition and its accompanying categorical framework emerged. In this study, a *transformative transition* refers to a particular kind of change in one's mathematical understandings that occurs as an individual encounters and integrates a new (to the learner) construct to his/her cognitive system. The change is of a particular kind in the sense that in order for the change to qualify as a transformative transition, (a) his/her existing understandings are transformed as a new construct is integrated into a learner's cognitive system and (b) the transformation involves a qualitative leap in the existing understandings. The use of the word *transition* in the transformative transition has to do with my goal of conceptualizing a notion that contrasts with a transition characterized by disconnection in one's understandings as in the double discontinuity problem.

The review of literature suggested that four constructs (breadth, depth, coherence, connectedness) were commonly used to typify the quality of one's mathematical understanding in the field. The four different qualities of mathematical understanding gave rise to each of the four categories as follows.

1. Extending category: A learner makes a transformative transition by increasing the boundary of contexts in which a set of existing understandings are situated in the individual's mind in the learning of a new construct.
2. Deepening category: A learner makes a transformative transition by increasing the depth level of existing understandings of a certain, single mathematical notion in the learning of a new construct.
3. Unifying category: A learner makes a transformative transition by increasing the extent to which, in the learning of a new construct, seemingly disparate concepts are viewed as instantiations of an overarching idea.
4. Strengthening category: A learner makes a transformative transition by increasing the strength of the link between existing understandings of more than one mathematical construct in the learning of a new construct.

Note that, in this EDUS framework (Lee, 2019; Lee & Heid, 2018), the deepening category captures how one's understandings of a *single* mathematical notion such as function may deepen, whereas the extending, strengthening, and unifying categories capture how existing understandings of *multiple* notions may form a different relationship in one's mind. Figure 1 depicts how each of the extending, strengthening, and unifying might be reflected in the change of the relationship between existing understandings in one's mind.

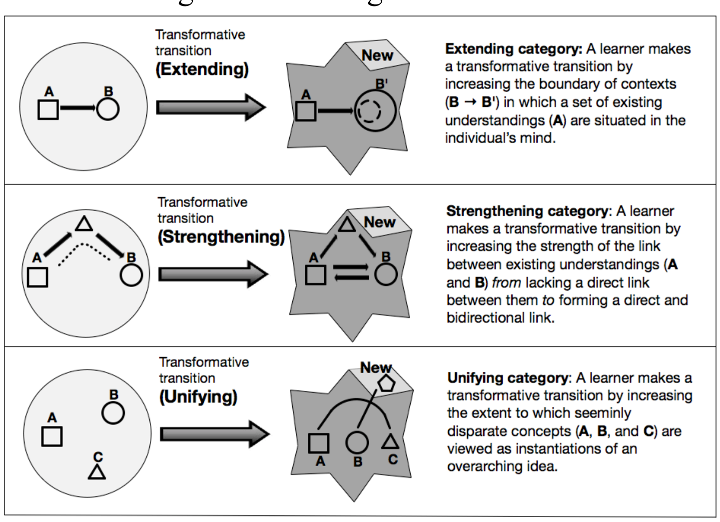

Figure 1. Extending, strengthening, and unifying: Different types of relationship change

To describe a qualitative leap in a transformative transition, levels for identifying student's progress in the deepening category were adapted from APOS theory (Arnon et al., 2014; Dubinsky, Dautermann, Leron, & Zazkis, 1994); for the other three categories, levels were adapted from Piaget and Garcia's triad (1983/1989) (see Table 1). This set of categories with corresponding levels under each category provided a lens to analyze and describe transformative transitions in the current study.

Table 1: Four categories for transformative transitions and levels in each category

Categories for transformative transition	Levels (as descriptors of transformative transition)	Levels adapted from …
Deepening	Action → Process → Object	APOS theory
Extending	Intra → Inter → Trans	Piaget and Garcia's triad
Strengthening	Intra → Inter → Trans	
Unifying	Intra → Inter → Trans	

Due to limited space, this paper focuses on the process of *unifying* and explains the three levels in the unifying category using the mathematical context involving factorization. From a knowledgeable outsider's perspective, a decomposition aspect of factorization can be seen as an overarching notion underlying many different mathematical procedures, such as factoring a whole number, factoring a polynomial, and factorization of a matrix which underlies Gaussian elimination. However, a learner might have disconnected conceptions on factorization of integers and factorization of polynomials and work with them in an entirely different mindset without recognizing an overarching notion underlying them. Such a tendency to work with them in isolation from each other indicates *intra-object level understanding of factorization*.

As the learner begins to coordinate and find similarities between the two factorizations (and, even further, among decompositions of other objects such as functions under composition and matrices under multiplication), s/he comes to understand that some aspects of several constructs can be explained by the same, overarching idea, but it may be still difficult to coordinate all relevant constructs related to factorization in order to produce the desired results in the problem given. This intermediate step of starting to see some similarities but not being able to coordinate them fully indicates *inter-object level understanding of factorization*.

Finally, at the *trans-object level*, when the learner attempts to compare sets of polynomials and integers or to describe his/her meaning of factorization, s/he demonstrates a coherent schema of his/her constructs related to factorization by building his/her explanations on specific overarching ideas which underlie factorization of integers and factorization of polynomials. There exist several overarching ideas that serve as connectors of the two constructs: decomposition, reducible, irreducible, unit, uniqueness of complete factorization, and so on.

Methods

The data were gathered using (a) AiC-based teaching interviews [TI] and (b) pre- and post-TI clinical interviews. To be specific, Hershkowitz and colleagues (2001) define teaching interview as an interview in which the interviewer asks questions with didactic purposes: "(a) to cause [student] to explain what she was doing and why and (b) to induce her to reflect on what she was doing and thus possibly progress beyond the point she would have reached without the interviewer" (p. 204). The design of TI in this study built on an assumption that instructional context grounded in the AiC framework (Dreyfus et al., 2015) can support university students to reflect on and reorganize their school mathematics knowledge in their learning of collegiate mathematics. AiC framework explains construction of new knowledge as the result of vertically reorganizing existing understandings—e.g., discovering regularities in solving different types of linear equations. AiC framework extends the line of research on knowledge construction by providing a model for abstraction process.

To put it simply, their model suggests that a learner's existing understandings serve as building blocks for constructing new knowledge and, through a chain of nested epistemic actions (so called recognizing, building-with, and constructing actions), his/her existing understandings are reorganized to produce a new construct. In their model, a new construct emerges in a problem context in which existing understandings of a learner, as they are, are not sufficient for the problem-solving or fully justifying the learner's own claim. In such a context, existing understandings that were recalled and activated are assembled and reflected in a novel (to the subject) way in building-with actions. In this process, I view that the learner's existing understandings co-develop as a new construct is being developed in his/her constructing actions.

The mathematical tasks in the TI of the current study were designed so that participants can build on what they had previously known about polynomials, factoring, and equation-solving to construct abstract algebra ideas, such as irreducible, reducible, units, and the unique factorization theorem of a polynomial (for example, see CORE-Q4 in Figure 2). Also, in pre- and post-TI interviews, I asked a similar set of questions to observe changes in their school mathematics understandings of polynomials, factoring, and equation-solving before and after a series of TIs (for example, see PRE/POST-Q8 in Figure 2).

CORE-Q4: Please factor completely the same set of polynomials over each of the number systems, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and record your results in the following table.

	$P_1(x) = x^4 - 12x^2 + 36$	$P_2(x) = x^5 - x^2$	$P_3(x) = 4x + 16x^3$	$P_4(x) = 12$	$P_5(x) = 4x^2 + 2x - 20$	$P_6(x) = 2x^3 - 10x^2 + 8x$
Over $\mathbb{Z}$						
Over $\mathbb{Q}$						
Over $\mathbb{R}$						
Over $\mathbb{C}$						

How would you describe what happened in this table in a general way? Or, how can you make some general statement about your answers in this table?

PRE-Q8 (and POST-Q8): According to the Common Core State Standards high school students need to *Understand that polynomials form a system analogous to the integers*. In what sense do you think they are analogous?

Figure 2. Examples of Tasks used in TI and pre- and post-TI interviews

Four mathematics majors (Jason, Calvin, Helen, and Andy; pseudonyms) and two secondary mathematics education majors (Sam and Lucy; pseudonyms) at a large Mid-Atlantic university who had not taken an abstract algebra course at the time of interview participated in this study. Each of the participants engaged in five to nine 60-minute interviews. All interviews were video-recorded, transcribed, and coded based on the levels and categories in the EDUS framework. In this paper, however, findings and discussions are centered on the unifying category due to limited space.

Findings

The diagram presented in Figure 3 provides an overview of how participants' understandings of factorization were advanced in (and distributed among) the three levels: intra (U1), inter (U2), and trans (U3). Specifically, five of the six participants (except for Helen) demonstrated understandings at U2 or U3 during POST-TI interviews, whereas only two participants (Jason and Calvin) did so during PRE-TI interviews. This trend of shifting from U1 to U3 implies that participants might have come to view previously disparate ideas (e.g., factorization of integers and factorization of polynomials) as instantiations of an overarching idea (e.g., decomposition) as a result of engaging in the teaching interviews and, perhaps, by participating in the pre-TI interview.

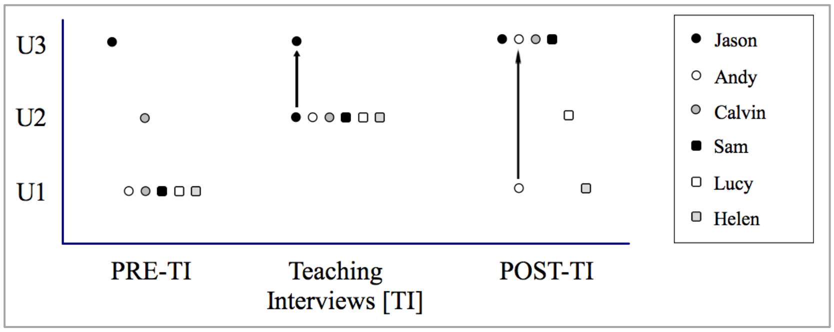

Figure 3. Participants' understandings of factorization with respect to unifying

The current section provides descriptions of transformative transition in the unifying category by building on Jason's, Andy's, and Sam's cases. Although, at a glance, the unifying process might be viewed as a linear process from U1 to U2 to U3, the data revealed that the process is not necessarily linear. Jason's understanding of factorization, for example, shifted from U3 to U2, then back to U3. In **PRE-Q8**, Jason demonstrated his unified understanding of factorization as decomposition, which he applied to the set of integers and the set of polynomials, in order to explain how polynomials form a system analogous to integers (suggesting U3). However, in **CORE-Q4** during TIs, Jason's understanding of factorization shifted back to U2 in his attempt to coordinate the case like $4x$ with the statement that he assumed to be true (i.e., the product of two irreducible polynomials is a reducible polynomial). He attempted to resolve this issue but was not successful at fully coordinating them until he figured out the idea of unit plays a crucial role in avoiding trivial factorization. During TIs and in **POST-Q8**, he concluded "multiplying by units doesn't change reducibility" and observed if a polynomial cannot be reduced in a nontrivial way, it needs to be considered irreducible. This allowed him to explain factorization across different expressions in a unified and coherent way, suggesting Jason's understanding of factorization shifted to U3.

Piaget's constructivist approach to a learner's schema development explains this phenomenon of stepping-back-and-forward as a growth in one's understanding. It is possible when a learner encounters a new experience, the subject struggles to assimilate the new experience and accommodate the existing schema to reach equilibrium status. It seemed that, for Jason, explaining factorization of polynomials with a leading coefficient $\neq 1$ and using the concepts of reducible or irreducible in that explanation was an unfamiliar experience and, through this experience, he further developed his existing schema of factorization with a new component of unit concept. This kind of schema development is a characteristic of the unifying process because the additional component to the schema (i.e., unit idea) allowed him to use it as an overarching idea to see the sameness in complete factorization across different expressions. Hence, the shifts from U3 to U2 to U3 can be considered ultimately a unifying process.

Whereas Jason's unifying process took almost one interview for him to get to a solid conclusion, some unifying processes seemed to happen immediately. For example, an immediate shift from U1 to U3 was observed in Andy's response to **POST-Q8**. The following excerpts show how Andy arrived at a conclusion that "polynomials are, it's a system where they're composed of smaller polynomials just like integers."

Andy: We've talked about this before and I had no idea what that meant [*this refers to the question in* **POST-Q8**].

Interviewer: How about now?

Andy: How about now? Understand that polynomials form a system. [*Andy reads the problem in a whisper*] It still doesn't make sense to me.

Interviewer: Can we think back to what we did so far in our past five interviews?

Andy: Well we talked about factoring, what it means to be completely factored, and over certain domains how things change. So I guess **polynomials form a system, and now it's the integers in the sense that some polynomials can be factored, whereas some integers can be factored**. And some polynomials are composite the same as some integers are composite. So you can have like prime polynomials like you have prime integers. You have composite polynomials; you have composite integers. So **polynomials are–it's a system where they're composed of smaller polynomials just like integers.** There are numbers that are composed of smaller integers. I guess in that sense–they're analogous in that sense.

Andy's explanation about the analogous nature between the set of integers and the set of polynomials centered on overarching ideas such as decomposition, reducible, and irreducible. With the ideas activated at that moment (i.e., polynomials, integers, factoring, reducible, irreducible, etc.), Andy seemed to provide a fully developed articulation about relationship among those ideas, especially about how they are analogous (suggesting U3). However, his initial response, "still doesn't make sense", indicates that what came to his mind first was no association between them (suggesting U1).

From an observer's perspective, one's understanding is only inferable from his/her articulation of the insights into the given problem in the moment. Hence, Andy's understandings as inferred from his articulation in **POST-Q8** does suggest a shift from U1 to U3. However, one might wonder to what extent such an immediate shift from U1 to U3 that skips an intermediate level of U2 provides evidence of a shift in his actual cognitive structure (which is not directly accessible). A possible explanation is that the *sudden* shift from U1 to U3 in Andy's responses could have been the matter of recognizing the relevance between factorization and the problem context of **POST-Q8**, rather than the function of an actual change in his cognitive structure relevant to factorization. Reflection on the past TIs (prompted by the interviewer) could have triggered him to bring in previously established relationships in his mind between factorization of integers and factorization of polynomials.

While Andy's data only *tentatively* suggested a growth of his understanding from U1 to U3, other participants' data suggested more convincing evidence that they came to develop a decomposition view in both factorization of integers and factorization of polynomials (as an overarching perspective). For example, the comparison of Sam's responses to **PRE/POST-Q8** and the data triangulation afforded by his retrospective statements strongly suggested a growth in his understandings. While Sam responded to **PRE-Q8** that he is "not sure how to interpret that standard," his response to **POST-Q8** reflected his overarching understanding of factorization that applies to both the set of integers and the set of polynomials. In **POST-Q8**, he responded, "I think it is saying… that their students need to understand that the set of polynomials and the set of integers kind of carry similar rules." He used examples of irreducible elements from each set, 5 and $x + 4$, to show how the rule of reducibility applies to both. He also explained how factorization similarly applies to both sets as follows: "whenever I see an expression and it gets factored, ... I am going to see now that it was split up into two or possibly more things that can be multiplied together to get back your original expression." This seemingly basic decomposition view that might be helpful to obtain trans-object level, however, may not naturally develop in students.

Several retrospective statements made by Sam (as follows in the bullet points) illustrated an old way he had thought about factorization and suggested his previously held view that separates factorization of polynomials from factorization of integers.

- "I had only ever thought of factoring in terms of polynomials and not in terms of numbers";
- "It was weird to think about [factoring a number and factoring a polynomial altogether] for the first time, I guess, but I'm actually seeing more similarities between the two now than **differences**, now that I think about it";
- "Whenever I saw it like this [*this refers to a problem, "How can you factor 60?"*], the term 'factor' had a much **different meaning** to me than if I was given the quadratic equation."

This indicates his understanding of factorization was at intra-object level originally. It seems possible, in school mathematics, students come to activate different schemas of factorization depending on the appearance/inscriptions of problems that students are solving. The series of questions during the TIs influenced participants to take a much deeper look into factoring and define it more carefully so that it explains all cases consistently and comprehensively. As such, the questioning seemed to have provided a chance for them to develop a coherent and unified approach to factoring in general.

Discussions

Freudenthal (1978/2004) noted that "Mathematics exercised on a lower level becomes mathematics observed on the higher level" (p. 61). This implies collegiate mathematic learning can provide a perfect chance for university students to illuminate their existing understandings from a different angle and reorganize them into a more coherent entity (not only as a chance to construct new knowledge). However, in a lecture-based approach, Hazzan (2001) observed that university students only partially internalized definitions, propositions, and proofs, or reduced the abstraction level of concepts by forcing them into a familiar context. Hazzan noted it was probably so because, from the world of their subjective knowledge, there was little sensible connection to the formally presented definitions, propositions, and their proofs. Based on Hazzan's study, I doubt my participants could have made the same transformative transitions if they were simply given formal definitions of abstract algebra ideas that I used in TIs (instead of constructing their own definitions). Hence, I suggest collegiate mathematics instructors to look for, develop, and integrate in their instruction the context in which university students can actively revisit, observe, reflect on, and interrelate their existing understandings at a higher level and from a different angle, to help them develop their mathematical knowledge into a coherent entity.

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Math and Moral Reasoning in the Age of the Internet: Undergraduate Students' Perspectives on the Line Between Acceptable Use of Resources and Cheating

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This study examined how eight students in an introduction to proof (ITP) course viewed a "cheating scandal" where their peers submitted homework containing solutions found on the web. Drawing on their weekly log entries, the analysis focuses on the students' reasoning about the difference between acceptable and unacceptable use of internet resources in learning mathematics. One pattern was that students' view of the relationship between beliefs about mathematics and the work of learning mathematics grounded their views of "cheating." Specifically, some felt that an implicit didactical contract required that model solutions should be available when one learned new material. The case raises the general issue of the relationship between the process of learning mathematics and the appropriate use of external resources. It suggests that instructors may need to re-examine the role of homework, especially its assessment, in their courses, so that productive struggle is valued, not avoided.

Keywords: Proof, Homework, Academic Integrity

There exists a broad literature on cheating at the collegiate level (Kleiner & Lord, 1999; Lathrop & Foss, 2000; McCabe, Butterfield & Treviño, 2012). Cheating is typically defined as an action of fraudulently deceiving or violating rules, including (1) use of paid or unpaid surrogate to complete assignments or assessments (2) unauthorized collaboration on assignments or assessments and (3) unauthorized coaching on assessments (Lathrop & Foss, 2000). This definition is broadly consistent with those typically printed in course syllabi and that are aligned with university academic honesty policies. According to McCabe and colleagues who led the International Center for Academic Integrity (ICAI) student surveys on academic integrity, 68% of undergraduates admit to cheating on tests or written assignments (McCabe, Butterfield, and Treviño, 2012). According to a 1999 U.S. News and World Reports Study, 80 % of students report having engaged in some form of cheating and 90% of those who reported cheating said they had never felt negative consequence for having done so. One of the factors related to the urgency of the issue is the "shifting sociocultural view on the morality of cheating" (Trenholm, 2007). Trenholm reports that a societal shift in focus towards products (degree, GPA, grade on homework) as opposed to processes (grappling with difficult material until one understands it) provides a context in which cheating is seen more as a pragmatic than moral issue (Kleiner & Lord, 1999).

Though there is a wide literature on cheating in collegiate classes generally, the case of upper division mathematics is of interest because both the nature of the work and the student population differs from the case of collegiate work in general (Smith et al., 2017). The character of out-of-class mathematical work required of students in upper division courses has typically involved the experience of exploring novel tasks and formulating arguments, which is a shift from presenting students with opportunities to practice using concepts or algorithms they have learned in class. Under these assumptions, homework is often allocated a substantial part of course grades to encourage students to spend significant time on working on tasks outside of class time. However, the availability of online sources of mathematical work, both open access and for-profit, has exploded in recent years, making the issue of students' access to these sources an issue for mathematics instructors at all levels when they design and assess student work. Understanding what students say about the phenomenon can help instructors of such courses consider how to re-shape their courses so as to respond to both (1) the learning needs of their students around the role of the work of others in learning mathematics and (2) the reality that so much of the traditional canon of problems exists now online, both for free and for purchase.

Setting

This paper considers data of students making sense of the phenomenon of "cheating" by using unauthorized online sources and turning them in as their own within an Introduction to Proof (ITP) course at a large midwestern university. The data was collected as part of a larger NSF-funded grant that involves collecting longitudinal data on a sample of mathematics students as they progress from their ITP course through their upper division mathematics coursework.

The project seeks to understand how ITP students navigate challenges such as completing homework sets that demand a different kind of organization of work and a different character of mathematical work than they have engaged with in the past. In particular, we are interested in the growth and development of students' autonomy and agency in their mathematical work as they navigate upper division coursework in mathematics. Thus, when it became clear to the instructors and the research team how widespread the practice of using unauthorized online sources was in the class (upon scrutiny, every section turned up multiple instances of the use of unauthorized, paid, internet sources on one homework set), we sought to understand students' perspectives regarding the use of online sources for homework.

To give some context for the analysis in this paper, about halfway through the semester (following a particularly long homework assignment), one of the instructors for the course alerted the instructional team that four students had turned in exactly the same solution to a homework task. The solution differed from the previously distributed solutions and the instructor suspected that the online student work subscription service Chegg was the source of the work in question. In response, the course coordinator and instructors opened their own Chegg account in order to monitor activity on the site around the homework questions and to assess the depth of materials available on the site. They confirmed that Chegg was the source of the suspicious student solutions in question and then systematically examined other papers for similar anomalies. Accordingly, the instructional team decided to change the homework grading policy for the remainder of the semester by focusing on homework completion.

The syllabus for the course contained two explicit academic integrity statements. The first statement came from the university code of honor: "I will strive to uphold values of the highest ethical standard. I will practice honesty in my work, foster honesty in my peers, and take pride in knowing that honor in ownership is worth more than grades." The second came from the course Academic Honesty statement: "…unless authorized by your instructor, you are expected to complete all course assignments, including homework, lab work, quizzes, tests, and exams, without any source… Students who violate academic integrity rules may receive a penalty grade, including a failing grade on the assignment or in the course."

In one section of the ITP class, an instructor explicitly talked with students in his section (one of five) about the academic honesty policy in the syllabus, while also encouraging joint work and collaboration outside of class and emphasizing the need for students to come away from a collaborative work experience with a personal understanding. One other section was observed by the project team to have had a similar discussion, if brief, about the intentions and role of out-of-class collaboration in completing class work. By design in this course, students had ample access to external supports: access to a mathematics learning center (MLC), the online platform Piazza, and instructor office hours. In addition, they had access to an extensive "examples document" that provides the tasks worked on in class. Further, the solutions to each assignment were released immediately *following* students turning it in.

Participants

The eight focal participants for this analysis come from two sections (of five) of the ITP course. With respect to the topic at hand, we have the most detailed evidence from these students' homework logs. As part of our broader project goal focusing on students' experience in ITP courses, our participants regularly wrote reflections about their process of navigating challenges in the ITP course. Thus, when the cheating scandal broke, we added a reflection entry prompting them to offer their views of it. The text of our prompt appears in Figure 1. Additionally, we also have corroborative evidence from regular observations conducted during the semester and interviews with the course coordinator.

> In posing these questions (below) to you, I am <u>not</u> asking how you personally choose to use or not use information from the web. I am interested in your views of the issues involved.
> - Do you think that looking on the web for solutions to homework problems and submitting them as your own work is "cheating"? Why or why not?
> - What do you think of the reasoning of the instructional team (including your instructor) that including someone else's solution does not help you prepare for the exams?
> - What do you think about the penalty imposed for "cheating": zero for the homework and you can't drop that particular HW grade?

Figure 1. Prompt given to students inquiring about their views on "cheating" and use of external resources.

As suggested by the questions we asked them (above), we were interested in (1) nuance in their perspective around using online sources, (2) the role of homework in learning the course content with some degree of autonomy (indexed by preparedness for exams) and (3) their perspective on the instructional response in the ITP class, including alternative options.

Theoretical Framework

In teaching mathematics, students, teachers, and other stakeholders associate expectations with what mathematics is taught and how it is taught (i.e. obligations that each undertakes and benefits from, and the means by which they envision satisfying the obligations, as well as the consequences for not satisfying them). A *didactical contract* is an interpretation of the set of these expectations and obligations (Brousseau & Warfield, 2014). We view our observations and interviews as providing different and complementary windows into the didactical contract in effect in one particular introduction to proof course. Further, we propose that this theoretical tool is a useful lens for examining students' interpretations of their rights and responsibilities more broadly.

Speaking to the source of these rights and responsibilities, many development researchers have distinguished between heteronomy and autonomy in moral reasoning (Kohlberg, 1976; Piaget & Inhelder, 1969). Heteronomy is the orientation that the source of direction for right behavior comes from dictates or principles from outside (e.g., an adult, parent, or recognized powerful authority). Autonomy, by contrast, is the view that the source of guidance for right behavior is the individual's own principles or sense-making. In the present context, we have external authority that is represented by the university guidelines for academic honesty/dishonesty outlining what is acceptable and unacceptable practice in completing and presenting one's academic work. These guidelines were developed well before the emergence and wide use of "internet resources" like Chegg. On the other hand, students in this course had access to many related sources of "good mathematical reasoning" on specific tasks (e.g., guidance from instructors in the MLC and office hours). These "legitimate" sources of the content and format of "good proofs" raise reasoning challenges for students: What distinguishes acceptable forms of assistance from unacceptable?

Analysis and Findings

For each prompt, participants' responses were grouped into thematic units that captured the essence of each part of their responses to the questions. Following the initial process of thematic grouping, the themes were re-organized to collect common themes across participants together. In this way, the analysis converged on a set of common themes that still preserved the essence of the diversity of themes represented in the data. The first issue our analysis addresses is the range and variability around where participants drew the line around what constitutes acceptable versus unacceptable use of internet resources. A second focus is the source of morality in terms of the role that external directives versus internal convictions played in students' reasoning about what constitutes acceptable/unacceptable behavior. A third focus is how students conceptualized the relationship between learning/understanding and internet resources. Students' logs also prompted them to consider the instructional response in their class, and some went on to discuss their perspective on potential instructional responses that address the issues of fairness and learning. Our analysis of students' perspectives provides a springboard into the discussions and conclusions of this paper, including potential instructor responses to the reality of resources like Chegg being part of the instructional ecology.

Defining Boundary Conditions

To give a sense for the complexity of students' reasoning about what constitutes cheating, consider a vignette that participant G made up to help himself think through the moral issues involved in the use of outside resources in homework completion:

> "Student A and B both go to MLC at the same time, working on the same problem such that the same teaching assistant helps both of them on the problem at the same time on the blackboard. Student A just continues writing on his own, while student B is copying what he saw. Then student C comes in, and without asking anything, but only reading what is on the board, he gets the answer. Then TA teach[es] the same stuff to student D who comes in later. They all have the same way of proof. Suppose that student E saw the work, not on the board but on a website instead, and complete his homework. Who is cheating?"

All eight participants responded with nuance around the issue of using internet-based resources for homework, all citing conditions under which using internet-based resources would and would not be considered cheating. For most, the important distinction was whether a student engaged in original thinking in using the resource, especially after having gotten stuck in working on one's own (participant M) and "not brainlessly copying down answers" (participant F, with a similar sentiment in the logs of participants O, K, G, H, D, M, N). Under the condition that one is not intentionally copying for the purpose of passing off another's work as one's own, participant K went so far as to say that looking [online] for an answer and using it to get one's own can even be better than powering through on one's own. With respect to the use of particular resources, one student explicitly discussed Chegg as a "gray area" (participant N) because it is a paid service as opposed to a public one, where two others found no problem with Chegg because it is a resource that students theoretically have available to them (participants F and D). One participant (M) preferred getting help from resources tied to the course such as Piazza or the Math Learning Center to seeking assistance online. However, M's reasons concerned the lack of information about how one's specific instructor grades proofs. Participant F offered a contrasting view where internet resources were preferable to use when stuck because they were anonymous and didn't bother the professor, teaching assistant, or other students.

Internal Versus External Orientation Concerning Impact of Copying

Participants were divided in their focus on whether copying could help/hurt with *external performance* (participants H: "problems on exam could be different so copying can hurt", O: "getting closure on HW problems could help solve future problems", G: "looking at others' solutions can help you prepare"). Instead, other participants focused on the negative impact of copying on personal understanding (e.g., N and F). The preparation issue extends beyond the academic context. Some students appeared to be skeptical about whether finding other solutions undermines preparation, for the class, but also more generally in the world of work. For example, consider the following segment from a classroom observation completed just after the use of Chegg was discovered:

> At the beginning of class, a male student is discussing differences between working in industry in academia with his peers. He claims that 90% of students in the class will be going into industry. Another student argues that it is ridiculous that everyone in industry uses calculators and googles things – why are calculators not permitted in classes? A third student then claims that the professors in this class keep saying that they need to go through rigorous and technical proofs, but in industry people just google things. The student recognizes that in academia people need to come up with new stuff that can't just be googled. However, why can't we use google since 90% of us are going into industry [referring back to the first student's claim]. People use computers and "cheat" all the time. He closes by agreeing that the situation [of not being allowed to use internet resources] is kind of annoying.

Connections between views on internet resource usage and views of mathematics learning

Explicit in two students' logs (O and D) were references to using online sources in order to obtain models for the purpose of understanding assigned problems. Participant O went so far as to say that if one had not been taught specific techniques to solve assigned problems then one "had" to google the problem in order to find out what techniques they should use. Such an assertion belies a commitment to the view that mathematics learning involves merely learning to apply a suite of algorithms or techniques appropriately. That is, assigning tasks for which one needs to develop a chain of reasoning or a technique that has not been explicitly taught is seen as a violation in expectations about the nature of mathematics learning (and thus, what mathematics teaching should entail). The purpose of homework in this model of teaching and learning is to practice known techniques. While O's log was notable for its explicit invocation of this view, this perspective on mathematics learning has been a common theme in our prior interviews of ITP students reflecting on the differences between their lower division mathematics experience and proof-intensive upper-division courses (Author, 20xx). In contrast to O's perspective, participants H, M, and K invoked the perspective that mathematics learning is about engaging in reasoning and thus copying solutions from the internet undercuts the purpose of the course (reasoning). Two other students (F and N) thought copying solutions was detrimental to learning (participant N: Chegg is bad for learning because you are not truly synthesizing the material; participant F: Writing new solutions is helpful because you recognize how to solve new problems in the future). However, the nature of "understanding" was never unpacked further in these students' logs. Thus, it is possible they believed that copying using Chegg doesn't help one learn the material, but that "learning the material" still could focus on known algorithm application.

Views on Instructional Response

With respect to our third question, participants were split on their level of sympathy to those who engaged in using unauthorized internet resources. Some (D and F) offered reasons why students may engage in this behavior (participant D: "necessary" if one can't make it to office hours; participant F: helpful to get closure if one is struggling, especially when the grade is based on correctness). Participant G remarked that students cheat because they want their grades, and they already don't have enough time for their work and sleep. When the homework seems too hard and there is no help through sanctioned methods on the weekend, then students naturally turn to the internet. In contrast, participant K was strongly unsympathetic to those who had been caught with copied solutions because of inequity in grading relative to students who had been playing by the rules. One theme that repeatedly emerged was students' worry over whether it could be possible for a student to be falsely identified as cheating when, because there is a canonical proof approach, their work could look like someone else's or an online source (H, O, and K). With respect to the move to completion as opposed to correctness grades, K remarked that he still planned to give his best effort on subsequent homework, and the move to completion made him feel like he didn't have to get everything exactly right. (However, we note that in a later interview with K, he mentioned that in practice, he had been less motivated to put in as much effort on the homework following the shift to grading for completion).

Discussion and Conclusions

Instructors may assume that students and instructors have a shared understanding of how to appropriately use sources, cite sources, and collaborate. Our modest analysis shows that these issues are more nebulous and complex for students than they may appear on the surface, especially as pertains to their work in upper-division mathematics, which has a different character than their lower division coursework and also other courses that require academic writing. A discussion of student reflections on cheating is important for course instructors because without it, instructors might try, as one TA did, to admonish their students to "Stop cheating!" However, this approach is unlikely to be successful because students may (1) honestly not know how to think about the role of the work of others in their learning activity in upper division courses and (2) it leaves completely open what exactly the students are supposed to stop. In fact, without further elaboration, instructors could be admonishing students to stop forms of activity that may actually be instrumental in their learning. Indeed, our participants did by and large seem to agree that a boundary around acceptable usage is that some understanding of the proof had to be generated by the student through sense-making. Backwards engineering or processing a proof that they found was different in their view than "brainless copying."

Since students have access to written academic integrity policies and many other forms of external guidance and yet copying solutions persists, we do not believe the most effective response is to make sure students understand the ground rules about what is and is not sanctioned. Instead, our perspective is that platforms such as Chegg are a part of the ecology of resources available to students that we need to be aware of and be more proactive in engaging with and designing experiences for students that still achieve our goals and maintain a level playing field. In a similar vein, Lang (2013) urges professors to move away from considering the dispositional factors that may make cheating more likely and instead focus on contextual factors that influence cheating. In his synthesis of cheating at the collegiate level, he identified four classroom-level conditions that can induce cheating: (1) Emphasis on performance, (2) High stakes on the outcomes of the performance, (3) Extrinsic motivation for success and (4) Low expectation of success.

The work of Mejía-Ramos et al. (2012) in delineating a framework for assessing proof comprehension provides some ideas in the direction of other forms of questions instructors can ask that tap into a deeper synthesis of the material than the commonly employed approach of asking students to reproduce complete arguments (that may be more possible to search and copy wholesale from the web). Asking students more open-ended questions or to generate their own questions can both quell cheating and also communicate a different image of the discipline to them—that they *should* be asking questions. With references to the findings of the analysis, it is worth considering whether extensive access to "model" tasks (as was the case in this particular ITP course) has the effect of reifying a view of mathematics learning as being primarily about learning to recognize and adapt model solutions. This is particularly salient in a course like Introduction to Proof where one of the meta goals for the course is to prepare students to engage in productive struggle (Hiebert & Grouws, 2007) and to become more comfortable grappling with tasks for which they will *not* have a model solution to base their work on.

In the context of collegiate mathematics, and calculus in particular, Reinholz and colleagues have been experimenting with a technique called Peer-Assisted Reflection (PAR) (Reinholz, 2015; 2016). Empowering students with the responsibility as a class to generate their own criteria for what makes a good proof and then engaging students in judging the quality of their own proofs or proofs of their peers may help shift disciplinary authority to the students. While turning in assignments and receiving a score can communicate that the teacher decides what is and is not good mathematical work, the PAR approach can help students self-assess their own work, understand flaws in work they might encounter on the internet, and also send the strong and disciplinarily authentic message that grading and proof quality is not a judgment up to the mercurial whims of one professor.

While this study is modest in its scope, it is hoped that it will spark a discussion about the ways in which upper-level mathematics instructors can respond to the fact of the widespread availability of solutions on the internet in ways that promote the learning goals of these courses.

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A Conceptual Analysis for Optimizing Two-Variable Functions in Linear Programming

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Secondary mathematics curricula predominantly present linear programming by introducing the corner point principle and formulating steps for a solution. The teaching and learning of this topic often overlook justification for the principle and procedures. Drawing on six clinical interviews with an in-service teacher, we illustrate how her covariational reasoning supported conceiving quantities' multi-variation entailed by an objective function and its constraints (a system of inequalities). We build upon this teacher's understandings to propose a conceptual analysis for optimizing two-variable objective functions in the context of linear programming.

Keywords: Linear Programming, Two-Variable Function, Covariational Reasoning

A linear programming problem is a common type of optimization problem in mathematics. Secondary mathematics curricula often introduce linear programming in two-dimensions that involves finding the maximum or minimum value of a two-variable linear function under a domain restricted by a system of two-variable inequalities. Undergraduate mathematics curricula expand this type of problems to *n*-dimensions, which requires more complicated algorithms (e.g., numerical optimization methods) to locate optimal solutions efficiently. Regardless, linear programming is a rich context for creating problems that provide students opportunities to apply mathematical knowledge to model authentic real-world situations.

Despite its merits, linear programming has received a general lack of attention from the mathematics education community. Understanding and solving linear programming problems at a secondary level requires a student to draw on a web of mathematical ideas, including finding intersections of two lines, reasoning with multi-variations of quantities, representing quantitative relationships (both inequality and equality relationships) algebraically and graphically, and so on. In this paper, we focus on the optimization of two-variable objective functions. In particular, we expound upon the question: *What mental actions and operations are involved in constructing productive meanings for optimizing an objective function?* By "productive," we refer to meanings that allow an individual to provide a justification for why a procedure works and that allow an individual to make sense of a variety of situations and ideas in coherent ways.

Background

An objective function in the context of two-dimensional linear programming is a two-variable function in the form of $z = f(x, y)$, where $f(x, y) = ax + by$. Throughout this paper, we maintain the convention of treating x and y as representing input values of the objective function and z as representing output values. Given extant literature on students' difficulty with coordinating two co-varying quantities in various contexts (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, Johnson, & Moore, 2013; Johnson, 2015; Moore, Stevens, Paoletti, Hobson, & Liang, 2019), we conjecture it is also challenging for students to simultaneously coordinate three quantities and anticipate the variation of value z in relation to the covariation of x and y, given the coefficients a and b. Moreover, reasoning with an objective function in the context of linear programming is more cognitively demanding because it requires an individual to go beyond describing quantities' multi-variation qualitatively to locate the optimal output of the objective function under certain constraints of x and y.

Extant secondary curricula often simplify the treatment of this topic as evaluating an objective function at corner points of a feasible region and comparing their outputs, often called vertices method (Figure 1, left) (e.g., Larson, Boswell, Kanold, & Stiff, 2001; Long & Ashley, 2008; Ryan, Doubet, Fabricant, & Rockbill, 1993; Senk et al., 1993). An underlying assumption of this approach is the *corner point principle*, which states that if there is a unique optimal solution to a linear programming problem, it must lie at a vertex of the feasible region. Most secondary curricula omit the proof of this principle and, instead, formulate steps for a solution using the vertices method (e.g., Larson et al., 2001; Long & Ashley, 2008; Ryan et al., 1993). One deficiency of this treatment is that it does not focus on the logico-mathematical operations (Piaget, 1976, 2001) that will afford students understandings of properties of the principle (e.g., why is it sufficient to only attend to corner points but not other points in the feasible region?).

Another approach introduced in textbooks and literature is parameterizing the value z and graphing a family of parallel lines (or level curves) of $y = -\frac{a}{b}x + \frac{z}{b}$, where $\frac{z}{b}$ is the y-intercept of the line (Figure 1, middle) (Edwards, Chelst, Principato, & Wilhelm, 2015; Fey, Hirsch, Hart, Schoen, & Watkins, 2009; Moon, 2019; Ryan et al., 1993; Senk et al., 1993; Shama & Dreyfus, 1994). Each line represents a collection of x-y pairs that satisfy the relationship for a fixed z value. Hence, the optimization problem is transformed into finding the x-y pair(s) that allow(s) the level curve to have a maximum or minimum y-intercept (and, thus, z). A third approach is visualizing the feasible region and the objective function in a three-dimensional coordinate system and describing the optimal solution as existing on the maximum (or minimum) vertical projection of the feasible region (Figure 1, right) (Brzezinski, 2017).

Detailed analysis and comparison of these methods are beyond the scope of this paper, and we consider each approach can be productive if students understand its rationale and underlying quantitative meanings. In the following, we aim at proposing an alternative approach to optimizing objective functions from a covariational perspective and discussing its affordances.

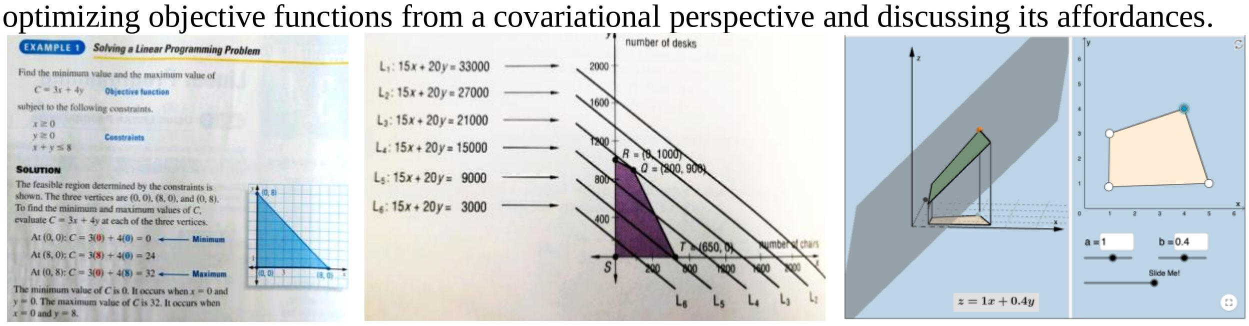


Figure 1. Curriculum examples of the vertices method (Larson et al., 2001, p. 164), level curve method (Senk et al., 1993, p. 288), and 3D graph method (Brzezinski, 2017).

Methodology

We adopt the epistemology of radical constructivism, believing that each individual constructs her or his knowledge in ways idiosyncratic and viable to herself or himself. We do not have access to anyone else's personal knowledge and, at best, can construct *hypothetical models* of others' knowledge that prove viable in our interactions with them. In proposing the following *conceptual analysis* (Thompson, 2008), we draw on our models of an in-service teacher's thinking and hypothesize productive meanings for optimizing objective functions based on our interactions with her. Specifically, the first author conducted six one-hour long clinical interviews with Jaime, who graduated from a secondary mathematics education program in the southeastern United States. At the time of the study, Jaime had completed her first year of full-time teaching at a public high school. Over the year of her teaching, Jaime taught Geometry

Support and Accelerated Geometry/Algebra II, and linear programming was the last topic of Algebra II. The interviews occurred during the summer following her first-year teaching. The interviewer started by asking Jaime to discuss her lesson goals, tasks used, and experience with working with students, and then shifted to ask her to engage in novel problems. While we acknowledge that such interactions could result in shifts in Jaime's thinking, our goal with the interviews was to interact with her in ways that focused on her mathematical meanings. Namely, although the interviewer continually attempted to elicit and perturb Jaime's thinking in order to construct a profile of her available meanings, she did not attempt to support Jaime in reconciling her experienced perturbations by providing systematic intervention to engender learning.

In reporting on our results, we primarily drew on Jaime's activities in two tasks (Figure 2): (1) A Trip to Sugar Bowl, a task Jaime used in class, and (2) Election Advertising, a task we adapted from the Core-Plus Curriculum (Fey et al., 2009).

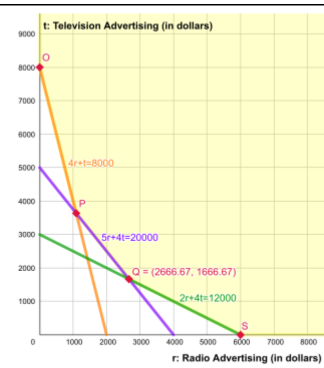
Task	Solution
A Trip to Sugar Bowl. A tourism agency can sell up to 1200 travel packages for the Sugar Bowl college football postseason game in New Orleans. The package includes airfare, weekend accommodations, and the choice of two types of flights: a nonstop flight or a two-stop flight. The nonstop flight can carry up to 150 passengers, and the two- stop flight can carry up to 100 passengers. The agency can locate no more than 10 planes for the travel packages. Each package with a nonstop flight sells for $1200, and each package with a two-stop flight sells for $900. Assume that each plane will carry the maximum number of passengers. *What is the maximum revenue for the given constraints and give the combination of flights that achieves this maximum?*	x=number of nonstop flights; y=number of two-stop flights **Constraints**: $x + y \leq 10$; $150x + 100y \leq 1200$ **Objective function**: $P(x, y)$ $= 1200 \cdot 150x + 900 \cdot 100y$ **Optimal solution** occurs at $(x, y) =$ (8,0) and $P(8,0) = 1200 \times 150 \times 8 + 900 \times 100 \times 0 = 1{,}440{,}000$  Note: Due to certain reasons, Jaime set up the objective function as $P = 1200x + 900y$, which corresponded to an optimal solution of (4,6). We discussed the following data in terms of her solution.
Election Advertising. When candidates plan their campaigns, they have choices to make about spending for advertising. Suppose that advisers told one candidate that radio and television ads might reach different audiences in the following ways: • Every dollar spent on radio advertising will assure that the candidate's message will reach 5 Democratic voters, 2 Republican voters, and 4 Independent voters. • Every dollar spent on television advertising will assure that the candidate's message will reach 4 Democratic voters, 4 Republican voters, and 1 Independent voter. The candidate's goals are to reach at least 20,000 Democratic voters, 12,000 Republican voters, and 8,000 Independent voters with a minimum total advertising expense. *What is the advertising plan that can minimize total advertising expense? What is the expense?*	r=dollars spent on radio; t=dollars spent on TV **Constraints**: $5r + 4t \geq 20{,}000$; $2r + 4t \geq 12{,}000$ $4r + t \geq 8000$ **Objective function**: $P(r, t) = r + t$ **Optimal solution** occurs at $(r, t) = (2666.67, 1666.67)$ and $P(r, t) = 2666.67 + 1666.67 = 4333.34$ 

Figure 2. A Trip to Sugar Bowl and Election Advertising.

The Case of Jaime

In this section, we illustrate Jaime's meanings for optimizing objective functions in three aspects: (1) her meanings for the relationship between an objective function and its constraints, (2) her meanings for the corner point principle, and (3) her meanings for an optimal solution in relation to quantities' amounts of change. We note that our intention is to provide vignettes of her thinking that inform the conceptual analysis we will propose in the next section; our intention is not to depict a development or progression of Jaime's thinking during the six interviews.

Relationship Between an Objective Function and its Constraint

In A Trip to Sugar Bowl, Jaime discussed her meanings for the objective function, $P = 1200x + 900y$, in terms of the three quantities' variation:

So I mean, one unit of increase of, like one extra nonstop flight, is going to make you more money than adding one extra two-stop flights because it's worth more. So the P will increase more, so when you have more nonstop flights. So it's a matter of like, do we just buy a bunch of nonstop flights, even though that means you are selling no two-stop, or do you need some sort of a balance there?

Building upon her reasoning, the interviewer questioned her why the "balance" could exist given the fact that each nonstop flight could produce more revenue. Then she responded:

Jaime: There's got to be a balance to how many you sell. So like, let's say, at most, you sell all 10 flights to be nonstop flights, that means that at most you are making $12,000. That means that you are assuming that you can't make any more than 12,000. Like there's not a way to combine the amount of nonstops and two-stops, that you won't go over that.

[The interview asked her why there is such a potential of selling less nonstop flights but having higher revenue.]

Jaime: Because if for every two nonstop flights you would sell [*5s pauses*], not sure how to say this, two-stops [*8s pauses*], trying to take like there is a, there is a point where, let's say you knock out [*3s pauses*], like it becomes more beneficial to have more two-stop flights, because like those times the 900 is worth more than the amount you would have gotten with nonstop flights. I can't think of that number off the top of my head.

Interviewer: Yeah, but that won't happen, right? Because you always multiply it by 900.

Jaime: Yes, but, I don't know.

As the conversation continued, Jaime remained perturbed in terms of why the optimal solution did not occur when the number of nonstop flights is ten. In fact, although $(x, y) = (10,0)$ satisfies $x + y \leq 10$, it was outside the feasible region due to the presence of the other constraint, $150x + 100y \leq 1200$. However, we did not observe Jaime raising this constraint to explain. We note that this excerpt happened after she applied the vertices method to find the optimal solution. Thus, we inferred she likely was not aware that it was simultaneously taking into account both constraints that led to an optimal solution occurred at (4,6) instead of (10,0). Her success in carrying out a series of calculations to evaluate the objective function at each corner point did not necessarily lead to an understanding about how multiple constraints interact to impact the output of an objective function. However, we acknowledge that Jaime's meanings for $P = 1200x + 900y$ and $x + y \leq 10$ were supported by aspects of quantitative and covariational reasoning; she discussed how the three quantities vary together and anticipated having ten nonstop flights might result in a greater P. We illustrate later how she leveraged these ways of thinking to reason with quantities' variation with respect to *multiple* constraints.

A "Justification" of the Corner Point Principle

Observing that Jaime's solution to Election Advertising was compatible with the vertices method, the interviewer asked her why it was sufficient to only attend to the corner points but not other points within the feasible region. She explained:

Because each one satisfies a pair. So O is where this [*pointing to "*$4r + t \geq 8000$*" she wrote on a paper*], you have r and t, that's, that works for $4r$ plus t equals to 8000 and for r is equal to zero. And then P does the same thing for this one [*pointing to "*$5r + 4t \geq 20{,}000$*"*], and this one [*pointing to "*$4r + t \geq 8000$*"*]. Q does it for this one [*pointing to "*$2r + 4t \geq 12{,}000$*"*] and this one [*pointing to "*$5r + 4t \geq 20{,}000$*"*], so on. The corner points are showing you like this is the most amount or the minimum value that will satisfy at least two equalities at once. Okay, so you are not having to think about like, okay, this minimum value plus all these other ones, so you have to worry about the other ones, you can just take a look at that minimum.

We interpret that Jaime was making two implicit assumptions here. First, she considered each corner point simultaneously satisfied the *equation* versions of two inequalities (either involving the minimum or maximum output of an inequality), which necessarily led to an optimal solution relative to those two inequalities. Second, by comparing the solutions of different equation pairs, she could obtain a final optimal solution regarding the entire system of inequalities. In fact, an optimal solution relative to two inequalities can occur at non-intersection points (e.g., Point P is the interaction of $5r + 4t = 20{,}000$ and $4r + t = 8{,}000$, but Point Q leads to a more optimal solution, see Figure 2), and sometimes intersection points are not in a feasible region (e.g., the intersection of $4r + t = 8000$ and $2r + 4t = 12{,}000$). We argue that Jaime's explanation was a contraindication that she was consciously aware of the rationale underlying the vertices method. Although her explanations appeared to be compatible with the calculational procedures she carried out, e.g., equating two equations to find the coordinate values of the intersection point, the mathematical assumptions she made were incoherent to the vertices method from our perspective. More importantly, we note that her explanation did not account for aspects of the objective function. We inferred that she conceived an optimal solution more as a result of finding extremes regarding a system of constraints than as a result of an interplay between the coefficients of the objective function and those of the inequalities. Next, we illustrate how reasoning with quantities' amounts of change (the mathematical meanings of the coefficients), supported Jaime's anticipation of how an objective function's output varies in a feasible region.

A Justification Using Amounts of Change

In the same context of Election Advertising, the interviewer drew her attention to the segment OP and asked her to describe the variation of P with respect to the segment. Jaime reasoned that because from Point O to Point P, we were spending $1090.91 more on radio and saving $4363.64 on television (see her annotations in Figure 3, left), the total amount of money spent should decrease in general. Observing that Jaime used the coordinates of Point O and Point P to make her claim, the interviewer asked her to discuss points in between Point O and Point P, whose coordinate values were not displayed at the time. Jaime claimed the relationship should hold, explaining "the amount that you are saving and spending is the same because you are moving along the slope [*gesturing along the segment OP*]." Then, she transformed each equation into the slope-intercept form (Figure 3, right) and described:

Because this is saying for every one increase, one unit increase in television, there's one-fourth decrease in radio. So [*7s pauses*] that means that if you decrease the television by four, then you are adding one to the radio. If t is negative four, so if I'm spending $4 less on television, that means I'd saved a dollar on the radio [*referring to "*$1 = -\frac{1}{4}(-4)$*" in Figure 3, right*]…So if you want r equals to 1 and t has to be this [*referring to the "*$-\frac{5}{4}$*" of "*$1 = \frac{-4}{5}(-\frac{5}{4})$*"*], so you are still saving more than you are spending because you are saving a dollar and 25 cents and you are spending 1 dollar. And so Q is better than P. So, for every dollar we are spending on the radio, we are only saving 50 cents [*referring to "*$1 = -2(-\frac{-1}{2})$*"*]. So we are, we are spending more than we are saving at that point. So now S is not as beneficial as Q."

Jaime's meanings for slope in terms of amounts of change in two quantities supported her comparison of the outputs of the objective function with respect to the boundary points, by which she justified the optimal solution should occur at Point Q. We contrast her thinking here with that discussed in the previous two sections, where she did not simultaneously account for quantities' amounts of change implied by the objective function and its constraints.

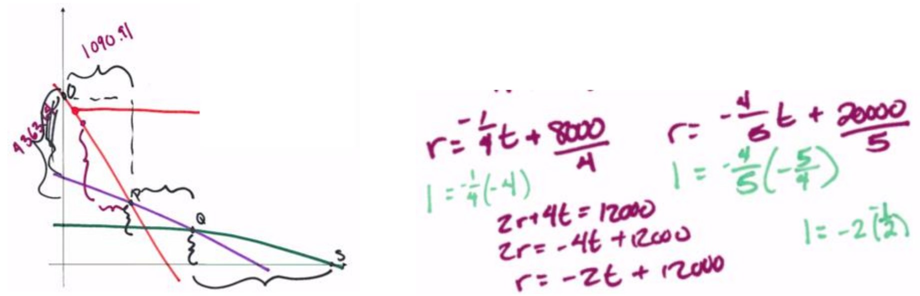


Figure 3. Jaime's (left) graphical and (right) algebraic illustration of amounts of change in Election Advertising.

However, Jamie did not continue to apply this way of reasoning to discuss other points of the feasible region. When asked to discuss points in the interior, she responded:

> Well, these [*pointing to the horizontal red segment in Figure 3, left*] are all the places where r and t, the combination of r and t is greater than this inequality [*referring to* $4r + t \geq 8000$]. Then that's giving us more than we need. And so for r plus t, we just want the minimum amount that we would have to spend, the minimum amount we have to spend is the equality part, not the greater than part. So that's all the combinations of r and t that's greater than 8000, which means that we are not looking at like a minimum amount that we would be spending on r and t for that function, the least amount that we could spend and still make this true, is all of the points on that line. So our r and t, like the point of the P equals r plus t is to figure out all of the places where like you are just satisfying that equality.

For Jaime, points satisfying the strict inequality necessarily led to less optimal (or higher) P values when compared to points satisfying the corresponding equality because "that's giving us more than we need." We interpret this as being consistent with her previous explanation that only points satisfying an equality were worth taking into account. Together, we hypothesize that she likely conflated the minimum of the output of an inequality and that of an objective function or, at least, made an association between them. Due to time constraints, we did not follow-up on her to test this hypothesis with other contexts, such as contexts including a combination of greater- than and less-than inequalities or including an objective function involving a minus sign. We are also unsure whether she could reason with the covariation of r and t entailed by the red segment and its influence on the output value P.

A Conceptual Analysis and its Implication

In summary, leveraging quantities' amounts of change afforded Jaime an understanding of how the output of an objective function varies with respect to points on a feasible region's boundary. Although she did not apply a similar way of reasoning to discuss points in the interior, we conjecture that it can serve as a conceptual tool for constructing productive meanings for optimizing objective functions. We outline a conceptual analysis as the following:

Consider an objective function, $P(x, y) = 2x + 3y$, under the constraints of $x + y \leq 3$ and $x + 2y \leq 4$, and the goal is to maximize P (see Figure 4 (left) for an illustration):

- Understanding that any point within the feasible region is less optimal than a reference point on the boundary:

- Choosing a point (x_1, y_1) in the interior of the feasible region, there is always a reference point, (x_1, y_2), on the boundary such that $y_1 < y_2$, and thus $P\ (x_1, y_1) < P(x_1, y_2)$; similarly, we can always find a reference point, (x_2, y_1), on the boundary such that $x_1 < x_2$, and thus $P\ (x_1, y_1) < P(x_2, y_1)$.

- Understanding that any non-endpoints on the boundary of $x + 2y = 4$ (i.e., Segment P_1P_2) are less optimal than one endpoint (i.e., Point P_1 or P_2) of the same boundary:
  - Understanding of $x + 2y = 4$ or $y = -0.5x + 2$: for any change in x, say Δx, y always changes by $-0.5\Delta x$
  - Understanding of $P\ (x, y) = 2x + 3y$: when x changes by Δx and y changes by $-0.5\Delta x$, P changes by $2 \times \Delta x + 3 \times (-0.5)\Delta x = 0.5\Delta x$.
  - Understanding that P increases from Point P_1 to Point P_2 (because Δx is positive) or P decreases from Point P_2 to Point P_1 (because Δx is negative).
- Applying the same way of reasoning to the boundary of $x + y = 3$ to anticipate P increases from Point P_3 to Point P_2 or decreases from Point P_2 to Point P_3.
- Coordinating direction of change in P with respect to points on the entire boundary to determine Point P_2 as producing the highest P value.

Critical to this conceptual analysis is an image of how P varies as x and y co-vary in the interior and along the boundary of a feasible region (Figure 4, left). This conceptualization extends the current theorization of covariational reasoning (Carlson et al., 2002), specifically amounts of change reasoning, from the context of single-variable functions to two-variable functions. Moreover, it does not require an individual to find an optimal solution by evaluating the objective function at *all* corner points. This affordance is significant in that it is coherent to an approach to solving n-dimensional linear programming problems. When dealing with n-dimensional problems, evaluating an objective function at each corner point of a feasible region (a polyhedron) is inefficient. In 1940s, George Dantzig, introduced the simplex method in numerical optimization as an algorithm for walking along the edges of a polyhedron to a corner point with a more optimal solution (Figure 4, right). Although the algorithm for determining the direction of walking is more complex in n-dimensions than what we have proposed for two-dimensions, we hypothesize that our suggested ways of thinking could provide students with a foundation for reasoning with more advanced optimization problems and algorithms.

We call for future research to continue to test and refine this model by interacting with students and teachers in similar or different linear programming contexts. Mathematics teachers can also foster students in constructing such an image of multi-variation of quantities regarding an objective function. These efforts will allow us to gain further insights into the affordances and limitations of such meaning for optimizing two-variable objective functions.

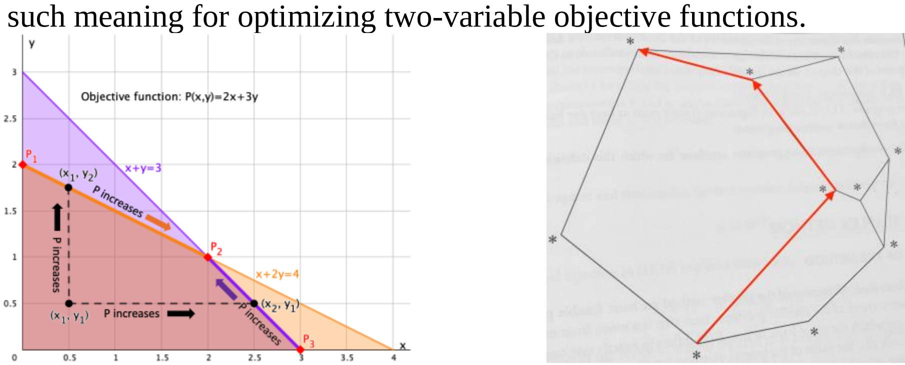


Figure 4. (left) An illustration of optimizing objective function from a covariation perspective and (right) the simplex method applied on a polyhedron (Nocedal & Wright, 1999, p. 373, annotated).

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Making Sense of Irrational Exponents: University Students Explore

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We investigate how students make sense of irrational exponents. The data is comprised of 32 interviews with university students, which revolved around a task designed to examine students sensemaking processes involved in the understanding of the concept. Both the task design and data analysis relied on the concept of sensemaking trajectories, blending the notions of sensemaking and (hypothetical/actual) learning trajectories. The findings focus on three kinds of reasoning utilized in the participants' sensemaking trajectories while working with irrational exponents: even/odd numbers and functions; range of exponent values; and exponentiation as repeated multiplication. These findings reveal students' conceptual development of irrational exponents, and could in turn be used for the refinement of tasks aimed at promoting students' comprehension of the topic.

Keywords: irrational exponents, sensemaking, learning trajectories

Mathematics education research has devoted considerable attention to both topics of irrational numbers (e.g., Kidron, 2018; Sirotic & Zazkis, 2007) and exponents (e.g., Levenson, 2012; Pitta-Pantazi, Christou, & Zachariades, 2007). However, students' understanding of the concept that emerges from the combination of the two – that is, irrational exponents – has not yet been sufficiently explored (e.g., Wasserman, Weber, Villanueva, & Mejia-Ramos, 2018). In this paper, we continue our research on irrational exponents (Marmur & Zazkis, 2019) and investigate how university students make sense of $f(x) = x^{\sqrt{2}}$. In particular, we examined students' work on a task involving different graphs of $y = x^{\alpha}$, where $\alpha \in \mathbb{Q}$ and "close" to $\sqrt{2}$.

Theoretical Framing

Our theoretical framing is informed by two constructs: *hypothetical learning trajectory* (HLT; Simon, 1995) and the notion of *sensemaking* (e.g., Weinberg, Wiesner, & Fukawa-Connelly, 2014). According to Simon (1995), an HLT describes a prediction of a route in which students' learning may occur. It is comprised of three parts: a learning goal, instructional activities planned to fulfill the goal, and a conjectured learning process predicting students' thinking and understanding during the different stages of these activities. When implemented with students, an HLT could then be compared with the learning process that occurs in reality, referred to as an *actual learning trajectory* (ALT; Clements & Sarama, 2004). The construct of learning trajectories has been used to bridge the work of researchers and teachers, as it "supports the content-specific documentation of common milestones and learning environments that support students' progression" (Andrews-Larson, Wawro, & Zandieh, 2017, p. 809).

Students' thinking and learning is closely related to the notion of sensemaking. That is, when students are facing a novel task or unfamiliar content, they attempt to make sense of what is involved by restructuring knowledge structures that are founded on prior mathematical knowledge (Scheiner, 2016). Informed by research in psychology and information systems, Weinberg et al. (2014) described sensemaking in the mathematics classroom as "the process by which students interpret and construct meaning for the activities, personal interactions, and discourse in which they participate; sense making also focuses on the individual student's mental schemas that shape this process of meaning construction" (p. 169). In our view, this description is not restricted to a classroom and is applicable to any engagement with mathematics.

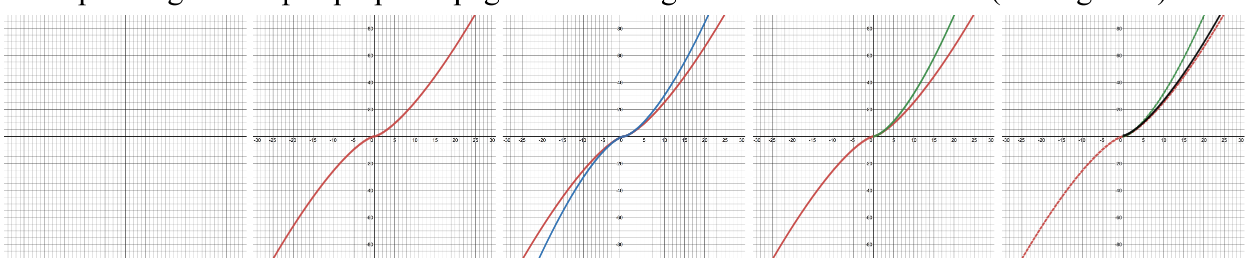
Accordingly, we focus on one of the sensemaking frames suggested by Weinberg et al. (2014) – *content-oriented sensemaking* – which describes a situation where students encounter gaps about the meaning of certain mathematical content or concepts. The sensemaking of the mathematical situation subsequently involves a learner "bridging the gap" by "drawing upon his or her ideas, thoughts, conceptions, attitudes, beliefs, intuitions, and knowledge" (ibid, p. 170).

We draw on both constructs of sensemaking and learning trajectories and suggest the blended concept of *sensemaking trajectory*, which involves a sensemaking goal, activities or tasks towards the goal, and a hypothetical/actual sensemaking path through the stages of the activities. Accordingly, we ask the following research questions: *What are students' sensemaking trajectories as they work on a task that involves an irrational exponent? In particular, how do they make sense of the graph of* $f(x) = x^{\sqrt{2}}$ *and associated mathematical ideas involved in the construction of the graph?*

Method

The Task and Considerations in its Design

How does the graph of $f(x) = x^{\sqrt{2}}$ look like? This question typically presents a novelty to students who are familiar with irrational numbers and operations on rational exponents, however have not yet considered the meaning of an irrational exponent. Accordingly, we designed a task based on our hypothetical sensemaking trajectory (HST) for interpreting an irrational exponent of x. The task was presented in an individual interview setting and included five stages corresponding to five pre-prepared pages that were given to the interviewees (see Figure 1).



(a) Empty graph paper (b) $y = x^{1.4}$ (red) (c) $y = x^{1.48}$ (blue) (d) $y = x^{1.5}$ (green) (e) $y = x^{\sqrt{2}}$ (black)

Figure 1: The five stages of the task

In Stage-1 of the interview, the participants were given an empty graph paper and asked to sketch $f(x) = x^{\sqrt{2}}$ (see Figure 1a). We conjectured that in the initial exploration stage, the participants would draw on their familiarity with the graphs of $y = x^n$, $n \in \mathbb{N}$, and based on $\sqrt{2}$ being between 1 and 2 would sketch a concave and monotonically increasing function in the first quadrant, starting from $(0,0)$, and "in between" $y = x^1$ and $y = x^2$. Subsequently, we suspected the participants to have some question marks regarding the meaning of $f(x) = x^{\sqrt{2}}$ for $x < 0$, and accordingly accounted for different options in their potential sketches (see first row in Figure 2 from left to right): a "cubic-like" graph, a "parabola-like" graph, claiming that the graph does not exist for $x < 0$, or leaving the graph blank for $x < 0$ as a result of being unsure of how to sketch it or whether it exists there. Alternatively, we conjectured participants might utilize other ideas for making sense of the task, such as attempting a table of values or calculating the derivative of $f(x) = x^{\sqrt{2}}$, which we did not expect would serve as productive avenues (and accordingly classified these as "Other").

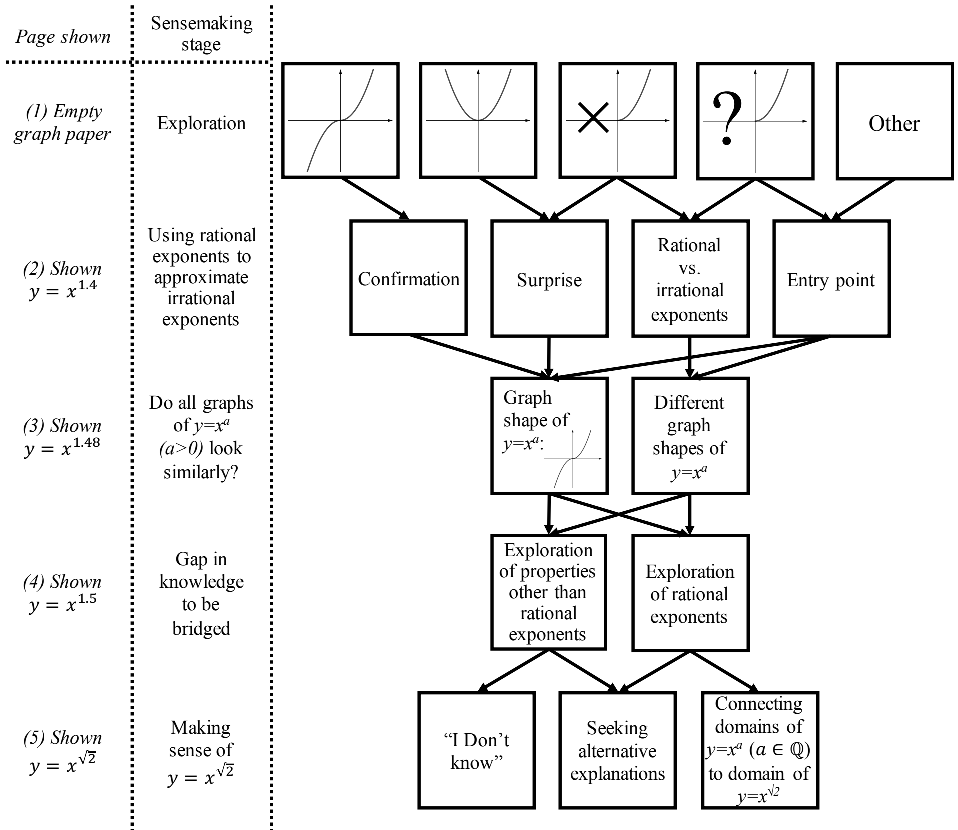


Figure 2: Hypothetical sensemaking trajectories for the $f(x) = x^{\sqrt{2}}$ task

Regardless of the initial response, in Stage-2 of the interview the participants were presented with the graph of $y = x^{1.4}$ (Figure 1b) and asked if this helped in constructing the required $f(x) = x^{\sqrt{2}}$. In case the participant was struggling in how to approach the task in Stage-1 of the interview, we expected the presented graph to serve as an "entry point" for making sense of $x^{\sqrt{2}}$ by using a rational exponent to approximate $\sqrt{2}$. In case the participant questioned whether $f(x) = x^{\sqrt{2}}$ is defined for $x < 0$, we expected the presented graph to focus his/her attention to differences between rational and irrational exponents. Alternatively, we expected the presented graph could confirm the participants' initial thoughts in case they had sketched a "cubic-like" function in Stage-1 of the interview; or be perceived as a mathematical surprise in case they had anticipated a different graph earlier. As a general rule, we anticipated the interviewees would sketch $f(x) = x^{\sqrt{2}}$ similar to $y = x^{1.4}$, an expectation that was further established in our design of the task, when the participants were presented with both graphs of $y = x^{1.4}$ and $y = x^{1.48}$ in Stage-3 (Figure 1c). Note that the exponent 1.48 was chosen purposefully, as this is the only number between 1.4 and 1.5 with two digits after the decimal point for which the graph has a similar shape to $y = x^{1.4}$. According to our conjectured HST, the interviewee would assume that $f(x) = x^{\sqrt{2}}$ is "in between" the graphs of $y = x^{1.4}$ and $y = x^{1.48}$ as $1.4 < \sqrt{2} < 1.48$. Alternatively, we suspected some interviewees may suggest different looking graphs based on other exponent choices.

Subsequently, in Stage-4 of the interview we presented the graphs of $y = x^{1.4}$ and $y = x^{1.5}$ (Figure 1d), illustrating that as opposed to $y = x^{1.4}$, the graph of $y = x^{1.5}$ is not defined for $x < 0$. Had this not been attended to earlier, we suspected the difference between the two graphs to invoke a cognitive conflict (e.g., Fischbein, 1987) that would accentuate the gap in the participant's knowledge, consequently highlighting the need for bridging and sensemaking of the mathematics involved. This gap could be bridged by reinterpreting the rational exponents involved (that is, $x^{1.4} = x^{\frac{7}{5}} = \sqrt[5]{x^7}$ and $x^{1.5} = x^{\frac{3}{2}} = \sqrt{x^3}$), attending to the domain of the related functions, and then extrapolating the conclusions to irrational exponents as a limit of rational ones. Alternatively, we suspected the participants might focus on more "peripheral" properties, other than rational exponents, when trying to make sense of the graphs. Regardless of the participants' reasoning, in the last stage of the interview they were presented with the graph of $f(x) = x^{\sqrt{2}}$ alongside $y = x^{1.4}$ and $y = x^{1.5}$ (Figure 1e), which was intended to confirm or confront previous conclusions on the shape, as well as allow them another opportunity to make sense of $x^{\sqrt{2}}$. Figure 2 presents a visual illustration of our conjectured HSTs.

Data Collection and Analysis

The participants of the study were 32 university students from STEM fields (25 undergraduate students and 7 graduate students), who were individually interviewed following the five stages as detailed above. The interviews were typically about 30-minute long, though ranged from 15-50 minutes. All interviews were audio-recorded and subsequently transcribed.

In the analysis we first explored all Actual Sensemaking Trajectories (ASTs) individually, and in particular how these compared to our conjectured HSTs. Subsequently, we focused on the reasoning the participants provided as they attempted to make sense of the task, which led to a categorization of the different ASTs based on prominent repeating themes.

Findings

While the interviews inquired into students' sensemaking of irrational exponents, the analysis additionally revealed the participants' reasoning and sensemaking of related mathematical ideas, such as rational exponents, approximation of irrational numbers by rational ones, even/odd functions and numbers, and the meaning of exponentiation in general. Given the page limit for this paper, we focus here on three repeating themes of reasoning we found in the sensemaking trajectories. Illustrations of ASTs in these themes are presented in Figure 3.

Even/Odd Reasoning

Students demonstrating what we call "even/odd reasoning" attempted to make sense of the different graphs by conflating the idea of even/odd numbers with the concept of even/odd functions. For functions in the form $f(x) = x^n$ ($n \in \mathbb{N}$), $f(x)$ is an even (odd) function if and only if n is an even (odd) number. It seems this idea was overgeneralized to any exponent in order to make sense of the newly encountered mathematical situations. A recurring theme found in the data was that students perceived a number as even (odd) if the last digit in its decimal representation was even (odd). Interpreting the number in the exponent accordingly, students then concluded on the evenness/oddness of the function. Based on this line of reasoning, $\sqrt{2}$ was not perceived as even (or odd) as it does not have a last digit in its decimal representation, and this was then used to explain the graph of $y = x^{\sqrt{2}}$. We illustrate this kind of reasoning by focusing on the case of Emma (see Figure 3a), who used even/odd reasoning in Stage-1 and Stage-4 of her interview.

In Stage-1 of the interview, Emma attempted to find different function values of $f(x) = x^{\sqrt{2}}$ for various values of x. In the following interview excerpt Emma uses even/odd reasoning to justify why $f(-1) = -1$ in her opinion, after having explained that $f(1) = 1$. In particular, note her explanation for $\sqrt{2}$ being "not even" and the graph of $y = x^{\sqrt{2}}$ behaving "like odd":

Emma: The y is going to be negative because we see that it's not like an even power so […] x is going to be -1 and the y is going to be -1. […]

Interviewer: Ok. Can I ask how you know it's not an even power?

E: […] Because it [$\sqrt{2}$] never ends like you don't know if it ends with like 3 or 2 or like with an even or an odd number because it's ever going. You can always add a 2 and make it even and you can always add a 3 and make it uneven.

I: Ok. So you don't know if an irrational is even or odd because it never ends?

E: Yeah. So it's definitely not even but it's also not odd so you don't know what to do with it so I just decided that it's maybe, maybe it's like odd.

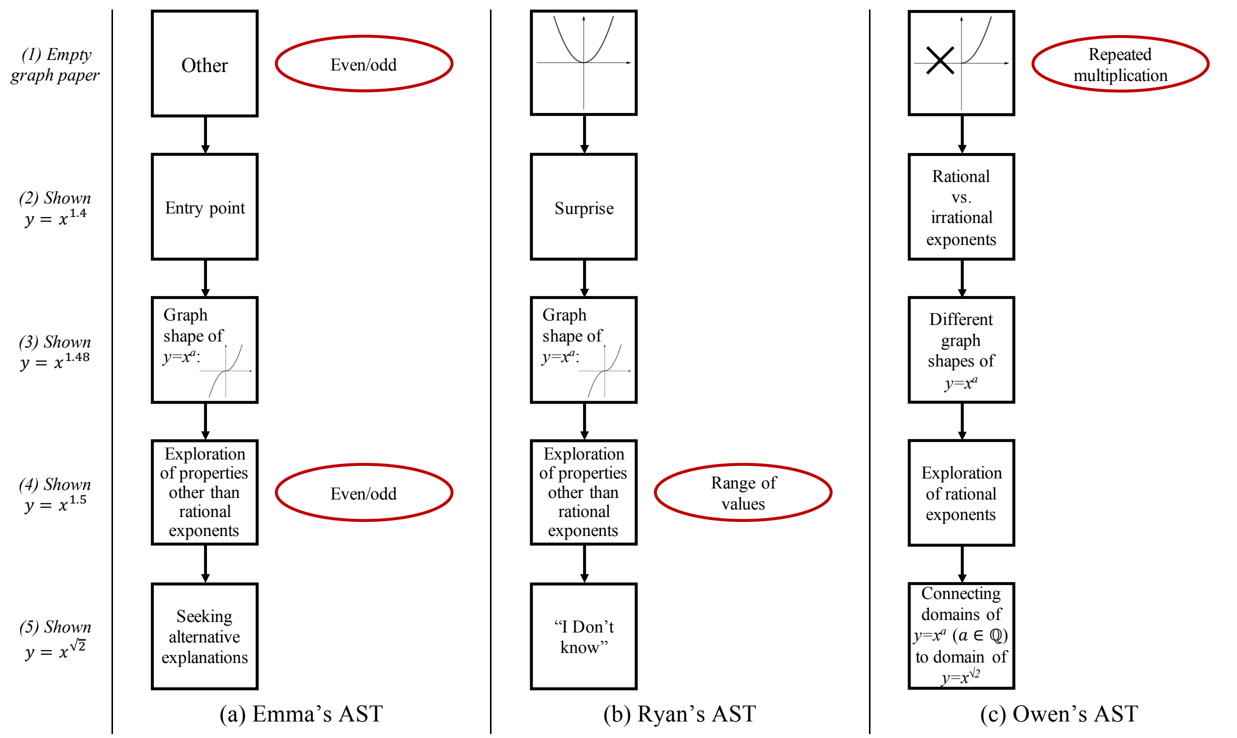


Figure 3: AST reasoning-themes: (a) Evenness/oddness; (b) Range of exponent values; (c) Repeated multiplication

In Stage-2 of the interview, when presented with the graph of $y = x^{1.4}$ and asked to draw the graph of $y = x^{\sqrt{2}}$, Emma sketched a similar graph to $y = x^{1.4}$, explaining that "it's just gonna follow where it's already going". Continuing this line of thought, in Stage-3 she drew the graph "in between" $y = x^{1.4}$ and $y = x^{1.48}$. However, Stage-4 introduced a cognitive conflict for Emma, for whom it did not make sense that the graph of $y = x^{1.5}$ was not defined for $x < 0$, which she again explained based on even/odd considerations:

E: To me it doesn't make sense that it stops [the graph of $y = x^{1.5}$]. […] If we look at these numbers and we […] multiply the power by 10. And we have 14 and 15. 14 is an even number. Which means the red [$y = x^{1.4}$] is an even graph. 15 is not an even number which means it's like an odd powered graph. […] Like to me it's like why is the red one continuing and the green one [$y = x^{1.5}$] not continuing? Like if, if this is an odd?

Lastly, in Stage-5, when shown the actual graph of $y = x^{\sqrt{2}}$, she struggled to make sense of it and returned to the idea she had expressed earlier regarding the "never-ending" decimal representation of $\sqrt{2}$ as an underlying explanation for the graph.

Reasoning based on a Range of Exponent Values

The way in which participants in this reasoning theme made sense of the graphs was by providing ranges of exponent values for which the graphs behaved similarly, alongside a particular value for which there was an abrupt change in how the corresponding graph would behave. To elaborate, one of the repeated claims was that functions in the form $y = x^{\alpha}$ would look similarly for α values between 1 and 1.5, and have a different shape for α values between 1.5 and 2. As illustration, we focus on Ryan's AST (see Figure 3b). In line with our original expectations regarding the participants' HSTs, it was only in Stage-4 of the interview, which introduced a cognitive conflict, that Ryan re-evaluated his initial sensemaking reasoning – in this case by suggesting the explanation stems from a particular range of exponent values.

In Stage-1 of the interview Ryan sketched a "parabola-like" graph, and was subsequently surprised when presented with $y = x^{1.4}$ in Stage-2. Ryan attempted to look for underlying explanations, and when examining the graphs of $y = x^{1.48}$ and $y = x^{1.4}$ in Stage-3, he described a dynamic change that occurs in the graph behaviors as the exponents increase in value: "So, it gets steeper as it gets closer to [exponent] 2." This may already be regarded as an initial attempt to make sense of the graphs by alluding to a range of exponent values. This idea was further developed by Ryan in Stage-4, when trying to explain the difference in the graph of $y = x^{1.5}$, as illustrated in the following excerpt (we note that also Ryan used the words "odd" and "even"; however, as opposed to the previous theme, his terminology was more accurate, and "odd"/"even" were discussed only regarding functions):

R: I suppose there is a range, or a limit in that it behaves odd, like an odd function only at a certain range, so I suppose if you change 1.4 to 1.48, in that it would just be like a range here, and then a range here, and those ranges behave like odd. So when, for example, x to the power n, if n is between those two [pointing to the graphs of $y = x^{1.48}$ and $y = x^{1.4}$], then it behaves like an odd function. Anything else, it loses the odd function property.

[…]

R: $y = x^n$ [also writes this], so if n is in between, say lower band is number here, equals to, n is between 1.4999 whatever, then this thing [x^n] behaves like an odd function. [writes: $\square \leq n \leq 1.4999 \rightarrow$ odd]

I: Is $\sqrt{2}$ in between these two numbers?

R: It's 1.4… [says "dot dot dot"], yeah. […] So, I don't know what the lower band is, hypothetically say it's around 1.39 or something.

Based on this "range-of-exponent-values reasoning", Ryan subsequently drew the graph of $y = x^{\sqrt{2}}$ between the graphs of $y = x^{1.4}$ and $y = x^{1.5}$ in the first quadrant, and continued the graph in the third quadrant – creating a graph of an odd function. However, when presented with the actual graph of $y = x^{\sqrt{2}}$ in Stage-5 of the interview, Ryan could not figure out the reasoning for the shape of the graph, and only referred again to his idea of ranges of exponent values as a way to make sense of the graphs.

Repeated-Multiplication Reasoning

In this reasoning theme, students referred to what we regard as a prevalent concept image of exponents, when understood only as repeated multiplication. This resulted in struggles when trying to make sense of irrational exponents, demonstrated by claims such as "If something is an irrational number, how can you multiply something by itself a certain number of times, like by an irrational number?" While such reasoning often hindered the participants' sensemaking process, we illustrate this theme with the case of Owen (see Figure 3c), who managed to overcome this issue by referring to mathematical formalism.

In Stage-1 of the interview, Owen claimed $y = x^{\sqrt{2}}$ is not defined for $x < 0$, though justified this incorrectly based on a view of exponentiation as repeated multiplication: "Because the exponents are multiplied, like for example 2^2 is just 2 multiplied by itself two times [...] As we have a negative number to an irrational number like $\sqrt{2}$, you don't necessarily know how many times it's multiplied by itself." When subsequently asked about $(-1)^{1.4}$, Owen again used repeated-multiplication reasoning to discard the option of this number being defined.

However, when shown that $y = x^{1.4}$ is defined also for $x < 0$ in Stage-2, Owen took a long pause, after which he wrote $x^{\frac{7}{5}}$ on the page, converted it to $\sqrt[5]{x^7}$, and provided proper explanations for the function domain. In the following two stages of the interview, Owen succeeded in explaining the presented graphs based on rules of rational exponents. Additionally, it seemed that being exposed to several graphs in the form of $y = x^{\alpha}$, $\alpha \in \mathbb{Q}$, supported Owen's sensemaking process of using these graphs to approximate $y = x^{\sqrt{2}}$. Lastly, when shown the graph of $y = x^{\sqrt{2}}$ in Stage-5 of the interview, Owen provided decent reasoning for $y = x^{\sqrt{2}}$ not being defined for $x < 0$: "So, because there are so many approximations for $y = x^{\sqrt{2}}$, and all of them don't necessarily agree with each other, some of them might exist there, some of them might not exist – [...] you can't be certain that the graph exists for x negative."

Conclusion

The motivation for this study stemmed from a gap of knowledge students have regarding the meaning of irrational exponents. The tendency to overlook irrational numbers during students' mathematical education has been recognized by Wu (2011), who noticed that in school mathematics, any property that is valid for rational numbers is automatically considered valid also for real numbers, without mathematical justification. Our findings, pointing towards students' difficulties to provide meaning to $x^{\sqrt{2}}$, suggest this "irrational gap" extends to tertiary mathematics education as well.

Overall, the ASTs the participating students went through as they were working on the $y = x^{\sqrt{2}}$ task, were generally in line with our conjectured HSTs, though revealed particular and elaborated reasoning, as well as areas of struggle. As hypothetical learning processes and the design of learning activities have a symbiotic relationship (Simon, 1995), it is the detailed analysis of students' reasoning that provides a better understanding of students' conceptual development, which could subsequently be used for the design of instructional tasks. In the findings presented herein, we focused on three kinds of reasoning students utilized when attempting to make sense of $y = x^{\sqrt{2}}$: even/odd reasoning (conflating even/odd numbers and functions); reasoning based on a range of exponent values (where the graph gradually changes as the exponent value grows); and repeated-multiplication reasoning (understanding exponentiation as repeated multiplication). These provided insight not only into the participants' comprehension of irrational exponents, but also into their understanding of rational exponents, number properties, function graphs, and the meaning of exponentiation in general. These findings call for further research that inquires into students' conceptual development of irrational exponents.

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Promoting Instructor Growth and Providing Resources: Course Coordinator Orientations Toward their Work

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One of the ways in which university math departments across the country are making efforts to improve their introductory math courses is by implementing or increasing the level of course coordination for their Precalculus to Calculus 2 sequence. This not only entails creating uniform course elements across different sections but also includes efforts to build a community among the instructors of the course. While many coordinators have the common goal of improving student success, we explore what guides their actions to see this accomplished, what we refer to as their orientation toward coordination. The orientation of a coordinator encompasses their beliefs, values and knowledge of mathematics and teaching. In this proposal we introduce and elaborate on two orientations toward coordination that arose from interviews with course coordinators from a variety of institutions across the country. We also discuss the importance of both orientations as they relate to drivers of change.

Keywords: Course coordinators, orientation, leadership, change

Introduction

Course coordination for multi-section introductory mathematics courses such as precalculus and calculus is one way in which universities across the country are attempting to improve instruction and the consistency and quality of students' learning experiences. Because multi-section introductory mathematics courses are often taught by a range of instructors (including graduate students, career line faculty, and ladder rank faculty), course coordination can help mitigate against uneven student experiences that can disadvantage students in different sections of the same course. Such uneven experiences include different content emphasis or coverage, different grading schemes, and different quality enactments of active learning. These differences in learning experiences are potentially problematic because they offer different opportunities for students to learn the intended content, and hence be adequately prepared for subsequent courses. As such, course coordination can be an important contributor to student success.

One of the first studies of course coordination in mathematics departments investigated the *coordination system* at five mathematics departments identified as having relatively more successful Calculus 1 programs (Rasmussen & Ellis, 2015). The phrase coordination system is used to evoke the image of coordination that goes beyond surface features of uniform course components (e.g. syllabus, textbook, homework, exams) to include efforts to build a community of instructors working together to create rigorous courses and high-quality learning experiences for students. In this study the authors identified concrete actions that the course coordinators took to provide both logistical support that promotes greater course uniformity and hence more equitable student experiences as well as just-in-time professional development support for teaching difficult topics, implementing active learning, pacing, etc. Rasmussen and Ellis (2015) liken the role of course coordinator to that of a *choice architect*, which comes from the work of Thaler and Sunstein's (2008) work in behavioral economics. A choice architect is someone who is able to structure choices for others in ways that can "nudge" them to make particular choices while still maintaining the feeling of independence. For example, one of the things that course coordinators at the five mathematics departments, studied by Rasmussen and Ellis, did was to make instructors' lives easier by providing a range of default options, including homework sets, class activities that actively engage students, pacing guides, etc. Instructors had leeway in how they made use of these options and thus maintained pedagogical autonomy. They further argue that this framing of a coordination system is consistent with effective change strategies identified by Henderson, Beach, and Finkelstein (2011).

In ongoing work at a different set of mathematics departments, Rasmussen et al. (2019) conducted five case studies of mathematics departments that have successfully initiated and sustained active learning in their Precalculus to Calculus 2 (P2C2) curricula. These researchers highlight the different ways that coordinators across the five sites make instructors' lives easier and build community among instructors. Williams et al. (2019) further analyzed these five sites to highlight the ways that coordinators can function as change agents by leveraging the following three key drivers for change: providing materials and tools, encouraging collaboration and communication, and encouraging (and providing) professional development. An important contribution of the work by Williams and colleagues is the strong connection between ongoing mathematics department change efforts and the substantive and growing literature focused on change in higher education (e.g., Shadle, Marker, & Earl, 2017).

One thing that is common (and abundantly clear) from this prior work is the critical role of the course coordinator in a coordination system. Hence, a better understanding of what values, beliefs, dispositions, etc. toward this work is needed. In conceptualizing these aspects of coordinators, we are inspired by the work of Thompson, Philipp, Thompson, and Boyd (1994), who examined the influence that teachers' conceptions have on their implementation of innovative curricula. In particular, they identified two contrasting orientations toward mathematics teaching: calculational orientation and a conceptual orientation. They illustrated how these different orientations have significant consequences for how teachers interact with students and content and hence offer different opportunities for learning. Similarly, we were curious to better understand coordinators' conceptions toward coordination because such beliefs and understandings profoundly influence how they interact with their colleagues and the consequent opportunities for professional growth. Thus, the research question that drives the analysis presented here is: *What orientations do course coordinators take toward their work?*

The potential contribution of this analysis is both pragmatic and theoretical. Pragmatically, a deeper understanding of the orientations of course coordinators offers mathematics departments a language for thinking about what their goals of coordination are and who, either in their department or new hires, would have the perspective on coordination that is likely to be able to enact their goals. Theoretically, this work contributes to conceptualizing the role of coordinators and coordination systems more generally.

Theoretical Background

To frame course coordinator orientations we draw on Philipp's (2007) comprehensive review of mathematics teachers' beliefs and affect, where beliefs are described as the "lenses through which one looks when interpreting the world," and affect is thought of as "a disposition or tendency one takes toward some aspect of his or her world" (p. 258). Our use of the term “orientation” encompasses both beliefs and affect as described by Philipp. In his chapter, Philipp attends to the differences and similarities between a teacher’s affect, beliefs, belief systems, conceptions, identity, knowledge and values as these terms are inconsistently used in the literature. Each has a unique impact on the way a teacher interacts with their classroom and can provide researchers with new perspectives on how to measure teacher development. While these terms require a localized focus, Philipp also steps back to discuss the existence of a teacher’s orientation as it encapsulates a variety of the localized terminology and requires a broader focus from a researcher’s perspective to better understand teacher impact in the classroom.

As described in Thompson et al.’s (1994) paper, varying teacher orientations can produce markedly different discussions in the classroom due to what the teacher considers valuable information. For example, a teacher with a calculational orientation will consider a procedural answer to the question, “How did you get that answer?” as all that is needed, whereas a teacher with a conceptual orientation is more interested in how the student is thinking about the quantities that are used and the relationships between them (Philipp, 2007; Thompson et al., 1994). The orientation of a teacher emphasizes the goals and intentions of the teacher as enacted through their actions and discourse in the classroom. We draw a parallel between the orientations of a teacher and the orientations that a coordinator may have, as their goals and intentions for how the course should be run are enacted through their actions and influenced by their beliefs, knowledge and values.

Methods

This study is part of a larger national study investigating Precalculus through Calculus 2 (P2C2) programs and student supports at the post-secondary level. As part of this larger study a census survey was conducted of all mathematics departments that offer a graduate degree in mathematics (Rasmussen, et al., 2019) and twelve institutions were selected as case study sites based on what the research team viewed as noteworthy or otherwise interesting features of their P2C2 programs. These features included ones previously identified as being associated with successful Calculus 1 programs, one of which being course coordination (Hagman, 2019; Rasmussen, Ellis & Zazkis, 2014). After the project team’s initial site visits and data collection, seven sites were identified as leveraging a coordination system that went beyond simply implementing uniform course elements (e.g., syllabus, textbook) to also include intentional efforts to build a community among instructors. In order to answer our research question, we conducted 13 interviews (2018-2019 academic year) with 19 P2C2 coordinators across the seven sites. We conducted 10 individual interviews and three group interviews that included two or more coordinators.

Interviews were audio-recorded and transcribed for analysis. We conducted a thematic analysis (Braun & Clark, 2006) to identify orientations coordinators take towards their work. Each researcher opened coded the transcripts for three sites, with at least two researchers coding the same site and comparing codes to reach consensus. The research team met to discuss and revise codes and group them by theme, reaching consensus on the grouping and descriptions of the categories. This phase of analysis resulted in 11 categories (henceforth referred to as themes) that shed light on these coordinators’ approach to their role. Each theme consisted of three or more codes from the first round of coding. The research team then engaged in further axial coding and identified two orientations towards coordination that encapsulated 10 of our 11 themes (with the theme of Personal Qualities not fitting into either orientation).

Findings and Results

Our analysis of the coordinator interviews resulted in identification of two distinct orientations to coordination. We refer to these two orientations as a Humanistic-Growth Orientation and a Knowledge-Managerial Orientation. We next illustrate each of these orientations, using interview excerpts that were selected to be representative of each respective theme within the orientation.

Humanistic-Growth Orientation

Five themes were identified during analysis that we later grouped to define the broader category that we call Humanistic-Growth Orientation toward coordination. These five themes are: a) intentional instructor support, b) interested in relationships, c) community builder, d) attends to student experience, and e) flexible. Together, these themes describe the orientation of a coordinator that incorporates humanistic values and a belief in the potential for professional growth of the instructors under their purview. For the purposes of this proposal we highlight three of these themes: intentional instructor support, community builder, and flexible.

Intentional instructor support. This theme goes beyond providing resources and materials for the instructors of the course to make their lives easier (which aligns more with a Knowledge-Managerial Orientation). All of the actions categorized under this theme are deliberately made by the coordinator to support instructors' improvement of their teaching. One example of this is exhibited by a coordinator describing their goals and intentions for coordination:

> The coordination is to try to get them [instructors] up to speed for thinking about how students learn math, how to help students be successful, how to help students connect to the ideas that are being taught in this specific class, but also for them to think a little bit more carefully about how they present things.

This coordinator is not only attending to student experiences from a content perspective, but is addressing the ways in which they can intentionally help instructors think about how to provide a more thoughtful and enriching experience for the students in the classroom. The following quote describes the level of intentionality of a coordinator that provides this type of support:

> But to the extent that I have been effective as a coordinator... I think it's been as a result of my intentions to influence instruction and influence the instructors' confidence with respect to teaching. I don't think that that view of coordinating is shared amongst others necessarily. I think the others really do view their role as being not only including, but limited to the managerial aspects. And that is very much secondary in my view.

While these quotes describe just two aspects of intentional instructor support, we noticed other actions of the coordinators that reflect this theme such as providing professional development opportunities, observing instructors' classes and giving feedback, supporting instructors to be reflective practitioners, as well as willing to be the "scapegoat" (as opposed to letting the instructor take the heat) when students are upset with how the course is being run.

Community builder. There was evidence of various community building efforts in all 13 interviews. Some of the actions that we identified to build community were: valuing contributions and feedback, getting people to work as a team, getting instructors excited about the course, and generating buy-in for the philosophy of the course. Some of these efforts are characterized well by a coordinator that had the following to say about coordination, "Coordination is not autonomy. It's about a team effort and setting up best practices that everyone follows." Many of the coordinators from our interviews reflected similar beliefs and viewed the coordination practices as a collective effort. A related aspect of community building was an intentional effort by the coordinator to distribute power amongst the instructors of the course. For example, a coordinator at a large research university reflected on their own work as coordinator:

> I do my best to structure those meetings to give the impression, not entirely artificial, that we're kind of engaged in a collective enterprise to improve all of our students' learning. So, I truly try to position myself as a co-participant in that process. Not somebody who's necessarily dictating to everyone else, you know, what to do or how to teach, but, you know, I'll pose particular questions or issues and invite people to offer their own perspectives and that sort of thing. And again, I'm sort of trying to nudge things along in particular directions and buy things in particular ways. But, I want individual instructors to feel like they have some agency over the direction of the course for everybody. And I think that this would result in kind of a sense of, at the very least, sort of codependence amongst the instructors where they are all like, we'll have lunch together, that sort of thing.

The efforts to build community vary from coordinator to coordinator, but the goal to establish a community is central to this theme.

Flexible. Most coordinators lead nonhomogeneous groups of instructors. In many circumstances, the heterogeneity of the instructors exists in the experience that they have teaching the course or teaching in general. As a means to provide the necessary support for the instructors as a collective, we saw that some coordinators would adhere to varying levels of coordination practices as described by a coordinator when asked about instructor autonomy:

> The degree of autonomy that instructors want when they're teaching the course is directly related to how many times they've taught the course or their experience with the course. The [graduate] student that's teaching Calc 2 for the very first time doesn't want any autonomy. They want to come in and they want to talk to me about here's what I'm doing next. 'How do you do this? What are the things that you emphasize?'… so, usually the greener the teacher, the less autonomy they want. Whereas the person that's taught the course over and over again has got- they have a good handle on it and they tend to not- they just have it down.

By incorporating a flexible approach to coordinating, the coordinator is able to provide a tailored experience for each instructor that has the potential to generate more buy-in from the instructors and foster a collaborative team environment.

Knowledge-Managerial Orientation

The themes from this analysis that shed light on a Knowledge-Managerial Orientation to coordination include the following aspects of coordination: a) course content and curriculum, b) organizing and attending to the details of the course, c) communication, d) knowledge of the course history (including department and university structure), and e) knowledge of teaching the course. While every coordinator described performing actions of one form or another from this orientation, in this proposal we only detail the themes of course historians and communication.

Course historian. Coordinators who discussed their role as a course historian demonstrated a rich knowledge of both the coordination structure and history as well as knowledge of the larger departmental and university system in which coordination is embedded. Notably, coordinators leveraged this knowledge to work towards sustaining and facilitating change because they knew what worked well and what has been met with resistance. For example, one coordinator said:

> We don't give ... a common exam. And I was sort of toying with the idea of maybe we should give a common exam, and I was told ... that would require a departmental vote. Only because it's calculus and people care about what calculus is… Because I will have tenured faculty teaching, often there is … a limit that's been made, not explicit, but implicitly clear to me about like you can't just take total control of this course. … It's not like it [a common exam] would never happen, but it would not be as simple thing that I could just decree that that's going to happen. So, it would take a lot of work.

This excerpt highlights an understanding of some of the departmental barriers to change and includes an understanding of ways to work within the system to facilitate changes for a course. Being a course historian also requires a continuous involvement within the coordination structures so that one's understanding and knowledge remains current and relevant. A participant highlighted this when they said a coordinator must be embedded within the department and ask, "'Hey, how are you? How are things going? Do you want to teach again?' or like, 'What are you doing now?' Like you have to be able to be part of the social network of the department in a way." The coordinator's involvement within the department is integral to their effectiveness. In addition to the importance of having this knowledge of the course history to make content or policy changes, coordinators that demonstrate a Knowledge-Managerial Orientation to coordination also draw on this knowledge when communicating department and university policies to instructors who are likely less familiar with this information.

Communication. The communication aspect of the Knowledge-Managerial Orientation to coordination includes both communicating important content and logistics about the course to instructors and being responsive to student and instructor emails. Some coordinators created a document or a set of examples to communicate important content, saying things like:

> We have these 62-page documents that are the expected learning outcomes for our calculus course that I developed. And it was so I can just be like, 'Hey grad student, this is the course, and it's a lot of high-level things. Students should be able to do blank... all organized in some hierarchical way. And that took a lot of experience to write that thing and now it, it's a lot of detail and it's all organized, and then it's communicated and disseminated.

Other coordinators communicated key content by drawing attention to it during formal or informal meetings/discussions with instructors. One coordinator acknowledged that he likes to allow room for instructors to have agency in the course in addition to clearly communicating important content, saying:

> If there's a certain thing that I really, really want to test students on ... I might, like say to them, 'Hey, try to implement something in your class, try to do something like problem number 25 on page 381.' Yeah, I might say something like that, but I try not to, I try not to overstep that with other people.

All of the themes encapsulated by the Knowledge-Managerial Orientation to coordination illuminate an approach that allows for the coordination structure to be implemented in an organized way, clearly communicating the coordinated elements and expectations to instructors. Coordinators who embrace this orientation leverage their knowledge of the students and their experience teaching the course to create appropriate resources and coordinated elements. Additionally, this approach allows for a coordination system that is well-informed by the course history, and departmental culture/policy surrounding it.

Discussion

All of the coordinators in our study demonstrated aspects of Knowledge-Managerial Orientation to coordination, highlighting the importance of being familiar with the course they are coordinating as well as creating and sharing resources with instructors teaching the course. This is not surprising since uniform course elements are a key component of coordination. The coordinators that also demonstrated a Humanistic-Growth Orientation, however, tended to focus more on this aspect of their role when discussing their perspectives and experiences coordinating P2C2 courses, often framing the Knowledge-Managerial aspects from a Humanistic-Growth Orientation. It is important to note that while not every coordinator demonstrated a Humanistic-Growth Orientation toward their coordination work, those that did were deliberate and prioritized personal and professional growth to improve the quality and effectiveness of their P2C2 courses.

We see a similar level of intention from the coordinators in the study by Williams et al. (2019) as various coordinators deliberately take action to improve student success by acting on three drivers of change to implement and sustain more active learning in their P2C2 sequences. These drivers, providing materials and tools, encouraging collaboration and communication, and encouraging (and providing) professional development nicely align with the two orientations presented in this proposal. Providing materials and tools is an action taken by coordinators with a Knowledge-Managerial Orientation while encouraging collaboration and professional development are two actions taken by coordinators that approach their work with a Humanistic-Growth Orientation. Thus, by encouraging coordinators to initiate change through an approach to coordination that incorporates both the Humanistic-Growth and Knowledge-Managerial Orientations, math departments across the country could reap the potential benefits of increased active learning in P2C2 classes.

By attending to these drivers and orientations, math departments now have the language and research evidence to support their goals of improving or implementing active learning and coordination. Drawing on the data from a census survey sent to all Ph.D. and master's granting institutions across the country, we know that there is a need for the improvement of professional development support as well active learning practices in the classroom (Rasmussen, et al., 2019). Math departments reported valuing active learning and professional development, but also reported not being very successful at each. In fact, 44% of math departments saw active learning as very important, 47% saw it as somewhat important and 9% did not see it being important. However, when asked about how successful they were at implementing active learning, only 15% of the 199 math departments reported that their program was very successful. Similarly, with graduate teaching assistant (GTA) professional development, 50% and 32% of the math departments saw it as very and somewhat important (respectively), while only 29% of the respondents reported being very successful at it. Clearly, math departments across the country are looking for ways to improve their active learning and professional development efforts, and effective course coordination is one opportunity to achieve this goal.

Our hope is that by bringing awareness to coordinators' orientation(s) we are not only supporting math departments in search of coordinators but are also encouraging coordinators themselves to reflect on how they approach their role and how they can act on the available drivers for change at their institutions. By providing this perspective towards coordination, we also hope that this empowers math departments across the country to improve their active learning and professional development efforts. The next step in our work surrounding P2C2 coordinators' orientations will be to analyze the instructor and GTA interviews to compare and contrast what is valued in terms of effective coordination.

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Get That Basket! Deciphering Student Strategies in the Linear Algebra Game Vector Unknown

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We present results of a grounded analysis of individual interviews in which students play Vector Unknown - a video game designed to support students who are taking their first semester of linear algebra. We categorized strategies students employed while playing the game. These strategies range from less-anticipatory button-pushing to more sophisticated strategies based on approximating solutions and choosing vectors based on their direction. We also found that students focus on numeric and geometric aspects of the game interface, which provides additional insight into their strategies. These results have informed revisions to the game and also inform our team's plans for incorporating the game into classroom instruction.

Keywords: Linear Algebra, Game-Based Learning, Inquiry-Oriented Instruction

Linear algebra is an important course for students in the STEM disciplines because of its unifying power within mathematics and its applicability to areas outside of mathematics. Research in linear algebra has evolved over the last twenty years from the pioneering work at the turn of the century (e.g., Dorier & Sierpinska, 2001; Harel, 1999; Hillel, 2000) to more recent work on improvements to teaching and learning using Tall's three worlds (Stewart & Thomas, 2010), models (e.g., Trigueros, 2018), dynamic geometry systems (e.g., Sinclair & Tabaghi, 2010) and everyday examples (e.g., Adiredja, Bélanger-Rioux, & Zandieh, 2019). Our work is most closely aligned with the curriculum design work of the Inquiry-Oriented Linear Algebra (IOLA) group (e.g., Andrews-Larson, Wawro, & Zandieh, 2017; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Zandieh, Wawro, & Rasmussen, 2017). The goal of our project is to explore Linear Algebra in the context of a video game developed from the IOLA curricular materials. In recent years, game-based learning (GBL) has become popular because of its ability to motivate students. Bakker vanden Heuvel-Panhuizen, and Robitzsch (2015) found that students who played mathematical games tended to spend more time on these games when compared to the amount of time spent on their normal math homework. Part of the reason can be attributed Gee's (2003) concept of skills as strategies, which states that players are more likely to practice necessary skills when they are seen as strategies for a larger goal. We view video games as providing an avenue to support and motivate students' development of understanding Linear Algebra concepts drawing on theories from Realistic Mathematics Education (RME), IOLA, and GBL (Authors, 2018, Authors, 2019).

Literature Review

Realistic Mathematics Education

This project leverages the activities of the IOLA curriculum (http://iola.math.vt.edu), which is a set of research-based curricular and instructional materials based on RME design principles and designed to move students through the different levels of mathematical activity from an experientially realistic situation to a formal mathematical understanding (Wawro, Rasmussen, Zandieh, & Larson, 2013). The curriculum begins with a task called the Magic Carpet Ride (MCR) task, which asks students to use two forms of transportation (represented by the vectors <3,1> and <1,2>) to reach Uncle Cramer's house located at (107, 64). The primary goal of the remaining tasks in this first unit is to support students' generalization lessons learned in their earlier activity and develop the formal constructs of linear independence and span.

Game–based learning (GBL) is the use of games (including video games) for educational purposes. While GBL has gained popularity in recent years (Gee, 2003; Gresalfi & Barnes, 2016; Foster & Shah, 2015; Shih & Shih, 2015) there are few games for higher levels of mathematics. Gee (2005) notes that well-designed games present players with well-ordered problems that allow the players to learn crucial skills necessary for gameplay. According to Gee, players also engage in cycles of expertise where they encounter difficult problems, solve them, practice and master them, and then are confronted with new problems that will challenge the assumptions of the initial problems, requiring them to apply their skills in a new context. Williams-Pierce (2017) notes that players of educational mathematics games go through several stages beginning with understanding how the game works and culminating in predictive behaviors that are optimal for higher-level mathematical activity. This difference between lesser-anticipatory and more anticipatory strategies can be seen in mathematical context in Hollebrands (2007) who noted that students engaged with Geometer's Sketchpad (GSP) in two key ways, proactive and reactive. Students who engaged in a proactive way used GSP to test theories, while reactive users reacted to the information presented to them on screen, but were not able to anticipate how their actions would change the on screen environment. This indicates that understanding how students anticipate using mathematical tools and games is key to interpreting their use of tools/games.

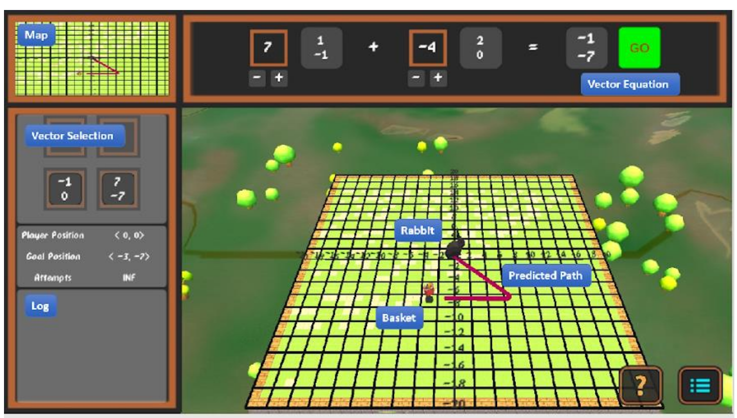

Figure 1. Sample Screen from the Game Vector Unknown.

Vector Unknown Gameplay

The game currently consists of five levels with our analysis investigating gameplay of levels 1, 2, and 5. In each level, four vectors are generated at random under the constraint that there are two pairs of linearly dependent vectors. To play, students guide the rabbit to the basket by selecting one or two vectors from the Vector Selection, placing them into the vector equation, adjusting the scalars, and pressing GO. During Level 1, players have a Predicted Path which highlights the path of the Rabbit before the player presses GO, providing players with a geometric representation of the Vector Equation. Level 2 is designed to force players who solely rely on the Predicted Path to consider the Vector Equation by removing the Predicted Path. On level 5, the Predicted Path returns, but players must now obtain several keys to unlock a lock that covers the goal forcing them to consider vectors originating from points other than the origin.

Methods

Data Collection

We interviewed five participants from a multi-purpose regional university in the southeastern United States. These participants were selected from 11 interviews comprising the total data corpus because they had not completed a first course in linear algebra. The participants came from a diverse background, including two black men, two black women, and one Asian woman. Each participant was interviewed for approximately one hour, during which they were asked to complete three levels of the *Vector Unknown* game. As needed, the interviewer provided help on how to navigate the game's screens and use the controls. The interviewer asked scripted questions along with impromptu follow-up questions. Impromptu questions were asked to further clarify and explore the participants' thinking about gameplay as well as any mathematical insights or strategies the participant developed during gameplay.

Analysis

Team members transcribed the interviews and documented student gameplay during the interview by noting significant interactions with the game's interface. Specifically, we documented moments in which the player chose or replaced vectors or when the player changed the scalar amounts for vectors in the vector equation. Consistent with grounded theory (Strauss & Corbin, 1994), team members then conducted an iterative grounded analysis of the interviews. This process began with multiple cycles of open coding and discussions during which team members agreed on interpretations of participants' statements and actions.

Table 1: Examples of codes from the first round of coding

Categories		**Examples**	
Focus on the vector equation	**Erp**: replaces a vector	**Es**: manipulates scalars	**Erv**: Removes a vector
Features of the game	**Fdl**: Data Log	**Fpp**: Player Position	**Fca**: Camera Angle
Focus on gameplay	**Pg**: Presses Go	**Pcomp**: Anticipates gameplay complexity	**Pu**: Undoing a Move
Conceptualizing a Vector	**Vd**: Testing the direction of a vector	**Vp**: Visualization and trace of vector path	**Vq**: Use of scalars or vectors components to reach quadrants
Student-expressed strategy	**Sc**: Connecting b/t representations	**Sp**: Previously used strategy	**Sr**: Retrospectively expresses a strategy
Focus on graphical game components	**Gdir**: Displacement (Direction)	**Gdis**: Displacement (direction and displacement)	**Gpp**: Refers to the predictive path

After initial coding, the research team grouped the codes into six broad categories (Table 1). The first category - Focus on the vector equation - includes student actions such as the introduction and removal of a vector or the manipulation of a scalar in the game interface. A second category - Features of the game - includes instances in which the student mentions using various features included in the game's interface, such as referencing the Data Log or the locations of the player or goal. The third category - Focus on gameplay - contains codes of student actions as they play the game, such as pressing GO or undoing a move. The fourth category - Conceptualizing a vector - includes codes about how the student expresses the ways in which they are Conceptualizing a vector, such as defining what the components of a vector are or testing its direction. Another category denotes Student-expressed strategies, such as focusing on one coordinate at a time or reusing a previous strategy. The final category - Focus on the graphical game components - such as the direction of the purple projected path or the graphical displacement from the goal.

Results

The initial codes are of a relatively fine grain size, but were important pieces into the analyzing the student strategies. Knowing what specific components of the game the student attended to while playing the game as well as how the student understood the vector equation and graphical component at different points in gameplay allowed us to discover several themes. These themes are seen across multiple students and can also be used to describe the strategies the student uses as they are introduced to the game and how their strategies evolve as they progress with each level. In this section, we use excerpts from four participants' gameplay to discuss the themes we developed. For each theme, we recognized that students were focusing to varying degrees on the geometric and numeric aspects of the game's interface. When a participant's strategy focused primarily on the vector equation without mentioning the Predicted Path or the graphical component of the interface, we called the strategy ***numeric***. When a strategy focused primarily on either the Predicted Path or the graph, we called the strategy ***geometric***.

There were several themes we found when analyzing student strategies. We characterized the first theme, ***button-pushing***, as rapidly adjustment of scalars and switching vectors. We denoted button-pushing as ***less-anticipatory*** when the player expressed surprise when reacting to a consequence of their actions. For example, Mouse[1] after quickly alternating the scalar keys made the following comment "what if I press the button that's ... the rabbit's going to go down. ohhh, yeah." Here, the tone of Mouse's "ohhh, yeah" comment indicated his surprise which lead us to describe his activity as *less-anticipatory* ***button-pushing***. His reference to going down with the Predictive Path indicated that he was attending to the *geometric* aspects of the game. Another student Gwen, employed a less-anticipatory button-pushing strategy as evidenced from her describing her gameplay as "messing around", "mindlessly clicking", and "throwing in numbers to get the answer" while quickly adjusting vectors and scalars in Levels 1 and 2. She did not reference the Predicted Path, but instead frequently referenced the Vector Equation and its components, indicating that her ***button-pushing*** was *less-anticipatory* and *numeric*.

As participants played the game more extensively, participants began to engage in *more-anticipatory* ***button-pushing*** which involved switching out vectors and scalars, but not as quickly and the sense of surprise is replaced by a knowledge of the results prior to pushing a button. Mouse's gameplay shifted from less-anticipatory to more-anticipatory throughout the interview. This is evidenced by a decrease in the number of times Mouse switched between increasing and decreasing the scalar for a given vector when he was button-pushing. For instance, in Level 1, Mouse expressed surprise when the predictive path of the linear

[1] Students chose their own pseudonyms.

combination moved toward the goal. In contrast, while playing Level 5, Mouse put in the first vector he clicked on the "+" scalar and then uttered "Oh -" before clicking the "-" scalar. We see this as a more-anticipatory button-pushing because his actions indicate he is expecting *one* of the scalars (+ or -) to move the predictive path closer to the goal position.

We characterized ***quadrantal*** themed strategies as strategies that involve choosing vectors based upon the signs of the coordinates. This theme manifested as participants choose a starting vector in the same quadrant of the goal, or more generally a vector that could be scaled in a direction heading towards the goal. For example, Latia noted the following:

> So I know that the basket is on <-6,6>, and my initial thought was use, to start with either my <-3,9> or my <-1,3> because it is in the second quadrant and then, since I already know I'm at negative six, six, I was trying to think of what I could use so that if I multiplied those two numbers I could get to my basket.

Here, Latia made her initial vector choices by matching the signs of the goal position with that of the potential vector choices placing her strategy in the *numeric* category.

The third type of strategy was ***focus on one coordinate*** where the participant focused on trying to match one coordinate of the result of the vector equation with one coordinate of the goal position. Lance expressed this strategy as follows:

> Uhhmm, so when I started, when I saw my vector choices, I started to see the, um, almost all connections of like how certain, um, how um, I was manipulating them individually based off of like ys first and then the xs. And so I was trying to find paths and where, where I get my first, um, my first um, coordinate and my y.

Here Lance focused on reaching the y-coordinate of the goal first. The strategy is *numeric* as he did not have the Predicted Path when playing this level and did not refer to any geometric components.

Another student, Zo, noted that when she was trying to get to the basket at <-18,-1> from <-18,-9>:

> I have to go up eight but I don't know how I'm about to do that. There's no, like, straight line in these numbers so I can't do that. But, uh, I'm just going to trial and error it 'cause I don't know how else to do this.

Zo's strategy was clearly to match the -18 and then go up 8 indicating she was attending to the *geometric* aspects of the game.

The fourth strategy observed was a ***focus on one vector*.** In this strategy, participants chose one vector, scaled it as close to the goal as possible - potentially going past the goal - and reducing the scalar, and then chose another vector and either scaled it to the answer or alternated adjusting scalars to reach the goal. In contrast, Mouse on Level 1 scaled one vector <-5,10> by 1 and then by 2 to reach the goal at <-11,8>. After which he commented that he had gone too far (referring to the Predicted Path) and reduced the scalar by 1. He then began to scale the other vector. Mouse's focus on the Predicted Path categorizes his strategy as *geometric*. Latia and Gwen also employed this strategy with the Predicted Path. While we did not observe a *numeric* example of ***focus on one vector***, we can imagine that a student would focus on adjusting the scalar on one vector until the result of the vector equation is close to the goal.

Examples of Strategies as Seen in Latia's Interview

Latia began with **less-anticipatory *button-pushing*** expressed by Latia as "playing with numbers", but quickly figured out that there was a link between the scalars and the geometric support noting that the predictive path doubled when she increased the scalar on a vector from 1 to 2. During the first level she relied on the geometric support to find the goal, but has clearly began to connect the two most notably stating that she made vector and scalar choices based upon the quadrant of goal expressing ***quadrantal*** reasoning. Also, of note, she mentions that she eliminated a vector choice because she did not want to go past the goal. Latia also ***focuses on one vector*** as indicated by choosing one vector, scaling it to what she believes is an appropriate distance to the goal, and then putting in a second vector and adjust it. This is followed by an adjustment to the first vector to reach the goal.

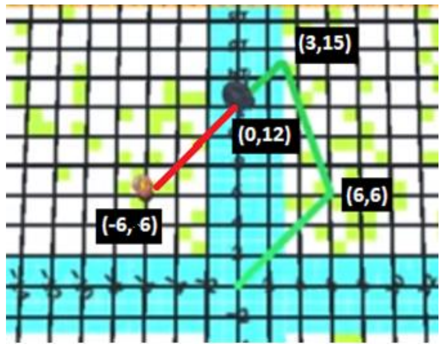

Figure 2. Latia Gameplay during Level 2.

During the second level (see Figure 2), Latia chose the vector <-3,-3> and the scalar -2 and pressed GO to take her to <6,6>. She then used a ***quadrantal*** strategy to chooses the vector <-1,3> and scales the vector by 3 attempting to ***focus on one vector*** to get close to the goal. She assumed that bunny will always start from zero and is surprised to find out it does not return to the origin after pressing GO. Noting that she wanted to go down and to the left, she utilized **more-anticipatory *button-pushing*** to test the direction of the vector <-1,-1>. This is an extension of **quadrantal reasoning** to a point that is not the origin. She noted that rabbit has to go down 6 and to the left 6 and scales <-1,-1> by 6 to reach the goal connecting the vector equation and its ***geometric*** interpretation without a reliance on predictive path. Latia replayed Level 2 with a goal at <-2,10> and mentioned that her strategy is to go past the goal and return, **repeating the strategy** used in a previous level playthrough and defying her intuition from the Level 1 playthrough. After not being able to visualize a solution with the given vectors, Latia solved the system of equations on a piece of paper indicating a flexibility in thinking and approach to completing the game.

Comparing conditions: Standard Basis Vectors

Both Mouse and Latia experienced Standard Basis Vectors (SBVs) on Level 2 and responded differently. On Level 1, Mouse relied more on the predictive path that resulted from ***button-pushing*** and looked at (and solved) the vector equation produced in the Data Log upon completion of the level. When he moved to Level 2, he sought to learn how the vector equation worked and focused on how the scalars related to the direction the rabbit traveled. The inclusion of SBVs allowed Mouse to focus more on the numerical components and see how the scalars affect the resultant vector. When asked about how he solved the level, he responded:

> I saw the position which is three and the other position which is negative five. So, I kinda made it to where I'm trying to get it to where the vector has three and five […]. I'm guessing that if I multiply negative one to negative three it will get three and negative one again to negative five.

Latia completed Level 1 and one attempt at Level 2 before she encounters SBVs. On opening Level 1, she noticed the vector equation and began to make sense of how the scalars and the equation relate to the graphical component of the game. On Level 2, she continued to explore this connection and gained a working understanding of how the resultant vector relates to the rabbit's motion, adjusting her strategy when starting away from the origin. When given SBVs, she ***focused on one vector*** and explained that the orientation of the vector does not matter because the negative scalar can flip directions. Latia said that the inclusion of the SBVs made the level easier to manage, which allowed her to use both vector slots comfortably.

Both students used the SBVs as a way to validate their understanding garnered from previous levels. Mouse used them to gain a better understanding on how the vector equation operates while Latia uses them to incorporate two vectors into the equation. Although this is the first time Latia used both vector slots, she began to gain the understanding in her first attempt at Level 2 when she has a starting position off the origin and needed only one vector to reach the goal from her second position.

Discussion

Throughout our analysis we found that students began with less-anticipatory gameplay. This early gameplay resembled the GBL version of reactive strategies (Hollenbrands, 2007) or Zone 1 and 2 gameplay (Williams-Pierce, 2017) which focuses on students understanding how the controls of the game function. Over time, most students shifted to more-anticipatory gameplay allowing them to develop complex strategies aligning with a shift to the GBL version proactive strategies (Hollenbrands, 2007). This is in line with Williams-Pierce's (2017) Zone 3 where players begin to anticipate what is desirable in their gameplay. This desirability is expressed in our themes as players making initial choices that they know will get them closer to the goal such as ***quadrantal strategy***, ***focusing on one vector,*** and ***focus on one coordinate***. Eventually, we found that at least one student Latia made a hypothesis about the direction of a vector and utilized the game to test the hypothesis. This aligns with students using GSP to test hypothesis (Hollenbrands, 2007) as a tenant of proactive strategies and of the characterization of Zone 4 gameplay (Williams-Pierce, 2017). Finally, Latia's reuse of a previous strategy is a key indicator that a student has moved to Zone 5 of mathematical gameplay which is ripe for mathematical discovery and advancement. A key result of our analysis is that as students spend more time playing they game they were able to anticipate what would happen in the game allowing for a greater variety of theories to form which could be valuable points of departure for educators using the game with students either inside or outside of class. about gameplay and used the game to test their theories.

In addition to analyzing student thinking and gameplay, this study served to inform further development of the videogame. For example, a new iteration of the game was created that included a difficulty setting of easy, medium, and hard. The entire easy level includes pair of vectors that are multiples of a standard basis element while higher difficulties exclude standard basis vectors. Further exploration will include interviewing more students to find some of the strategies that we speculated would exist such as the *numeric* ***focus on one vector*** strategy. Finally we wish to explore how the game can be integrated in an IOLA classroom with a particular focus on building on the student strategies discovered.

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Calculus Students' Epistemologies of Mathematics: The Case for a Dynamic Interpretation

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Students' epistemologies (beliefs about mathematical knowledge) play a significant role in how they learn mathematics. Much prior work in this area describes student epistemologies as unitary, where evidence of a particular epistemology is taken to represent that student's singular epistemology of mathematics. Most argue that these epistemologies stem from their experiences across their mathematics classes. This study set out to examine students' epistemologies in introductory calculus and how those epistemologies influence and reflect students' experiences in mathematics courses. To a significant extent, it confirms results from prior studies. However, through one-on-one it finds more variability in students' epistemologies than most of the literature has described. That variability is significant for helping educators think about how to help students learn mathematics in general, and calculus in particular.

Keywords: calculus, epistemologies, beliefs, context

Introduction

An important yet often implicit aspect of mathematics courses is what students learn about the nature of mathematics knowledge and what it means to do mathematics. These beliefs about the nature of mathematics and mathematical knowledge are an important part of learning (Schoenfeld, 1985). I use "epistemologies of mathematics" to mean a person's set of beliefs about the nature of mathematical knowledge, including their beliefs about what counts as valid mathematical knowledge, where mathematical knowledge comes from, and what it means to do mathematics. In this paper, I focus on the epistemologies of mathematics held by students taking introductory calculus courses. Even high-performing students leave calculus courses with weak understandings of key calculus concepts (Selden, Selden, Hauk, & Mason, 2000). It is possible that students' epistemologies play a role in this, but currently no systematic data has been collected and analyzed to examine this possibility. This study addresses this research gap by examining calculus students' epistemological beliefs about mathematics.

The research question I focus on is: What kinds of epistemologies of mathematics do beginning calculus students hold? Through analysis of in-depth interview data, I will provide evidence that (a) Individual students exhibit multiple, seemingly contradictory epistemologies of mathematics, and (b) when students experienced conflicts between their formal mathematical knowledge and their own intuitive knowledge, they did not attempt to reconcile the conflict.

Theoretical Framework

A core idea that underlies this study is that students' mathematical thinking is often dynamic and sensitive to context, rather than stable and unitary. Research supports the dynamic and context-dependent nature of students' cognitive understandings, so a natural extension of these theories is that students' epistemological beliefs may also be dynamic and dependent on context. This stance builds broadly on earlier work that establishes knowledge as constructed by individuals in social context, (e.g., Piaget, 1966; von Glaserfeld, 1989; Vygotsky, 1978).

While some studies have examined undergraduate students' ideas about mathematics (e.g., Richland, Stigler, & Holyoak, 2012), relatively little is known about epistemological beliefs in the specific case of beginning calculus students. Muis' (2004) review of the literature on epistemologies of mathematics specifically calls for investigation on students' epistemologies in calculus and whether and how they may differ from their epistemologies in other areas of mathematics. However, a relatively robust body of literature (e.g., Carey & Smith, 1993; Elby & Hammer, 2001; Hammer, 1994; Hammer & Elby, 2003) has investigated student epistemologies in introductory college physics courses and found evidence that students who view physics as a coherent system of ideas learn the core ideas of introductory physics more successfully (in the sense that they have more correct conceptual understandings of basic physics phenomena) than those who view physics as a set of facts and formulas to be memorized.

Examining how calculus students' epistemological beliefs vary and shift in different contexts gives insight not only into the nature of their beliefs but also into the relationship between those beliefs and the types of mathematical reasoning in which students engage. This has implications for both research and instruction. If researchers do not attend to the context of their data collection, they may mischaracterize student epistemologies as stable beliefs. Instructors may attempt to *confront and replace* "unproductive epistemologies" when it may be more appropriate to encourage students to *activate different epistemological resources* that they already have.

Methods

Course Setting and Participants

The data for this study are drawn from a first-semester calculus course (Calculus I) at a small research university in the northeastern United States. The course was a standard introductory-level calculus course on polynomial, rational, trigonometric, exponential, and logarithmic functions of a single real variable. The specific topics covered were limits and continuity, derivatives and techniques of differentiation, related rates, definite integrals, and the Fundamental Theorem of Calculus. The course did not include any specific pedagogical innovations nor elements explicitly aimed at examining or changing students' epistemologies of mathematics. All 41 students from one section of the course were invited to participate in a one-on-one interview. Nine students volunteered and all of them participated. The students were a mix of first-year and second-year students, had a variety of intended majors, and had varying levels of success in the course. Three had final course grades in the A range, three in the B range, two in the C range, and one withdrew from the course.

Interviews

The semi-structured, videotaped, one-on-one interviews were centered around mathematical tasks and were conducted near the end of the semester. The explicit goal of the interviews was to uncover student reasoning, rather than to teach mathematical content. The interview tasks for the data reported in this paper were centered around the "Car Problem." The first part of the Car Problem (see Figure 1) is adapted from Carlson's (1998) study. Students were presented with graphs of the speed of two cars that left from the same place and traveled forward in the same direction. They were asked to decide which car was ahead, which was traveling faster, and which was accelerating faster at $t = 2$.

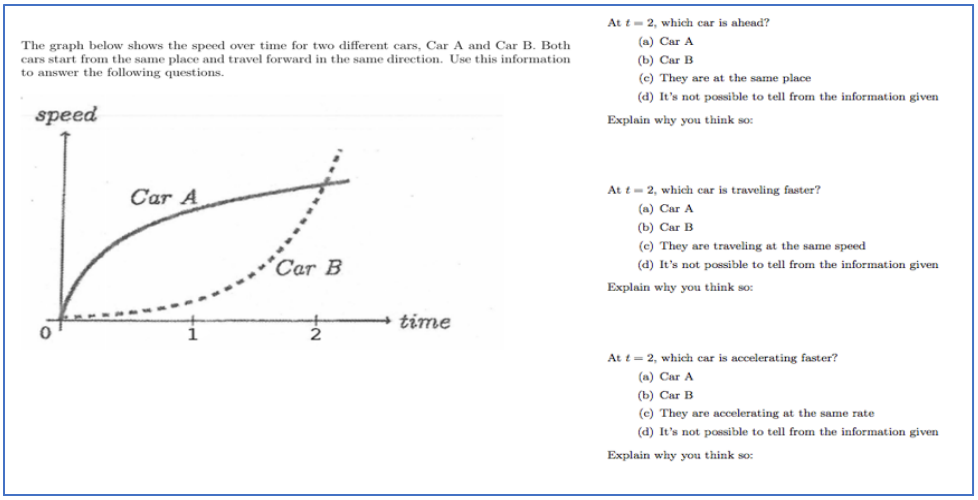

Figure 1. The first part of the "Car Problem."

In the second part, students were asked to decide which car had a greater average speed and to describe the relative position of the cars during certain time intervals (see Figure 2). These tasks were chosen because they addressed core introductory calculus concepts such as rate of change, average value, and accumulation. These tasks also did not include explicit functions to manipulate. While the intent of these tasks was to uncover students' conceptual ideas, the ways students approached the tasks and their verbal commentary during the interviews also gave strong insights into their epistemologies of mathematics.

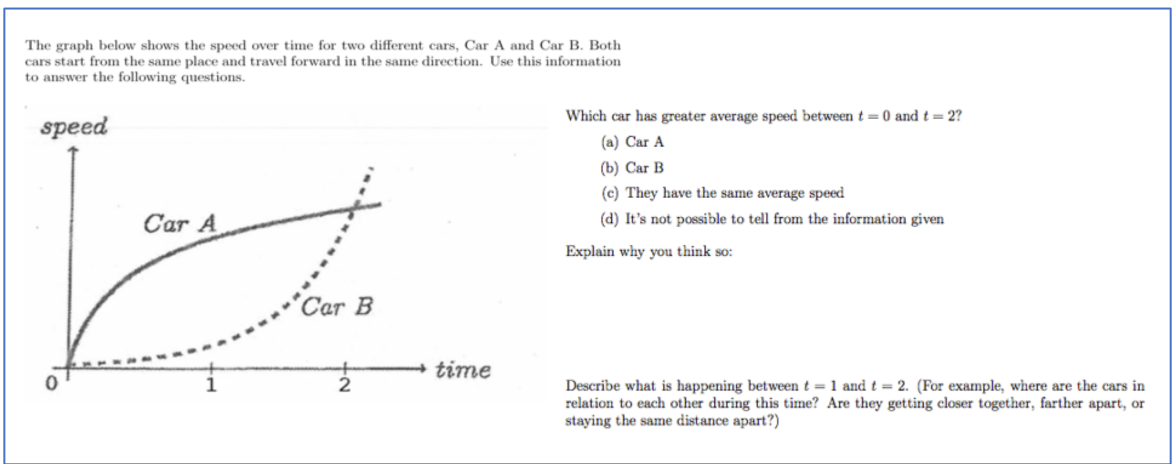

Figure 2. The second part of the "Car Problem."

Coding and Data Analysis

All nine interviews were fully transcribed and examined to identify moments that were evidence for students' beliefs about the nature of calculus specifically or mathematics generally. Some of these moments occurred when students spontaneously spoke about their beliefs while others occurred in the context of working on the interview tasks or responding to interviewer questions. I iteratively developed categories that balanced capturing detail about an individual student's beliefs with capturing similarities across individuals. The categories I developed in this analysis focus on factors that are readily observable in research interview settings. Table 1 shows the categories and codes in the framework used in this study. For example, if a student made a statement, comment, or question that showed evidence of what their apparent goal was, it was coded first into the "Apparent Goals" category and then either the "Correct Answer" or "Makes Sense" code. Note that the categories and codes are not mutually exclusive.

Table 1. Description and example of each code.

Code	Description	Example
Apparent Goals (what the participant was trying to do)		
Correct Answer	Apparent goal is to obtain a correct answer, regardless of the method by which it is obtained	"I would guess Car A 'cause frankly *the only thing I haven't look at in the graph yet* is area." – Mary [4:08]
Make Sense	Apparent goal is to make sense of the problem and/or arrive at an answer that feels satisfying	"But that, like *doesn't make sense* because it [Car A] is going faster the whole time." – Ash [8:35]
Trust (kinds of knowledge that the student viewed as valid)		
External Trust	Trusts external knowledge such as memorized formulae, facts from class, information from textbook, etc.	"I think it's C because *in the formula* you would only use the two points and those are the same." – Gloria [13:43]
Intuitive Trust	Trusts his/her own intuition	"I *feel like* there has to be an endpoint. Yeah, so it must be here." – Yuki [22:07]
Sources (type of intellectual sources students draw upon)		
External Sources	Drawing upon external sources such as formulas, symbols, memorized facts, etc.	"We're going over average values *in class now* so I'm gonna use that." – Lynn [6:29]
Internal Sources	Drawing upon internal sources such as conceptual reasoning, own understanding, intuition, etc.	"Well *if I'm thinking about the water*…you would just pour and pour and would go up faster and faster." – Nate [38:08]

While the set of six codes captures each individual's epistemologies of mathematics in more detail, collapsing the codes into two thematically similar code groups allows us to (a) more clearly examine similarities and differences in epistemologies between and across individuals, and (b) track large-scale shifts in the type of epistemological resources an individual used throughout different parts of the interview. The first code group is "Formal Resources" which includes Correct Answer, External Trust, and External Sources, indicated in green in Table 1. The second code group is "Intuitive Resources." which includes Make Sense, Intuitive Trust, and Internal Sources, indicated in orange in Table 1.

Results

Shifting and Varied Epistemologies

Students display multiple, seemingly contradictory epistemologies of mathematics. These ideas about the nature of mathematical knowledge appear to be dynamic and context-dependent. All nine participants in the study drew on elements (apparent goals, trust, and/or sources of knowledge) from both the Formal Resources and Intuitive Resources code groups while working on the different interview tasks. The Car Problem provides a clear example of students drawing on epistemological elements from both the Formal Resources and Intuitive Resources code groups across different question contexts. Each of the three question contexts within the Car Problem (position question, speed question, and acceleration question) were similar in structure, as they all asked about a characteristic of the data represented by the same problem setup and graphs, all had the same multiple-choice formatting, and all had the same answer choices. Comparing student responses to the three question contexts illustrates how each individual utilized different resources at different times. Table 2 displays the primary code group (Formal or Intuitive) for each participant while working on each question. Eight of the nine students used primarily Intuitive Resources for some questions and primarily Formal Resources for others. There was not a strong relationship between which question the student was working on and the type of resource used..

Table 2. Primary resource type used by students in the first part of the Car Problem.

Participant	Position Question	Speed Question	Acceleration Question
Ash	Intuitive	Intuitive	Formal
Gloria	Formal	Intuitive	Intuitive
Ian	Intuitive	Intuitive	Formal
Jane	Formal	Formal	Formal
Lynn	Intuitive	Formal	Formal
Mary	Formal → Intuitive	Formal	Formal → Intuitive
Nate	Formal → Intuitive	Intuitive	Intuitive → Formal
Sonia	Formal → Intuitive	Intuitive	Intuitive → Formal
Yuki	Intuitive → Formal	Intuitive	Intuitive → Formal

The arrow notation (→) denotes a significant shift in the primary resource type used within a question.

The case of Mary working on the Car Problem provides a case study of shifts in the type of epistemological resources she drew upon. Mary initially spent a total of two and a half minutes on all three questions in the first part of the Car Problem. She relied primarily on Formal Resources to arrive at her answers, including memorized-from-class ideas (e.g., "slope = velocity" and "speed is acceleration") and process of elimination ("well, the only thing I haven't used yet is area, so it's probably that.") Without prompting, she stated that she was unsure of whether her answers to the questions about the cars' relative positions, speeds, and accelerations were correct. She said "on a test, I'd just, like put down this answer and move on." She then asked if she could move on to the next part of the interview task. The interviewer responded by asking her to explain how she knew the answers she had written down. After this, Mary spent eighteen minutes reworking all three parts of the Car Problem. Eventually, she decided to label the axes with units (miles per hour on the vertical "speed" axis, and hours on the horizontal "time" axis) to figure out what the area and slope should represent. She stated that "slope is *y* over x" and calculated that miles per hour divided by hours would be miles per hour squared, which she determined was a type of unit that would represent acceleration. She said to find the area you would take "length times width," so it would be "miles per hour times hours, which is miles" and reasoned that the area under the curve would represent miles traveled. When asked to explain her thinking Mary spent more time on her work, drew on Intuitive Resources for the first time in the interview, and eventually corrected her erroneous initial responses based on her own reasoning that did not rely on memorized procedures. A possible explanation for this is that the interviewer pointing out the "explain why you think so" part of the question implied to Mary that "explaining why" is an epistemologically valid and valuable thing to do, and the kind of thinking needed to do it was a different kind of thinking than what she originally used.

Resolving (or Not) Conflicts Between Intuitive and Formal Resources

During the interviews, students often used both Formal Resources and Intuitive Resources while solving a problem. A clear example of this is students' work on the average value question in the second part of the Car Problem (see Figure 2). Of the eight students who worked on this problem, six (75%) used both Formal Resources and Intuitive Resources and verbalized an apparent conflict between the two approaches (see Table 3). Using Formal Resources led all six of these students to the conclusion that Car A and Car B would have the same average speed from $t = 0$ to $t = 2$. Conversely, using Intuitive Resources led all six students to the answer that Car A would have a higher average speed in this interval. All of these students then proceeded to select which resource type (and hence, which answer) they thought was correct. All three students who prioritized their Intuitive Resources ultimately answered that Car A had higher average speed (answer choice "A") while all three who prioritized Formal Resources concluded that Car A and Car B had the same average speed (answer choice "C").

Table 3. Resource type(s) used and answer selected by students on the average value part of the Car Problem.

Participant*	Resource type(s) used	How conflict between resources types was resolved, if applicable	Answer selected
Ash	Both	Formal	C
Gloria	Both	Formal	C
Ian	Both	Intuitive	A
Jane	Intuitive	N/A	A
Lynn	Both	Intuitive	A
Mary	Both	Formal	C
Nate	Both	Intuitive	A
Sonia	"Complete guess"	N/A	A

*Yuki did not work on the average value question due to time constraints.

Ian's work on this problem is an illustrative example, as his reasoning shared similarities with several other students and he articulated clearly the conflict between his intuition and use of formal mathematics. Parts of Ian's interview that were coded as Formal Resources are indicated in green text in the transcript below. Intuitive Resources are indicated in orange.

Ian: I'm trying to recall the formula for average speed, and right now I think it's the speed at time-2 and the speed at time-0... I mean, the speed at time-2 minus the speed at time-0, divided by 2 because time is 2. Well, based on that, I would say the average speed is the same. [circles the answer choice "(c) Car A and Car B have the same average speed"]. But intuitively, that's not how I think. Car A has higher speed because it has more area [below the curve].

Interviewer: Tell me more about both of those ideas. So one was a formula, and the other you said intuitively you think it should be Car A, but the formula tells you they're the same.

Ian: I know that the distance is speed times time. I don't know. I would say this [points to his writing of 'distance = speed · time'] is certain. The distance of a speed-time graph would be the area under the graph. So based on that, Car A, it would travel farther. If Car A traveled further, then it would make sense to say that it would have higher average speed over the time that it's been traveling, because that's the only way for it to travel further. So yeah, I guess just based on that reason, I can choose Car A. I don't know. [Ian erases his earlier choice of answer (c) and instead chooses "(a) Car A has greater average speed."]

Interviewer: What about the formula you had that told you they're the same?

Ian: This [pointing to the writing of 'distance = speed · time' is somehow here in my head. That formula was taught to me before somewhere. Yeah. But I trust my intuition more."

Ian abandons the formula in favor of his intuition, even though the formula was the first thing he attempted to use to solve the problem. He does not attempt to explore *why* the formula gave a different answer than his intuition or why it might not have been an appropriate formula to use in this case. Furthermore, his statement that the formula is "somehow here in my head" and the lack of attempt to use the meaning of the formula in thinking about this problem suggests that this piece of formal mathematical knowledge is not linked to his intuitive knowledge, at least in this case for this particular student. Furthermore, he does not *expect* that the two should be linked. Like Ian, none of the students who experienced conflict between formal mathematical knowledge and their own intuitive knowledge made any attempt to reconcile them. This suggests that students view these intellectual resources as quite separate and independent, rather than resources that could be useful in conjunction with each other.

Discussion

The main conclusion from this study is that many students employ varied epistemologies when doing mathematics that can sometimes be in tension. They may at times consciously or unconsciously suppress their own abilities to reason intuitively about mathematical phenomena because they do not believe that these intellectual resources are valuable when doing mathematics. For mathematics education research, this suggests that evidence of students' beliefs about mathematics gathered in one context (on a survey or an interview, for example) should not be interpreted as indicative of the full range of their epistemologies. For instruction, there is not necessarily a need to replace students' epistemological beliefs with different ones, but rather a need to supporting and reinforcing the productive beliefs that are already there. Last, noting that students' epistemologies of mathematics play a role in how they approach mathematical tasks gives instructors a potential opportunity to leverage shifts in epistemologies to support students' learning of calculus content.

This is a small sample that only included participants who volunteered to be interviewed. While it is possible that these findings are indicative of the kinds of epistemologies that some beginning calculus students hold, these results should not be interpreted as broad claims about calculus students in general. In addition, further research could explore how different instructional strategies in calculus courses influence students' epistemologies.

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A Confirmatory Factor Analysis of EQIPM, a Video Coding Protocol to Assess the Quality of Community College Algebra Instruction

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Evaluating the Quality of Instruction in Post-secondary Mathematics (EQIPM) is a video coding instrument that provides indicators of the quality of instruction in community college algebra. Following an extensive revision, and the coding of a set of 84 videos from 40 instructors, we report the results of a confirmatory factor analysis, that suggests that the instrument captures three distinct hypothesized dimensions of quality of instruction in community college algebra classes. Due to the nature of the instruction observed, the items in the instrument needed to be treated as categorical and some had to be removed from the analysis. We use the data to provide a first glimpse of the quality of algebra instruction that we found in this data set.

Keywords: Algebra, Instruction, Video Coding, Community Colleges

Quality of mathematics instruction matters for students' learning and motivation towards pursuing further studies in a particular discipline. Interestingly most research primarily documents the negative effects of poor quality of instruction on various aspects of students' experiences: learning, affect, beliefs, identity, perception of self, etc. (Maltese & Tai, 2011; Seymour, 2002; Watkins & Mazur, 2013). The interest in documenting classroom activity has been growing, because of mounting evidence that when 'active learning' is implemented in classrooms, students perform better (Freeman et al., 2014). Most of these reports, however, are not based on evidence collected from large-scale observations of classroom teaching. In part, this is because teaching is a complex practice that is also highly contextualized, not only by the content itself, but also by the available resources (e.g., textbooks that support inquiry, online sites), and the institutional features in which it occurs (teaching at an institution with selective admissions is different from teaching at a community college that is open access). In recent years, more scholars have been scrutinizing teaching practice with the goal of making claims regarding its connection to student learning. We contribute to this body of work, with an instrument that attends to instruction in algebra courses at community colleges. Our work builds on foundational contributions made to understand the quality of mathematics instruction in elementary and middle school settings and its connection to student achievement, and complements current work to describe the instruction needed to support faculty in teaching inquiry-oriented materials. We also posit that a reliable and valid method to fully describe how instruction occurs can bring to light the complexity of instructors' work in post-secondary settings and contribute to the development of faculty professional development opportunities that can be focused on the classroom interactions that matter for student learning (Bryk, Gomez, Grunow, & LeMahieu, 2015). As part of a larger project that investigates the connection between the quality of instruction and student learning in community college algebra courses, we have been working on an instrument, Evaluating Quality of Instruction in Postsecondary Mathematics

(EQIPM), that seeks to characterize instruction in community college algebra.* In this paper, we present the results of a confirmatory factor analysis that supports the hypothesized structure of quality of instruction that we envisioned. We note that at the time of our project there were no instruments that could reliably be used to ascertain quality of instruction in post-secondary settings, in particular at community colleges.

Theoretical Perspective

We assume that teaching and learning are phenomena that occur among people enacting different roles—those of instructor or student—aided by resources of different types (e.g., classroom environment, technology, knowledge) and constrained by specific institutional requirements (e.g., covering preset mathematical content, having periods of 50 minutes, see Chazan, Herbst, & Clark, 2016; Cohen, Raudenbush, & Ball, 2003). We focus on instruction, one of many activities that can be encompassed within teaching (Chazan, et al., 2016), and define it as the interactions that occur between instructors and students in concert with the mathematical content (Cohen et al., 2003). Such interactions are influenced by the environment in which they occur and change over time. The EQIPM instrument, designed with the goal of assessing the quality of the interactions defining instruction, specifically the Quality of Instructor-Student interaction, the Quality of Instructor-Content Interaction, and the Quality of Student-Content Interaction, supported by the quality of *Mathematical Errors and Imprecisions in Content or Language* that are present in a lesson. Figure 1 illustrates the theorized structure of the coding instrument by showing individual items within the three constructs. The items under *Segment features* help characterize the segment (i.e., Nature of Mathematics, Modes of instruction, Technology use). See the Appendix for the definition of the items

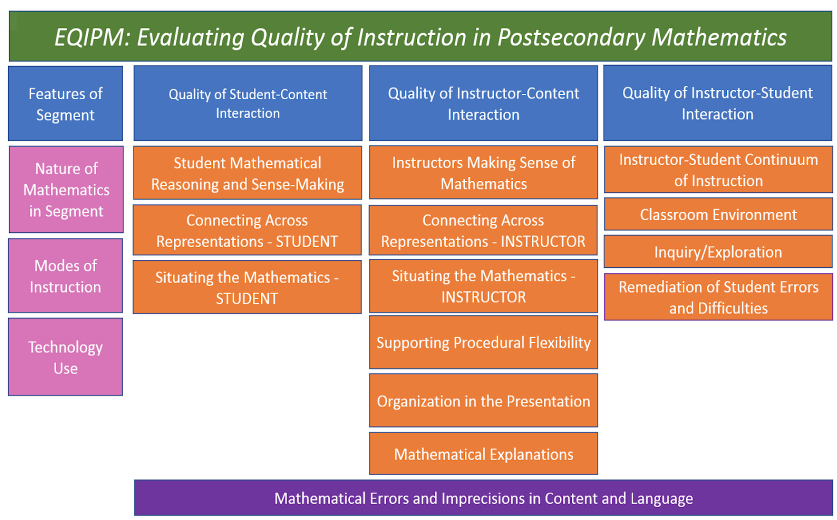

Figure 1. Dimensions and codes for the EQIPM instrument.

Methods

In the Fall 2017 semester we video-recorded 131 lessons in intermediate and college algebra classes from two different community colleges in each of three different states. The lessons ranged in duration between 45 and 150 minutes, and were taught by 40 different instructors (44 different unique courses video-recorded; 4 instructors taught 2 sections of a course). The lessons covered one of three topics: linear equations/functions, rational equations/functions, or exponential equations/functions. These topics were chosen because they offer us opportunities to

* Our instrument is different from another classroom observation tool, *Equity QUantified In Participation* (EQUIP) developed by Reinholz and Shah (2018) which assesses equitable participations in classrooms.

observe instruction on key mathematical concepts (e.g., transformations of functions; algebra of functions) and to attend to key ways of thinking about equations and functions (e.g., preservation of solutions after transformations; covariational reasoning), which are foundational algebraic ideas that support more advanced mathematical understanding (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The development of EQIPM was similar to the process used by Hill and colleagues (2008) and by Litke (2015). Their instruments describe and qualify instructional practices from video-recorded lessons by subdividing lessons into 7.5-minute segments and rating all segments within a lesson. For the analysis reported here, we coded 44 lessons of each of the unique courses video-recorded.

Using Version 4 of EQIPM (AI@CC Research Group, 2018), each 7.5-minute segment within a lesson was coded by one member of a team of 15 researchers using a rubric that defined the items. For each item, coders were asked to identify evidence that matched each item. If no evidence was found, coders entered a 0. If there was evidence, to support the item, the coders were asked to rate its quality on a 1 to 4 scale according to the rubric. In all items except one, a rating of 1 or 2 meant that the evidence in the segment weakly represented the description of high-quality for the item, whereas a rating of 3 or 4 meant that the evidence within the segment strongly represented the description of high-quality for the item. The item *Mathematical Errors and Imprecisions in Content and Language*, was rated in the reverse (a low rating was assigned for minor imprecisions and errors; a high rating was assigned for major imprecisions and errors). Each coder provided a justification for their rating. Each coder independently coded 3 to 4 consecutive segments of a lesson to minimize bias due to familiarity with the instructor or the lesson. Fourteen percent of segments were randomly chosen for double-coding by two coders. Each pair of coders held calibration meetings to discuss their ratings when one of the following situations arose: (1) when one rater assigned a 0 and the other rating was not 0 and (2) when the discrepancy between non-zero ratings by the coders was greater than one point. In each case a single rating was assigned after a discussion of the evidence. When it was difficult to decide, the coding was discussed in a meeting with all the coders.

Thirteen percent of segments (150 out of 1122) were removed from the data set before conducting the confirmatory factor analysis (CFA) because they were marked as having no mathematics taught in the segment. The variables in our model were continuously scored 0- to 4, but their distributions were not uniformly distributed, which is an assumption for typical CFA. For this reason, we decided to use ordinal CFA and reduced the rating scale into three ordinal categories. The 0 ratings remained as 0 to indicate no evidence; ratings of 1 or 2 were reclassified as a 1, to indicate low quality of instruction, and ratings of 3 or 4 were reclassified as a 2, to indicate high quality of instruction. The final sample included 972 segments, representing a balanced proportion of segments by course (intermediate algebra, n = 515 segments; college algebra, n = 457 segments) and diverse representation by topic (linear equations, n = 107 segments; rational equations, n = 283 segments; exponential equations n = 539 segments; and other topics, n = 43 segments).

We used our theoretical model that hypothesized that the items fell into 3-factors, *Student-Content Interaction, Instructor-Content Interaction, and Instructor-Student Interaction,* and tested the structure using confirmatory factor analysis (CFA). CFA is a statistical technique used to verify the factor structure of a set of observed variables, when there is a robust hypothetical model. Using modification indices (see below), that included removing problematic items, the model was adjusted until a properly fitting model was found. This process was based on commonly used criteria for CFA models (Hu & Bentler, 1999).

Preliminary Findings

We found that the 3-factor theoretical model was an adequate good fit to the data: CFI = 0.967, RMSEA = 0.039, SRMR = 0.079. The item *Mathematical Errors and Imprecisions in Content and Language* was excluded from the modeling, because, consistent with other studies, we hypothesized that the item would potentially load in all the factors. In addition we removed *Remediation of Student Errors and Difficulties* because its loadings (0.158) were below the level of acceptability. We added two modifications based on suggestions from modification indices which allowed the error variances between two pairs of variables to be correlated: *Student Connecting Across Representations* and *Instructor Connecting Across Representation* as well as *Instructor Connecting Across Representations* and *Inquiry/Exploration*. This model had both acceptable model fit indices and acceptable standardized factor loadings; we did not pursue further modifications. Our CFA model supported three meaningful factors using 12 of the 13 items (see Table 1). We found that two factors, *Student-Content Interactions* and *Instructor-Student Interactions* are positively correlated ($r = 0.981$, $p < 0.001$) whereas the other two are not (*Student Content Interactions – Instructor Content Interactions,* $r = 0.264$, *n.s.*; *Instructor-Content Interactions – Instructor Student Interactions,* $r = 0.231$, *n.s.*).

Table 1. Standardized Factor Loadings for the 3-Factor CFA Solution.

Construct	Item	Standard Estimate	SE
Factor 1: Student-Content Interaction			
	Student Mathematical Reasoning & Sense-Making	0.798	0.041
	Student Connecting Across Representations.	0.603	0.041
	Student Situating Mathematics	0.439	0.084
Factor 2: Instructor-Content Interaction			
	Instructor Sense-Making	0.791	0.059
	Instructor Connecting Across Representations	0.377	0.049
	Instructor Situating Mathematics	0.284	0.049
	Supporting Procedural Flexibility	0.256	0.054
	Organization in the Presentation	0.394	0.057
	Mathematical Explanations	0.397	0.051
Factor 3: Instructor-Student Interaction			
	Continuum of Instruction	0.874	0.047
	Classroom Environment	0.736	0.049
	Inquiry/Exploration	0.660	0.049

Discussion

Factor 1 seems to capture all three hypothesized items of **quality of student-content** in the second column of Figure 1 with reasonably strong loadings (factor loading greater than 0.6) for *Student Mathematical Reasoning and Sense-Making* and *Connecting Across Representation* and adequate loading (greater than 0.3) for *Students Situating Mathematics*. This item had 97% of segments rated as not present. *Students Situating Mathematics* contributes less to the quality of student-content interaction than the other two variables and relates less to the other two variables in this factor. Further discussion focused on refining this item may be warranted.

Factor 2 also appears to capture all six items hypothesized for **quality of instructor-content interaction** (third column of Figure 1); this factor has a wider range of loading. Only *Instructor Making Sense of Mathematics* has a reasonably strong loading. Three other items, *Instructor Connecting Across Representations*, *Organization in the Presentation*, and *Mathematical*

Explanations, have adequate loadings (greater than 0.3); two items, *Instructor Situating the Mathematics* and *Supporting Procedural Flexibility* have weak loadings. The weak loadings may speak directly about how instructors manage mathematical content attending to discrete topics rather than situating the topics within broader mathematical ideas (67% of 972 of the instances were rated 0, no evidence) and presenting mathematics with one efficient procedure rather than attending to flexibility (71% of 972 of the instances were rated 0, no evidence). These two items have limited contributions to the quality of instructor-content interaction at this time.

Finally, Factor 3 for **quality of instructor-student interaction**, embedded three of the four items under the fourth column of Figure 1 that were meant to address how students and the instructor were working together. The three items have reasonably strong loadings, all greater than 0.6. *Remediation of Student Errors and Difficulties* had to be removed from the analysis because of poor fit. One problem in rating this item stems from when instructors had opportunities to remediate, which occurred mainly during sessions in which students worked individually on problems at their desks. In many instances, it was difficult to ascertain what remediation was occurring, because it was unclear how the remediation was happening; thus, it was not possible to assess its quality. How instructors remediate errors and difficulties students have is crucial for high quality instruction, in particular in 1-1 interactions with the students. The item may need to be further refined to account for these cases.

These CFA results are encouraging. We will use the constructs to model student outcomes (percent grade in course), controlling for variables at the student and instructor level (demographics, beliefs, attitudes, prior knowledge) to determine if there is a difference in student performance that can be associated with high quality instruction. The next step would be to scale these individual variables into composite scores for use in further analysis.

Being able to identify three distinct factors that can be used to describe the quality of community college algebra instruction is promising for the field in that each of the constructs suggest specific areas for supporting the work of instructors in teaching community college algebra. These results also support previous research in K-12 that models learning via assessing the quality of instruction defined as the interactions between teacher, student, and content. The factors used will be included in the full model of our data to determine links between instructional qualities and student performance. Moreover, the instrument has substantial promise for the design of future professional development efforts with community college faculty that would promote high quality instruction.

Acknowledgement

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Appendix: Definition of the Items

Student-Content Interaction
Student Mathematical Reasoning and Sense-Making: Assesses student utterances that showcase reasoning and sense-making about mathematical ideas
Connecting across Representations-STUDENT: Assesses the connections that students express within, between, and across representations of the same mathematical problems, ideas, and concepts
Situating the Mathematics-STUDENT: Assesses how students make connections to other aspects of the algebra curriculum, related topics, or the broader domain of mathematics, situating and motivating the current area under study within a broader context.
Instructor-Content Interaction
Instructors Making Sense of Mathematics: Assesses how instructors leverage known and new mathematical ideas or students' personal knowledge or experiences, in order to make meaning of the mathematics presented.
Connecting across Representations-INSTRUCTOR: Assesses the connections that instructors make within, between, and across representations of the same mathematical problems, ideas, and concepts
Situating the Mathematics-INSTRUCTOR: Assesses how instructors make connections to other aspects of the algebra curriculum, related topics, or the broader domain of mathematics, situating and motivating the current area under study within a broader context.
Supporting Procedural Flexibility: Assesses how instructors support the development of procedural flexibility, by identifying what procedure to apply and when and where to apply it.
Organization in the Presentation: Assesses how complete, detailed, and organized the instructor's presentation content is when outlining or describing the mathematics, or describing the steps used in a procedure.
Mathematical Explanations: Assesses how instructors provide mathematical reasons and justifications for why something is done.
Instructor-Student Interaction
Instructor-Student Continuum of Instruction: Assesses the degree to which either the instructor or the students contribute to the development of the mathematical ideas (abstract concepts, formulas, notation, definitions, concrete examples, pictorial examples, and rules/properties). It captures who is responsible for the development of those ideas.
Classroom Environment: Assesses how instructor and students create a respectful and open environment in their classroom and in which expectations for high quality mathematical work are the norm.
Inquiry / Exploration: Assesses the degree to which mathematics exploration and inquiry occurs
Remediation of Student Errors and Difficulties: Assesses remediation (either for the whole class or with individual/small groups) in which student misconceptions and difficulties with the content are addressed
Cross-Cutting Item
Mathematical Errors and Imprecisions in Content or Language: Assesses mathematically incorrect or problematic use of mathematical ideas, language, or notation.

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Analyzing Collegiate Mathematics Observation Protocols: Attending to the Instructional Triangle and Inquiry-Based Mathematics Education Practices

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Typically, departments and universities evaluate and provide feedback to college mathematics instructors on their teaching by conducting classroom observations. When these observations are guided by an observation protocol, the protocol provides a particular lens that focuses the observer's attention on certain aspects of instruction. To understand the way in which mathematics departments are currently evaluating teaching, we investigated the structure, focus, and alignment with inquiry-based mathematics education practices of items on 25 observation protocols. The results suggest that protocols mainly include closed-response items that are evaluative, focus more on the instructor than the students or the content, and rarely attend to inquiry-based mathematics education practices. These results have implications for both those who develop observation protocols and those who use observation protocols as a measure of teaching practices. In particular, our study highlights the need for careful consideration to be taken when observations are used for performance measures.

Keywords: Classroom Observation Protocols, Instructional Triangle, Inquiry-Based Mathematics Education

At its core, a classroom observation protocol (OP) has an agenda. It guides the observer to explicitly attend to certain aspects of teaching, learning, and the classroom environment while disregarding others. To better understand the purpose of OPs and identify different aspects of teaching and learning they attend to, we analyzed OPs that were developed by and used "in house" in mathematics departments as well as OPs that were developed by education researchers and used more broadly. The purpose of this study is to inform our understanding of the current (explicit or implicit) stance that mathematics departments take on evaluating teaching in order to identify where progress must be made with observations and provide direction for teachers of undergraduate mathematics education. The following research questions guided our analysis.

RQ1. How are classroom observation protocols structured?
RQ2. On what aspects of instruction do classroom observation protocol items focus?
RQ3. To what extent do classroom observation protocol items attend to the four pillars of Inquiry-Based Mathematics Education (Laursen & Rasmussen, 2019)?

Literature Review

Classroom Observation Protocols

OPs are often used as a way to provide teachers with specific feedback and evaluations of their instructional methods and classroom practices. OPs may be designed as descriptive or evaluative instruments. The principal aim of descriptive OPs is to capture and describe what is happening in a class, whereas evaluative OPs measure (with ordinal, cardinal, or scalar measurements) the quality of a class against a theory of instructional quality. Moreover, OPs may also be holistic (encompassing the whole lesson) or segmented (separately rating shorter segments of the lesson). Segmented protocols usually capture structure, while holistic views can be more appropriate for capturing quality (Tatto, Burn, Menter, Mutton, & Thompson, 2018;

Tatto et al., in press). For example, a lengthy whole-class discussion may span several segments that are treated separately. A holistic protocol, on the other hand, might focus on how an error made early in a lesson is featured in the lesson closing, clearing up students' misconceptions.

Additionally, protocols will likely vary in the degree of structure, whether they require high or low inference by observers, and on what they focus observers' attention. Although much can be learned from a content-general OP (e.g., Classroom Assessment Scoring System, Mikami et al., 2011), specific indicators of mathematics teaching and learning are important for capturing quality of mathematics instruction. Multiple research projects (e.g., Joe et al., 2013; Mihaly et al., 2013) and reviews (Bostic et al., in press; Boston et al., 2015) synthesize the major OPs developed for teaching mathematics at the K-12 level. These reports provide the field with syntheses of measures for K-12 mathematics teaching, but there is a lack of overarching syntheses of protocols for measuring the quality of undergraduate mathematics instruction.

The Instructional Triangle

In order to analyze what aspects of instruction OPs focus on, we chose to use the instructional triangle framework. Cohen, Raudenbush, and Ball (2003) defined teaching as "what teachers do, say, and think with learners, concerning content, in particular organizations and other environments, in time" (p. 124). In their definition, four critical aspects of teaching become apparent: teachers, learners, content, and environments. Cohen et. al took these four aspects and situated them in a model that represents instruction as interaction (Figure 1). This model is often referred to as the instructional triangle, since it situates the teacher, learners, and content in direct interaction inside of environments.

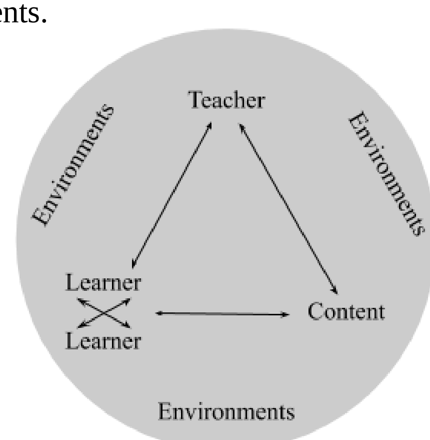

Figure 1. Cohen, Raudenbush, and Ball's (2003) instructional triangle.

Inquiry-Based Mathematics Education

Over the past two years, undergraduate mathematics educators in the U.S. have brought together the ideas of inquiry-based learning (Laursen, Hassi, Kogen, & Weston, 2014) and inquiry-oriented instruction (Rasmussen & Kwon, 2007) to align with international work in inquiry-based mathematics education (IBME, Laursen & Rasmussen, 2019). In Laursen and Rasmussen's seminal work, IBME does not try to claim its own scholarly territory, but collaboratively complement the use of inquiry for a more unified and robust model to aid undergraduate mathematics education. Drawing from secondary and collegiate views of inquiry about student engagement and student learning internationally, IBME prescribes four critical pillars for student learning (p. 138):

1. Students engage deeply with coherent and meaningful mathematics,

2. Students collaboratively process mathematical ideas,
3. Instructors inquire into student thinking, and
4. Instructors foster equity in their design and facilitation choices.

The intent of these four pillars is to "account for student learning and thus offer guidance to instructors seeking to develop their teaching practice" (Laursen & Rasmussen, 2019, p. 138). Therefore, these four pillars provide guidance towards evidence-based effective teaching strategies that align with the Mathematical Association of America's (2018) Instructional Practices Guide and offer a prescriptive model to look at teaching. This new model for teaching undergraduate mathematics education offers a student-centered focus on inquiry, but is this currently a measurement used by higher education? To answer this question, this study uses the IBME pillars as a tool to determine to what extent OPs already incorporate these pillars. As a result of this study, we aim to provide the field with a baseline to understand what mathematics departments attend to and focus on in OPs to aid in further study of growth in teaching undergraduate mathematics (Miller et al., 2018).

Methods

Data Collection

This study grew out of a session from the College Mathematics Instructor Development Source (CoMInDS) Workshop on Improving the Preparation of Graduate Students to Teach Undergraduate Mathematics at the University of Tennessee, Knoxville in 2019. Participants at the workshop came from universities across the U.S. and were involved with providing professional development for college mathematics instructors in their respective mathematics departments. During the workshop session, the organizers asked participants to upload OPs to a shared folder. A total of 25 OPs were uploaded by 14 individuals from 13 different universities, the majority of which were developed and used internally by individual mathematics departments (N = 17). However, participants also uploaded some OPs developed by universities (N = 3) and researchers (N = 5), which we also analyzed. Although the majority of OPs uploaded were for observing graduate students, some were designed for observing faculty members or instructors at large. Also, the majority of the OPs were specifically for observing mathematics classes (N = 19), but two were for observing STEM classrooms and four were for observing any classroom. Since this sample of OPs provided us with the opportunity to analyze department observation protocols that we normally would not have access to, we chose to focus on only these OPs for this study. In doing so, we aimed to understand the nature of the OPs that were identified by faculty who are invested in improving mathematics instruction in their departments.

Data Analysis

In order to answer our research questions, we chose to analyze the individual OP items along three main dimensions: structure, item focus, and attention to the four pillars of IBME (Laursen & Rasmussen, 2019). To analyze the structure, we first recorded the Item Type (Scalar, Checkbox, Numeric, or Open Response). As we began coding, we added additional structural codes to note if an item (b) had Additional Space for Comments even though it was a closed-response item, (c) was primarily used for Observation Record Keeping (e.g., observer name, date, number of students present), or (d) provided an opportunity for the Observee (to have a) Voice by leaving comments. For the second dimension, we analyzed if the item was Teacher-Focused, Student-Focused, Lesson/Content-Focused, or Classroom Logistics/Physical Environment-Focused. Although it was possible that an item may have attended to multiple aspects of instruction (e.g., both the teacher and students), we chose to only code for an aspect if it was the primary focus of the item. We also recognized that social or cultural environment was also sometimes a focused, but coded these items as either Teacher- or Student-Focused since they involved mostly human interactions. Finally, for the third dimension, we analyzed whether or not each item attended to Student Engagement, Student Collaboration, Instructor Inquiry, and Equity. We decided to apply a broad interpretation of what counted as Student Engagement and Equity. For example, we coded items that attended to whether or not the classroom environment was respectful as attending to Equity and included any item that mentioned student engagement, even if the item did not explicitly attend to whether or not "students engage deeply with coherent and meaningful mathematical tasks" (Laursen & Rasmussen, 2019, p. 138).

Table 1. Examples of Item Focus and IBME Pillar codes.

Example Item	Codes	Justification
The teacher used questioning strategies to encourage participation, check on skill development, and facilitate intellectual engagement and productive interaction with students about important science and mathematics content and concepts.	Teacher-Focused Student Engagement Instructor Inquiry	The item mentions student engagement, but is mainly focused on the teacher's classroom facilitation.
Interactions reflected collegial working relationships among students (e.g., students worked together productively and talked with each other about the lesson).	Student-Focused Student Collaboration	This item primarily focuses on evaluating how well students worked together during class.
The lesson was well organized and structured (e.g., the objectives of the lesson were clear to students, and the sequence of the lesson was structured to build understanding and maintain a sense of purpose).	Lesson/Content-Focused	Although the teacher directs and organizes the lesson, this item focuses primarily on evaluating the lesson and content.
The classroom is organized appropriately such that students can work in groups easily and get to lab materials as needed, and the teacher can move to each student or student group.	Classroom Logistics/Physical Environment-Focused	This item explicitly focuses on the physical (as opposed to social or cultural) environment of the classroom.

Results

In total, there were 656 items on the 25 OPs. However, this number includes 125 items that we categorized as Observation Record Keeping. Although some of these items recorded information that may be relevant to evaluating student engagement (e.g., number of students enrolled, number of students present, number of students tardy), we decided to categorize these items separately and focus only on the other 531 items in the OPs. In particular, all of the numbers and percentages we mention in the rest of the report are focused on only the 531 non-

Observation Record Keeping items. Due to space constraints, we cannot report our results for each individual OP. Instead, we group OPs by who developed them (departments, universities, or researchers) and compare across the groups. In Table 2, the percentages in each column are calculated based on the last row (Total Number of Items) in that column.

Table 2. Comparison group summary of code frequencies.

Code Categories	Department (N = 17)		University (N = 3)		Researcher (N = 5)		Total (N = 25)	
Item Type*								
Scalar-Categorical	84	(32%)	111	(87%)	33	(24%)	228	(43%)
Scalar-Continuous	6	(2%)	0	(0%)	30	(22%)	36	(7%)
Checkbox	40	(15%)	0	(0%)	26	(19%)	66	(12%)
Numeric	1	(0%)	0	(0%)	39	(28%)	40	(8%)
Open Response	135	(51%)	17	(13%)	9	(7%)	161	(30%)
Item Focus								
Teacher	161	(61%)	46	(36%)	63	(46%)	270	(51%)
Student	43	(16%)	14	(11%)	44	(32%)	101	(19%)
Lesson/Content	48	(18%)	48	(38%)	31	(23%)	127	(24%)
Classroom Logistics/ Physical Environment	20	(8%)	7	(5%)	11	(8%)	38	(7%)
IBME Pillar								
Student Engagement	45	(17%)	10	(8%)	56	(41%)	111	(21%)
Student Collaboration	15	(6%)	4	(3%)	19	(14%)	38	(7%)
Instructor Inquiry	18	(7%)	27	(21%)	33	(24%)	78	(15%)
Equity	18	(7%)	21	(16%)	15	(11%)	54	(10%)
Total Number of Items	266		128		137		531	

**Item Type is the only code group where category columns will add up to 100%, because every item was required to be categorized into a distinct item type. However, items could be coded with none, one, or multiple Item Focus and IBME Pillar codes, so these rows are not distinct and category columns do not total to 100%.*

The OPs contained between 7 and 93 items (N = 25, M = 21, SD = 17.68). Of the three comparison groups, department OPs tended to be the shortest; containing 7 to 36 items (N = 17, M = 16, SD = 9.26). Researcher OPs had the most consistent number of total items; containing 22 to 33 items (N = 5, M = 27, SD = 4.62). University OPs had a wide variance of total items (9, 27, and 93), although the longest OP did say that the observer should only pick 20 items to focus on. The majority of items on the OPs were either Scalar-Categorical (43%) or Open Response (30%). Looking across the comparison groups (Table 2), we noticed that department OPs had a much higher percentage of items coded as Open Response (51%) in comparison to university (13%) and researcher (7%) OPs. However, university OPs did not include Scalar-Continuous, Checkbox, or Numeric items, while researcher OPs had a more even distribution of the items that were closed-response.

In addition to looking at the total number of items and item types, we considered whether or not closed-response items had Additional Space for Comments to be left by the observer. On the one hand, researcher OPs had the highest percentage of closed-response items (93%) and the highest percentage of closed-response items that had Additional Space for Comments (68%); so they still had a balance of quantitative and qualitative feedback. Department OPs, on the other hand, only provided Additional Space for Comments 15% of the time and the three university OPs never included Additional Space for Comments. The final structure that we coded for is whether or not the OP contained items that provided the observee with an opportunity to leave feedback or comments, which we called Observee Voice. This was not a common code, but department OPs had the highest percentage of Observee Voice items (14%) in comparison to university (2%) and researcher (5%) OPs.

For RQ2, we coded for the four aspects of the instructional triangle: teacher, student, content, and environment. The majority of items across all protocols were Teacher Focused (51%) and few items were Classroom Logistics/Physical Environment Focused (7%). However, considering the three comparison groups, we see each group had a different focus. Department OPs had the highest percentage of Teacher Focused items (61%), while university OPs had the highest percentage of Content/Lesson Focused items (38%) and researcher OPs had the highest percentage of Student Focused items (32%). Moreover, university OPs had almost an even split of Teacher (36%) and Content/Lesson (38%) focused items and researcher OPs had more Teacher (46%) than Student (32%) focused items.

For RQ3, the final dimension we analyzed was whether or not items on OPs attended to the four pillars of IBME (Laursen & Rasmussen, 2019). Although the IBME Pillars were only recently identified, we hoped that many items would attend to them since these pillars are based on established best practices and recommendations in undergraduate mathematics education. However, we found that the minority of the items (42%) attended to the IBME pillars, and ironically this percentage drops to 36% if we consider only the OPs that were explicitly designed for observing mathematics classrooms. Of the four pillars, Student Engagement was attended to the most (21%) and Student Collaboration was attended to the least (7%). Out of three comparison groups, the researcher OPs had the highest percentages of items that attended to Student Engagement (41%), Student Collaboration (14%), and Instructor Inquiry (24%). Few of the items on university OPs attended to Student Engagement (8%) and even fewer attended to Student Collaboration (3%), but they did have a higher percentage of items that attended to Instructor Inquiry (21%) and the highest percentage of items that attended to Equity (16%). Finally, we found that items on department OPs sometimes attended to Student Engagement (17%), but rarely attended to Student Collaboration (6%), Teacher Inquiry (7%), or Equity (7%).

Discussion

From this analysis, we found that the majority of items included in OPs provided quantitative measures of instruction, which suggests that the majority of items analyzed were designed as evaluative measures, not descriptive. Given that OPs are often used to evaluate teaching, this result was not surprising, as quantitative data is often the primary way that performance is measured in academia (Yee, Deshler, Rogers, Petrulis, Potvin, & Sweeney, 2019). Many items focused primarily on the teacher, which may suggest that OPs still have a strong tendency to be teacher-focused as opposed to student-focused. Even though the majority (N = 19) of these OPs were developed specifically for observing mathematics classrooms, few of the items attend to the IBME pillars. This result may suggest that although the field of mathematics education has recognized the importance of student engagement, student collaboration, instructor inquiry, and equity, our evaluative methods have not.

One implication of our results is that OPs tend to be more evaluative than descriptive, which may lead to the introduction of bias or inconsistency in interpreting results. In particular, many of the Scalar items did not include descriptors for the categories or comparisons that were being made. Therefore, there is no way to judge whether or not observers are interpreting and rating items in the same way. This lack of descriptors is especially concerning given that OPs can be used as performance measures for academic raises, performance reviews, and promotion decisions. Additionally, although mathematics education research has described instruction as having three main components (teachers, students, and content), OPs often focus primarily on one or two of these components, which puts the instructional triangle off-balance. For instance, in Figure 2 we include a black triangle that represents an even split of items from an OP focusing on the teacher, students, and content. The triangle in red represents the department OPs, which focused mostly on the teacher and very little on the other aspects. The triangle in blue represents the university OPs, which split the focus on the teacher and content, but paid little attention to the students. Finally, the green triangle represents the researcher OPs, which did attend to instructors more than content, but were the closest to an even split among the three aspects of instruction. Ultimately, our results indicate that mathematics departments need to carefully consider redesigning their OPs to better align with current recommendations in order to realize the vision for IBME at the collegiate level.

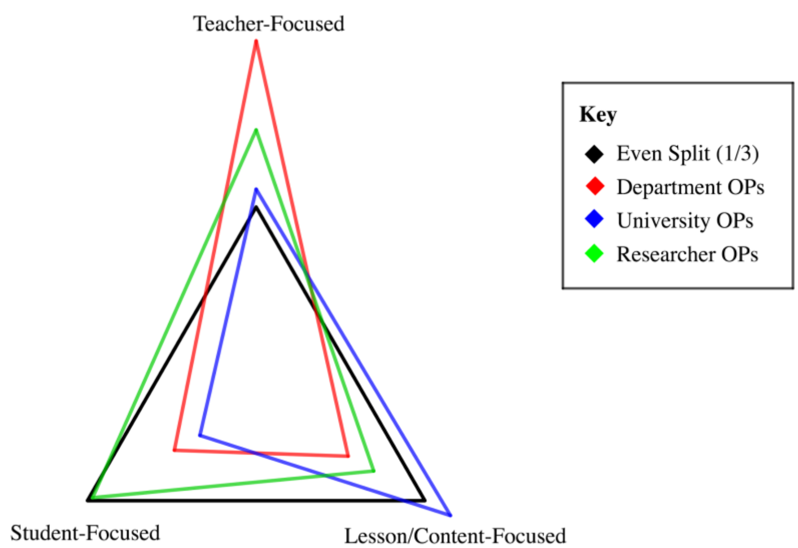

Figure 2. Distribution of Item Focus for each comparison group.

One limitation of our study is the limited sample size. In particular, we analyzed a small number of OPs developed for internal use by universities and those developed by researchers for external use. Future work could increase the sample size for these two categories to make a more robust comparison of the three groups, but this initial study provides us with an understanding of what is currently being used when observing collegiate instruction at 13 different universities. Additionally, all department OPs analyzed were shared by participants who chose to attend a workshop specifically focused on collegiate mathematics instruction. In particular, many workshop participants described their involvement in the development or revision of the department OPs that they shared. This background information means our sample of department OPs comes from individuals who have a vested interest in and commitment to improving and evaluating instruction at the undergraduate level, and therefore may be biased. However, this bias of educators who value and care about instruction shows how much more work is still needed as we see such low references to the IBME. This is not to suggest we cannot make progress, but rather the field needs to recognize where we are versus where we want to be.

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Students' Understanding of Partial Derivatives

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This paper extends work in the area of quantitative reasoning at the undergraduate level, in addition to proposing a conceptual framework for interpreting partial derivatives in different contexts. Task-based interviews were used to examine third-semester calculus students' reasoning about partial derivatives in two tasks, one situated in a mathematics context (Alajmi, 2012) and the other in a diet and exercise context. Findings of this study indicate that interpreting partial derivatives, especially in the latter context, was problematic for a majority of the students. Overall, findings of this study suggest that student difficulties with interpreting partial derivatives in real-world contexts are similar to student difficulties with interpreting ordinary derivatives. Implications for calculus instruction are included.

Keywords: partial derivatives, quantitative reasoning, calculus

Research on student reasoning about several calculus topics at the undergraduate level is well documented. However, much of this research has focused on topics in univariate calculus and has not examined student reasoning about topics in multivariable calculus (e.g., Henderson & Britton, 2013; Rasmussen, Marrongelle, & Borba, 2014). Student reasoning about ordinary derivatives is one area that has received much attention in the research literature (e.g., Berry & Nyman, 2003; Habre & Abboud, 2006; Jones, 2017; Siyepu, 2013; Zandieh, 2000). An ordinary derivative (denoted by $\frac{d}{dx}$) describes how a real-valued function of a single independent variable changes with respect to the independent variable. A few studies have examined student reasoning about partial derivatives (cf., Martínez-Planell, Gaisman, & McGee, 2015; Thompson, Manogue, Roundy, & Mountcastle, 2012; Weber, 2012). A partial derivative (denoted by $\frac{\partial}{\partial x}$) describes how a real-valued function of two or more independent variables changes with respect to one of the independent variables by treating the other independent variable(s) as a constant.

In response to the scarcity of studies that have examined student reasoning about calculus topics in multivariable calculus, the current study reports on undergraduate students' reasoning about partial derivatives. Our study was guided by the following research question: What do calculus students' responses to application problems involving partial derivatives reveal about student understanding of partial derivatives? Partial derivatives are covered in every first course in multivariable calculus (commonly known as Calculus III) in the United States. Although we acknowledge that the ideas of ordinary and partial derivatives can be used in connection with complex functions as well, in the present study, we limit the use of these ideas to real-valued functions. In this study, the term "quantity" refers to a measurable attribute of an object (Thompson, 2011). Examples of quantities in this study include the weight (of a person-the object) and time spent on daily exercise.

Literature Review

Students' Understanding of Partial Derivatives

As previously noted, studies that have reported on student thinking about partial derivatives are scarce. Of the few available studies, Martínez-Planell et al. (2015) examined student understanding of several topics, including partial derivatives, in differential calculus of real-valued functions of two variables. The study participants were 26 science and engineering majors who had recently completed a multivariable calculus course. These researchers reported that approximating the value of a partial derivative at a point when given a real-valued function of two variables in tabular form was problematic for 10 students. In addition, determining the sign (positive, negative, zero) of a partial derivative at a point when given the graph of a real-valued function of two variables was difficult for several students. The present study provided opportunities for students to interpret positive and negative partial derivatives in different contexts. Martínez-Planell and colleagues posited that findings of their study show that students' misconceptions and difficulties with partial derivatives originate from a weak understanding of ordinary derivatives.

Thompson et al. (2012) found that students struggle to represent partial derivatives algebraically in a thermodynamics context, in addition to lacking an understanding of what it means to keep all (but one) independent variables of a real-valued function of several variables constant when calculating a partial derivative. Weber (2012) used a teaching experiment to assess and support student thinking about rates of change (partial derivatives) of real-valued functions of two variables. This researcher found that quantitative reasoning supported students' ability to generalize their understanding of rates of change of real-valued function of a single variable to that of real-valued valued functions of two variables. Weber argued that the results of his study created a need to investigate how mathematicians and mathematics educators think about rates of change (partial derivatives) of real-valued functions of two variables. In response to this argument, we examined how experts think about partial derivatives in different contexts. Details of this examination are provided in the next section-A Conceptual Framework for Partial Derivatives.

Students' Understanding of Ordinary Derivatives

Although the current study is focused on students' thinking about partial derivatives, it is important to discuss the research literature on students' understanding of ordinary derivatives for comparison. Specifically, we use the literature base on students' understanding of ordinary derivatives to identify trends in students' reasoning about ordinary derivatives that may be comparable to the results of the current study. There are two themes that emerge from the research that has looked at students' understanding of ordinary derivatives: (1) students' difficulties and misconceptions and (2) teaching interventions for developing students' understanding. Following is a discussion of these themes.

In the first theme, several studies have reported on students' difficulty to write ordinary derivatives in symbolic form (e.g., Klymchuk, Zverkova, & Sauerbier, 2010; Villegas, Castro, & Gutiérrez, 2009; White & Mitchelmore, 1996). Much research has reported on students' tendency to confuse ordinary derivatives ("rate quantities") with "amount quantities" when solving application problems that have real-world contexts (e.g., Mkhatshwa & Doerr, 2018; Mkhatshwa, 2019; Prince, Vigeant, & Nottis, 2012; Rasmussen & Marrongelle, 2006). For instance, Rasmussen and Marrongelle (2006) reported on a student who did not make a "distinction between rate of change in the amount of salt [a rate quantity] and amount of salt [an amount quantity]" (p. 408) in the context of a differential equations modeling activity. A related line of research shows that interpreting negative ordinary derivatives is problematic for undergraduate students (e.g., Beichner, 1994; Orton, 1983). Many studies have found that while ordinary derivatives are often well understood by students in kinematics contexts, making sense of ordinary derivatives in non-kinematics contexts is often difficult for students (e.g., Cetin, 2009; Ibrahim & Rebello, 2012).

In the second theme, research by Carlson, Larsen, and Lesh (2003) shows that the integration of a models and modeling perspective (Lesh & Doerr, 2003; Lesh & Zawojewski, 2007) in the teaching of calculus could lead to an improved student understanding of ordinary derivatives in real-world contexts. Other research has reported that using different modes of reasoning such as covariational reasoning and performing physical enactments of dynamic situations in calculus instruction enhances students' understanding of ordinary derivatives (e.g., Carlson, Jacobs, Coe, Larson, & Hsu, 2002). Berry and Nyman (2003) used technology in the form of motion detectors to help students develop robust understandings of ordinary derivatives in a kinematics context.

A Conceptual Framework for Interpreting Partial Derivatives

This framework explains how experts (mathematicians and mathematics educators) interpret partial derivatives in different contexts. In addition, the framework identifies things that experts pay attention to when interpreting partial derivatives. Precisely, the framework describes what it means to have a complete understanding of the concept of a partial derivative in calculus education. The framework emerged from analysis of three mathematics and one mathematics education professors' written responses to the two tasks shown in the methods section. Most of these experts had recently taught at least a section of multivariable calculus. At the time of the study and based on the Carnegie Classification of Institutions of Higher Education, two of the experts were affiliated with different mathematics departments in R1 institutions and the other two experts were affiliated with different mathematics departments in R2 institutions.

A thematic analysis of the experts' responses to questions about the meaning of the partial derivative in the aforementioned tasks revealed that depending on the context of the task, all the experts interpreted partial derivatives as "the slope of a tangent plane" in Task 1-a task with a mathematics context. Figure 1 illustrates how one of these experts interpreted the partial derivative in part b of Task 1 as the slope of a tangent plane. We state as a remark that one of the experts interpreted partial derivatives first as a "rate of change", and then as "the slope of a tangent

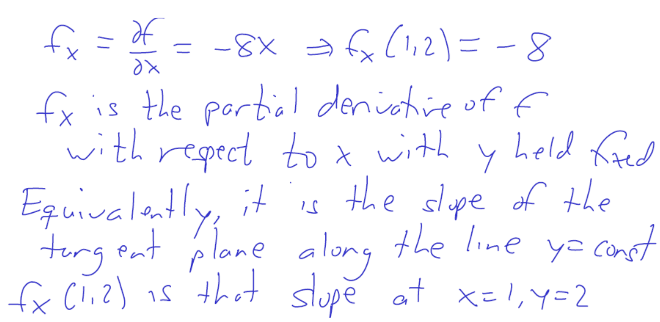

Figure 1. An expert's interpretation of the partial derivative as slope of the tangent plane.

plane" in Task 1. All the experts interpreted partial derivatives as a "rate" in Task 2-a task with a diet and exercise context. When interpreting partial derivatives in Task 2, all the experts consistently attended to three elements, namely constant and varying quantities, direction of change, and quantifying (Thompson, 1994d) the rate of change or slope. The following response to part b(i) of Task 2 written by one of the experts exemplifies the experts' attention to the aforementioned elements when interpreting partial derivatives in Task 2, in addition to interpreting a partial derivative as a rate:

> This is the <u>rate</u> [interpreting a partial derivative as a rate] that weight <u>increases</u> [attention to direction of change] with respect to calories consumed when calorie input is 2000 and with 15 minutes of exercise. So, <u>if my exercise stays constant</u> [attention to a constant quantity-time spent exercising] at 15 minutes a day, then <u>for every calorie that I consume</u> [attention to varying quantity-calories consumed] over 2000, I would expect my weight to increase by about <u>0.02</u> [quantifying rate of change] pounds.

When interpreting partial derivatives in Task 1, the experts only mentioned two elements, namely constant and varying quantities and quantifying the slope. This can be seen in one of the expert's response to part b of Task 1 shown in Figure 1.

Methods

This qualitative study used task-based interviews (Goldin, 2000) with 19 students. The interviews contained five tasks. In this paper, we report on how three students reasoned about two of the tasks, herein referred to as Task 1 and Task 2:

Task 1. Let $f(x,y) = 16 - 4x^2 - y^2$.

a. Calculate $f(1,2)$ and explain what it means.
b. Calculate $f_x(1,2)$ and explain what it means.
c. Calculate $f_y(1,2)$ and explain what it means.

Task 2. Suppose that your weight, w (in pounds), is a function of two variables:

$$w = f(c,n)$$

where c is the number of calories you consume daily and n is the number of minutes you exercise daily.

a. $f(2000, 15) = 175$.
   i. What does this tell you?
   ii. What are the units of the number 175?

b. $w_c(2000,15) = 0.02$.
   i. What does this tell you?
   ii. What are the units of the number 0.02?

c. $w_n(2000,15) = -0.025$.
   i. What does this tell you?
   ii. What are the units of the number -0.025?

The students worked through the tasks while the interviewer asked clarifying questions about their work. After a student concluded their work on a task, the interviewer asked the following questions about the task and the content of their solutions: (a) Have you seen a problem like this before? (b) What was the easiest part when solving this problem? (c) What was the challenging part when solving this problem? Two students acknowledged having seen a problem similar to Task 1 prior to participating in the study. All the students stated that they had not seen or solved problems similar to Task 2.

Setting and Participants

The study participants were three undergraduate students (pseudonyms Ava, Kim, and Cody) at a research university in the United States. At the time of the study, the students were enrolled in three sections of a traditional calculus III course taught by three different professors who had previously taught multiple sections of the course. Two sections were taught in spring 2018, and the other section was taught in spring 2019. The students were recruited via email using class rosters obtained from their calculus professors. In addition, the students were chosen based on their willingness and availability to participate in the study. The students were familiar with partial derivatives through course lectures and the course textbook (Stewart, 2016). We state as a remark that the students in this study had limited exposure to problems that involve interpreting partial derivatives in real-world contexts such as in Task 2 through classroom instruction and/or homework assignments. At the time of the study, two students were mathematics majors, and the other student was a statistics major. We note that this is a required course for mathematics, engineering, and physics majors, and that students outside the aforementioned disciplines rarely take this course as an elective. All the students were sophomores.

Data Collection and Analysis

Data for the study consisted of transcriptions of video-recordings of the task-based interviews and work written by the three students during each task-based interview session. Data analysis was done in two stages. In the first stage, we used a priori codes that were suggested by the conceptual framework discussed earlier. Specifically, we read through each interview transcript and coded instances where each student: (1) interpreted partial derivatives (e.g., as slopes or as rates of change), (2) identified (or did not) constant and varying quantities when interpreting partial derivatives, (3) commented (or did not) on direction of change when interpreting partial derivatives, and (4) quantified (or did not) the rates of change (ROCs) or slopes when interpreting partial derivatives. In addition to the a priori codes from the conceptual framework, we used three additional a priori codes, namely confusing quantities, easy part, and challenging part.

Confusing quantities. As noted earlier, several studies have reported on students' tendency to confuse rate quantities with amount quantities (e.g., Mkhatshwa, 2019; Rasmussen & Marrongelle, 2006). Since partial derivatives can be interpreted as rate quantities, we coded instances where students confused partial derivatives with amount quantities. For example, a student who assigned units of pounds instead of pounds per calorie to the partial derivative in part b (ii) of Task 2 was coded as having confused a rate quantity with an amount quantity.

Easiest part. This code was informed by one of the questions we asked students after completing each task: What was the easiest part when solving this problem? We coded instances where students reasoned about what came easy when solving each of the two tasks.

Challenging part. This code was informed by one of the questions we asked students after completing each task: What was the challenging part when solving this problem? We coded instances where students reasoned about what was difficult when solving each of the two tasks.

In the second stage of the analysis, we looked for patterns in each of the codes identified in the first stage of the analysis. These patterns included trends in the students' understandings, or difficulties they had in connection with each of the a priori codes identified in the first stage. The common understandings or difficulties in students' reasoning found in the second stage of our analysis provided answers to our research question.

Results

There are three findings, summarized in Table 1, from this study. First, all the students' interpretations of partial derivatives varied with the context of the tasks they were given. Cody interpreted the partial derivative $f_x(1,2) = -8$ as "the slope of the surface [$f(x, y) = 16 - 4x^2 - y^2$] in the x direction at (1,2)" in Task 1, and the partial derivative $w_c(2000,15) = 0.02$

as “the rate of change of one’s weight with respect to calorie consumption” in Task 2. Ava interpreted the partial derivative $f_x(1,2) = -8$ as “the slope of the tangent vector in the x direction at the point (1,2)” in Task 1, and the partial derivative $w_c(2000,15) = 0.02$ as weight gained “daily due to calorie intake” in Task 2. When asked about the units of the number 0.02, Ava said it would be in pounds, suggesting that she interpreted the rate quantity $w_c(2000,15) = 0.02$ with units of pounds per calorie as an amount quantity with units of pounds. Kim interpreted the partial derivative $f_x(1,2) = -8$ as “the derivative function evaluated at the point (1,2) with respect to x” in Task 1. When asked to elaborate on this, she said “if the function is a curve [pointing at the equation $f(x,y) = 16 - 4x^2 - y^2$ in Task 1], then the derivative is a line. She went on to say, “when you are evaluating it [pointing at the expression $f_x(1,2)$ in Task 1] at the point [(1, 2)], I think you get the value of the line, so the -8 [from $f_x(1,2) = -8$] is the value of the line.” When probed about the idea of a line, she said “from Calc I, when you take the derivative of a curve [pointing at the equation $f(x,y) = 16 - 4x^2 - y^2$ in Task 1], you get a line.” Kim interpreted the partial derivative $w_c(2000,15) = 0.02$ as “the average weight gain per day…with respect to calories.”

Table 1. Summary of results.

Student	Task	Interpretation of Partial Derivatives			Constant(s)	Variables	Direction of Change	Quantifying ROC/Slope /Other
		Slope	ROC	Other				
Ava	Task 1	x					x	x
	Task 2			x			x	x
Kim	Task 1			x				x
	Task 2			x			x	x
Cody	Task 1	x			x	x	x	x
	Task 2		x		x	x		x

Second, two students (Cody and Kim) did not comment on direction of change in at least one of the two tasks. For example, when asked to explain what $w_c(2000,15) = 0.02$ means in Task 2, Cody responded by saying “assuming one’s daily amount of exercise is held at 15 minutes, the rate of change of one’s weight with respect to daily calorie consumption at 2000 calories is 0.02 pounds per calorie.” In his response, Cody did not comment on direction of change in that he did not indicate whether or not the person’s weight will be increasing or decreasing. Cody is the only student who remarked on constant and varying quantities in the two tasks when interpreting partial derivatives. Specifically, Cody identified time spent on daily exercise as a constant quantity when he commented, “assuming one’s daily amount of exercise is held at 15 minutes,” in his interpretation of the partial derivative $w_c(2000,15) = 0.02$ in Task 2. It can be argued that he identified number of calories consumed per day as a varying quantity when he stated, “…the rate of change of one’s weight with respect to daily calorie consumption,” in his interpretation of the same partial derivative. All the students successfully quantified the ROC/Slope/other quantity they used as a numeric value for the partial derivatives they were asked to interpret, especially in Task 1 where these values were not given.

Third, all the students either identified calculating the output of a real-valued function of two variables such as the function in Task 1 as the easiest part when solving the two tasks, or interpreting the output of such functions. For example, when asked about the easiest part when solving Task 1, Ava commented, “finding the partial derivatives and plugging in the values of x and y [i.e., the point (1,2)] was the easiest part.” All the students stated that interpreting partial-derivatives in real-world contexts such as in the context of Task 2 was the hardest part when solving the two tasks. For example, in response to the question about the challenging part when solving the two problems, Cody said, “interpreting partial derivatives in real-world context.” He added, “we have not done problems like this in class.” The other students expressed similar sentiments about the opportunity they had to learn about partial derivatives in real-world contexts during course lectures.

Discussion and Conclusions

Overall, findings of this study indicate that students have a weak understanding of partial derivatives, especially when applied in real-world contexts. To some extent, the students’ difficulties with interpreting partial derivatives observed in this study reflect the opportunity they had to learn about applications of partial derivatives in real-world contexts through classroom instruction or homework assignments. As noted by one of the students (Cody) above, the students were not exposed to applications of partial derivatives in real-world contexts through classroom instruction. In fact, the presentation of partial derivatives during course lectures closely followed the presentation of partial derivatives in the course textbook (Stewart, 2016). We argue that there is a limited number of practice problems on interpreting partial derivatives in real-world contexts provided in the textbook. Specifically, the textbook (Stewart, 2016) contains a total of 114 practice problems, consisting of 9 examples in the expository section on partial derivatives and 105 exercises at the end of the aforementioned section. Of these practice problems, only 13% (i.e., 15 problems) involve interpreting partial derivatives in different contexts, namely physics, mathematics, biology, and economics.

Findings of several studies suggest that mathematics instructors often closely follow course textbooks in their presentation of content during course lectures (e.g., Alajmi, 2012; Begle, 1973; Kolovou, van den Heuvel-Panhuizen, & Bakker, 2009; Törnroos, 2005; Reys, Reys, & Chavez, 2004; Wijaya, van den Heuvel-Panhuzen, & Doorman, 2015). For instance, Reys et al. (2004) argued “that the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn” (p. 61). We recommend that calculus instructors supplement the examples and practice problems given in calculus textbooks to include more examples and exercises with applications of partial derivatives in different real-world contexts in order to maximize students’ opportunity to learn from such tasks which are rare in the course textbook used by the students in this study. In addition, calculus textbook authors need to include more examples and exercises on applications of partial derivatives in real-world contexts to help students gain a deeper understanding of partial derivatives. Future research might examine the opportunity to learn about partial derivatives provided by classroom instruction and calculus textbooks in the United States.

Partial derivatives have numerous applications in many fields of study such as in thermodynamics, electrostatics, and magnetostatics (Thompson et al., 2012). Future research might involve interviewing engineering instructors to determine the meanings/interpretations of the partial derivative (e.g., slope of a tangent plane or rate of change) they would want students to master upon completing a multivariable calculus course, and prior to enrolling in an engineering course such as electromagnetics for which knowledge of multivariable calculus is expected. One student (Ava) confused a partial derivative (a rate quantity) with an amount quantity in a diet and exercise context. Many studies have reported similar results in other real-world contexts (cf., Mkhatshwa, 2019; Rasmussen & Marrongelle, 2006).

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Using RME to Support PSTs' Meanings for Quadratic Relationships

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Quadratic relationships are an important topic in algebra through calculus. In this report, we describe how a task designed using Realistic Mathematics Education (RME) principles supported students in developing meanings for quadratic relationships via their covariational reasoning. After describing the task design and sequence, we present student activity highlighting shifts in their meanings before, during, and after a six-week teaching experiment. We highlight the role of interactions amongst students, task, and the teacher in supporting such shifts. We conclude with a discussion and areas of future research.

Keywords: Quadratic Relationships, Realistic Mathematics Education, Pre-Service Teachers, Covariational Reasoning

Quadratic relationships are an important topic in algebra through calculus. However, much of literature examining students' meanings for quadratic relationships point to misconceptions rather than accounts of students' developing productive meanings for quadratic relationships (e.g., Zaslavsky 1997; Lopez et al., 2016). The few studies (e.g., Ellis & Grinstead, 2008; Ellis 2011) providing accounts of students' developing productive meanings for quadratic relationships via their covariational reasoning examine the activity of middle and high school students. We build on this research by examining the possibility of engendering similar meanings in pre-service teachers (PSTs), which is an important population as their meanings are likely to impact their future students' meanings (Thompson, 2013; Thompson, 2016). In this report, we address the research question, "How can tasks designed using Realistic Mathematics Education (RME) design principles support students in developing meanings for quadratic relationships?"

Theoretical Perspective: Covariational Reasoning and Quadratic Growth

We ascribe to Thompson's (1994) characterization of quantitative reasoning as a person's analysis of situations into quantitative structures which include a network of quantities and quantitative relationships. Relevant to this report, quantitative reasoning can involve both numerical and non-numerical reasoning (Johnson, 2012), but the essence of quantitative reasoning is non-numerical, having more to do with the comprehension of quantities in a situation and how they relate to each other (Smith & Thompson, 2007).

Extending Thompson and colleague's characterization of quantitative reasoning, Carlson et al. (2002) defined *covariational reasoning* to be the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). They described five mental actions students may engage in when reasoning covariationally that allow for a fine-grained analysis of student activity. The mental actions include coordinating *direction of change* (area and height both increase; MA2), *amounts of change* (the *change* in area *increases* as height *increases in equal amounts*; MA3), and *rates of change* (area *increases at an increasing rate* with respect to height; MA4-5). Several researchers (e.g., Paoletti & Moore, 2018; Moore, 2014) have highlighted how engaging in these mental actions can support students in developing productive meanings for various secondary and post-secondary mathematical ideas.

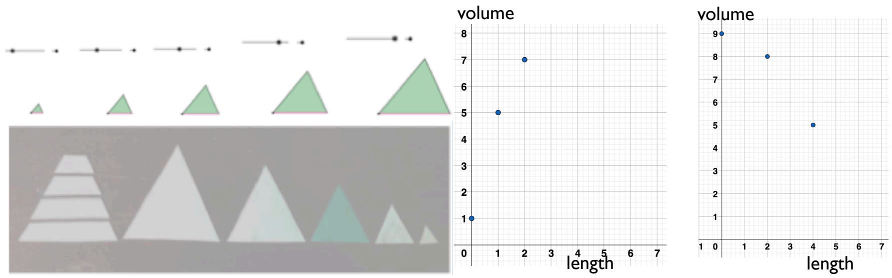
Because our goal was to support students in developing meanings for quadratic relationships via their covariational reasoning, we first intended to support students in reasoning about the amounts of change of one quantity with respect to a second. After this, we hoped to support students in identifying the amounts of change of one quantity increase by a constant amount for equal changes in the second quantity, which is the defining characteristic of quadratic relationships (Ellis, 2011; Lobato et al., 2012). Ellis (2011) highlighted how middle-school students are capable of developing such understandings via their covariational reasoning. We were interested in examining the possibility of supporting PSTs developing similar meanings.

Realistic Mathematics Education: An Outline with Examples

RME is an instructional theory for mathematics (Van den Heuvel-Panhuizen & Drijvers, 2014) that aims to connect what students already know to what they do not know yet (Gravemeijer, 2008). RME focuses on opportunities for students to re-invent mathematics by organizing *experientially real* situations (Cobb et al., 2008; Gravemeijer, 2008); tasks are *experientially real* to students if they can engage in a personally meaningful mathematical activity and need not refer to some ‘real-world’ situation or context.

Horizontal mathematization refers to the process of a student engaging in and developing understandings for a constructed experientially real context. These initial *models of* (Linchevski & Williams, 1999) the situation(s) are specific to the context but should support students in developing informal strategies and representations that will be useful as they begin to generalize to other situations or contexts. After developing models of situations via horizontal mathematization, students can begin the process of elaborating and extending their informal mathematical activity, which is referred to as *vertical mathematization*. Gravemeijer and Doorman (1999) referred to vertical mathematization as “mathematizing one’s own mathematical activity” (p. 117), which may involve making a drawing, defining, or using conventional notations (Cobb et al., 2008). Vertical mathematization should result in a shift to *models for* a mathematical idea.

We designed the *Growing Triangle Task* (Figure 1 (a, top), https://ggbm.at/t6d63cun) with RME principles and our theoretical perspective in mind. In this task, we have students interact with a dynamic Geogebra applet showing an apparently smoothly growing scalene triangle with the intention of providing students an experientially real context to begin to mathematize. To support the students in attending to area and one other quantity (i.e. reason covariationally), we highlighted the base length of the triangle in pink. The base length grows (apparently) smoothly as the longer slider increases (apparently) smoothly. Further, with amounts of change (Carlson et al., 2002) in mind, we included a second smaller slider which allows students to increase the increment by which the pink base length increases (e.g., to larger equal chunks versus apparently smoothly). Finally, we provided students with several paper cut-out manipulatives (Figure 1(a, bottom)) including five triangles with equal changes in base length as well as trapezoidal cutouts that represent the amounts of change of area for equal increases in base length. We intended for these manipulatives both to support students in constructing an experientially real context and non-numerically examining how the amounts of change were changing (i.e. to support students in developing a *model of* the area growing such that it has a constant second difference).



x	y
1	3
2	12
3	27
4	48
5	75

(a) (b) (c)

Figure 1.(a) (top) Several screenshots of Growing Triangle Task and (bottom) Paper Manipulatives, (b) two graphs with three coordinate points used during in class session, and (c) a table of values task used during pre-interviews

After the PSTs had the opportunity to identify constant second differences in the growing triangle task, we presented them with several graphs representing imaginary length-volume situations, each with three coordinate points (see Figure 1(b) for examples) and the instructions "We know the **differences in the amounts of change of volume are constant** with respect to the length of a side. Complete each graph." Our goal in this activity is to support students in beginning the process of mathematizing their own mathematical activity.

After providing students with several opportunities to represent covarying quantities that had a constant second difference, we were interested in exploring the extent to which the students may identify such a relationship as quadratic. If they did not, we intended to define such a relationship as quadratic. Hence, we intended for the results of the students' vertical mathematization to be a *model for* quadratic relationships such that the defining characteristic was the amounts of amounts of change of one quantity being constant with respect to equal changes in the second quantity.

Methods, Data Corpus, and Analytical Perspective

In order to better understand PSTs developing meanings for quadratic (and other) relationships via their covariational reasoning, we conducted a six-week whole-class teaching experiment (Cobb et al., 2003). Prior to the teaching experiment, we conducted four task based clinical pre-interviews (Clement, 2000; Goldin, 2000) which allowed us to examine several PST's initial meanings for quadratic growth. We interviewed all four PSTs who volunteered for about 45 minutes. Additionally, we conducted post-interviews with two of these PSTs to examine shifts in each PST's meanings for quadratic growth.

The teaching experiment occurred in a course that was specifically for PSTs and covered both mathematics content and pedagogy. There were 19 PSTs enrolled in the course, each of whom had taken a calculus sequence and several additional advanced mathematics courses. Each session of the teaching experiment lasted 75-minutes, with the majority of time spent with students working in small groups on large whiteboards. After small group work, the course instructor (CI) led whole class discussions to present various groups' work and to summarize key concepts. We video-recorded one group during group work and captured all whole class interactions. Class content prior to the two sessions addressing quadratic growth included activities intended to support students in reasoning covariationally, including the Ferris Wheel Task (Moore & Carlson, 2012; Moore, 2014) and Power Tower (Moore et al., 2014).

To analyze the data, we adopted a social constructivist perspective (Vygotsky, 1978; Ernest, 1998). Principally, Vygotsky (1978) accepted that sense-making happens on an individual level but asserted the social interaction which initiated the individual's sense-making should not be ignored. As such, social constructivism does not propose to explain an individual's cognitive activity that occurs during the learning process but rather proposes to take the individual-in-social-action as a unit of analysis to better understand additional facets of the individual learning

(Cobb, 1994). Particular to this report, we are interested in examining the ways in which the interactions among the students, teachers, and tasks may have supported students developing more sophisticated meanings for quadratic growth.

With respect to student-to-teacher interactions, the teacher serves as a facilitator, or more knowledgeable other (MKO), in the learning process (Ernest, 1998; Vygotsky, 1978). As the teacher observes student activity, she is responsible for providing learning opportunities to students during classroom discourse (Bozkurt, 2017). Further, as the MKO, a mathematics teacher's responsibility includes enculturating students into a broader mathematical community (Cobb, 2007), including supporting students in developing meanings for ideas consistent with the larger community so they can be participants in this community.

Results

Results from the Pre-Clinical Interviews

During each of the four pre-interviews, none of the PSTs maintained meanings that supported her in identifying quadratic change when prompted to identify the relationships represented in a table. As an example, consider Jennifer's activity addressing the table in Figure 1(c). She first determined the slope between the first two and second two points to determine the relationship was non-linear. She then continued, "Not sure if this means anything but I'm pretty sure all the *y*'s are multiples of threes… and since they're rapidly increasing maybe it can be exponential growth." In this and the other cases, each PST's meanings did not support her in identifying quadratic growth from a table. Hence, each PST's meanings did not support them in identifying second difference to characterize a relationship as quadratic.

Results from the In-Class Teaching Sessions

The Growing Triangle Task: Horizontal Mathematization. Four weeks after the pre-interviews, the class instructor (CI) presented the whole class with the series of tasks around the Growing Triangle Task. He provided them a link to the task, and prompted them to "Sketch a graph representing how the green area of the triangle changes with respect to the base (in pink).". During all group work, we video-recorded Jennifer and Rebecca as they worked together with a teacher-researcher (TR) who was not the CI. The pair quickly identified the area as increasing as the pink base length increased (MA2). When the TR prompted them to consider how the amounts of change of area were changing for successive equal changes in base length, both PSTs quickly responded "by more" each time (MA3). Justifying their conclusion, and reasoning non-numerically, the PSTs pointed to the successive trapezoids (Figure 2(a)) and stated these represented the "area added" to make the next triangle. Attempting to support them in describing how the amounts of change were changing (i.e. examining second differences), the TR asked, "Are they [*pointing to the trapezoids*] increasing by more, less, or by the same amount?" Responding, Rebecca overlaid the trapezoids (Figure 2(b) and (c)), and again reasoning non-numerically, used her fingers to denote the amounts of change in area from one trapezoid to the next, and stated the trapezoids "are all increasing by the same [amount]." The TR, as the MKO, facilitated and provided students opportunities to reason covariationally about an experientially real situation and to identify the constant second difference with respect to base length. Hence, we infer the combination of a carefully designed task sequence along with a TR, provided Jennifer and Rebecca opportunities to develop more sophisticated meanings for quadratic relationships.

(a) (b) (c) (d) (e)

Figure 2. Trapezoids (a) one below the other (b) overlaid and (c) recreated showing equal amounts of change, (d) the table in the on-line homework, and (e) the student work during micro-teaching.

After each of the groups constructed and graphically represented the relationship between the growing triangle's area and base length, the CI led a whole class discussion summarizing important findings intending to support the students in horizontal mathematization. Specifically, he highlighted the amounts of change of area increased for successive increases in side length and that the amounts of amounts of change increased by the same amount. Further, he presented some groups' graphs, which accurately reflected this relationship prior to class time ending.

Addressing the second task: Beginning Vertical Mathematization. Intending to support the PSTs in starting the vertical mathematization process (i.e., beginning to formalize and generalize their activity), the CI began the next class by having PSTs determine new points in several graphs representing volume and length values as described above (e.g. Figure 1(b)). As part of this process, we intended for the students to have an opportunity to consider how to operationalize their non-numeric reasoning about constant second differences with the trapezoids (e.g. Figure 2(c)) to a situation with numeric values.

As the PSTs engaged with this task, they used numeric constant second differences to determine new points (i.e. further mathematize their own mathematical activity). Specifically, they made tables of values from the given points, identified equal changes in the Length values, and then identified first and second changes in the given Volume values. Based on the CI's observations, most groups used the constant second difference to determine the next first difference prior to determining the next volume value. We take such activity as indicative of the students beginning the process of vertical mathematization.

Although students were able to use the constant second differences, none of the students, often when prompted, made any conclusions regarding the type of relationship represented by the points. Hence, after engaging in a whole class discussion in which groups described how to find additional points from the given points, the CI, as MKO, attempted to enculturate the PSTs into the broader mathematical community by defining a relationship with a constant second difference as representing a quadratic relationship.

CI: [*after the students have described how to plot additional points and connected these points*] Have you ever seen things that looks like this before? Yes? No? What do they look like?

S1: [Inaudible] negative parabola?

CI: A negative parabola? Are the other two [*pointing to the graphs created for two other situations*] parabolas?

S2: Pieces.

CI: Could be pieces of parabolas?

S3: Half a parabola?

CI: And so I actually put those first two [*pointing to two graphs*] in there to see if someone commented on the general shape. If you are just looking at the first two [*graphs*], what could they be? Could be half parabolas or could they be anything else?

S4: Exponential?

CI: Could be something more exponentially? … So, it turns out, whenever the amounts of change, change constantly… basically you have +2, +2, +2 for that second difference… then the relationship is quadratic. That's what it means to be quadratic.
[*The CI then presented a slide defining "Whenever the* ***amounts of change*** *change constantly (i.e., we have a second constant difference), the relationship is* ***quadratic"***].

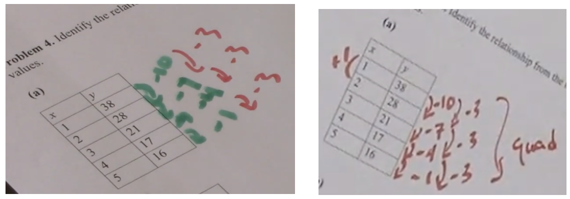
Whereas prior to this whole class interaction none of the PSTs identified the relationships as quadratic, and none of the four PSTs in the pre-interview identified a given relationship represented tabularly as quadratic, immediately after this classroom interaction there were indications students had begun to reorganize their meanings for quadratic relationships. Specifically, in an on-line homework due two weeks after this interaction, we asked the students to identify the type of relationships shown in Figure 2(d) (with options linear, quadratic, cubic, or not enough information). Ten of the 19 PSTs correctly identified the relationship as quadratic on their first attempt. An additional four PSTs correctly identified the relationship on their second attempt. We take this data to indicate many of the PSTs had engaged in vertical mathematization as they generalized properties of the relationship they inferred in the Growing Triangle Task (i.e. their horizontal mathematization) to quadratic relationships more generally, and began to understand constant second differences as a defining characteristic of quadratic relationships. Hence, we infer the classroom interactions (including among students, TR, CI, and the task) supported many students in developing more sophisticated meanings for quadratic growth.

Student activity after the Growing Triangle Task: Evidence of Vertical Mathematization. We present data from a later classroom session and from the post interviews to provide evidence of the lasting impact of shifts in the PST's meanings. Four weeks after the last in-class teaching session addressing quadratic relationships, most of the PSTs were working in groups pretending to be high school students as a pair of their peers were acting as teachers (i.e. micro-teaching). The 'students' were working on a task asking them to identify the number of distinct regions in the plane created as they added non-parallel lines, no three of which were collinear. Two of the PSTs (Sonya and Nina) were working together to create the table representing the number of lines (L) and regions (R). They created the table shown in Figure 2(e) then quickly moved to identifying first and second differences. Immediately after writing the '1' representing the second difference between the first differences 3 and 4, they said:

N: So our change, our change of our change is constant. So what does that mean?
S: [*interjecting*] It's a quadratic equation.
N: [*continuing*] It's a quadratic equation!

Both Sonya and Nina immediately identified a relationship with a constant second difference as quadratic. We take this as evidence that each student's meanings for quadratic growth at this point in the semester included this defining feature of quadratic change.

As additional examples, Jennifer and Rebecca each maintained meanings that supported her in determining if a relationship, represented tabularly, was quadratic in individual post interviews. When prompted with the table in Figure 3(a) and asked to find the relationship, Jennifer immediately mentioned she needed to find the first and second differences. As she was finding the first differences (in green) and second differences (in red), Jennifer stated, "It's a quadratic, uhhh… yes, quadratic. 'cause the amounts of amounts of change are the same. Quadratic!" Similarly, Rebecca quickly identified first and second differences (Figure 3(b)) to determine the relationship was quadratic.



(a) (b)

Figure 3. Task in Post-Clinical Interview. (a) Jennifer's work (b) Rebecca's work

Discussion, Implications, and Future Research

We described a series of tasks designed with RME principles in mind aimed at supporting PSTs' developing more sophisticated meanings for quadratic relationships via their numeric and non-numeric covariational reasoning. We described how the students constructed an experientially real situation, then began the process of horizontal mathematization as they reasoned non-numerically about amounts of change and constant second differences in the *Growing Triangle Task*. The students initially developed a model *of* a situation which the PSTs further mathematized as they addressed additional tasks. As shown in the later interactions, for many students, the initial model *of* the Growing Triangle transformed to a model *for* quadratic change, which is the result of vertical mathematization (Gravemeijer & Doorman, 1999; Linchevski & Williams, 1999). Hence, we add to the growing body of literature (Paoletti & Moore, 2018; Moore, 2014) showing ways PSTs can develop productive meanings for quadratic relationships via their covariational reasoning.

In this paper, we focused on the ways in which a carefully designed task can be implemented in a whole class setting by one or more MKOs to support PSTs in developing more productive meanings for quadratic relationships. These findings lead us to suggest several different areas for future research. First, a limitation of this study was that we conducted it with one class in one setting. Future researchers may be interested in using similar tasks with PSTs in other settings to examine the efficacy of this instructional sequence in these settings. Second, as we adopted a social constructivist lens, we do not make claims regarding the underlying reasons for the observed shifts in students reasoning (e.g., the mental operations that support students in achieving such shifts). Future researchers may be interested in using these or similar tasks as a means to explore PST's mental operations to develop models of such changes. Such research may support further task design, development, and implementation.

Finally, we note that our tasks are similar to those used by others (Ellis, 2011; Lobato et al., 2012) but that our tasks first supported PSTs identifying constant second differences non-numerically, which is the essence of quantitative reasoning (Smith & Thompson, 2007). We conjecture such reasoning may also be productive for middle or secondary students first learning about quadratic growth. We call for researchers to consider using tasks similar to those presented here as a means to further explore ways to support these students in developing foundational understandings of quadratic growth. Hence, we intend for these tasks and results to serve as a starting point for additional investigations into students of various levels developing foundational understandings of quadratic growth via their quantitative and covariational reasoning.

Acknowledgements

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Quantitative Reasoning and Symbolization Activity: Do Individuals Expect Calculations and Expressions to Have Quantitative Significance?

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This paper describes ways of thinking students employ when they choose to use calculations or produce algebraic expressions to respond to mathematical tasks and their expectations regarding the meanings of what they produce. My findings suggest that students' reasoning in symbolization activity is often guided by perceptual features of tasks, such as the numbers explicitly given in prompts and key words students identify. I describe the construct "emergent symbolization" as a potentially productive way of thinking in symbolization activity based on synthesizing prior work and the results of this study. I close by making connections between similar work in analyzing students' ways of thinking about graphs and discussing how my work contributes to the common instructional goal of promoting connections between representations.

Keywords: quantitative reasoning, student thinking, mathematical representations

Math standards commonly include the goal that students make connections between representations of mathematical relationships. For example, The American Association of Two-Year Colleges (2018) standards state that "Students should be provided opportunities to represent and communicate mathematical ideas using multiple representations such as numerical, graphical, symbolic, and verbal" (p. 25). However, intending that students make such connections does not guarantee they occur. Whether a student is successful in recognizing what a representation conveys about how two quantities are related and changing together relies on such things as the ways of thinking she engages in and the connections she is positioned to make.

Moore and Thompson (2015) studied students' ways of thinking about graphs and described advances derived from taking seriously how students interpret the meaning of graphs. Their work highlights complexities in students' reasoning about one kind of representation (graphs) and suggests potential benefits for students who envision graphs as emerging from coordinating pairs of covarying quantities they have conceptualized for a given context. In this paper I report results from studying students' reasoning about symbolization activity (the production of mathematical expressions and formulas that use operations and algebra to represent relationships). My results suggest that a key component of students' fluency and flexibility in representing relationships algebraically is also the degree to which they conceptualize quantities and quantitative relationships in a situation and foreground their activity based on these conceptualizations.

It seems likely that students inclined to engage in quantitative reasoning (Thompson, 1990, 1993, 1994, 2011), with a focus on coordinating relationships between quantities including using variables and variable expressions to represent the values of quantities, are well-positioned to make productive connections between mathematical representations. However, testing this hypothesis, and understanding the mechanisms by which students successfully make connections between representations, requires continued empirical studies designed to recognize and articulate students' ways of thinking about representations. This paper contributes to that work.

Theoretical Perspective

Quantitative reasoning (Thompson, 1990, 1993, 1994, 2011) describes a way of reasoning involving identifying objects' measurable attributes (*quantities*) and conceptualizing relationships between quantities. *Quantification* is the process of deciding on a measurement system for producing quantities' values. Quantities exist within the mind of an individual, and thus when talking about quantities, quantitative relationships, quantification, and so on, I take the position of describing them from an individual's perspective (either actual or hypothetical). As a lens for considering students' meanings and reasoning, I use Thompson and Harel's (Thompson, Carlson, Byerley, & Hatfield, 2014) descriptions that are extensions and elaboration of Piaget's (2001) genetic epistemology. An individual's *meaning* for an idea describes the set of implications inherent to the scheme(s) triggered by an in-the-moment *understanding* of a context or stimulus. Meanings may be temporary or they may be stable over time. A *way of thinking* describes a pattern in how an individual uses certain meanings. Descriptions of students' ways of thinking are useful for characterizing mental processes essential for specific types of mathematical reasoning (e.g., quantitative reasoning, proportional reasoning).

Emergent Symbolization

In O'Bryan and Carlson (2016) we described our work with a teacher (Tracy) involving a professional development intervention, classroom observations, and clinical interviews. Our analysis suggested a set of expectations she developed about the meanings for the calculations she performed and algebraic representations she generated that she independently established as mathematical learning goals for her students. These expectations included explaining what quantity she intended to calculate (represent) before performing a calculation (writing an expression) and how the order in which calculations are performed or the order of operations used to evaluate expressions reflects details about how individuals conceptualize a situation.

I have since encapsulated aspects of my understanding of Tracy's expectations and ways of thinking into the construct *emergent symbolization (or emergent symbol meaning*) (O'Bryan, 2018). Emergent symbolization refers to actions motivated by one or more of the following expectations.

1. An expectation that performing calculations or generating expressions should reflect a quantification process for quantities that the individual conceptualizes.
2. An expectation that demonstrating calculations and producing expressions are attempts to communicate an individual's meanings. Thus, when given a set of calculations or expression/formula, we can hypothesize how the individual conceptualized a situation based on analyzing the products of their reasoning.
3. An expectation that the order of operations used to perform calculations, evaluate expressions, and solve equations "reflects the hierarchy of quantities within a conceptualized quantitative structure" (O'Bryan, 2018, p. 234).

I hypothesize that emergent symbolization is a potentially productive way of thinking for students and teachers and that targeting its development is a worthwhile instructional goal. Part of the important initial work in testing this hypothesis is describing nuances in how students reason about, and their expectations regarding, performing calculations or generating algebraic expressions in response to mathematical contexts and prompts.

Research Questions

Given that students develop stable ways of thinking over time, understanding nuances in these ways of thinking is important for educators and researchers. The work reported here focuses on results from considering two research questions.

1. When a student chooses to perform a series of calculations or generate an algebraic model to respond to a task, what does the student believe her work represents?

2. Do students' ways of thinking reflect an attempt to organize mathematical processes according to generalizations of quantitative relationships or is there a different set of expectations driving their mathematical activity?

Methods

During Spring 2018 seventy students enrolled in a precalculus course at a large public university in the United States using our research-based online materials (O'Bryan & Carlson, 2016). All students completed a pre-/post-test including multiple-choice items from the APCR and CCR exams (Madison, Carlson, Oehrtman, & Tallman, 2015) at the beginning and end of the course. In addition, I selected five students (Gina, Lisa, John, Marcus, and Shelby) for recorded clinical interviews (Clement, 2000) representing a wide range of majors and pre-test scores. I conducted interviews with each student at the beginning of the course and during the last month of the course using the same subset of questions from the pre-/post-test and the same interview protocols. During these interviews the multiple-choice options were hidden. I coded student responses to identify when students did or did not reference quantities, units, and quantitative relationships as a foundation for their calculations and symbolization activity.

Results

I share the following results from pre-/post tests and interviews, and to highlight common expectations and ways of thinking I observed. In this paper I am not focusing on shifts that may have resulted during the course, but rather on characterizing stable meanings and ways of thinking students exhibited.

Pre-Test and Pre-Interview Results

Students answered the Tomato Plant A task in Figure 1 on the test and during interviews. I chose this task to assess students' recognition of the need to update the reference quantity from one interval to the next (that is, update the answer to the question, "Percent of what?").

A tomato plant that is 4 inches tall when first planted in a garden grows by 50% each week during the first few weeks after it is planted. How tall is the tomato plant 2 weeks after it was planted?

a. 5 inches b. 6 inches c. 8 inches **d. 9 inches** e. 12 inches

Figure 1. The Tomato Plant A task (Madison et al., 2015).

On the pre-test, 54% of students selected the correct response. During interviews, three of the five students answered the question correctly, and all responses contained two similarities. First, all students focused on the amount to *add* each week and none of them described the relative size of the new height compared to the old height when explaining their reasoning. Second, all students described a percentage value as a number that results from moving a decimal place two positions rather than describing a measurement process. Of the two students who answered

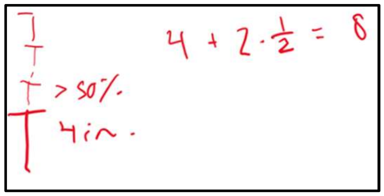

Figure 2. Marcus's work for the Tomato Plant A task.

incorrectly, Marcus' response proved interesting relative to later observations. See Figure 2.

Marcus's answer (8 inches) assumed that the weekly 50% increase was always based on the plant's initial height. But what is most interesting is the expression he used to represent his answer. Apart from being an incorrect statement (which he recognized), all five students utilized the structure "initial value + (change in independent value)(1-unit percent change as a decimal or fraction)" to create a general model in the Tomato Plant B task. See Figure 3.

> José plants a 7-inch tomato plant in his garden. The plant grows by about 13% per week for several months. Which formula represents the height h of the tomato plant (in inches) as a function of the time t in weeks since it was planted?
>
> a. $h = 7(0.13)^t$ b. $h = 7 + 1.13t$ c. $h = 7(1.13t)$ **d. $\underline{h = 7(1.13)^t}$** e. $h = 7 + 0.13t$

Figure 3. The Tomato Plant B task (Madison et al., 2015).

During the pre-test, 20% of students selected the correct answer, but <u>none</u> of the students interviewed provided the correct response. In their interview responses, all five students generated a formula equivalent to "$h = 7 + 0.13t$" where h is the tomato plant's height (in inches) and t is the elapsed time (in weeks) since the initial height measurement. The important observation here is not that students failed to provide (or select) the correct algebraic model. Rather, *the most important result is what their reasoning tells us about their expectations regarding the quantitative significance of the calculations they perform and the expressions they write.* Figure 4 shows Shelby's work in completing the Tomato Plant B task.

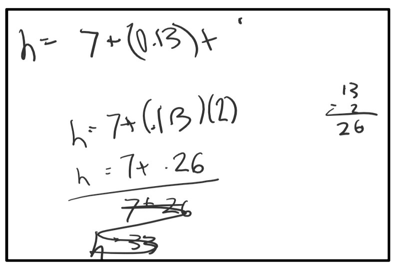

Figure 4. Shelby's work on the Tomato Plant B task.

Shelby: Okay so we're trying to find the height [*she underlines "height h" in the problem statement*] and then t is gonna be a variable [*she underlines "t in" in the problem statement*]. So height [*she writes "h="*] and then we start with seven [*she writes "7"*]. So we start at seven, so that's just gonna keep increasing [*she writes "+" after h = 7*] so and then put a plus. Um [*she writes "(0.13)t" after h=7+*]. Yeah, I'm gonna go with that.

At this point I asked Shelby to determine the plant's height after two weeks. She evaluated her formula but hesitated after writing "7+.26". She considered that maybe her answer was wrong and tried "7+26" based on moving the decimal two places as an alternative. She decided that "33" was too large, so her original answer must be correct and settled on "7.26" as her answer.

In Shelby's final answer we see a form similar to what appeared in Marcus's response to the Tomato Plant A task, and several aspects of her response appeared throughout interviews with all students. First, Shelby was not inclined to check the reasonableness of her solution without prompting. Second, she appeared to be using keywords to guide her work (such as "increase" means "add"). Third, her criteria for judging the accuracy of her solution was based on whether the values it produced seemed reasonable. None of the students explained how parts of their algebraic representations modeled relationships between quantities within the situation or explained the meaning of the term "0.13t"

Consistently in the Tomato Plant B task, as well as similar tasks such as modeling a bacteria colony doubling over set time intervals, students appeared to establish a goal and then create a mathematical representation as a literal translation of this goal from English. See Figure 5.

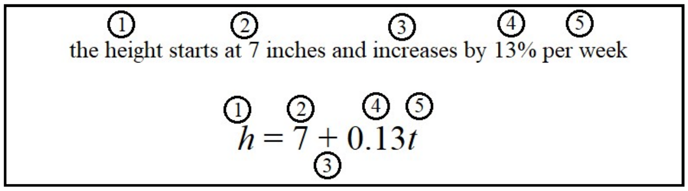

Figure 5. Students may produce the formula h = 7 + 0.13t as a literal translation of a goal statement written in English focusing on key words.

Supporting this idea is the consistent absence in students' responses of any numbers not explicitly stated in task prompts. For example, the correct model for the Tomato Plant B task requires using "1.13", and neither "1.13" nor "113%" appear in the prompt. Similarly, I observed students model a doubling-time situation with a function formula missing the number two when the prompt included the word "double" but did not include "2" in the prompt.

The students interviewed exhibited similar behaviors during symbolization activities throughout the course. Students rarely mentioned units when referencing numbers given in the problem text and rarely referenced the quantities being measured. In addition, students were not inclined to justify their answers unless prompted and, when I did prompt them, judged their models based on whether they produced numerical values the student deemed reasonable rather than referencing the structure of quantitative relationships the student conceptualized.

Post-Test and Post-Interview Results

All five students interviewed successfully completed the class, and all earned at least a "B" on the course final exam. Therefore, based on grades and exam performance, interviewees all met course requirements and objectives. However, data from the post-test, post-interviews, and analysis on students' interactions with course materials shows that the ways of thinking identified in the pre-test and pre-interviews was remarkably stable for many students.

Post-Results for Gina, Marcus, and Shelby. Gina, Marcus, and Shelby all continued to display many of the same ways of thinking during their post-interviews. All three had difficulty explaining how claims they made about features of one task or representation appeared in other tasks or representations and their reasoning often involved generating sets of potential answers then picking from among the results. For example, I asked Gina to determine the percent change from a price of $131 to $195. Gina first evaluated the difference "195 – 131 = 64". She then said, "I kinda remember how to do this" and used a calculator to compute "64/195 = 0.328", then "ans * 100 = 32.8", then "195/64 = 3.047", then "ans * 100 = 304.7", and finally "195/131 = 1.489". After reviewing these results she told me that the price increased by 32.8%. I asked her what that number was a measurement of (what quantities were being compared). Rather than answering my question, she computed "131/195 = 0.672" and said that her answer should have been "67%". "Sixty-seven percent of what?" I asked. She responded "131".

Analyzing these students' behavior during course lessons showed that all three tended to require high numbers of attempts to complete tasks in the lessons I examined. When their initial attempt was incorrect, they required seven or more attempts over one-third of the time, and nearly 20% of those instances they required 11 or more attempts before finally completing or abandoning tasks. For example, one lesson task asked students to determine a percent change from one value of a quantity to another value. The correct answer was 40%, but Shelby entered 140%. She then completed *19 additional attempts* in quick succession (240, 1.40, 139, 239, 299, 399, 499, 199, 240, 1.4, 39.9, 29.9, 49.9, 2, 200, 100, 140, 14, and 40) before finally entering the correct answer. In these instances the students were not coordinating their reasoning by first conceptualizing the quantities in a situation and the quantitative relationship they wanted to communicate and then considering how to represent or evaluate that relationship.

Finally, all three students tended to revert to the general form "$a + rt$" when completing tasks like Tomato Plant B. While Marcus and Shelby often caught themselves and modified their answers to exponential models, their continued challenges in justifying aspects of their models quantitatively suggest that this might be "pattern-matching" behavior rather than shifts in their expectations for the meaning and goals of generating algebraic representations.

The case of John and Lisa. Both John and Lisa exhibited shifts in describing the quantitative significance for the calculations they performed and the representations they generated. Since the focus of my paper is not on analyzing the results of a teaching intervention, I will not dwell on these changes beyond demonstrating some differences I observed in their reasoning and expectations during the post-test and post-interviews.

Figure 6 shows Lisa's work on the Tomato Plant B task during the post-interview.

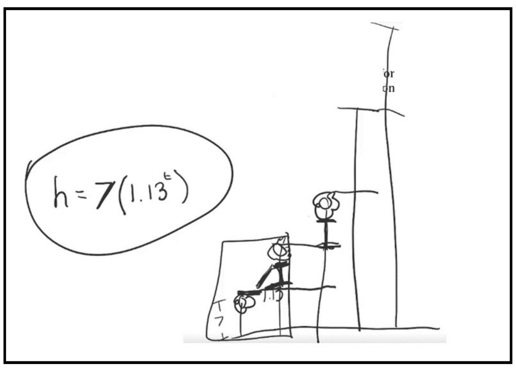

Figure 6. Lisa's work on the Tomato Plant B task (post-interview).

In justifying her work, Lisa provided a clear explanation for the meaning of each value in her model and supported her explanations with drawings where she elaborated on how she imagined the relationship between h and t, as modeled by the formula, also being modeled using diagrams.

An example from John's post-interview is also worth sharing. In the Tomato Plant A task John was the only student who spontaneously represented the plant height as an exponential expression rather than calculating the amount to add each week and performing addition. He wrote "$y = 4 \text{ x } (1.5)^2$" [*"x" representing multiplication*] as his answer and then determined that value with a calculator. In this process he first evaluated $(1.5)^2$ to get 2.25 and multiplied this by four. I asked him if 2.25 was an important number to understand or if it just represented an intermediate step in the calculation process. He explained that 2.25 represented the growth factor for a two-week change in time elapsed. This was an idea never mentioned by the other students I interviewed but seems important in terms of John's fluency with writing and explaining exponential models at the end of the course.

These examples hint at a connection between students' fluency with mathematical representations and the degree to which they expected these representations to reflect their conceptualization of relationships between quantities, at least in familiar contexts.

Discussion

Moore (2016) proposed the distinction between *operative thought* and *figurative thought* (Piaget, 2001; Steffe, 1991; Thompson, 1985) as a useful lens for understanding differences between students' ways of thinking about graphs. Figurative thought describes thought that relies on perceptions directly accessible during the reasoning process. One implication of a person limited to figurative reasoning within a context is the inability to extend beyond that context to organize his thinking relative to general relationships and connections to other contexts and ideas. Operative thought suggests a degree of control and coordination in the reasoning process that extends beyond perceptual features of a given context. An implication of operative thought is that the individual makes conscious decisions throughout his reasoning process and is aware of how work within one context connects to work in other contexts.

I see parallels in Moore's analysis with students' symbolization activity in the answers to my research questions. Students' symbolization activity was primarily driven by their assimilating interpretations of the tasks at hand to schemes where the resulting actions (performing specific calculations, generating algebraic representations, etc.) were not motivated by reflections on quantities, general relationships, or connections to other tasks. Their activity seemed subordinate to perceptual features of the tasks (such as the specific numbers given in a problem statement, their identification of key words, and perhaps even the structure of students' native language). The students whose ways of thinking did not shift during the course struggled to describe quantitative meanings for their calculations, often produced sets of potential responses and picked from among these choices rather than foregrounding their activities in conceptualizing the quantities' values and relationships they wanted to represent, and typically did not or could not make connections between tasks or between different representations.

Emergent symbolization is an example of operative thought "involv[ing] mental representations of actions and consideration of the consequences of those actions that allows students to make propitious decisions about next steps in their reasoning process and how those steps connect to conclusions already made" (O'Bryan, 2018, p. 315). My results do not prove that John and Lisa developed the full set of expectations that drive emergent symbolization reasoning (that would require a different study). They do show that students who were able to explain quantitative meanings for steps in their calculation processes and components of their mathematical expressions were also able to explain how aspects of their calculations or algebraic reasoning related to reasoning in other tasks and might also be represented in other forms.

Moore and Thompson (2015) argued that studying students' shape thinking is critical for providing perspective on how to foster students' connections between the variety of possible representations for mathematical relationships. It is important for "researchers to be clearer about what a graph represents to a student, and thus what students understand multiple representations to be representations of" (p. 784). This paper is an attempt to begin answering this call relative to students' symbolization activity with the long-term goal of better understanding how students see connections between mathematical representations. More work is needed to continue to flesh out the ways of thinking students employ in symbolization activity. I am particularly interested in examining students' symbolization activity in novel contexts to understand the potential implications of students' expectations regarding the models they generate and the meanings they identify in those models in less familiar contexts.

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Development of Students' Shared Understanding in Guided Reinvention of a Formal Definition of the Limit - from Commognitive Perspective

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This paper examines the mechanism behind students' guided reinvention process of a formal definition of the limit of a sequence by looking at how students' shared understanding develops and what discursive rules enable such development using the commognitive approach focusing on the relationship between routines (discursive rules) and changes in endorsed narratives (shared understanding). A Calculus II instructor conducted a teaching experiment in which 11 students reinvented a formal definition of the limit starting from their initial statement about the sequence convergence, mainly including dynamic language such as "approaching." In our analysis, we identified the routines about the application of the definition in examples and non-examples, and the consistency between written and illustrated form of their definition, and the readers of the definition, and how those routines contributed to the changes in students' endorsed narratives about sequence convergence during the guided reinvention.

Introduction

Research has documented students' difficulty in understanding formal limit definitions and attempted to make these definitions more accessible to students through guided reinvention teaching experiments with pairs of students (Oehrtman, Swinyard, & Martin, 2014; Swinyard & Larsen, 2012). This study expands on this body of research by implementing a teaching experiment in class where the instructor asked groups of students to reinvent a formal definition of sequence convergence. Our goal was to examine the mechanism behind students' reinvention by looking at their interactions with instructor's prompts and other students' work in class through the lens of routines and endorsed narrative.

Theoretical Background

Guided Reinvention of a Formal Definition of Sequence Convergence

A typical formal definition for a limit with an illustration can be seen in Figure 1. Several studies have pointed out that this definition involves challenges involving multiple nested quantifiers such as the universally quantified ε, the index N depending ε, and absolute values, and inequalities, and developed instructional approaches that involve students' engagement with such challenges (e.g., Cory & Garofalo, 2011; Swinyard & Larsen, 2012). Some of such approaches have adopted a guided reinvention iterative refinement process (IRP, Oehrtman, Swinyard, & Martin, 2014) approach based on principles from Realistic Mathematics Education (Freudenthal, 1973) in which learners "come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer, 1999, p. 158). During an IRP in creating a sequence convergence definition, students write a definition, evaluate their definition against graphical examples of sequences converging to and not converging to a specified limit value, identify problems, discuss potential solutions, revise their definition, and then initiate another iteration (Oehrtman, Swinyard, & Martin, 2014). Within the IRP, *problems* are defined as conflicts raised by the students between their descriptions and illustrations for the limit versus their stated definition. Once students realize a problem, students discuss potential solutions and their definition eventually evolves to reflect adopted solutions.

During this process of realizing, engaging in, and resolving problems, students develop ownership over the process and product of the IRP.

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

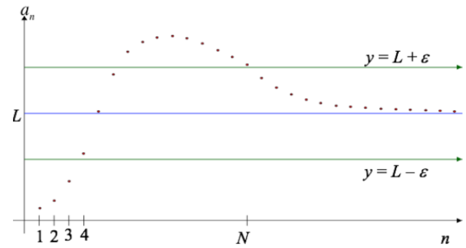


Figure 1: Example of formal sequence convergence definition (Stewart 2008, p. 677)

Endorsed Narratives and Routines

To examine the mechanism behind students' guided reinvention of a formal definition of the limit, we adopted endorsed narratives and routines from Sfard's (2008) commognitive approach. Sfard (2008) defined *endorsed narratives* in discourse as shared "descriptions of [mathematical] objects, of relations between objects, or of processes with or by objects," which the discourse community endorses as true (p. 134). Sfard (2008) defines a *mathematical object* as a collection of realizations sharing the same endorsed narratives and a *realization* as any perceptually tangible, visual or vocal, means of communicating a noun element of an endorsed narrative. In our analysis, we included modifiers of a noun element as realizations due to the important role that modifiers play in the quantification of such elements (e.g., one, any, and every attached to epsilon). Endorsed narratives are evidenced by students' adoption of the narrative as true and their rejection of any violations to the narrative that may get "called out" by the discursive community. During reinvention's IRP, what students create is not only written definitions, but also the problem/solution pairs that explain how the endorsed narratives contributing to the changes in their definition have evolved.

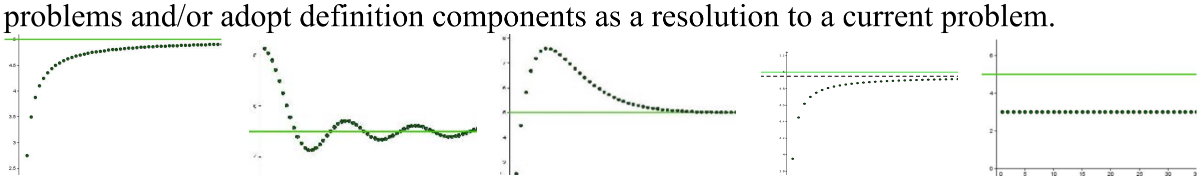
Routines refer to meta-discursive rules (i.e., metarules) that define repetitive discursive patterns in speakers' use of words and visuals, or while creating and endorsing narratives. Metarules are different from object-level rules; object-level rules "are narratives about regularities in the behaviour of objects in the discourse, whereas metarules define the patterns in the activity of" the people who share a discourse, "trying to produce and substantiate object level narratives," which is a form of endorsed narratives (Sfard, 2008, p. 201). This paper adopts this definition of routines as metarules that define, govern, and derive the repetitive discursive patterns, which enabled students' progress in reinventing a formal definition of the limit of a convergent sequence. According to Sfard (2008), there are three types of routines: certain courses of actions, metarules for when a certain course of actions are appropriate to apply, and metarules for when the application of the course of actions is concluded. For example, in a guided reinvention context, a students' course of action could be to engage the IRP – write a definition, test their definition against example and non-example graphs, find a problem and the solution, and revise their definition, and repeat this cycle again. Students could decide to apply this IRP course of action when the instructor asked for an illustration, or after revising their definition. Their course of action could be considered as concluded by the students when they explicitly express their satisfaction with their definition and stop going through revision process. Students could also decide that a course of action is concluded if they observe an instructor approving a change made in their definition or even if an instructor merely re-voices what students said, did, or wrote. Metarules are not specific to certain realizations, but determine certain discursive actions in multiple contexts. Since routines are metarules of discursive actions aiming to produce object-level rules, routines directly impact a generation of or changes in endorsed narratives (Sfard, 2008). Using routines and endorsed narratives as the theoretical lens, we address the following questions:

- What were the routines established during the guided reinvention? How were the routines initiated, established, and adopted by students?
- How did the endorsed narrative change during the guided reinvention? How did the routines support the changes in the endorsed narratives that afforded progress toward a formal definition of sequence convergence?

Research Design

The data for this study was collected from a second-semester calculus class at a medium-sized public university in the U.S., where the instructor conducted a guided reinvention teaching experiment with two groups of four students and one group of three students. The experiment was conducted over five class sessions that ranged from 50 to 75 minutes in length. Students had just completed sections on sequences, series, power series, and Taylor series using Stewart (2008). The activity of one group of four students was audio and video recorded. We refer to the videoed group as GTE, and the other two groups as G1 and G2. Another camera captured whole class interactions with the instructor. We will refer to individual students within GTE using pseudonyms: Becky, Jamie, Trevor, and Rachel. All their written work was collected and digitized.

The teaching experiment began when the instructor asked students to create example and non-example graphs of sequences converging to 5 (Figure 2). Following the creation of these graphs, the students were asked to create a formal definition for sequence convergence by completing statements like "$\lim_{n\to\infty} a_n = 5$ means that …" The students were encouraged to create a definition for sequence convergence to 5 that captured all of their example and excluded all of their non-example graphs. Following the prompt, the instructor guided groups through the IRP. Between the first three teaching experiment sessions, students completed homework by reflecting on their IRP progress during class and attempted individual definitions. The students prepped for the fourth and fifth sessions by viewing other groups' definitions to identify problems and/or adopt definition components as a resolution to a current problem.



Example A Example B Example G Non-example 5 Non-example 7

Figure 2: Selected graphs from 12 graphs used during the TE (Solid lines show 5).

We transcribed all recorded data, created lists of definitions written or verbally stated by individual GTE students or by GTE as a whole, and catalogued GTE's interactions with the instructor and with other groups, their problems, proposed solutions, and resulting changes to their definitions. Then, we focused on changes in their definitions and illustrations, and came up with their endorsed narratives which drove those changes. Then, we examined the interactions and discussions around those changes to identify the discursive patterns and the routines that govern them. Then, we examined how routines contributed in changes in students' endorsed narratives and definitions.

Results

Routines

In our data, the IRP is the course of action, and we were interested in the metarules that determine when this IRP is initiated and applied, and when it is considered as concluded by GTE and the whole class. In our analysis, we found four routines:

Routine 1. A definition should include all examples and exclude all non-examples.
Routine 2. Each component in a written definition should have a place on graphs.
Routine 3. Graphical observation and illustration of a written definition on graphs should be captured in the definition.
Routine 4. A written definition should make sense to everyone.

Routine 1 defines and derives the course of action, IRP, in which students consistently revise their definitions, test it against examples and non-examples, identifying the problems, and coming up with solutions, and iterating the cycle by revising their definition again. Routine 1 not only governs the initiation of an IRP, but also defines a closure for the IRP. We see such a closure when students are in-the-moment satisfied that their definition includes all examples and excludes all non-examples.

Routines 2 and 3 define and derive discursive patterns where students constantly moving back and forth between their written form of definitions and illustration of their definitions. Specifically, Routine 2 defines the discursive pattern of students placing the components in the written definition on example and non-example graphs. This was closely tied to the quantification of the components in the definition (e.g., the word “ultimately” in “ultimately approaching 5” was drawn on graphs). Routine 3 defines the discursive pattern of students examining if their illustration and their observation on example and non-example graphs are captured in their written definition. This was closely tied into the quantification and relationships their illustration reflects based on how and in what order they placed those components (e.g., students always drew ε first, and then identified *N*, but their written definition did not clearly state ε as an independent quantity).

Routine 4 defined and derived students’ discursive pattern of continuously revisiting their use of words and symbols in the definition. For example, in the beginning of the TE, the instructor used a third person to point out the problems with their definitions (e.g., “I could see a student who hadn't seen your definition before, and saying…terms go away from 5, and then they come back and then they go away and come back, that the terms in [example graph] B, the values of them, a_n, they don't always approach 5,” Instructor, Day 1), had groups of students critique each other’s definition by asking them to list problems and suggest solutions, and then groups were to address those critiques. This discursive pattern was identified multiple times in students’ utterances while they express concerns about how they use and relate the terms and symbols, which might not be clear to others (e.g., “The way they [the students in another group] are reading it is that the error bound is coming off the n_c… To us it doesn't say that, but to other people it does”, Rachel, Day 5).

In general, we found a trend that in the beginning of the guided reinvention, those discursive patterns were mainly initiated by the instructor, but as the guided reinvention progressed, those discursive patterns naturally occur within students’ discussion without the instructor being present, which shows that the students adopted the routines that govern those discursive patterns. When a discursive pattern occurred at least three times in students’ discussion without the instructors’ presence, we concluded that students adopted the routine.

Endorsed Narratives

We found four endorsed narratives that led to changes in GTE's definitions:

Endorsed Narrative 1. Dynamic language is insufficient.

Endorsed Narrative 2. Convergence behavior needs to be explained with relevant vertical distances and their relationships.

Endorsed Narrative 3. Convergence behavior involves designating a quantitative relationship between vertical distance and *n*.

Endorsed Narrative 4. Epsilon comes first.

For each endorsed narrative, multiple routines were involved. Due to the limited space, we will discuss the routines that were involved in the generation of Endorsed Narrative 1 in detail and provide a summary for the rest. Endorsed Narrative 1 was initiated with GTE students' problem with dynamic language such as "goes to" and "approaching" used with "terms," "errors," and "error bound" in their definition. Routine 1 was dominantly applied in students' discussion of potential solutions, but Routine 4 was also applied. They mainly used two graphs. Non-example 5 converges to a number less than 5, but still approach 5, and the limit of example B is 5 but its terms do not always approach 5. The inclusion of graph B and subsequent terms in graph B as "not approaching" and the exclusion of graph 5 and its terms "approaching," provided the impetus to produce conflict for the students between the students' use of "approaching" within their current definition and Routine 1. The following excerpt from Day 1 shows an example of one such conversation involving the problems initiated first by the instructor [1,3], and then by GTE students [4, 6, 11, 13], and GTE students' proposed solutions to the problems [2, 5, 6, 12, 14]. This excerpt also shows the GTE students' repeated narrative of rejection of related dynamic language coordinated with testing such language against the necessity of capturing graphs B as convergent ([4, 11,13]). Routine 4 was also applied first by the instructor pointing out this conflict using a third person [3], and then by the students when they were deciding the word for the distance between the terms and the limit as "error" or "distance" [7-10].

1 *Instructor*: They're [pointing to dots in Example B] not approaching, they're getting further away [Becky: nodding while pointing to the same part of Example B].

2 *Becky*: That's what the ultimately is for.

3 *Instructor*: I could see a student who hadn't seen your definition before saying … since you described it as terms go away from 5, and then they come back and then they go away and come back, that the terms in B, the values of them, a_n, they don't always approach 5…[going away]

4 *Jamie*: He is right, it's like not approaching 5 right here [pointing to Example B]

5 *Trevor*: So if we said something about the distance between a_n and 5, instead of approaches?... I know it goes further from here [showing distance with fingers on Example B where errors are increasing] but then ultimately there's never a distance that far away again. [pointing to a minimum value]

6 *Becky*: I mean that's kind of the idea that the error is decreasing

7 *Becky*: but we're not supposed to say that, right? Definitions?

8 *Jamie*: Cause I don't know if. So we're telling this to like dumb people, right?

9 *Rachel*: Yeah, people who don't understand.

10 *Jamie*: So we can't say error. We just have to say distance.

11 *Becky*: Well that's not true either cause the distance is increasing at the last point.

12 *Trevor*: It increases, increases right here [pointing to the 1st minimum], but it's never going to be further away than that. So ultimately, the distance is still getting shorter, so if ultimately worked for G then I feel like it should work for B.
13 *Becky*: I'm not sure if that's true either. Like this one is closer to 5 than that one [where the dots are going further away from 5]. And that's as n is increasing.
14 *Trevor*: [continuing his sentence] Period.

To resolve the initial conflict, GTE students introduced several new words to their definition. First, they considered the word "ultimately" ([2 ,5, 12]), but later stated that "ultimately" was not a solution as they came to realize a non-terminating pattern of terms "going away and coming back" again and again in Example B (e.g., Rachel: "It's still the same thing because the value is going away and it comes back and it goes away again and it comes back. Ultimately, obviously not cover that," Italics added, Day 2). The word "ultimately," later evolves to an n-value (equivalent to N in a formal definition, Figure 1) to rule out bad early behavior of the terms when Routine 2 was applied. Then, the distance between the term and 5 ([6, 11]) was considered and used with "ultimately," which Becky called an "error." With the observation of the behaviors of the terms in Example B ([5, 12]), which "increases, increases…but never going to be further away than" the first extreme value of the term ([12]), they also introduced the word, "period" as a solution ([14]). The "period" was used to establish an error bound dependent upon the maximum value of the terms within each oscillation of the sequence [13], which later they rejected by applying Routine 1 about the definition capturing all the example graphs because "not all of the graphs have periods" [Trevor, Day 1]. Although introducing those words did not directly resolve the conflict in Day 1 or 2, their realizations of those words evolves over time, and eventually used to resolve the conflict.

Endorsed Narrative 2 was supported by their problem about using the description about the location of the terms on graphs. For example, they used the "error bound" as "over minus under" [the vertical distance between consecutive terms: one above 5 and one below 5] to graph B, but found that this was a problem due to Routine 1 because this definition does not apply to all the terms of the sequence or other example graphs. As a solution, they decided to move toward comparing the vertical distance between the limit (5) and the term (a_n) (i.e., $|a_n–5|$) and the error bound as the distance between bounding lines (ε) around the limit. It should be noted that the instructor pointed out the conflict between their use of "error bound" as "over minus under" and how "error bound" is used in Lagrange Remainder formula, which students had seen in class. However, students' follow-up conversation did not address this conflict.

Endorsed Narrative 3 was initiated with GTE students' problem with using words describing the convergence based on the behavior of the dots on graphs rather than the quantitative relations. For example, with example G, GTE students used the word "ultimately" in their early definitions such as "a_n ultimately approach 5 as n increases" or "the distance between the terms of a_n and 5 is ultimately decreasing as n increases." By applying Routine 2, placing the components in the written definition on example and non-example graphs, GTE students located "ultimately" on graph G, and finally denoted the start of "ultimately" with a specific value N where error between a_n and the limit 5 is less than the error bound, $|a_n–5|<\varepsilon$, is satisfied for $n>N$ combined with Endorsed Narrative 2.

Endorsed Narrative 4 started with the problem of GTE students' illustrations of the definition on the graphs were inconsistent with their definition. For example, students were drawing the error bound, ε first, but their definition not capturing this process (e.g., GTE's definition 16. If $|a_n–5|\leq \varepsilon$ for any value of n, $n_c < n$ with error $\rightarrow 0$, then a_n converges to 5 (n_c = some n)). The instructor initiated the problem based on the Routine 3: the written definition should capture the illustration by asking "What did you always draw first?", and then "Do you feel like your definition…captures that?" multiple times. As a solution, GTE group first rearranged the statement "$|a_n–L|\leq$error bound" to read "error bound$\geq|a_n–L|$," and then added "within a given error bound". After receiving a critique from another group about the relationship between "error bound" and "n_c" (Problem: "Error bound for an n_c?" Suggested Solution" Should be error bound for $\{a_n\}$") to GTE's definition 18 (If error bound $\geq |a_n–5|$ for n_c being an n that all terms after n_c are within a necessary given error bound, then $\lim_{n\to\infty} a_n = 5$), GTE students applied Routine 4 (e.g., Rachel: "the way they're reading it [our definition] is that the error bound is coming off the n sub c…To us, it doesn't say that, but to other people it does"), and changed their definition by defining n_c based on the error bound given for the terms (GTE's Definition 21: $\lim_{n\to\infty} a_n = 5$ means that within a given error bound for a_n there is an n_c such that if $n>n_c$ then $|a_n–5|\leq$ the given error bound).

Discussion and Conclusions

Building on the body of research that examined students' epistemological development of a formal definition of the limit of a convergent sequence through guided reinvention, this study revealed what enabled such development. The results of this study provide the mechanism behind the guided reinvention with the analysis of what types of the routines were initiated by the instructors, how they were eventually adopted by students, and how those routines applied to students' problem/solution pairs which contributed to the changes in endorsed narratives, which led to the development of their definitions. In our results, students' adoption of the routines, and the problems as their own was key to development of their endorsed narratives and definitions. Specifically, when students realized the conflict between adopted routines and their definitions, they often identified it as a problem and attempted to solved it. However, when the conflict came from outside the routines that they adopted (e.g., the content covered in class), they did not realize it as a problem even if the instructor pointed out the conflict. This shows that what students have created through guided reinvention – adopted routines and shared endorsed narratives – had more weight than the instruction given in class.

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Students' Interpretations of Expressions from Calculus Statements in the Graphical Register

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Graphs of real-valued functions figure prominently in the study of calculus. While the use of graphs in instruction has shown promise for supporting student thinking, previous research has also shown that students may interpret graphs in ways that differ from an instructor's intention. This study aimed to investigate how students interpret expressions from calculus statements in the graphical register. To this end, I conducted 150-minute clinical interviews with 13 undergraduate mathematics students who had completed a calculus course. In the interviews, students evaluated six calculus statements for various real-valued functions depicted in graphs in the Cartesian coordinate system. I describe the characteristics of four distinct interpretations of expressions from these statements in the graphical register that students used in this study, which I refer to as (1) nominal, (2) ordinal, (3) cardinal, and (4) magnitude. I discuss some implications of these findings for teaching and directions for future research.

Keywords: Students' Interpretations, Graphical Register, Graphs of Functions, Cartesian Coordinate System, Calculus

In the modern approach to the teaching of calculus, graphs of real-valued functions tend to figure prominently. Key results in calculus, such as the Intermediate Value Theorem (IVT) or the limit definition of a derivative are often illustrated with graphs as figures in textbooks and curricula (e.g., Stewart 2012). Calculus instructors, when surveyed, reported a significant portion of their assessments included graphical interpretations of central ideas of calculus (Burn & Mesa, 2015). The inclusion of such graphs in the teaching of calculus, by both instructors and in texts, is in line with initiatives to incorporate visual representations into mathematics curriculum (e.g., Arcavi, 2003). In fact, recent calculus curriculum reform efforts have often sought to leverage the power of technology to illustrate the coherence of Calculus concepts using animated graphs of functions (e.g., Active Calculus- Boelkins, Austin, & Schlicker, 2018; DIRACC-Thompson, Byerley, & Hatfield, 2013).

Graphs have been shown to afford students opportunities for reasoning about concepts in calculus (e.g., Kidron & Tall, 2015) and may even promote an understanding of key definitions and theorems in calculus (Roh & Lee, 2017). While graphs may provide a powerful support to students, they are only effective insofar as students interpret them in conventional ways. In fact, research has shown that students encounter a variety of issues when interpreting graphs of functions provided to them or those of their own construction (e.g., Bell & Janvier, 1981; Frank, 2016). Some students may conceive of graphs as static shapes on the page with certain properties due to that shape (Moore & Thompson, 2015). Other students may fail to conceive a point as a coordination of two pairs of values (Thompson & Carlson, 2017), and instead interpret points as designating a location in space, labeled as outputs of a function (David et al., 2019). Students' interpretations of graphs, when unconventional, may have far-reaching consequences at the undergraduate level. As Dawkins and Epperson (2014) found, the students who withdrew from Calculus classes typically lacked an understanding of graphical representations. In this study, I sought to extend the current work that has been done in understanding students' interpretation of graphs by addressing the following question: *How do undergraduate students interpret expressions from statements in calculus on graphs of functions in the Cartesian plane?*

Theoretical Perspective

For the purposes of this study, I adopt radical constructivism (Glasersfeld, 1995) as my guiding theoretical perspective. In this view, graphs do not inherently contain a certain set of information; rather, the way in which an observer might interpret them is subject to his or her action schemes, as well as where he or she places attention in his or her perceptual field (Glasersfeld, ibid). I also adopt the view that interpreting graphs is semiotic in nature, consistent with Duval (1999). By this, I mean that interpreting a graph involves cognitive activity of processing a system of symbols intended to represent certain mathematical properties. To distinguish between students' interpretations with expressions on graphs from their reasoning with numerical values, I also use Duval's (1999) language of *registers* and refer to the first as an interpretation in the *graphical register* and the second as an interpretation in the *numerical register*. In this study, I focus on students' interpretations of expressions in the graphical register of the Cartesian coordinate system, specifically.

Methods

As part of a larger study, I conducted 150-minute clinical interviews (Clement, 2000) with 13 undergraduate mathematics students from a large public university who had taken at least one calculus course. These students were selected from a group of 82 students who volunteered to be pre-screened using a survey. I analyzed students' responses to survey items and selected 13 students to interview, with a goal of optimizing the variety of students' interpretations of expressions in the graphical register. This participant selection was in line with the method of theoretical sampling (Corbin & Strauss, 2014; Emmel, 2013).

In the clinical interviews, the selected students were asked to evaluate six statements related to key definitions and theorems from calculus for a variety of real-valued functions, displayed as graphs in the Cartesian plane. These statements were written to be either true or false, depending on the function to which they referred. The six statements were related to the following calculus topics: (1) the Intermediate Value Theorem, (2) the definition of a strictly increasing function, (3) the definition of an injective function, (4) the definition of continuity at a point, (5) the difference quotient, and (6) the Mean Value Theorem. As the students explained their evaluation, I asked them to label the graph of the function for which they were evaluating the statement. The placement of these labels, as well as the student's words and gestures served as a basis for the model I began creating of the student's thinking as the interview progressed. I also asked the student follow-up questions as needed to refine this model, in response to the different possible responses from students.

The preliminary data analysis took place both during each interview as well as immediately following each interview. Once all the interviews have been conducted and initially analyzed, I began to re-analyze each interview, drawing on the patterns observed in the first round of analysis. The observed patterns, developed through a process of open coding (Corbin & Strauss, 2014), in turn developed into a set of codes that I used to re-analyze each interview. The nature of the development and use of these codes was cyclical. As more codes emerge from analyzing interviews, these codes were applied retroactively to previously analyzed interviews. The cycle was complete once no new codes were needed to characterize each student's thinking with each graph and statement pair relative to the research question. The final set of codes was used once more to re-analyze each interview to ensure consistency in coding and refine descriptions of each code as necessary.

Results

From the instances in which students operated with expressions in the graphical register with provided or self-drawn graphs, I observed four different ways in which students assigned and used these expressions, which I refer to as students' *interpretations* of expressions in the graphical register. The four ways of interpreting expressions in graphs I refer to as follows: (1) *nominal*, referring to the use of an expression as a label; (2) *ordinal*, referring to the use of an expression as indicative of an ordering; (3) *cardinal*, referring to the use of an expression as a count of some quantity; and (4) *magnitude*, referring to the use of an expression to indicate an amount of a quantity regardless of unit.

Nominal Interpretation of Expressions

Throughout the interviews, I observed some students in some instances place expressions on graphs to indicate a certain position within the graph, typically on an axis or along the trace of the graph of the function. These students then reasoned about these expressions as referring to these positions they identified in the graph. In doing so, they tended to allow only a single label to be ascribed to one position and used two different labels for two different positions in the graph. These students used such labels for comparison of the positions they had marked to see if these positions were the same or different, as a means of determining whether expressions were equal or unequal. In these cases, the students did not refer to the value of these expressions numerically, or as indicating a certain ordering, an amount of units, or a measurement, but purely as a label. When students reasoned about expressions as labels for positions on the graph in this way, I refer to their interpretation of expressions as *nominal*.

Micah, a student who had completed Advanced Calculus, was one such student who interpreted expressions nominally in the graphical register when explaining his evaluation of Statement 3 (For all real numbers c, d in (a, b), if $f(c) = f(d)$, then $c=d$.) with Graph 2. With this task, Micah used expressions as labels of positions, and then compared these positions as a way of comparing the expressions. Micah explained that in his counterexample labeled on the graph (Figure 1), "$f(c)$ does indeed equal $f(d)$, but c does not equal d. If they [c and d] were [equal], they would have to be the exact same point. So d would have to equal c (*writes in '=c' next to d on the x-axis*). They would both have to be here, but that's not true."

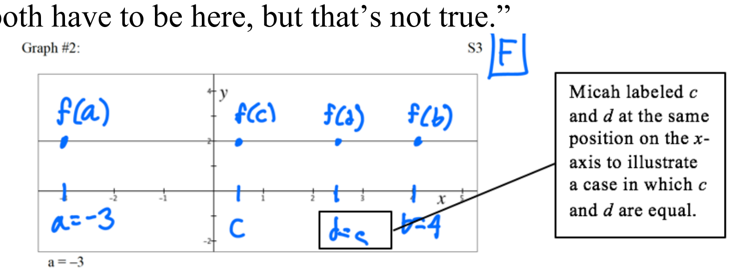

Figure 1. Micah's labels on Graph 2 when evaluating Statement 3.

Based on his explanation of his counterexample, which he labeled on the graph, Micah evaluated Statement 3 as false for the function shown in Graph 2. Micah claimed that for the $f(c)$ and $f(d)$ he chose, c and d were not equal to each other. The justification that he used to arrive at this claim was that c and d were in different positions along the x-axis. He confirmed that this was the reason for his conclusion by explaining that the only way for c and d to be considered equal would be for the two values to be located at "the exact same point." To show what he meant by being at "the exact same point," Micah used the label of "d=c" to indicate the case when these two expressions would be the same, i.e., when they would refer to the same position on the x-axis. In reasoning this way, Micah equated two expressions being placed on the same position on the x-axis with the expressions being equal to each other. For this reason, I categorized Micah's interpretation of the values c and d as nominal.

Ordinal Interpretation of Expressions

I observed other instances in which students ordered expressions by using a spatial relation of positions in the graph (i.e., a position is above/below a position, or a position is to the left of/to the right of another position). Students using this interpretation would label expressions at positions in the graph and then spatially relate these positions to each other to describe the order of the expressions. Often, students used this interpretation of expressions on the graph to claim that one expression was less than, equal to, or greater than another expression. Unlike students using a nominal interpretation of expressions, students interpreting expressions in this way described the *ordering* of expressions using the positions labeled in relation to each other on the graphs. I refer to this way of interpreting expressions on graphs as *ordinal*.

Jess, a student in a senior-level Topology course at the time of the interview, interpreted expressions on the graph through a spatial relation of positions when evaluating Statement 2 (For all real numbers c, d in (a, b), if $c < d$, then $f(c) < f(d)$.) for the function in Graph 1. Jess first labeled c and d on the x-axis, and then labeled the corresponding $f(c)$ and $f(d)$ on the y-axis (Figure 2). Jess claimed that the example of $f(c)$ and $f(d)$ she chose disproved Statement 2 because $f(c)$ was greater than $f(d)$. When I asked how she knew that $f(c)$ was greater than $f(d)$, she replied that "vertically [f of] c is higher than [f of] d."

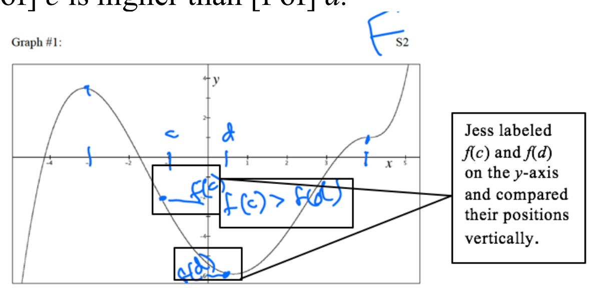

Figure 2. Jess' labels on Graph 1 when evaluating Statement 2.

With this task, Jess first identified the positions of $f(c)$ and $f(d)$ and labeled them on the y-axis. To make a comparison of $f(c)$ and $f(d)$, Jess compared the vertical orientation of these positions on the y-axis to each other. Specifically, Jess concluded that $f(c)$ was greater than $f(d)$ because the position labeled $f(c)$ was higher "vertically" on the graph than $f(d)$. In reasoning about the expressions in this way, Jess interpreted $f(d)$ as a relative reference point from which she compared the position of $f(c)$. By describing $f(c)$ as higher than $f(d)$, and thus, greater, Jess used a spatial ordering of these expressions. Jess made this visual comparison about the positions she had labeled as $f(c)$ and $f(d)$ to draw conclusions about the comparison of two expressions, rather than referring to numerical values of $f(c)$ and $f(d)$. For this reason, I classify her interpretation of expressions in the graphical register in this instance as ordinal.

Cardinal Interpretation of Expressions

I also observed some students interpreting expressions as numerical values measuring a portion of an axis or trace of the graph between two reference points. These students counted discrete units which they perceived or ascribed to portions of axes or traces of the graph to measure them. Some students counted the tick marks on the axes on the provided graphs as units to measure portions of the axes. In other cases, students ascribed units which were discrete points along the trace of the graph that they could count. Figure 3 illustrates ways students interpreting expressions in this way may have ascribed units and counted along a horizontal axis, a vertical axis, or the trace of the graph. I refer to this interpretation of expressions in the graphical register as a *cardinal* interpretation.

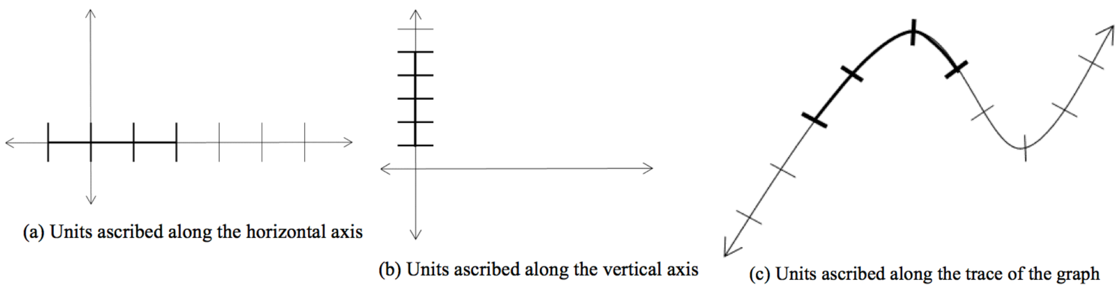

Figure3(a)-(c). Units ascribed and counted along a portion of an axis or graph–A key characteristic of a cardinal interpretation of expressions on graphs.

Annie, enrolled in a Calculus III course at the time of the interview, encountered the limitation of a cardinal interpretation of expressions in the graphical register when evaluating Statement 6. Annie interpret a difference in inputs as a count of units along the *x*-axis and also attempted to interpret a difference in outputs as a count of units along the graph. When working with Graph 1, Annie described her interpretation of the differences "*f*(*b*) – *f*(*a*)" and "*b*–*a*" in Statement 6 (There exists a real number *c* in (*a*, *b*) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$). Figure 4 shows Annie's labels on Graph 1 when interpreting Statement 6.

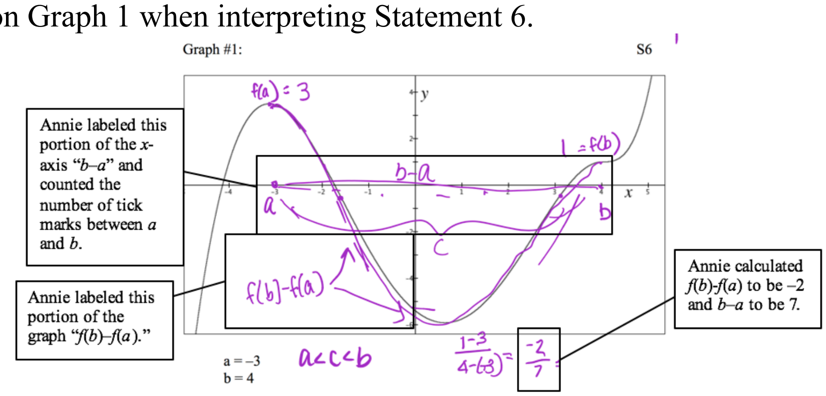

Figure 4. Annie's labels on Graph 1 with Statement 6, showing her labels of differences on the graph.

While working on this task, Annie spoke about the differences "*f*(*b*) – *f*(*a*)" and "*b*–*a*" in the statement as "changes in the values." When I asked her to represent those changes on the graph, Annie drew a line from *b* to *a* on the horizontal axis and claimed "this is *b*–*a*." She also traced over the graph from the point she had labeled *f*(*a*) to the point she had labeled *f*(*b*) and claimed "this whole thing is *f*(*b*) – *f*(*a*)." She described the values she calculated for the quotient in the statement, as –2/7. She explained this was related to slope and the "changes in *y* values over the changes in the *x* values," motioning to the portions of the graph she had marked off. She then explained that the changes in the *x* values were "all of these values between our *b* and *a* [*motioning over x-axis from a to b*], which is 7." When I asked about the meaning of 7 in reference to those values, Annie claimed "Well, there's 7 of them, like, in between, like if you were to count them out there's 7." As she said this, she tapped on the tick marks on the *x*-axis between *a*, labeled at –3, and *b* at 4. I then asked about her meaning for –2 in the graph, the value she calculated for "the change in *y*." Annie responded, "So, like, yeah we can get that 7 because there's like 7 real numbers in between that um but for this (*points to f(b)–f(a) label on graph*) it's not like there's –2 numbers in between (*long pause*) I don't know."

In this episode, Annie used a cardinal interpretation of the value of the difference "*b–a,*" which she calculated to be 7, as a count of units along a portion of the *x*-axis. Her words and motioning indicate that she was thinking of the *x*-axis as consisting of units which she could count. While Annie verbally claimed to be counting in units of "real numbers" between –3 and 4, she motioned to count the integer values located at the discrete tick marks along the *x*-axis. Because Annie's interpretation of a difference was a measurement of a portion of an axis between two reference points in additive units, I categorized Annie's interpretation of the expression as cardinal.

Annie's explanation of "$f(b) - f(a)$" indicated an attempt to interpret this difference on the graph in a similar manner, but was met with conflict. In this case, Annie interpreted $f(b)$ and $f(a)$ at positions on the trace of the graph. While she placed these output labels on the graph, Annie found the numerical values associated with these outputs by projecting the points to the *y*-axis, and calculated the difference to be –2. Annie was confused when trying to interpret $f(b) - f(a)$ similarly to her interpretation of *b–a* described above. Annie was trying to make sense of "a change in values" which she claimed to be –2 as related to a number of units, between the positions she labeled $f(a)$ and $f(b)$ on the graph. Annie had previously interpreted *b–a* in terms of additive units, which were represented by tick marks, and did not express any conflict in explaining the meaning of the numerical value of "7" in that instance. However, Annie was unable to reconcile her interpretation of $f(b) - f(a)$ on the graph with a negative value. Because she was attempting to think about the units additively along the graph, which she anticipated counting up from 0, a numerical value of –2 was a source of conflict. At the end of the episode, Annie was unable to say any more about what the "–2" meant in the graph beyond "the change in the values [$f(b)$ and $f(a)$]." Thus, Annie's cardinal interpretation of the expression of a difference on the graph was limited to whole number values.

Magnitude Interpretation of Expressions

Unlike the other interpretations, some students in some instances interpreted an expression as representing an amount of distance on a portion of a graph that could be measured from a fixed reference point on the graph. These students used distance or measurement language and labeled segments which they interpreted as having the length of the value to which they were referring. Students in these instances labeled expressions at the end of the portion of the graph with the specified length, or with a curly bracket to indicate a portion of the graph. Some students even expressed that the interpreted an expression as a directed distance, in which the sign of the numerical value of the expression indicated a direction between a starting and ending position on the graph. In these instances, I refer to these students as interpreting expressions as *magnitudes* on the graph.

Micah was one such student who conceived of expressions as distances from 0 along the axes when interpreting Statement 2. Micah claimed that the statement "may hold for some values of *c* and *d* but definitely not for all," drew a graph of a linear function, and labeled *c* and *d* on the *x*-axis and $f(c)$ and $f(d)$ on the graph (Figure 5). He explained that in his example, "*c* is less than *d* but $f(c)$ is not less than $f(d)$." When I asked him to further explain how he knew this, he replied "so *c* is less than *d* so I guess the distance here from 0 to *c* [*draws in horizontal curly bracket from the origin to c on x-axis*] is less than the distance from 0 to *d* [*draws in horizontal curly bracket from the origin to d on x-axis*]." To explain that $f(c)$ is not less than $f(d)$, he drew two vertical curly brackets on his graph, one from the origin to $f(c)$ on the *y*-axis, and one to the right of the point he labeled $f(d)$ from the origin up to this point. He explained, "then now $f(c)$ we're, we're doing vertically so this distance from 0 to $f(c)$ is …smaller than the distance from 0 to $f(d)$, because we go from 0 when we count."

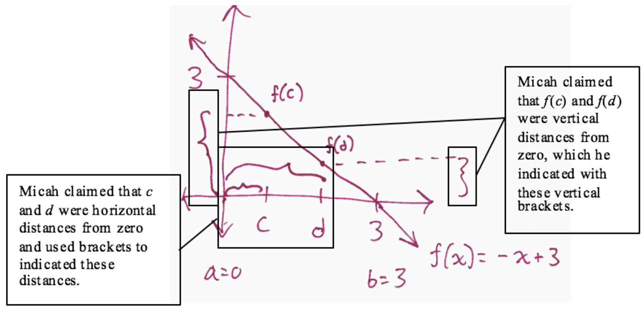

Figure 5. Micah's self-drawn graph to illustrate an example where he claimed Statement 2 was false, with distances marked as curly brackets.

To compare the pairs, c and d and $f(c)$ and $f(d)$, Micah compared the distances from 0 on their respective axes. In doing so, Micah revealed that he was reasoning about these expressions as magnitudes on the graph. Micah drew horizontal curly brackets from the origin, which he referenced as 0, to c and d, and explained that he was comparing the size of these distances to illustrate that c was less than d. Similarly, Micah drew in vertical curly brackets to compare the distances from 0 at the origin to $f(c)$ and 0 to $f(d)$. Although Micah mentioned the notion of counting when describing 0 as a reference point, Micah did not count by units when describing the distances in the graph. Instead, Micah compared the magnitudes of the distances, represented as segments on the graph. For Micah, comparing the value of pairs of inputs and outputs was connected to comparing distances from 0 in the direction of the axes on the graph. For this reason, I classified Micah's interpretation of expressions in the graphical register as magnitudes in this episode.

Discussion & Conclusion

The findings of this study characterize ways in which students may interpret expressions from mathematical statements in the graphical register. The findings reported in this study have implications for the teaching of calculus, specifically related to expressions involving differences. Many calculus theorems and definitions are stated using differences, which are illustrated as distances in the graphical register. In this study, the students who interpreted differences of expressions as cardinals on graphs did not at any time interpret these as magnitudes on graphs, and vice versa. One possible explanation for the mutual exclusion of the use of these interpretations is that students who used a cardinal interpretation of expressions may have had no other interpretation of expressions to draw on. If this is the case, students may benefit from opportunities to reason about magnitudes on graphs in their instruction. These reasoning opportunities ought to encourage students to conceive of measurements in the Cartesian plane as more than additive, possibly including fractional or irrational values rather than whole numbers.

Further research in this area may examine the role of each of the four interpretations of expressions in the graphical register identified in this study in other mathematical contexts or within various coordinate systems. Investigating how students interpret expressions in the graphical register in these various contexts may shed light on students' understanding of related concepts. Another direction for future research may be to develop interventions to support certain interpretations of expressions in graphs for contexts in calculus. For instance, interventions may be designed to support students in interpreting expressions as magnitudes, to develop their understanding of slope in the graphical register as a ratio of magnitudes.

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Folding Back in the Arzelà-Ascoli Theorem

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While learning is often characterized as a process of abstractions from robust understandings, students' reasoning in advanced contexts can be a more dynamic process of reflection across multiple layers of complex cognitive structures. To explore the ways that real analysis students' understandings of abstract mathematical concepts evolve in the context of multi-faceted proofs, we observed students' attempts to prove the Arzelà-Ascoli Theorem. We describe the understandings from which students built to engage in more complex reasoning and demonstrate their folding back (Pirie & Kieren, 1994) to facilitate the co-evolution of their foundational and advanced understandings.

Keywords: real analysis, metric spaces, formal mathematics, folding back

Introduction and Review of the Literature

As students learn mathematics in progressively more advanced settings, their understandings and ways of reasoning are supported by increasingly complex cognitive structures that rely on more basic understandings developed from earlier courses. This is especially true in abstract and formal mathematical settings such as real analysis and abstract algebra, specifically in courses beyond transition to proof or first semester introductions to advanced topics like algebra, analysis, or topology. To investigate students' co-evolution of both advanced and basic understandings of abstract and proof-based mathematics, we conducted a series of task-based clinical interviews (Hunting, 1997) with two pairs of students in the context of functional analysis. We report on the sessions wherein the students attempted a proof of the Arzelà-Ascoli Theorem, which required the coordination of multiple complex concepts in metric and function spaces. Observing their sense-making in this advanced setting allows us to characterize the complex nature of learning at an advanced level, attending to students' folding back (Pirie & Kieren, 1994; Gravemeijer et al., 2000) to not only support their understandings of advanced mathematics, but also to reinforce their understandings of the more basic concepts to which they folded. This results in a "thickening" (Pirie & Keiren, 1994) of their cognitive structures at multiple levels. While studies abound characterizing students' abstraction of knowledge at the undergraduate level (e.g., Cook, 2018; Gravemeijer & Doorman, 1999; Lockwood, Swinyard, & Caughman, 2015; Oehrtman, Swinyard, & Martin, 2014; Rasmussen, Zandieh, King, & Teppo, 2005; Zandieh & Rasmussen, 2010), much less has been documented about the role that folding back plays in students' progressive development of complex understandings, especially as students engage with formal mathematics.

We also explore ways that students understand key concepts in real analysis. While real analysis is a pillar of advanced mathematics, there are still few studies that explore students' reasoning about its fundamental topics. Real analysis has been utilized as a setting for inquiries into various tangential topics (Alcock & Weber, 2005; Lew et al., 2016; Weber, 2009; Troudt, Gulden, & Oehrtman, 2015; Zazkis, Weber, & Mejía-Ramos, 2016; Edwards & Ward, 2004; 2008) and basic concepts such as formal convergence (e.g. Adiredja, 2013; Cornu, 1991; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Gass, 1992; Oehrtman, Swinyard, & Martin, 2014; Roh, 2008; Swinyard, 2011; Swinyard & Larsen, 2012; Tall, 1992;

Tall & Vinner, 1981; Williams, 1991) have been more heavily investigated, yet few studies (Reed, 2017; 2018; 2019; Caglayan, 2018a; 2018b; Strand 2017) investigate students' understandings of key anlaytical topics.

Reed (2017) demonstrated the impact of a real analysis students' reversed quantification of real number convergence on his construction of pointwise function convergence. The student, Kyle, had adopted a domain-first perspective (Swinyard & Oehrtman, 2014) of real number convergence which he then abstracted to point-wise convergence of functions, stating that the error bound ϵ was a function of x and the index N, writing $\epsilon(x, N)$. Reed (2017) attributed this to his understanding of convergence being rooted in his internalization of the informal notation $\lim_{n\to\infty} f_n(x)$ rather than any attention to the implications from the definition's quantification. We build on this work to investigate the degree to which students draw from informal understandings to generate meaning for definitions with multiple quantifiers interacting with constrained sets within a definition. We thus address the following research question: *How do students leverage and develop relevant background knowledge in advanced proof activity.*

Theoretical Perspectives

We employ a radical constructivist (Glasersfeld, 1995) perspective to characterize students' understandings and infer aspects of their cognition by observing their utterances and mathematical activity. We specifically attend to the students' engagement in goal-oriented activity as indications of their understandings (in the sense of Thompson et al., 2014) of the analytical concepts relevant to the Arzelà-Ascoli Theorem. While we attend to their development of understandings in terms of their assimilations to schemes (Thompson et al., 2014), we also attend to other more broad aspects of their cognitive progression. Specifically, we pay special attention to the instances when students *fold back* (Pirie & Kieren, 1994), drawing from more concrete understandings to reinforce their developing advanced knowledge.

Folding Back

The theory of Realistic Mathematics Education (Gravemeijer, Cobb, Bowers, & Whitenack, 2000) characterizes students' engagement in activity and representation that progresses from concrete to abstract. Students' "models are initially tied to activity in specific settings and involve situation-specific imagery" (p. 243), but evolve along with their purpose to interpretations independent of situation-specific imagery, and ultimately to conventional mathematics independent of support from prior models and applicable "as an entity in its own right." Though Gravemeijer, et al. acknowledge their levels "of activity clearly involve a developmental progression," they note that activity may often "fold back" to prior levels.

> The construct of folding back was originally defined by Pirie and Kieren (1994) as activity, vital to growth of understanding, which reveals the non-unidirectional nature of coming to understand mathematics. When faced with a problem or question at any level, which is not immediately solvable, one needs to fold back to an inner level in order to extend one's current, inadequate understanding. This returned-to, inner level activity, however, is not identical to the original inner level actions; it is now informed and shaped by outer level interests and understandings. Continuing with our metaphor of folding, we can say that one now has a 'thicker' understanding at the returned-to level. This inner level action is part of a recursive reconstruction of knowledge, necessary to further build outer level knowing. (p. 173)

Underlying Mathematics

We focus on our subjects' attempts at constructing a proof of the Arzelà-Ascoli Theorem, specifically the version stating that a subset S of the continuous real-valued functions defined on a closed interval, $C([a,b],\mathbb{R})$, with the supremum metric is totally bounded if and only if it is equicontinuous and bounded. The supremum metric on this space is given by $d_{\sup}(f,g) = \sup_{x\in[a,b]}|f(x) - g(x)|$.

A set S in a metric space is bounded if there exists an element of the set $x \in S$ and a radius r such that $S \subset B_r(x)$. In the case of $(C([a,b],\mathbb{R}), d_{\sup})$, a bounded set is equivalent to a uniformly bounded collection of functions, i.e. a set S where there is an upper bound $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $f \in S$ and $x \in [a,b]$.

A totally bounded set S is one such that for every $\epsilon > 0$, there exists a finite collection of points $\{x_i\} \subset S$ such that $S \subset \cup B_\epsilon(x_i)$. We call such a collection of ϵ-balls a finite covering of S.

An equicontinuous set of functions S is one such that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $f \in S$, if $|x - t|<\delta$ then $|f(x) - f(t)|<\epsilon$.

These properties of sets are often visually depicted in distinct meaningful ways. One common method is to represent the metric space as a region on the page with points inside the region representing elements of the space, as if the space were a subset of $\mathbb{R}^2$. An ϵ-neighborhood is then represented by a disk in the diagram. We call such a representation a generic metric space diagram. While such a representation emphasizes metric properties of the set, a function space represented in this way loses detail about the ways in which such a metric is measured, e.g., as a supremum of output differences for two functions across all input-values. A second common representation of a function space is to draw their graphs on a common set of axes. In this representation, an ϵ-neighborhood around a function can be thought of as a tube extending vertically above and below its graph, and any function whose graph lies completely inside this tube is thus in the ϵ-neighborhood. We call such a representation a graphical diagram. While this representation emphasizes the structure of functions and details about the ways that the metric is defined, it can obscure some of the general metric space properties, for example thinking of functions as elements rather than points. We were interested in the representations students would draw from to generate meaning in the context of proof-construction, and the impact of these choices on their reasoning.

Methods

The episodes we present were part of a larger collection of task-based interviews in which we investigated the ways that students engaged in sense-making in the context of real analysis. We recruited students from a two-semester, metric-space-focused advanced calculus sequence intended to prepare advanced undergraduate math majors (and graduate students with less background in analysis) for a graduate-level analysis sequence. The course covered sequence convergence in Euclidean spaces and general metric spaces with applications to basic function spaces, completeness and compactness, differentiation and integration in single and multiple variables, and differential forms.

Clinical Interviews

We recruited four students from the class to participate in a series of five task-based clinical interviews (Hunting, 1997) each lasting about 90-minutes. One pair of students were statistics graduate students preparing for real analysis, while the other two were masters-level mathematics students. In this report we discuss the results of the latter sessions with the two statistics students, Adisa and Fenfang (pseudonyms), wherein we observed the students attempting to prove the aforementioned version of the Arzelà-Ascoli theorem.

We frequently asked students to explain and justify arguments they made, describe how they understood ideas they were utilizing, and how they understood any visual representations they made. All interventions were exploratory in nature, meaning that we were interested in seeing how the students would reason with these tasks rather than trying to achieve a specific learning trajectory.

Data Analysis

Both researchers reviewed video recordings of all sessions to identify episodes that demonstrated the prior mathematical understandings the students drew from to construct arguments, the ways their understandings evolved, and instances of folding back. Author 1 made descriptive timestamps of the video records attending to the aforementioned criteria, and then both authors met to discuss the relevant episodes identified in the timestamps and to characterize themes that emerged throughout the sessions. The authors then returned to the videos to code relevant episodes according to the initial collection of emergent themes, analyzing students' folding back and further documenting their development of understandings. The authors then met again, resulting in an iterative process that continued until the researchers had identified and agreed on the major themes that emerged from the data, and also the students' in-the-moment understandings (Thompson et al., 2014).

Results

We now describe the ways that Adisa and Fenfang drew from the structures and representations of general metric spaces when reasoning about function spaces, and then in greater detail present a key moment of folding back for Adisa, wherein his understandings for function continuity and equicontinuity co-evolved through targeted graphical investigation.

Generic Metric Space Diagrams vs. Graphs

Early in their attempt to prove totally bounded implies bounded, Adisa and Fenfang looked up their textbook's definition of totally bounded. Reading off the phrase "A set $A \subset M$ is totally bounded if for each $\epsilon > 0$ there exists a finite covering of A by ϵ-neighborhoods," Adisa represented the function space $C([a, b], \mathbb{R})$ as a generic metric space with a square on the board and a covering represented by smaller overlapping circles. He commented that, "Every function we have is in one of these balls," while Fenfang stressed that the centers were functions. Adisa then began an inequality, $|f(x)| \leq |f(x) - f_i(x)| + |f_i(x)| \leq \epsilon + \ldots$ and left the remaining portion blank as they wrestled with how to work with bounding the values of $f_i(x)$. Unsure how to proceed, Adisa redrew their generic metric space diagram with center dots that he called $f_1, f_2, f_3, \ldots$ then stated, "I don't know why I keep drawing it this way. This is referring to sets." He erased the generic metric space diagram and redrew it as a graph of several functions with tubes around centers f_1, f_2, and f_3, saying "It's more like I enclose the function in an ϵ-tube."

Thus, Adisa began to interpret the general phrase of an ϵ-neighborhood as meaning an ϵ-ball for a set of points and an ϵ-tube for a set of functions, remaining adamant about this distinction throughout the interviews. His image of enclosing a function in an ϵ-tube on a graph conceptually entailed that the center of the neighborhood was a function, but in which nearby were points in $[a, b] \times \mathbb{R}$, rather than other functions in $C([a, b], \mathbb{R})$. Adisa described this both graphically and algebraically, saying "That's the problem. You can't relate these to together [points to metric space and graph diagrams]. Here [points to metric space diagram], you are just covering points. Here [points at several locations along a graph], you are covering a **sequence** of points, a formula line." Adisa's emphasis on always attending to, and representing, multiple x-values of a function rather than attending to functions as objects resulted in corresponding difficulty engaging in visual reasoning throughout the interviews.

While focusing singularly on the meaning of a totally bounded subset $\mathcal{E} \subset C([a,b],\mathbb{R})$, Adisa was able to conclude that any function $f \in \mathcal{E}$ would have to be in one of the ϵ-neighborhoods, and since ϵ-neighborhoods are ϵ-tubes graphically, that f should lie entirely in one of the ϵ-tubes. However, as Adisa focused on other tasks involving using the idea of a totally bounded set of functions, he would revert to reasoning about points being close to functions. In one episode, he began questioning, then asserting, that a covering of $\mathcal{E}$ meant that the ϵ-tubes only needed to cover $[a,b] \times [-M,M]$ so that a function might not be in any single ϵ-tube but cross through many, as long as its graph was covered. Adisa eventually resolved the conflicts arising from this reasoning by recalling that he had earlier decided that any function must lie entirely in one ϵ-tube, but without ever revisiting the reason why.

Fenfang, on the other hand consistently viewed the generic metric space diagrams precisely as generic images. She responded to Adisa's confusion by explaining,

> I think A is just a set. We don't need to think a set of what. It's like a general version. A is a set. If we only talk about a property of the set, we don't need to look at what is in the set. Like we learned in the first semester, we just look at the general version of a set of compact or complete or something. I think in the set can be function or something or point or whatever. But if we look at this specified question, should be reasonable in the ϵ-tube because in the set is a function. So how a function is to the other function is in the ϵ-tube. Here [points at set diagram] it's just a general version. Here [points at function inequalities] it's more special case.

Her ability to easily see the same things and translate between a generic metric space diagram and a graph supported flexible reasoning that required Adisa to fold back and reason through more carefully. For example, Adisa eventually had the insight into how to complete their inequality, recognizing that the collection of centers, f_i, were finite so that they could just consider a uniform bound for all of them. He, however, had to consistently reason about that bound pointwise for each function, $-M_i \leq f_i(x) \leq M_i$, then considering the maximum M_i while Fenfang was able to more efficiently characterize these bounds in terms of the sup metric on $C([a,b],\mathbb{R})$, saying for example "Each of the infinite functions [f] will be close to one of the finite functions [f_i]." This account of the interplay between their generic metric space representations and their graphical representations demonstrates the kinds of background images that were meaningful to each student, and the degree to which both students explicitly or implicitly drew from general structures to inform their reasoning in metric spaces. Moreover, this demonstrates the students' folding back to basic images, algebraic representations, and principles of general metric spaces to inform their argument construction.

Folding back to continuity

While attempting to prove that a uniformly bounded, equicontinuous collection of functions is totally bounded, Adisa and Fenfang first conjectured the role that boundedness played by representing functions graphically. This representation led the students to visually assess that the graphs restricted to $[a, b] \times [-M, M]$ would eventually be covered by finitely many ϵ-tubes. Their strategy then became to attempt successive adding of epsilon tubes in some type of boot-strapping manner until the entire collection of functions was covered. As the representation did not force them to distinguish between covering the points in $[a, b] \times [-M, M]$ from covering the space of functions, they did not recognize the difficulty introduced by vertical variation of the functions. Correspondingly, they observed that they could not conceive how equicontinuity played a role and that they needed to better understand this concept. Thus they intentionally engaged in a session of folding back to interpret this definition.

Adisa and Fenfang first turned to a graphical representation of equicontinuity in their textbook, attending to regions of "higher changes" in the graphs and the observation that for "every function in the set, the . . . output will be close if the input is close enough." They did not, however, relate this interpretation to the formal or logical structure of the definition. Since the diagram in the book was ambiguous in its representation, Author 2 suggested investigating the collection of functions, $\{x^n\}$ on $[0, 1]$, already familiar to the students. After drawing a few functions in this sequence, the students continued offering vague interpretations of amounts of vertical change corresponding to small changes in the domain, disconnected from the formal structure of the definition of equicontinuity. After a few moments of thinking about their graphs, Adisa expressed that he was "done" reasoning about the picture, and that he "understood" the written characterization of equicontinuity, but did not understand what a picture would look like. We infer that he saw a need to further develop his understanding of equicontinuity to incorporate into their current proposed way of arguing, but was unsure how to do so.

Author 2 then further engaged the students in connecting their imagery to the definition by specifying a value $\epsilon = 1/2$ and asking whether there was a δ such that, within that δ distance from $x = 1$, all outputs of the functions would be within $1/2$ of $f_n(1)$. The students affirmed that no such delta existed, and Adisa drew a graph in the sequence with large n and depicted the $\epsilon = 1/2$ as an epsilon tube around this function. He then demonstrated that picking a domain value close to 1 would result in an image distance of greater than $1/2$ from $f_n(1)$.

Drawing from visual analogy to the behavior of the set of $\{x^n\}$ near $x = 1$, Adisa then claimed there was no δ that would work for his single x^n with such a large value of n. However, when pressed whether his function was continuous he laughed nervously, confessed confusion over the discrepancy, and lamented that he could not revisit all of this more fully as an undergraduate. Author 2 then guided the students through graphically identifying a δ-neighborhood of $x = 1$ that would result in Adisa's x^n being within $\epsilon = 1/2$ of the output 1.

Author 2 then asked the students to find another function in the sequence for which their identified δ did not satisfy the constraints. Fenfang drew an appropriate function, corresponding to an even larger n-value, and Adisa remarked "Oh another one in between." Fenfang then suggested the presence of another δ corresponding to her new function, which then served as the basis for their developing a more robust understanding of equicontinuity. Both students then made comments suggesting an understanding that, for a given ϵ and point of continuity, each function had an associated δ because of continuity, but the set $\{x^n\}$ was not equicontinuous because further iterations of the sequence would require increasingly smaller δs, and no one δ would work for the entire sequence.

Adisa then restated the definition of equicontinuity, emphasizing aspects that were now made relevant for him "So if you can find *one* δ that works ***for all of them***, then you can say it's uniformly [equicontinuous]" (Adisa's emphases). While he previously had uttered those words when reading the written definition, we infer from his new emphases and the context of the discussion that he had developed a new (and thickened) understanding of equicontinuity. Specifically, Adisa (and Fenfang as well) understood that the role of the singular δ was associated with the continuity of each function in the collection.

Later, Adisa drew a different example of a function set that was not equicontinuous, and his description of the sequence attended to the variation in the δs associated with the uniform continuity of each function. The students' folding back facilitated both of their' development of two kinds of thicker understandings related to equicontinuity. First, the students could construct visual depictions of function collections that were not equicontinuous and could articulate the structural elements of those sets that resulted in this lack of equicontinuity. Second, the students developed a more robust understanding of the relationships between the variables within the definition, specifically the role that δ plays in establishing the uniformity of the equicontinuity.

Discussion and Concluding Remarks

Addressing our research question, we have documented Adisa and Fenfang's varied reliance on general metric space understandings and representations when reasoning about function spaces, and also demonstrated a key moment of folding back in the students' coordination of various complex ideas while proving in real analysis. Adisa and Fenfang had completed a proof-based advanced calculus course and were proficient in drawing from useful results and making sophisticated arguments at the formal level, but still benefited from engaging in more informal ways of reasoning to thicken their formal understandings. In these instances, it was the setting aside of more "advanced" concepts (such as equicontinuity) to spend time reinforcing the foundational understandings that contributed to their productive mathematical activity. While Adisa and Fenfang could productively reason with quantifiers, we also found that they were less proficient at attending to the implications of the quantifiers on the more complex set-based relationships occurring within definitions. Instead, they drew from concrete images such as the general metric image, the idea of an epsilon neighborhood, or a nonspecific relationship between vertical and horizontal changes in functions.

We have documented the phenomenon of folding back as a mechanism for learning in the midst of complex goal-oriented activity. Adisa's thickening of equicontinuity coincided with his development of a more robust understanding of function continuity, a concept he originally learned in undergraduate real analysis. These students' folding back represents a nonlinear development of understanding that has been underrepresented in characterizations of abstraction and generalization in more unidirectional terms.

We find it reasonable that many advanced mathematics students have developed complex understandings of important mathematical topics that are supported by more basic understandings that are not as robust as we would expect. Further investigation can explore ways to support students' learning in the presence of such understandings. Specifically, we are interested in examining how hypothetical learning trajectories and other constructs of design research can explicitly anticipate and support effective folding back. Attention to opportunities in proof-based courses for students to grapple with and coordinate various basic understandings meaningfully in their constructions of arguments could improve this natural co-evolution of foundational and more advanced mathematics.

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How problem posing can impact student motivation: A case study

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This case study utilizes self-determination theory (Deci & Ryan, 2000) to explore the motivational development of three students who participated in problem posing as part of an introduction to proofs course. The experience of these students illustrates several ways in which problem posing can create new more integrated habits of motivational regulation. However, the differences of these cases highlight how students may benefit from experiencing problem posing in a way that makes explicit the purposes of student problem posing in relation to other educational goals. This research underscores the motivational benefits of making the connections between problem posing, problem solving, learning, and self-regulation more explicit earlier in students' mathematical education.

Keywords: problem posing, motivation, introduction to proof

In improving the teaching and learning of mathematics, there is need to understand the factors that promote the development of student motivation (Pantziara & Philippou, 2013). Savelsbergh et al. (2016) conducted meta-analysis of 65 independent math and science teaching experiments concluding that for the five types of teaching approaches considered (inquiry-based, context-based, computer-based, collaborative learning strategies, and extra-curricular activities), all demonstrated positive effects on general attitude and interest toward the subject. However, there was "little clarity about what interventions cause effects on what outcome, and under what conditions." (p. 159). The evidence in mathematics instruction was sparser, focusing primarily on classroom atmosphere and teacher behavior, with only one case studying problem posing.

While problem posing been suggested as a motivational tool in mathematics education (Hošpesová & Tichá, 2015), there is little research explaining *how* problem posing impacts the motivation of students studying mathematics. This paper contributes to the research on motivation in mathematics, firstly, by studying the types of motivational regulation experienced by students engaging in problem posing, and secondly, by exploring the ways in which problem posing appears to have supported their motivational development.

1. Literature Review

1.1. Problem posing

According to Silver (1994), problem posing refers to the generation of new mathematical problems as well as the re-formulation of given problems. Problem posing has been variously described as a "practice deeply embedded in the activity of problem-solving" (Brown & Walter, 1983), an important part of research mathematics (Silver, 1994), a means of learning mathematics (Kilpatrick, 1987), a means for fostering mathematical creativity (Silver, 1997), an integral part of mathematical exploration (Cifarelli & Cai, 2005), a formative assessment tool for instructors (Kwek, 2015), and an aspect of general mathematics instruction (NCTM, 1991). This paper focuses on this later aspect of problem posing and its use as it relates to teaching and learning mathematics. For this purpose, I consider student problem-posing as the activity of students authoring their own mathematical questions (Walkington, 2018).

There is a wide variation to the ways problem posing can be fostered (Silver, 1994; Stoyanova & Ellerton, 1996). Various techniques have been offered as to how to facilitate student problem posing in the classroom, including Brown and Walter's "what-if-not" strategy (1983) and personalization via student's writing their own story-problem (Winograd, 1990). Problem posing is also considered a natural part of inquiry-oriented instruction in teaching students to use self-questioning and self-regulatory techniques (Collins, 1986).

However, Cai, Hwang, Jiang, and Silber (2015) assert that, "Few researchers have tried to describe carefully the dynamics of classroom instruction where students are engaged in problem-posing activities" (p. 22). Cai et al. continue, "because classroom instruction is generally complex, with many salient features that can be investigated, researchers will need to identify those features that are most relevant for problem posing and which may be the most influenced by introduction of problem posing" (p. 23).

Problem posing has been described as a means of improving student disposition toward mathematics (Silver, 1997), of which motivation toward math is a significant aspect. Thus, a better understanding of the relationship between problem posing and student motivation may contribute toward answering the question offered by Cai et al. (2015), "what are the key features of effective problem-posing instruction in classrooms?" (p. 23).

1.2. Motivational Regulation

A central question in teaching is how to maintain or improve intrinsic motivation for learning (Valås & Søvik 1994). Motivation in education has been researched from many perspectives (Bern, 1972; Wigfield & Eccles, 2000). In particular, social learning theory (SLT; Bandura, 1982) and self-determination theory (SDT; Ryan & Deci, 2000) explain motivation as mediated by perceived competence, or self-efficacy. Both SLT and SDT provide valuable perspectives to understanding the impact of problem posing on motivation by highlighting the role of the internalization of standards or values that regulate activity.

While SLT focuses on *intrinsic interest*, or the predisposition or tendency to engage in an activity, SDT considers motivation in a slightly broader sense as "energy, direction, persistence and equifinality—all aspects of activation and intention" (Ryan & Deci, 2000, p. 69). This study utilizes this later perspective, taking the view that people are motivated or "moved to act by very different types of factors, with highly varied experiences and consequences." (p. 69). Rather than measuring students' interest toward mathematics, this project seeks to understand the broad reasons why students are motivated to act in relation to mathematics.

SDT also offers two theoretical perspectives that can guide the study of student motivation toward mathematics. Firstly, SDT characterizes extrinsic and intrinsic motivation not as binary variables, but on a continuum based on by perceived locus of causality (external vs. internal). This offers finer-grain analysis of student motivation than other motivational theories. Secondly, SDT (Ryan & Ryan, 2000) describes three psychological needs that foster the development of self-motivation – competence, autonomy, and relatedness.

1.2.1 SDT Continuum of Motivational Regulation. SDT describes three intermediate forms of "extrinsic" regulation between purely external and purely internal (intrinsic) regulation—introjected, identified, and integrated. These types of regulation vary by the degree to which an external regulators or reason for acting has been internalized (or "taken in") and subsequently integrated (accepted as one's own). In this way, *external regulation* describes behavior "performed to satisfy an external demand or reward contingency" (p. 72), *introjected regulation* describes behavior that results from "taking in a regulation but not fully accepting it as one's own" (p. 72) and is enforced through internal pressures, such as guilt, anxiety, or regulated self-esteem dynamics; *identified regulation* "reflects conscious valuing of a behavioral goal or regulation, such that the action is accepted or owned as personally important" (p. 72); and *integrated regulation* describes reasons for acting that are "fully assimilated to the self, which means they have been evaluated and brought into congruence with one's other values and needs" (p. 73). While integrated regulation shares many qualities with *intrinsic regulation*, or behaviors done for their "inherent satisfactions" (p. 72), they are still considered separate because integrated regulation refers to actions done "to attain separable [external] outcomes" (p. 73).

According to SDT, as one increasingly identifies with or takes ownership of a reason for acting, the perceived locus of causality (PLOC) for that action will increasingly stem from oneself. As external regulators are increasingly internalized and integrated, one experiences greater autonomy in action. In this way SDT explains the development of self-motivation from external regulation and introjected regulation to identified regulation and integrated regulation. Studies have shown that people demonstrating higher levels of identified and integrated regulation experience greater interest and enjoyment, exert greater effort, and utilize more positive coping styles (Ryan & Connel 1989).

1.2.2 Psychological needs that support self-motivation. SDT also describes three innate psychological needs—competence, autonomy, and relatedness—which are the basis for self-motivation. *Competence* refers to one's sense of their ability to do things successfully, which is described by SDT as facilitating intrinsic motivation. However, developing a sense of competence is not a sufficient condition for experiencing self-motivation; according to Deci and Ryan (2000), "people must not only experience competence or efficacy, they must also experience their behavior as self-determined." This need to maintain a sense of self-determination, or ability to take action, is referred to as *autonomy*. This is described in attributional terms as an "internal perceived locus of causality" (deCharms, 1968). Deci and Ryan further state that "this requires either immediate contextual supports for autonomy and competence or abiding inner resources (Reeve, 1996) that are typically the result of prior developmental supports for perceived autonomy and competence" (p. 70).

The third psychological need, *relatedness* refers to the need to maintain a secure, interpersonal relational base. Deci and Ryan (2000) describe that experiencing a sense of relatedness is critically important to internalization since the "primary reason people initially perform [externally motivated] actions is because the behaviors are prompted, modeled, or valued by significant others to whom they feel (or want to feel) attached or related" (p. 73).

1.3 Research Question

This paper studies the question: how does problem posing impact students' motivation for mathematics? To begin answering this question, this paper explores the following three sub-questions. In the context of students' experience of problem posing in an introduction to proofs course: RQ1-What types of regulation characterize a student's motivation for mathematics over time? RQ2-To what degree does a student attribute changes in motivational regulation to problem posing? RQ3-In what ways does problem posing appear to support a student's sense of competence, autonomy, relatedness in Discrete Mathematics?

2. Methods

This qualitative research project serves as an illustrative case study of students experience problem posing in an introduction-to-proofs course, taught in fall of 2017. Each class period of this course was observed and video-recorded by the author for prior research (Regier & Savic, 2019). The course was taught utilizing an inquiry-based teaching pedagogy (Laursen et al. 2014) which involved both student problem posing in class, as well as several assignments which provided explicit opportunities for problem posing out of class, collected as data.

Two students, given pseudonyms Fred and Frank, were recruited due to their participation in interviews in December 2017. Fred had taken a Calculus II the prior semester with the same instructor, Dr. F. A third student, Aaron, was recruited voluntarily from among the remaining students that agreed to be contacted for follow-up research. Each student had demonstrated active participation in class. At the end of the semester, Frank, Aaron, and Fred received grades of A, B, and C, respectively, and all indicated that they had worked hard and were proud of the grade they received.

Interviews were conducted in the spring of 2019 using a semi-structured interview methodology. Questions included “Why did you take Discrete Mathematics?”; “What motivated you in class?”; “What took away motivation?” Each student’s write-your-own test assignment was shown to students and used to ask questions about problem posing. Students were asked if and how problem posing impacted them in general, if and how it impacted their approach to Discrete Mathematics, as well as if and how it impacted their sense of competence, autonomy, and relatedness. Finally, students were asked to describe their motivation (for engaging in class, solving problems, and for posing new problems) since Discrete Mathematics.

Each interview was transcribed, removing phrases such as "like," "um," and "kind of" for clarity, and coded in a qualitative data analysis software package (NVivo) in three separate cycles as described below. The first cycle involved reading and coding each interview transcript for evidence of motivational regulation for *learning* and *doing* mathematics, as expressed in the student views concerning their engagement toward mathematics. This followed Hannula’s (2006) view that motivation “is observable only as it manifests itself in affect and cognition, for example as beliefs, values and emotional reactions” (p. 165). Then, anything coded for evidence of regulation was then sub-coded these into the five types of regulation detailed in **Section 1.2.1**. This utilized the student’s self-described locus of causality (PLOC) for acting and followed Ryan & Connel’s (1989) research perspective that "the status of variables as real causes or motives is not directly relevant," rather the “focus is more on how persons understand and describe their own purposes for acting and the relation of such purposes to a continuum of autonomy" (p. 750).

The second coding cycle involved coding for student experience of, or instructor support for, any of the three psyhcological needs—competence, autonomy, and relatedness—that serve as the basis for self-motivation (see **Section 2.2.2**). Because the sub-codes used for these first two cycles were previously defined by SDT, this coding style can be characterized as provisional coding (see Saldaña, 2013, p. 144).

The third cycle involved applying content-based codes for anything related to problem posing, that is, students authoring their own mathematical questions. These codes primarily fell into the following four categories: student’s own problem posing (in or out of class), peers’ problem posing (in or out of class), instructor (or teaching assistant) support for student problem posing, student support for others’ problem posing. This coding method can be described as structural coding (Saldaña, 2013, p. 84) since these codes served to categorize experiences involving problem posing for analysis across other codes.

Results

The three research questions (section 2.4) were explored through three respective steps of analysis. First, each students’ motivation development toward mathematics in and since Discrete Mathematics was summarized based off quotes coded for “regulatory style” (RQ1). Second, the degree to which each student attributed their level of (or a change in) regulatory style to problem posing (RQ2) was studied by looking at the intersection of coding for “problem posing” and “regulatory style.” Third, the degree to which problem posing appeared to have supported the integration of regulation (RQ3) was analyzed by considering the intersection of the codes “problem posing” and “support for self-motivation.” A summary of the results for RQ1-3 for each student are provided in Table 1 below. For space considerations, only brief segments of the analysis of Aaron's interview regarding RQ1 and RQ2 are provided below, with the coding for **regulatory style** given in **bold**.

Table 1. Summary of Results for Fred, Aaron, and Frank

Research Question	Fred	Aaron	Frank
1. What types of regulation characterize a student's motivation for mathematics over time?	Described consistent development from external to internal (identified and integrated) regulation for *doing* mathematics.	Described development from external to internal (identified and integrated) regulation for *doing* and *teaching* mathematics.	Described little change in regulation, appearing to be largely internally motivated from the start of college.
2. To what degree did a student attribute changes in regulation to problem posing?	Attributed this change in regulation, in part, to his *own* experience problem posing.	Connected problem posing with this change in regulation and with taking ownership of his own learning.	Did *not* attribute a change in regulation to problem posing.
3. In what ways does problem posing appear to support a student's sense of competence, autonomy, relatedness?	Described increased confidence for exams due to his own problem posing, an increased sense of autonomy in relation to his own problem posing and the instructor support for problem posing, and an increased sense of relatedness from his own and peers' problem posing.	Described gaining a sense of competence and being able to "choose where his learning went" from both his own problem posing and from the way class was set up in support of problem posing, and gaining a sense of relatedness from the problem posing of his peers.	Described problem posing as having little impact on his sense of competence, autonomy, and relatedness in class.

Aaron described at the beginning of the semester, having "**a big assignment for another class** one week," saying "I didn't apply myself in Dr. [F]'s class and that took a hit to my motivation in that class because I would go to class and **I'd feel like an idiot**." This was coded for **external regulation** since Aaron described another assignment as the reason for not participating, and **introjected regulation** since feeling "like an idiot" describes a self-esteem dynamic.

Later, Aaron described an experience of playing basketball at the gym and realizing he had an idea for a proof. He said "I pulled out my phone and **I started writing up a proof on it. It hit me just then. I was like, "hmm, oh." I just had to start writing notes and stuff on it."** Since solving the proof was important enough to him that he stopped playing basketball to start writing down his ideas for a proof, this was coded for **identified regulation**. Quotes like this contributed toward demonstrating a shift in Aaron's motivation (RQ1).

Aaron's interview further provided evidence that this shift in regulation was linked to his problem posing activity. When asked "what did you take from Discrete Math?" Aaron mentioned exploration several times, saying things such as, “I like how he gave us like things to explore before we ever covered content in math.” At this point I asked about his experience problem posing, to which Aaron commented on.

> *Aaron*: That’s one thing that I struggled with in that class because I hadn’t done that yet, but I feel I’ve kind of gotten better at it because [every] homework he had us ask two questions. It was always do this problem and ask two questions. A lot of times I was like, what the heck am I supposed to ask? **What I found myself doing a lot of the times is I would ask a question and then I would answer it myself.** I’d be like, I can’t put that question because I just answered it. So, it was actually like **I was learning by myself and learning [from] what *I thought* of rather than learning some kind of prescribed material.** I was thinking creatively and asking my own questions and answering them.

Aaron learning by himself from what he “thought of rather than learning some kind of prescribed material” was coded for **integrated regulation**, since this described his learning as stemming from himself. Since this was in association with Aaron’s experience asking questions on homework, this was considered evidence that a shift in regulation was linked to his problem posing activity (RQ2).

4. Discussion

Dr. F’s support of problem posing appeared to have increased Fred’s sense of relatedness through other students’ problem posing, which in turn contributed to his willingness to pose his own problems. The carrying out of those actions, i.e. problem posing, then provided opportunities for integration of regulation related to those actions. An example of such integration of regulation can be seen from Fred describing problem posing as part of the larger process of “learning” that mathematics is not replicated but created. Rather than “just plugging numbers basically and memorizing formulas and plugging in numbers” as Fred described his experience in primary and secondary school, Fred shifted toward seeing mathematics as “coming to your own setup and equation and solution on our own.”

Aaron also described how habits of thinking involving problem posing contributed to developing greater integrated regulation toward mathematics. Aaron described utilizing problem posing in his approach problems: “I try to break it into pieces I do understand and think about ‘what I do understand?’ and ‘how I can tackle the problem if I understand this bit of it or this bit of it?’” This perspective conveys how problem posing enabled Aaron to gain an internal orientation (PLOC) toward mathematics.

Aaron described problem as contributing to his confidence for mathematics and helping him “feel like a mathematician,” as well as his sense of being related to other students. At several points in the interview, he ascribed his change in motivation to his peers’ problem posing above his own problem posing experience. This demonstrates how the development of self-regulation is impacted by the social context in which students experience or engage in problem posing.

In contrast to Aaron and Fred, Frank described little change in motivational regulation in Discrete Mathematics. In fact, he described lower motivation for the problem posing assignments than for "very challenging math problems." While he recognized the value of problem posing, he did not perceive it as challenging enough to merit his effort. He said that, “unfortunately the downside of assignments that are this kind of open-ended and this free-form is that, really, you don’t have to push yourself more than you want to.”

SDT explains Frank’s behavior in that, without a sufficient level of perceived challenge, the problem posing tasks used failed to meet Frank’s need for developing a greater sense of competence toward mathematics. Ryan and Connel (1989) found that the identified and intrinsic regulation defined by SDT correlate with an intrinsic motivation component of a binary motivational model (Harter, 1981) called *mastery motivation*, characterized by an internal PLOC. It appears Frank was more self-motivated for challenge in mathematics than he was for exploration and self-discovery. SLT offers a further explanation of Frank's behavior by describing how "self-development is aided by a strong sense of self-efficacy to withstand failures, tempered with some uncertainty (construed in terms of the challenges of the task rather than fundamental doubts about one's capabilities) to spur preparatory acquisition of knowledge and skills" (Bandura, 1986, p. 394). Accordingly, Frank appeared to not judge the uncertainty of the problem posing tasks as useful to the acquisition of knowledge and skills.

In spite of being one of the most mathematically advanced students in the class (from an assessment standpoint), Frank said "I haven't developed the mathematical confidence to feel like my problems are valuable." In contrast to Fred and Aaron, Frank did not view himself as posing "valuable" mathematical questions. Why was Frank hesitant to see the value problem posing in the context of his class' learning, problem solving, and proof writing?

One explanation for this has been offered by Silver (1994) who described that, for students already successful under more directed instruction, "there may be little desire or motivation to alter the existing power relations in the classroom, or to alter the hierarchical assumptions underlying current conceptions of mathematical performance" (Silver, 1994, p. 25). Thus, for students like Frank, explicit discussion of the fundamental assumptions of how mathematical knowledge and skills are or can be obtained may be needed in order to promote views of problem posing as worthy of effort. Therefore, to benefit the most students possible, problem posing may need to be presented in a way that makes explicit the purposes of problem posing in relation to other educational goals. This also highlights the importance of framing problem posing and uncertainty as useful to the development of mathematical knowledge and skills.

5. Conclusion

This study confirms the use of problem posing as a motivational tool beyond simply connecting students to their own existing interests (Silver, 1994). Problem posing has the potential to create new more integrated habits of regulation. Aaron's experience demonstrates how by asking his own questions, by posing his own problems, he gained internal orientation to his own mathematical activity. As someone (student, instructor, researcher) poses problems that are increasingly internally obtained, they gain greater access to the resources within them, perhaps primarily, the motivation or energy to continue pursuing something new, difficult, and unknown.

One limitation of this research is that the results are based on students' self-reported reasons for acting, and thus, there may be bias in the explanation of their actions. However, the results for each student do appear internally consistent, and as the interviews were voluntary and over a year after their participation in Discrete Mathematics, there was little external reason to give biased responses. Another limitation of this paper is the lack of explicit characterization of instructor actions used to foster or promote student problem posing. Such a characterization would further the study of the impact of instructor actions for fostering problem posing on student development of self-motivation. There is a need for both an explicit characterization of the key features of problem-posing instruction (Cai et al., 2015) as well as the ways to measure teaching characterized by those features to be able to better study the impact of teaching to foster problem posing on student regulation toward mathematics. By better explaining the ways in which problem-posing can impact student regulation, this research can contribute toward allowing students to develop autonomy, ownership, and self-motivation for studying mathematics.

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Varieties of Sameness: Instructors' Descriptions for Themselves and Students

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Group isomorphism and homomorphism are topics central to abstract algebra, yet research on instructors' views of these concepts is limited. This study examines two instructors' views of group isomorphism and homomorphism and how they drew on notions of sameness to describe them. Differences between instructors' descriptions in interviews and instruction are explored.

Keywords: Isomorphism, Homomorphism, Abstract Algebra, Instructional Practice

Experts have identified isomorphism and homomorphism as two of the central topics of Abstract Algebra (Melhuish, 2015). Although some research has been done on how students and mathematicians approach isomorphism, research on understandings of homomorphism is more limited. Furthermore, research on mathematicians and, specifically, instructors is limited. This paper focuses on two instructors' views of isomorphism and homomorphism and on differences in how they described these concepts in different settings.

Literature Review and Conceptual Framework

Previous work has examined conceptions of isomorphism more than homomorphism and largely from students' perspectives. Early studies focused on students' approaches to determining if groups were isomorphic. Dubinsky, Dautermann, Leron, and Zazkis (1994) indicated that students focused on cardinality of groups when looking for isomorphisms, but not whether the homomorphism property was satisfied. Leron, Hazzan, and Zazkis (1995) also noted students' tendency to check cardinalities, but their students tested other properties too. Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to theorems related to isomorphism and to prove or disprove specific groups were isomorphic. While both doctoral and undergraduate students were able to prove simple propositions, doctoral students had continued success with more sophisticated propositions. Melhuish's (2018) replication study suggested doctoral students' success rested on their relational understanding of properties of isomorphism, though knowledge of properties and relational use of them could be distinct.

In isomorphism-focused studies, some research has been conducted on homomorphism. Nardi (2000) noted students' struggles in proving the Fundamental Homomorphism Theorem (FHT) stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping, and not recognizing the purpose of sections of the proof. Much like the isomorphism context, Weber (2001) observed that despite undergraduates' ability to recall relevant theorems, they struggled to move past "definition unpacking" techniques when trying to prove theorems related to isomorphism and homomorphism, whereas doctoral students experienced success by invoking their holistic understanding of concepts. Melhuish, Lew, Hicks, and Kandasamy (2019) examined relationships between students' understanding of function and their abstract algebra knowledge, including understandings of isomorphism and homomorphism, by examining properties, metaphors, examples, representations, and different views of functions in abstract algebra.

Limited work has been conducted on mathematicians' views of isomorphism. Weber and Alcock (2004) noted algebraists' notions of isomorphism as meaning groups were "essentially the same" (p. 218) or that groups being isomorphic meant "one group was simply a re-labelling of the other group" (p. 218). However, non-algebraist instructors' views remain unknown.

A theoretical lens for analyzing mappings is the conceptual metaphor construct (e.g. Lakoff & Núñez, 1997). Conceptual metaphors utilize one's structured knowledge of a source domain to inform one's view of a related target domain. Because conceptual metaphors reveal the structure of thought, they are a natural lens for studying the abstract concepts of isomorphism and homomorphism. Conceptual metaphors have previously been used to examine students' reasoning about functions in algebra (Zandieh, et al., 2016; Rupnow, 2017). Zandieh et al. (2016) examined students' notions of function in linear algebra. Rupnow (2017) examined students' metaphorical expressions in the context of isomorphism. In this study, conceptual metaphors are used to frame responses to two research questions: (1) how did instructors draw on notions of sameness to describe isomorphism in interviews and instruction and (2) how did instructors draw on notions of sameness to describe homomorphism in interviews and instruction?

Methods

Data for this paper are drawn from classroom video and interviews of instructors. Classroom data was collected when isomorphism or homomorphism were discussed in class. Only segments focused on isomorphism or homomorphism were completely transcribed.

The participants were two faculty members teaching an introductory abstract algebra class. Instructor A used the Inquiry-Oriented Abstract Algebra (IOAA) materials in class. Instructor B used a mixture of lecture and activity days. Both had taught the course at least once before and taught isomorphism before homomorphism. Neither does research in algebra. Instructors were recruited at the beginning of the semester from that semester's abstract algebra teachers. Participants engaged in semi-structured interviews (Fylan, 2005) lasting roughly one hour each. The interview with each instructor occurred as they began teaching isomorphism and focused on definitions and descriptions of isomorphism and homomorphism, as well as their explanations when teaching. Interviews were audio and video recorded and any written work was collected.

The interviews and videos were transcribed and coded using thematic analysis (Braun & Clarke, 2006). This included multiple iterations of coding (Anfara, Brown, & Mangione, 2002); first, transcripts were open-coded for vivid, active words that could indicate conceptual metaphors; next, statements were viewed holistically for mathematical approaches being conveyed by statements; finally, codes were generated and refined by repeating the previous stages. These codes were influenced by Zandieh and colleagues' (2016) and Rupnow's (2017) work. Codes related to the formal definition and mapping also arose (Rupnow, 2019), but only codes related to sameness or combining sameness and mapping are reported here in Table 1.

Results

Results address isomorphism (research question 1) and then homomorphism (research question 2). Metaphor codes are abbreviated to generic sameness, same properties (isomorphism) or similar properties (homomorphism), matching, renaming (Instructor A) or relabeling (Instructor B), disembedding, and equivalence classes.

Isomorphism

Instructor A. Many examples of sameness language were invoked in both interview and classroom settings, though more variety appeared in class.

Interview setting. When discussing isomorphism, Instructor A often invoked generic sameness. Their initial description of isomorphism was: "When I think about groups, if they're isomorphic, it means that they are the same group just notated with different names or notated with a different operation, but that the groups are essentially the same." They went on to link this generic idea of sameness to the specific idea of finding a renaming function to demonstrate sameness: "I would verify that two things are the same by finding a renaming function, an isomorphism between the two groups. So I kind of feel like that is the test...."

Table 1. Metaphor code descriptions and examples.

Metaphor Code	Code Description	Example
An isomorphism/A homomorphism is a mapping between groups that are the same/similar.	Generic references to groups being the same or similar	"So going back to the two different groups of just things...you would have to have the same, like, relationship equivalencies between the one and the other group."
An isomorphism/A homomorphism is a mapping between groups with the same/similar properties.	Use of properties shared by isomorphic groups (e.g. size) or in a homomorphism (domain mapping to a range with the same or fewer elements)	"...they're two different sizes so there can't exist an isomorphism between them."
An isomorphism/A homomorphism is a matching.	Emphasizes aligning similar objects with each other	"I guess I would say [isomorphism]'s a correspondence that...matches like things with like things."
An isomorphism/A homomorphism is a renaming/relabeling.	Aligns similar elements while recognizing names are arbitrary	"If you just took these elements and attached these other labels instead of the labels you originally had and you get the same exact structure..."
An isomorphism/A homomorphism is a disembedding.	Extraction or highlighting of properties from domain to codomain	"A homomorphism is just a function that preserves the structure…not necessarily...all of the structures…Like the integers map to Z mod 2…, that could preserve the structure of…the evens and the odds, but it destroys a lot of the other properties of the integers."
A homomorphism is a mapping defined by equivalence classes.	Leverages shared structures of groups to find similar elements in the domain	"So my idea for the homomorphism is just taking it from Z mod 6 to Z mod 3 sort of directly so I mean like, theta of 4 mod 6 would equal 1 mod 6 because they'd both be 1 mod 3..."

When asked if they thought about isomorphism the same way they described it to students, they related their thinking about the formal definition to renaming again:

> I want them to get to the formal definition, but even then, I want them to understand the formal definition as like a renaming function. I feel like that was not at all obvious to me as a student. And so it was really hard to unpack...why this...seemingly arbitrary function would prove that two things were the same.

In addition to standard descriptions given to describe isomorphism, Instructor A related isomorphism to naming stuffed animals when asked how they would explain isomorphism to a child: “I might say…each of their stuffed animals has a name, but it would be the same bear even if I called it a different name....It’s the same bear if I call them Fred or Sam.” Comparing this intentional metaphor to their standard language for isomorphism, we see that the choice highlighted sameness once again. Specifically, they highlighted the arbitrariness of the name of the bear in keeping with the renaming metaphor.

Classroom setting. In teaching, Instructor A used sameness when defining isomorphism and when thinking about how to approach whether or not groups were isomorphic (e.g. “Always start these with…are they the same?”). Same properties language was also incorporated, especially when verifying groups were not isomorphic (e.g. different sizes implies not isomorphic).

Renaming language was only used when first presenting the formal definition as a colloquial way to understand the idea. However, matching was used often before the formal definition was given to reason about whether groups in Cayley tables were the “same” in some way:

> If I look at the same equation, those line up…and those should line up, but when I look at my equation, those don’t line up. On the mystery table, this equals D. So it’s like I can check the equation on the mystery table and D_6 and see if I get the same answers.

Note this matching language placed value on the specific results of computations, whereas the renaming language used names but focused more on underlying properties.

Summary. In both interview and classroom settings, Instructor A highlighted generic aspects of sameness as well as the specific idea of renaming. In class, matching and same properties language were added. Same properties were used to identify specific aspects that were or were not the same when addressing specific groups, which had not been a prompted activity in the interviews. Instructor A’s use of matching language often seemed to replace renaming in class.

Instructor B. Like Instructor A, many examples of sameness language were used in interview and classroom settings, but more variety appeared in class.

Interview setting. Instructor B used isomorphism language focused on general sameness as well as relabeling. When first asked about the words or phrases that came to mind to describe isomorphism, they provided three images: “structure-preserving map, equivalence of structures, relabeling of elements.” Matching was incorporated when considering how to describe isomorphism to a child: “It’s a correspondence that matches like things with like things.” When pressed for a preferred way of thinking about isomorphism, they wove together relabeling and sameness ideas with the formal definition:

> I really prefer to think of it as a relabeling so that…from an algebraic point of view, there’s really no difference between these structures, and so...if you just took these elements and attached these other labels instead of the labels you originally had…you get the same exact structure. So that’s the idea I try to get across more than...that you have a...bijective function that...preserves such and such operation.

They incorporated disembedding as well as general mapping and sameness language when expanding on what relabeling meant to them:

> The isomorphism itself can just sort of disappear to the background and you can...really just identify...these structures….And you could start talking about *the* cyclic group with n elements….And you don’t need to know...what cyclic group, you don’t need to…know that there could possibly be two different cyclic groups hanging around, and…a function mapping elements to another. You can just say, ‘Well, if...I had a different instance of a cyclic group with n elements, if I wanted to, I could just change those labels to these labels,’ and so really it’s the same underlying structure.

Of note, their concept of isomorphism was about a shared structure, allowing us to see the groups were *isomorphic*, as opposed to a focus on the function that connected them (*isomorphism*).

Classroom setting. In class, sameness was used to refer to groups being "essentially the same" on a number of occasions, especially when the definition was initially given. Same properties were used especially during the activity day when students were being prompted to look for properties that should hold in both groups if they were isomorphic. Relabeling was used numerous times to refer to the isomorphism being used to show groups were isomorphic and on the activity day, it was used to prompt approaches for students when working with discrete mappings. Matching was used a few times as students were prompted to look at which element was "matched up" with each element when creating a discrete mapping.

Summary. In both interview and classroom settings, Instructor B highlighted generic aspects of sameness as well as the specific ideas of matching and relabeling. In class, same properties language was added. Like Instructor A, same properties language was used to identify specific aspects that were or were not the same when addressing specific groups, which had not been a prompted activity in the interviews. In both interview and class contexts, Instructor B used relabeling language more than matching language.

Homomorphism

Instructor A. Although Instructor A often used sameness-related ways of thinking about homomorphism in the interview, they used these infrequently in the classroom setting.

Interview setting. Instructor A frequently spoke in terms of equivalence classes, specifically with a goal of addressing which elements were behaving in a similar way. For example, when initially describing homomorphism, they focused on equivalence classes as a way to identify which elements from the domain group were somehow the same:

> I think equivalence classes; like the idea that I could pick one representative for a set, and the homomorphism…gives me a way to think about what's equivalent.…Apply the same name to a group of things that are equivalent or collapsing a set into a single element…

Thus, for homomorphism, much like isomorphism, they viewed sameness as central to their understanding. However, in this case the sameness focused within groups, to see which elements acted in the same way, instead of globally identifying the two groups as being the same. When asked if their view of sameness in isomorphism and homomorphism was the same, they clarified:

> With isomorphism I mean this collection and the operation is the same as this collection and operation, and the only thing different about them is the names that I chose for the elements in operation… In homomorphism... I'm saying what things in my domain are the same under that mapping. So it's kind of like I take my domain, call everything the same, and then that collection of same things maps to a single element in the range.

When asked to describe a homomorphism to a ten-year-old, they struggled to find an adequate picture, but finally decided the 'class of same name' idea could be captured:

> So going back to stuffed animals…I could sort things into bears and dogs…, but the idea that it partitions the set, or that these are equal sizes…there it quickly falls apart with that collection of stuffed animals. But I could get at...that they function the same.

In this intentional analogy, we again see a focus on creating equivalence classes, much like the majority of the time spent discussing homomorphism. When asked if they thought about homomorphism in the same way they tried to relate it to students, they believed they spent more class time on the formal definition than ideas of sameness in class.

Classroom setting. Their perception of sameness language used in class given in the interview was relatively accurate. Equivalence classes were highlighted on the last day of the unit in the context of the FHT. While discussing this theorem, pictures were drawn on the board illustrating bands that would be mapped to specific places. The similar properties metaphor was added in class, but largely used indirectly as properties of homomorphisms were being derived. Most other discussion around homomorphism centered on formal definition language.

Summary. In the interviews, Instructor A's view of homomorphism focused on sameness, especially through equivalence classes. They emphasized a view of localized sameness, where elements that were the same in the domain would be collapsed to a single representative. This image was shared in class, but on the last day of the unit. Thus, most of the sameness language that infused the interview were absent until the final day of the unit.

Instructor B. Although Instructor B used some sameness-related language in the interview, notions of sameness were clearer in their teaching.

Interview setting. When discussing homomorphism, Instructor B initially used formal definition language. However, when asked to explain further, they contrasted their view of homomorphism with their view of isomorphism as a relabeling: "Since you...lose the bijectiveness, you sort of lose this…other way of thinking about it as just...being able to take an element here and then just attach the label that you were using over here instead of…the original label." After contrasting with isomorphism, they created an image focused on collapsing by mapping that fit the equivalence class type: "I guess you could sort of view it as threads condensing into a single...element in the codomain and...then those would become equivalence classes modulo the kernel of the map." In this description, a set of elements from the domain are mapped to a single element in the codomain. Each such set forms an equivalence class, and these sets all have the same size as the kernel, which is the equivalence class mapped to the identity.

When asked how they described homomorphism to students, they noted it shifted over the course of the semester. Initially they would focus on homomorphisms as isomorphisms without bijectivity. However, later in the semester, more details would emerge:

> When you look at the seven elements that get mapped to a particular element, then what we really have is this…equivalence class modulo the kernel, and then we can...if we mod out by the kernel, then we can take any one of those things as a...representative. So I think...by...the first isomorphism theorem [FHT], then I'm sort of describing to them what I'm thinking about when I think about a homomorphism.…Kind of don't really initially see how the... structure within the...domain group is reflected in the...co-domain whereas with isomorphism we...see that right away.

Note Instructor B would introduce the term homomorphism soon after introducing isomorphism, but because students did not have an understanding of quotient groups to draw upon, the initial motivation for homomorphism was showing it was distinct from isomorphism. However, this initial distinction shared with students was not Instructor B's structural view of homomorphism. Once students learned about quotient groups and could understand the FHT, the structural view of homomorphism would become accessible and part of instruction through equivalence classes.

Classroom setting. In class, the similar properties metaphor was used indirectly as different properties of homomorphism were derived and as the FHT was used to find all possible homomorphisms. Disembedding was used in the context of the FHT as a way to relate groups:

> If you've got a surjective homomorphism, then the range H essentially is already living inside of G somehow. All the information about H is already here, and in fact we can recover H purely in terms of G by taking the factor group of G mod the kernel.

Another way of saying this is if we have an onto homomorphism, then the structure of the range exists in the domain somehow and can be extracted through quotient groups. This metaphor was also used in practice as a template for finding all possible homomorphisms between groups. Equivalence classes were modeled through specific examples that used congruence as a potential homomorphism mapping. For example, the instructor showed a homomorphism existed from Z_6 to Z_3 by using congruence classes as the mapping, though the rationale for choosing this function was not stated. After the FHT was given, equivalence classes were suggested as the "big picture" takeaway: "the range of the homomorphism has to be a factor group of the original group."

Summary. Instructor B did not use generic sameness language to talk about homomorphism and used limited sameness-based language in the interview. Yet, sameness-based language was invoked through equivalence class and disembedding language. In class, equivalence class and disembedding language was used throughout the unit when structuring approaches to finding homomorphisms and to explain the FHT. Thus, despite not immediately recognizing sameness in their thinking about homomorphism, it still structured much of their teaching.

Discussion

Both instructors intentionally drew upon ideas of sameness to discuss isomorphism in interview and teaching contexts. These included calling isomorphic groups "essentially the same" and using renaming or relabeling to talk about how the isomorphism function showed sameness. In class settings, Instructor A used more matching, a less abstract version of renaming. This did not appear to be an intentional shift but could have been influenced by the instructional materials or a desire to give students a more concrete understanding of isomorphism.

For homomorphism, differences were more obvious between interview and class contexts. Instructor A provided a detailed view of localized sameness through equivalence classes in the interview but only used this imagery on the last day of the unit after introducing the FHT. Instructor B did not initially use sameness-based language to describe homomorphism in the interview but used equivalence classes to structure finding homomorphisms both before and after the FHT and used disembedding to interpret the FHT.

These shifts in language also highlight the difference in the conceptual difficulty of isomorphism and homomorphism. Whereas both instructors directly noted sameness was at the heart of their understanding of isomorphism, only Instructor A stated a sameness connection in homomorphism. Even then, it required further explanation to articulate what type of sameness was intended and how it could be distinguished from the stronger sameness of isomorphism.

Conclusions and Future Work

The instructors drew upon different ideas related to sameness to discuss both isomorphism and homomorphism. Instructors' views of isomorphism, including general ideas of sameness and renaming or relabeling, were similar to algebraists' recorded views (Weber & Alcock, 2004). However, it is unknown whether instructors who are not algebraists and trained algebraists have different views of the more conceptually difficult idea of homomorphism.

Of note, there were subtle shifts in language for isomorphism and homomorphism in the two contexts. The instructors' sameness language for homomorphism especially shifted in interview and classroom contexts. One instructor used extensive sameness language in the interview but only used such language on one class day. The other instructor struggled to give non-definition views of homomorphism in the interview but threaded sameness-based ideas throughout their teaching. This suggests that instructors may not always employ language reflecting their own understanding in the classroom; other instructors may not be able to articulate their understandings equally well in all contexts. Furthermore, this raises questions about what students take away from instructors' lessons: frequently used language or the last language used.

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Metacognition in College Algebra: An Analysis of "Simple" Mistakes

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Students' characterizations of their errors have the potential to direct their future efforts in mathematics, both in and out of the classroom. This study investigates the errors made on one exam by three college algebra students who identified "simple mistakes" as the primary reason they were not satisfied with their test grade. We developed codes for the types of errors made and classified each as "simple" or "not simple." We then calculated the percentage of points lost due to simple mistakes for each of the three students. Results showed that for each student, approximately 20-40 percent of the errors made on this test could be classified as simple mistakes. We present hypotheses connecting the students' frustration to these results and provide directions for future research that may test these hypotheses and illuminate differences between student and instructor definitions of simple mistakes.

Keywords: Metacognition, College Algebra, Error Analysis, Simple Mistakes

Introduction

Math education research has largely focused on the cognitive dimension of learning, exploring students' understanding of specific concepts and the ways in which instruction can better serve students in their pursuit of mathematics learning. While incredibly important, this focus alone does not explain or assist with challenges that go beyond or precede understanding of the content. For students to learn most effectively and thoroughly, they must be able to assess their level of understanding accurately to know where to focus their studying. Metacognition, defined as "knowledge and cognition about cognitive phenomena" (Flavell, 1979, p. 906), is not a fixed trait, but rather knowledge that can be taught and expanded. It is possible for mathematics instructors to help students develop metacognitive knowledge and skills in the classroom (Donker, De Boer, Kostons, Ewijk, & van der Werf, 2014; Schneider & Artelt, 2010). Moreover, this is a worthwhile effort because of the link between metacognition and performance. Specifically, meta-analyses have shown that students with stronger metacognitive skills outperform their peers (Ohtani & Hisasaka, 2018; Schneider & Artelt, 2010). However, despite calls for efforts toward this end (Pintrich, 2002) little attention has been directed towards equipping mathematics instructors with methods to develop students' metacognition.

This paper discusses part of a larger study that examines the impact of metacognitive instruction for college algebra students. Here we classify the types of errors made by students who describe their primary reason for poor exam performance as making "simple," "careless," or "small" mistakes. We are motivated by our expectation that students' views of their errors may impact their mathematical confidence, their willingness to persist, and the ways in which they focus their study efforts in the future. Specifically for those students, we ask: How closely do students' perceptions of their mistakes align with their actual mistakes? To answer this question, we discovered the types of errors students made on a college algebra exam and then classified those errors as "simple" or "not simple." We then determined the proportion of lost points that could be attributed to “simple mistakes.” Finally, we considered whether, without simple mistakes, the students may have achieved their desired grades, and ultimately whether their disappointment with their performance could accurately be attributed to these kinds of error.

Review of Literature

Research of student metacognition has considered both knowledge and regulation of cognitive processes (Brown, 1978; Desoete & De Craene, 2019; Garofalo & Lester, 1985). Most have agreed it is difficult to completely distinguish metacognitive knowledge from self-regulation. Social cognitive theory now includes metacognitive processes, motivation, and self-efficacy as part of self-regulatory knowledge. Self-regulation skills can be taught and use of these skills improves student performance (Cook, Kennedy, & McGuire, 2013; Zimmerman, 2015).

Mathematics educators, focusing on metacognitive skills, recognize the key role of metacognitive monitoring processes in problem solving (Carlson & Bloom, 2005; Desoete, 2003; Goos & Galbraith, 1996; Schoenfeld, 1992; Silver, 1982). As Garofalo and Lester (1985) point out, others have studied “examining one’s own knowledge and thoughts” without using the term metacognition. They highlight Skemp’s “reflective intelligence” and Piaget’s “reflexive abstraction” as two key examples.

Some have begun trying to improve metacognitive function through in-class instruction, which is appropriate, as studies have fairly consistently shown a strong link between metacognition and performance (Pintrich, 2002; Zimmerman, 2002; Zimmerman, Moylan, Hudesman, White, & Flugman, 2011). The call to provide metacognitive instruction in context, rather than detached from mathematics, is not new (Lester, Garofalo & Kroll, 1989). Many students lack metacognitive awareness when they come to college (Pintrich, 2002) and could greatly benefit from metacognitive strategies being taught in the context of a specific course (McGuire, 2015; Tanner, 2012). Zhao, Wardeska, McGuire, and Cook (2014) report on a 55-minute introductory chemistry classroom intervention that presented to students the results of a learning strategies survey and introduced students to the Study Cycle developed by Frank Christ (1998). Similarly, Tanner (2012) devoted time in a biology course to instruction on planning, monitoring, and evaluating strategies. In math education, recent studies show the success of teaching students to use self-explanations in learning mathematics (Hodds, Alcock, & Inglis, 2014; Rittle-Johnson, Loehr, & Durkin, 2017). There have also been some attempts at teaching other various study skills in the college mathematics classroom (Taylor & Mander, 2003). The meta-analysis of Donker and colleagues (2014) found metacognitive knowledge instruction had an effect size of 0.66 in the field of mathematics.

Psychologists Kruger and Dunning (1999) note that people who are unskilled in a particular domain tend to overestimate their abilities in that domain; they attribute this miscalibration to a lack of metacognition. The Kruger-Dunning effect, or “meta-ignorance” suggests that students don’t know what they don’t know; thus, it is possible to label an error as a “simple mistake” when in fact the error was due, at least in part, to lack of conceptual understanding. A 2017 meta-review by Panadero, Jonsson, and Botella, shows that self-assessment contributes to self-regulated learning and self-efficacy. In order to improve student self-assessment, Zimmerman (2015) has called for more research on how self-regulatory knowledge can be included in all levels of instruction. Studies are needed to specifically look at the relationship between self-calibration and undergraduate mathematics performance and whether in-class metacognitive instruction can improve self-calibration and thereby improve performance.

We study student calibration by determining how closely students' prediction or assessment of their performance aligns with the actual errors they make on exams. Following the focus of math educators on the role of metacognition in problem solving, early error analysis classified errors according to the level of thinking used in Polya's (1945) four problem solving phases: understanding the problem, devising a plan, carrying out the plan, and checking the solution (Abdullah, Abidin, & Ali, 2015). Others have increased the breadth of error analysis to include mistakes made on typical high school problems (Zaslasky, 1987). These broad characterizations of errors led to more detailed analysis of errors for specific concepts from those as fundamental as slope to those as applied as optimization (Cho & Nagle, 2017; LaRue & Infante, 2015). We have built on and modified these error analysis techniques to focus on college algebra concepts.

Methods

This is part of a larger study about the impact of metacognitive instruction within the mathematics classroom. Twenty College Algebra students were identified as at-risk and enrolled in a support co-requisite course (co-course), which has traditionally been used to provide additional practice on content and an opportunity for students to ask questions in a smaller classroom setting. Attendance in the co-course was voluntary and the majority of students enrolled did not attend regularly. After exam one, approximately one day each week was devoted to activities designed to improve students' metacognition.

Seven students completed a Post-Exam Self-Assessment after exam one was returned and answered the question, "Are you happy with your score; why or why not?" Three students, whose exam one scores ranged from 51% to 71%, replied with some version of the statement "No, because I made lots of simple mistakes."

We analyzed all incorrect problems on exam one for these three students. Some exam questions were graded with partial credit based on student work and others were multiple choice. We analyzed graded exams where points lost for each error were clearly marked. We developed a coding scheme for the types of errors made. This was an iterative process of creating codes, discussing them among team members, revising and recoding. We did not restrict ourselves to an a priori list of codes, (Braun & Clarke, 2018; Braun & Clarke, 2006) but did refer to a list of error types on our exam analysis activity which students used to determine the reason for missing exam questions. This list included 13 reasons for errors such as "I made a copy error", "I could not remember the steps needed", "I made an arithmetic error," etc. We tested the coding system by individually coding exam questions for the three participants. In subsequent discussions, codes were added, combined and refined.

Once the list of codes was finalized, each error code was classified as "simple" or "not simple." We defined a "simple" error as one that could be made accidentally, would likely not be repeated, or violates a mathematical convention rather than a rule (i.e. not reducing coefficients). A "not simple" error arises from a lack of conceptual understanding. We do not know if this is how students define "simple" and "not simple" errors as this is beyond the scope of this study and is an area for future research. We note that in making decisions about things like convention, which may vary across different mathematical environments, we relied heavily on one author's knowledge of the course expectations. For example, many college algebra problems ask students to simplify their answer, and the definition of simplified may vary across classrooms.

Some errors were coded as "either" simple or not simple because of a lack of information about the student's thinking about written work. This led to results being reported as an interval rather than a fixed number. For example, students were not required to show work for multiple choice questions. For any multiple choice question with no work, the student could have simply bubbled in the wrong letter, a simple mistake, or had a conceptual misunderstanding, a not simple mistake. Therefore, multiple choice questions with no work were automatically coded as “either.” 8-14% of the missed points were on multiple choice problems with no work shown. For other problems, surrounding student work was sometimes used to determine the type of error. For example, if the student had performed a similar action correctly multiple other places, it was coded as a “simple” copy error. However, in the absence of evidence a student could perform a correct action, the error was coded as a “not simple” algebra error. In the future, interviews could be conducted to classify “either” errors as “simple” or “not simple.”

Once the codes were labeled as “simple” or “not simple,” an independent coder who was familiar with the course was presented with instructions for coding. The codes developed by the authors were listed as examples for the independent coder, but the independent coder was merely asked to classify each error as “simple” or “not simple.” Of 108 points lost between the three students, the authors classified 89 of those as either “simple” or “not simple.” The remaining 19 points were classified as “either.” For the 89 points not categorized as “either,” inter-rater reliability was 86.5%. The few instances of disagreement between the independent coder and the authors were balanced; that is, sometimes the authors classified an error as simple while the independent coder chose not simple, and in other cases it was reversed, so that the overall percentages of simple versus not simple were very similar.

Finally, we calculated metrics to determine how “simple” errors impacted student performance. Because any one problem may have contained multiple errors, the annotated graded exams were used to determine how many points were lost for each error and were coded accordingly. The percentage of missed points due to “simple” errors was then calculated. Additionally, the possible grade range without “simple” mistakes was calculated for each student and compared to their desired grade.

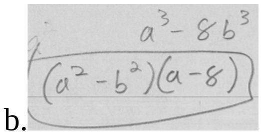
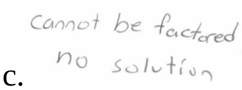
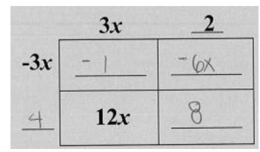
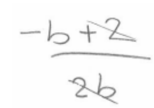
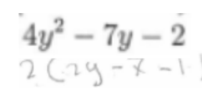
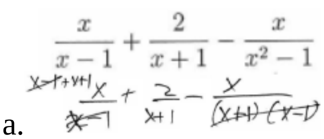
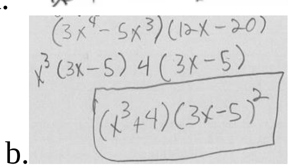
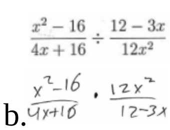
Results

The coding process ended with nine categories of errors, some of which were divided into subcategories. Tables 1 and 2 present these categories with student examples. We note that these categories emerged from analysis of a single exam; thus, the problem types examined were limited and we expect the categories would need to be expanded to be comprehensive even just for college algebra. Three of these categories were classified as “simple” and six as “not simple.”

The designations of these specific error types as “simple” or “not simple” are made specifically with college algebra coverage and expectations in mind. Errors designated as “not simple” for college algebra may be considered “simple’ for students in a higher level course. For example, a mistake in finding a common denominator when adding fractions involving variables would not be “simple” for college algebra as this is part of the course content for college algebra, but for a calculus student, such a mistake would likely be “simple.”

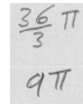
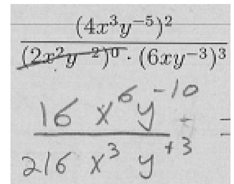
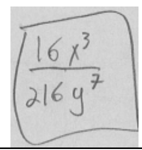
Table 1 provides examples of the six different types of errors classified as “not simple” mistakes. These errors were classified as “not simple” because there was an apparent lack of knowledge or understanding of college algebra or prerequisite content. We do not provide totals for the number of each type of error since we expect these results would depend significantly on the exam used and the problems given on the exam. Rather, we used these codes to classify each error as “simple” or “not simple” and report the percentages of each type for each student.

Table 1. Not Simple Error types

Not simple Error	Description	Selected Student Example
Formula	a) uses wrong formula for situation b) incorrectly memorized formula c) unaware a formula is needed d) incorrectly plugs into formula	b.  c. 
Misunderstanding	misunderstands question vocabulary, format, or instructions	
Arithmetic to Algebra	unable to translate arithmetic ideas to situations with variables	
Algebra (single step)	errors in factoring, distribution, exponent rules, like term collection, etc.	
Algebra (process)	a) incorrect application of memorized multistep process (wrong situation) b) memorized process incorrectly displaying no comprehension of concept	a.  b. 
Gets stuck	a) no work shown b) started but abandoned problem	b. 

Errors of arithmetic, transcription, and convention were classified as "simple" mistakes. These are displayed in Table 2. In each of these types of errors, we recognize first that the simple mistake is not likely to be repeated, and in classifying student errors, we confirmed this when possible by considering the student's surrounding work. We also noted that these were mistakes that could easily be made by an expert who was limited by time and did not check over their work. That is, the mistake could be made by someone who had a complete conceptual understanding necessary to solve the given problem.

Table 2. Simple Error types

Simple Error	Description	Selected Student Example
Arithmetic	Mistake in one of four arithmetic operations	
Transcription	Copy error not occurring as a pattern of misunderstanding	
Convention	Answer is correct but does not adhere to conventions	

We found that between 20-40% of the lost points on Exam One for these students can be attributed to simple mistakes, such as a one-time arithmetic error or copying a formula or expression incorrectly. This percentage is somewhat small when viewed in light of how the participants were selected: all three participants identified simple mistakes as the reason they were unhappy with their grade. Table 3, below, shows the percent of simple errors for each participant and shows their potential grade in the absence of simple errors.

Table 3. Results by Student

Student	Exam Score (%)	Percent of errors that are "simple"	Grade without "simple" errors	Desired Grade
1	68	18.75-40.63	77.08-84.38	80
2	51	20.40-36.73	63.54-71.88	70
3	71	20.69-37.93	80.20-85.42	90-95*

*Student 3 was absent when the class was asked about desired grades. This is the student's predicted score rather than desired score.

For each student, approximately 20-40% of mistakes were due to simple errors. Student 1 could have made their desired grade of 80% and student 2 their desired grade of 70% if they had not made these "simple" mistakes. Therefore, it is possible students do not believe all (or even the majority) of their mistakes are simple, but only enough to make the difference between the grade they got and the grade they wanted.

However, we note that the majority of the students' errors were not simple, yet this is the type of error that these three students identified as preventing satisfaction with their grade. It is also possible that students define a simple error differently than we have defined it and that they would have classified a larger portion of their mistakes as simple.

Limitations and Future Research

Analysis was conducted solely with written student work. This required making assumptions about student thinking and what students did or did not know or understand. This led to uncertainty in our results and having to present the results as intervals. Future work would benefit from incorporating student interviews, conducted shortly after exams, to distinguish more clearly between cases of simple and not simple errors.

Additionally, our results are not generalizable due to the small number of participants and the restriction of analyzing a single exam in a single course. Subsequent studies may ask all students in multiple courses to estimate the percent of points lost on multiple exams due to simple mistakes. These studies should also collect and compare student and instructor definitions of simple mistakes; one or more may differ significantly from the definition used here. We suspect that some students define mistakes as simple if the error is easily identified and understood after looking at the solution.

Another area of future research is on the effect of classifying errors as “simple” on student motivation and study effort. Misclassifying an error as “simple” when it was due to a conceptual misunderstanding could result in overconfidence. Ferla, Valcke, and Schuyten (2010) showed that overconfidence causes decreased motivation to understand course material, which results in decreased study effort and poorer course performance.

We pose a final question: If student definitions of a simple mistake do align with our definition, why are these students seemingly more frustrated with their simple mistakes than their not simple mistakes? We suspect the reverse may be true for instructors, particularly since a decrease in not simple errors would be an indication of learning gains as opposed to test taking skills. Future research may assess the impact of addressing this issue directly with students. In particular, instruction in how to direct their studying efforts as well as self-assessment training may cause students to focus their attention more on their not simple errors and how they might prevent these errors on future exams.

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Affective Pathways of Undergraduate Students While Engaged in Proof Construction Tasks

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The transition to proof is a known point of struggle for many undergraduate mathematics majors. This work examines this difficult transition point from an affective lens, specifically students' affective pathways while working on proof construction tasks. A series of four semi-structured interviews were conducted with 11 undergraduate students enrolled in a transition to proof course, and affective pathways were studied using non-traditional instruments: an emotion word selection task and an emotion graph. Open coding and constant comparative methods were used to identify common affective pathways within the emotion words and graphs. These pathways reveal common student experiences while working on proofs, which may be used to fine tune external learning conditions to positively influence engagement and experience.

Keywords: Emotions, Beliefs, and Attitudes; Reasoning and Proof; Problem Solving; Methodological Tools

This transition to proof-based mathematics is known to be difficult for undergraduate students; students struggle with learning how to prove, as a shift from mostly computational to now problem-solving work involving writing arguments (Moore, 1994; Selden & Selden, 1987). In order to help students through this, there is a need to understand the student's experience. We know much about the mathematical struggles students encounter (Alcock & Weber, 2010; Harel & Sowder, 2007; Selden & Selden, 1987), but to fully examine experience, especially those factors that drive experience, we need to understand affectively what is happening to students as they engage in proving (Selden & Selden, 2013).

Affect may be defined as "the wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition" (McLeod, 1992, p. 576). Three major types of affect are beliefs, attitudes, and emotions (McLeod, 1992). As we have increasingly paid attention to affect over the years (Middleton, Jansen, & Goldin, 2017), it is clear that affect is an integral part of learning, especially problem solving (McLeod & Adams, 1989). In thinking about what parts of a student's affect are malleable, it may be emotions they are open to change, as they occur in-the-moment (Middleton, Jansen, & Goldin, 2017). By understanding the sequence of students' emotions, we can develop and fine-tune tasks and lesson sequences to foster opportunities for positive affective experiences.

The research question guiding this work is: What are the affective pathways undergraduate students experience when constructing proofs, while learning how to prove? The primary purpose of this work is to examine the affective pathways of undergraduates' emotions while proving. A secondary purpose is to test novel instruments for collecting data about affective pathways: selecting emotion word cards and graphing their emotions.

Background and Theoretical Frameworks

There are different perspectives from which to approach studying students' proving (Stylianides, Stylianides, & Weber, 2017). I take the perspective of proving *as a form of problem solving*. Students' tasks can also be broken into three categories of activity: constructing, reading, and presenting arguments (Mejía-Ramos & Inglis, 2009). This study focuses on the task of *constructing a proof* for a novel argument, where statements to be proven are new to the student and additionally designed to cause an impasse, drawing from the problem solving perspective. Research on proof construction is common in the field (Mejía-Ramos & Inglis, 2009), and I take an affective view on the activity.

Historically, affect has been conceptualized as separate from cognition. Three types of emotions often discussed in mathematics education literature, dating back to McLeod (1988), are beliefs, attitudes, and emotions. These can be compared along the dimensions of *magnitude, valence (positive or negative), duration, level of awareness*, and *level of control* (McLeod, 1988; Gómez-Chacón, 2000). Early research focused on attitudes (e.g. Higgins, 1970) and later years brought a focus on students' beliefs about math and the influence of said beliefs. Studies on emotion, however, are rarer. Emotions are more difficult to study as they are much shorter in duration and thus fleeting and hard to capture, compared to attitudes and beliefs which have been measured using large-scale questionnaires. Due to my focus on emotions, I focused on capturing *magnitude* and *valence*.

DeBellis and Goldin (2006) discussed the notion of an *affective pathway* as a sequence of feelings, e.g. *bewildered* to *frustrated* to *surprised* to *happy* over the course of working on a task. This framework explains how certain emotions can be positive or negative depending on where they lie in the pathway and what function they fulfill. For example, *frustration* is commonly seen as negative but it can also lead to searching for a better solution so it can serve a positive function. In other words, "negative" emotions can serve a positive purpose. As such, the goal is not to avoid negative emotions. The affective pathway is one strand of a larger *engagement structure* (Goldin, Epstein, Schorr, & Warner, 2011), which describes a student's pattern of engagement in the classroom.

It is vital to study affective pathways, as a piece of engagement structures, because they occur in-the-moment and so can be influenced by the teacher, task, environment, etc. (Middleton, Jansen, & Goldin, 2017). By understanding affective pathways better, we can more precisely and with more understanding fine-tune the external conditions that influence learning that are under our control. In-the-moment affective constructs, like affective pathways, are thereby a vehicle for student change. The purpose of this work is to identify students' affective pathways while constructing proofs.

Method

Participants, Context, & Data

A series of four semi-structured interviews were conducted with N=11 undergraduate students taking a transition to proof course at a large Midwestern university. In each interview, students were given two proof construction tasks to work on (see Figure 1 for example).

We say that two integers, x and y, have the same <u>parity</u> if both x and y are odd or both x and y are even. Prove the following statement: Suppose x and y are integers. If $x^2 - y^2$ is odd, then x and y do not have the same parity.

Figure 1. Proof task from 1st interview.

Two instruments served as stimuli for students to describe their experiences: an emotion word selection and an emotion graph task. Originally meant as stimuli, one larger goal of this study is to examine whether the data from these instruments can be analyzed on their own. Self-report was an appropriate method because a subject's perception of their experience is by definition their experience, in what they take forward from them, drawing from phenomenography (Marton, 1981). There were N = 88 graphs and word selections in total.

Emotion word selection. Participants were shown eleven cards with an emotion written on each. Five pairs of positive-negative and one neutral emotion word were chosen based on the literature about emotion and math to capture range: annoyed, curious, disappointed, surprised, sad, joyful, frustrated, satisfying, ashamed, proud, and indifferent. Participants were asked to select which (if any or other) words they experienced on each proof task.

Emotion graphs (adapted from Smith et al., 2017; McLeod, Craviotto, & Ortega, 1990). Each participant was given a blank graph, where the x-axis was time and the y-axis represented valence of emotion (positive and negative). Participants were asked to sketch how their emotions varied in intensity over the entirety of each proof task. The graph allowed for a temporal look at the ups and downs in emotion over the course of the experience. Graphing emotions is a technique that can be used to describe variations in students' emotional responses while solving a problem (McLeod, Craviotto, & Ortega, 1990). Participants were asked to mark on the horizontal axis and/or annotate their graph with short captions at the points where they felt emotions shift.

Data Analysis

To identify affective pathways, I used qualitative bottom-up methods, specifically forms of open coding and constant comparison, to find groupings of conceptually meaningful categories. For the emotion graphs, I first categorized graphs by their end behavior, which revealed how the task ended: six graphs ended with negative emotion, ten ended with baseline emotion, and 72 ended with positive emotion. I then used the following algorithm: First, I sorted graphs by Overall (monotonically) Positive, Overall Negative, Flat or little variation, and Other. I separated monotonic graphs, especially positive ones, because dips were likely to represent hardships and hence different experiences. Lastly, I looked within the Other category for meaningful patterns and found the Concave Down and Concave Up categories; the rest stayed categorized as Other.

While analyzing the word selection, the emotion words fell into one of three categories: Negative, Neutral, or Positive. Originally considered positive words, curious and surprised were in fact *off-balance states*; they could be spoken about positively or negatively, depending on the context and surrounding emotions. For these reasons, curious and surprised were categorized along with indifferent now as Neutral. Many participants naturally listed emotion words in the order in which they felt them, despite no prompting to list them this way. I coded the temporal order for the shifts in valence (positive/negative/neutral).

Negative	**Positive**	**Neutral**
annoyed		curious
disappointed		surprised
sad	joyful	indifferent
frustrated	satisfying	
ashamed	proud	

Figure 2. Positive, negative, and neutral emotion words.

Results

First, I share an example of a positive and negative affective pathway. These examples were chosen because they illustrated the pathways well, both in graph and words. Then, I report on the types of affective pathways found looking across all the entire dataset of graphs and words.

Example: Positive Affective Pathway

Positive affective pathways are characterized by a positive end behavior and tend to overall trend positive. Figure 3 shows an emotion graph that depicts a positive affective pathway.

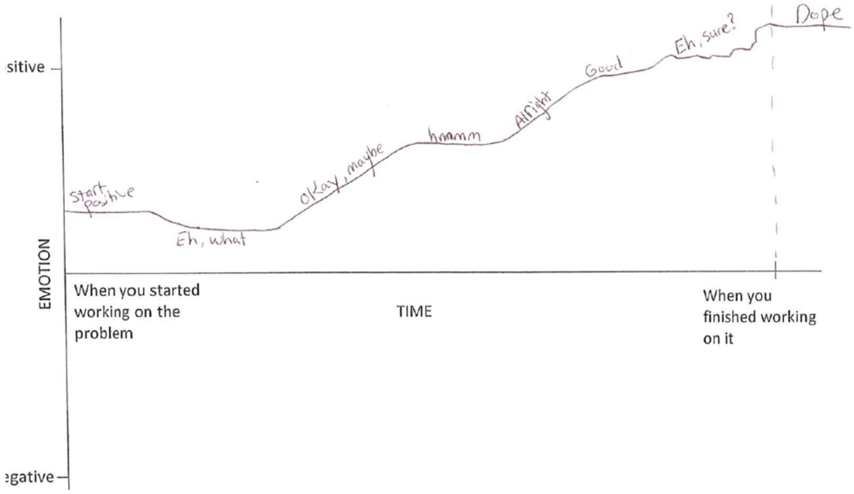

Figure 3. Leonhard's emotion graph for Interview 2, Task 2.

Leonhard chose the following emotions words, in this order: curious, befuddled (his choice), surprised, proud, satisfying, and joyful. One can see how the start behavior, dip, and then ensuing increases in his emotion graph match the order of his emotions: his annotations of "starting positive" may correspond to feeling curious, "eh, what" to feeling befuddled, and his annotations become increasingly positive of his work as do his emotions.

Example: Negative Affective Pathway

In contrast, a negative affective pathway typically ends with negative emotion and tends to overall trend negatively as the task goes on. The start behavior may be positive and there may be local peaks but the end behavior is always negative. Figure 4 is an emotion graph that depicts a negative affective pathway.

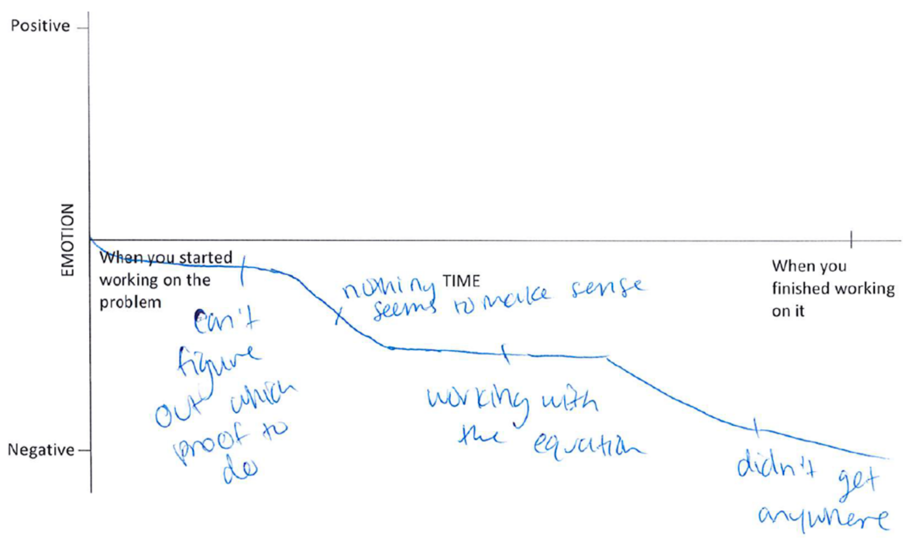

Figure 4. Gabriella's emotion graph for Interview 4, Task 2.

Gabriella started out at a neutral emotional level, based on her graph starting at y=0. Gabriella said she typically would approach proofs by looking for which proof method (direct proof, proof by contrapositive, proof by contradiction, etc.) to apply. Soon into the task, she didn't know which one to use. Based on this graph, we can see how not only were her emotions all negative after the start but also the magnitude of her negative emotions grew. There was a brief plateau as she was working with the equation given in the statement, but ultimately not coming to an answer felt extremely negative.

Gabriella's emotion words, chosen in this order, were curious, frustrated, annoyed, and sad. The feeling of curiosity is neutral, as it can in general be positive or negative depending on the context. Given that it was the first word she chose, the graph starting at the zero axis corroborates the idea of curiosity as neutral. From there, her string of negative emotions aligns with her graph showing increasingly negative feeling.

Having examined two examples of affective pathways via emotion graphs and words, we turn to analyzing the corpus of graphs and words to identify more complicated pathways.

Affective Pathways: Graphical Analysis

The emotion graphs reveal a number of different affective pathways, corresponding to different experiences while proving. Table 1 shows the affective pathways found within the emotion graphs. Of the 88 graphs, 39% were concave up, 18% were overall positive, 13% were overall negative, 10% were flat, 9% were concave down, and 11% fell into the Other category.

Table 1. Percentages are rounded.

Frequency Percentage of Affective Pathways via Emotion Graphs

Graph Behavior	Meaning	#	%
Overall Positive	Went well	16	18%
Overall Negative	Went badly	11	13%
Flat	No or little influence	9	10%
Concave Down	Started well, ended badly	8	9%
Concave Up	Overcame struggle	34	39%
Other	N/A	10	11%

Overall Positive, Overall Negative, and Flat graphs. Overall positive graphs represented experiences that went well, with no hardships to the extent that they negatively influenced emotions. These experiences may not be true problem solving, i.e. encountering a problem; these were experiences where the student was successful throughout. In comparison, overall negative graphs were where the problem solving experience went poorly, usually with the participant perceiving they did not know what to do or had not done it correctly.

Flat graphs were those where the participant drew a straight line, with little to no variation. Flat graphs showed experiences with little influence on and/or change in emotion, depending on if the graph were completely flat and above the x-axis because the problem felt easy or along the x-axis because the student was unsure the entire time.

Concave Up vs. Concave Down. Concave up and concave down match their mathematical namesakes: concave up graphs started at base-line or positive and had at least one dip but overall ended positively. These graphs suggested that the student overcame struggle(s). In comparison, concave down graphs were characterized by a high but then falling at the end. Concave down graphs were experiences that started well but where the student got stuck and could not resolve it. Based on the end behavior, concave up graphs tended to align with self-perceptions of success while concave down were with self-perceptions of failure.

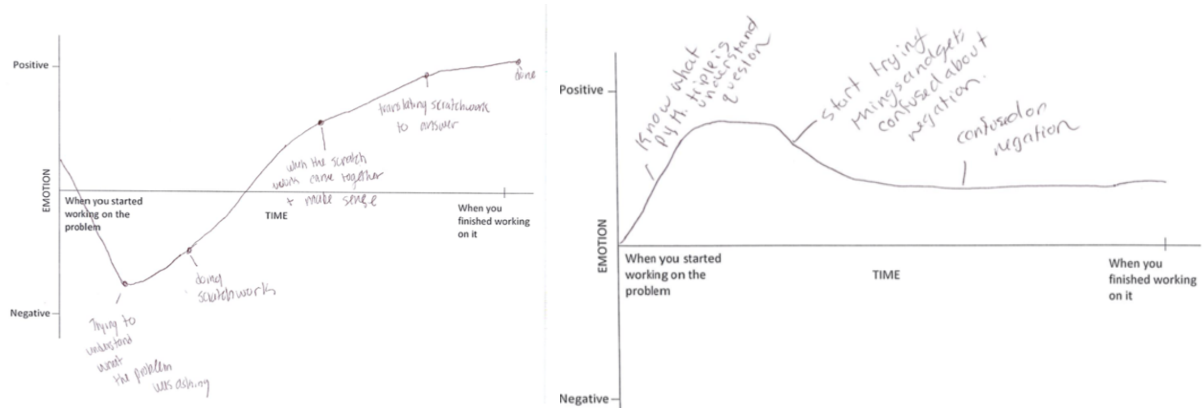

Figure 5. Examples of a Concave Up graph on left and Concave Down graph on right

Other. The Other category consisted of emotion graphs that did not fit the existing categories. The graphs had either of the following general patterns, shown in Figure 6.

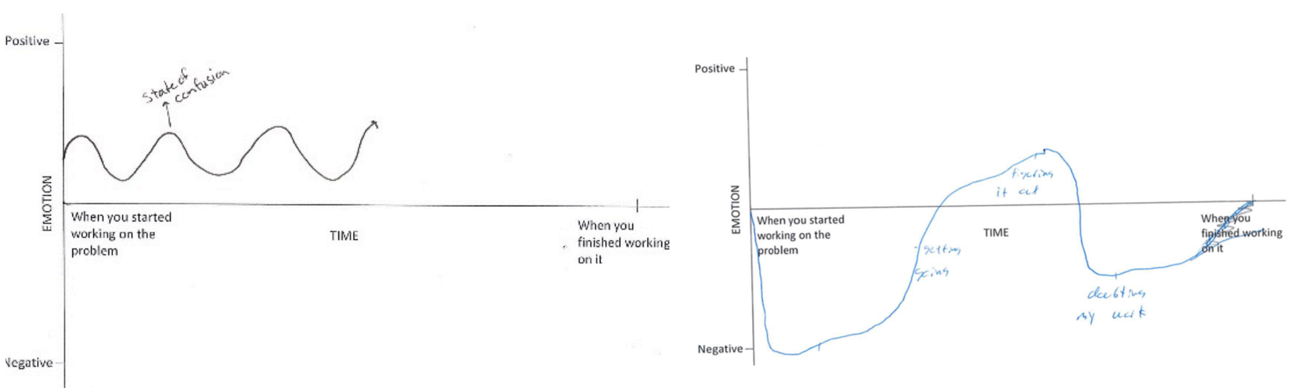

Figure 6. Examples of the Other category

Two emotion graphs reflected feeling confused (left graph of Figure 6). The rest had a number of peaks and falls (right graph of Figure 6), such that they could not be squarely *Concave Up* or *Concave Down*. These latter graphs reflect how proving (and problem solving in general) can bring about a number of emotional highs and lows for a student, in the space of a single task.

Affective Pathways: Word Analysis

There were more categories of affective pathways found in the emotion words, because they were a string of discrete items (in contrast to the continuous visual nature of graphs) and there was no maximum number of emotions. Table 2 lists the types of affective pathways of positive-negative-neutral emotions that occurred in the dataset with an example. This includes times where students did not explicitly say the emotions were in order of occurrence. This is a limitation, but I argue the order in which students stated them may be naturally revealing.

Table 2. + is positive, - is negative, N is neutral, ellipses (...) show additional emotion shifts, forward slash (/) indicates two emotions occurring simultaneously, and semicolon (;) indicates an emotion felt over the entirety.

Frequency and Percentage of Affective Pathways via Emotion Words, by Start and End Valence

Pathway	Description	#	%	Example Pathway
+	All Positive	11	13%	satisfying→joyful
-	All Negative	8	9%	confused
N	All Neutral	7	8%	indifferent
-/N...+/N	Emotions Rise	33	38%	overwhelmed/ashamed→curious→satisfying
+/N...-/N	Emotions Drop	14	16%	satisfying→curious→disappointed→indifferent
-...-	Negative Start & End	4	5%	nervous; ashamed→frustrated→indifferent→ disappointed
+...+	Positive Start & End	2	2%	satisfying→overwhelmed→surprised→annoyed →satisfying
N...N	Neutral Start & End	9	10%	curious→confused→surprised

The most common affective pathway was Emotions Rise (n=33, 38%). These experiences started out negative or neutral (curious/surprised/indifferent) and then ended well. These were likely true problem-solving experiences in that students likely did not know what to do at the start, in contrast to All Positive, where participants had positive emotions the entire way. The second most common affective pathway was Emotions Drop, (n=14, 16%) showing the opposite behavior, where the experience started well but then negative emotions occurred in the middle of the problem. These are instances where students experienced unexpected challenge and did not overcome them, perceiving their work to be wrong.

Discussion

These affective pathways illustrate the myriad of emotions that students experience while working on a proof. For some proof tasks, emotions are straightforwardly positive or negative. But more often than not, there were quite a few highs and lows over a problem as students searched for a solution, and these sharp transitions paint a detailed picture of students' mathematical experiences while proving. The emotion graphs provide a continuous portrait of the intensity and valence of students' emotions, while the emotion words give the nature of each emotion, as discretized by the individual. These tools together provide different information but can, if used carefully, be used to corroborate each other. One limitation is that the researcher searched for alignment; it may be beneficial to ask students themselves to align their words with their graphs, for validity. Future analyses will integrate graphs and words more closely.

Examining affective pathways contributes to understanding students' emotional responses to mathematics, specifically on proof tasks but also problem solving in general. By identifying common sequences of emotions, this work helps inform curriculum developers on how to design mathematical tasks and lesson sequences in order to foster opportunities for positive, satisfying mathematical experiences for all.

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A Tale of Two Sides: Students' Feedback Preferences and Professor's Feedback Provisions

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The purpose of this study is to investigate students' engagement with and utilization of the proof feedback from their professor, and their feedback preferences. We interviewed seven participants about their general experience with feedback on proofs. We then continued interviewing four of these students periodically about the feedback they received on specific selected proofs to see how they interpreted the feedback they received and how they utilize the feedback when asked to revise their proof production. The results showed that the majority of participants had a strong preference towards explicit feedback and, in contrast, that professors provided primarily implicit feedback. Furthermore, we found that students did not read professors feedback when they received satisfactory grades on their proofs, they usually only utilized the feedback when working subsequent assignments or studying for exams, and they did not use the Professors' feedback to rewrite their proof production.

Keywords: feedback, proofs, implicit, explicit, revision.

It is well documented that undergraduate students have difficulty writing proofs (Weber, 2001; Moore, 1994; Sowder & Harel, 1998) and that proof-writing is a primary competence that mathematicians want students to learn in upper-level mathematics classes (Weber, 2004; Zorn, 2015). In order to help students to improve their proof writing, professors provide feedback on students' proofs with the goal that students will use the feedback to improve their proof writing (Byrne, Hanusch, Moore, & Fukawa-Connelly, 2018; Moore, 2016; Miller, Infante, & Weber, 2018). It is important to investigate professors' feedback on students' proofs because feedback can have a greater impact on students' understanding and writing of mathematical proofs than other instructional practices (Black & William, 1998; Henderson, Yerushalmi, Kuo, Heller, & Heller, 2004; Hattie & Timperley, 2007).

Byrne et al. (2018) found that many students do not use professors' feedback in any significant way, if they use the feedback at all. One reason that students may not use professors' feedback in any significant way (or altogether) may be because they (usually) are not required to address the feedback or revise their proofs once assignments are returned. While we would argue that feedback can aid students' proof-writing skills and proof comprehension, we do not know much about students' preference for proof feedback and the type of feedback professors' provide on students' proofs.

Conceptual Framework and Related Literature

Researchers in a variety of disciplines have found that professors' (corrective) feedback on students' assignments is an important component of teaching and learning in the university setting (e.g., Ackerman & Gross, 2010; Hattie & Timperley, 2007; Hounsell, 2003; Mulliner & Tucker, 2017). Van Beuningen, De Jong, and Kuiken (2008) defined corrective feedback as "the marking of students' error by the teacher" (p. 280). Multiple studies have classified feedback in a variety of categorizations (for example Carroll & Swain, 1993; Ellis, 2008; Lyster & Ranta, 1997; Van Beuningen et al., 2008; Vardi, 2008) and for brevity, we will only mention two of these categorizations. These two are the ones that we used to analyze feedback in this study.

Several authors have described a feedback classification according to the explicitness of the feedback: *explicit* and *implicit* feedback (i.e. Carroll and Swain, 1993; Ellis, Loewen, & Erlam, 2006, Kim & Mathes, 2001). *Implicit corrective feedback* is defined as feedback that has no overt indicator that an error has been committed, whereas *explicit corrective feedback* refers to the feedback with an overt indication of the errors (Ellis, Loewen, & Erlam, 2006). In other words, in terms of proof feedback, implicit corrective feedback is instructors' feedback on a student's proof production that does not provide a clear indication that an error has been committed and explicit corrective feedback would clearly indicate a proof production error.

Research findings on *implicit/explicit* feedback have either found that explicit feedback is more effective than implicit feedback or that there is no difference. For example, a number of studies in language acquisition found the superiority of explicit feedback compared to implicit feedback (Carroll & Swain, 1993; Ellis et al., 2006). Other studies found no difference between the effectiveness between *implicit* and *explicit* feedback (Kim & Mathes, 2001). In a meta-analysis from 33 studies of *explicit* and *implicit* feedback, Li (2010) found that *explicit* feedback is more effective on a short-term scale and *implicit* feedback is more effective on a long-term scale.

Several authors also have defined a feedback classification considering the presence of correction towards the error: *input-providing* and *output-prompting* feedback (i.e. Ellis, 2008; Adams, Nuevo, & Egi, 2011, Lyster, 2004). *Input-providing feedback* is defined as feedback that supplies the learner with the correct form (or model) of the error, and *output-prompting feedback* is feedback that does not supply a model (Adams, Nuevo, & Egi, 2011). Most "research on implicit versus explicit feedback has not taken into account the *output-input* dimensions of feedback" (Adams, Nuevo, & Egi, 2011, p.45).

Other studies have found that *output-prompting* feedback is more effective than *input-providing* feedback (Ammar, 2008; Ammar & Spada, 2006; Lyster, 2004; Lyster & Ranta, 1997; Sheen, 2004). Lyster and Ranta (1997) and Sheen (2004) state that *output-prompting* feedback will most usually spur students to respond to the feedback. Ammar (2008), Ammar and Spada (2006), and Lyster (2004) found that students who received *input-providing* feedback (in the form of *recast*) performed no differently from students who received no feedback. Finally, Lyster and Izquierdo (2009) found that both *input-providing* and *output-prompting* feedback are equally effective in the context of dyadic interaction.

Regarding students' preference of feedback, Kim and Mathes (2001), Nagata (1993), and Amrhein and Nassaji (2010) reported that students clearly prefer *explicit* feedback. Moreover, Amrhein and Nassaji (2010) also found not only students would prefer *explicit* feedback, but they also prefer *input-providing* feedback. In contrast, the students think that *implicit* and *output-prompting* feedback would not be useful for them. Amrhein and Nassaji (2010) mentioned

> The students also showed approval of having their errors explicitly marked and corrected with WCF (written corrective feedback) such as error correction with a comment and overt correction by the teacher [...]. Most students explained that a clue with no correction is not useful because students need more specific advice (p. 115).

Finally, no prior studies have investigated professors' feedback on students' proof production in undergraduate mathematics based on *implicit/explicit* or *input-providing/output-prompting* feedback. However, there have been a few studies that have examined professors grading of proofs (Byrne et al., 2018; Miller et al., 2018; Moore, 2016) and Byrne et al. (2018) also examined professors' feedback on proofs with regards to Lyster and Ranta (1997) classification. Byrne et al. (2018) found that students had a lot of difficulty describing normatively correct logic for the changes that professors were expecting students to make when implementing their feedback on proofs. That is, for the most part, students could use the professor's feedback to rework their proofs, but had great difficulty justifying why the change was imperative.

We focused on the combined feedback categorization (both implicit/explicit and input-providing/output-prompting) for several reasons. First, Byrne et al. (2018) found that students have better understanding towards the feedback when it is supplied with a correction (recast and explicit correction). This suggests that feedback in the form of *input-providing* is superior than *output-prompting* in terms of eliciting corrections from students during proof revision. Secondly, prior literature in language acquisition has shown that students prefer *explicit* feedback while teachers find that *implicit* feedback is more desirable (Amrhein and Nassaji, 2010). Finally, language acquisition studies have found that both explicit and output-prompting feedback is more effective than implicit and input-providing feedback, respectively.

Since professors spend a considerable amount of time and energy providing feedback on students' proof productions (Moore, 2016) and we believe that professors want students to learn from the feedback that they provided, it would be well worth the time, especially at a point in the undergraduate mathematics curriculum when students are first learning about proof, to find out if professors' feedback are supporting students as they learn to write and understand proofs. In this paper we investigate students' preference and utilization of professors' feedback and the types of feedback that professors provide while grading proofs in an introduction-to-proofs class. The research questions that guided our investigation are stated as follows:

Research Questions:

1. What are students' preferences for the proof feedback they receive from their professor?
2. How do students use their professors' proof feedback?
3. In terms of *implicit/explicit* and *input-providing/output-prompting*, what type of feedback do professors give on students' proof productions in an introduction-to-proof class and how does that compare with students' feedback preferences?

Methodology

Participants

Seven students in an introduction-to-proof class at a large state university in the United States consented to participate in this study during the Fall of 2018. All participants had completed at least the second semester of Calculus. We anonymized the data by referring to the first interviewed student as S1, the second as S2, and so on, and assigned a feminine pronoun for each student.

Material and Procedure

Each participant (S1 through S7) met with the first author and was videotaped and audiotaped for the first phase of the interview that asked participants semi-structured open-ended general interview questions about their views and use of feedback on proofs, such as: Why do you think professors write feedback on your assignment?, Do you think feedback is important?, Is there any type of feedback you prefer or not prefer?, How do you use feedback?.

For the second phase, four participants (S1 through S4) agreed to meet with the first author to further interview about their interpretations of the feedback on a few of their graded proofs. The selected participants for this phase varied in performance (from below to above average) for the first part of the semester in the course with content covered in three chapters on basic set theory, logic, and basic introduction to proofs (direct, contrapositive, and contradiction proof techniques). For each participant, the first author, in consultation with the second author, selected two to three proofs on homework and exams during the remaining portion of the course for which participants had been given numerous items of feedback from the professor and participants' proofs contained a number of errors. The interviews occurred periodically as the proof artifacts were identified by the authors. For each selected proof, the participants were then asked by the first author to review their proof, discuss each feedback item that they received and the professor's intentions for the feedback, and finally revise their proof, articulating their thoughts as they worked on their revisions. This process was repeated for one to two more proofs for each participant, totaling 11 participant interviews.

All participants' revision work and their verbal thoughts about their revision were recorded on a LiveScribe notebook and backed up using video and audio recordings. After the participants finished working on the revisions, participants were asked to describe their approach to revising their proof to get their overall thoughts on their revision work. Once the interview was completed, the first author rewrote the revised proof to anonymize the participants' written proofs and names, and then asked the professor of the course (second author) to provide feedback and regrade the proof. The interviewer asked the students for another revision only if the revision submitted received a grade of 70% or lower. In the case that the participants needed to revise their proof again, the first author only asked them to revise their proof to see if they would use the feedback. This was followed by asking the participants to describe their approach to revising their proof again.

Analysis

All interviews were transcribed. The authors engaged in thematic analysis (Braun & Clarke, 2006) in the following way. First, both authors individually read over all the transcribed data while identifying and highlighting passages that might be of theoretical interest concerning feedback. Both authors met to discuss and compare their findings and developed a list of phenomena for further investigation. Next, each author reread the transcripts again, searching for further participant comments related to each phenomena of interest; we then put these excerpts into a file of comments related to that phenomenon. Through this process, we developed criteria and descriptions of these themes. Finally, the authors individually went back through the data one last time to identify passages that would satisfy the criteria for each of these themes.

In addition, the professors' feedback was categorized using the *input-providing/output-prompting* and *implicit/explicit* categories. Each comment was classified with regards to the two properties: is a correction presented and is there an overt indication that an error had been committed. To code this, we followed these steps:

1. Did the professor provide the correct form in the comment? If so, then it is *input-providing* feedback; otherwise, it is *output-prompting* feedback.
2. Did the professors give an overt indication of the error? If yes, then it is *explicit* feedback; otherwise, it is *implicit* feedback.

After each author categorized all the feedback, they meet to compare their feedback categorizations and discussed each time they disagreed until they came to a consensus. The result of this categorization can be found on Table 1 in the results section.

Results

Students Engagement with Professors' Feedback

Disregard proof feedback when awarded satisfactory grades. The first finding was that the majority (5 out of 7) of the participants did not pay attention to the professor's feedback when they are satisfied with their grade, regardless of the feedback. For instance, S5 talked about the circumstance that she would ignore the feedback, "I guess when I got a hundred on it [a proof], I might not read it [the feedback] because I figured I did well." S4 also mentioned getting good grades and prioritizing importance were reasons not to look at feedback when she said, “If I've got a lot of papers back and my grade is pretty good and there are other papers that need more attention, I'll ignore the feedback on the better ones.” Finally, S1 remarked about not paying close attention to feedback on a good performance; she gave an indication of the possibility that she might look at it later even with a good grade, “If I get like high enough grade I'll probably just like brush it [the feedback] off and wait to look at it for later.” In contrast, a minority (2 of 7) of the participants claimed to read the feedback more regularly. For example, S2 asserted that “I always read it [the feedback], I'm not quite certain if I always if I always read into it what I should but I always I always try.”

This evidence suggests that the grade can be an indicator of how much students engage with professors’ feedback.

Students’ utilization of feedback for future reference. The second finding is that students generally do not use the feedback to revise their work. This was illustrated by S3 when she noted, “I just remember that [the feedback], I usually don't write anything on paper. I know it's probably a good thing to do but hmm, I don't know I just I just look at it and then put it up usually.” Only one (of seven) student, S7, claimed to use the feedback to revise the work when she explained, “let's practice studying rewriting the whole proof and then change the parts that he advised me to change.”

As part of this finding, we found that most students (5 out of 7) professed they used feedback to help them with their next assignments. For example, S3 voiced, “if I have a similar problem in another homework then I might go back and look what I missed previously but usually I just looked at it for 5 minutes or so and put it up.” Moreover, S6 also stated that she would use the feedback for reference on future assignments, declaring “like getting ready for a final or like a big test after like the little quizzes I like to look at those [previous work with feedback] and make sure like remember this little thing that you messed up on.”

However, one out of seven students, S7, claimed that she would fix the part she messed up by redoing the work. She stated,

> if I messed up like part of the proof, I feel like okay let's practice studying rewrite the whole proof and then change the parts that he advised me to change and see if after that it looks a little bit better and then go from there.

In summary, the participants, generally, do not find professors’ feedback to be a prompt for them to conduct a revision on their own. This suggests that revising proofs as a response to the feedback they received does not seem to be a norm among students. However, this data suggests that the feedback had an impact on students’ future assignments; they would use the feedback as a reference when proving a similar statement on homework and when studying for an exam.

Students Preferences on Proof Feedback

First, the majority (5 out of 7) of the participants preferred *explicit* feedback. For example, S1 mentioned that “I think the best feedback is just tell me straight up what's wrong in it [...] I mean not like this can be or this could be right, but I'd rather just get a straight yes-or-no answer.” However, she did not state any preference for input-providing or output-prompting feedback.

S2 described that good feedback needs to be *explicit*, but also in the form of an output-*prompting*. This is illustrated as S2 stated,

> good feedback *tells you what you did wrong* and usually *has a hint* of how you could improve it [...] because if it is more detailed, if it's spelling it out for us, we're not as likely to think about it as much.

The preference for an *explicit* feedback was also emphasized S3 when the interviewer asked about her preference on feedback with the answer of "I guess any feedback that tells you why, you know, the underlying reason as opposed to just putting an X.*"* S3 did not state any preference for either *input-providing/output-prompting* feedback.

S6 emphasized the need for *input-providing* feedback, but no preference for implicit or explicit feedback when she said,

> I like whenever they actually like write out what the right answer would have been, not just saying you should have done this but like I like to see the steps and kind of compare like what I might have been thinking and didn't put to what it should have actually looked like.

Overall, the participants showed a strong view towards their preference for *explicit* feedback and did not particularly express a preference for input-providing or output-prompting feedback.

The Professor's Proof Feedback

Table 1 below shows the result of the classification of the feedback of participants' proofs that we used according to *implicit/explicit* and *input-providing* (IP)/ *output-prompting* (OP) categorization. We see that the majority (70%) of the professor's feedback were classified as *implicit* feedback and almost all (89%) of the feedback were classified as *output-prompting*. Therefore, the professor generally did not give feedback that explained the errors in the students' written proof, nor did the professor fill in the details or outline how to correct the error.

Table 1 ***Feedback type distribution on each of students' proofs. Each number represents the number of the professor's comments that belong to the types.***

Feedback Type		S1-1	S1-2	S1-3	S2-1	S2-2	S2-3	S3-1	S3-2	S4-1	S4-2	Quantity & Percentage
IP	Implicit					1				1	2	4 (9.09%)
	Explicit			1								1 (2.27%)
OP	Implicit	4	3	2	9			3	4		2	27 (61.36%)
	Explicit	1	3	1	1	2	1	1		3	1	12 (27.27%)

Conclusion and Discussion

There are four important results from this study. First, students' engagement with feedback was affected by grades. We found that grades affected the way students perceive the feedback and that students would dismiss the feedback when their performance is satisfactory. One possible effective way to improve students' engagement with feedback is refraining from assigning the grades (Black and William, 1998; Jackson and Marks, 2016). Jackson and Marks (2016) found that grade-withholding resulted in a significant improvement in feedback utilization which resulted in improvement on students' achievement. We hypothesize that professors provide feedback on proofs even when they assign a high grade to help students improve and reinforce their proof-writing, and understanding the norms of proof-writing.

Second, we found that most of the participants preferred *explicit* feedback from the professor, which reinforces language acquisition students' preferences (Amrhein and Nassaii, 2010; Kim and Mathes, 2001; Nagata, 1993). We also found that 70% of professors feedback was *implicit* which reinforced research by Amrhein and Nassaji (2010). However, we were unable to obtain results for students' preference for *input-providing/output-prompting* feedback. Since we found that: 1) the majority of the professors feedback was *output-prompting*, 2) some participants expressed they preferred *input-providing* feedback while others preferred *output-prompting* feedback, and 3) participants did not vocally express complaints about their professors predominating *output-prompting* feedback during the interviews, we believe that students do not have strong preference about *input-providing/output-prompting* feedback

Third, generally, we found that students do not use feedback to revise their proofs on their own and, for the most part, referred to feedback when working on similar proofs on subsequent assignments or when studying for exams. We find this to be a very positive result and an excellent use of feedback since students were not asked to revise their proof after they were graded, and that students referred back to the feedback when they were working on proofs in subsequent homework, when studying for exams, or both. Students may benefit from addressing the feedback on their proofs, or with an opportunity to either revise their proof or implement the feedback to prove a similar statement.

Finally, students' preferences on feedback did not align with what the professor liked to provide in terms of explicit/implicit feedback. Most participants preferred explicit feedback, but 70% of the professor's feedback was implicit. To balance students' preferences on feedback and professors' provisions on feedback, professors may need to consider giving more explicit feedback so that students obtain an overt reason on why the proof is flawed or why a correction is needed.

We were not able to determine all students' preferences with respect to *input-providing* and *output-prompting* feedback, but we did notice that 89% of the professors' feedback was output-prompting feedback. Finally, with respect to the combination of these two feedbacks, we found that the majority (61%) of the professor's feedback was *implicit output-prompting* feedback. These findings are in line with Amrhein and Nassaji (2010) where they found that teachers valued feedback that is less explicit such as *clues* or *comments* with no corrections, because they believed it is important for students to know how to correct their errors, so they understand it better.

Limitations and Future Directions

The findings in this paper may not accurately represent the behavior of mathematics' professors in general since our data only comes from one professor. Another limitation for this study is every participant was not interviewed on the same problems since we needed student proofs that had numerous errors, and this varied for each participant. For future research, more careful attention should be given to proofs produced by the participants over the course of the study to find ones where most participants had numerous errors.

Future research on feedback with a larger number of professors is needed to further investigate our results. Other needed studies may focus on the type of feedback professors provide, the purpose for each type of feedback, what they want students to learn from their feedback, and how their purpose impacts what feedback they provide.

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Students' Language about Basis and Change of Basis in a Quantum Mechanics Problem

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We explored how 12 quantum mechanics students from two universities discussed basis and change of basis as they performed probability tasks, one of which required a change of basis. We found that students' utterances referred to a person, a calculation, or a vector being in a basis, or a vector being written in a basis. Students discussed change of basis as changing the form of a vector, writing the vector in another form, making a vector become a new vector, and switching bases. We performed a discourse analysis on the situated meanings of students' phrases.

Keywords: linear algebra, basis, change of basis, quantum physics, discourse analysis

A course in linear algebra is commonly required for undergraduate physics students, given that several topics covered in linear algebra, such as vector and matrix manipulation, change of basis, normalization, and eigentheory are also addressed in physics courses, including quantum mechanics. Concepts included in both linear algebra and quantum mechanics courses are sometimes discussed and used differently in the two different contexts. One such example is that bases are typically orthonormal in quantum mechanics contexts (McIntyre, Manogue, & Tate, 2012), which is not the case in linear algebra generally. Furthermore, a change of basis in a quantum mechanics context may be performed differently than a change of basis in a linear algebra context, such as through formulaic substitution of commonly used bases rather than a change of basis matrix. Because instructors want students to have a cohesive understanding of basis and change of basis across these interdisciplinary contexts, it is useful to explore how students describe (change of) basis and perform calculations in a quantum mechanics context.

In this study, we explore how quantum mechanics students discuss linear algebra concepts in the context of two spin-½ system probability problems, the second of which required a change of basis. In particular, we explore how quantum mechanics students discuss basis and change of basis by conducting a discourse analysis of the situated, contextually-dependent meanings of their utterances that refer to basis and change of basis (Gee, 2005). Our research question is: how do students discuss basis and change of basis as they perform these tasks, and what are the meanings of students' utterances regarding basis and change of basis?

Literature Review

There is a growing body of literature related to student understanding of basis (e.g., Bagley & Rabin, 2016). Some studies described students' productive ways of intuitively reasoning about basis (Adiredja, Belanger-Rioux, & Zandieh, 2019; Adiredja & Zandieh, 2017; Zandieh, Adiredja, & Knapp, 2019). Adiredja and Zandieh (2017) investigated students' conceptual metaphors for basis. Students in their interview study described real-life examples of basis, including examples related to recipes, fashion outfit choices, Legos, marching band, and religious teachings. As students explained their examples, their verbs related to bases describing or generating a space, and their adjectives described bases as minimal, maximal, representative, essential, different, and non-redundant. Adiredja et al. (2019) described a class implementation of a task that prompted students to give real-life examples of basis and explain how they relate to the formal definition of basis. This task allowed students to connect their intuitive understanding of basis with formal mathematical language. Using the same task, Zandieh et al. (2019) found that their students often used real-life examples of basis that illustrated the roles of a basis as generating, structuring, and traveling, and the characteristics of a basis as different and essential.

Stewart and Thomas (2010) found that when students in their study reasoned about basis, they mainly focused on symbolic matrix manipulations such as row-reduction but often did not seem to understand how the calculations were related to finding a basis. In addition, the students often did not connect span and linear independence with basis as they created concept maps, nor did they attend to embodied conceptualizations of basis. Schlarmann (2013) found that two students focused on linear independence as they determined a basis for a particular subspace of $\mathbb{R}^3$ and on span as they verified their set actually formed a basis. We found few studies that focused on student understanding of change of basis. One exception is Hillel (2000), who found challenges students may face with the algebraic notion of change of basis; Hillel stated that students who understand a vector as a string of numbers may have difficulty may have difficulty understanding how one string of numbers can be equivalent to a different string of numbers.

Theoretical Framing

Spoken utterances have situated meanings; they take on specific interpretations in specific contexts relative to the culture of the people communicating (Gee, 2005). Situated meanings of phrases are negotiated through communicative social interaction, and they are assembled as people communicate in a given context based on their perception of that context. Language and context have a reflexive relationship in that the language used influences how people construe the context, and the context influences how people interpret the meaning of the language; discourse analysis involves exploring these relationships. In this study, we perform a discourse analysis of students' phrases referring to basis and change of basis within their problem-solving approaches to a quantum mechanics problem in an interview setting. We explore the situated meaning of students' utterances while considering the contexts in which the language was used.

Of particular interest to our study was that of Ochs, Gonzales, and Jacoby (1996), who performed a discourse analysis of physicists' utterances as they collaborated in a lab setting. They identified three main ways that physicists' technical parlance was grammatically constructed: 1) physicist-centered, 2) physics-centered, and 3) indeterminate. In physicist-centered constructions, the physicist is either an active agent who directly affects physical entities or an experiencer of those physical entities. Physics-centered constructions highlight the physics as being the subject of interest and consider relevant physical entities as either active or sentient in some fashion. Somewhere between these two concrete and more grammatically sensical constructions are the indeterminate constructions, which Ochs et al. found to be ubiquitous in physicists' utterances. These constructions are characterized by a personal pronominal subject completed by an inanimate predicate (i.e., a clause that grammatically should not apply to a personal pronoun), such as "when I come down I'm in the domain state." Ochs et al. supported the theory that the pronominal subjects here have a "blended identity composed of both the animate physicist and the inanimate physical entity" (p. 348); in other words, physicists used language that seemed to sympathize with the inanimate objects they were trying to understand, blurring their identity with that of the physical object.

Other studies recognized this kind of language in mathematicians' and students' utterances. As mathematicians describe concepts, they sometimes use talk, gestures, and diagrams "in ways that blur the distinction between the mathematical and physical world" (Sinclair & Tabaghi, 2010, p. 223), and mathematics and physics students sometimes use language that blur this distinction. Rasmussen (2001) exemplified a differential equations student who used language indicative of positioning himself as an actor in a direction field: "If I were unstable, I'd try to get stable as much as I could…if you're stable, you won't want to go away from that line" (p. 82). The student seemed to blur his own identity with that of a graphical solution to a differential equation. Gire and Manogue (2011) found that two quantum mechanics students used non-standard terminology in their explanations of operators. Although the students' indeterminate verbal constructions were not the study's focus, we noticed students made utterances, such as "when you measure something you are acting on it… you act on that state which changes the state" (p. 4), in which they seemed to blend their identities with that of an operator. This language may seem informal, but students' identification with a physics entity has been shown to be productive way of making sense of physics concepts (Scherr, Close, Close, & Vokos, 2012).

Methods

Semi-structured interviews (Bernard, 1988) were conducted with 12 quantum physics students; eight were in a junior-level course at a large public research university in the northwest US, and four were in a senior-level course at a medium public research university in the northeast US. Both courses used McIntyre et al. (2012) as the main text. The interview aimed to elicit evidence of student understanding of linear algebra concepts used in quantum mechanics. In this study, we analyze the students' responses to the questions in Figure 1. A relevant follow-up question was: "How do you see this problem relating to basis or change of basis?"

Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$.

a) Calculate the probabilities that the spin component is up or down along the *z*-axis.
b) Calculate the probabilities that the spin component is up or down along the *y*-axis.

Figure 1. The interview questions analyzed in this paper.

The data collected include video recordings and transcripts of the interviews and copies of student work. For each student, all mentions of basis and change of basis in the interview transcripts were extracted. Based on these excerpts, we performed a grounded theory analysis of how students talked about basis and change of basis (Strauss & Corbin, 1998). We performed multiple rounds of coding, in which we produced in vivo codes, used the constant-comparative method to identify similarities among the codes across all the transcripts, and grouped codes together to create axial codes that identified the different ways in which students tended to talk about basis and change of basis. These codes are discussed in the results section.

Some student utterances that we coded, such as "we're in the *z*-basis," seemed to have ambiguous meanings, yet we assumed these utterances were sensible to the speaker. Because many of these phrases were ubiquitous throughout the interview transcripts, we assumed the meanings of these phrases are culturally shared among these quantum mechanics students. We examined the situated meanings (Gee, 2005) of students' utterances regarding basis and change of basis to unpack what the students likely meant in these utterances. We inferred the utterances' meanings by considering the context in which they were spoken. Finally, we used Ochs et al.'s (1996) characterization of indeterminate grammatical constructions to guide our interpretation of the meanings of the students' utterances that referred to a person being in a basis.

Brief Physics Background and Literature

Quantum mechanical systems and all knowable information about them are represented mathematically by normalized kets, symbolized in Dirac notation as $|\psi\rangle$. Kets mathematically behave like vectors, and a ket's complex conjugate transpose, called a bra, is symbolized as $\langle\psi|$. Spin is a measure of a particle's intrinsic angular momentum and is represented mathematically by an operator such as $\hat{S}_z$ (where the z indicates the particle's axis of rotation). In a spin-½ system, there are two possible results for the S_z measurement: $\pm\frac{\hbar}{2}$; they correspond to $|+\rangle$ and $|-\rangle$, which comprise a set of orthonormal basis vectors called the S_z basis. Any quantum state $|\psi\rangle$ is a linear combination of them: $|\psi\rangle = a|+\rangle + b|-\rangle$ with corresponding bra $\langle\psi| = a^*\langle+| + b^*\langle-|$.

Interview question (a) asks students to determine the probability of obtaining $\frac{\hbar}{2}$ or $\frac{-\hbar}{2}$ in a measurement of the observable S_z on a system in state $|\psi\rangle$. This is calculated by $P_\pm = |\langle\pm|\psi\rangle|^2$, where $\langle\pm|\psi\rangle$ is an inner product between one of the basis kets and psi. Because $|\psi\rangle$ is written as a linear combination of the two vectors that comprise the z-basis, solving this problem requires no change of basis. The analogous information can be determined for other axes of rotation, such as y. To complete question (b), a change of basis is involved because the given state vector $|\psi\rangle$ is written in terms of the z-basis but the prompt asks for the probability that the spin component is up along the y-axis. The two main approaches are to either change $|\psi\rangle$ to be written in terms of the y-basis (denoted $|\pm\rangle_y$), or change the y-basis vectors to be written in terms of the z-basis. In either change of basis approach, one would need to utilize the relations $|\pm\rangle_y = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$.

Some studies have investigated student understanding of probability in quantum mechanics contexts (e.g., Close, Schiber, Close, & Donnelly, 2013; Passante, Sadaghiani, Pollock, & Schermerhorn, 2018; Wan, Emigh, & Shaffer, 2019), but we have not found any studies that specifically focus on students' understanding of basis or change of basis within physics contexts.

Results

We present our results in three sections. We first describe the codes of how students talked about basis and the meanings of students' utterances associated with the codes. Then we present the codes of how students talked about change of basis and the meanings of those utterances. We conclude with a description of how one student talked about (change of) basis in various ways.

Meanings of Students' Utterances Involving Entities "In a Basis"

We examined the situated meanings (Gee, 2005) of students' utterances that contained the phrase "in a basis" as they appeared in the students' responses to the interview questions. In this subsection, we describe the ways in which students talked about entities being "in a basis" and discuss the situated meanings of those utterances, given the context in which they were spoken.

A person is in a basis. When students used phrases such as "we are in a basis," we coded the object of focus in the student's language as a person. According to Ochs et al. (1996), the phrases can be understood as indeterminate grammatical constructions that use a personal pronoun subject with a predicate involving the experience of an inanimate object. In these phrases, it may seem linguistically unclear what entity the pronouns I, you, and we refer to. These phrases do not seem to literally mean that the speaker is in the basis, so we explore other meanings.

Five students spoke utterances that were coded as referring to a person being in a basis. Two of them, A6 and A25, made these utterances as they began question (a). A6 explained, "So, because we're in the *z*-basis, I already know that the probabilities- my plus *z* is going to be the square of the norm of my first component." A25 claimed, "So we are in the *z*, we are in the *z*-basis, so this makes this pretty easy." In both instances, the students referred to "we" as being in a basis as part of their justification for why they could perform the "easy" procedure of squaring the norm of the coefficients. A6 and A25's utterances of "we are in the *z*-basis" seem to mean that they interpret the problem to involve vectors that are all expressed in terms of the *z*-basis.

A6 and A8 used phrases that were coded as a person being in a basis as they justified the change of basis procedure they performed for question (b) in Figure 1. When the interviewer asked A6 if she thought about basis for this problem, she claimed that the change of basis was the first step and said, "you can't do anything until you're in the same basis." In her utterance, it seems that a person being in a basis was a condition necessary for an inner product calculation to be performed. A6 seemed to blend her identity (Ochs et al., 1996) with that of a basis vector. Here, "you" in "you're in the same basis" figuratively refers to basis vectors. Given the context of this utterance, the phrase referring to "you" being in a basis seems to mean ${}_{y}\langle +|$ and $|\psi\rangle$ are expressed in terms of the same basis, which allows one to calculate the inner product.

A calculation is in a basis. Two students' (A21 and A8) utterances were coded as having an object of focus as a calculation when talking about basis. They used phrases such as "we do this in a basis" or "we are working in a basis." These phrases seemed ambiguous, so we explored what the students' utterances mean within the context of the interview task setting. We look at A21 as an example. After A21 correctly computed the probability in question (b), he explained how expressing $|+\rangle_y$ in terms of the z-basis allowed him to make assumptions about the inner products $\langle +|-\rangle$ and $\langle -|+\rangle$ being 0 (because the basis vectors are orthogonal). Then he claimed,

> But if I were to do this in the y-basis…if I were to just like write this out as like let's say y plus, but against all of this [points to $|\psi\rangle$], then I can't make any assumptions about that [orthogonality], so I don't really know how to calculate that in bra ket language.

A21's phrase "if I were to do this in the *y*-basis" referred to calculating the inner product of ${}_{y}\langle +|$ and $|\psi\rangle$, and it implied a basis can be a context in which calculations can be performed.

A vector is in a basis. Every student referred to a vector as being in a basis, but the implied meanings were varied. One meaning of the phrase "a vector is in a basis" is that the vector is an element of a set of basis vectors (e.g., A6's utterance, "I mean I can just call this a plus *y* [referring to $|+\rangle_y$], in the y basis"). Another meaning of the phrase "a vector is in a basis" is that a vector is expressed in terms of basis vectors. For example, A6 claimed, "I needed plus *y* [$||+\rangle_y$] in the *z*-basis, because [$|\psi\rangle$] was in the *z*-basis." In this case, neither of the vectors $|+\rangle_y$ or $|\psi\rangle$ are actually elements of the z-basis. Therefore, A6's utterance can be interpreted as her needing $|+\rangle_y$ to be a linear combination of *z*-basis vectors because $|\psi\rangle$ is a linear combination of *z*-basis vectors. Most students' utterances about a vector being in a basis seemed to have this meaning.

Some of the students used language consistent with a vector being in a basis when they explained the task setting or as they justified their choice of using a certain strategy for performing the task. For instance, as A21 justified his change of basis, he explained, "Yeah okay, so if I wanna compute this [pointing to $\left|(\frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|)|\psi\rangle\right|^2$]... I wanna be able to read off those coefficients really easily and do this in bra-ket notation if these are in the same uh basis." By "these," A21 seemed to be referring to the vectors in the inner product each being a linear combination of basis vectors from the same basis. Vectors being in the same basis seemed to be a condition allowing him to perform inner products using bra-ket notation.

A vector is written in a basis. Six students spoke utterances that referred to a vector being written in a basis. Some used this language as they justified the need for a change of basis. C5 said, "If I'm doing the inner product of the positive *y* [$|+\rangle_y$] with that [pointing to $|\psi\rangle$], I need that to be written in the *z*-basis or to do those inner products to be nice." C5's phrase "I need that to be written in the *z*-basis" means that he needs $|+\rangle_y$ to be written as a linear combination of *z*-basis vectors. Some students referred to a vector being written in a basis as they described the result of the change of basis. For instance, A32 claimed, "that's the quick thing we can see is that now this is, this is written in the *z*-basis." When A32 claimed a vector is "written in the *z*-basis," he seemed to mean that a vector is written symbolically as a linear combination of basis vectors.

Meanings of Students' Utterances Involving Change of Basis

Utterances relating to change of basis were identified by verbs indicating some form of alteration of the basis or a vector. These phrases fell into distinct categories:

A) Phrases in which the verb's connotation deems its object (the vector) as concrete or (figuratively) tangible, and the verb implies a change in that physical form
B) Phrases in which the verb's usage alters its object (the vector) titularly
C) Phrases in which the verb relates two distinct objects, both of which are vectors
D) Phrases in which the verb's object is the basis itself.

Category A. A student's phrase was coded in category A if the student used language which indicated the vector was "change[d]", "convert[ed]", or in some other way altered as a result of the basis change. Six students used this type of language, such as "I *changed this* from the *y* basis to the *z* basis" (A6), "I could've *turned this* into the y basis" (A11), and "I'm going to *convert the y plus or minus bra* into the z basis" (A30). These phrases follow the language structure for category A. Due to the connotation of verbs used in this category, we propose that these students see the vector as having a physical form, and the change-of-basis process alters the vector's physical form. Of particular note is that without exception, students refer to the vector being changed into a basis, not another vector. These phrases seem to mean that the vector is being changed to be in terms of a different basis, rather than the vector actually changing into a basis.

Category B. A student's phrase was coded in category B if the student used language which indicated an explicit titular change in the vector; for example, C3 said "So, from that you just rewrite it like that, and that's essentially changing the basis of plus y into the z basis." From this statement, we infer that C3 sees what might be considered the "name" of the vector (i.e., the coefficients of the basis kets) changing to be the result of the change of basis. The most common verbs used in this category were "rewrite" and "express"; the objects of these verbs were always vectors, never bases. The meanings of these phrases can be taken somewhat literally because the verb's relationship to its object is clear from the verb's inherent meaning. For example, the verb "rewrite" creates a setting in which the vector is being altered into something of the same form.

Category C. A student's phrase was coded in category C if the student used language which indicated that the pre-change of basis and post-change of basis vectors were two distinct vectors. Unlike category A, this language does not imply that the post-change of basis vector is the pre-change of basis vector with some alterations, but rather implies that the post-change vector is a different vector entirely. Only one student, A8, used phrases coded in this category. In each instance, he said something to the effect of, "we can represent our φ going to the φ' in the y-basis." Of particular interest in this phrase is that A8 gave the pre- and post-change of basis vectors two different names, indicating that, to him, the two vectors are objectively distinct.

Category D. A student's phrase was coded in category D if the student used language which positioned the basis, not the vector undergoing the change, as the object of the verb. Only A13 made utterances that were coded in this category, with utterances such as "switch the basis" and "change your *y*-basis to your *z*-basis." These utterances seem to have a literal meaning.

Example of a Student's Utterances Regarding Basis and Change of Basis

A8 talked about basis in a variety of ways as he reasoned through the interview questions. As he engaged in question (a), his utterances exemplified three of our "in a basis" meanings:

> Because *this is written in z*, oh and this along the *z* axis, so because this is written along the *z* axis, I'm assuming that *we're working in the z basis*...Then you do plus with, norm squared of plus with psi, and by the same rule I talked about earlier … you get 4/13 and 9/13. So you literally just square the components if you want it like that…It's super

simple *if you're working in the basis that you're in.*

By "this is written in *z*," A8 seems to mean the vector is written in terms of the *z*-basis. Given this fact and that the problem asked for the probability that the spin component of the state vector was up along the *z*-axis, he assumed "we're working in the *z*-basis." Here his language seemed to position the *z*-basis as a context for performing the probability calculation. By "we're working in the *z*-basis," he seems to mean every vector involved in the inner product was expressed in terms of the *z*-basis. This justified his procedure choice of squaring coefficients because he knew that having every vector in terms of the same *z*-basis lets him take advantage of the orthogonality of the basis vectors. After performing this procedure, he said, "It's super simple if you're working in the basis that you're in." He refers to "you" as working in the basis and "you" as being in a basis. He seems to mean that the computation is simple if the vectors involved are in terms of the same basis. As A8 worked on question (b), he described two methods for changing basis:

> So… one [method] is to *put this vector in some φ' that's in the y basis,* and then just do y plus phi prime y cause it makes calculations, and it follows the same rules as this. Um, the other possibility is to do, is to *take the spin up y and go to whatever it is in the z, in the z basis,* cause we have *this in the z basis*. Um, they're both equivalent.

A8's utterances seemed to refer to change of basis as changing the vector ($|\psi\rangle$ or $|+\rangle$) into another vector ($|\psi'\rangle_y$ or $\frac{1}{\sqrt{2}}(|+\rangle + i\,|-\rangle)$). He referred to the post-change of basis vector as a different vector than the original (i.e., Category C) and referred to the result of the change of basis as a vector that is in a basis. Later, in detailing how he thought about basis in this problem, he said, "[Y]ou can't actually perform this calculation *when you're … in the different bases*." He again used "you" being in a basis as a situation in which one can perform the calculation. Overall, A8 referred to a vector being written in a basis or a person working in a basis as he explained the problem situation, he referred to a vector being in a basis as the result of the change of basis, and he referred to a person being in a basis when he justified his procedure.

Conclusion

In this study, we explored the ways in which quantum mechanics students discussed basis and change of basis as they performed probability tasks, of which one required a change of basis. The students' utterances referred to a person, calculation, or vector being in a basis, or a vector being written in a basis. Students talked about change of basis as changing the form of a vector, writing the vector in another way, making a vector to become a new vector, and switching bases.

We examined the situated meanings (Gee, 2005) of students' utterances that referred to basis and change of basis to unpack what the students meant. Even though phrases, such as "we're in a basis," may seem ambiguous, these phrases make sense to the speaker. The meanings of these utterances are socially constructed and contextually dependent. "Words have different specific meanings in different contexts of use…The meanings of words are also integrally linked to and vary across different social and cultural groups" (Gee, 2005, p. 53). The meanings of those utterances are likely shared among members of the quantum physics students' culture. Understanding the context of students' utterances helped us gain insight into their meanings. Future research can explore ways in which students socially construct meaning for something to be "in a basis" and can investigate how instructors' utterances influence the phrases students use.

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The Decision to use Inquiry-Oriented Instruction: Why Don't Beliefs Align with Practice?

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In this study of inquiry-oriented instruction (IOI), we explore the relationship between beliefs, professional obligations, and inquiry-oriented practice among college mathematics instructors. Professional obligations consist of the responsibilities that instructors have towards various stakeholders, including the institution, the individual students, mathematics as a discipline, and society (Herbst & Chazan, 2012). Past studies have reported inconsistencies between beliefs and practice; professional obligations may help explain why instructors cannot always realize IOI practices in the classroom. We operationalize these constructs in a set of surveys and use structural equation modeling to explore the hypothesized relationships.

Keywords: Inquiry-Oriented Instruction, Professional Obligations, Decision-making, Beliefs

Across the nation, students are required to take introductory mathematics courses as part of their undergraduate degree for STEM and other fields. More than 60% of students intending to complete a STEM degree end up switching to a different major or not completing a degree, with many of these students citing uninspiring, unwelcoming mathematics courses as the reason for their departure (PCAST, 2012). There have been widespread calls for mathematics faculty to use evidence-based teaching methods that include more active learning for the sake of student retention (PCAST, 2012), more authentic understanding (Johnson, Caughman, Fredericks, & Gibson, 2013), and more equitable outcomes (Laursen, Hassi, Kogan, & Weston, 2014). We use the term *inquiry-oriented instruction* (IOI) in this study to describe any or all instructional practices aimed to increase the students' engagement, with the goal of discovering what inquiry-oriented practices are reportedly being used and why. In this study, we expand upon the work done with the INQUiry-oriented Instructional REview (INQUIRE) instrument (Shultz & Herbst, 2019) to name what practices are being used by undergraduate mathematics instructors, and investigate possible reasons why those practices are being used.

Literature and Framing

We review the literature on the connection, or lack thereof, between beliefs and practice. The studies motivate the theoretical framework because we pose that there are rational justifications why instructors do not always enact their beliefs. Beliefs have been studied because of their central role in guiding judgements and decision-making (Bandura, 1986; Philipp, 2007). Though links have been found between teacher beliefs and teacher practice, researchers have also found inconsistencies between beliefs and practice. For example, Wijaya, van den Heuvel-Panhuizen, & Doorman (2015) found that junior high school teachers in Indonesia reported in questionnaires that they found context-based tasks beneficial for students, but observations revealed the teachers did not implement them. Raymond (1997) followed an elementary teacher and found that her practices were consistent with her beliefs about mathematics content, but not about mathematics pedagogy. DeFranco & McGivney-Burelle (2001) observed 22 graduate teaching assistants in a mathematics department throughout enrollment in a mathematics pedagogy course and found that, though their written journal reflections demonstrated that they adopted a new set of beliefs, their classroom practices did not change.

Instead of accusing instructors of being unmotivated to change, we agree with Leatham's (2006) notion that "teachers are inherently sensible rather than inconsistent beings" (p. 92) and pose that instructors must attend to social and contextual factors. Sztajn (2003) illustrated this phenomenon by studying two teachers with similar beliefs about mathematics but differences in instruction. The two teachers taught in separate contexts, and Sztajn (2003) could only explain the differences in instruction once she considered the contexts in which they were teaching and their broader beliefs about children, society, and education. Skott (2001) studied a teacher who believed that mathematics learning happened best when he could engage students in an unobtrusive manner but was prevented from always enacting this belief due to conflicting priorities such as building students' confidence and managing the classroom. While he wanted to support students to work and investigate mathematics independently, such as by not offering too much scaffolding and not evaluating student responses by letting them find out if their suggestions are correct on their own, he sometimes used direct instruction due to the classroom environment. Engeln and colleagues (2013) found that while the 917 teachers from 12 countries all had, on average, positive attitudes towards inquiry-based learning, the implementation varied more widely country to country. This suggests that the implementation was less dependent on individual beliefs than it was on contextual and cultural factors.

Practical Rationality

Herbst and Chazan (2011, 2012) proposed the framework of *practical rationality* to describe the ways that practicing teachers make decisions based on both social and individual resources which is useful to account for the environment in which the relationships among teachers, students, and content exist. This theory of decision-making provides a lens for how teachers account for and rationalize their decisions, rather than explaining the cognitive processes involved in making the decision. Herbst & Chazan (2011) hypothesize that a source of justifications for decisions in institutional environments where mathematics instructors work come from four *professional obligations* that mathematics teachers must respond to as professionals: towards creating a socially and culturally appropriate environment for students to share space and resources in a class (interpersonal), representing the discipline of mathematics appropriately (disciplinary), respecting institutions such as the department, school, or state in matters including curriculum, assessment, and policy (institutional), and to treating individual students as persons with unique assets and needs (individual). The obligations correspond to various stakeholders in the educational system, namely the community, mathematics as a field, the institution, and the students, respectively (Herbst & Chazan, 2012). The theory provides a lens to address the "multiple and sometimes conflicting educational priorities" (Skott, 2001, p. 18) that arise in the practice of teaching.

We are interested in factors that can explain instructors' use of IOI practices, and the framework of beliefs with practical rationality can help balance the individual and social factors that go into making those decisions. The following questions guide our study:

1. Is there a relationship between the beliefs of instructors and inquiry-oriented practices in college mathematics instruction?
2. If so, do professional obligations help explain the relationship between beliefs and inquiry-oriented practices in college mathematics instruction? If so, how?

Methods

Data Collection

We administered four self-report instruments to collect data for this study: inquiry-oriented instruction review (INQUIRE), professional obligations (PROSE), beliefs (Clark and colleagues, 2014), and a background survey. Self-report surveys are widely used and researchers have found ways to argue for their validity. Hayward, Weston, & Laursen (2018) found that self-report surveys were trustworthy when used to describe what is happening in instruction, in comparison to observation protocols designed to measure the same happenings. Similarly, Desimone, Smith, and Frisvold (2010) found that there were only small differences between student reports of teacher behavior and teacher's self-reports, that disappeared when controlling for background variables.

We collected data from a national sample with participants representing 94 different mathematics departments across 37 states, plus Washington, D.C. We compiled a comprehensive list[1] of Research I and Research II mathematics department emails, along with a handful of high-ranking liberal arts colleges. Participants qualified for the study if they had at minimum one year's teaching experience, including tenure or tenure-track faculty, adjunct faculty, and graduate student instructors. There were 252 finishers of the INQUIRE instrument, and 194 participants that completed all instruments in the study[2].

The INQUIRE instrument was designed using different practices found in the literature on IOI, and organized around the interactions represented in the instructional triangle (Cohen, Raudenbush, & Ball, 2003; Shultz & Herbst, 2019). The practices measured by the instrument are organized and defined in Table 1. Each practice is measured by at least three items. Instructors were asked to consider a typical day in a lower-division course and respond to questions on a Likert-type frequency scale from 1-Never to 6-Multiple times each class.

The PROSE instrument was a four-part multimedia questionnaire created using the Lesson*Sketch* platform (www.lessonsketch.org). Each part of the questionnaire consisted of fifteen to eighteen scenarios that featured an undergraduate college mathematics classroom where an instructor chose to act on behalf of one of the professional obligations. Participants were then shown a statement about the instructor's action (e.g. "The teacher should stick to the mathematics at hand, rather than take class time to make connections to other mathematical ideas.") that they respond to on a 6-point Likert-type scale from strongly disagree to strongly agree.

For the beliefs instrument we used all published items from Clark and colleagues (2014). They developed a survey on a six-point scale from 1-Strongly disagree to 6-Strongly agree. Their study included an exploratory factor analysis that yielded three main factors consisting of 21 items grouping into the following three categories: Allowance for student struggle with problems (TASSP), teaching modeling for incremental mastery (TMIM), and awareness of students' mathematical dispositions (TASMD). Their items were tested with 259 upper-elementary and

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[1] The list was compiled from an original list acquired by Pablo Mejía-Ramos of Rutgers University and updated by the GRIP lab.

[2] The INQUIRE instrument had a flawed implementation: In Qualtrics, if a user responded to the question, "How often do you ask students to revise a definition?" with "1-Never", they were skipped past the remaining sections of the survey (all student-student and teacher-content questions) due to an error in the survey logic. Of the original 252 finishers, 143 chose "1-Never". We recovered data for 85 participants, who retook the rest of the instrument. There are 58 participants that did not retake it.

184 middle-grade mathematics teachers. To keep the text of our results as transparent as possible, we use the names *Struggle, Model, and Awareness,* in place of the acronyms TASSP, TMIM, and TASMD, respectively. Though college mathematics instructors may have different beliefs than K-12 teachers, we expected that the general factor structure would still hold.

Table 1: Definitions and organization of constructs measured by the INQUIRE instrument

Triangle Relationship	**Constructs**	**Description**
Student-Content	Open problems	Posing problems that either have multiple solutions or multiple nontrivial ways of arriving at a solution
	Constructing/Critiquing	Posing tasks that ask students to create or critique conjectures or claims
	Proving	Using previously established results to construct formal arguments for mathematical claims
Teacher-Student	Interactive lecture	Instructing to the full class while asking for feedback from students, asking questions of students, and having students engage with the mathematics
	Hinting without telling	Guiding a student to work productively without directly telling the student a correct way to proceed
Student-Student	Group work	Creating an environment where students work together on mathematical tasks or problems
	Student Presentations	Having a student or students present completed or in-progress work to the class

We chose this instrument because of its alignment with common categorizations of mathematics teacher beliefs on teaching and learning and because we hypothesized categories of struggle, model, and awareness would predict inquiry-oriented practices. We suspected that the belief that students should struggle would predict more frequent use of inquiry practices, while belief that teachers should model to help students incrementally master techniques would predict less inquiry-oriented practices. Instructors that interact with their students more through active learning strategies like group work, or hinting without telling, might perceive themselves as more aware of their students' dispositions. Finally, all participants took a background information instrument that collected basic information such as years of teaching experience, status in the department, general research interests, and type of institution.

Data Analysis

We explore the relationships between constructs using a structural equation model (SEM). SEM refers to a family of related techniques, including factor analysis and structural regression modeling (Kline, 2016). It is a causal inference method that takes qualitative hypotheses and questions about causal relationships and outputs numerical estimates of hypothesized effects (Kline, 2016). An advantage of using SEM over ordinary least squares regression (OLS) is that OLS assumes each item contributes equally to a given construct, while

SEM considers that not all items have the same amount of error. Once the dimensionality of each instrument was established using factor analyses in MPlus, we checked the internal consistency of each factor by calculating Cronbach's alpha and mean inter-item correlations in Stata.

Results

Reliability and Validity

We conducted confirmatory factor analyses for each instrument to confirm that items grouped as hypothesized. The reliability statistics for each item grouping are shown in Table 2. Most groupings had satisfactory reliability statistics, following the guidelines of inter-item correlations (IICs) between .15 and .50 (Clark & Watson, 1995) and alpha scores above .7 (Kline, 2016). Kline (1999) noted that for some psychological constructs, lower alpha scores can be appropriate due to the diverse nature of what is being captured.

Table 2: Inter-item correlations and alpha statistics for items in each construct included in our models

Instrument	Constructs	IIC	α
INQUIRE	Constructing/Critiquing	.48	.90
	Proving	.45	.87
	Solving open problems	.38	.71
	Interactive Lecture	.25	.62
	Hinting without telling	.38	.65
	Group work	.58	.90
	Presentations	.54	.87
Beliefs	Struggle	.38	.75
	Model	.31	.76
	Awareness	.42	.74
PROSE	Individual	.19	.74
	Interpersonal	.26	.86
	Disciplinary	.25	.85
	Institutional	.18	.66

We created a structural equation model for each targeted practice of the INQUIRE instrument. We separated them into individual models because the study is aimed to investigate how beliefs and obligation predict specific practices rather than how practices are related to each other. Following the recommendation by Little, Cunningham, Shahar, and Widaman (2002), we used parcels after exploring the dimensionality of the items to be parceled. Using parcels, or aggregated groups of items, as indicators for latent constructs has the advantages of being more likely to fulfill the distribution assumptions inherent in SEM, and creating a model with fewer free parameters so that the sample size is sufficient (Little et al., 2002). All seven models had satisfactory fit statistics (CFI>=.91, TLI>=.90, RMSEA<=.05). These fit statistics show that beliefs and professional obligations are reasonable indicators for modeling the inquiry-oriented practices that instructors choose to implement.

Predicting Inquiry-Oriented Practices

Professional obligations significantly predicted five of seven inquiry-oriented instructional practices: the use of open problems, having students work with each other in group work, having students make presentations, inviting students to construct and critique work, and using interactive lecture. We illustrate only three models here due to space limitations (see Table 3).

The instructors use these three practices more or less than their beliefs alone would indicate. We found that professional obligations could be in direct opposition with the actions that their avowed beliefs would recommend.

Table 3: Structural equation models showing how beliefs and professional obligations predict inquiry-oriented practices of including group work, presentations, and open problems, coefficients are STDYX standardized

		Group Work		**Presentations**		**Open Problems**	
Instrument	Latent construct	Loading	SE	Loading	SE	Loading	SE
Beliefs	Struggle	.46**	.17	.63***	.18	.36*	.17
	Model	-.04	.19	.31	.20	-.18	.19
	Awareness	.27**	.10	.15	.11	.32***	.10
Professional Obligations	Individual	.28*	.14	.64***	.14	.42**	.16
	Interpersonal	-.01	.16	.04	.17	-.21	.17
	Disciplinary	-.13	.17	-.53**	.18	-.25	.18
	Institution	.28*	.11	.21†	.12	.37**	.12

† p≤0.10, *p≤0.05, **p≤=.01, ***p≤.001

Choosing to have students participate in *group work* with their peers was significantly predicted by the belief that students should be allowed to struggle (ß=.46, p<.01), awareness of students' mathematical dispositions (ß=.27, p<.01), the individual obligation (ß=.28, p<.05), and the institutional obligation (ß=.28, p<.05). If an instructor held more of the two belief constructs, they would be more likely to use group work. If an instructor tended to recognize the individual and institutional obligations, they would be even more likely to use group work. The individual obligation makes sense, because the type of patience and empathy required to attend to individual students' needs is also necessary for attending to individuals as they engage with group work. We are not certain why recognition of the institutional obligation was positive, as we hypothesized compliance with institutional policies would predict less inquiry-oriented practices.

Having students give *presentations* was significantly predicted by the belief that students should be allowed to struggle (ß=.63, p<.001), the individual obligation (ß=.64, p<.001), and the disciplinary obligation (ß=-.53, p<.01). While the individual obligation is in line with the belief that students should be allowed to struggle, the disciplinary obligation is in direct opposition with it in predicting the practice of having students present their work. Many of the high-loading items in the instrument measuring the disciplinary obligation had to do with correcting student work or choosing to highlight more elegant mathematical work. So an instructor who believes student presentations are beneficial still might not use them due to recognizing the disciplinary obligation, wanting to expose students to accurate or elegant mathematical work.

Presenting students with *open problems* to solve was significantly predicted by the belief that students should be allowed to struggle (ß=.36, p<.05), awareness of students' mathematical dispositions (ß=.32, p<.01)., the individual obligation (ß=.42, p<.01) and the institutional obligation (ß=.37, p<.01). The two beliefs have positive coefficients, indicating that if an instructor holds more of those beliefs, they are more likely to give students open problems. Recognition of the individual obligation and intuitional obligation could be used to understand why even more instructors use open problems than just their beliefs would indicate.

Beliefs played a significant role predicting the constructs of *proving,* and *hinting without telling,* and professional obligations did not hold a significant relationship with either of them. We conjecture that these two practices are not as distinct from more typical forms of instruction, e.g., they can be done within a lecture format without changing anything in the curriculum or lesson plan. The comparatively small deviation from typical instruction might explain why recognition of various professional obligations do not have a significant impact on the decision to use them or not.

Discussion

While policymakers might be frustrated that their research-based recommendations are seemingly ignored, a consideration of the complex environments within which instructors operate may help them understand why that happens. Our results indicate that beliefs often do not align with practice because there are other environmental and social factors, as modeled here by professional obligations. For example, even if instructors are convinced that having more peer interaction with group work would create the most learning opportunities, they might decide to have more whole-class discussions for the sake of individual students or exposing students to all the content within the scheduled time. Having students give presentations might align with an instructor's desire to have students experience struggle, but not enough to risk that the discipline of mathematics would be represented poorly.

Conversely, our results give insights into which practices might be more feasible to implement: including open problems, having students work in groups, including more opportunities to prove, and hinting without telling. As professional obligations were not significant negative predictors of these practices, if instructors are convinced these practices are beneficial, they might be more inclined to act on it. For the practices that are not as feasible, our results have implications for the types of professional development or policy changes that might accompany an initiative to include more IOI. For example, if a department wanted instructors to include more student presentations, they might hold workshops that emphasize how to do so in a way that would allow students to help each other represent the mathematics accurately. While the results suggest that the disciplinary obligation acts as a constraint on enacting beliefs, the recognition of the institutional and individual obligations works in tandem with beliefs for the use of student presentations. Departments might leverage instructors' individual and institutioanl obligations to encourage the use of such practices.

This study offers a conceptual framework for understanding instructors' decisions and a methodology (SEM) to operationalize it. For future work, we would like to expand the lens from the instructor perspective to gauge the impact of each individual practice on students' experiences. If instructors choose to not use student presentations due to their individual obligation, what if we were able to give them data about how it impacts the student experience? What if we could give instructors data about how to best implement inquiry-oriented practices in ways that would build students' confidence instead of potentially embarrass them? We hypothesize that additional data on the student experience with particular aspects of IOI, rather than IOI as a whole, would be useful information for both instructors and researchers.

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“What Happened to the Vector?”
Felix’s Emerging Conception of Translations

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Felix initially used a metaphor in considering translations between algebraic representations of systems during an individual clinical interview. I investigated the mathematical underpinnings for his metaphor by asking him to perform translations in the vector space register. That investigation precipitated a discussion where Felix developed mathematical understanding for a translation between the vector space register and the linear systems register. My analysis is based on the Theory of Quantitative Systems which I developed as a result of my study of Duval’s Theory of Semiotic Representation Registers (1999, 2006, 2017).

Keywords: systems of linear equations, registers of representation, supporting students’ learning, metaphor, structure sense, teaching experiments

Introduction: Study Specifics

Herein I report on part of my dissertation study (Sipes, 2019) entitled *Undergraduate Students’ Conceptions of Multiple Analytic Representations of Systems (of Equations)*. For the study I conducted semi-structured individual clinical interviews which I video recorded and analyzed. The content of this report is based on my interview with Felix, a university student enrolled in a junior-level applied linear algebra course.

The theoretical lens for my analysis was a theory I developed, the Theory of Quantitative Systems, which was motivated by Duval’s (1999, 2006, 2017) Theory of Semiotic Representation Registers. (Note that in addition to this contributed report, I have submitted a theoretical report delineating my theory.) Briefly, in my theory I declare that mathematical systems such as matrix algebras, the algebra of real numbers, and vector spaces fit Duval’s definition of registers of representation (systems of notation with specific symbols and rules). Translations of algebraic expressions within the same register are referred to as treatments, while translations of algebraic expressions between differing registers are referred to as conversions. Duval hypothesized that two sources of incomprehension in mathematics are: 1) the kind of cognition required to perform treatments and how it contrasts with 2) the kind of cognition required to perform conversions.

The need for attending to notational distinctions in linear algebra was documented by Hillel (2000), Pavlopoulou (1994), and Artigue, Chartier, & Orier (2000). Hillel (2000) studied video-recorded sessions of lecturers teaching on the topic of eigenvectors and eigenvalues. He noted that the experts moved fluidly between various representations, giving little notice to the nuances of notation and meaning. Using “graphical”, “symbolic”, and “tabular” as categories of registers of representation, Pavlopoulou’s (1994) dissertation study, experimental in design, allowed her to obtain important and convincing results in linear algebra (Artigue, Chartier, & Orier, 2000). Artigue et al. (2000) called for investigations that would check the consistency of Pavlopoulou’s results in contexts that are richer and more complex. I am aware of only one such follow-up study; Sandoval & Possani (2016) considered students’ conceptions of vectors and planes in $\mathbb{R}^3$. While their study centered strongly on the geometric, I focus on the algebraic.

My semi-structured interview with Felix unfolded in such a way that I could address two questions. First, I set out to address the question “What is the nature of Felix’s understanding of vector space operations?" Next, I considered "How might I influence Felix's thinking about changes in register?" In terms of the Theory of Quantitative Systems (TQS) and the Theory of Semiotic Representation Registers (Duval, 1999, 2006, 2017), the first question is about Felix's work with treatments. In terms of the theories, the second question addresses Felix's work with a conversion. Each question is addressed following a discussion of Felix's initial use of metaphor in our interactions.

The task that I had Felix consider was comparing and contrasting various representations that may come up in solving a system of linear equations. Ten students who participated in individual video- and audio-recorded clinical interviews were initially asked to compare and contrast the first three representations shown in Figure 1. Next, I presented participants with the fourth representation shown in Figure 1 and asked them to comment further. I refer to the representations, from left to right, as the augmented matrix, the matrix equation, the vector equation, and the linear system.

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 4 & -1 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad x\begin{bmatrix} 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \qquad \begin{cases} x + 2y = 7 \\ 4x - y = 1 \end{cases}$$

Figure 1. Four representations of system.

Felix's Metaphor

Felix largely described the various representations in terms of how they appear on paper. For instance, he described the augmented matrix as "before *x* and *y* is placed". He described the matrix equation as "where it was before we multiply it" (referring to the linear system that results from performing matrix multiplication). He described the vector equation as "showing" multiplication not element-by-element, but column-by-column. In addition, Felix described the representations in terms of "columns", "individual pieces", and "one piece". He was apparently describing the coefficient vectors in the vector equation as "columns", the individual coefficients as in the augmented matrix as "pieces", and the coefficient matrix in the matrix equation as "one thing". In other words, Felix acknowledged various units in the representations. In addition to his visual descriptions, he referenced various actions involved in translating between representations, and those actions were not particularly mathematical. In at least four instances, Felix referred to the action of "cutting". Other actions he mentioned were "gluing", "throwing", "tilting", and "multiplying downwards".

Overall, Felix described the various representations in terms of differing units (sub-pieces of algebraic expressions) that get put together in various ways. I would say that Felix was using a *cut-and-paste metaphor* for translating between representations. That is, he seemed to group symbols into different pieces, some larger and some smaller, while describing the action of translation in terms of rearrangement. This regrouping and rearranging of units might be referred to as imagistic (Clement, 1994) since Felix seemed to act (mentally) on the representations using dynamic imagery. In other words, Felix used dynamic imagery as a means to account for what happens between the source representation and the target representation. The cut-and-paste metaphor allowed Felix to view the translations as transparent, where the target representation is obvious from the source representation, given his mental activities. With his cut-and-paste metaphor, Felix seemed to be rearranging various units (pieces) of notation without much consideration of the mathematics. That is, I questioned whether Felix had good *structure sense*, which I take as the parsing of algebraic notation into appropriate mathematical units. (Structure sense is an area of interest for me; I use the term differently than it is used in existing literature.)

Felix gave some indications of mathematical foundations for his cutting and pasting by frequently referring to multiplying and using mathematically correct terminology like “vector”, “columns”, and “element”. However, while I was conducting the interview, I was uncertain about what mathematics might or might not underlie his metaphor. Opportunities that allowed me to address my uncertainty presented themselves. As a result, I was able to investigate the mathematical underpinnings for his cut-and-paste metaphor in two ways. First, I set out to investigate his understanding of vector space operations. Next, I worked to influence his thinking about a change in register.

Felix’s Developing Conceptions

While Felix did not initially attend to mathematical rigor in his descriptions of the various translations, I got the impression that reflection and deep thinking were mathematical practices for him. For instance, he mentioned that his experience of being introduced to row reduction left him with the impression that it was an entirely new topic. In trying to make sense of the morass of calculations involved in row reduction, he was able to tie the process back to the elimination method he had learned earlier in his schooling. In time he came to see row reduction as an extension of the elimination method.

Another instance that seemed particularly insightful was Felix’s comparison between coordinate systems and the four representations shown in Figure 1. He compared the four representations, which he viewed as representative of a linear system (the last representation in Figure 1) to using polar coordinates instead of Cartesian ones; he seemed have in mind how varying systems of notation can be used to indicate the same thing. The two instances I have described along with my overall experience of our interview suggested to me that Felix could provide potentially valuable insights into students’ thinking about vector space operations and changes of register. I followed up on my impression as described in the next two sections.

Working with Treatments (Considering Vector Space Operations)

Possibly, the four representations under consideration were so obviously transparent (visually similar) to Felix that it did not occur to him to describe the translations in terms of mathematics. In the moment of the interview, his descriptions did not make it apparent that he was doing more than visually manipulating the symbols. I was able to get a more complete picture of Felix’s conceptions by discussing the vector space operations of scalar multiplication and vector addition. (Our latter interactions support that transparency and visual symbol manipulation were, in all likelihood, both parts of Felix’s conception.) Of the first three representations in Figure 1, Felix stated: “This way (pointing to the augmented matrix) you can solve with row-reduced echelon form. And this one (pointing to the matrix equation), if it’s invertible, we can just do an inverse matrix. And for this one (pointing to vector equation)…” (long pause) “… I don’t think I know of a way to solve it with this notation”. I saw Felix’s admission as a perfect opportunity to divert from my interview protocol.

I asked Felix to solve the system using the vector equation (the third representation in Figure 1) after he expressed doubt in his ability to do so. (I take his admission as evidence that, to some extent, he had previously been engaging in visual symbol manipulation rooted in his cut-and-paste metaphor.) I asked Felix, “If x is a scalar, how do you multiply it by the vector?” Felix responded, “Just multiply by each element”. The written work he produced upon answering appears in line 2 of Figure 2. Note that I felt it might be necessary to described x as a scalar in case Felix had a surface view of x as a variable without attributing to it the additional role of scalar within a vector space; further, the possibility existed that Felix had only seen scalars that were real numbers.

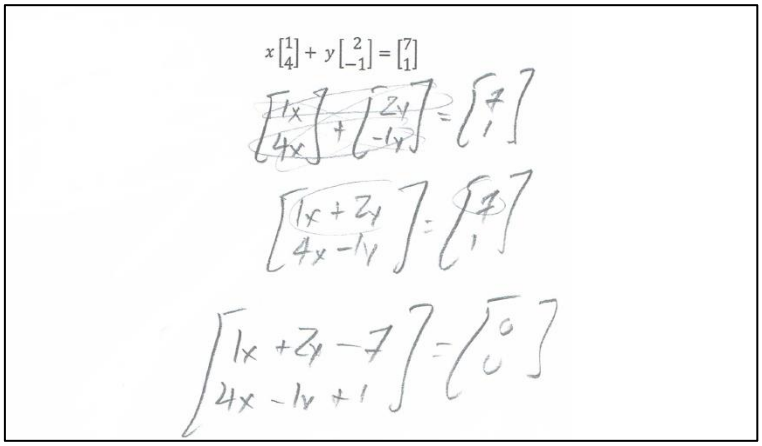

Figure 2. Felix's Written Work.

Next, I framed Felix's written result in line 2 of Figure 2 as the sum of two vectors. I pointed to the left-hand side of the equation in line 2 of Figure 2, and asked, "How do you add two vectors?" Felix stated, "So just add the corresponding elements." Then he mumbled and wrote line 3 of Figure 2. My expectation was that Felix would next write a system of equations; however, he did not. Rather, he stated, "You could even go as far as like…go one step further." He then mumbled while writing line 4 of Figure 2. In terms of my and Duval's theories, Felix performed a treatment in the vector space register rather than performing, as I expected, a conversion between the vector space register and the linear systems register. His work in line 4 is evidence of a propensity to continue working in the same register rather than changing registers. An open question is if students in general exhibit such an inclination to perform treatments rather than conversions. Sandoval & Possani (2016) found some such evidence.

In summary I set out to investigate Felix's understanding to address what I saw as a less-than-impressive display of formal mathematics in his cut-and-paste metaphor. To be clear, I see Felix's metaphor as an enlightening and potentially productive as a way of thinking. However, in the moment that I was conducting the interview, his descriptions were confusing mixtures inconsistent with other profound statements he had made. What I found was that with framing in terms of mathematical terminology and ideas that Felix connected to, mathematical underpinnings that were absent in his metaphorical descriptions were revealed. Specifically, I framed the problem by suggesting that *x* is a scalar and that the result of the scalar multiplication is the sum of two vectors. Further, I encouraged Felix with a statement and a question: "You know how to multiply by a scalar" and "How do you add two vectors?" Felix accurately described and enacted the vector space operations of scalar multiplication and vector addition. In other words, Felix was able to perform a series of treatments in the vector space register as shown in Figure 2. This suggests that, while Felix's descriptions tended to be metaphorical and imagistic, his conceptions were not without mathematical foundations.

Supporting a Conversion Mathematically

My expectation was that Felix would write a system of equations from line 3 of his written work (Figure 2). Since he performed a treatment in the vector space register instead, I took the opportunity to investigate if he could develop a mathematically logical foundation for the translation from the vector register to the linear systems register. I note that Felix was interested in, and therefore able to, engage in the conversation. Not all interview participants (10 students) seemed interested in investigating and developing connections. However, Felix's interest in making connections had become apparent during our interview as evidenced by the explanations he often constructed for himself. Recall, for instance, his connection between row reduction and the elimination method, and his comparison of the representations shown in Figure 1 to Cartesian versus polar coordinates. My attempt to influence his thinking is documented in our dialogue shown in Figure 3.

When I suggested (line 1, Figure 3) to Felix that some people go directly from what he wrote in line 3 of Figure 2 to writing a system of equations, Felix acknowledged that the translation "seems kind of weird" (line 6, Figure 3) since "we're just kind of getting rid of the vector". While I described the translation as getting "rid of the brackets" (line 7, Figure 3), Felix spoke in terms of getting "rid of the vector" (lines 6 and 8, Figure 3). I assumed that his coursework had required him to enact the translation, so I saw Felix's statement that the translation was "weird" as evidence that he had become uncomfortable with the "disappearing vector" approach. Additional evidence of his discomfort/curiosity is supported by his subsequent enthusiastic reaction, asking a question in three ways (line 8, Figure 3): "Why did you get rid of the vector? What happened to the vector? Why is the vector not there anymore?"

1	*Interviewer*:	So some people go from this (pointing to line 3 of Figure 2) straight to writing a system, which we have written here somewhere.
2	*Felix*:	Yeah, like this. (pointing back to the systems representation on some of his earlier work.)
3	*Interviewer*:	Why can I write a system from this? (pointing to line 3 of Figure 2) When I multiply these out, I should be getting matrices, right? Or these are actually vectors? Do you see those as vectors? (I tend to think in terms of matrices, but Felix had been calling them vectors.)
4	*Felix*:	Um-hmm.
5	*Interviewer*:	How can I go from having a vector to having a system? How can I explain that mathematically?
6	*Felix*:	Yeah, trying to explain it from just looking at this (pointing to line 3 of Figure 2) seems kind of weird because we're just kind of getting rid of the vector.
7	*Interviewer*:	You just get rid of the brackets, right?
8	*Felix*:	Yeah, you just get rid of the vector. But then the question comes up, why did you get rid of the vector? What happened to the vector? Why is the vector not there anymore?
9	*Interviewer*:	(I laugh.) Yeah, that's my question!
10	*Felix*:	Yeah!
11	*Interviewer*:	What if you think about it as a matrix rather than a vector? Does that help any?
12	*Interviewer*:	What makes two matrices equal? How do you know two matrices are equal?
13	*Felix*:	If the corresponding elements equal to each other.
14	*Felix*:	So this would equal that!
15	*Felix*:	Yeah, but that IS the definition of a matrix, and if they know how to identify if they are the same.
16	*Interviewer*:	What does matrix equality mean?
17	*Felix*:	Yeah, *matrix equality*.

Figure 3. A dialogue about a change in register.

Next, I asked Felix if thinking in terms of matrices would help resolve the issue (line 11, Figure 3). I then followed up by asking, “What makes two matrices equal?” (line 12, Figure 3). I did this since I supposed that he might be more familiar with the tenets of matrix algebra than with the tenets of vector spaces. Specifically, I thought the idea of “matrix equality” might be a more familiar idea than that of “vector equality”. Felix responded, “If the corresponding elements equal to each other” (line 13, Figure 3). At that point Felix seemed to have an epiphany. In an animated way, he circled the first entry of each of the vectors/matrices as shown in line 3 of Figure 4. While circling the top two components, he stated “So this would equal that!”(line 14, Figure 3). Next, Felix extended his written work by drawing an arrow from line 3 and writing the first equation of the system as shown in the final line of Figure 4.

Felix had apparently realized that the equations for the linear system result from equating corresponding elements of the matrices (vectors). In his next statement, Felix seemed to be describing the essence of matrix equality (line 15, Figure 3): “but that IS the definition of a matrix, and if they know how to identify if they are the same”. I gave Felix a name for his description by asking (line 16, Figure 3): “What does matrix equality mean?” He latched onto the terminology, stating “Yeah, *matrix equality*.”

$$x\begin{bmatrix}1\\4\end{bmatrix} + y\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$$

$$\begin{bmatrix}1x\\4x\end{bmatrix} + \begin{bmatrix}2y\\-1y\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$$

$$\begin{bmatrix}1x+2y\\4x-1y\end{bmatrix} = \begin{bmatrix}7\\1\end{bmatrix}$$

$$\begin{bmatrix}1x+2y-7\\4x-1y+1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$1x+2y=7$$

Figure 4. Felix’s Addendum to His Written Work

As a result of my influence, Felix seemed to have constructed mathematical support for the conversion from the vector register to the linear systems register. In other words, Felix became able to mathematically justify why a system of equations can be extracted from a representation involving only vectors (or matrices depending on one’s perspective/language). I assert that for Felix, an unreflected, taken-for-granted step in a solution process had become a translation rooted in mathematical logic.

A Note on Methodology. I would describe my work with Felix on the change in register as an impromptu, mini teaching experiment. By *impromptu*, I mean I acted on an opportunity that presented itself during the clinical interview and took a diversion away from my intended interview trajectory, a practice encouraged by Hunting (1997). I use *teaching experiment* in the sense that Steffe & Thompson (2000) prescribed; I tend to think of these experiments as being individual and/or small group (not whole class experiments) and cognitive in nature. Steffe and Thompson elaborated the idea of a teaching experiment as a living methodology; the experiment may occur over several sessions with adaptions taking place between sessions. Such a teaching experiment involves a great deal of thinking and planning to conduct a conceptual analysis of a math concept prior to the development of an experimental teaching sequence. The process includes considering how to influence the student's thinking in productive ways, and the researcher must be adaptive both within and between sessions when the student responds in ways that are not expected. Clearly, my work with Felix occurred within a single session. While I did not have a formal conceptual analysis and experimental instructional sequence written out, I had ones in mind from several years of repetitively teaching multiple sections of College Algebra. I believe recognizing the opportunities that presented themselves in my interview with Felix was only possible given my prior experience with teaching students to solve systems using matrices.

As I see it, the appeal the of teaching experiment methodology is that inherent in it are pedagogical suggestions. While documenting students' thought processes through individual clinical interviews is a productive undertaking in itself, pedagogical recommendations can only be hypothesized subsequent to the investigation. In contrast, the teaching experiment, puts on display a pedagogical approach that data from interviews supports as productive, unproductive, or needs improvement.

Conclusions

Students mathematical conceptions may not be mathematically rigorous for a variety of reasons. While students may use visual techniques, heuristics, metaphors, and rehearsed algorithms, experts can mathematically support such approaches when they recognize that students are using them. This is what I did with Felix in two distinct ways.

Initially, Felix accounted for translations primarily by using dynamic imagery which I have designated the *cut-and-paste metaphor*. With support, however, he was able to voice his knowledge of vector space operations and to carry them out. Further, a change of register that he had seemingly taken for granted prior to our interview became a matter of importance for him. With support Felix was able to construct mathematical justification for the conversion and resolve the conflict he encountered.

Since I have argued that teaching experiments have direct pedagogical implications, I will say a bit more about my interaction with Felix. I believe I did three key things to support Felix's mathematical realization. First, I drew his attention to the change in register (line 5, Figure 3). Apparently, he had not previously acknowledged it, but I sensed that he might be capable of considering the distinction. Next, I changed my language to match his language (lines 3 and 5, Figure 3). I was thinking in terms of matrices; but he was speaking in terms of vectors. Third, I attempted to help him shift from his vector perspective to my matrix perspective (line 11, Figure 3), believing that my language might enlighten him in ways that his language did not. The key, I believe, was in navigating between the student's language and conception (acknowledging where he was) and attempting to steer him to what I believed might be a more productive perspective based on my experience of teaching students about matrices. As a result of my influence, Felix seemed to have constructed mathematical support for the conversion from the vector register to the linear systems register.

I make one additional note that Felix is an existence proof that students can have their interest piqued in such things as changes in register. I found other instances of students attending to differences in register in my larger study of university students enrolled in a junior-level applied linear algebra course.

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Linear Algebra Thinking in the Embodied, Symbolic and Formal Worlds: Students' Reasoning behind Preferring certain Worlds

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To understand linear algebra concepts, one needs to be familiar in many different modes of thinking. Mathematicians often move between these modes of thinking fluently and expect that students will pick up the main ideas along the way. Yet, a majority of students do not have the cognitive framework to perform the move that is so natural to mathematicians. Employing Tall's (2013) three-world model, in this research a set of linear algebra tasks were given to students in order to encourage them to move between Tall's embodied, symbolic and formal worlds of mathematical thinking. Our working hypothesize is that by creating opportunities to move between the worlds, students may be exposed to multiple modes of thinking which results in richer conceptual understanding. The results of a survey revealed that a majority of students preferred the symbolic world. Their reasoning for their choices will be discussed.

Keywords: linear algebra, embodied, symbolic, formal

Introduction

Hillel (1997, p. 232), outlined three types of languages in linear algebra: "The language of the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.); The language of the more specific theory of R^n (n-tuples, matrices, rank, solutions of the system of equations, etc.); The geometric language of 2- and 3- space (directed line segments, points, lines, planes and geometric transformations)". He believed that these languages are interchangeable, but are not equivalent. In Hillel's (1997, p. 233) views "knowing when a particular language is used metaphorically, how the different levels of description are related, and when one is more appropriate than the others is a major source of difficulty for students". Dreyfus (1991, p. 32) believed that, "one needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take… Teaching and learning this process of switching is not easy because the structure is a very complex one." Duval (2006) believed that students' difficulties with comprehending mathematical ideas is due to their lack of flexibility between moving between registers, as most students do not have the cognitive structure to perform the switch that is available to the expert. In his views, "changing representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum" (p. 128). To understand the nature of movements between modes of thinking of a group of linear algebra students in this study we employed Tall's (2013) three worlds of embodied, symbolic and formal mathematical thinking.

Theoretical Background

Tall (2010) defined the worlds as follows: The embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognise properties and patterns...and other forms of figures and diagrams" (p. 22). Embodiment can also be perceived as giving body to an abstract idea. The symbolic world is based on practicing sequences of actions which can be achieved effortlessly and accurately as operations that can be expressed as manipulable symbols. The formal world is based on “lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure” (p. 22).

Using Tall’s model, Stewart, Thompson and Brady (2017) investigated a mathematician’s (and co-author) movements between Tall’s worlds while teaching algebraic topology. In this study, the instructor reported that students experienced the most difficulty in moving from the embodied world into the formal world. Believing the struggle would stimulate mathematical growth in his students, this instructor “refused to give students proofs that were pre-packaged. More specifically, he wanted to provide students with intuitions and pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it” (p. 2262).

Although, moving between the worlds is a natural part of doing mathematics for many mathematicians, our studies suggest that many students are not as eager to move. For example, Stewart, Troup and Plaxco (2019a) created a model of instruction to investigate a linear algebra instructor’s movements between Tall’s (2013) three worlds of mathematical thinking. The study revealed that despite instructor’s intentions in moving the class between the worlds, his students were reluctant to move. In a case study, Hannah, Stewart and Thomas (2014) found that a majority of first-year linear algebra students preferred to stay in the symbolic world. Other studies in linear algebra have considered the relationship between formal and other approaches and have demonstrated that developing teaching approaches that promote formal ideas is valuable. For example, Wawro, Sweeney, and Rabin (2011) considered students’ concept images of the notion of subspace and found that students made use of geometric, algebraic and metaphoric ideas to make sense of the formal definition. Zandieh, Wawro, and Rasmussen (2017) analysed data of students engaging in symbolizing a variety of linear algebra tasks in classroom. In a study by Stewart (2018), nine linear algebra tasks were created to encourage the learners to move inside and in between the three worlds. The creation of suitable tasks is one of the crucial aspects of helping students to move between different modes of thinking.

In this paper, we will investigate, students’ reasoning in moving between Tall’s worlds via a survey after doing a task designed by the instructor in class. The research questions guiding this study were: (a) Which was the most comfortable world for students and why? (b) What were some of the factors causing the students to move between Tall’s worlds, if any?

Method

As part of a larger study (Stewart, Epstein, & Troup, 2019b), this research involves an introductory linear algebra class at a Southwestern research university. There were 38 students, mostly engineering majors in their second or third years of study, and the instructor was a mathematician specializing in differential geometry. This paper focuses on a period of approximately one week following the first midterm exam, which covered matrix algebra, linear systems, Gaussian elimination. During the first class period following the exam, Tall’s (2013) Three Worlds were presented by the instructor. This involved a description of each world followed by a representative piece of mathematics, and a diagram, taken from Tall’s (2013) book demonstrating how the worlds are not necessarily disjoint, but instead weave together creating mathematics as we know it. In order to see what effect empowering the students with knowledge of Tall’s worlds would have they were given an activity which offers the opportunity to inhabit or move among all three worlds. We describe this activity now.

The activity explores the connection between the number of solutions to a linear system, the configuration of the lines or planes determined by the system, and the reduced row echelon form of the system's associated augmented matrix. In the first part of the activity (see Figures 1a and 1b), students are provided with a GeoGebra file which displays a linear system with two

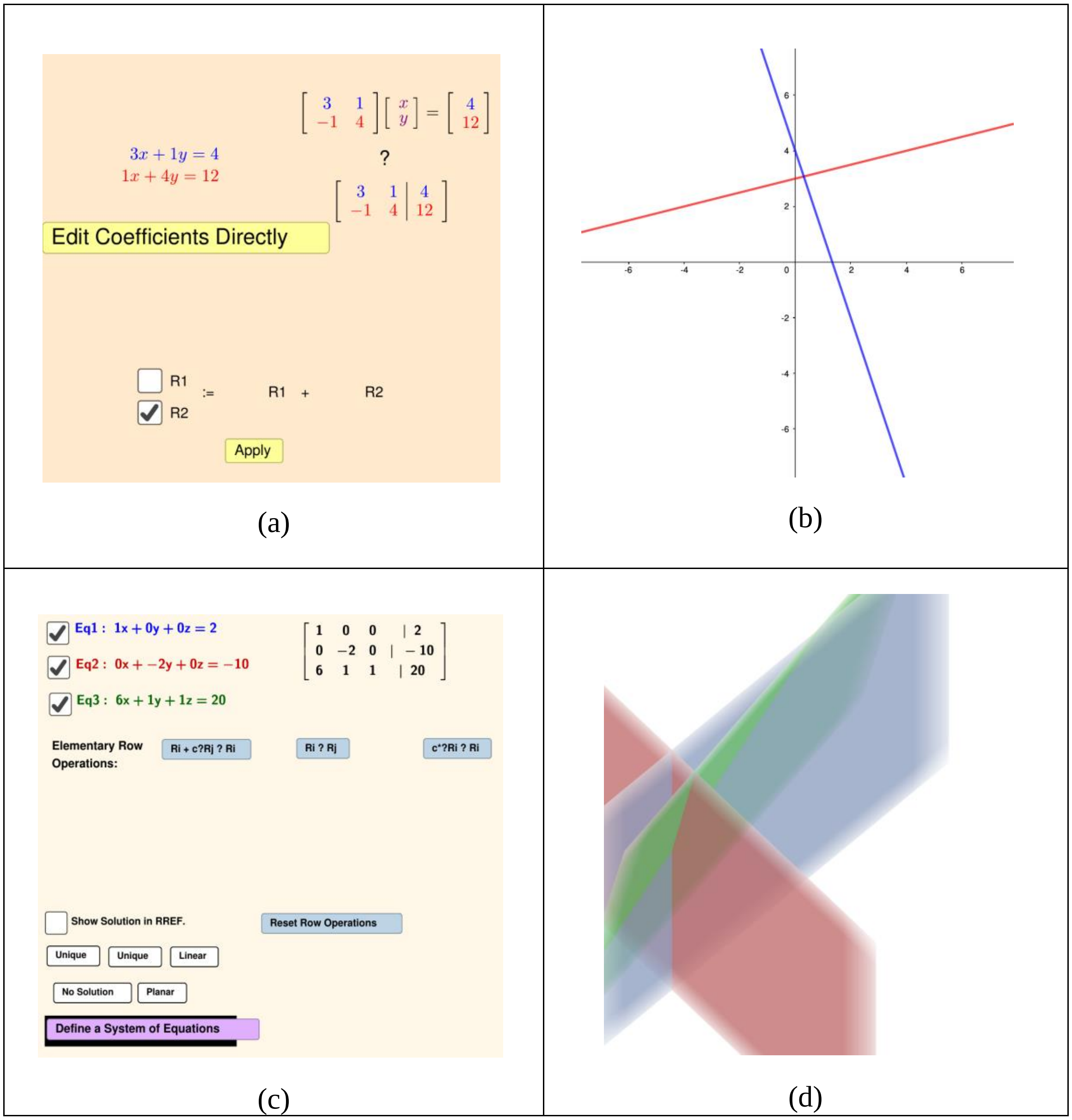

Figure 1. Geogebra tasks.

unknowns and two equations in three ways (as a linear system, in matrix algebraic notation, and as an augmented matrix), while also displaying the two lines determined by the system on the right-hand side. The worksheet also allows one to apply row operations to the system, immediately updating all the notations and graphics. After familiarizing themselves with the GeoGebra file, the activity asks the students:

1. Use Gaussian elimination to solve the system. For each step in the process:
    a. decide what row operation is needed;
    b. express the row operation in our notation;
    c. perform the row operation to the system in the GeoGebra worksheet;
    d. and sketch the graphs of the equations that result.
2. Imagine that the graphs from each step are frames in a movie (like a flip-book). What remains constant throughout the movie? What changes throughout the movie?

In the second part of the activity, the students use a similar GeoGebra file (by James Factor) to explore the effect of Gaussian elimination on linear systems with three unknowns. The interface for applying row operations is shown on the left-hand side in Figure 1c, while the three planes determined by the systems are shown graphically on the right-hand side in Figure 1d (by James Factor). For four systems $\mathbf{A}x = \mathbf{b}$, the students are asked to "examine the graphs of the planes and deduce the reduced row echelon form of the augmented matrix $[\mathbf{A}|\mathbf{b}]$. As in class, use the symbols 0, 1, $*$ to express your answers. If necessary, use row operations to help determine your answers."

While working through this activity students can choose to inhabit any of the three worlds, or move between them. The symbolic world is available via the algorithmic procedures of Gaussian elimination and row reduction. The graphical component of the activity, provided by GeoGebra, gives access to the embodied world. Lastly, the questions regarding abstract reduced row echelon forms allow students to inhabit the formal world by systematizing observations and making general statements about entire classes of linear systems. The following week a survey was administered. The survey questions were: Q1: With your understanding of the definition of the three worlds, which of the worlds (possibly more than one) were you inhabiting while working on Worksheet 8 last week? Q2: Which of the worlds are you most comfortable inhabiting? Why?

In this study the research team converted the worksheet results into Excel spreadsheets to expedite an open coding (Strauss & Corbin, 1998) and sorting the data to search for themes after coding. The analysis of the survey from 15 students who agreed to take part in this study is discussed.

Results

The data was coded according to which worlds were indicated by the students. If a response indicated that one world was primarily inhabited while a another was secondary, it was coded for both of the worlds, etc. For each of the six possible combinations of worlds and for each of the two survey questions, Table 1 shows how many responses indicated that combination.

Table 1. The codes.

Tall's (2013)Worlds	Survey Question 1	Survey Question 2
Embodied (E)	0	2
Symbolic (S)	2	10
Formal (F)	0	0
ES	7	3
EF	2	0
SF	2	0
ESF	2	0

In response to the first survey question, we observed that S2 predominantly commented on the embodied world aspect of the activity, but also remarked on the symbolic facets of the task. Whereas, S9 felt that she or he originally was in the symbolic world and then was forced to go into the embodied world.

S2: Mostly the Embodied world. Using the Geogebra software we could physically see how the concepts we were analyzing manifested themselves as manipulable intersecting lines. There was also an element of the symbolic world involved as we manipulated the equations using the elementary operations.

S9: I think I was in the symbolic world because we were given matrices. I think I was then forced into the embodied world by comparing the matrices with the corresponding visuals.

On the other hand, S11 was completely content to carry on in the symbolic realm, despite the fact that the visual aspects of the task could have potentially added to her or his knowledge.

S11: While working on worksheet #8, I was inhabiting mostly the symbolic world, bordering on the embolic and formal world. My mind is more computationally driven, so I started immediately manipulating the matrix into reduced row echelon form. I occasionally checked the graphs on the right to see what was going on in the embolic world. Then after completing my computation, I took a step back at the big picture, how what I was doing computationally could be formalized for all dimensions beyond R^2 and R^3.

Interestingly, both S5 and S14 did not make any comments about the embodied aspects of the task. S14 was only concerned with the symbolic parts of the task, however in general felt more comfortable in the embodied world.

S5: I think worksheet 8 is based on the symbolic. Yes, there are axioms and laws to follow while working on the worksheet, but the work is pretty cut and dry and does not need much interpretation.

S14: 1 seemed to be more in the symbolic while 2 was also symbolic knowing the terminology would have been more in the formal. The symbolic was due to the following of the steps and the formal was just due to needing to know what the gaussian elimination.

Inhabiting in the Symbolic World

We noted that, many students in this study preferred the symbolic world. Their justifications for favoring this world included comments such as: “very interesting way of looking at things mathematically” (S1), “makes sense and follows a concrete and fathomable set of rules” (S2). Three students referred to their past experiences and argued that: “where most of my recent math education has been focused” (S6), “Ever since I was little, the symbolic world was easier to relate the number” (S8), and “because in a long time I was study in that way. When I was in China they do the practicing sequences of actions a lot” (S13). Another student pointed out that: “it is far more intuitive than the formal world” (S9). One student went as far as saying that: “In the symbolic world, I can clearly and effectively express my ideas through symbols that are agreed upon and understood by the mathematical community” (S4). It seems that these students desired working in this world based on their past experiences and the fact that their view of mathematics could mainly include calculations and following rules. Furthermore, since students can be successful in manipulating the symbols, there is no need to consider other ways of tackling a problem. For example, the following students call the formal world harsh and believe that maybe the embodied world would be somewhat limiting, and more complex ideas are difficult to express in this world.

S1: The embodied is almost meaningless without some structure, and the formal is a bit harsh, at least mentally.
S4: Mathematical concepts can quickly run out of space in the embodied world.
S15: The embodied world is relatively straightforward, but more complex ideas are very difficult to express in this form.

Inhabiting in the Embodied World

Only two students expressed their tendencies for the embodies world and strikingly linked that to understanding. S14 also acknowledged a need for knowing more formal understanding as well.

S7: I personally am a very visual learner so I am most comfortable inhabiting the embodied world because it shows information in the form of figures, diagrams, and other visual aspects. The embodied world is easier for me to understand.
S14: I am more comfortable with the embodied form. I am more of a visual person that needs to see how an equation relates to real life situations or at least many examples so I can understand what something means. I know I need more of a formal understanding because I am taking flight controls next semester in which will rely heavily in the formal.

Inhabiting in the Embodied and Symbolic Worlds

The results also revealed two students' willingness to dwell in both embodied and symbolic worlds simultaneously. S3 admits that she or he is *still* living in these worlds. The main rational for this is the need to have a picture and the fact that the formal world is too abstract.

S3: I am still in embodied and symbolic world. I would say formal world would be too abstract for me to have a picture on my mind. Embodied is easier because it is fun and real. Symbolic can help manipulate the embodied.
S12: I am by far the most comfortable with the embodied and symbolic worlds. I am a very visual learner, and I struggle with taking abstract ideas and applying them to new situations. I learn best by repetition, which lends itself to both the embodied and symbolic worlds.

Discussion and Concluding Remarks

As the researchers were seeking to understand how and why students move between Tall's worlds (2013), they decided to empower the students by giving them the three-world framework and terminology in class. The activity presented them with a situation in which they could inhabit and move between the worlds, and their reflections are a window onto how they viewed themselves operating within the three-world framework. To the extent that mathematicians operate in all three worlds, what inhibited or promoted simultaneous inhabitation of the three worlds? In line with our previous findings (Hannah et al., 2014), most students remained primarily in the symbolic world during the activity and stated that they are most comfortable there. However, several students gave illuminating reasons. The finiteness of the embodied world present obviously limitations, when the goal of the mathematics is to create general statements or theorems. On the other hand, it is difficult to articulate those general statements without a fluency in the use of symbols. Hence the student preference for the symbolic world may be caused not only by the ease of rote symbolic manipulations, but also by the fact that the symbolic world plays a role in the other two. In Tall's views, "As mathematical ideas progress into more sophisticated levels, the balance between embodiment and symbolism changes" (2013, p. 408). He also adds that "Over the longer term, embodied strategies may give insightful meaning at various stages of development, but as the mathematics becomes more complicated, symbolic strategies offer greater power and precision" (p. 407).

This study suggests a way of encouraging more organic movement between the worlds. By investing time preparing the students to express embodied notions and to articulate general statements in symbols, this natural barrier to moving from the embodied and formal worlds to the symbolic could be diminished. We also believe that once students arrive to a new world, in order to mathematically grow they need to be supported and nurtured.

Acknowledgements

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Reasoning Covariationally about Constant Rate of Change: The Case of Samantha

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We present the results of a teaching experiment designed to foster a pre-service secondary teacher's construction of a scheme for constant rate of change through engendering reflecting and reflected abstractions. Although the research participant developed a productive conception of rate of change as an interiorized ratio, images of chunky continuous covariation imposed an obstacle to her ability to reason efficiently across a variety of contexts. The participant's rate of change scheme was grounded in reflected abstraction, which enabled her to become consciously aware of its essential aspects and to appreciate its applicability across a variety of contexts.

Keywords: Algebraic thinking, Reflected abstraction, Pedagogical content knowledge

Introduction

We previously reported the results of a study that examined the effect of a pre-service secondary teacher (Samantha) having constructed a scheme for constant rate of change grounded in reflecting and reflected abstractions (Tallman & Weaver, 2018). Specifically, we concluded that the conscious awareness of mental actions and conceptual operations constituting Samantha's scheme for constant rate of change—which resulted from having engaged in reflected abstraction—had positive implications for her image of effective teaching and for the pedagogical actions she envisioned enacting to conform to this image. The central findings of Tallman and Weaver (2018) challenge the assumption that *pedagogical content knowledge* (Shulman, 1986, 1987) is a synthesis of knowledge of content and knowledge of pedagogy (Depaepe, Verschaffel, & Kelchtermans, 2013). Tallman and Weaver did not, however, document the specific features of the mathematical intervention that enabled Samantha to construct a scheme for constant rate of change implicative in her instructional goals and supportive of her pedagogical approach. Such is the focus of this paper.

We present the results of a teaching experiment designed to foster Samantha's construction of a scheme for constant rate of change through engendering *reflecting* and *reflected abstractions* (Piaget, 2001). Our objective in doing so is to clarify how mathematics faculty at the undergraduate level might support pre- and in-service teachers' construction of mathematical schemes that enable effective pedagogies, rather than to maintain the traditional practice of teaching content and pedagogy separately with the expectation that these distinct knowledge domains will unify in the context of practice to support effective instruction.

Theoretical Background

Piaget proposed abstraction as the mechanism of scheme construction and refinement and distinguished five varieties: *empirical*, *pseudo-empirical*, *reflecting*, *reflected*, and *meta-reflection* (Piaget, 2001). We summarize only *pseudo-empirical*, *reflecting*, and *reflected abstraction* because of their unique role in the construction and refinement of mathematical schemes and because these three types of abstraction served as design principles and analytical constructs for our teaching experiment.

Pseudo-empirical abstraction occurs when a subject abstracts properties of objects that have been modified by or created through the subject's actions and enriched by the properties drawn from their coordination (Piaget, 1977). In pseudo-empirical abstraction, the subject does not disassociate the action from the object nor the object from the action. What the subject abstracts, then, is not simply a property of the object (as in empirical abstraction) nor the action that produces or modifies the object, as in reflecting abstraction (see below), but a property of the object that is produced by and which, crucially, *represents* the action of the subject.

Reflecting abstraction involves the subject's reconstruction on a higher cognitive level of the coordination of actions from a lower level, and results in the development of *logico-mathematical knowledge*, or schemes at the level of *operative thought*. Reflecting abstraction is thus an abstraction of actions and occurs in three phases: (1) the *differentiation* of a sequence of actions from the effect of employing them, (2) the *projection* of the differentiated action sequence from the level of activity to the level of representation, or the reflected level, and (3) the *reorganization* that occurs on the level of representation of the projected actions (Piaget, 2001). Once a subject differentiates actions from their effect and coordinates them, she is prepared to project these coordinated actions to the reflected level where they are organized into cognitive structures, or schemes.

Reflected abstraction involves operating on the internalized actions that result from prior reflecting abstractions, which results in a coherence of actions and operations accompanied by conscious awareness. It is the act of deliberately operating on the actions and operations that result from prior reflecting abstractions that engenders such cognizance. As a result of the awareness of internalized actions that occurs as a byproduct of reflected abstraction, the subject's ability to purposefully assimilate new experiences to the reflected level provides evidence that she has engaged in reflected abstraction.

Methods

We conducted a teaching experiment (Steffe & Thompson, 2000) to construct a model of Samantha's scheme for constant rate of change and to characterize the evolution of this scheme as she engaged in instructional experiences designed to promote reflecting and reflected abstractions. The fundamental goal of a teaching experiment is to construct a model of another's mathematical realities that is viable with the researcher's interpretation of students' observable behaviors. In the context of a teaching experiment, the schemes that students construct through spontaneous development are brought forth through exploratory teaching, and it is the responsibility of the researcher to discern how students modify their cognitive schemes as they experience specific teaching actions.

Our teaching experiment occurred during Samantha's student teaching semester while she was enrolled at a large university in the Midwestern United States. Samantha participated in eight teaching episodes that each lasted between 60 and 105 minutes. Samantha also attempted problems between teaching episodes and discussed her solutions during the following interview. All teaching episodes were video recorded and selectively transcribed. Members of the research team met between teaching episodes to discuss provisional hypotheses about the current state and development of Samantha's scheme for constant rate of change and to modify tasks for subsequent teaching episodes.

We employed grounded theory procedures (Corbin & Strauss, 2008; Strauss & Corbin, 1990) to analyze the teaching experiment data. Specifically, we began by performing an iteration of open coding to identify instances in which Samantha revealed characteristics of her meanings for constant rate of change and associated ideas (e.g., slope). We then conducted an iteration of axial coding to construct and refine categories of episodes identified with particular codes from the initial open coding. For each category, we articulated the conceptual operations that appeared to inform Samantha's language and actions. Finally, we identified shifts in the meanings Samantha demonstrated and described the instructional actions that appeared to occasion these shifts.

Conceptual Analysis of Constant Rate of Change

In this section, we briefly describe the particular conception of constant rate of change we designed our teaching experiment to support. This conception, together with our image of the actions and abstractions required for its construction, constitutes our central research hypothesis.

A mature rate of change scheme relies upon productive conceptualizations of ratio, rate, and continuous variation. A *ratio* is a multiplicative comparison of the measures of two constant (non-varying) quantities while a *rate* defines a proportional relationship between varying quantities' measures (Thompson & Thompson, 1992). Constructing a rate therefore involves images of *smooth continuous variation* (Thompson & Carlson, 2017), as well as the expectation that as two quantities covary, multiplicative comparisons of their measures remain invariant. A rate is a reflectively abstracted constant ratio (Thompson & Thompson, 1992, p. 7). As such, its construction involves internalizing the coordinated actions entailed in multiplicatively comparing the measures of continuously covarying quantities. *Rate of change* is a quantification of two covarying quantities and results from a multiplicative comparison of changes in the quantities' measures. The rate of change is constant if these changes are proportional. Conceptualizing changes in quantities' measures as quantities themselves involves a *quantitative operation* (Thompson, 1990) with attention to a point of reference (Joshua et al., 2015).

Thompson and Thompson (1992) describe four levels in the development of students' ratio/rate schemes:

> The first level, *ratio*, is characterized by children's comparison of two taken-as-unchanging quantities according to the criterion "as many times as." The second level, *internalized ratio*, is characterized by children's construction of co-varying accumulations of quantities, where the accrual of the quantities occurs additively, but there is no conceptual relationship between ratio of accumulated quantities at iteration x and the ratio of accumulated quantities at iteration $x + 1$. The third level, *interiorized ratio*, is characterized by children's construction of co-varying amounts of quantities, where the amounts vary additively but with the anticipation that the ratio of the accumulations does not change. The fourth level, *rate*, is characterized by children's conception of constant ratio variation as being a single quantity—the multiplicative variation of a pair of quantities as a measure of a single attribute. A *rate* is a reflectively-abstracted conception of constant ratio variation. (Thompson & Thompson, 1992, p. 7-8)

What distinguishes these levels in the development of students' ratio/rate schemes is how they conceptualize the variation of quantities upon which they operate to construct the ratio/rate. In other words, these levels reflect various levels of sophistication in students' *covariational reasoning* (Carlson et al., 2002; Thompson & Carlson, 2017).

Results

Teaching Episode 1: Transition from *Internalized Ratio* to *Interiorized Ratio*

We began the first teaching episode by engaging Samantha in a series of tasks situated in the following context: "While training for a triathlon Clara rides her bike on a long, straight road at a constant rate. She passes under a stoplight while riding." To discern how Samantha conceptualized the covariation of Clara's distance from the stoplight and the amount of time elapsed since she passed the stoplight, we asked her to describe how these quantities vary together.

> *Interviewer*: How are those two quantities—the distance that she is from the stoplight and the amount of time elapsed since she passed the stoplight—how are those two quantities changing together?
>
> *Samantha*: Um, as more time passes she's getting farther from the stoplight.
>
> *Interviewer*: … Could you be more specific?
>
> *Samantha*: … Since she's going a constant rate she'd go the same distance for each change in time. … So if she goes an hour she goes the same amount of miles that she goes the next hour.

Samantha's response reveals her conception of constant rate of change as a correspondence of additive changes in accumulated distance and elapsed time. This meaning for rate is consistent with Thompson and Thompson's (1992) second level (*internalized ratio*) in the development of students' ratio/rate schemes. Samantha appeared to have dissociated elapsed time as an independently varying quantity, but remained focused on discrete accumulations of distance that correspond to one-unit accruals of elapsed time.

We later asked Samantha to describe what it means to say that Clara rode her bike at a constant speed of 12 miles per hour. Her response further supports our claim that chunks of accumulated distance corresponding to one-unit of elapsed time were dominant in her initial meaning for rate:

> Every hour, she goes 12 miles. … You go 12 miles in an hour and then like you're going multiples of 12 I guess for every hour. So you'd take 12 times one, 12 times two. So it'd be 12 times x … where x would be your time in hours.

To perturb Samantha's understanding of speed as a distance and time as a ratio, and to begin to support her conception of an invariant multiplicative relationship between continuously covarying quantities, we prompted her to consider the amount of time it would take Clara to ride 25 miles if she rides at a constant speed of 10 miles per hour. Since the solution is not a whole number of hours, Samantha was positioned to (pseudo-empirically) abstract a constant multiplicative relationship between accumulated distance and elapsed time. She correctly determined that it would take Clara 2.5 hours to travel 25 miles if riding at a constant speed of 10 miles per hour. Samantha justified the 0.5 hour required for Clara to travel the final 5 miles as follows: "If you're assuming that she's goes the same speed the whole time, then you have– a whole hour will take 10 (miles) so since she's going the same speed then a half an hour will take five." Samantha's response suggests an initial step towards the generalization that if Clara travels a fraction of the distance she rides in one hour, it takes that same fraction of an hour, thus revealing a transition towards a conception of rate as an *interiorized ratio*.

We subsequently asked Samantha to determine the amount of time it would take Clara to travel 17 miles if she rides at a constant speed of 13 miles per hour. Her immediate response was to divide 17 by 13, which appeared to have been motivated by a pseudo-empirical abstraction from her activity on the previous task. Although Samantha struggled for several minutes to formulate a coherent explanation for why division is the appropriate operation, aspects of her five-minute response revealed her tentative understanding that a fractional part of a time unit corresponds to the same fractional part of the distance traveled in that time unit (and vice versa).

Samantha's response to a subsequent task ("How long does it take Clara to ride 5 miles from the stoplight if she rides at a constant speed of 9 miles per hour?") further demonstrated her reasoning in a way compatible with Thompson and Thompson's (1992) description of an interiorized ratio. She drew a line segment to represent an elapsed time of one hour and partitioned this line segment into nine pieces that represent consecutive changes in distance of one mile (see Figure 1). This image itself represents Samantha's understanding that one mile (being 1/9th of the distance Clara travels in one hour) corresponds to an elapsed time of 1/9th of an hour. She explained, "There should be nine of them (miles) in an hour. … say she goes five (miles), that's five out of the nine miles, and so since I broke my bar up into ninths, then I have five out of nine of an hour." It is noteworthy that Samantha's solution relied upon having partitioned one hour into the number of miles Clara travels in an hour.

Figure 1. Uniform partition of one hour into one-mile accumulations of distance.

To support Samantha's understanding that the correspondence of fractional parts of a time unit and equivalent fractional parts of the distance traveled in that time unit does not require having first imagined a uniform partition of an hour (where the number of partitions is determined by the number of miles Clara travels in an hour), we directed her focus to Clara's accumulated distance as a whole and prompted her to multiplicatively compare this distance with the corresponding fractional part of an hour. Specifically, we asked, "Five miles is what fraction of the number of miles Clara travels in one hour?" We intended this question to promote her realization that this multiplicative comparison exists without having to construct a uniform partition of one-unit of elapsed time—an understanding essential to constructing a general rate of change scheme to which Samantha might assimilate subsequent tasks. In essence, we attempted to support Samantha's anticipation that *any* distance from the stoplight is some fraction of the distance Clara travels in one unit of time, and that it therefore takes Clara that same fraction of one time-unit to travel that distance. After struggling for several minutes to formulate a response to the question we posed (and various restatements of it) Samantha finally admitted, "I don't understand." The difficulty she experienced stemmed from her conception of the variation of accumulated distance and elapsed time as occurring in chunks (i.e., as changing discretely by intervals of fixed magnitude), a way of thinking Thompson and Carlson (2017) reify as *chunky continuous covariation* (p. 435). We demonstrate how Samantha's chunky continuous covariational reasoning persisted throughout the remainder of the teaching experiment and illustrate its effects on the development of her understanding of rate of change below.

Teaching Episodes 2-5: Solidification of *Interiorized Ratio*

Samantha independently reexamined the tasks from the first teaching episode prior to the second. The second teaching episode began with Samantha describing what she had been thinking about since the conclusion of the prior interview. She read aloud a sentence she wrote to summarize the relationship between accumulated distance and elapsed time for an object traveling at a constant speed: "The fraction (or multiple) of the distance traveled in one hour is the amount of time it takes to go a given distance." Samantha independently applied this understanding to work through the tasks from the first teaching episode, and was ultimately surprised by its general applicability.

After having summarized her activity between the first and second teaching episodes, Samantha responded to the task, "Suppose Clara rides her bike at a constant speed of 16 miles per hour. How far is Clara from the stoplight 0.38 hour after she passed the stoplight?" Samantha reasoned that since Clara rode for 0.38 of an hour, she traveled 0.38 of the distance she would have traveled in an hour, or 0.38 times 16. Her solution to this problem, and several others like it, demonstrates that Samantha had conceptualized rate as an interiorized ratio. We later asked Samantha to consider the following task:

> A swimming pool that is only partially full has 10,800 gallons of water. Water is added to the swimming pool 10:00 AM at a constant rate of 24 gallons per minute. Let Δt represent any change in the number of minutes elapsed while the pool is being filled and let Δv represent the corresponding change in the number of gallons of water in the pool. Write an equation that expresses the relationship between Δt and Δv.

Samantha wrote the equations "$\Delta v = 24(\Delta t)$" and "$\Delta t = {}^{1}/_{24}(\Delta v)$" to express the relationship between Δt and Δv. To justify these equations, Samantha explained, "Since it's 24 gallons per minute, any number of minutes that I'm given, I would multiply that by 24 because I have change in *t* of them, and that would give me the number of gallons in the pool." Samantha's response—as with many others that preceded it—suggest that her conception of rate was grounded in a correspondence of discrete accumulations in covarying quantities' measures, as opposed to being based on images of smooth continuous covariation constrained by a proportional relationship.

We prompted Samantha to explain how her the second equation above expresses the relationship between the change in time elapsed since water started being added to the pool and the corresponding change in the volume of water in the pool. She replied,

> This is the hard part. I mean for every one minute you get 24 gallons, so if you take any chunk of the gallons added, it doesn't matter which chunk you take since 10:00 AM—you could choose 10:30 to 11:00, you know—then you would multiply that by 1/24. It's hard to figure out exactly how you would get there though. … So, in this one (*pointing to the equation* "$\Delta v = 24(\Delta t)$) it's like 24 gallons per minute and for however many minutes, and so that gives you your chunk of gallons, so since you have that then you know that your time would then have to be 1/24$^{\text{th}}$ of the gallons, the chunk of gallons, change in time I guess would be. I'm thinking of them as chunks in my mind. Like that's how I'm visualizing them. Then change in volume as like a chunk. I don't know why.

Samantha's statement and prior activity indicate that she pseudo-empirically abstracted from previous tasks the multiplicative relationship between a whole-number of minutes elapsed and the corresponding number of gallons of water added to the pool. She then made a generalizing assimilation of this relationship to non-integer values of elapsed time and inverted the proportionality constant to describe the reciprocal relation between Δt and Δv. There was no evidence that Samantha conceptualized Δt and Δv as quantities that covary smoothly and continuously. We therefore suspect that Samantha had not constructed a *rate* as Thompson and Thompson (1992) described it—as a reflectively abstracted constant ratio. Her conception of rate was instead the product of a pseudo-empirically abstracted constant ratio, which explains the tenuousness of her conception of the proportional relationship between Δt and Δv.

Teaching Episodes 6-8: Engendering Reflected Abstraction

In the final three teaching episodes, we asked Samantha to compare and contrast the reasoning required to solve various pairs of tasks from the instructional sequence that guided the teaching experiment. Our intention was to engender reflected abstraction by prompting Samantha to perform mental operations on the products of her prior reflecting abstractions. Continually reenacting, comparing, and contrasting her prior reasoning necessarily brought her conceptual activity into conscious awareness and resulted in her rate of change scheme becoming progressively coherent and refined. We share one illustrative example.

During the sixth teaching episode, we prompted Samantha to compare and contrast the reasoning required to solve the following two tasks:

1. While training for a triathlon Clara rides her bike on a long, straight road at a constant rate. She passes under a stoplight while riding. How long does it take Clara to ride 17 miles from the stoplight if she rides at a constant speed of 13 miles per hour?
2. A swimming pool that is only partially full has 10,800 gallons of water. Water is added to the swimming pool 10:00AM at a constant rate of 24 gallons per minute. How long will it take to add 1,725 gallons of water to the pool?

Discussing the meaning of rate of change that might enable students to solve these two problems, Samantha explained,

> I think they have to think about for any, like, (*long pause*) for any change in one quantity, how that relates to the other quantity. They need to think about, um, just because your change in distance is, you know, 13 times as large as your change in time, also thinking about it works the other way too. Your change in time also has a relationship to your change in distance. It's $1/13^{\text{th}}$. … Like being able to go both directions.

We asked Samantha to express her point in the general context of "a varies at a rate of r with respect to b." She clarified,

> They would need to understand that it varying at a constant rate means that like any change in a is going to be r times as large as the change in b. … They have to think the change in a is r times as large as the change in b or change in b is $1/r^{\text{th}}$ the change in a. … They have to be able to think about both of those.

Discussion

Although Samantha ultimately developed a productive conception of rate of change as an interiorized ratio, and thus was able to coordinate corresponding fractional changes in covarying quantities' measures, images of chunky continuous covariation (Thompson & Carlson, 2017) imposed an obstacle to her ability to reason efficiently across a variety of contexts. Still, Samantha demonstrated a sophisticated conception of rate of change that enabled her to reason productively about an assortment of novel tasks. Additionally, Samantha became consciously aware of essential aspects of her scheme for constant rate of change by engaging in reflected abstractions, which contributed to her recognition and appreciation of the general applicability of the meaning she had constructed. Our results suggest two important implications for the content preparation of pre-service secondary teachers at the undergraduate level: (1) images of smooth continuous covariation are fundamental to a mature rate of change scheme, and possibly other mathematical ideas where the covariation of two (or more) quantities is constrained by multiplicative relationships; and (2) engaging in reflected abstraction is essential for supporting pre-service teachers' construction of coherent and refined mathematical schemes, and should thus be a central principle of instructional design for content courses serving prospective secondary teachers.

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The Role of Lines and Points in the Construction of Emergent Shape Thinking

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In this study, we focus on a student's meanings for lines and points in the context of graphing covarying quantities. Specifically, we illustrate a student conceiving a line as representing a direction of movement of a dot on a coordinate plane. Consequently, the student did not conceive a dot moving in the coordinate plane as leaving a trace of infinitely many points; similarly, points on a line did not exist until they were physically and visually plotted. We conclude that the student's meanings for lines and points had a significant impact on his graphing activities, in particular, on his construction of emergent shape thinking.

Keywords: Graphical Shape Thinking, Quantitative and Covariational Reasoning

Graphing is critical for understanding various ideas in mathematics (Kaput, 2008; Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Despite its importance, students experience a number of challenges in interpreting and making sense of graphs that may affect their learning of many topics in algebra and calculus (e.g., Clement, 1989; Leinhardt et al., 1990; Moore & Thompson, 2015). In addressing those challenges and difficulties, a number of researchers have suggested that quantitative and covariational reasoning is productive for students' construction of graphing meanings (e.g., Carlson, 1998; Frank, 2017; Johnson, McClintock, & Hornbein, 2017; Saldanha & Thompson, 1998; Thompson, Hatfield, Yoon, Joshua & Byerley, 2017). Specifically, some researchers (e.g., Frank, 2017, 2018; Moore & Thompson, 2015) have emphasized that a productive meaning for a graph is conceiving it as an emergent trace of points that unite two covarying quantities' magnitudes and/or values (i.e., *emergent shape thinking*). In this paper, we contribute to extant theorization of emergent shape thinking by demonstrating how a student's meanings for lines and points influenced his construction of emergent shape thinking. In particular, the student's meaning for a line as indicating a direction of movement of a dot was a contraindication of him conceiving of a graph as an emergent trace of a point, and an implication of his meaning for a line is that it constrained him from thinking of a line as consisting of infinitely many points.

It is worth noting that, in this study, we situated students' learning about quantities' relationships within contexts emphasizing the quantities' *magnitudes* independent of numerical *values*. By a quantity's magnitude, we mean the general sense of the quantitative size of an object's measurable attribute (e.g., length), whereas, by quantity's value, we mean the result of measuring that attribute. Researchers have argued that reasoning with quantities' magnitudes supports students in understanding quantities' covariational relationships (Liang, Stevens, Tasova, & Moore, 2018; Thompson, Carlson, Byerley, & Hatfield, 2014).

Background

Emergent Shape Thinking

Moore and Thompson (2015) introduced the notion of *emergent shape thinking* to describe a person who envisions a graph "*simultaneously* as what is made (a trace) and how it is made (covariation)" (p. 785). This conception involves (1) representing two inter-dependent quantities' magnitudes and/or values varying on each axis of a coordinate system, (2) forming a multiplicative object (Saldanha & Thompson, 1998; Thompson, 2011; Thompson et al., 2017) by uniting those two quantities' magnitudes or values as a single object, and (3) assimilating the process of a multiplicative object moving within the plane in ways invariant with the two covarying quantities as generating a graph, or conceiving a given graph as an emergent record of all instantiated moments of the simultaneous coordination of two covarying quantities. In this study, we report on a student who demonstrated compelling evidence of the first two elements of emergent shape thinking that are listed above; however, demonstrated a contraindication of the third element of emergent shape thinking due to his meanings for points and lines.

Students' understanding of points on a line

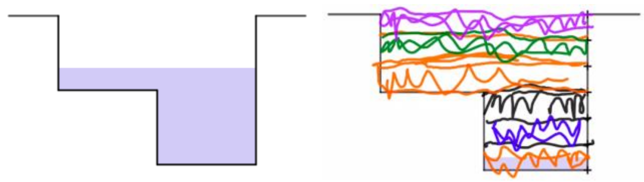
Researchers (e.g., Kerslake, 1981; Mansfield, 1985) have investigated how students conceived points and lines in the context of graphing. They have revealed that students (especially secondary level students) tend to "see" points on a line only if they are visually marked on a graph. For example, Manfield (1985) reported that some secondary and undergraduate students did not perceive points in between two marked points on a line, and some students only perceived endpoints of a straight line on a paper or points on the vertices of a zigzag line. These authors also reported on students who believed no line has points until they are placed on the line. Similarly, Kerslake (1981) reported that about 89% of secondary students (N=1798) did not conceive of infinitely many points on a line. When asked how many points are on a straight line that goes through three points, several students answered "three" or the number of places where the line and a coordinate grid intersect. Although there were some students being aware that there were many points (e.g., "lots" or "hundreds" p. 123) on a line, Kerslake reported that their conception was still constrained "by the physical constraints of actually drawing the points" (p. 123). For example, a student said there are points "as many as there is room for" on the line in between two points plotted (p. 123). In this study, we found a student whose meanings for points on a line was similar to that of the students reported by these aforementioned researchers. We contribute to these findings by discussing how such meanings constrained his construction of emergent shape thinking.

Method

We conducted a semester-long teaching experiment (Steffe & Thompson, 2000) with four secondary students. In the teaching experiment, we aimed to investigate the mental actions involved in the students' conceiving situations quantitatively and representing particular quantitative relationships on number lines and coordinate systems. In this paper, we focus on one of the four students, Zane, since his meanings for the lines and points were consistent and clearly described by him throughout the teaching experiment. We believe it is important to document his ways of thinking in order to add nuances to our models of students' thinking in a graphing activity in terms of emergent shape thinking.

Zane participated in 16 one-hour long videotaped teaching experiment sessions over the course of seven weeks. The first author was the teacher-researcher (TR), and the second author served as the witness-researcher (WR). Before conducting the teaching experiment, the TR developed an initial sequence of tasks by considering particular design principles focused on graphing covarying quantities (e.g., Frank, 2017; Moore & Thompson, 2015; Thompson & Carlson, 2017). The TR revised and implemented those tasks based on on-going inferences and analysis of Zane's thinking. Each task was designed with a dynamic geometry software and displayed on a tablet device. We recorded all sessions using two video cameras to capture Zane's work and his gestures and a screen recorder to capture Zane's activities on the tablet device. We transcribed the video and digitized Zane's written work for both on-going and retrospective conceptual analyses (Thompson, 2008). Our analysis relied on the generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to developing working models of Zane's thinking based on his observable and audible behaviors.

In this paper, we report data from Zane's activity in the Swimming Pool Task adapted from Swan (1985). We presented Zane a dynamic diagram of a pool (see Figure 1a), where he could fill or drain the pool by dragging a point on a given slider. We designed the task to support Zane in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool.



(a) (b)

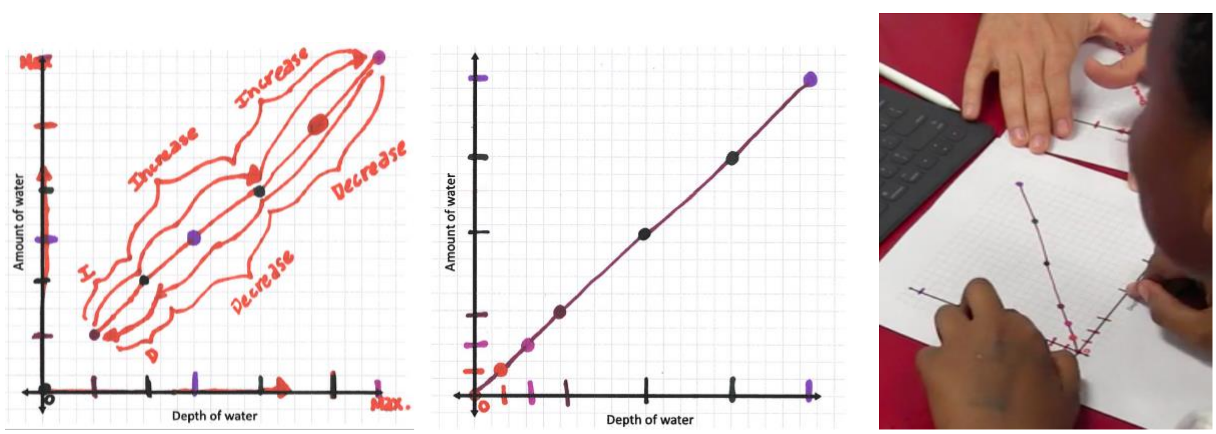
Figure 1.(a) A diagram of the pool (b) illustration of Zane's partitioning activity.

Analysis and Findings

In this section, we illustrate Zane's meanings for his graphs, including the tick marks, points, and lines as he perceived on the graphs. Then, we discuss how these meanings influence his assimilation of what we perceive to be an emergent trace of a point.

Zane's construction and interpretation of his displayed graphs

We asked Zane to sketch a graph that shows the relationship between AoW and DoW as the pool fills up. Zane started with drawing tick marks on each axis and plotting points corresponding to two related tick marks (see his color-coded points and tick marks in Figure 2a and 2b), then he connected those points with line segments. He initially constructed Figure 2a and adjusted his graph to Figure 2b to represent bigger increments at the top half of the pool. He also drew arrows to show "increase" and "decrease" in both quantities (Figure 2a).



(a) (b) (c)

Figure 2. (a) Zane's first draft, (b) Zane's second draft, and (c) Zane moving his fingers on axes.

Meanings for tick marks. When questioned about his tick marks, Zane referred to the quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis. He also used his fingers to simulate the quantities' variation as the TR played the animation to fill the empty pool. He initially placed his left and right index fingers at the origin saying "I started from zero" and then moved his left index finger up along the vertical and his right index finger to the right along the horizontal axis (Figure 2c). While he was moving his fingers, we inferred that he wanted to make sure both fingers hit each corresponding tick marks at the same time so as to match AoW and DoW as he perceived in the animation.

In order to determine if Zane perceived quantities' magnitudes *in between* his tick marks, the TR asked him if he moved his fingers by jumping from one tick mark to another. He responded that he moved his fingers continuously and described an intermediate state:

> Because, I mean, on the thing [*pointing to the pool in Figure 1b*], it is not like very jumping up [*moving up his finger very fast from the bottom of the pool*]. It is really just, like, because the water can be [*pointing to the orange shaded area at the bottom of the pool, Figure 1b*] also a half of it too [*pointing to the water level in Figure 1b*].

In summary, we infer that Zane could simultaneously coordinate both quantities' variations on the Cartesian coordinate system. He conceived of the distance from each finger to the origin as representing the magnitude of AoW or DoW, and he could keep track of the two quantities' variations simultaneously and continuously, including intermediate states between tick marks.

Meanings for points. As the conversation continued, the TR tried to gain insights into the extent to which he coordinated those tick marks on the axes to construct meaning of points on the graph. The TR asked Zane to show the point on his graph that shows the AoW and DoW when the pool is full. Zane first pointed to the far right and top purple tick marks on each axis (see Figure 2b, also see his gesture illustrated in Figure 3a), and then, he pointed to the corresponding purple point on the plane (see Figure 2b). His actions showed that he could associate these two tick marks on each axis to the point on the plane. Then the TR asked him to move his fingers correspondingly on each axis as we played the animation. The following excerpt demonstrates his activity:

TR: I am gonna take out water. You are gonna

Zane: Go down [*moving his right and left index finger to the left and down, along the horizontal and vertical axis respectively. Then, he put his finger back in their original position at the very end tick marks on each axis, see Figure 3a*].

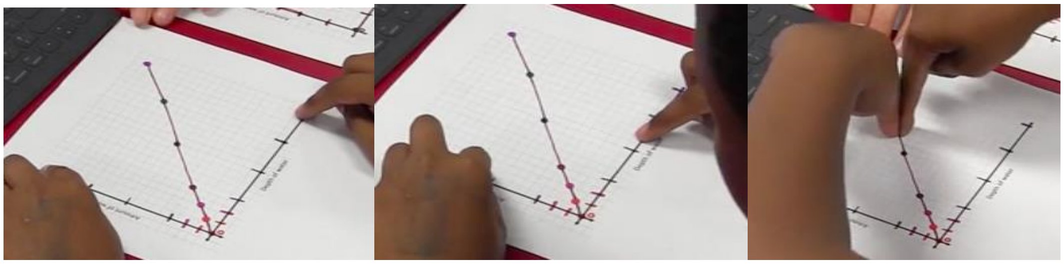
TR: Yes. But, when you do this, you gotta imagine what happens to this point [*pointing to the corresponding point*] ... when I start changing, you are gonna move your fingers and imagine what happens to the corresponding point.

Zane: [*We played the animation and Zane moved his left and right index fingers smoothly on each axis to the left and down, respectively. The TR stopped the animation where the water level is within the area that are shaded in green [see Figure 1b], and he immediately stopped moving his both fingers, see Figure 3b*].

TR: Okay. Where is the corresponding point?

Zane: [He *simultaneously moved his left index finger to the right horizontally and right index finger up vertically, and stopped when the two fingers met; see Figure 3c*] like right here. [*Then he plotted a point on his graph and added two corresponding tick marks on each axis*].

We interpreted that Zane was able to conceive of a point on the plane as a multiplicative object that simultaneously unites the two quantities' variations. Later, he described a point moving up and down along the line as representing *both* quantities' increases and decreases at the same time, saying "the dot represents both." That is, he held in mind two quantities associated with a point and imagined variation of the two quantities as the point moved.



(a) (b) (c)

Figure 3. (a) the location of Zane's fingers before the animation started, (b) the place where Zane stopped, and (c) Zane pointing to the corresponding point where both fingers met.

Meanings for lines. When questioned why he connected the points with lines, Zane responded that the line shows "where the dots go." Additionally, he said, "it also helps to person who comes in, they will understand that the line is probably moving down and up." We infer that Zane conceived of a line on the coordinate plane as showing a path of movement of a dot in either direction. We hypothesized that this meaning for a line might be related to his meaning constructed outside a graphing context prior to the study. Therefore, the TR drew a straight line on a blank paper and asked him how he would define a line. He responded, "a point, hmm, something that goes and never stops." The TR followed up, "What is that goes and never stops?" After an eight-second pause, he said "hmm, from a start point [*placed his right index finger on the left end side of the line*] to an end point [*moves his finger to the right end side of the line*]. We inferred that Zane conceived a line as describing how an object moves from one location to another, and this was compatible with his understanding of lines in the graphical context.

Based on this inference, the WR hypothesized that he might not conceive of a line as consisting of infinitely many points. To test this hypothesis, the WR asked him, "how many points do we need to plot in order to fully describe what is going on here?" Zane added three additional points on each segment between the points that he originally plotted on his graph (see Figure 2b) and said "24". The WR then asked him to compare two graphs, a graph with 24 dots plotted and another graph that includes a line, and discuss how they were similar or different. He replied "wait, did the person who has the line also have 24 points too?" We inferred from his activity that, for him, a line with dots and a line without dots are different graphs and the dots are critical components that convey additional information to a line.

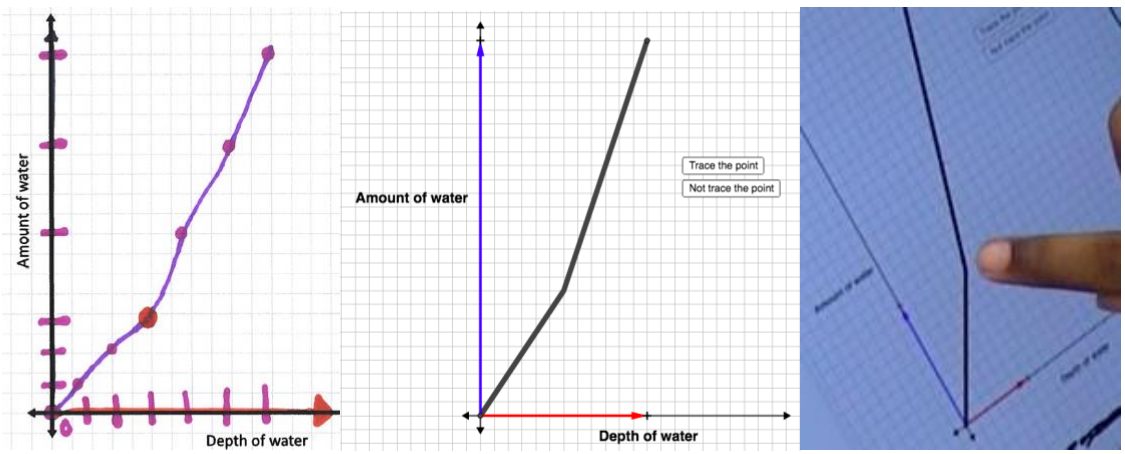
As an additional evidence, when asked whether his graph (see Figure 4a) shows every single moment of how the two quantities vary in the situation, Zane said no because in order to show it, you need to plot an additional point. We infer that, for Zane, lines do not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. That is, he did not conceive of his line as showing these extra moments. In the next section, we demonstrate that such meaning for a line played a role in his construction and interpretation of what we perceived to be in-progress trace.

Zane's Interpretations of an Emergent Trace

Given these interpretations of Zane's meanings for tick marks, lines, and points, the TR hypothesized that he likely did not interpret his prior finger activity (Figure 2c and Figure 3) as generating infinitely many coordinate points. To test this hypothesis, the TR showed him an animation on an tablet device (see https://youtu.be/97EOENUM_co) and asked: "is this trace [*Figure 4b*] showing us the relationship between depth of water and amount of water for this pool?" He replied "no" and struggled to make sense of what the animation was showing, which suggested that he did not perceive the animation as a simulation of his prior graphing activity on paper.

The WR asked Zane whether those dots[1] on his paper (see Figure 4a) are "part of the line on the computer." Zane replied, "there is only one dot," pointing to the animating dot that produced the trace (see his gesture in Figure 4c). When asked "is there any other dots on this graph?" he shook his head. Moreover, he interpreted his graph (Figure 4a) as having more dots than the one produced in the animation (Figure 4b), commenting that mine is better because "mine have more dots".

Zane also claimed that he could not construct his graph in the same way as the animation did due to physical constraints of human, saying, "well, I cannot do that, because, like, can you do dots and dots [*tapping his right index finger very fast along his graph shown in Figure 4a*] and trace it?" This is an additional contraindication that he conceived of graphing a line as a way to represent infinitely many points.



(a) (b) (c)

Figure 4. (a) Zane's last draft and (b) an instance of the animation, and (c) Zane pointing to the "only" point on the trace.

In summary, despite his success in the finger activity and being able to conceive of a point as a multiplicative object, Zane assimilated his activity as well as the animation as one dot moving along a line path instead of one dot generating infinitely many points by leaving a trace. We claimed that his meaning for a line as describing a direction of movement (as opposed to consisting of infinitely many points) played a critical role in his construction and constrained him from conceiving a graph as an emergent, in-progress trace (i.e., the third component of emergent shape thinking).

Discussion

Moore and Thompson (2015) and Frank (2017, 2018) emphasized the important role of forming a multiplicative object in students' development of emergent shape thinking. What we have found in this study is that multiplicative object is a necessary but not sufficient condition

[1] We use the word *dot* instead of *point* when we have evidence of Zane referring to a visual circular object that is plotted on the plane, but do not have explicit evidence of him holding the two quantities *in mind* (i.e., conceiving a multiplicative objective) at the moment. By using the word "dot", we are also genuine to Zane's language in this activity.

for emergent shape thinking. We found that Zane constructed a meaning for individual points as multiplicative objects, and he could do the finger tool activity (i.e., moving index fingers on each axis) smoothly to represent continuous, simultaneous co-variation of two quantities. However, he interpreted his activity of connecting points on his graph as representing a direction of movement of one point from one location to another; hence he did not imagine a segment or a line as including infinitely many points and representing every moment of the quantities' covariation. In turn, he did not perceive an emergent trace generated by a continuously moving point as a line. Thus, we claim that Zane's meanings for lines and points constrained him from developing a meaning of graphs consistent with emergent shape thinking (i.e., a graph as an emergent trace of points). Given the results provided by this study and other researchers (e.g., Manfield, 1985; Kerslake, 1981) regarding students' understanding of lines and points, it is important that we take into account students' meanings for lines and points constructed in other contexts and consider how those meanings may influence students' construction of graphing meanings.

Although we report on findings from a secondary student, the implication of this study is important for both the secondary and the undergraduate mathematics education community. We are drawing on the construct, emergent shape thinking, that was initially developed from undergraduate students. The results of this study regarding a secondary student's meanings afford us a better understanding of students' graphing activity, including their construction of emergent shape thinking. Prior researchers (e.g., Frank, 2017) have reported on students' difficulty with constructing an emergent meaning for graphs, and the case of Zane suggests that one source of such difficulty can be students' meanings for lines and points in general. If students conceived of a line as showing *one* point's movement, it would be difficult for them to assimilate a line graph as an emergent trace that includes *infinitely many* points, despite being able to conceiving each point as a representation of two quantities' magnitudes (and/or values) at a particular moment. Consequently, they would need to physically plot points to show quantities' magnitudes (and/or values) at any specified moments (i.e., a pointwise meaning for a graph). Thus, we conjecture that meanings for lines and points consistent with Zane's might help explain students' difficulties with constructing and representing smooth and continuous images of covariation in graphical contexts (Castillo-Garsow, Johnson, & Moore, 2013).

Our analysis supports the affordance of Castillo-Garsow's (2012) distinction between "a problem situation, the method used to solve it, and the reasoning that derives or selects that method" (p. 56) when characterizing students' covariational reasoning. Zane conceptualized two quantities' covariation continuously regarding the pool situation; however, he used a discrete method (i.e., plotting points) when constructing his graph due to a constrain implied by his meanings of lines. We argue that it is important for researchers to be aware of such a discrepancy and be aware that a pointwise graph activity (i.e., plotting the points first, and then connecting them) does not necessarily imply that their images of quantities' covariation is discrete, and vice versa. When characterizing students' covariational reasoning, it is important for us to take into account students' activities and reasoning in various context (e.g., situations, graphs in different coordinate systems, and number lines) before we make claims about their covariational reasoning. We believe identifying this inconsistency and its causes can allow us to better advance students' understandings of graphs.

Acknowledgments

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You Don't Want to Come Into a Broken System: Critical and Dominant Perspectives for Increasing Diversity in STEM among Undergraduate Mathematics Program Stakeholders

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As attention grows towards the disparities between majority groups and underrepresented minorities within undergraduate STEM education, there is a need for understanding where different university stakeholders stand on the subject of increasing diversity. This paper aims to juxtapose categorizations of stakeholder motivations and dominant and critical perspectives to propose a framework for analyzing stakeholder attitudes and informing increasingly productive conversations on eliminating inequity in STEM fields.

Keywords: Diversity, Equity, STEM, critical perspective, dominant perspective

The STEM achievement and enrollment disparities between populations based on gender, race, ethnicity, and class are well documented; so well documented, in fact, that we choose to take this shared knowledge as fact and do not provide documentation, treating this as a claim that does not necessitate warrant. We instead focus on the varying ways that communities and individuals respond to this shared knowledge, both in how they make sense of this knowledge within their own frameworks and what they propose to do with this knowledge. Mathematics, and specifically calculus, is an integral component of all math-intensive STEM degrees, and has been repeatedly shown to contribute to the disparities in STEM (Chen, 2005; Ellis, Fosdick, & Rasmussen, 2016; Hagman, 2019). In this paper, we focus on how different stakeholders involved in the precalculus and calculus programs that serve math-intensive STEM programs make sense of the lack of diversity in STEM and the role of precalculus and calculus programs in that disparity.

A recent national conversation among calculus coordinators and math department chairs from 56 research-focused mathematics departments indicates that this population is aware of the need to attend to diversity, equity, and inclusion within their precalculus and calculus programs, but currently rely on university-level programs to support a diverse student population (Apkarian et al., 2019). While these university-level programs can be a very important support for students from historically marginalized populations, it is also important for there to be an integration of academic and social supports (Tinto, 1997); in other words, it is important for mathematics departments to explicitly take ownership of issues related to diversity, equity, and inclusion and integrate attention to these principles into their precalculus and calculus programs (Hagman, 2019). It is also important to recognize where various stakeholders within the mathematics programs (such as coordinators, chairs, instructors, and students) currently are in their own understanding of issues of diversity, equity, and inclusion so that we can "meet them where they are" to support growth. Together, these motivations lead us to our research question: *What are the implicit and explicit motivations of various stakeholders at the university for why it is important to attend to diversity in STEM education?*

Theoretical Framing

To begin to understand various stakeholders' views regarding diversity, equity, and inclusion in undergraduate mathematics, we draw heavily on an analysis conducted by Basile and Lopez (2015) in which they explore the motivations for increasing diversity in STEM articulated by different STEM policy documents. They performed this analysis using a critical race theory lens, which enabled the authors to be especially attuned to the underlying and potentially subversive components of these documents; in other words, rather than taking any argument for an increase in diversity in STEM as good, the authors critique the motivations articulated, allowing for a more nuanced (and honest) discussion of why and how various stakeholders argue for attention to students of color (the primary population attended to in their study). We both narrow and extend this analysis by exploring the motivations articulated by various stakeholders related to a precalculus and calculus program at one institution for attending to diversity in STEM.

Basile and Lopez's (2015) analysis provides an essential starting place for our analysis by critically considering the different motivations articulated in policy documents for increasing diversity in STEM. Basile and Lopez identify three motivational categories as instrumental in distinguishing why individuals and organizations see value in changing representation in STEM: *economic benefit to state*, referencing the goal of supporting the United States to be globally competitive; *diverse perspective as a benefit to STEM enterprise*, referencing the different perspective that diverse populations bring to the STEM disciplines; and S*TEM experience as a direct benefit to Students of Color,* referencing the benefit to students of experiencing the joy of learning the knowledge and skills of STEM.

Data Collection

The data used in this analysis was collected as part of the Progress through Calculus (PtC) grant, spanning multiple universities and many individuals within those universities. For this analysis, we used purposeful sampling to identify three samples of data from a highly selective private university, herein referred to as Dunshire University. At Dunshire, we identified a local liaison familiar with PtC who could connect us to individuals and groups that were involved with diversity-in-STEM related initiatives on campus, as well as individuals who identified with historically underrepresented groups in STEM fields.

Our data collection consisted of individual interviews and focus groups. The three interviews we chose from this process were selected not only due to the different university stakeholders they represent (students, faculty, and administrators), but also because we felt they exemplified adaptations of Basile and Lopez's categories within a collegiate context. In order to gain insight into student perspectives on increased diversity, we identified a student group on campus focused on fostering inclusivity and providing feedback on university actions, and invited members of that student group, which we will refer to as the Diverse Student Organization (DSO), to participate in a focus group. We followed a similar focus group structure for faculty members involved in an on-campus program designed to increase persistence of undergraduate underrepresented minority students through faculty mentorship. We will refer to these individuals as Faculty Education Mentors (FEM). In addition, we conducted an individual interview with the advisor of DSO, who is considered an administrative director at Dunshire.

In addition to asking each interviewee about their role on campus and their own interactions with diverse populations in a collegiate setting, we asked them about the need to increase diversity in STEM and their reasoning regarding why such an increase was necessary. During our interviews, we left the meaning of the term "diversity in STEM[1]" up to the interpretation of each interviewee, allowing the interviewees to dictate the populations to which they wanted to attend according to their own personal and professional experiences. While this was intended to allow everyone an entry point into the discussion, we recognize that it also limits the amount of discussion regarding the experiences of individuals within STEM and has the potential to enact essentialism within our own work.

Thematic Analysis

After collecting the interview data and transcribing the interviews, we conducted a multiphase thematic analysis (Braun & Clark, 2006) to understand the motivations of various undergraduate mathematics program stakeholders for attending to diversity. The results of this analysis are twofold: first, a data driven theoretical framing for perspectives on diversity in STEM in a university context, and second, the application of this framing on our three interviews giving insight into the perspectives of different stakeholders.

First Level Analysis

In the first phase of our analysis, we conducted a deductive analysis drawing on Basile and Lopez's (2015) three motivations, while also inductively using the data to inform adaptations and expansions of these motivations. Our adaptations of these categories encapsulate what each category might look like when primarily applied to the reasoning of university stakeholders as well as what was actually shared by the interviewees. This resulted in four first level themes, the first three which parallel the original three categories, and then a fourth which emerged from the data, as summarized in Table 1.

We adapted Basile & Lopez's "economic benefit to the state" code to "financial or cultural benefit to the university." While Basile & Lopez analyzed policy documents in which stakeholders would have been primarily concerned with the economic welfare of the nation on a broad scale, a university setting yields more concern regarding specific university funding and reputation. We added a cultural component after recognizing the consistency of stakeholders desiring not only a quantitative university benefit, but a qualitative benefit to the overall university culture as well. This theme is referred to as "*Cultural or financial benefit to the University*." Incorporating Basile and Lopez's assertion that motivation can come from a desire to provide diverse perspectives to existing questions within STEM fields, "diverse perspectives as a benefit to STEM" remains relevant in a university context as underrepresented minority (URM) students will become the primary contributors to their selected STEM industry as they graduate. This category acknowledges the unique contributions of URM students, faculty, and administrators to the fields they support through both economic and entrepreneurial contributions. Diverse perspectives are extremely valuable when addressing current questions posed in the STEM community, and when posing new questions that may not have been previously considered by STEM contributors. This theme is referred to as "*Economic and entrepreneurial benefit to the STEM enterprise*." While Basile & Lopez's third category

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[1] While putting forth these questions, we used terms such as "underrepresented" and "minority/ies", which Basile & Lopez identify as fairly generalizing and, "while potentially appearing to be inclusive, [are] an example of racial essentialism" (p. 532). We recognize that these terms have the capacity to disregard individual experience and perspective, especially considering the broadness in spectrum of the experiences of individuals of differing race, gender, sexual orientation, cultural background, or disability status and the intersectionality that may occur amongst those identities.

discusses the direct benefit to students, the primary assertion in their categorization was that "all individuals deserve to experience the joy of learning STEM and benefit from the knowledge and skills, especially groups that have been traditionally excluded" (p. 544). We have broadened the category to include any motivator that has a direct and positive impact on the college experience or long-term outcome for a STEM student of a traditionally underrepresented identity. This theme is referred to as "*Direct benefit to underrepresented minority students in STEM fields*." Through our analysis, there emerged only one additional motivation that was not captured by the above three categories. Several of the interviewees cited benefits that related to the broader population of individuals for whom an increase in diversity in a college setting might be beneficial by way of influence, connections, or availability of opportunities. The two major populations mentioned by our data were the families of underrepresented minority students, and overall minority communities, and this theme is referred to as "*Societal and economic benefit to underrepresented minority families and communities as a whole*."

Table 1. *Descriptions of first level themes.*

Theme Category	**Description:** *Representation in STEM should be increased due to the:*
Cultural or financial benefit to the University	Cultural or economic benefit for the university at which individuals of underrepresented minorities are involved, whether as students, faculty, administrators, or volunteers.
Economic and entrepreneurial benefit to the STEM enterprise	Economic benefit it creates for the industries into which those individuals will enter as members of the workforce, through URM contributions to existing or difficult questions, or as URM participation allows for new questions and broader population inclusion.
Direct benefit to underrepresented minority students in STEM fields	Direct benefits - social, economic, and personal - that it fosters for the students pursuing those STEM fields at a university level.
Societal and economic benefit to underrepresented minority families and communities as a whole	Increased positive outcomes it creates for disadvantaged URM communities and families by way of increased income, a broader spectrum of opportunity, and general quality of life.

Second Level Analysis

After all four members of the research team agreed on the first level themes and their application to the three transcripts, we revisited the coded segments and recognized that there was nuance within the segments that was not captured through the first level themes. To help us make sense of this nuance, we revisited literature that explicitly addresses different perspectives on equity within mathematics education. We found Martin's (2003) paper discussing the rhetoric related to the concept of "mathematics for all" and Gutiérrez's (2009) paper defining equity along a critical and dominant perspective as particularly relevant to our data, and synthesized these perspectives to create our second level themes.

Martin (2003) conducted a critical analysis of the language used within mathematics education researchers and practitioners related to equity, which is well aligned with Basile and Lopez's analysis of the language related to equity used within STEM policy documents. Martin distinguishes between a top-down perspective and a community-up perspective. The top-down perspective on equity is driven by those in power (policy makers, researchers, administrators, teachers) and draws heavily on the idea that equity in mathematics education will be achieved when achievement gaps between white students and students of color are eliminated. This perspective frequently draws on “mathematics for all” language. Martin emphasizes that this is a worthy goal, but cannot be the only goal. He contrasts this approach with a community-up perspective, which is driven by the voices and needs of students, parents, and communities of color, and is viewed as a process rather than a finite goal:

> the second conceptualization of equity—as a process—highlights the fact that the necessary hard work will be ongoing and even when gains are made, a high degree of vigilance will be necessary to ensure that needs of marginalized students are attended to and that our definitions of equity are responsive to who these students are, where they come from, and where they want to go in life (p. 14).

Martin’s distinctions provide a link between the analysis of policy documents and a more theoretical grounding of equity in mathematics education provided by Gutiérrez. Gutiérrez (2009) defines equity in mathematics education along two axes, the dominant axis and the critical axis. The dominant axis entails attention to access to mathematics education and how this affects achievement. Discussions of equity from the dominant axis frequently attend to differences in access and achievement between white students and students of color, and focus on supporting students of color to “play the game”, working within the existing systems to achieve goals set forth by stakeholders with power in the discipline. This axis is contrasted with the critical perspective on equity, which attends to people’s identities and how these identities affect power. Discussions of equity from the critical axis emphasizes ways to support students to “change the game” and critically reflect on and challenge the entity of mathematics. Gutiérrez emphasizes that both dimensions are necessary to achieve equity in mathematics education. We note that both Martin and Gutiérrez emphasized that discussions on “mathematics for all” and attention to removing access and achievement gaps are not inherently bad; instead, these are important components of discussions related to equity when combined with discussions that attend to the experiences of the students in these systems and an awareness that these systems themselves create and sustain the inequities we are seeking to dismantle.

While these frameworks are explicitly about equity in mathematics education, and our emphasis is on motivations for increasing diversity in STEM, we view them as compatible. One way to view discussions of increasing diversity in STEM is through a STEM for all/increased access and achievement lens. Another way to view these discussions is by examining and questioning the system that has created these disparities. These distinctions, informed by Martin (2003) and Gutiérrez (2009), lead to our two second level themes, in which we use Gutiérrez’s language of dominant versus critical.

The *dominant perspective*, in working toward eliminating inequity in STEM, does so by operating within the current structure of the STEM enterprise. When applied to a university setting, it asks students, faculty, and administrators to alter their practices in a way such that they work toward ‘success’ definitions that have already been set out by the STEM status-quo, such as ensuring better grades for students, increasing access to research opportunities, or applying their knowledge by working to solve questions that are already being asked by the STEM community. In contrast, the *critical perspective* frames change as an increase in identity leading to an increase in power for the marginalized group in question. The critical perspective shifts focus from how to obtain better outcomes within the current STEM system to how eliminating inequity can redefine and challenge the STEM system, resulting in comprehensive, wide-spread change to STEM as an entity, whether that be through asking previously unconsidered questions, changing common perceptions of who can be successful within STEM, or creating differentiated spaces for marginalized populations in a newly defined STEM atmosphere. It interrogates what STEM can be by recognizing that the definition of the field itself is fluid and thus can be challenged and molded by diverse perspectives.

Reasons given by stakeholders for increasing diversity in STEM

This framework allows us to examine not only the reasons for increasing diversity from the perspective of different stakeholders, but also how to frame an argument that pushes the field and reframes the argument from a student/community-focused perspective. In answer to our research question, we share the integration of our first and second level coding and provide sample examples from our data set. In Table 2, we share the overlay between our first and second level codes, driven by our data and the literature and theory described previously. To help exemplify the interaction between these layers and highlight the difference of dominant versus critical perspective within them, we examine how these differences show up in data related to the benefit to the STEM enterprise. The first example we highlight comes from a student participating in DSO, in which they explain how a more diverse population can solve tasks set forth in the current STEM environment better than a homogeneous group:

> *DSO Student:* I've read reviews where, like [professional] groups are working on like just problem solving tasks and the more diversified group is the one who usually does the best just because they have different perspectives…. So it's just interesting that people who think differently are from different backgrounds just overall do better on tasks.

A contrasting example of a critical perspective on the benefit to the STEM enterprise comes from the advisor of DSO, who shares the examples of advances in videography and cinematography by now knowing how to film black skin by utilizing blue lighting:

> *Advisor of DSO:* That means that there had to be people of color to know, to come in and say no this is how you do this or had to discover that and broaden awareness and you know make it more widely acceptable, like, this is how you do this ...there's lots of things that students- that people can benefit from by being able to have a seat at the table because things can be made with not someone else in mind but with everyone in mind.

Table 2. *Integration of first and second level codes.*

	Dominant Perspective	**Critical Perspective**
Economic and entrepreneurial benefit to the STEM enterprise	Bringing in diverse perspectives will help solve existing problems and answer questions that already exist in STEM industries.	Bringing in diverse perspectives changes what STEM asks and what it can solve, and alters the methodology of solving STEM-focused problems by making that methodology more inclusive.
Cultural or financial benefit to the University	Increasing diversity provides more representation in faculty and students, which helps with recruitment and allocates more pre-existing resources to diverse populations on campus.	Increasing diversity will culturally shift STEM environments at the university and is needed to address systemic hiring inequalities.

Direct benefits to students in STEM fields	Increasing diversity provides role models and helps other students by increasing representation and creating spaces within existing systems that are inclusive.	Increasing diversity creates a more positive identity for students in STEM fields, and thus increases their power within the discipline.
Societal and economic benefit to families and communities as a whole	Increasing diversity fosters success for traditionally underrepresented groups within society's existing structures.	Increasing diversity recognizes and challenges the way underrepresented minority communities and families and their motivations and attitudes about STEM are perceived.

Discussion

Our work has synthesized a categorical analysis of STEM policy documents with theoretical perspectives on equity within mathematics education, yielding a framework from which to situate perspectives on increasing diversity among math stakeholders within a broader theoretical field. Our framework enables detailed themes to be identified in terms of where certain stakeholders may stand regarding their motivation to increase diversity, as well as an idea of how best to format productive discussions that center around the need to increase diversity.

An example of how this application might take place may be through a trend that we as a research team noticed when analyzing this smaller data set. Within our transcribed focus group with the Faculty Educational Mentors, we noticed that they, as faculty, typically communicate through a dominant perspective. We conjecture that this is due in part to the fact that the faculty *are within* the system, and so can advocate for improving the system related to diversity, equity, and inclusion, but it may be too far outside the scope of their lived experiences to recognize a need to change the system that they have been successful within. By applying this framing to various stakeholders' perspectives, we can begin to identify which arguments are more comfortable or natural for various groups, and then can seek ways to connect with them at those levels and also use the framing to help further their thinking into other realms. For example, it may be beneficial to support faculty who come from a dominant perspective regarding benefits for students to engage in those benefits from the critical perspective, or perhaps it is more beneficial to remain within the dominant perspective but broaden the reach of the perceived benefits to additional first level categories. More research needs to be done to understand which approach is most constructive, for whom and for what goals.

We cannot dismiss the fact that STEM can be a very empowering system for students from marginalized populations, and so only working to dismantle this system does not honor those students and their achievements. So, just as only coming from a dominant perspective is alone not enough, only coming from a critical perspective has the potential to devalue the benefits of the current STEM system to a diverse population of students. By developing a framework that takes both the dominant and critical perspective into account and applies them to four major categories of motivation, we hope to further the quality and depth of discussions and actions taken to increase diversity in STEM education by undergraduate mathematics program stakeholders.

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Tasks to Foster Mathematical Creativity in Calculus I

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Fostering students' mathematical creativity necessitates certain instructional actions - one of which is designing and implementing tasks that foster creativity. Drawing on the literature on mathematical creativity, we describe existing research-based features of tasks for eliciting student creativity, or creativity-based tasks, and provide suggestions for implementation of such tasks. Based on these features, we analyzed two instructors' first experiences designing and implementing creativity-based tasks in Calculus I. Both instructors' frequent use of the multiple-solutions feature suggests that this feature could be an entry-point for designing and implementing creativity-based tasks for other instructors seeking to foster creativity.

Keywords: Calculus, creativity-based tasks, mathematical creativity, task design

The importance of mathematical creativity in mathematics and mathematics courses is documented in numerous research studies, policy and curriculum-standard documents (e.g., Borwein, Liljedahl, & Zhai, 2014; CUPM, 2015; Levenson, 2013; Moore-Russo & Demler, 2018; NSB, 2010; Silver, 1997; Sriraman, 2009; Tang et al., 2017; Zazkis, & Holton, 2009). Askew (2013) points out that "[c]alls for creativity within mathematics and science teaching and learning are not new, but having them enshrined in mandated curricula is relatively recent" (p. 169). For example, in its latest guidelines for majors in mathematical sciences, the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM) states that "these major programs should include activities designed to promote students' progress in learning to approach mathematical problems with curiosity and *creativity* [emphasis added] and persist in the face of difficulties" (Schumacher & Siegel, 2015, p. 10). In this paper, we focus on mathematical creativity in Calculus I, a course that is commonly offered for majors in these mathematical sciences programs.

Students' experiences in Calculus I play a critical role in their persistence in science, technology, engineering, and mathematics (STEM) programs (Rasmussen et al., 2019). The lack of or limited exposure to class materials (e.g., tasks, homework problems, exam questions) that promote conceptual discussions is one of the reasons reported by Calculus I students for switching out of a STEM major (Johnson, Ellis, & Rasmussen, 2014). In fact, a textbook analysis (Lithner, 2004) concluding that 70% of Calculus exercises at the end of the section were about mimicking previously done examples of the same section is one indication to students' limited exposure to conceptual ideas. Though there are Calculus reform projects that address this particular issue by advocating for new curriculum materials for more conceptual discussions (see Bressoud et al., n.d.), we believe it is still a priority to explicitly value and foster students' mathematical creativity in Calculus I classes. In particular, we argue for the creation and implementation of tasks that are designed not only for conceptual understanding but also for enhancing students' mathematical creativity. In this paper, we discuss research-based features of tasks that promote mathematical creativity and their potential implementation. Additionally, we share analysis of two instructors' uses of these features in the design and implementation of the tasks in their Calculus I classes.

Theoretical Perspective and Background Literature

In our work, we define mathematical creativity as a process of offering new solutions or insights that are unexpected for the student with respect to their mathematics background or the problems they have seen before (Liljedahl & Sriraman, 2006; Savic et al., 2017). This process-oriented definition (Pelczer & Rodriguez, 2011), in contrast to examining final products (Runco & Jaeger, 2012) of those processes, provides a dynamic view of creativity rather than a static one. We focus on valued mathematical actions (Cuoco, Goldenberg, & Mark, 1996; Schoenfeld, 1992) such as taking risks and making connections that can lead to creativity in mathematics (Karakok et al., 2015; Leikin, 2009). Our definition also encompasses creativity relative to the student versus creativity relative to the field of mathematics (Beghetto & Kaufman, 2013; Leikin, 2009). Finally, this particular definition identifies creativity specific to the domain of mathematics rather than domain-general creativity (Baer, 1998).

Fostering students' mathematical creativity, as we define it, necessitates certain instructional actions that are encapsulated by Sriraman's (2005) five theoretical principles. The *Gestalt principle* discusses the importance of giving time to allow incubation to occur (Hadamard, 1945). The *Aesthetic principle* highlights explicitly valuing the beauty and uniqueness of solutions or methods. The *Free-market* and *Scholarly principles* emphasize creating a safe environment where students can present and defend their work, and allowing students to build off one another's work, respectively. The fifth principle, *Uncertainty*, focuses on tolerating ambiguity and knowing that it is acceptable to not know a solution. Levenson (2011) provides empirical support for these principles and adds "choosing appropriate tasks" (Levenson, 2013, p. 273) as one of the roles of teachers for the promotion of creativity.

We conjecture that intentionally implemented tasks designed to align with Sriraman's principles have the potential to enhance mathematical creativity. For example, a task that is aligned with the Gestalt principle needs to be challenging enough for students that they would need time to incubate. An intentional implementation of such a task that demonstrates the Scholarly principle would include an instructor giving students the opportunities to discuss their approaches and build off of one another's work; all the while, students' mathematical creativity is at the forefront of such discussions.

Task design

We adopt Henningsen and Stein's (1997) definition of a mathematical task as ‘‘a classroom activity, the purpose of which is to focus students' attention on a particular mathematical concept, idea, or skill'' (p. 528). For a task to promote creativity, it needs to have additional features that provide students opportunities to push their mathematical processes toward new solutions or insights that are unexpected for them. For practical reasons, we use the term *creativity-based tasks* to describe such tasks.

We situate our discussion of creativity-based tasks and task design within two perspectives of creativity: *Developmental* and *Problem Solving and Expertise-Based* perspectives (Kozbelt, Beghetto, & Runco, 2010). The *Developmental* perspective posits that creativity develops over time (i.e., process-orientation) in an environment where students are provided authentic tasks and opportunities to interact with others. The *Problem Solving and Expertise-Based* perspective emphasizes problem-solving processes, heuristics, and tasks, underscoring the use of tasks to challenge students' thinking processes and provide opportunities to solve problems in various ways. A key component of our work is developing tasks that would allow such processes. We base these tasks on Skemp's (1976) relational understanding framework and Lithner's (2008) creative mathematically-founded reasoning. As relational understanding relates to students' development of conceptual structures, a creativity-based task promotes both making connections between concepts and taking risks by students to become independent thinkers. Additionally, creative mathematically-founded reasoning involves novel (with respect to students) mathematical arguments. It is noted by Boesen, Lithner, and Plam (2010) that students use creative mathematically-founded reasoning to solve unfamiliar, nonroutine tasks. In this sense, creativity-based tasks can be viewed as unfamiliar and non-routine.

Features of Creativity-Based Tasks

"Recall and apply" tasks are important in developing procedural fluency in Calculus I, but to foster mathematical creativity, instructors need to design tasks that require "evaluating mathematical statements; example generation (constructing an instance); analyzing reasoning; conjecturing; generalizing; visualization; using definitions" (Breen & O'Shea, 2011, p.87). Tasks with these features can promote conceptual discussions and making connections between seemingly different ideas and concepts.

Silver (1997) discusses the importance of the interplay between problem posing and problem solving to creativity and states "[i]t is in this interplay of formulating, attempting to solve, reformulating, and eventually solving a problem that one sees creative activity" (p. 76). We believe it is important for tasks to engage students in problem posing and problem solving not only in order to promote creativity but also to enable "teachers and students to become subjects of the educational process by overcoming authoritarianism and an alienating intellectualism" (Freire, 1999, p. 8). The need for posing problems can be facilitated by assigning tasks that are ill-defined, ambiguous, or open-ended. Kwon et al. (2006) define an incomplete or an open-ended problem as "a problem which does not define clearly what the question asks for, therefore allowing many possible solutions" (p. 52). Thus, another feature of a creativity-based task is providing opportunities for students to pose problems and questions, then to seek answers to these problems and questions (Haylock, 1997; Silver 1997). Experts describe the ability to identify key research questions as part of their creative work (e.g., Hadamard, 1945; Mansfield & Busse, 1981).

Relating to many possible solutions, Leikin (2013) defines a multiple-solution task as one that "explicitly requires students to solve a mathematical problem in different ways" (p. 388) where different solutions are determined by: "(a) different representations of some mathematical concepts involved in the task, (b) different properties (definitions or theorems) of mathematical objects within a particular field, or (c) different properties of a mathematical object in different fields" (Leikin, 2013, p. 388). Thus, multiple-solution tasks can also promote utilizing other representations (verbal, symbolic, gestures) as well as connecting certain aspects of different representations in a way that fosters deeper mathematical thinking. Multiple-solution tasks not only value students' individual approaches, but they also allow for originality and novelty in using certain standard or less standard tricks.

Examples

One of the tasks that we designed for Calculus I, the *Circle Task* poses the questions "Is there anything in real-life that is a perfect circle? How do you know if you have a perfect circle?" This task involves the mathematical concepts of infinitesimals, (possibly) limits, integrals, and arc length. As a creativity-based task, its open-ended nature provides opportunities for students to take risks in exploring novel ideas and to make conceptual connections to fundamental aspects of Calculus.

As another example, we modified a typical "find the limit" question to "Consider the limit $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$. Evaluate the limit in as many ways as possible." We designed this *Limit Task* as a creativity-based task that involves the mathematical concepts of limits and derivatives. By asking students to evaluate the limit in more than one way, students are pushed to think of a solution beyond one that they are mathematically inclined to provide. As the question did not specify how students are to approach the problem (algebraically, graphically, using a table, etc), the task carries the open-ended feature. The task fosters making connections between various concepts as students could view this limit as a slope of the tangent line, which could be computed by the derivative at 1.

Task Implementation

Stein, Grover and Henningsen (1996) noticed that tasks that were designed to be cognitively demanding (e.g., involving conjecturing, justifying, generalizing) became less demanding because they "became routinized, either through students' pressing the teacher to reduce task ambiguity and complexity by specifying explicit procedures or steps to perform or by teachers' taking over the challenging aspects of the task" (p. 479). That is to say, the implementation of a task plays a crucial role in fostering creativity.

To mitigate the possibility of a reduction in cognitive demand, we suggest using Sriraman's (2005) five principles as guidelines for implementing creativity-based tasks. To bridge theory and practice, the authors (Cilli-Turner et al., 2019) investigated one teacher's actions with the five principles and suggested numerous teacher actions stemming from these principles to potentially foster students' mathematical creativity. Building on the results from (Cilli-Turner et al., 2019), we suggest that when implementing an open-ended task such as the Circle Task, instructors could assuage students' discomfort in task ambiguity (Uncertainty) by assuring students that multiple solutions exist. The instructors can also provide additional time for students to incubate (Gestalt) on what it means to have a perfect circle and a real-life circle. Following this incubation, instructors can give students opportunities to present their findings (Free Market and Scholarly). Similarly, the Limit Task affords different possible implementations because it has a "call-back" feature where instructors can use it progressively at various points throughout the semester as new material is covered, deepening students' incubation period (Gestalt). Aside from (re)using this task while covering graphs, limits, and derivatives, the task can be revisited after covering L'Hospital's Rule by asking students to find the limit in four different ways and giving students opportunities to share (Free Market and Scholarly) these different ways. Instructors can facilitate the Aesthetic principle by valuing the novelty of using the trick of factoring linear terms in this limit.

Research Methods

As part of a larger research study that explores mathematical creativity in Calculus I classes, our research team designed the two creativity-based tasks: the Circle Task and Limit Task. These two tasks and their features were shared with two instructors, Jo Parker and Juniper Travers (pseudonymous), at a South-Midwest regional university who participated in the larger study. Prior to the start of the 'research' semester, we had 2 two-hour online professional development (PD) sessions with a two-day break between the sessions. At these sessions, we introduced some goals of the research project, features of creativity-based tasks, and the Limit Task. The sessions also included discussions on various ways to explicitly value and foster mathematical creativity and implement creativity-based tasks. The two instructors designed a task or modified an existing one to include some creativity-fostering features. To provide extended support, the online PD continued on a weekly basis throughout the semester for an hour to support participants' instructional practices that explicitly foster mathematical creativity and task design. We facilitated open-ended discussions concerning how to assess student work on creativity-based tasks. Both instructors were asked to implement the Limit and Circle tasks in their Calculus I classes and also to develop and implement at least four other creativity-based tasks.

In this paper, we share results from our preliminary analysis of collected data to address the research question: What features do instructors use in their creativity-based task design and implementation processes? We collected instructors' creativity-based tasks, their journal entries for the PD in which they reflected on their design and implementation processes, their classroom video-recording of the days on which these tasks were used in class, and their Calculus I material from previous teaching experiences. At the end of the semester, we conducted semi-structured interviews that included questions regarding their tasks and task design processes. Instructors' tasks, journal entries, and interview transcripts were analyzed with a deductive approach using task features as codes (Patton, 2002).

These codes or features (in *italic*) were: *open-ended*; allows *multiple solutions, multiple representations, different approaches leading to one answer, posing problems and questions;* promotes *making connections between different concepts, evaluation/justification, generalization making, incubation;* allows for *originality/novelty, conjecturing, use of a trick or a less standard algorithm*, and *uncertainty*.

Analysis

We analyzed the tasks that instructors used as creativity-based tasks in their courses. Some of these tasks were only one problem in a longer activity sheet or an assignment set. The implementation of these tasks varied from task to task and instructor to instructor. We observed that both instructors most frequently used the "multiple solutions" feature in their tasks. These multiple-solution tasks also afforded multiple representations (algebraic, symbolic, graphical). For example, Juniper developed this extrema task:

(a) Sketch or write the equation of a function for which the 2nd derivative test is inconclusive at $x = 1$. Provide justification as to how you know the 2nd derivative test fails.

(b) What would be a next step for finding extrema if the 2nd derivative fails?

We coded this task as a *multiple-solution* task that could afford *multiple representations* (algebraic and/or graphical). This task also included features such as requiring students' *originality* in creating such a function, promoting *uncertainty* as students might need to try few possibilities before they find a function that works. We coded the task as one that fosters students' *evaluation* skills as it required them to justify if the test fails. We coded Part (b) of the task as *an open-ended* question where the instructor did not provide specific directions to students.

Similarly, Jo's task on integration "Find a non-trigonometric function f and domain $[a, b]$ such that $\int_a^b f(x)dx = 0$" was coded as a task that has *multiple solutions* and *multiple representations* (algebraic and/or graphical). It was also coded as a task requiring students' *originality* in creating such a function and promoting *uncertainty* as students might need to try a few possibilities before they find a function that works. As a contrast to the extrema task, we did not code this task as an open-ended one because the question instructed students to find such a function, and hence students knew, before tackling the task, that there will be at least one. If the task was phrased "Is there a non-trigonometric function…? If so, find at least one (or two)" then it would have carried the open-ended feature.

In many of Jo's creativity-based tasks, she asked for creation of a function that satisfies (or does not satisfy) certain criteria. With this structure, the most common feature Jo utilized was *multiple solutions* affording *multiple representations*. The *open-ended* feature was utilized on fewer occasions. The feature of *different approaches leading to one answer* was also utilized but minimally. Most of Jo's tasks fostered *making connections between different ideas and concepts* and promoted Sriraman's *uncertainty* principle.

One of the most noticeable changes between materials from her previous Calculus teaching to the 'research' semester was the adaptation of creativity-based tasks on her final exam review sheet. On the final review sheet in the previous semester, we coded 15 questions as "*routine*" exercises in Calculus I as they resembled the typical exercises. On the other hand, the final review sheet for the research semester had five "non-routine" creativity-based tasks with the following combination of features: *multiple solutions* and *representations*, *open-ended*, and *different approaches*. The tasks also fostered *making connections between different concepts* and aligned with the *uncertainty* principle. She also added a creativity-based task on her final exam.

Juniper's creativity-based tasks were developed for each of the following content areas: limits, continuity, relative extrema, absolute extrema, and integration. Many of these tasks asked to create a function that satisfies (or does not satisfy) certain criteria, which were similar to tasks developed by Jo. With this structure, we noticed that Juniper utilized *multiple-solution* tasks affording *multiple representations* most frequently. The *open-ended* feature was also used frequently. Juniper included tasks that required students to *pose problems and questions*. There were questions on tasks that explicitly asked students to *evaluate* or justify their answers or approaches. It was noticeable that her final exam did not include any of her creativity-based tasks and it was very similar to her final exams from previous semesters. However, she did use a creativity-based task on Exam 2.

Participants' journal entries were also coded for features of tasks. For example, Jo Parker implemented the Circle task as a writing assignment. In her teaching reflection journal, she stated that she wanted to "provide students with the opportunity to practice written communication skills while also gaining experience in answering a more open-ended mathematical question. I told the students that there was not a single correct answer to the prompt…" In this reflection, it seems that Jo Parker embraces the *open-ended* feature of this task to facilitate an additional mathematical process, written mathematical communication skill. Juniper's reflections hinted that she wanted to emphasize *incubation* in her implementation of tasks. She assigned students to work on some of the tasks outside of class and asked them to bring their work to the next class for discussion.

We also triangulated our coding of participants' tasks and journal entries with their interview transcripts. When asked "was there any particular feature that was most important to you when you were designing or thinking about these tasks," Jo referred to the features of *multiple solutions* and *posing questions*. In her implementation of her task in which she asked students to explore the conditions for which Rolle's theorem holds, she was happily surprised that even after students submitted their answers, they were posing questions and discussing ideas with each other in class. Referring to the Limit task, she said, "I love the fact that you can approach it in many different ways…I think the traditional way that students think of it is multiply by the conjugate. But, I mean they can factor in it. I mean it's just a little trick…I think it's cute. It's a little bit outside the box but it's still within their realm of knowledge.”

On the other hand, Juniper referred to the extrema task as a memorable task, stating that “[it] sticks out in my mind because that's where I actually saw [students] construct another concept that we haven't gotten to yet. So that it's like a huge leap where creativity had taken them further in mathematics content.” The most common task feature she stated was justification because, “[it] is always important in my mind. They can come up with an answer, but they have to be able to explain… I think underlying all of it was ‘do these questions really highlight the underlying mathematical concepts?” Although this was coded for *evaluation/justification* elsewhere in her tasks, the *multiple solutions* feature was coded most often in Juniper’s tasks. It could be that in her implementation of these tasks, she provided additional questions during discussions that facilitated this feature of evaluating and justifying.

Discussion

Raising instructors’ awareness of mathematical creativity and advocating for change in our pedagogies to promote creativity are the driving purposes of this project. This change can be initiated with small adjustments in task design. The results of our initial analysis of data demonstrate these adjustments made by two instructors. As noted in the analysis, the multiple solutions feature was most commonly used in tasks designed by the instructors. This result mirrors Levenson’s (2013) findings that many of her participants advocated for the use of multi-solution tasks. Levenson further noted that the collection of tasks that were chosen by five instructors who thought these tasks fostered mathematical creativity had fewer open-ended tasks and no problem-posing tasks. Our results indicate a similarity, which we believe is due to the fact that this project was the first time for our participants to set out to explicitly value creativity in their classroom; hence there seems to be a limited attempt to having open-ended and problem-posing tasks. Though Levenson (2013) highlights the importance of open-ended tasks and problem-posing tasks claiming that they “may afford additional opportunities for developing students’ mathematical creativity and it is important to raise teachers’ awareness to the variety of tasks which may serve this purpose” (p. 288).

An intentional selection or creation of tasks that have multiple solutions or that can be approached in different ways seem to be a feasible entry point into fostering mathematical creativity. Allowing students to pose problems, generate examples, make conjectures, and so forth, are other ways to direct students’ thought processes towards creative mathematically-founded reasoning (Lithner, 2008). According to Beghetto (2017), instructors can alter their tasks slightly, such as asking students to come up with their own approaches, or substantially by asking students pose their own problems and ways of solving those problems.

According to Sriraman, teaching practices that are aligned with the five principles have the potential of fostering students’ creativity. In our analysis of creativity-based tasks, we noticed that the Gestalt and Uncertainty principles can be viewed as features of such tasks. However, we claim that the Free Market, Scholarly, and Aesthetic principles are important aspects of the implementation of such tasks that can further promote students’ creativity. In fact, these aspects of implementations differentiate creativity-based tasks from non-routine tasks. We plan to investigate this claim in future research to study potential implementations that complement creativity-based tasks in fostering students’ creativity.

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Factors That Influence Graduate Student Instructors' Pedagogical Empathy

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Interpersonal relationships are central to the teaching and learning of mathematics. One way that teachers relate to their students is by experiencing empathy for them. In this study, I explore the phenomenon of pedagogical empathy, which is defined as empathy that influences teaching practices. Specifically, I examine how graduate student instructors (GSIs) conceptualize empathy as it relates to the teaching and learning of mathematics and identify factors that influence this pedagogical empathy. The findings presented in this paper emerged from interviews with 11 GSIs in which they answered questions designed to elicit their thoughts about empathy. The participants provided complex and diverse notions of how empathy relates to teaching mathematics. Conclusions drawn from this paper may help inform professional development efforts targeted at novice and experienced instructors at the post-secondary level.

Keywords: Empathy, Graduate student instructors, Professional development

Interpersonal relationships are central in the teaching and learning of mathematics. One way that teachers relate to their students is by experiencing empathy for them (Cooper, 2011). This empathy is often situated at the intersection of teacher and student affect and has the potential to influence teaching practices; yet, little research has been conducted in this area (Zembylas, 2007). Exploring how empathy influences teaching can lead to a deeper understanding of teaching practices and have far reaching implications for the broader education community.

This study aims to explore how graduate student instructors (GSIs) conceptualize empathy as it relates to the teaching and learning of mathematics. The conclusions drawn from this study will add to the emerging body of literature about GSIs' knowledge, beliefs, and teaching practices (e.g., Speer, Murphy, & Gutmann, 2009). In addition, this research can be used to inform professional development efforts for both novice and experienced instructors.

Theoretical Perspective

Empathy is a nuanced concept. Researchers have described empathy in various ways and have developed different characterizations depending on the context in which empathy is referenced (Cuff, Brown, Taylor, & Howat, 2016). It is therefore important to provide a working definition of empathy as it relates to the context of teaching. Drawing on the work of Cooper (2002, 2004, 2011), I define empathy as "a highly complex phenomenon, closely associated with moral development, which develops over time and with frequency of interaction and which is highly dependent on the actors and the context of the interaction" (Cooper, 2004, p. 12). In this paper, I use the term *pedagogical empathy* to refer to the empathy that a teacher experiences for a student or group of students. I further propose that pedagogical empathy has the potential to influence teaching decisions and involves how teachers use their understanding of students' perspectives and experiences to inform the actions they take in the classroom.

The findings presented in this paper also draw on Zembylas's (2007) framework, which establishes a connection between pedagogical content knowledge and emotional knowledge. Zembylas (2007) contends that "teacher knowledge is a form of knowledge ecology—a system consisting of many sources and forms of knowledge in a symbiotic relationship" including "content knowledge, pedagogical knowledge, curriculum knowledge, knowledge of learners, emotional knowledge, knowledge of educational values and goals and so on" (p. 356). Specifically, Zembylas (2007) narrows in on emotional knowledge and defines the construct of emotional ecology to explain the relationship between emotional knowledge and pedagogical content knowledge. Building on his work, I conceptualize pedagogical empathy as a filter that connects pedagogical content knowledge and emotional knowledge. The findings presented in this paper describe three factors that are critical in the development of this connection.

Purpose

This paper stems from a larger study that explores the ways in which pedagogical empathy might influence teaching practices. As a first step, it is important to consider how instructors conceptualize empathy and identify factors that have the potential to influence pedagogical empathy. Thus, this paper focuses on the following research question: *What factors might influence the ways in which mathematics GSIs experience empathy for their students?*

GSIs are an important and unique population to study because they are not only instructors of undergraduate mathematics courses, but also students in graduate mathematics courses. Having these simultaneous roles may make it more likely for GSIs to experience pedagogical empathy for their students as they may have recently had similar or shared experiences as students themselves, an important factor that has been shown to be associated with empathy in other contexts (Eklund, Andersson-Stråberg, & Hansen, 2009). Leaders of GSI professional development can use insights from this study to help GSIs reflect on their experiences and increase their awareness of pedagogical empathy.

Methods

Eleven mathematics GSIs at a large Midwestern university participated in this study. Each GSI was the instructor of record for a precalculus course, although they had varying levels of teaching experience. None were international GSIs. Participants selected their own pseudonyms, which are used below. Data were collected during the summer and fall of 2018 through multiple interviews and classroom observations of the GSIs. The findings in this paper emerged primarily from analysis of the initial interviews, which were designed to explore how GSIs conceptualize empathy. Participants were initially asked to write down their definition of empathy as it relates to teaching mathematics and provide an illustrative example. Participants then discussed their definitions and responded to questions such as: "Where do you think empathy comes from?" and "Is it necessary for math teachers to have empathy for their students?"

Interviews were transcribed and analyzed using MAXQDA. Since pedagogical empathy is a novel theoretical construct, inductive coding was initially used to generate a set of preliminary conceptual codes relating to how GSIs conceptualize empathy within the context of teaching and learning mathematics (Miles, Huberman, & Saldaña, 2014). Each interview was tagged using these preliminary codes, and each coded segment was summarized. These segments were then sorted using pattern coding (Miles, Huberman, & Saldaña, 2014). During this process, the conceptual codes were refined and further developed into themes. Finally, these themes were organized into two categories that relate to how the participants described their experiences with empathy within the context of teaching and learning mathematics.

Findings

Two main categories of themes emerged from coding the initial interviews: drivers of pedagogical empathy and conditions related to the enactment of pedagogical empathy. The themes that fall under these categories highlight elements that have the potential to influence how GSIs experience pedagogical empathy. Due to space considerations, only the drivers of pedagogical empathy are described in this paper. Table 1 provides an overview of the drivers along with a short summary of each theme and a representative quote from a participant.

Table 1. Drivers of Pedagogical Empathy

Theme	Summary	Representative Quote
Past Experiences	Drawing on past experiences as a way to empathize with students	*"I think I draw from those experiences, my own personal experiences as a student to kind of guide me in how I should interact with my own students."* - Oliver
Context of Mathematics	Considering how the disciplinary context of mathematics might impact pedagogical empathy	*"Students come in from a lot of backgrounds, many in which they're taught to hate math or to think of themselves as bad at math."* - Mark
Non-mathematical Factors	Experiencing empathy for students in ways that go beyond the scope of teaching and learning mathematics	*"People have come up to me with problems about their life, just in general, and you have to be able to relate to that also, which isn't necessarily math-related."* - Bill

Drivers of Pedagogical Empathy

During analysis, several drivers emerged. These drivers appear to motivate pedagogical empathy and can be grouped into three main themes: personal experiences, the context of mathematics, and non-mathematical factors. Specifically, GSIs discussed the importance of drawing on personal experiences, considering the disciplinary context of mathematics, and accounting for non-mathematical factors when experiencing pedagogical empathy for students. These themes illustrate how pedagogical empathy can be influenced by different factors and are described in depth below.

Drawing on personal experiences. All of the participants discussed how experiencing empathy for students can be impacted by past personal experiences. In some cases, these experiences helped GSIs relate to their students and were seen as a way to empathize with students. In other cases, GSIs saw having a lack of similar past personal experiences as a potential barrier to experiencing empathy for their students. A few GSIs also reflected on how they had developed empathy over time by learning from their experiences as both teachers and students.

Oliver was one of the GSIs who discussed relating to his students by drawing on his past experiences as an undergraduate student. He recalled being in a physics class and constantly needing to use sine and cosine, even though he "barely understood what they meant at the time." He went on to explain how "this moment created something" that he could reflect back on and use when trying to empathize with his students. Oliver also gave a specific example of how his experiences as a student have influenced his teaching practices:

> I think often I don't click with instructors that don't have this [empathy]. I feel like certain instructors, they get frustrated. I feel like there's a sense of frustration with you when you don't understand something quickly, and I've never been the person to learn things really quickly. So I think I draw from those experiences, my own personal

> experiences as a student to kind of guide me in how I should interact with my own students and how not to be frustrated with their slowness or their inability to comprehend something that maybe seems trivial to me.

Amanda was another GSI who talked about understanding what it feels like to be a student who is struggling with something. When asked where empathy might come from, she reflected on her experiences as a first-year graduate student. In particular, Amanda said that she remembered how it felt to be "sitting in a classroom listening to a lecture and have no idea what's going on," and that sometimes she will see that same look of confusion in her students' eyes. Amanda went on to say that her empathy comes from wanting students "to understand that they can get to an understanding of the material," and that she does not "want anybody to leave a classroom thinking that they're bad at math."

While some GSIs found it relatively easy to connect with students in their precalculus courses, others discussed how their life experiences might be dissimilar to those students, making it harder to empathize with them. In particular, Jack talked about how it is easier to empathize with situations that are more "proximal" and how it "takes skill to empathize with people who are further away from you in life experience." He explained how it might be difficult for some GSIs to empathize with their students:

> I do think it's probably harder for people to empathize with students who take the lower math courses. There are a lot of experiential barriers between someone who's in graduate school in mathematics teaching an Intermediate Algebra student. So I do think that it is more difficult in that situation for the instructor to empathize.

Jack also discussed how it was easier for him to empathize with his students in a Calculus II course that he had recently taught because it was easier to "simulate the student's emotions" and "detect what then needs to be done."

Beth expressed similar thoughts about why it might be difficult for GSIs to experience empathy for their students: "From my experience with other grad students teaching courses, for some of them, they didn't struggle with it. They don't see a reason why their students are struggling with it." She went on to explain that sometimes it is difficult for her to figure out how to explain something to a student because "it's hard to come up with those sorts of reasonings for very basic things," and that GSIs who are teaching lower level math courses "might struggle more because it's been a long time since they've seen it and they think it's easy."

A few of the GSIs with more teaching experience discussed their belief that empathy could be improved over time because of their past experiences with teaching. Specifically, Mark discussed how he was less cognizant of what his students were experiencing when he first started teaching:

> I think it's fair to say I was less empathetic, less aware of my students' feelings the first few semesters I was teaching than I am now because I just didn't realize...the extent to which people did actually come in dreading these math classes, and it's not because they're like lazy or something, but just because they've been taught to hate them.

Mark also talked about how one can develop empathy by recognizing "something that you've seen a lot in other students." Likewise, Aaron discussed not having had similar experiences to many of his students and developing empathy by being exposed to people's stories, reading articles, and absorbing information about people's college lives.

Joe was another GSI who talked about empathy as a skill that comes from "training and practice." He also explained how teachers might "start" in different places when it comes to experiencing empathy:

> I think a lot of empathy comes from practice, but I think also I'd say people don't start in the same place. Like some people, the first class they ever teach, they're like really good at empathizing with students, and some people don't get that and maybe I'm guessing some people never develop that and some people develop that while they go on and practice more and work at it, if they work at it.

In the past, Joe had been "less empathetic to students who didn't really care about their math class" because he had always liked math and thought it was interesting. He discussed trying to "contextualize" that for himself by thinking about having to take a driving course, which he did not enjoy. After making this connection between his own experiences and his students' experiences, Joe described being more aware of how his students might feel and trying to use this newfound understanding in his interactions with them.

Considering the disciplinary context of mathematics. In addition to discussing how empathy can be influenced by past experiences, several participants reflected on how the disciplinary context of mathematics might impact pedagogical empathy. Many GSIs talked about how students might feel or think about mathematics, often linking back to cultural stereotypes and students' prior experiences with mathematics. Participants also acknowledged that teachers and students could hold opposing beliefs about mathematics and that it is important to consider these different perspectives as they have the potential to influence how teachers empathize with their students.

A few GSIs discussed how students might feel or think about mathematics and the resulting impact that might have on their experience in the course. Aaron brought up empathizing with a student in his class who he described as "hesitant" and "scared." He talked about "trying to think of ways to figure out more about how this student is feeling about math" and how he could "bring her more confidence while she approaches problem solving." Another participant, Mark, emphasized the importance of knowing that "students come in from a lot of backgrounds, many in which they're taught to hate math or to think of themselves as bad at math" and suggested that in order to "reach all students" instructors need to consider students' perspectives of mathematics as a discipline.

In addition to considering student perspectives of mathematics, a few of the GSIs discussed their own perspectives and experiences with mathematics. Beth talked about how one of her goals as a teacher is to get her students to "appreciate math and appreciate that they can do it" because she wants them to feel the way that she feels about mathematics. She went on to explain how this view impacts her teaching:

> I think if you close yourself off from how your students are feeling and just like this is the material I need to teach and I'm going to teach this material at this pace regardless of how my students are learning, you're going to cut a lot of people off from learning math and actually wanting to pursue it. I don't know. I mean people think I'm crazy for wanting to do math, and I want more people to think, "oh I could do this. I could go on to Calc 3 if I wanted to or even further than that."

Mark also brought up his history with mathematics, although he discussed having mixed feelings about it: "I did go through a few years in high school where I didn't like my math classes, but even then I didn't really feel like I disliked math. I just disliked the math class because it was boring." He went on to share how Calculus had changed his perspective and how that influenced his view of the course as an instructor:

> Calculus, for me, had been just such a joyful class. It was sort of like what reignited my interest in math, reminded me that math class was a cool thing. So when I was teaching it

> the first semester, it didn't really even occur to me that my students could legitimately be worried about this and frightened about this.

Other GSIs openly discussed their negative experiences with math. Aaron explained how he shares a personal story with his students to tell them about his "perspective on the culture of math and how there's a fear-based culture of math in many cases." Aaron's story involves being a student in an undergraduate math class where he was handed a sheet of problems to solve:

> I didn't know how to even start any of them because I hadn't taken the necessary class prerequisites. So I was just sitting there waiting, not really able to do anything, and the professor stood in front of me and just started staring at me, at my blank sheet of paper for like five minutes. He finally broke the silence by asking, "Do you understand?" And it made me feel really bad.

By sharing this story with his students, Aaron hopes that they will be able to recognize and acknowledge their own fears and anxieties about math and can better relate to him as an instructor.

Accounting for non-mathematical factors. While many participants talked about how the context of mathematics influences empathy, several of them discussed how instructors are likely to experience empathy for their students in ways that go beyond the scope of teaching and learning mathematics. Many GSIs addressed this fact by acknowledging that students' experiences outside of the classroom can profoundly affect their learning and emphasized that it is important for instructors to consider non-mathematical factors when dealing with certain situations.

Bill brought up these nonacademic factors during his interview and suggested that part of being a teacher is empathizing with students outside of the classroom:

> There's also parts of it that aren't even about math, right, that are…I mean even just in the couple of years that I've been teaching, people have come up to me with problems about their life, just in general, and you have to be able to relate to that also, which isn't necessarily math-related. I guess you have to be able to be empathetic, have empathy for students even outside of an academic setting.

Other GSIs echoed this sentiment. Mark talked about all of the "stuff students are going through, inside and outside of our classrooms that affects their ability to learn and get something out of the class." Similarly, Henry mentioned being aware of and understanding that there might be lots of different reasons why a student may not be doing well in a class: "It might be something to do with their understanding of the material...[or] with how hard they're working. It might be something completely outside of those things involving the student's life, and keeping that in mind is probably beneficial."

Aaron also discussed the importance of understanding and relating to what students are going through outside of the classroom. Specifically, he noted the importance of considering a student's "history in a "holistic sense" and "conditions like what their schedule looks like, whether they have a supportive family environment or a demanding family environment, whether they have a commute to school that's long, all of these sorts of things." He went on to say that it is necessary for teachers to be aware of these external pressures and have an understanding of "how they impact the student's learning outcomes."

Another GSI, Jane, talked about teaching first-year college students who might be adjusting to a new environment: "you're empathetic to the fact that they just got to college and that they're not only taking your course, but they're transitioning into this new experience, these new classes that are way different than their high school classes." She also clarified that experiencing pedagogical empathy in this way is "more general" and not "necessarily specific to math" but still important, especially for students in their first year of college.

Discussion

The data presented above highlight several factors that have the potential to influence how GSIs experience pedagogical empathy for their students. While these factors can greatly impact interactions between instructors and students, instructors often have little to no control over many of them. In this section, I discuss how the three emergent themes address my research question and pose further questions worth investigating in this area of research.

As illustrated by the findings, past personal experiences can support or hinder the ways in which GSIs experience pedagogical empathy for their students. Seeing similarities between your own personal experiences and those of a student can make it easier to empathize with that student, while lacking similar experiences might make it more difficult. This result brings up questions related to equity: Are GSIs more likely to experience empathy for students with whom they more closely identify? Could this tendency lead to inequitable teaching practices? Are there ways to identify and avoid this occurrence? If so, how can professional development be structured to acknowledge this fact and expand GSIs' perspectives to include a broader awareness of student experiences?

Another question prompted by these findings is whether GSIs might be able to transfer their experiences from one context to another in order to empathize with students. One GSI, Joe, was able to "contextualize" what his students might feel by recalling his experience of taking a driving course that he did not enjoy. Joe's story suggests that this transfer of empathy might be possible. Thus, leaders of GSI professional development should design activities that encourage GSIs to reflect on their past experiences as *both* teachers and students in a variety of contexts. Engaging in this reflection might help GSIs better understand how to leverage past experiences—which may or may not be related to mathematics—in order to empathize with a wider range of students in more equitable ways.

In addition to past experiences, mathematics itself can be a driving influence of pedagogical empathy. As Mark (and other GSIs) acknowledged, students "come from a lot of backgrounds, many in which they're taught to hate math or to think of themselves as bad at math." Experiencing pedagogical empathy can increase GSIs' awareness of these different backgrounds and help them understand how their students might feel towards mathematics. If GSIs are aware of these negative perceptions or fixed mindsets, they can then take steps to reframe them and encourage their students to view mathematics in a more positive light.

Finally, it is important to recognize that empathy can be elicited in ways that are beyond the scope of teaching and learning mathematics. As many of the GSIs noted, students deal with a variety of issues in their daily lives, and their experiences outside of the classroom can influence the learning that happens in the classroom. Pedagogical empathy can help instructors understand the challenges that students may be facing and better attend to individual student needs.

Many facets of pedagogical empathy remain unexplored. Further research is needed to better understand the concept of pedagogical empathy and its connections to different types of knowledge that inform teaching decisions. This work sheds light on a few of its motivating drivers and provides insights for leaders of GSI professional development to consider. By encouraging GSIs to reflect on both their own and their students' experiences, leaders of professional development can increase GSIs' awareness of pedagogical empathy and help them identify different ways to relate to their students, thus supporting their growth as undergraduate mathematics instructors.

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Dimensions of Variation in Group Work within the "Same" Multi-Section Undergraduate Course

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This paper reports a qualitative study of how small group problem solving was enacted differently across sections of a multi-section undergraduate introduction to proof course. Common course materials, common guidelines for instruction, and weekly instructor meetings led by a faculty course coordinator supported similar instruction across sections, including an emphasis on in-class group work. But within that shared structure, classroom observations revealed important differences in how group work was introduced, organized, and managed. Our results focus on differences in the time allotted to group work, the rationale for group work, the selection and organization of groups, and aspects of student activity and participation. We suggest that these differences shaped different opportunities to learn proof writing in small groups. These results have implications for the design and teaching of collegiate mathematics courses where group work is a regular element of classroom work.

Keywords: Proof and proving, Group work, Undergraduate

Broad support now exists for using "active learning" methods in collegiate mathematics and science courses, where lecture has previously been the dominant element of instruction (Braun, Bremser, Duval, Lockwood, & White, 2017; Ernest, Hodge, & Yoshinobu, 2017; Freeman, et al., 2014; Laursen & Rasmussen, 2019). Where "active learning" has generally been described in inclusive terms—embracing a wide range of instructional activities (e.g., Braun, et al., 2017), most presentations include small group problem solving work as a central element. These presentations have emphasized mathematically challenging tasks ("problems," not "exercises") and peer collaboration in solving them. Prior publications focusing on cooperative learning in undergraduate mathematics have provided the rationale for small group work as well as practical advice on implementation in classrooms (Ahmadi, 2002; Dubinsky, Mathews, & Reynolds, 1997; Mathematical Association of America, 2017; Rogers, Reynolds, Davidson, & Thomas, 2001). But research has not yet examined how implementation can differ within multi-section courses that are common in collegiate STEM programs, much less the impact of those differences on students' learning.

The paper reports differences in the enactment of group work in four sections of one multi-section undergraduate mathematics course. Designed to introduce students to the nature of proof and proof writing, the course committed substantial class time to small group work on proof tasks. The paper first summarizes the common elements of content and pedagogy that united the four sections. Efforts to align teaching and learning across the sections had substantial effect. It is within that commonality that differences in how group work was introduced, undertaken, and managed were observed. Our data do not allow us to make the claim that differences in the enactment of group work had clear and definite impact on either students' views of group work or their growing competence with mathematical proof. But the results do show that a different set of affordances and constraints for working productively in groups on challenging proof tasks existed in the four sections. We hope this paper becomes a contribution to scholarship that identifies the issues that instructors and course coordinators of multi-section undergraduate courses should attend to, discuss, and manage in their efforts to maximize the positive impact of group work in undergraduate mathematics.

The Course (as Research Setting)

The study was part of a larger research project that has examined the nature of students' experience as they adjust from computation to proof in collegiate mathematics, with particular attention to the development of their mathematical agency and autonomy in challenging proof and proving work. One locus of the study is one mathematics department's introduction to proof (ITP) course. The course is required for mathematics majors and minors, is taken after students complete at least one semester of calculus, and is populated by majors and minors in about equal numbers. Minors typically major in some other STEM discipline. The department routinely offers 13 sections of the course each year (of three semesters), each enrolling between 20 and 25 students. ITP is a four credit-hour course; sections meet either three or four times each week.

The course introduces students to proof as the central process for establishing the truth of mathematical statements; to different proof methods (direct, cases, induction, contradiction, and contrapositive); and to concepts and statements in set theory, real analysis, and number theory as "content." Homework—assigned, graded, and returned weekly—includes short-answer tasks, informal arguments, and full proofs. As the semester moves along, full proofs become the dominant task. In spring and fall semesters, four exams are given, and students' grades depend primarily on exam and homework performance. Outside of class, students have access to a course-specific on-line *Piazza* forum—where they can ask questions (e.g., about homework tasks) and read suggestions from instructors and other students—and to the department's Math Learning Center (MLC) which provides a separate room and staffing for the ITP course.

In the semester of the study, an instructor and a teaching assistant were assigned to each section. The faculty member who coordinated the course taught one section, which was not observed. The other instructors were mathematics graduate students; all had prior experience as teaching assistants in the course. The teaching assistants were also mathematics graduate students. The instructional team (coordinator, instructors, and assistants) met weekly on Mondays to plan/review activities for the week and address emerging issues.

Four sections (A, B, C, and D) were observed in the semester of study. In the shared instructional plan, content was presented on Mondays in all sections. In sections A, C, and D, group work was the focus on Tuesdays and Thursdays. In section B, content was presented on Mondays and Wednesdays, with group work on Tuesdays and Fridays. In addition to lecture and group work, two other instructional activities were review for exams and interactive presentation of proofs of particular statements. In the latter, instructors presented a proof but also solicit suggestions (and questions) from students as part of the presentation.

In place of a textbook, work in the course was supported by a lengthy "Examples document" that had been compiled and revised over many prior semesters. All students were given a physical copy and on-line access to a digital copy. The digital version provided students with model solutions to each task. The Examples document provided the tasks for group work; instructors typically initiated segments of group work by writing a list of those tasks on the board. In addition, the course coordinator provided six tasks to introduce group work in the first two weeks. But for all but one, those tasks did not involve proof.

Data Collection

Part of the data collected for the larger project were classroom observations; these became the main data for this study. Observations were conducted roughly twice each week in the first five weeks to see how patterns of activity, including group work, were established. Thereafter, frequency decreased to once per week. The observations were "passive," but focused; observers did not engage in classroom activity. The objective was to document the nature and duration of the instructional activities, including but not limited to group work. Early in the semester, the focus was the class as a whole (students and instructors). Later in the semester, we focused on how study participants (3 to 5 per section) engaged in class activities, particularly group work. The observers were a faculty member with expertise in mathematics education and three graduate students in a mathematics education doctoral program. All four are authors.

Observations were completed on a common template used to characterize different segments of the class sessions (e.g., nature and duration). The template also included common focal questions that observers addressed at the end of each session. All observations were posted to a central project repository. Observations in all four sections included some "lecture" days but more "group work" days. The timing of observations was constrained by observers' schedules. For example, though we recognized that the first few class meetings were particularly important to observe, we were not always able to do so. Table 1 summarizes the number and timing of the observations carried out in the four sections.

Table 1: Summary of Observations in the four sections A, B, C, and D

	A	**B**	**C**	**D**
Class Meetings	M,T,Th @ 80 mins each	M,W,F @ 50 mins; T @ 80 mins	M,T,Th @ 80 mins each	M,T,Th @ 80 mins each
# of observations	12*	12	12	14
# of obs. by day	4 (M), 2 (T), 6 (Th)	3 (M), 9 (T)	5 (M), 7 (Th)	4 (M), 10 (T)
# of obs. in 1st 2 weeks	2	2	3	4
1st class observed?	Yes	No	Yes	Yes

* Two of the 12 observations in Section A were exam days

Whole-class activities (e.g., lecture and interactive presentation) were relatively unproblematic to document. In contrast, interactions involving students during group work were sometimes difficult to observe and document due to challenges of distance and audibility. Nevertheless, some important aspects of interaction and activity were observable in segments of group work. These included: (1) the physical location of groups, (2) students' physical positions and gaze as indicators of engagement, (3) the distribution of vocal participation in the group, (4) the actions of instructors and teaching assistants, and (5) often, the groups' methods for completing assigned tasks (e.g., working together on each task versus distributing tasks among group members [i.e., division of labor]).

Data Analysis

Attention to differences in how group work was enacted across sections arose in project meetings in the semester of study. Preliminary work identified some initial dimensions of contrast such as timing and duration and how groups were formed. After the semester ended, the first author read each summary (N = 50; each five to nine pages of single-spaced text within the template) and wrote summaries that characterized the group work segments of each observed session. Those summaries in turn supported the (a) development of a framework of seven dimensions used to characterize the enactment of group work (Table 2 below) and (b) analysis to address the question: *How did the enactment of group work differ across the four sections?*

Table 2: Framework for Analyzing Differences

Dimension	**Some Relevant Foci**
Frequency & Duration	How often and for how long did group work happen?
Rationale	What rationale was given, orally and in writing, about the purpose of group work?
Selection/Organization	How were groups selected/formed? How were adjustments to groups made?
Location & Integrity	Where were groups located in the classroom? Once formed, did groups stay together?
Content	What tasks were assigned to groups? Did instructors bring in additional tasks?
Student Activity & Participation	Did groups have access to shared work space, and did they use it? What were the patterns of participation in the groups?
Instructor Assistance	What did instructors do while groups were working?
Feedback & Evaluation	Did instructors seek and report back students' views of group work?

The observations produced clearer and more complete data on some dimensions than others. For example, data on the frequency and duration of group work was nearly always reported, and the time stamps provided a strong basis for comparison across sections. Similarly, because many observed class sessions included group work, we had extensive data on how instructors selected and organized their groups. By contrast, data on student activity and participation was less complete and robust due to the number of groups and the distance between observers and those groups. Also, we were unable to observe section B in the first week of the semester, limiting what we can say below about that instructor's rationale for group work.

Results

We present our results to highlight that important differences in the enactment of group work across the four sections existed *within* the inter-section commonalities that justified the sections as different instantiations of "the same course." As outlined above, these commonalities resulted from sustained work by the course coordinator to align content and pedagogy within the course. Important common elements included: (1) a single course syllabus and schedule of sessions, (2) a shared macro-level approach to instruction that devoted substantial time to group work, (3) a shared commitment to groups of three to four students, (4) shared roles for instructors and teaching assistant during group work (i.e., circulate, listen, question, and/or direct/correct; engage all groups), and (5) a common source of group work tasks (the Examples document). Relative to the framework outlined in Table 2, one other commonality united the four sections: The course syllabus did not state the purpose of group work in the course (Rationale).

Differences in the Enactment of Group Work

We focus on four dimensions—frequency & duration, rationale, selection/organization, and student activity & participation—where we observed marked differences in how group work was implemented. We also found differences in location & integrity and content, but we judged these of lesser impact, as differences in location & integrity correlated to differences along the four dimensions. Issues related to student activity & participation are very complex, and our presentation below addresses only some aspects of that dimension of group work.

Frequency & duration. Despite the shared commitment to group work on two days each week, the sections could still differ in the time allotted to group work. Table 3 summarizes the time devoted to group work on instructional days when that activity was observed. Row three entries present the number of observations that reported some group work. Entries in rows four through six present measures of central tendency and range for total group work time.

Table 3: Frequency and duration of group work

	A	B	C	D
# of Overall Observations	12	12	12	14
Observations of Group Work	10	9	8	11
Mean Duration (min)	37	58	32	48
Median Duration (min)	31	56	29	52
Range (min)	23-52	36-80	18-55	24-80

The entries in Table 3 indicate substantial differences in time allotted to group work; Sections B and D devoted substantially more time to that activity than Sections A and C. But the amount of time devoted to group work is independent of its character and quality. As the paragraphs below reveal, more minutes devoted to group work do not necessarily map onto more productive opportunities for student interaction (see Selection/Organization below).

Rationale. We found no direct reference to why group work was a central activity in the course, either in the syllabus or the observation data. But observations of sections A, C, and D in the first two weeks showed that instructors introduced group work differently—suggesting quite different purposes for group work in learning mathematics. (Recall that section B was not observed during the first week of the semester.)

The instructor in section A posed three questions on the first day of class—*why do group work, what does good group look like, and what does bad group work look like?* In the ensuing discussion, that instructor emphasized three points to students: (1) don't judge people as smart or not, (2) think about your body language in group interactions, (3) don't separate yourself from the group. The instructor did not, however, present any personal experience with or stance toward group work. In contrast, the instructor in section C stated that most people do not like group work and he himself "hated it," before adding that everyone recognizes that students should be good at it. Shortly thereafter, that instructor sent students an optional survey to learn their views of group work. The instructor in section D said nothing about group work generally in the first week, indicating only that the goal of the introductory group tasks was for students to get to know each other. These introductions expressed quite different messages about the purpose/rationale for group work: Group work can work for you (A), group work is a pain—if somewhat useful (C), and no opinion/not worth discussing (D). These introductory rationales suggest caution in interpreting the entries in Table 3 with a simple "more minutes are more productive" lens.

Selection/Organization. An important set of implementation decisions involve the composition of groups: How large groups will be, how will they be formed, and how will instructors address "problems" (e.g., late and absent students, students sitting alone, group non-interaction)? Though our data showed that all instructors initially formed groups of three to four students, their adherence to that principle varied widely. Their methods for composing groups also varied, as did the consistency in their application of those methods. These variations had direct consequences for whom students interacted with, daily and through the semester.

Group size. In section A, the instructor created and maintained groups of four students for the entire semester, using the same method (see below). Early in the semester, groups of four were sometimes broken into two pairs and paired work was observed, but that practice was limited to the first three weeks. The instructors in the other sections initially composed groups of four, but at different points in the semester, stopped attending to group size. The instructors in section B and D stopped focusing on group size by weeks 2 and 4 respectively, and the size of their groups varied widely thereafter (from one to five students). The instructor in section C maintained groups of four longer but then permitted smaller groups, including pairs and individuals, and intervened only when group size got large (e.g., six students).

Methods of composition. The instructor in section A used a transparent and random process (common features of playing cards) to compose groups, with new groups generally formed at the beginning of each week. In section B, the teaching assistant, subbing for the instructor, formed the first groups randomly by having students count off by remainder under division by 4. Thereafter, groups were composed by proximity: The instructor asked students sitting near each other to forms groups. Once this practice started, the constituency of the groups in Section B changed little, and two students worked on their own for nearly the entire semester. The instructor in section D pursued a similar practice. The first groups of four were composed by proximity, and in week 2 students were asked to remember their groups. Most students did not change their group membership for the rest of the semester (in groups that varied widely in size). The instructor in section C used transparent but non-random methods to compose groups of four in the first two class sessions (ordering students by their height and then their birthdays). But by week 3, students picked their own groups and by week 4 could work in groups or not.

These two issues, when combined, resulted in quite different patterns of "work with others" in the four sections. Students in Section A had no choice but to work with all (or nearly all) of their peers at least once in the semester. Students in the other three sections had much more control over whom they worked with and as a consequence of their choices, much more limited experience of work with others. In these sections, some students' experience of work with others was limited to single peers; others who worked alone had no experience of work with peers.

Student activity & participation. Group work in the four sections also differed in how much mathematical work was carried out in private versus public space. We use "public" to designate space that all group members can access visually and "private" to designate space used by some group members that affords limited visual access of others. With the commitment to groups of four, greater use of public space affords greater likelihood that groups will generate collaborative solutions. Access to public space in classrooms depends on the number and size of available whiteboards (or blackboards), as these afford wide visual access. By contrast, work carried out in students' paper notebooks and tablet computers is more private because it limits visual access. So, one aspect of student activity & participation in group work involves the extent of use of public vs. private space, and that access in turn is shaped by the physical properties of classrooms, specifically the amount of board space.

In Section A, the work of all groups was carried out in public space—on whiteboards and blackboards. One classroom for that section had sufficient board space to accommodate the work of six groups of four, where board space was limited in the other—accommodating only four groups of four. The instructor removed that limitation by attaching two transportable whiteboards to an empty wall of that classroom. In section B, sufficient board space was available for three days of the week (M-W-F) but was limited in the Tuesday classroom where the focus was group work. As might be suggested above (methods of composition) much less use of public space was observed in section B—though this may not have been simply a function of existing board space. In sections C and D, public space in the classrooms used on both "group work" days accommodated only four groups. As with section B, the groups' use of public space in these two sections was spotty. Many "groups" in sections B, C, and D completed their work in private space, sharing with one other student and generally when snags arose.

Discussion

In one semester of one course at one university, we have shown that the group work undergraduate students of mathematics experience differed in significant ways in different sections of the same coordinated course. Where this result could seem obviously true when multi-section courses are taught without much focused attention to coordination across sections, the results take on more importance in this case, because of the sustained efforts to align content, pedagogy, and assessment across sections. We do not interpret our results as evidence that course coordination failed. Instead, we see our results as identifying the challenges inherent in supporting equally productive group work in multi-section courses, even when coordinated.

The core issue explored in this paper—factors that contribute to college students' experience of and learning in small group work—is understudied at the level of detail we have examined and reported it. Research on the dynamics of group work over time—within the "same" group over time, within a single section, or within a multi-section course—is needed to understand and advance the promise of "active learning" and inquiry-based mathematics. More observation-based descriptive studies of group work enactment are needed, as well as studies that take on the challenge of linking the dynamics of that enactment to students' thinking and learning—about mathematics and peer interactions. For such linking studies, observations alone will not be sufficient. For both types of studies, we hope that the framework used here is a modest contribution, though we make no claim for its completeness or sufficiency.

With respect to practice, the study raises important questions and challenges for coordinators of multi-section courses who seek to provide all students with roughly equivalent and productive learning experiences. For example, given the inevitable differences in instructors' orientation to group work, how best to recruit and select them? In the scarce minutes before the start of the semester, how best to discuss instructors' personal experiences with group work and how to introduce it to students? Given the many issues that call for attention in course meetings, how much time should be allotted to discussing the "health" of groups and strategies for addressing problems in them? This study has raised our awareness of the challenges inherent in coordinating small group work across sections.

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Differentiating between Quadratic and Exponential Change via Covariational Reasoning: A Case Study

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Quadratic and exponential relationships are important topics in school mathematics. However, there is limited research examining students' understandings of these relationships. In this paper, we present data from a semester long teaching experiment with a pre-service teacher in which we examined ways in which she could leverage covariational reasoning to differentiate between quadratic and exponential change. Whereas in the pre-interview, the student did not have meanings that supported her in differentiating between the relationships, her experiences in the teaching experiment supported her in developing more robust meanings. By the end of the teaching experiment, the student could differentiate between quadratic and exponential change.

Keywords: Quadratic Change, Exponential Change, Teacher Education-Preservice

Authors of both the Common Core State Standards for Mathematics and the NCTM Standards (2000) recommended that students can identify different types of change that occur in various relationships including situations that can be modelled by exponential and polynomial growth. Despite the importance of these types of relationships, researchers found that undergraduate students struggled recognizing change as exponential (Presmeg & Nenduradu, 2005) and explaining the features of exponential relationship (Weber, 2002). College students' limited understandings of quadratic and exponential change is particularly problematic if these students are pre-service teachers (PSTs) as researchers (e.g., Copur-Gencturk, 2015; Thompson, 2013; Thompson, 2016) have argued that students are unlikely to learn mathematical concepts coherently if their teachers do not maintain robust meanings for the ideas. As such, there is a particular need to investigate secondary PSTs' meanings for quadratic and exponential relationships as they are likely to teach these topics to their future students. In this paper, we address the research question, 'How can a PST leverage her covariational reasoning to differentiate between quadratic and exponential change?'

Theoretical Framework: Covariational Reasoning

Several researchers have offered different perspectives of covariational reasoning (see Thompson & Carlson, 2017 for a synthesis). Drawing on several of these researchers' characterizations, Carlson and colleagues (2002) described five mental actions a student might engage in when reasoning covariationally. The mental actions include coordinating *direction of change* (area of a square increases as the square's side length increases; MA2), *amounts of change* (for equal changes in the side length, the successive changes in the area increase; MA3), and *rates of change* (the area is increasing at an increasing rate with respect to the side length; MA4-5). Researchers have characterized students leveraging these mental actions as they developed productive understandings of various mathematical ideas, including various ideas critical to precalculus and calculus (e.g., Paoletti & Moore, 2017; Moore, 2016).

Engaging in the mental actions described by Carlson et al. (2002) is a powerful way for students to determine and graphically represent relationships between quantities' values. However, such reasoning does not provide students with the tools necessary to further characterize the relationship as all relationships that increase at an increasing rate are characterized the same way. Particular to this study, we are focused on students extending their covariational reasoning to differentiate between quadratic and exponential change. In what follows, we describe our operationalization of quadratic and exponential change from a covariational perspective based on the extent literature. We synthesize key findings from studies that examined students' understandings of quadratic and exponential change against the backdrop of these characterizations.

Covariational Reasoning and Quadratic and Exponential Change

Quadratic change. Researchers have emphasized that a productive way to understand quadratic relationship is to conceive that the rate of change of the rate of change is constant (Ellis, 2011; Lobato, Hohensee, Rhodehamel, & Diamond, 2012). For example, Ellis (2011) examined middle-school students' ways of reasoning about the lengths, heights, and areas of a growing rectangle. Students in this study determined the first and second differences and identified constant second differences to identify the quadratic change in the area of the growing rectangle. Consistent with these researchers, we adopted the perspective that understanding quadratic change means understanding one quantity has constant second differences with respect to equal changes in a second quantity.

Exponential change. Drawing on their characterizations of covariational reasoning, Confrey and Smith (1994, 1995) and Thompson (2008) explained different ways students might conceive of exponential change. Confrey and Smith (1994, 1995) described exponential growth as a juxtaposition of values of a variable in a geometric sequence with values of a variable in an arithmetic sequence. They reported that students calculated either the differences or ratios of successive values of one variable for constant unit changes in the other variable to conceive of exponential change. Students noted that while the differences increased, the ratios remained constant. Confrey and Smith proposed this conceptualisation of 'increasing additive rate' and 'constant multiplicative rate' as fundamental to understanding exponential growth. Drawing on this view, Ellis and colleagues (2015) investigated the activity of middle school students who reasoned about the height of a plant varying with respect to time. They found that, students conceived exponential growth by coordinating the multiplicative growth of the height values with constant unit additive growth in time.

Differing from Confrey and Smith's approach of coordinating sequences, Thompson (2008) explained, "a defining characteristic of exponential functions is that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument" (p. 39). Drawing on this view, Castillo Garsow (2012) presented tasks to high school students in the context of interest bearing bank accounts. Castillo-Garsow reported that one student conceived that as the account's value increases in time, the rate of change of the value of the account at a moment remains proportional to the account's value at that moment. For our study, we adopted Thompson's view and intended to support students in understanding exponential change as occurring when the rate of change of a quantity at a certain moment is proportional to its value at that moment.

Methodology

This study is situated within a larger study examining two PSTs' developing meanings for quadratic and exponential relationships via their covariational reasoning. In this report, we focus on one of the two students, Rebecca (pseudonym), who was enrolled in a secondary mathematics teacher education program at a university in the northeast United States. Rebecca had already completed a minimum of two semesters of calculus sequence and was enrolled in a course designed to explore high school mathematics topics from a quantitative and covariational reasoning perspective. We first conducted a one-on-one semi-structured pre-clinical interview (Clement, 2000; Goldin, 2000; Hunting, 1997) to gain insight into Rebecca's covariational reasoning and her current meanings of quadratic and exponential relationships. We then conducted 11 paired teaching episodes (Steffe & Thompson, 2000). Finally, six weeks after the last teaching episode, we conducted one-on-one semi-structured post-clinical interview to examine any shifts in her meanings. The teaching episodes occurred during the in class sessions of the course. For these teaching episodes, while certain tasks were designed to engage the PSTs in reasoning covariationally and quantitatively, (e.g., the Ferris Wheel Task as described by Carlson & Moore, 2012; Moore, 2014), other tasks (which we describe below) were designed specifically to support their reasoning about quadratic and exponential relationships.

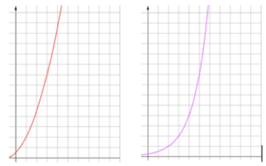
In order to analyze the data, we engaged in conceptual analysis, "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008). To do this, we analyzed data with generative and convergent purposes (Clement, 2000). Initially, we looked for instances that provided insights into Rebecca's meanings of the relationships and built initial models of her thinking by analyzing her observable words and actions. When evidence contradicted our models, we made new hypotheses to explain her ways of reasoning, including the possibility that Rebecca's operative meanings had changed. We revisited the data with the new conjectures in mind to refine the models. We used this iterative method to construct viable models of Rebecca's reasoning.

Results

We first present excerpts of Rebecca's activity in the pre-interview which highlight her limited meanings for exponential and quadratic change. We then present her activity during the teaching episodes in which she began to develop more sophisticated understandings of each type of relationship via reasoning covariationally. We conclude by highlighting shifts in Rebecca's meanings in the post-interview. For the sake of brevity, we highlight instances of the teaching episodes that are particularly relevant to her developing understandings of quadratic and exponential change.

Results from the Pre-Clinical Interview

The first pre-interview task consisted of graphs and tables of values we intended to represent quadratic (see Figure 1a and c) and exponential change (see Figure 1b and d). We asked Rebecca to identify the relationship represented by each graph and table. For both graphs and tables, Rebecca's current meanings did not support her in differentiating between these relationships. Graphically, Rebecca relied on memorized facts associated with perceptual shapes of the graphs (what Moore and Thompson (2015) refer to as static shape thinking) rather than on the patterns of change. For example, Rebecca said, "I know, like the basic shapes of like an x squared or cubed" and identified the relationship represented in Figure 1(a) as an x squared function and Figure 1(b) as a cubic function. For the tables, she attempted to establish a correspondence rule between x and y, stating, "I am trying to see if we can multiply or add something to the x to get the y's." However, she was unable to determine any algebraic rule and therefore could not characterize the relationships in the tables. Hence, we infer Rebecca's current meanings did not support her in differentiating between the change patterns.



(a) (b)

(c)

x	y
1	3
2	12
3	27
4	48
5	75

(d)

x	y
1	6
2	18
3	54
4	162
5	486

Figure 1. Graphs that represent (a) quadratic growth (b) exponential growth and Tables that represent (c) quadratic growth (d) exponential growth

The Growing Triangle Task

The Growing Triangle Task and the Growing Quadrilateral Task (described in the next section), are adaptions of tasks implemented in previous studies that investigated students' reasoning about rate (Johnson, 2012) and quadratic growth (Ellis, 2011). In the Growing Triangle Task (see Figure 2a for several screen shots of the task), we presented students with a Geogebra applet showing an (apparently) smoothly growing isosceles triangle and asked them to sketch the relationship between the pink side length and the green area of the triangle.

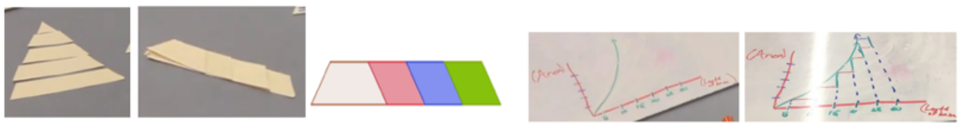
The applet (https://www.geogebra.org/m/pwzzdjaz) includes two sliders. The longer slider allows the PSTs to change the length of the pink base of the triangle and the shorter slider allows them to increase the increment by which the longer slider increased. Additionally, to support the PSTs in considering the *amounts of amounts of change* in the area of the triangle as a quantity, we provided paper cut-out manipulatives including: (i) a set of five triangles, each with an integer change in the base; (ii) four trapezoids representing the area added to each triangle to obtain the next triangle (see Figure 2b). By including the trapezoidal cut outs, we intended to support the PSTs in considering how the *amounts of amounts of change* increased by a constant amount as the area of the triangle increased for equal changes in the base.

(a) (b)

Figure 2. (a) Several screenshots of the Growing Triangle task and (b)a photo of the paper cut out manipulatives

Rebecca's activity in the Growing Triangle Task. Since our interest includes PSTs' general covariational reasoning, we engaged the students in four teaching episodes addressing other tasks that have been shown to support students reasoning covariationally including the Ferris Wheel Task (Carlson & Moore, 2012; Moore, 2014) and the Power Tower Task (Moore, et al., 2014). After these tasks, we provided Rebecca and the other student with the Growing Triangle Task. After exploring the applet, each PST coordinated how the pink segment and green area covaried in terms of direction of change (MA2) and amounts of change (MA3). To illustrate, after noting that the base increased by one unit as the triangle became larger, Rebecca arranged the trapezoidal cut outs one below the other (see Figure 3a) and claimed each trapezoid represents "the area added on each time…and they [*the areas of the trapezoids*] are increasing." Hence, we infer Rebecca reasoned about the direction and the amount of change to conclude that the area of the triangle increased by more as the base of the triangle increased.

Further, Rebecca described that the amounts of amounts of change are constant in the situation. For example, immediately after the above interaction, the first author asked, "Are they [*pointing to the trapezoids*] increasing by more, less, or by the same amount?" Responding to this, Rebecca overlaid the trapezoids (Figure 3b) and using her fingers to denote the amounts of change in area from one trapezoid to the next (Figure 3c), stated "they [*the trapezoids*] are all increasing by the same."



(a) (b) (c) (d) (e)

Figure 3. Trapezoids (a)one below the other (b) overlaid (c) recreation of overlaid trapezoids with the three colored trapezoids representing equal amounts of change in their areas (d) smooth curve representing area with respect to side length (e) color coordinated segments to explain the situation

When we asked Rebecca to represent the relationship between the area and side length of the triangle graphically, she drew a smooth curve and color-coordinated segments (Figure 3d-e). She described the green segments as representing the amounts of change in the areas of the triangle as "the base gets bigger" and claimed "they [the green segments] are increasing." When we prompted her to say more about the green segments, Rebecca returned to the stacked trapezoids, pointed to the changes in the areas of the trapezoids (colored trapezoids in Figure 3c), and claimed, 'the changes [in the areas of the successive trapezoids] are the same." We infer Rebecca understood the amounts of amounts of change were constant situationally and described how such a relationship could be represented graphically.

Despite identifying the constant second differences, Rebecca (or the other student) provided no indication she related this to a quadratic relationship. During a whole class discussion, the course instructor defined a relationship with a constant second difference as a quadratic relationship, which we infer provided Rebecca with an opportunity to reflect on her current activity identifying (and later using) constant second differences with her meanings for quadratic relationships. After this session, Rebecca consistently identified relationships with a constant second difference as quadratic, which we highlight in the next sections.

The Growing Quadrilateral Task

In the first iteration of this task, we presented students with an applet showing a growing quadrilateral (https://www.geogebra.org/m/mrusgb2v). The applet includes one slider and allows the PSTs to animate the quadrilateral (see Figure 4 for several screen shots of the task). Because we designed the task in such a way that, as the slider values increase by one unit, the area of the quadrilateral increases by one half of the present area, we conjectured this task had the potential to support students conceive of exponential growth consistent with Thompson's (2008) perspective; the rate of change of area is proportional to the area at that value.

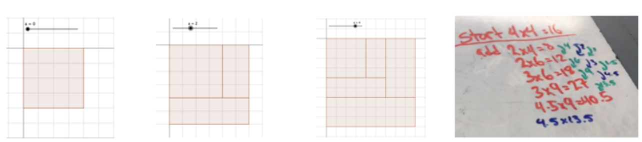


Figure 4. Animation Snapshots of the Growing Quadrilateral task and Rebecca's work of determining the differences in the areas

Rebecca's activity in the Growing Quadrilateral Task. Rebecca attended to how the quantities' values covary in terms of direction of change (MA2) and amounts of change (MA3). To illustrate, she watched the animation and explained "as the slider gets bigger" the area of the quadrilateral is increasing and "goes up maybe a little, little more, little bit more, a little little more, and then for the last one, it's a lot." Further, to characterize the growth in the area of the quadrilateral, she attended to the amounts of amounts of change. Rebecca determined numerical values for the amounts of area added, the differences in these amounts, and their differences (Figure 4). Pointing to the second differences she said, "…. like if [the second differences] were plus 2, plus 2, plus 2, then it would have been quadratic, but it's not." Rebecca's explanation provides evidence that she was leveraging her reorganized meanings of quadratic change to characterize the current pattern as not quadratic. Although we designed the task to support students in developing conceptions of exponential growth consistent with Thompson's (2008) descriptions, Rebecca attended to the increasing additive amounts of change which aligns with Confrey and Smith's (1994, 1995) characterization of exponential growth.

To support students in conceiving exponential growth consistent with Thompson's (2008) descriptions, over the next five teaching sessions we presented variations of this task. Rebecca did reason about the proportional change in the areas in conjunction with reasoning about increasing (or decreasing) additive changes and constant multiplicative changes (Confrey & Smith, 1994). For example, we presented an animated shrinking quadrilateral (see Figure 5 a-d) whose area decreases by one half the current area as the slider's values change continuously (https://www.geogebra.org/m/mbaa5pus). Rebecca initially attended to how the area decreased according to consecutive integer values, describing "[the area]'s decreasing by a small amount, uh, but like half of the previous area." She then drew a smooth curve with green vertical segments representing amounts of change (Figure 5e) and described, "the area is one half of the previous area and the change would be one half too." We infer that in the moment, Rebecca was both conceiving there was a constant multiplicative rate (e.g., area is always ½ the previous area) and conceiving the change in area was proportional to the current area.

When asked to identify the relationship, Rebecca sought to examine the additive differences in the values of areas to conclude the relationship was exponential. Rebecca created a table of values (Figure 5f), calculated the first and second differences in the area values and explained, "I think for the exponential they [*pointing to the second differences*] will never be the same, if I go to the next thing, they are never going to be the same. So I will have to say this is exponential." Hence, Rebecca determined a non-constant second difference, and used this understanding to conclude that the relationship must be exponential.

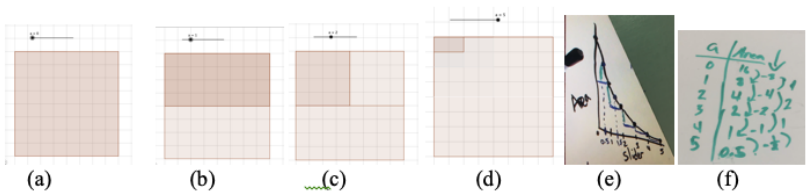
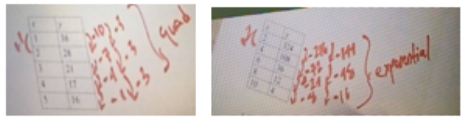


Figure 5. (a) to (d): Snapshots of the shrinking quadrilateral, (e) and (f) Rebecca's graph and table of values to represent of the situation

Rebecca's Activity in the Post-Clinical Interview

During the post-clinical interview, six weeks after the last in-class teaching session, Rebecca reasoned covariationally to identify the relationships represented by the graphs and tables of values. For example, addressing the tables of values task (Figure 6) she determined a constant difference in x value prior to identifying the first and second differences in the y values. When the second differences were constant she inferred the relationship was quadratic (Figure 6a).



(a) (b)

Figure 6. Rebecca's work to identify (a) quadratic and (b) exponential relationship

However, for the table in Figure 6b, she determined that the first and second differences were not constant and presumed that if she continued determining additional differences (e.g., third, fourth, fifth, etc.), they would not be constant either and concluded "since the amounts of amounts of change are not same till the end, it is exponential." Thus, Rebecca coordinated the amounts of change (MA3) as well as the amounts of amounts of change of one quantity with respect to changes in the second quantity to identify quadratic and exponential relationships.

Discussion and Closing Remarks

In this paper, we presented one PST's activity that indicated she reasoned covariationally to develop meanings of quadratic change and differentiate between quadratic and exponential change patterns. Hence, we add to the literature on college students' and PSTs' covariational reasoning and their developing more sophisticated mathematical meanings. Addressing the research question, we illustrated Rebecca maintained limited meanings for quadratic and exponential relationships at the outset of the teaching experiment. As the teaching episodes progressed, we noted shifts in her meanings for these relationships such that by the end of the teaching experiment she could distinguish between exponential and quadratic change.

Over the course of the teaching experiment, Rebecca had a multitude of experiences coordinating covarying quantities in terms of direction and amounts of changes (Carlson et al., 2002). We conjecture such reasoning was a prerequisite to Rebecca identifying and differentiating between quadratic (Ellis, 2011) and exponential change (Confrey & Smith, 1994, 1995; Thompson, 2008). Specifically, after determining how the amounts of change were changing, Rebecca could further analyze these changes. Such analysis supported her in reorganizing her meanings of quadratic relationships in ways that are critical for her which has the potential to impact her future students developing compatible meanings for quadratic change (e.g., Ellis, 2011; Lobato et al., 2012).

Particular to exponential change, we note Rebecca leveraged each of the characterizations of exponential relationships we described. We conjecture the interplay of these different ways of reasoning was due to different prompts and tasks providing varying opportunities for each type of reasoning to be useful (e.g., in a table of values looking at additive changes is useful). Hence, we conjecture students may be able to develop a richer conception of exponential relationships if they can leverage each of the different ways. However, the extent to which Rebecca's meanings for exponential growth entailed each by the end of the teaching experiment is an open question. Hence, we call for additional research examining ways to support PSTs, and college students generally, in developing more productive understandings of exponential relationships, potentially including developing an empirically grounded superseding theoretical framework (see Strom, 2008 for a theoretically grounded example) for understanding exponential relationships.

Acknowledgements

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A Case of Learning How to Use and Order Quantified Variables by Way of Defining

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Research literature suggests that students struggle in gaining fluency with mathematical language, and especially with statements with multiple quantifiers. I conducted a design study rooted in Realistic Mathematics Education in order to support students in developing such fluency. Drawing on the emergent model heuristic, I present a pair of students' reinvention of relationships between variables in statements with multiple quantifiers. This work is a case study for how students might engage in defining in order to learn how and in which order to use quantified variables. Relationships between variables first emerged as a model-of a pair of students' activity as they defined different concepts that are best defined using multiple quantified variables. Additionally, these relationships became more explicit to the students as they reflected on their definitions.

Keywords: Defining, quantification, emergent model, case study

It is well documented that students have difficulties with statements with multiple quantifiers, which are characteristic of many advanced mathematics definitions. There are two types of statements with multiple quantifiers that have received significant attention from the research community: statements in the form (1) *for all… there exists…such that…* (AE) and (2) *there exists…such that… for all…* (EA). When the mathematics community uses an AE statement like 'for all *x* in *X* there exists *y* in *Y* such that*…'*, we intend to portray the idea that 'every *x* in *X* has a corresponding *y* in *Y*'. On the other hand, an EA statement like 'there exists a *x* in *X* such that for all *y* in *Y…'* is meant to capture the idea that 'there is one *x* in *X* that corresponds to every *y* in *Y*'. Durand-Guerrier and Arsac (2005) explained that students sometimes do not see the dependency in AE statements. That is, for the previous AE statement, the value of *y* depends on the value of *x*, yet student may not notice that if you change the *x* values then you may change the *y* value. Additionally, Dubinsky and Yiparaki (2000) gave students statements with multiple quantifiers in the context of natural and mathematical language. They found the students tended to interpret EA statements as AE statements.

There have been several scholars who have designed instructional tasks or tools to address students' challenges with statements with multiple quantifiers (e.g., David, Roh, & Sellers, 2018; Dubinsky & Yiparaki, 2000; Dawkins & Roh, 2016; Roh & Lee, 2011). These researchers' efforts were focused on helping students *interpret* the statements. For instance, Dubinsky and Yiparak (2000) offer a game that they designed to help students recognize the problem with interpreting EA as AE. Additionally, Dawkins and Roh (2016) suggest that analogies from natural language can be a helpful tool for students' interpretation of statements with multiple quantifiers when the analogies are well thought out and rely on similar rules and conventions. The purpose of this research study is to pose an alternative approach for supporting students in learning about and using quantified variables. This approach not only helps students learn how to write statements with multiple quantifiers, but it should also support them to interpret statements with multiple quantifiers because they get the experience of trying to use them to express ideas (and distinctions among them) that are meaningful to them.

Theoretical Perspective

This study is part of a design experiment that drew on the theory of Realistic Mathematics Education (RME). RME is an instructional design theory that is based on Hans Freudenthal's principle that, "mathematics is a human activity" (Gravemeijer, 1998, p. 277). Guided reinvention is what Larsen (2018) described as the mission statement of any RME design study. The heuristic establishes that instructional activities are designed so that the formal mathematics emerges *from* students' informal mathematical activities, giving students a sense of ownership over the mathematics developed. For instance, instead of tasking students to interpret a formal definition to help them understand a concept, the researchers would carefully design tasks that support students to build on their ideas towards a more formal definition of the concept.

Broadly speaking, the RME heuristic *emergent models* describes how students' informal mathematical activity might develop into more formal mathematics. The 'model' in emergent models refers to a mathematical concept the researcher first recognizes in the students' activity and the students themselves eventually reinvent. There is a key shift when the students recognize and begin to make use of the mathematical idea. We say that a *model-of* informal mathematical activity becomes a *model-for* more formal mathematical reasoning (Gravemeijer, 1999). When the mathematical concept emergences as a model-of the students' activity, the researcher can explain the students' activity using the mathematical concept but the students are not aware that such a thing exists. When the concept transitions to a model-for the students' formal mathematical reasoning, the concept becomes explicit to the students and the students are able to use the concept as a tool.

My task design aimed to capitalize on students' experience with defining concepts that are best defined using quantified variables. I adopt Zandieh and Rassmussen's (2010) conceptualization of defining. Specifically, I take defining to involve formulating, negotiating, and revising a definition. Zandieh and Rassmussen noted that defining can include more than just creating a definition; defining can occur "as students are proving a statement, generating conjectures, creating examples, and trying out or 'proving' a definition" (p. 59). The tasks designed for this study engaged students in defining multiple concepts in a similar fashion to what Oehrtman, Swinyard, and Martin (2014) called a cyclic process of definition refinement, where their students first generated examples and non-examples of a concept and then entered into an iterative process of writing a definition, evaluated the definition against examples and non-examples, acknowledged and discussed conflict, and then discussed potential solutions.

The hope with my task design was that students would leverage their rich, shared concept image (Tall & Vinner, 1989) to help them articulate defining properties of several concepts. Through this defining process, students would quantify variables as a way to express their thinking. After defining several concepts that required the student to quantify variable(s), the students were tasked to reflect on their definitions and create an instruction manual that would allow someone to decide how, when, and in which order to use 'there exists' and/or 'for all' when defining a concept. This task is what Carlson, Larsen, and Lesh (2003) would call a model-eliciting activity.

Methods

I conducted a laboratory design study with two community college students, Lori and Ada. By laboratory design experiment I mean a design experiment (Cobb & Gravemeijer, 2008) with a pair of students outside of class rather than in a regular classroom setting. Both students had just completed a course where my research team began scaling up efforts for an inquiry-oriented curriculum for advanced calculus (for more information on the project see ASPPMIRE NSF

DUE No. 1916490; Larsen, Strand, & Vroom, submitted; Strand, 2016). Prior to this course, Lori and Ada excelled in their undergraduate mathematics studies, but had no formal proof experience at the undergraduate level. During the course, the students reinvented several root-approximation methods and then followed Cauchy's lead by repurposing their methods to reinvent a proof-outline of the Intermediate Value Theorem (IVT). Along the way, the students developed several key concepts. During this course the students had exposure to the phrases 'for all' and 'there exists' but an exit survey revealed that Lori and Ada had not yet mastered how or when to use them.

The laboratory design experiment spanned 11 sessions with each lasting 1.5 hours. The goals of the experiment were to learn how to leverage their reinvention context (real analysis) to (a) support students in learning about and gaining fluency with mathematical language and (b) support students in using definitions in proving. I served as the teacher-researcher and there was at least one witness during all of the sessions. The sessions were recorded on multiple devices; a video-camera captured Lori and Ada's gestures and screen mirroring software video-recorded Lori and Ada's written work on two tablets.

I conducted ongoing and retrospective analyses. Between sessions, I wrote detailed chronological descriptions of how the students engaged with the tasks. These descriptions were used to aid debriefing sessions between each teaching session with another researcher who was not present during the teaching session. This researcher would read the descriptions and ask clarifying questions and I would amend the descriptions as needed. After the completion of the design experiment, I reread the chronological descriptions and re-watched the corresponding video in order to create a timeline of ideas that emerged related to quantification. The timeline included the ideas and the tasks or student interactions that supported the idea generation. As I began creating the timeline, I realized that the emergent models heuristic was a powerful tool to understand what supported Lori and Ada to articulate the relationships between variables in AE and EA statements. In what follows, I argue that these relationships emerged as a model-of their informal mathematical activity as they defined several concepts that involved quantifying variables.

Results

In the following section, I will present two key episodes of Lori and Ada engaging in defining concepts that are best defined using quantified variables.. Through defining the concepts, ideas emerged about how and in which order to quantify variables. That is, the two episodes focus on the initial emergence of a concept as a model-of students' informal mathematical activity. In the presentation, I will also present an episode of students' reflecting on their definitions to articulate the relationships expressed in EA and AE statements. This episode will focus on the beginning of the model-of/model-for transition.

Defining Eventually Constant Sequence

The third session of the design experiment began with Lori and Ada engaging in a task that led them to generate two sequences: (1) $\{1, 1, 1, \frac{15}{16}, \frac{29}{32}, \frac{57}{64}, \ldots\}$ where $a_n = \frac{7}{8} + \frac{1}{2^n}$ for $n \geq 4$, and (2) $\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \ldots\}$, where $a_n = \frac{7}{8}$ for $n \geq 4$. They conjectured the first sequence would converge because it was decreasing and bounded (an idea that was discussed in their prior course) and the second sequence would converge because it was increasing and bounded, but it held an additional property. Lori explained that "it reaches its bound" (meaning least upper bound). This motivated Ada and Lori to define a new kind of sequence, which they referred to as "eventually constant" (EC).

To begin defining they created examples and non-examples of EC sequences. They decided that EC sequences did not require the sequence to be monotone and provided an example (see Figure 1A), explaining that "it's all wack here [referring to the terms that are circled] but as soon as you hit this one here [referring to the 8th term] it is all constant." They also discussed other types of sequences, deeming the sequences depicted in Figure 1A-D as EC and the sequences depicted in Figure 1E-G as non-examples of EC.

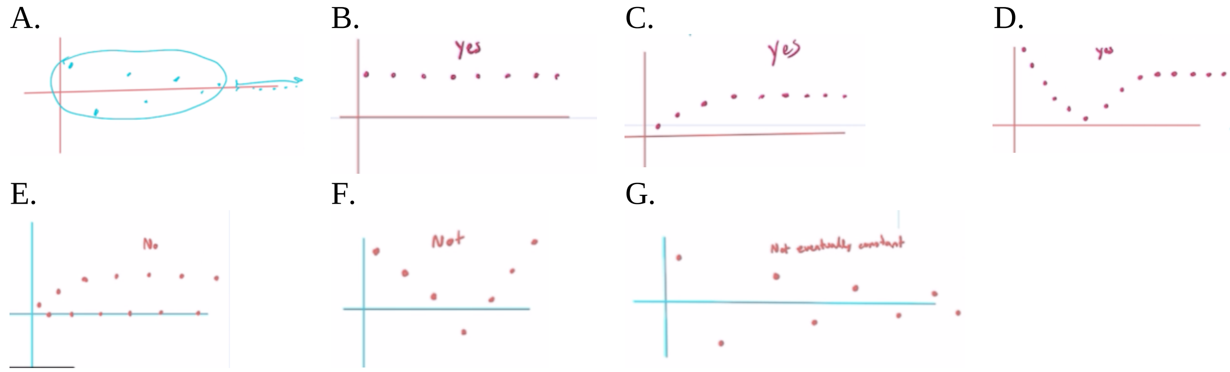

Figure 1. Examples and non-examples of EC sequences.

Once they were satisfied with their collection of examples and non-examples, they began to articulate a definition. Lori made the first attempt (see Table 1 Definition A). Ada expressed that she liked Lori's statement and explained, "that's kind of the same thing as 'if *k* is very large' so like - so the beginning of the sequence is beneath that largeness it does whatever it wants and then after it's reached the large portion of its life it will end constant." Notice Ada's words "beneath that largeness" treated the statement like there is a single point to be underneath, motivating Lori to refine the definition to include the existence of some number, L (See Table 1 Definition B).

Table 1. A table of definition refinements for EC sequence.

Definition A	If k is very large, a_k will equal a_{k+1}
Definition B	For a sequence a_k, there exist some number L beyond which $a_k = a_{k+1}$
Definition C	For a sequence a_k, there exist some natural number L beyond which $a_k = a_{k+1}$
Definition D	A sequence is eventually constant if: For a sequence a_k, there exist some natural number L, where for $k \geq L$, $a_k = a_{k+1}$"
Definition E	A sequence is eventually constant if: For a sequence a_k, there exist some natural number L, where for all $k \geq L$, $a_k = a_{k+1}$

Next, Lori turned to the example in Figure 1D and said: "so this way […] it can do all sorts of wacky stuff but there is some L [marking an L next to the ninth dot], and I don't know if it makes sense to have it a natural number or a member of the sequence…" After Ada expressed a reason for L to be a natural number, explaining, "I think it should be a natural number that way L isn't unique to any particular sequence," Lori changed the definition (See Table 1 Definition C). Then, Lori expressed the desire to relate L to the sequence and again refined the definition (see Table 1 Definition D).

At this point, Ada and Lori were seemingly satisfied and turned back to their examples and non-examples to evaluate their definition. During this, Lori changed one of their non-examples to the sequence depicted in Figure 2A and they had the following exchange:

Lori: There is something that I would like to do to quickly change the sequence (referring to sequence in Figure 1F) to keep it so that it is not [eventually constant] so if we make it 'momentarily constant' let's make sure […] it (meaning their definition) still accounts for that. So it seems like this is L (labeling L, see Figure 2B) and it fits here (labeling the forth term with a_k) and this a_k is equal to this a_{k+1} (labeling the 5th term with a_{k+1}) and fits here (labeling the 6th term with a_{k+2}) but it does not fit here - beyond here. So does our definition account for that?

Ada: I think it does. I think you can also put like 'where for all $k \geq L$' because it might be misinterpreted as 'for some $k \geq L$'. So we might want to put an 'all' in there so that we are very clear that everything beyond.

A. B.

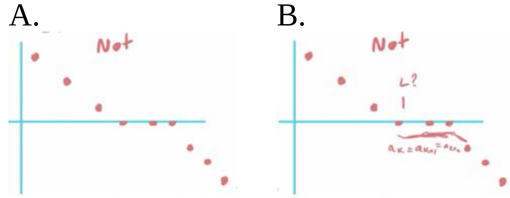

Figure 2. A "momentarily constant" sequence.

This motivated Lori to change their definition for the last time during this session (see Table 1 Definition E).

Discussion of episode. In this episode, Ada and Lori engaged in defining the concept of EC sequence by sketching examples and non-examples, writing an initial definition, and refining it in a cyclic manner. Through this process, they experienced the need for 'there exist' to encode the idea that after a certain point the terms were all the same. They also introduced 'for all' in order to clearly communicate that *all* the terms after a certain point needed to be equal, not just *some* of the terms. This idea emerged after Lori introduced a sequence that they did not want to classify as EC, but had the property that $a_k = a_{k+1}$ for some k.

Defining Prime Sequence

During the ninth episode, I tasked Ada and Lori to define a new type of sequence which I called "prime sequences." I explained a prime sequence contained all the prime numbers. Again, Ada and Lori began to negotiate what counted as an example and what counted as a non-example. They established that a prime sequence: (a) could contain other numbers that were not prime, (b) could have the same prime number listed more than once, (c) did not have to necessarily list the prime numbers in order, (d) was not necessarily increasing, and (e) could never be a constant sequence. They produced sketches of five examples (including $a_k = k$ and several non-monotone examples) and two non-examples (i.e., $a_k = k^2$ and $a_k = 3$).

Table 2. A table of definition refinements for prime sequence.

Definition A	A prime sequence is a sequence that must contain the set of prime numbers in its range.
Definition B	There exists an input *k* which maps to a prime number for all prime numbers in $\mathbb{P}$

Definition C	There exists at least one input k which maps to a prime number for all prime numbers in $\mathbb{P}$
Definition D	There exists at least one input k for all prime numbers in $\mathbb{P}$
Definition E	For all numbers in $\mathbb{P}$, there exists at least one input k whose output is in $\mathbb{P}$.
Definition F	For every prime number p in $\mathbb{P}$ there exists at least one input k where $a_k = p$.

Ada's first attempt at writing down a definition is given in Table 2 Definition A. Notice this definition is essentially the description that I gave them when I originally posed the task. They seemed satisfied that their definition could sort their sequences into examples and non-examples. At this point, I reminded them that many of their previous definitions used the phrases 'for all' and/or 'there exists' and encouraged them to rewrite their definition using the phrases. With this, Ada commented, "I feel like I definitely [want to] put the 'for all' with the prime numbers like 'for all prime numbers' or something like that." After some discussion about the set of prime numbers (which they denoted with $\mathbb{P}$), Lori wrote ''For all prime numbers' and then posed a question: "Where do we go from that? Like what do we put before and after?" which led to the following discussion:

Ada: That's a good question
Lori: And if we want to have a there exists we should probably...
Ada: There exists a natural number for all prime numbers something like that. Like you could just draw that connection.
Lori: Maybe a… (she starts writing)
Ada: Basically saying that each prime number has its own natural number something like that
Lori: Oh I think I get it - hang on - so like there exists a k an input k which maps to
Ada: Yeah
Lori: A prime number for all
Ada: Mhm - there is like one prime there's one natural number for all prime numbers

During this interaction, Lori wrote the statement given in Table 2 Definition B. Although Ada and Lori seemed to have a robust concept image of prime sequence prior to writing down a definition, Ada's descriptions go back and forth between the AE and EA meanings. For instance, Ada's idea that "each prime number has its own natural number" describes a relationship that is typically captured by an AE statement. Whereas her ideas "there exists a natural number for all prime numbers" and "there's one natural number for all prime numbers" seems to describe a relationship that is typically captured by an EA statement.

Quickly after Lori updated the definition to Table 2 Definition B, Ada requested: "Could you say at least one input for each k because that implies that there could be more than one input that would result in the same prime number but at least all of the prime numbers are accounted for once?" This resulted in Lori adding "at least one" to the definition (see Table 2 Definition C). Ada's request gives us some insight into her thinking about 'there exists an input k.' For her, this statement seemed to suggest that there was only one input that could map to a prime number but they agreed a prime sequence could have the same prime number listed more than once.

At this point, the students recognized two issues with their current definition. Lori explained the first: "… it's starting to sound like there's only one prime number in the whole set or in the whole sequence […] because it is saying there exists at least one input which maps to *a* prime number (emphasizing 'a') and then it jumps into for all." Ada addressed her concern by refining the definition (see Table 2 Definition D). This alteration highlighted the second issue.

The second issue was that their definition seemed to imply that all of the prime numbers were associated with one *k* value. They explained:

Lori: Okay our current problem is that we have this *k* guy here he must - if this is our definition for a prime sequence then our prime sequence looks [draws sketch in Figure 3] *k* is equal to one and two and three…

Ada: All on top

Lori: And that's not good

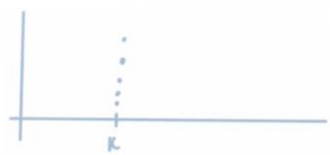

Figure 3. Issue with Table 2 Definition D

Ada then had an idea to use a different phrase, writing "for every prime number in ℙ" on the tablet beside their statement given in Table 2 Definition D. Unsure of what she was proposing, I asked her how I was supposed to read her new suggestion. This seemingly inspired Lori to suggest, "let's see what would happen if we swapped the order" and produced the statement in Table 2 Definition E. She explained, "so we're saying that you've got all of your prime numbers and it's saying you will always be able to find at least one k who will output to one of those prime numbers for all of the prime numbers." Ada liked the idea and later commented that the new definition, "sound[s] more like this situation more like each of them get their own."

Then, Lori explained that there is still more to fix, explaining "we kind of need more information now - who's this *k* guy and what does he have to do prime numbers?" This leads her to make the final refinement during this session (see Table 2 Definition F).

Discussion of episode. In this episode, Ada and Lori began to write a definition of prime sequence using the phrases 'for all' and 'there exists'. As an observer, I was able to see that the relationships between variables in AE and EA statements was a model-of their activity. Specifically, they realized that one of their refined definitions (see Table 2 Definition D) could be interpreted as all the prime numbers were to be mapped to one *k* value and switching the order of the quantifiers (and corresponding variables) made it clearer that each prime number was assigned to (at least one) *k* value.

Discussion

In this proposal, I have focused on the emergence of the relationships between variables in statements with multiple quantifiers for Lori and Ada. The students engaged in defining concepts that ultimately led them to AE and EA statements. As an observer, I could interpret their definitions as describing the normative relationships between the variables in AE and EA statements, but they were not yet aware that AE and EA structure captured these relationships. This supports the claim that the defining tasks given in the two episodes (as well as additional defining tasks that Ada and Lori engaged with), were helpful in evoking the relationships between variables in AE and EA statements as a model of Lori and Ada's activity. In the presentation, I will present these two episodes as well as an episode that began to promote the model-of/model-for transition. In this episode, I posed a model-eliciting activity in which Ada and Lori reflected on their collection of definitions. Through this task, the relationships between variables in AE and EA statements became more explicit for Ada and Lori. This case study provides promising evidence that engaging in defining can support students in learning about and gaining fluency with mathematical language.

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Student Reasoning with Graphs, Contour Maps, and Rate of Change for Multivariable Functions

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The purpose of this paper is to highlight the reasonings students use to move between representations of multivariable functions. We identify five ways students use of aligning graphical representations of functions, or surfaces, with contour maps. Subsequent tasks highlight the affordances one particular reasoning, called Contour Spacing Alignment, offers for students to further develop their reasoning. We analyze student reasoning highlighting the transition between model-of and model-for this reasoning.

Keywords: Realistic Mathematics Education, Representations of Multivariable Functions, Multivariable Calculus, Gradient

Multivariable calculus (MVC) introduces several representations of functions, including both contour plots and graphical representations of functions, which we refer to as surfaces. The rate of change concept, which is now dependent upon direction, can be investigated across these different representations. External representations of function contain different pieces of information, making some representations easier to use than others for certain tasks (Ainsworth, 1999; Ainsworth, 2006). Thus, it is important for students to be able to move fluidly between representations of functions in multivariable calculus.

Focusing primarily on contour plots and surface representations of functions, our primary question is "What reasoning do students develop for the alignment between the contour map and surface representation of multivariable functions?" With a more limited scope, we focus on the range of reasonings students develop for this relationship during the second day of the semester. We then highlight, with a broader scope of four days throughout the course, how one particular reasoning (The Contour Spacing Alignment) affords a progressive mathematization which is useful throughout the course.

Theoretical Background

The activities and materials utilized in this study were developed to provide students with tangible objects and realistic scenarios from which to discover new mathematics (Wangberg & Johnson, 2013). We utilize Realistic Mathematics Education (RME) for this analysis for several reasons: (1) It highlights the role emergent models play in individual students' learning and in the collective mathematical development of the classroom community (Gravemeijer, 2000). (2) The continual negotiations between the teacher and students support the emergence of taken-as-shared meanings in the classroom (Cobb & Yackel, 1996; Yackel & Cobb, 1996). (3) We utilize the horizontal and vertical mathematizing aspects of reinvention. In the case of horizontal mathematization, the classroom community develops informal, taken-as-shared ways of speaking, symbolizing, and reasoning as students attempt to mathematize starting-point problems, while vertical mathematization addresses when these ways of describing become subject to further mathematization. (Gravemeijer, 2000). Thus, we highlight the model-of / model-for distinction noting that at the referential level, models are grounded in students' understanding of experientially real settings (such as when working with the project's materials) while in general activity, the students' reasoning loses its dependency on situation specific imagery. (Gravemeijer, 2000).

An instructor's implied learning trajectory and involvement in whole class discussion are key features highlighted in the analysis; Space limits prohibit further discussion of RME principles.

Data Collection and Analysis

Data analyzed for this study involves video data collected from five (50 minute) classroom days in MVC involving 21 students at a medium-sized regional university in the midwestern United States in Fall 2016. Dedicated video and audio recording technology captured students working in small groups and whole class discussion. Nine cameras were used for each classroom period, providing 37.5 total hours of video. Additional data from day 2 for an MVC course in Spring 2016 (18 students) helped re-affirm the range of reasonings reported to be used on day 2.

The data analysis initially focuses on day 2, where students first encounter the concept of level curves, contour maps, and surface representations of functions. This day provides the first opportunity in the classroom for students to create models-of the relationship between the different representations. The four subsequent periods (day 8, 21, 23, and 36) provide opportunities to observe how student reasonings transition from being models-of to models-for the relationships between these representations.

Day 2 – The Park: Aligning Contour Maps and Surfaces

We briefly describe The Park activity, which occurs prior to any discussion of function representations in the course. Student groups are given one of six surface models whose height represents the concentration of lead (in g/m^2) in the topsoil for 'Rock City'. In their small groups, students locate points on the surface whose lead concentration matches a certain value. A brief whole class discussion lets students name these 'level curves' (or contours). The groups are then given a contour map, larger than the surface, which represents the same quantity for 'Rock County'. Students are asked to '*Find where the town is located in the county and place it there.*'

Small Groups Reasonings for Alignment between Surface and Contour Map

Video analysis shows students utilize five techniques to place the surface on the contour map. We present discussion from groups utilizing each technique.

Function-value alignment. The first group aligned the surface by focusing on values of the contours. After the third student places the surface on the contour map, the group proceeds:

S1: So, I'm guessing it is going this way. [S1 places surface 'backwards on the contour map.] [S1 points to a contour.] Does it get bigger? … No, it gonna be smaller....[S1 turns the surface around].

S2: No, I think that's ... [S2 looks at a value on the contour map.] Then these are negative over here. And these are big over here.

The students re-align the surface and proceed to comment on bigger and smaller values of the contours. Then, within about a minute, they arrive at a solution:

S1: What if we go... what if we turn it this way.... [S1 grabs the surface at places it in the right place.] 'Cause then this is at 1 over here and that's - that's pretty close, and [S1 points to the contour near a high point on the surface] this gets to about 6 here.

S2: Yep. [S2 checking value of contours nearby.]

S1: Down to about 2 [S1 points to the base of a peak], and this [S1 points to a nearby peak] this gets to about 6, and this [S1 points to a valley] is this about 1 right here? [S1 can't read the contour, but estimates using the surface]

S2: Yep. [S1 retracts their hand.] Perfect!

The students verify that high and low points on the surface correspond to high and low contour values. This technique is often used by groups using the blue, green, or purple surfaces.

Contour pattern alignment. Another alignment reasoning focuses matching repeating patterns in the contour map to the corresponding patterns on the surface.

S4: There's two lower spots, a higher spot, then two lower spots again.

S5: I think it goes like [S5 grabs the surface and places in correctly on the contour plot.]

All three students peer over the top of the surface and confirm that the repeating features on the contour map align with the repeating patterns on the surface.

S6: Looks pretty good.

This technique can be used by students using the red, green, blue, or purple surfaces.

Distinguished contour alignment. A third alignment strategy occurs when students match up a prominent contour on the surface with the corresponding curve on the map:

S7: So... [S7 puts the contour map next to the surface]

S8: This is there! [S8 points to a contour on the surface and finds the same contour shape on the contour map. S8 rotates the surface to align the curves, then gets flustered.]

S7: [S7 points to a region enclosed by a prominent level curve] There's this big region here.

S8: That matches up with that fairly well. [S8 points to a dip on the surface and map.]

After a brief discussion of the dimensions of the county and city, Student 9 grabs the surface:

S9: I'm trying to match this basic feature cause that's right there. [S9 points again to the prominent feature on the contour map and surface.]

The three students each then verify the location making sure peaks and valleys match up, value-wise, with those features on the surface. Students typically use this reasoning based on a distinguished contour shape when using the yellow, blue, or green surfaces.

Maxima/Minima and loop contour alignment. A fourth alignment strategy occurs when students use maxima and minima on the surface:

Student 12 moves the surface onto the contour map.

S10: Oh yeah, lay it on top. That's a good idea. I.. I can see that this ridge [S10 points to a maximum on the surface's interior] is long... [S10 looks down through the surface to the map.]

Student 12 rotates the surface into position. All three students look down through the surface to the map below.

S10: Yeah. Right there. [S10 and S12 point to minimums and maximums on the surface's interior] We've got all the dips and tops are above the ovals.

By working with the maxima and minima for the surface, the group eventually realized that they corresponded to ovals, or closed loop, contours. This type of reasoning is available to groups working with the red, yellow, or purple surface manipulatives.

Notice that none of the reasonings used to align surfaces work for the orange surface.

Whole-class Discussion about Alignment Reasonings

We highlight a part of the whole class discussion, in which the instructor called upon various groups to describe how they aligned their surface to their map. The instructor's progression of calling on groups finally lead to the group with Students 10, 11, and 12, who utilized the Maxima/Minima and loop contour alignment:

Instructor: How did you match up the city and county?

S12: We looked at the ... the max/mins of the, of our surface, and they ... we were... so maxes had circles. and mins had circles too. One max was a really long one, so we found this shape. And then lined them up with maximums and minimums.

Instructor: What were circles?

S10: The maximums and minimums

Instructor: The maximums and minimums were circles. Is that what they mean? Do you guys know what they mean?

S10: They're like, as it got like, like higher than a valley, it would be going around a ridge. [S10 draws on the surface] So it would be like going around an oval.

The instructor highlights how the contours loop around peaks and valleys, then addresses all the student groups:

Instructor: How many other groups used maximum and minimums?

Students in some groups raise their hands. Others do not, depending upon whether their maximum or minimums occur within the interior of their surface.

Contour spacing. Continuing the whole class discussion, the instructor then points to a group using the orange surface. They have not raised their hand for the previous technique.

Instructor: What technique did you use?

S13: We tried to match up the max's and minimums.

Instructor: Ok, so... the techniques... Did it work?

S14: Sorta...

S13: I feel like we're kind of in the area...

S14: Not quite there...

Instructor: [pointing to people in the group] Can you hold up your surface, and you hold up the contour map? So, on the contour map, there's a big feature... that... um, that is right there, in the middle, where there's not a lot of contours. What does that mean?

[many students are heard talking.]

A student from another group reports: It's not steep. It's flat.

S10, from the max/min group: It's gradual.

Instructor: So it's not as steep? Not changing as much. It could still be changing a bit, but it's not as steep, right? And, so, on the surface, where... is there a place on the surface that would be... not steep?

S13: Right there. [points to the surface's flat part.]

Instructor: So, one way to match up the surface is to look at where the contours are really spread out, or where they're really close together.

At this stage, the instructor asks students to see if this technique works for their particular case and report back. Contour Spacing Alignment is applicable in every case but less obvious than other strategies.

Discussion I: Aligning Contour Maps and Surfaces on Day 1

Students develop reasonings based upon their own starting place, which is dependent upon the surface assigned to their group. The whole class discussion gives an opportunity for groups to share their strategies with the class. It also provides an opportunity for groups to test out different types of reasoning. In this way, students receive implicit information about the generality of their reasoning. This is not evidence of a shift from model-of to a model-for, but this questioning does highlight that certain reasonings will be needed in more general cases.

The instructor utilizes the whole class discussion to shape the progression of the collective mathematization, or reinvention, of the relationship between contours maps and surface representations of multivariable functions. Even though the Contour Spacing Alignment reasoning is available to all students, it is not at first accessible to them. Notice that the instructor struggles with how to frame the questions to the class. This struggle resulted from the instructor's need to introduce this new reasoning for their intended learning trajectory, but also wanted to make sure students had some ownership of the reasoning. The instructor asked students if this alignment reasoning worked for their case in order to let them see that it works for many general cases.

Finally, the instructor notes that while these reasonings were helpful, they proved less helpful on a related task in which students were asked to draw the contours for a function represented with a table of data. The instructor noted that such a task required a concept of contours which separated function values into regions, where (for example) the function was larger on one side of the contour than on the other.

Broader Scope: Contour Space Reasoning in Subsequent Activities

Subsequent tasks in the course provides a broader perspective to the affordances for students to use and progressively develop these models for aligning the surface with the contour map. Due to space constraints, we focus on brief snippets involving class periods developing partial derivatives, constrained optimization, path integration, and surface area integration. We limit our focus to the model encompassed by the Contour Spacing Alignment reasoning.

Day 8 - The Hotplate Activity: Discovering Partial Derivatives

We focus on the last part of the activity, where students are asked to rank the value of a partial derivatives at three points on a contour map. Students had just used a measurement tool to measure partial derivatives (slope) on their surface using a ratio-of-small-changes concept.

When working with the contour map, students were often stuck until the instructor directed them to draw a vector in the desired direction of the partial derivative. This allowed students to form ratios of the change in output and input along that curve and to compute the partial derivatives. However, students did rank the partial derivatives without computation by noting how their partial derivative values are really positive (or really negative) when their vector crosses many contour lines and close to zero when it crosses few contours.

Day 21 - The Roller Coaster Activity: Discovering Constrained Optimization

In this activity, students draw a curve (representing a roller coaster) on a contour map and mark places where the curve is perpendicular (or parallel) to the level curves with an X (or O). After they transfer the curve and markings to the surface, students investigate “What happens at each of the marked points?” They are asked to explain what they find.

We look at discussion within a single group. (Groups have been re-organized since Day 2 and Day 8; We continue labeling students to distinguish them from earlier groups of students.) Student 16 proposes a conjecture while Student 18 draws the curve on the surface.

S16: Perpendicular places – that’ll be – slope of, er, it won’t be changing, then. Parallel places, it will be changing a lot.

S17: I think it’s the other way around.

S16: No cause the… remember when we did the [S16 puts one finger pointing forward, like a gradient vector, and another finger pointing perpendicular to it.]… when it’s going this way, it’s not changing. [Note: this is referring to an activity involving directional derivatives.]

S17: But the… going perpendicular to the level curve was the quickest – was the gradient.

S18: Perpendicular to the curve is the gradient – that’s the steepest direction.

Student 18 contributes this fact to the group while drawing on the surface. Once finished, Student 18 re-addresses Student 16’s conjecture by using the curve drawn on the surface:

S18: Yep, the slopes are like 0 pretty much at the O’s.

The small group discussion recognizing that the path is in the gradient direction at the X's occurs in several groups. In addition, students in many groups recognize that the O's occur at a locally flat place on the curve. For example, from a separate group:

S19: Kind of looks like where its… parallel, it kind of flattens out there just for a minute. Like here it just sits in the valley. Here it just kinda sits here. And here it just sits there.

S20: Uh-ha.

S19: Like it just kind of steadies out. Everywhere else it s perpendicular stuff is changing…

S20: Where ever it's perpendicular, it's… oh, so, it's… perpendicular the, like… the gradient is perpendicular to the level curve. It's pointing in the steepest direction.

S19: Oh yeah!

The instructor frames the subsequent classroom discussion to build upon the notions of crossing level curves or being parallel to level curves in order to mathematize these relationships using vectors and dot product relationships.

Day 23 - The Boysenberry Patch Activity: Discovering Path Integrals

In this activity, students are asked to rank four paths drawn on a contour map according to the amount of berries they could pick along each path. Paths (2), (3), and (4) have the same length and are more than twice as long as Path (1). Students can use the surface if they wish.

Due to space constraints, we focus on one group in which the students are reasoning using the spacing of the contours:

S23: So, we're just going by the more contour lines you cross?

S24: Yeah. Which…

S22: So the steeper you go the more boysenberries you get, right?

S24: yeah.

Student 22 is confusing the meaning of the level curves, which are a density of berries, with the concept of level curve spacing, or density, which indicates how the amount of berries changes. This is a common misunderstanding encountered by students. This group resolved the discussion by reasoning about the meaning of the values on the level curves. This same confusion was also recognized in another group. To resolve the confusion in that second group, a student traced out the paths on the surface in order to highlight that a path running along a high ridge has a lot of berries even though that path crosses very few contour lines. Earlier attempts to reason using just the surface were not enough to convince their groupmate of their error.

The instructor organized the whole class discussion to highlight this confusion. By carefully selecting groups, the instructor was able to first introduce correct results based upon incorrect reasoning. Subsequent groups, which used the correct reasoning, lead to debate amongst the students about the mathematics and meaning of 'density' for these contours.

Day 36 – The Putting Green Activity: Discovering Surface Area Integrals

This final activity requires students to rank four regions drawn on a contour map according to the amount of sod needed to cover each region. Students can use the surface, but more than half the groups are content working just with the contour maps.

Due to space concerns, we point out only that this activity utilizes reasoning which is opposite that used for The Boysenberry Patch: Classroom discussion again is organized to let students make conjectures tying large or small amounts to the value (misconception) or spacing of contours (correct conception) within a region.

Discussion II: Contour Space Reasoning throughout the course

The selected activities provide opportunities for students to build upon certain models relating the contour map and surface representations of multivariable functions.

In the partial derivative activity (The Hotplate), the ranking task involved a contour map which did not correspond to any surface which groups had used. Thus, when students reasoned about the ranking of the partial derivatives using the spacing of contours, this provides some evidence that students have changed from a model-of to a model-for relationships between contour plots and surfaces in a more general case.

The Roller Coaster activity provided evidence of students connecting notions of 'crossing level curves' to the concept of gradient. This indicates students are progressing away from the initial model relating only contour maps and surfaces, and are now connecting new concepts (e.g. gradient and gradient direction) to this model. The evidence suggests students are now transitioning into a more general use of their model, where it is now becoming an object of further mathematization.

In the last two activities involving integration, it appears that students struggle with the notions of 'density of contour lines' of a function and 'contour lines represent density' of a function. The instructional activities and instructor actions are designed to bring this discussion to the forefront. More importantly, the materials available to the students provide them opportunity to go back and reason in necessary ways with each other: One student goes back to the surface, aligns it to the contour map, and uses this as a tool to explain the misconception about 'density' for the contour lines, while students in another group are able to reason through the misconception using only the contour map.

Finally, we highlight the implied trajectory adopted by the instructor in the classroom discussion. While some of the reasonings between the contour map and surface could be useful in some of the activities, only the Contour Spacing Alignment was available in every situation. Further, knowing the importance of concepts like gradient direction enabled the instructor to capitalize upon certain concepts when students raised them in small group or whole class discussion.

Acknowledgments

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Student Meanings for Eigenequations in Mathematics and in Quantum Mechanics

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Students encounter advanced mathematical concepts in both mathematics classes and physics classes. What meanings do they develop about the concepts across the various contexts? Our research project investigates students' meanings for eigentheory in quantum mechanics and how their language for eigentheory compares and contrasts across mathematics and quantum physics contexts. We present students' interpretations of a canonical mathematical 2x2 eigenequation, a spin-½ operator eigenequation, and a spin-½ operator equation in which the operation "flips" the spin state. In individual, semi-structured interviews, 9 quantum mechanics students were asked to explain what the first two equations meant to them and then to compare and contrast how they conceptualize eigentheory in the two contexts. They were then asked to discuss the third equation. Using discourse analysis, results characterize students' nuanced imagery for the equations and highlight instances of synergistic and potentially incompatible interpretations.

Keywords: Linear algebra, student thinking, physics, eigentheory

Students encounter many of the same advanced mathematical concepts across mathematics classes and physics classes. For instance, within quantum mechanics alone, students are asked to reason about and use linear algebra content such as basis, normalization, projection, inner products, and matrix-vector operations. What meanings do students develop about the concepts across the various contexts? Our research project investigates students' meanings for eigentheory in quantum mechanics and how their language for eigentheory compares and contrasts across mathematics and quantum physics contexts. In particular, we present students' interpretations of a canonical mathematical 2x2 eigenequation, a canonical quantum mechanics eigenequation (specifically, for a spin-½ operator), and a spin-½ operator equation in which the operation "flips" the spin state (which is not an eigenequation). We characterize students' imagery for the equations and highlight instances of synergistic and potentially incompatible interpretations.

Literature and Physics Review

Research on students' understanding of eigentheory has grown over the past decade, and it provides several insights into the complexity of the topic, students' sophisticated ways of reasoning, and pedagogical suggestions for overcoming the challenges students face. Thomas and Stewart (2011) noted students' need to understand both how matrix multiplication and scalar multiplication on the two sides of the equation yield the same result. They also advocate for instructors to help their students develop a graphical conception of eigenvectors and eigenvalues, something they noted was weak in their study participants. Gol Tabaghi and Sinclair (2013) investigated students' visual and kinesthetic understanding of eigenvector and eigenvalue. They found that students' work with an interactive sketch promoted their flexibility between synthetic-geometric and analytic-arithmetic modes of reasoning. While this interpretation of eigenvectors and eigenvalues is important for understanding eigentheory in general, we demonstrate in this paper how eigenequations in quantum mechanics require students to think about eigenvectors and eigenvalues in different ways then they have learned in the past, and how there are some unique challenges that arise with interpreting the eigenequations in quantum.

Quantum mechanical systems and all knowable information about them are represented mathematically by normalized kets, symbolized in Dirac notation as $|\psi\rangle$. Kets mathematically behave like vectors. Spin is a measure of a particle's intrinsic angular momentum, which is related to the particle's magnetic moment. Spin is a vector quantity; a component of the spin vector can be represented mathematically by an operator such as $\hat{S}_z$ (where the z indicates the component axis). In a spin-½ system, there are two possible results for the S_z measurement: $\pm\frac{\hbar}{2}$; they correspond to $|+\rangle$ and $|-\rangle$, which comprise a set of orthonormal basis vectors called the S_z basis. The analogous information can be determined for other component axes, such as x. In that instance, the possible results of the S_x measurement are also $\pm\frac{\hbar}{2}$, which correspond to the set of orthonormal basis vectors $|\pm\rangle_x$ called the S_x basis. The relationship of these objects and quantities can be expressed symbolically in eigenequation form, e.g., $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$.

Methods

The data consist of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 9 students from a senior-level quantum mechanics course at a medium-sized public research university in the northeast US. The interviews took place in week 8 of the semester, after the completion of a unit on spin-½ systems with Dirac notation. For this paper, we focus on students' reasoning on one particular interview question which probed their thinking about three particular equations (denoted E1, E2, and E3), the first two of which are eigenequations (see Figure 1).

"I have a few equations prepared. For each one, I want you to explain what the equation means to you."

[E1] $A\vec{x} = \lambda\vec{x}$, where A is a 2x2 matrix, $\vec{x}$ is a 2x1 vector, and λ is a scalar

[E2] $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$

"You mentioned both related to eigentheory. Please compare and contrast how you personally conceptualize eigentheory in the two situations."

[E3] $\hat{S}_z|+\rangle_x = \frac{\hbar}{2}|-\rangle_x$

Figure 1. The interview questions relevant to the analysis.

To pursue our research question regarding what meanings students develop for the equations in Figure 1, we assume a theoretical stance consistent with the Knowledge in Pieces framework (diSessa, 1993; Smith, diSessa, & Roschelle, 1993). This assumed that students' intuitively held knowledge pieces are productive in some context, and knowledge change involves evolutionary refinement and reorganization of ideas. We conducted our analysis by iteratively examining the transcript for nuance in student imagery and noting potentially relevant discursive cues (Gee, 2005). We elaborate on the themes that surfaced across E1-E3 in the Results section.

Results

We organize the results by students' work on each equation individually; we also include a detailed comparison across E1 and E2 in particular.

Student Work on Equation 1

In the first part of the interview task, students were asked to read and explain $A\vec{x} = \lambda\vec{x}$. Most students read the equation by reading each letter or symbol ("*A x* is equal to lambda *x*"); some included "vector" with "*x*" and some added "times" in between. Only two students explicitly initially discussed this using "operate"/"operators" and "eigenvector"/"eigenvalue" language. All of the students subsequently used a form of "scale" or "scalar" in describing the equation, but only 6 of the 9 used a form of "operate" or "operator." All but one student explicitly identified the correct mathematical entities in the equations, i.e., A is a matrix, $\vec{x}$ a vector, and λ a scalar. One student only referred to the operators as such, rather than a matrix, for both Equations 1-2.

Students described the meaning of the equation in terms that also contained different interpretations of the role of the equals sign. For example, C3 explained the equals sign in this context by saying, "some *A* two-by-two matrix multiplied by an *x* vector can be represented by some *x* vector times a scalar." One student, C5, started by saying that "there is a scalar that can do what *A* does, in a way." C5 went on to say, "a matrix can be described as a scalar" but then followed up by saying that "matrix multiplication is equivalent to scalar multiplication" and showed that "what I get from doing a two-by-two on this two-by-one is gonna be equal to when I do [a scalar times a vector]." Other students described the equation as each side having something happen that leads to the same result, a relational meaning of the equal sign.

Student Work on Equation 2

The second part of the interview task asked students to explain what $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ meant to them. All students used "eigen" language in their explanations, indicating their recognition of the form of the equation. In general this was accompanied by other terminology, including "basis" and "operator." Most students described operating the S_x operator on the up-*x* state as either a measurement or a possible measurement. For example, C6 explained, "You're measuring S_x…not only are you mathematically operating, like it represents physically doing something to the system, which in this case is measuring the spin in the *x* direction…you're getting a measurement of $\hbar/2$ in the plus *x* direction." C11 said "it immediately registers as like a physical operation…like a mathematical representation of a physical measurement. So, I guess the first thing that comes to mind when I see this equation is that, um, h-bar over two is…a value of some measurement the, the spin operator represents." The main reasoning by students in these responses to describe operation as measurement is that the equation is a mathematical representation of a physical operation. The terms "operator," "operation," and "acting" may actually suggest measurement to these students.

Some students were clearer in their interpretation: describing an operation as a potential or possible measurement and relating what value would be obtained if a measurement were made of the quantity represented by the operator. Note that some students discussed operation both ways during the interview, as measurement and as possible measurement. For example, C1 said "This is a measurement on some, measurement of the state of some system…This eigenvalue is the value that you would measure um, when you operate this [$\hat{S}_x$] in this state [$|+\rangle_x$]." In C1's statement that their initial interpretation was as a "measurement of the state of some system" and later described the eigenvalue as "the value you would measure." C8 stated, "This makes me more think of taking a measurement…And when we take the measurement of that state, we get the scalar quantity." C8 initially described the operation as a measurement, but then their words suggest a more hypothetical measurement. It's possible that the subtlety between "measurement" and "possible measurement" may not be reflected in student responses.

This is arguably a more refined interpretation, often used by instructors, and is supported by the phrasing of Postulate 3 in the text: "The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator *A*" (McIntyre, 2012, p. 35). The phrase here is "possible result of a measurement" rather than "result of a possible measurement," but students may not discern that distinction. Gire and Manogue (2011) stated the role of eigenvalue equations in quantum mechanics is "to determine the eigenvalues (possible values of measurements) and eigenstates (possible states after a measurement is made)" (p. 195). A concern with this interpretation of an operation is that if a linear combination of (non-degenerate) eigenstates – known as a superposition state – were operated on, the result would not be an eigenvalue multiplied by the initial state. A measurement of a superposition state would yield a measurement of the quantity represented by the operator that corresponds to a single eigenstate that would describe the system after the measurement. Here we are only asking students about eigenstates being operated on, so we cannot gauge the generalizability of their statements about if they would treat an operator as a measurement for a superposition state.

Comparisons of Equations 1 and 2

As seen above, most students explicitly identified the components of both E1 and E2. Very few students seemed conflicted about relating their explanations of E1 and E2 or described a disconnect between the two equations. Here, however, we present an example of each of these.

Student C8 was categorized above as having a "possible measurement" interpretation of the operator in the eigenequation. But their comparison between the two equations came unprompted when they were asked to interpret E2. When C8 read E1, they said, "A acting on this vector is the exact same thing as this scalar quantity [λ] acting on the vector." When they subsequently were asked for their physical interpretation for E2, C8 said, "I guess everything I have said earlier [for E1] would apply to this, however, I don't really think of it in terms of that way when I see it in this notation [E2]. This makes me more think of taking a measurement." C8 demonstrated an explicit separation between interpreting the eigenequation in a mathematical context and in a quantum mechanics context. C8 did not express any confusion or concern about this distinction.

C1, on the other hand, expressed confusion about how the equations relate when prompted. They used "operate" language for both equations: the vector(s) x are "operated on by A," and "S_x operated on the plus-x vector." They also referred to x as the "solution" to the equation. They read E2 as "S_x operated on the plus-x vector is equal to h-bar over two, ah, or times the plus-x vector." However, C1 recognized and struggled to reconcile the disconnect between the geometric and quantum mechanical interpretations of the quantum mechanical eigenequation:

> I feel like I know what the individual pieces mean okay. Like this, this eigenvalue [$\hbar/2$] is the value that you would measure, um, when you operate this [$\hat{S}_x$] in this state [$|+\rangle_x$ on LHS]. I'm not sure what this [RHS] represents as a vector. Because to say that the, the state is scaled by a vector seems wrong because the particles aren't then in this state [entire RHS], they're just in this state [$|+\rangle_x$ on RHS], just the plus-x state.

C1 invokes the requirement of a state vector to be normalized to reject the idea that the operation scales the eigenstate.

In general, students did not invoke a general mathematical meaning of operator (a function that transforms an input into an output); instead they used a physical context for their description, tying operators to measurement of the quantity represented by the operator. Their reasoning, when teased out, centered around the idea that the equation is a mathematical representation of physical operation. These findings complement those of Gire and Manogue (2008); they found a tendency for students to associate measurements with operators when discussing sequential measurements on a superposition state, with similar reasoning that the operator acting on a state is a mathematical representation of a measurement of the quantity. We propose an explanation for why the idea of "operator as measurement" makes sense to students and is a reasonable expectation, given most of their past experience with equations in physics.

Physics equations typically provide symbolic representations of physical relationships between quantities: Newton's second law ($\Sigma\vec{F} = ma$) relates mass, acceleration, and net force. The allowable energies of a quantum mechanical harmonic oscillator – the eigenvalues of the Hamiltonian operator – can be expressed in terms of the energy level index (n), the angular oscillation frequency of the system (ω), and Planck's constant divided by 2π ($E_n = (n + \frac{1}{2})\hbar\omega$). An equation can be used to determine how a system's properties will change in response to a change in a different property, effectively demonstrating covariation or multivariation between physical quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Kuster & Jones, 2019).

Eigenequations can take on different valid meanings. For example, the eigenvalue equation $A\vec{x} = \lambda\vec{x}$ can have a graphical interpretation that the (infinitely many) eigenvectors of A are those that are only scaled by the operator. Research suggests this is a helpful conceptualization for students to develop in linear algebra (e.g., AUTHORS).

Our participants seemed to reasonably understand the meaning of the standard eigenequation. They also in general recognized the symbol template (Sherin, 2001) in the quantum mechanical version, correctly identifying the "beasts" (Karakok, 2019). But this equation does not have the same physical significance as other physics equations, nor does the geometric meaning apply, as pointed out by C1. Rather, this equation is useful to make a correspondence between physical quantities and mathematical entities, i.e., to interpret the physical meaning of individual terms in the equation. The real practical use is in computation of various quantities, especially expectation values, which provide the predicted mean value of identical measurements on identically prepared systems, e.g., $\langle\hat{S}_x\rangle = \langle\psi|\hat{S}_x|\psi\rangle$. When evaluating an expectation value for a superposition state (e.g., $|\psi\rangle = a|+\rangle_x + b|-\rangle_x$), one solution method is to operate on each eigenstate of the operator and then take the inner product of the "bras" and the kets. (The kets are unchanged if they are eigenvectors of the operator.) Thus the quantum mechanical eigenequation does not have the same meaning as either eigenequations in linear algebra or typical physics equations.

It is reasonable that this very different interpretation of a QM eigenequation can explain our results to some extent. Students are familiar with an eigenequation in mathematics, and some even have a reasonable geometric interpretation, but have a tendency to associate the operator and the equation overall with a measurement of the quantity represented by the operator. This interpretation of the equation is arguably consistent with their prior experience of physics equations to date, namely that the equation represents a symbolic expression of a physical relationship related to measurements.

Student Work on Equation 3

The third part of the interview question asked students to explain what $\hat{S}_z|+\rangle_x = \frac{\hbar}{2}|-\rangle_x$ meant to them. First, we explore student reasoning about if it was a true equation, and second, we explore student reasoning about if it was an eigenequation.

Student reasoning about if it was a true equation. Four of the ten students began by needing to reason about whether or not the given equation was even valid. Two of them used written calculations to eventually convince themselves of the eigenequation's validity, but two of them believed it to be an untrue equation. We include one example of each.

When student C3 saw the equation $\hat{S}_z|+\rangle_x = \frac{\hbar}{2}|-\rangle_x$, they repeatedly commented on wanting to check if the equation was correct:

> Now, I'm thinking, if this is correct, because we're in a different basis…Um, to check this I would s—rewrite the plus *x* state and the minus *x* state in the *z* coordinates, er the *z* basis, to check to see if this statement makes sense…My first thing is just to check it, to make sure it's correct essentially, 'cause I don't just see it just by looking at it…

C3 continued by writing the matrix representation of the $\hat{S}_z$ operator (in terms of the *z* basis) and used a reference sheet to know how to write $|\pm\rangle_x$ in terms of *z* basis. This information was given in Dirac notation on the sheet, but C3 wrote the vector notation form of the information to assist with their work on the problem. After they wrote a matrix-vector version of E3, they correctly carried out computations to confirm the equality of the two sides. They indicated a resolution to their work by then stating, "so these are the same," convinced the equation was correct. Similar to C3, student C2 seemed uncertain about the validity of E3:

> *C2*: This doesn't seem, I'm thinking about this not seeming to, this doesn't make the right sense. Because um, these [kets in E3] are not the same. And that's what I think about it [draws dash through equal sign to turn it into $\neq$]. Um, these are not the same eigenvalue--er, eigenvectors for one. Um, this matrix is not defined in this basis [points to $|+\rangle_x$]. So these eigenvectors, neither of them are the eigenvectors for this matrix…
>
> *Int*: So, because of those two things, that the, these two eigenvectors aren't the eigenvectors for this operator, and that the plus and the minus don't match, it makes you say that this is an untrue equation? Is that what you mean by the dash? Or do you mean?
>
> *C2*: Yes. Yep, Yep, that is not equal.

C2 recognized the mismatch between the operator and the basis vectors in Equation 3 and that the kets on the RHS and LHS were not the same, and concludes that the equation is not true.

Student reasoning about if it was an eigen equation. Eight of the ten students eventually determined that they knew that $\hat{S}_z|+\rangle_x = \frac{\hbar}{2}|-\rangle_x$ was not an eigenequation. All eight in some way discussed the two kets in the equation not matching, namely either by reasoning about co-existing distinct vectors (a static view of the equation) or reasoning about not getting same vector back (a dynamic view of the equation). The remaining two students displayed reasoning that indicated they knew Equation 3 contained some aspects that related to eigentheory but were not sure if it was an eigenequation. We share examples of each of these instances.

Reasoning about co-existing, non-matching distinct vectors was the most prominent way among the eight students of determining that Equation 3 was not an eigenequation. We think about this conceptualization as a static view of the equation. For example, C5 stated, "So not so much an eigenvalue equation because we don't have the same vector on either side." In this justification, C5 examined the structure of the equation, seemingly looking for a particular form – namely that of the same ket or vector on both sides of the equals sign. This is consistent with Sherin's work on symbolic forms (2001), which interprets students' understanding of equations in terms of pairing symbol templates with conceptual justifications for the structure of the equation. A second example of a student's reasoning why Equation 3 isn't an eigenequation that we categorized as static was C11's:

> I feel like it isn't, even though these are both eigenvectors. The down in the *x* and the up in the *x* are both eigenvectors. Uh, because they're different eigenvectors I feel like, I feel like this is saying if we operate, or if we project the spin *z* operator onto the um, up *x*, that we get a value of h bar over two. I think that it's a coincidence that this one is of the same form as these two [Equations 1 and 2].

It is noteworthy that C11 says "different eigenvectors," not "different vectors." We suspect that C11 knows that $|\pm\rangle_x$ are eigenvectors of the $\hat{S}_x$ operator and this is why they say they "are both eigenvectors." However, eigenvectors are special vectors of a particular operator, so if that operator isn't present, the vectors shouldn't be referred to as eigenvectors. Finally, we interpret C11's statement of Equation 3 coincidentally having "the same form as" Equations 1 and 2 to mean that all three have the symbolic form of *operator · ket = scalar · ket*.

Reasoning about not getting the same vector back as a way to determine that E3 was not an eigenequation was evidenced by two students. This conceptualization resembles a dynamic view of the equation, in which students are interpreting the role of the operator – namely, that of acting on or transforming the left-most "input" vector – and checking if they resulting "output" vector is the same as (or scalar multiple of) the input vector. For instance, C6 stated E3 wasn't an eigenequation "Because we're not getting the same thing back," and C8 stated, "Because the vector, the matrix acting on the vector is not the same vector that you get out over here." This highlights that the input/output imagery for function that is prevalent in K-12 and undergraduate mathematics courses can also be productively insightful in quantum mechanics.

Finally, the remaining two students displayed reasoning that indicated they were not sure if E3 was an eigenequation, although some aspects of it related to eigentheory. For example, C12 began by saying they found it "upsetting" the kets in E3 didn't match and remembered from class that "this [equation] flips the vector." Wanting to know if they thought of it as an eigenequation, the interviewer prompted further about if E3 had "eigen-things" for them:

> *C12*: So this is the, I mean this is the eigenvector, right? That's still an eigenvector. That's an eigenvalue of that. Um, but it doesn't like, I guess this one for me [points to E2] had a lot more like, yeah, that's the measurement. Whereas this one, if we're measuring down x, that's not going to be the measurement—it's going to be minus h bar over two. So whatever kind of analogy in my head you know I had like, 'this is the measurement. It's always the measurement.' I'm like, eh, um…yeah."

First, we notice C12's use of "the eigenvector" and "still an eigenvector." Similar to C11 above, C12 does not seem to attend to the paired nature of the operator-eigenvector relationship. Also, we note C12's mention of "that's an eigenvalue of that." It is true that $\hbar/2$ is an eigenvalue of the operator $\hat{S}_z$ from E3; however, it is actually an eigenvalue of *any* spin-½ operator $\hat{S}_n$, which the students interacted with frequently in class. Thus, it is understandable that the presence of a known eigenvalue would potentially be something a student considers when determining if that equation was an eigenequation. Finally, we note that C12's overarching deliberation about E3 seemed to be about measurement. Their explanation seems to indicate they have a strong sense of the spin eigenequation meaning measurement (with the eigenvalue being the measured value), but that is falling apart in E3 with the scalar not correctly matching the ket on the right hand side of the equation. It seems that C12 analogizes eigenequations with measurement in quantum mechanics, so they seem uncertain on how to think about E3 with respect to measurement.

Conclusion

Students often encounter the same concepts across mathematics and physics, yet they may develop varying meanings, notation, and interpretations of those concepts. We focused on analyzing students' imagery for eigenequations in linear algebra and quantum contexts, as well as how they compared the equations between the two contexts, noting language related to operating, (potentially) measuring and graphical interpretation. We also examined how students made sense of an equation that was symbolically similar to an eigenequation, noting both their attempts at physical interpretation and structural features of the equation. Our future work will employ a symbolic forms analysis to continue to synthesize results across equations and will develop subsequent implications for teaching based on student reasoning.

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In the Driver's Seat: Course Coordinators as Change Agents for Active Learning in University Precalculus to Calculus 2

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Mounting evidence of the effectiveness of active learning strategies has prompted some mathematics departments to engage in transformational change efforts. However, convincing instructors to change their teaching practices can be difficult. Some departments are using the coordination of multi-section courses as a vehicle to enact changes and have hired or designated course coordinators to oversee efforts to transform instruction via active learning. This study explores the role of coordinators as change agents for active learning using data collected from five university mathematics departments, all of which are successfully sustaining such efforts. Coordinators who act as change agents leverage three key drivers for change: providing materials and tools, encouraging collaboration and communication, and encouraging (and providing) professional development. Coordinators must also understand the local contexts and culture in order to engage in effective processes for supporting departmental change. Implications of this research are discussed for departments seeking to transform instruction.

Keywords: Course coordinators, departmental change, active learning

Introduction

In response to evidence that actively engaging students in learning is more effective than traditional lecturing (e.g., Freeman et al., 2014), more undergraduate departments are trying to change their teaching cultures. Transforming attitudes and practices related to teaching and learning--changing culture--is difficult, but possible in the presence of motivated and effective *change agents* (e.g., Kezar, 2014). Change agents provide the impetus for change while also attending to the change process and fitting the change efforts to the local culture.

One strategy that can help initiate and sustain changes is implementing course coordination for multi-section courses. Course coordination often starts with a common syllabus and textbook, and typically includes common assessments (homework, exams) and regular instructor meetings to discuss student learning and teaching strategies. Across the country, many universities have some level of course coordination for their introductory mathematics courses (Apkarian et al., 2019). Some departments have rotating coordinators, chosen from among instructors teaching that particular course that term; other departments have teaching faculty hired specifically to coordinate specific courses (typically Precalculus, Calculus 1 and Calculus 2). Given the direct access that coordinators have in regard to the introductory math courses, they are often well-positioned in departments to serve as change agents. In this study, we look closely at the roles of

Precalculus to Calculus 2 (P2C2) coordinators as change agents in five math departments who have successfully adopted and sustained active learning strategies in P2C2 courses.

Theoretical Framing and Purpose

To frame this study, we first carefully describe course coordination (and the roles of course coordinators) and then discuss theories of change and what is known about effective departmental transformation efforts. Course coordination describes a system for ensuring consistency across instructors and terms within a particular course, in order to support consistent experiences for students taking the course. Generally speaking, course coordination has two main components: (1) common course elements (e.g., exams, homework); and (2) regular instructor interactions around course instruction (Rasmussen & Ellis, 2015). Course coordinators, the focus of this proposal, are those charged with managing a course coordination system. While this role can be taken up in a minimal way, coordinators also have the potential to serve as powerful drivers of change. By creating course materials that lend themselves to active learning and steering conversations to a particular foci in course meetings, a coordinator can nudge instructors (and perhaps departments) toward using student-centered instruction and developing a community of practice which values teaching (Rasmussen & Ellis, 2015; Thaler & Sunstein, 2009; Wenger, 1998). In this proposal we explore the ways in which some coordinators have taken up their role to drive change within the courses they supervise.

Ongoing concern about the state of introductory university mathematics has brought about many change initiatives, which in turn has resulted in an increased focus on the study of change in higher education. Change initiatives (and studies of those initiatives) have focused on individuals, departments, and entire universities (e.g., Kezar, 2014; Henderson, Beach, & Finkelstein, 2011; Weaver et al., 2016). Many are beginning to suggest "the department" as a prime locus for change, as departments are relatively autonomous academic entities regarding their courses and their constituent members come from similar disciplinary backgrounds (Austin, 2011; Lee, 2007; Reinholz & Apkarian, 2018). Successful change initiatives need to be grounded in change research, understand the complexities inherent in transformational change, include a focus on the processes of change, and be adapted to fit local contexts (Henderson, Beach, & Finkelstein, 2011; Kezar, 2014; Kezar & Gehrke, 2015). The change initiatives in our study are centered on departments that have successful change initiatives that are focused on the institutionalization of active learning and have been sustained; course coordinators were an essential part of change efforts at all five of our sites.

Much of the research surrounding instructional change initiatives in postsecondary STEM departments emphasizes the challenges to such endeavors. In particular, limited or implicit theories of change do not always account for the complexity of the local context, and often result in superficial or unsustained changes (Kezar & Gehrke, 2015; Weaver et al., 2016). However, the sites we explore in this study all succeeded in institutionalizing active learning in their P2C2 course sequence, and so we consider them as examples of successful change.

We use recent work by Shadle et al. (2017) to frame the results of our study. Shadle and colleagues designed and enacted department-wide dialogues about changing instructional norms and culture with 12 different STEM departments at Boise State University. Each dialogue began with an introduction to STEM education followed by department members reading through and reacting to Boise State's vision about their culture of teaching and learning. Researchers' analysis of these conversations yielded categories of 18 barriers that faculty believed could stop (or were stopping) change within their specific contexts and 15 categories of drivers that could help (or were helping) overcome barriers and catalyze change. It is important to note that while some of the drivers can be viewed as ways of removing or getting over barriers, others are distinct and go beyond simplifying the change process. Many of the barriers that emerged are consistent with existing research, such as lack of resources (e.g., Weaver et al., 2016) or resistance to change (e.g., Kezer, 2014). Although less is known about the drivers of change in current research, the 15 drivers identified by Shadle and colleagues help illustrate and articulate how faculty view possible strategies for change, and their findings are consistent with research suggesting that the removal of barriers is not the same as supporting change (e.g., Johnson, Keller, & Fukawa-Connelly, 2018). Connecting Shadle et al.'s (2017) drivers for change and Rasmussen's and Ellis' (2015) characterization of course coordination in successful university calculus programs, we identified three key change levers coordinators are situated to take-up: Building common tools and resources; Encouraging collaboration and the development of shared objectives; and Promoting professional development

We use these drivers as a framework for examining data from five retrospective case studies of change in order to see how these speculative drivers were descriptive of how coordinators were influential as change agents in each story. In this paper, we describe how course coordinators leveraged their positions to act as change agents, focusing on drivers that are closely related to their roles. The research question guiding this study is: *In what ways have course coordinators leveraged their roles to drive the institutionalization of active learning in university Precalculus, Calculus 1, and Calculus 2 courses?*

Methods and Participants

The data for this qualitative analysis are drawn from a larger study, Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL), focused on identifying strategies for successfully implementing and sustaining active learning in P2C2 courses. Data were collected primarily during site visits in Spring 2017 through interviews with key stakeholders (e.g., course instructors, coordinators, department chairs, administrators, students, etc.) and observations of P2C2 courses. Audio recordings of the interviews were transcribed and analyzed in MAXQDA. Each interview was independently coded by at least two researchers, who then met to reconcile their coded segments (Creswell & Poth, 2018). The codebook was informed by institutional change literature, Bressoud, Mesa, and Rasmussen's (2015) characteristics of successful calculus programs study, and the site visits themselves. For this proposal, we selected interview segments related to the "coordination" code and analyzed these segments for evidence of drivers identified in Shadle et al. (2017).

The five universities in this study are all public R1 or R2 institutions with approximate undergraduate student enrollments between 19,000 and 31,000 students. Each of the five sites coordinates their P2C2 courses, although the degree and types of coordination vary by institution. We give the five institutions pseudonyms: All-in University (AiU), Long-Term University (LTU), Crossroads University (CU), Phased-Change University (PCU), and Critical Response University (CRU). Nearly all of the coordinators have (semi) permanent positions (at least three years as a coordinator for each course); a few coordinator positions are tenure-track faculty, but most are non-tenure track full time instructors. All of the institutions have multiple coordinators (typically one per P2C2 course). For more information about the larger study, including in-depth descriptions of the university contexts, see Smith et al. (under review).

Findings

Course coordinators are, by design, tasked with developing uniform course elements to be used across multiple sections of a particular course, facilitating communication among stakeholders, and overseeing professional development. These three aspects of the coordinators role are tightly tied to our focal three drivers: building common tools and resources, encouraging collaboration and shared objectives, and promoting professional development. We use these drivers to organize our results and include evidence of how course coordinators from each of the five sites utilized these drivers for change.

Building Common Tools and Resources

The details and responsibilities of course coordinators at each of the five sites evolved over time, but the nature of their role always involved *building common tools and resources* for instructors. It is in looking back at what participants said about coordinators that we are better able to identify the tools that were developed, how they changed, and why coordinators focused on these particular resources. We also see how these common tools and resources support instructors and can be used to implement instruction that promotes active learning.

One overarching reason pushing coordinators to build tools and resources was the goal that they would make the instructors' and students' lives easier. At CRU, the common homework, exams, and grading software are all linked to a common learning management system that was created and continues to be maintained by the course coordinator, which not only supports uniformity but also "makes it easier for each instructor too so they don't need to do much" (CRU Coordinator). At AiU, the coordinators are in charge of "designing all the midterms, all the finals, coordinating the rooms, proctors" (AiU Coordinator); similarly at LTU, "running the exams is one of the really big things. We [the coordinators] write the exams, we coordinate who's giving it where and run around keeping track of everything while it's going" (LTU Coordinator). Coordinators across sites also function as shared resources themselves, answering instructors' questions and handling issues related to grading, complaints, and general logistics. At all five sites, the centralized system of tools and resources made instructors' lives easier.

The common toolbox and resources provided by course coordinators are especially helpful for new and rotating instructors and coordinators, which is key for sustainability of these programs. Of the coordinator at PCU, another upper administrator noted that "she developed these courses in excruciating detail. And having done that, it's kind of in the can enough that's not hard to hand to somebody else who can then coordinate the course the way she designed it" (PCU). When asked about the value of the coordination system, the coordinator said,

> for uniformity in what's offered to the undergraduates in the courses, and also as a way [...] to support [GSIs] in reaching the level of teaching that we want [to] see happen and then in their own development. So they aren't out flying solo yet. (PCU coordinator)

This theme of consistency as a support for new instructors, be they new to the course or new to teaching entirely, was present in many interviews. A coordinator at LTU succinctly summarized the rationale for uniformity: "we want uniformity of [students'] experience and the standards, and part of the way we try to achieve that is by having a lot of support for the instructors, so even the brand new instructors, they have suggested lesson plans, suggested problems they can use."

Building common tools and resources are perhaps most obviously a driver for change when the resources nudge instructors toward particular teaching approaches. Course coordinators at these five sites know that helping instructors adopt active learning teaching practices is extremely difficult and can be overwhelming. As per the PCU coordinator, "to completely transform my classroom [...] that's too big a bite to eat and chew" and further that "it's too overwhelming" to have instructors who only have lecture experience overhaul their instructional practice in short order. To help instructors transition to using active learning strategies, coordinators provided them with resources geared toward active learning. For example, one coordinator at AiU noted that he designs discussion section activities so that "they are conducive toward having [students engage in] active learning," but also recognizes that "not everything can be turned into an active learning [task]." By creating these materials, this coordinator assists in the selection of group-worthy tasks, which is frequently a challenge for novice practitioners. In another case, the CRU coordinator provides new graduate students with a 6-step outline for running active breakout sections, including how much time to spend in small group work and guidelines for posing tasks and supporting student presentations. This document helps novice practitioners make complicated decisions in their classroom.

Encouraging Collaboration and Shared Objectives

Faculty members identify *encouraging collaboration and shared objectives* (Shadle, et al. 2017) as one of the most important drivers in upholding a shared vision for coordinating instructional practice and improving curriculum to support student success. In order to coordinate instructional practices, instructors need to be given the space, and have the desire, to engage in dialogue about teaching; coordinators at each of these five institutions played a critical role in facilitating these discussions, getting instructors to see the value in having these discussions, and centering these discussions on active learning principles. Furthermore, these coordinators encouraged instructors to collaboratively develop and improve course materials, contributing to an overall treatment of these courses as "community property" (Rasmussen & Ellis, 2015).

Although communicating logistical and administrative information about coordination is necessary, coordinators at each of the five institutions also deliberately communicate with instructors about teaching using active learning strategies. At most of the five institutions, coordinators lead weekly course meetings to communicate specific course objectives (e.g., content areas to highlight) and broader pedagogical objectives (e.g., encouraging student engagement, eliciting and building on student thinking). A coordinator at PCU explained, "I meet regularly with the instructors to clarify issues of content, scheduling, pace, any teaching issues that might arise, [and] help them with their personal growth as teachers." One critical advantage of these meetings is that teachers are growing alongside one another, forming a community of practice that supports a cohesive vision for these coordinated courses. In these meetings coordinators often set aside time for instructors to discuss challenges related to teaching and receive feedback from their peers about possible strategies for addressing these challenges. Because coordinators are the ones who lead these meetings, they are uniquely positioned to help establish an atmosphere of trust amongst instructors and also convey the importance of having these types of discussions in order to improve teaching practices.

Another way that coordinators help create a shared vision for courses is by including instructors in the development and continuous improvement of course materials. For example, coordinators at LTU actively encourage instructors to collaborate with one another. As one coordinator stated, instructors and coordinators "operate very much in a sort of team dynamic. There's not just one person who can do stuff, right? It's very collaborative." Encouraging instructors to treat coordinated courses as "community property" is critical in getting instructors to buy-in to active learning. As the department chair at AiU described:

> I think one of the biggest things is to get buy-in from the instructors, and allow the instructors to have some ownership in the entire process instead of just having it be top down...if the instructors don't feel like they have any ownership in the process then it's I think it's doomed.

By empowering instructors to take ownership of these coordinated courses, coordinators play an active role in ensuring active learning becomes integrated into departmental norms for teaching.

Promoting Professional Development

Across the five institutions, coordinator-led professional development to support the implementation of active learning took many different forms. All five sites offered some form of pre-semester teaching workshop for graduate students and other instructors (from a single day to a full week). The coordinators typically led or co-led these workshops. Some of the sites also had professional development workshops for faculty (single or multi-day). At PCU, the coordinator was able to bring an inquiry-based learning workshop to their campus; the convenience of the local workshop seemed to encourage more faculty to attend. Most universities also had some type of informal teaching mentoring, typically the coordinator observing instructors and providing constructive feedback. Regular instructor meetings led by coordinators can be a form of professional development, particularly when coordinators focus conversations on anticipating student difficulties or crafting activities and examples to meet specific objectives. One coordinator at LTU explained, "a lot of course meeting time was devoted to discussions of things that have happened in class... And we spent a fair amount of time [on] 'let's come up with five different ways you can explain what this arcsin is.'" Finally, having open discussions to listen to people's concerns can be a form of professional development. At CRU, a department leader explained that right before the semester, at the coordinator-led department meeting, they "broke out in smaller meetings and talked about the changes we were planning for that semester…. and I think that was very important to listen to everybody and to be thoughtful about it."

Coordinators at several institutions used co-construction and evaluation of assessments to provide professional development. At CRU, one faculty member noted the opportunities co-construction afforded: "the common exams and actually working on the exams, it seems like you have more collaboration than putting together the problems." In addition to assessments, coconstruction can also occur in curriculum development. In some instances more senior graduate instructors (under the supervision of coordinators) are even given the opportunity to write materials that are used across all sections of a course. Additionally, professional development content can be explicitly linked with the course content being taught.

Professional development for graduate students can have a slightly different focus compared to faculty members, since part of the mission of graduate school is to prepare graduate students for future (teaching) jobs in academia. Three of the universities have semester- or year-long courses for graduate student instructors, usually taught by coordinators to support graduate students' ongoing learning of effective teaching practices as they teach. A coordinator at CRU explained the purpose of the academic year graduate student course there: "The purpose of this seminar is to build your teaching skills and provide you with tools to be an effective teaching assistant. The course also offers insights and mentoring for a successful graduate career." The other two sites with academic year courses in teaching for graduate students had a comparable focus. One department leader at CU noted that the "pedagogy course has activities which are built on their experience in a classroom." Further, a coordinator at AiU explained how professional development is recognized as important for graduate students, in that it is necessary to train the graduate students "how to deal with students and to teach. Gets them ready for when they graduate. A lot want to be teachers." Effective professional development actively engages the target instructors; coordinators can help to frame the professional development as both useful and as a support for helping instructors realistically improve their teaching practices.

Discussion and Implications

As our findings show, the role of coordinator is positioned in ways to leverage different drivers for change. While we considered the drivers independently in our findings, effective coordinators must focus on multiple drivers to support active learning. For example, when coordinators are facilitating regular instructor meetings, they have opportunities to utilize each of the three drivers. As we saw with the coordinators at CU and LTU, coordinators may intentionally use regular instructor meetings to develop and revise coordinator materials, including worksheets of group-worthy tasks and outlines of active class sessions (*building common tools*). In this process, coordinators are promoting conversations among instructors about objectives for student learning (*encouraging collaboration and sharing objectives)* that naturally lend themselves to conversations about active learning pedagogies (*promoting [informal] professional development*). This is one of many examples where coordinators can leverage multiple drivers in their role to support the use of active learning in P2C2 courses.

We note that not every coordinator at every site actively leveraged all the major change drivers. However, across each team of coordinators, the group did collectively create and share resources and tools, encourage collaboration and communication, and provide professional development to instructors. Coordinators need to intentionally position themselves as change agents, and to be strategic about how to initiate, implement and sustain desired changes to infuse active learning across a department (Henderson, Beach, & Finkelstein, 2011). When coordinators act as change agents, their efforts are grounded in their local contexts. For instance, when one coordinator talked about starting with "low-hanging fruit," that person had taken the time to understand their local culture and contexts, and determine the best entry points to engage people in adopting active learning strategies.

Course coordination systems have previously been suggested as a way of transforming P2C2 programs from instructor-centered to student-centered by changing (or nudging) a set of courses all at once rather than individual instructors one-by-one (Apkarian et al., 2019; Rasmussen & Ellis, 2015). These systems must be led by effective change agents who proactively support transformation efforts and work to overcome barriers to change. Coordinators have the potential to serve as these change agents in departments that are looking to successfully integrate active learning into instructional practices. Thus, leaders in departments seeking to make these types of transformations through course coordination should carefully consider who they designate as course coordinators. Since coordinators are naturally positioned as change agents, departments should ensure that they have the power and resources that they need to leverage their role as coordinator. While the drivers presented in this paper are not necessarily "job requirements," our study specifies concrete ways in which coordinators can capitalize on their role to help shift instruction to support active learning. coordinators can capitalize on their role to help. By using drivers such as building common tools and resources, encouraging collaboration and shared objectives, and promote professional development, coordinators can help change the culture surrounding active learning practices in their departments.

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Interactions between Student Engagement and Collective Mathematical Activity

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In this small-scaled classroom teaching experiment we investigated the nature of six preservice secondary mathematics teachers' in-the-moment engagement and collective mathematical activity as they worked in pairs. Students participated in five 1-hour sessions focused on concepts of logarithms and relationships between linear and multiplicative change. We analyzed pair- and whole-group argumentation using Toulmin models (1958\2003) to understand collective mathematical activity and individuals' participation in argumentation. Self-reported data on student engagement were collected at two random times per session per student, which were further explicated through recall interviews. Results suggest active involvement in mathematical argumentation is not sufficient for high engagement. Instead, contributing to mathematical ideas functioning as-if-shared *which also* prompt deeper understanding of ongoing work align with states of relatively high engagement.

Keywords: Student Engagement, Mathematical Activity, Emergent Perspective, Flow Theory

Student engagement is a complex, metaconstruct consisting of emotional, behavioral, and cognitive components (Fredricks, Blumenfeld, & Paris, 2004; Middleton, Jansen, & Goldin, 2017). As defined by Middleton et al. (2017), engagement is "the in-the-moment relationship between someone and her immediate environment, including the tasks, internal states, and others with whom she interacts. Engagement manifests itself in activity, including both observable behavior and mental activity involving attention, effort, cognition, and emotion" (p. 667). Understanding student engagement and its role in educational settings has been key for explaining drop-out rates and persistence (Newmann, 1989), academic involvement (Finn & Cox, 1992), and "has been found to be one of the most robust predictors of student achievement and behavior in school" (Klem & Connell, 2004, p. 5). However, little research on student engagement in undergraduate mathematics classrooms has been conducted (cf. Peterson & Miller, 2004). Moreover, much of the work in K-12 mathematics settings focuses on engagement as an output or aims to associate student engagement with achievement in mathematics under the guise of grades or exam scores (e.g., Bodovski & Farkas, 2007; Lan et al., 2009; Martin, 2007; Rimm-Kaufman, Baroody, Larsen, Curby, & Abry, 2015; Robinson, 2013). This study extends research on mathematical engagement in undergraduate mathematics classrooms by investigating relationships between engagement and learning. We consider learning through developing mathematical conceptions and collective practices (Cobb & Yackel, 1996).

Theoretical Framework

Both student engagement (Middleton et al., 2017) and learning (Cobb & Yackel, 1996) have been conceived of as simultaneously social and psychological such that constructs from both planes are inextricably linked. That is, mathematical conceptions (psychological) and collective mathematical practices (social), for example, are co-constructed. Social and psychological aspects are interdependent. Thus, we propose appending a row for student engagement to the interpretive framework (Cobb & Yackel, 1996), offering a powerful means for aligning learning and engagement. Moreover, such an amendment to the interpretive framework makes posing significant research questions about relationships between student engagement and learning and interpreting results of such research possible. Figure 1 presents an aligned framework, where emotional and cognitive components of student engagement fall under the psychological column and behavioral engagement under social. Although these components make up aspects of engagement (Middleton et al., 2017), how these components are conceptualized requires theoretical backing. For this study, we view student engagement from the perspective of Flow Theory (Csikszentmihalyi, 1975, 1990), where engagement comprises of interest, enjoyment, and concentration (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003). From this perspective, interest and enjoyment are aspects of behavioral and emotional engagement, while concentration makes up cognitive engagement.

Construct	**Social Perspective**	**Psychological Perspective**
Collective & Individual Mathematical Activity and Learning	Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
	Sociomathematical Norms	Mathematical beliefs and values
	Classroom Mathematical Practices	Mathematical Conceptions
Student Engagement (flow theory)	Behavioral Engagement	Emotional & Cognitive Engagement

Figure 1. Theoretical alignment between Emergent Perspective and student engagement.

Research using the emergent perspective on learning (Cobb & Yackel, 1996) has traditionally investigated the reflexive relationships within rows of the interpretive framework (e.g., Yackel & Rasmussen, 2002). Within-row focused research has led to further development, yielding more robust understanding of collective and individual mathematical growth and gave rise to more in-depth research questions about the reflexive nature of social and psychological constructs (e.g., Rasmussen, Wawro, & Zandieh, 2015). Coordinating row-level constructs from the interpretive framework continues to be important; though, recently this body of work has led to examination of constructs across rows. Rasmussen et al. (in press) demonstrated reflexivity between social norms (social; top row) and emerging mathematical conceptions (psychological; third row) about infinite processes. Results from this work have two important implications. First, the study provides further evidence of the reflexivity between social and psychological development taking place in mathematics classrooms; and second, establishing cross column-row relationships between constructs of the interpretive framework allows for additional constructs with social and psychological correlates, such as student engagement, to be aligned with normative aspects of mathematics classrooms and learning. The present study coordinates analyses of individual student engagement with collective mathematical activity.

Methods

The purpose of this study is to investigate the nature of relationships and interactions between student engagement and learning. We address the research questions: (a) What are characteristics of collective mathematical activity during exceptional experiences of engagement? And (b) What is the nature of student engagement when contributing to collective mathematical activity?

Setting & Participants

To address these questions we conducted a small-scale classroom teaching experiment (Cobb, 2000). Six preservice secondary mathematics teachers – Uma, Sean, Alice, Frank, Ella, and Henry (pseudonyms) – participated in 5 one-hour sessions. Students selected a partner with whom to work; resulting pairs each consisted of one woman and one man. Alice was unable to attend the last two sessions. Her partner, Frank, worked individually in session 4 and with Uma and Sean in the last session.

The sessions focused on concepts of logarithms, where participants worked on the Timeline Problem (Confrey, 1991) in session 1. This task provides a list of 22 events ranging from "Now" at 0 years ago to "Big Bang" at 1.5×10^{10} years ago. Students are prompted to construct a timeline for these events, and then posed questions requiring them to use their timeline to compare various amounts of elapsed time. Subsequent sessions and prompts were designed based on students' emerging understandings of multiplicative and additive change. An important contextual note is that students developed models with linear scales within segments delineated by powers of ten – resembling scientific notation. In session two, students were presented with a hypothetical student's timeline which used a completely multiplicative scale with base 2.

Data Collection & Analysis

Sessions from the teaching experiment were video and audio recorded. These data were analyzed to document collective mathematical practices, following Stephan and Rasmussen's (2002) procedure to monitor mathematical ideas emerging during argumentation as they function as-if-shared. This procedure consists of three steps: 1) diagram arguments using Toulmin's (1958/2003) model, 2) develop an argumentation log to track the function of mathematical ideas as they function as-if-shared, and 3) organized such ideas into collective mathematical practices. Mathematical ideas are determined to function as-if-shared when either backings/warrants from one argument are no longer present in subsequent arguments, or when any element of an argument functions differently in future arguments without being questioned.

In-the-moment student engagement was captured through the experience sampling method (ESM) (e.g., Shernoff et al., 2003). Participants were sent a 5-item Likert-type survey to their phones at two random times each session, and were instructed to respond immediately upon notification. ESM data have nested structure, which allows for analyses to describe each participants' typical levels of interest, enjoyment, concentration (i.e. engagement), perceived skill and challenge as well as within- and between-student fluctuations. In-the-moment student engagement was taken to be the sum of interest, enjoyment, and concentration for each response. Technical issues rendered ESM data from session 1 unusable. Sean did not respond in session 5.

Lastly, ESM responses and video data were used to develop stimulated recall interviews, which took place after the teaching experiment. Participants were asked about their overall interest, enjoyment, and concentration from the study, and were asked to reflect on these aspects when showed three clips. The first clip showed an instance when the interviewee reported either personally high or low levels of engagement – operationalized as one standard deviation above/below their personal average. The second clip showed an instance where the interviewee made a meaningful mathematical contribution. The third clip showed an instance where women made a *potentially* meaningful mathematical contribution while working in pairs, which was either not pursued or ignored by her male partner. This clip was the same for each pair. Interviews were transcribed and coded to understand each participants' engagement. These analyses were merged to explicate links between engagement and mathematical activity. Results presented focus on discussions of the first clip. Ella chose not to be interviewed.

Results & Discussion

Table 1 presents student engagement reported on ESM surveys. Overall, Uma reported being less engaged than her peers for much of the teaching experiment and her engagement varied more than others. Uma's experience during session four appears to have been substantially different from other participants. On the other hand, Henry consistently reported being more engaged than others. We draw on ESM data to discuss mathematical activity taking place when students reported exceptionally high or low levels of engagement, operationalized as one standard deviation from individual means. These were instances selected for clip 1 and discussed during recall interviews. Also note the second portion of session 2 was the only time in which more than one student reported being less engaged than normal, warranting investigation of mathematical argumentation of that period. Pairs were: Uma & Sean, Alice & Frank, and Henry & Ella. Frank worked with Uma and Sean during session five (Alice only attended sessions 1-3).

Table 1. Engagement (sum of interest, enjoyment, & concentration) from ESM responses.

	Session 2		Session 3		Session 4		Session 5	
	R1	R2	R1	R2	R1	R2	R1	R2
Sean	4^L	12	14	12	15	12		
Uma	8	3^L	6	11^H	3^L	3^L	6	7
Alice	9	9	12	12				
Frank	12	9^L	10	13	12	11	14^H	11
Henry	15	15	11	13	15	14	15	15
Ella	10	11	10	11	10	10	12	14^H

Note: ^{H} Personal high; ^{L} Personal low

Mathematical Activity and Exceptional Experiences of Engagement

The first clips from interviews featured participants during times when they reported personal high or low points of engagement on the ESM survey. For Uma and Frank, these clips showed times when they reported above average engagement, while Sean and Henry were showed instances of below average engagement. Alice's ESM responses about engagement did not fluctuate much. Her clip depicted a moment when she reported being under-skilled.

Exceptional experiences of engagement in ESM results aligned with students' involvement in establishing mathematical ideas as-if-shared. High engagement reported by Uma and Frank corresponded with their active participation in collective activity. For example, Uma authored the mathematical idea *equivalent models,* which designates students' reasoning about similarities between their model (linear within powers of 10) and the hypothetical model (multiplicative). Figure 2 diagrams argumentation starting with Uma's claim and ending with Sean's use of *equivalent models* when responding to a teacher's questions. Uma and Sean were discussing how successive equal-length segments in their model had a 10:1 ratio in the amount of time represented and were investigating the same phenomenon in the hypothetical model.

In the example, Sean uses calculations to provide a warrant and backing to Uma's claim. Then, in the next argument, Sean responds to the teacher's question with a claim using the mathematical idea *equivalent models,* which was unchallenged by Uma. When first introduced, this idea was accompanied by a warrant and backing which were no longer needed, suggesting it was functioning as-if-shared within this pair. Moreover, discussions stemming from Uma's original claim led the pair to further investigate differences between their model and the hypothetical model. They later concluded that their model had similar qualities to the hypothetical model, but the linear relationship between distance and elapsed time in their model was a key difference between the two. Uma states, "So that is gonna make that relationship different from the one we found in ours. Ours is going to be lines of different slopes. The other one is gonna be- like a curve [Sean completes this sentence]." Thus, Uma's most engaging moment occurred when her involvement in collective mathematical activity established ideas that function as-if-shared *and* which prompted deeper understanding of previous work.

Claim 1: (*Uma*) In the hypothetical model, I would think that the same reasoning applies

Data 1: (*Uma*) …because between- the relationship should still translate from the hypothetical model to our model

Warrant 1: (*Sean*) Regardless of the base, 2 or 10, the same-

Backing 1: (*Sean*) Here we have $2^6 = 64$ and $2^4 = 16$ and $2^2 = 4$, so they are… there is 4:1 between those segments.

Claim 2: (*Sean*) All that truly changed is the base, the base 10 to 2. I think we have the same phenomena and that the same reasoning applies.

Figure 2. Argumentation during Uma's most engaging moment.

Similarly, Frank's reported highpoint in engagement took place when he was contributing mathematical ideas that were becoming as-if-shared and which prompted deeper understanding of ongoing work by his group. His highest reported level of engagement took place during session 5 when he was working with Uma and Sean. At the time, they were interpreting graphical representations of elapsed time and distance – two quantities from the timelines. Frank introduced the mathematical idea of *covariation* as backing to a claim about the relationship between these two quantities in the graphical representation of the hypothetical model. Later in the session, Sean used *covariation* as a claim to establish a functional relationship between elapsed time and distance, which was unquestioned by Uma or Frank. That *covariation* as a mathematical idea shifted function in arguments – from backing to claim – without challenge is indication that it was functioning as-if-shared by the group. In addition, Frank's initial contribution asserting *covariation* as a mathematical idea catalyzed a shift in the group's focus from reasoning about elapsed time alone to reasoning about simultaneous change in both quantities. In sum, similar to Uma's most engaging moment, Frank's also accompanied involvement in mathematical ideas functioning as-if-shared that also furthered group understanding of ongoing work. These characteristics of collective mathematical activity were absent during students' reported lowpoints in engagement.

When students reported low levels of engagement, mathematical ideas being established did not meet criteria for functioning as-if-shared (Stephan & Rasmussen, 2002) during the same session of the teaching experiment, regardless of whether students were actively contributing to mathematical argumentation. Additionally, when students were participating in collective mathematical activity, the nature of their contributions were based either on observations or calculations. By this we mean that data or warrants in argumentation derived from reading prompts or timelines (i.e. observations) or were grounded in calculations. This is not to say that observations and calculations are unimportant for collective and individual development. Instead, we take this to indicate mathematical ideas of these types did not *need* to become as-if-shared because of broader sociomathematical norms (Cobb & Yackel, 1996) these students were operating under as juniors in a secondary mathematics education program. For these students, participation in argumentation as observational or calculational was generally accepted and tended to be less engaging. Additionally, this result suggests normative aspects of mathematics classrooms may influence relationships between student engagement and learning.

Nature of Student Engagement when Contributing to Collective Mathematical Activity

The previous section described characteristics of collective mathematical activity taking place in moments when students reported personally exceptional experiences of engagement. In this section, we examine this relationship in the reverse direction. To begin, two primary themes emerged as factors contributing to students' engagement from recall interviews: *familiarity* and *working with a partner*. Familiarity manifested as *familiarity within the sessions*, which typically involved students expressing beliefs that the timeline context became repetitive. For example, Frank stated, "I'd say the only thing that got old was the repetitiveness of we were doing one task continuously." The second form was *familiarity from prior experiences*, which stemmed from students correlating prior experiences learning about exponentials and logarithms to their engagement. Although Uma expressed excitement about being "in an actual learning math class again", students tended to associate familiarity with decreased engagement. This may explain Frank and Uma's reported low levels of engagement during the second half of session 2.

Session 2 began with participants completing their timelines from session one and using these models to finalize their responses to prompts. In the latter half of the session, students were presented with a hypothetical student's model and posed similar prompts to address while reasoning about the new model. Familiarity might explain low levels of engagement reported by Frank and Uma at this time because of the repetition of prompts and context, in Frank's case; and for Uma because the hypothetical model was constructed with a logarithmic scale (though this was not explicitly mentioned to students at the time). In fact, for Frank, this association with his engagement might overrule effects of actively participating in collective mathematical activity.

Frank was the main contributor to mathematical argumentation at the time of his lowest reported level of engagement during session 2. At the time, Alice and Frank were using the hypothetical model to answer the prompt, "Consider events A, positioned at 16.005, and B positioned at 18.013. Which event took place longer ago and by how much? Explain." Figure 3 diagrams argumentation that took place between the pair when Frank responded to ESM survey.

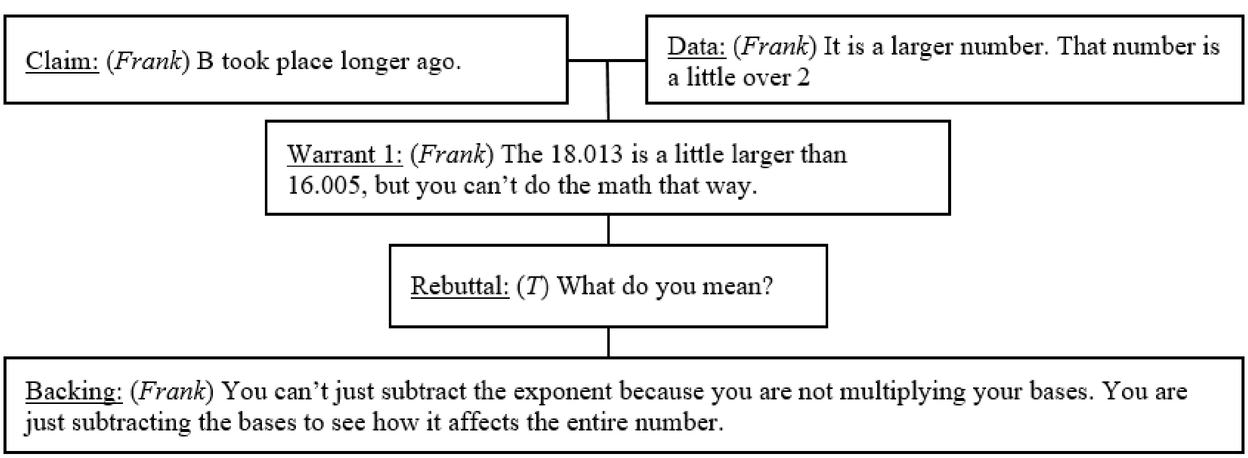

Figure 3. Argumentation for Alice & Frank during session 2.

That Frank was actively contributing to mathematical activity at the time of his lowest reported level of engagement suggests participation in mathematical activity is not sufficient for high engagement. Instead, factors associated with engagement, group dynamics or other normative aspects of classrooms may influence the association between student engagement and collective mathematical activity. On the other hand, Uma's engagement during the second half of session 2 was low, but unlike Frank's experience, Uma was not actively contributing to argumentation between her and Sean at the time. Thus, despite expressing high emotional engagement associated with being positioned to learn mathematics again, her lack of familiarity with underlying concepts explored in the teaching experiment may have contributed to her involvement in mathematical argumentation as well as her engagement. In fact, Uma describes feeling unconfident because of her lack of prior knowledge, "without [Sean], I probably kind of would have just sat there like staring… I would have liked picked some points and put them on there, but I wouldn't be very confident." This quote also demonstrates the impact of *working with a partner* on Uma's engagement and hints at this theme's influence on her involvement in mathematical activity. She further elaborates, "I didn't necessarily know what to put on my own papers, I was like before I do anything I wanted to like see where [Sean] was going with it."

Conclusion

Results from this study have demonstrated that states of in-the-moment student engagement are associated with collective mathematical activity. However, simply participating in mathematical argumentation is not sufficient for promoting high levels of engagement. Instead, when students in this study reported high levels of engagement, characteristics of mathematical activity included involvement in establishing mathematical ideas that functioned as-if-shared which also led to further understanding of ongoing work. Also, factors associated with student engagement emerging from recall interviews may explain the nature of students' involvement in mathematical argumentation. These associations suggest that student engagement and collective mathematical activity are reflexively linked, demonstrating "the *fundamental inseparability* of learning from the engagement through which learning takes place" (Middleton et al., 2017, p. 667). Additionally, that certain types of evidence provided in argumentation, in this case observational or calculational, correlate with lower levels of engagement provides evidence that sociomathematical norms may also influence the relationship between student engagement and mathematical activity. Although this result needs to be further explored, it would imply that engagement experienced by participants in this study was in fact associated with normative aspects of a *mathematics classroom*. This would further the contributions of Rasmussen et al. (in press) demonstrating the role of social norms on mathematical development. Finally, this study has contributed how amending the interpretive framework (Cobb & Yackel, 1996) to include a row for student engagement allows for research to investigate relationships and interactions between student engagement and collective mathematical activity.

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Calculus for Teachers: Vision and Considerations of Mathematicians

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Acknowledging the significant contribution of mathematicians to the mathematical education of teachers, we explore the views of mathematicians on an envisioned Calculus course for prospective teachers. We analyzed the semi-structured interviews with 24 mathematicians, using the EDW (Essence-Doing-Worth) framework (Hoffmann & Even, 2018, 2019); and subsequently, we adapted the framework by extending and refining the existing themes. The findings of our study indicate that the mathematicians believe the primary purpose of a Calculus course for teachers is to communicate the nature of mathematics as a discipline. By providing a variety of examples that could shape and expand the teachers' understanding of mathematics, the majority of the mathematicians participated in the study emphasized the value of mathematical investigation in an envisioned Calculus course for teachers, as well as connections within and beyond the subject.

Keywords: advanced mathematical knowledge, mathematicians as educators, calculus, teacher education

Calculus is one of the standard mathematics courses offered to a wide range of student populations in most universities. Special Calculus courses are usually designed for engineering, life sciences, or business majors. While prospective secondary mathematics teachers are often required to take one or two Calculus courses at university, a special Calculus course designed for teachers is rarely available. Research on the mathematical development of prospective teachers has shown that learning Calculus at the undergraduate level enriches prospective teachers' understanding of the mathematical concepts introduced in school (e.g., Keene et al., 2014; Wasserman & Weber, 2017). These research findings make us wonder how a Calculus course for teachers would look like. In this paper, we report on mathematicians' views of a Calculus course designed specifically for prospective teachers.

Theoretical Underpinnings

Research Background

The relevance and contribution of university-level mathematics to the education of secondary school mathematics teachers have been debated for decades. Nevertheless, a broad consensus reached among researchers is that mathematics teachers at the secondary level need to have insight into advanced mathematics (e.g., Dörfler & McLone 1986; Ferrini-Mundy & Findell, 2001; Murray et al., 2017; Winsløw & Grønbæk, 2014).

Zazkis and Leikin (2010) described advanced mathematical knowledge (AMK) as knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college. Mathematics as a scientific discipline taught at university has an axiomatic-deductive structure and focuses on the rigorous establishment of theory concerning definitions, theorems, and proofs (Klein, 2016). This approach is often adopted in mathematics courses for prospective teachers to help them gain advanced mathematical knowledge (AMK) and develop advanced mathematical thinking (AMT). Tall (1991) examined differences between elementary and advanced mathematical thinking as transitions from *describing* to *defining*, from *convincing* to *proving* based on abstract entities. These transitions are often considered challenging by prospective teachers. In fact, research on secondary mathematics teachers' conceptions of the role and usage of AMK in their teaching practice, has shown that while some teachers acknowledged its importance, others are unaware of the connections between advanced and secondary mathematics, and are often dismissive of their upper-level training (e.g., Goulding, Hatch, & Rodd, 2003; Even, 2011; Zazkis & Leikin, 2010).

Dreher Lindmeier, Heinze, and Niemand (2018) pointed out that the gap between school mathematics and the advanced mathematics taught at university is often too wide, and as a result, it is difficult for prospective teachers to make connections between the two. A few studies have sought possible connections related to mathematics content and disciplinary practices. Concerning mathematics content, Jukic and Brückler (2014) showed that Calculus tasks that connect embodied conceptions and symbolic manipulations promote flexibility in mathematical thinking of prospective teachers. Concerning disciplinary practices, Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos, and Weber's (2017) research explored how the study of proofs in real analysis could be used to enhance the teachers' ability to engage in quality instructional practices at the secondary school level. Other studies shed light on how abstract algebra might support teachers to unpack particular secondary mathematics topics and how they might shape pedagogy in the secondary classroom (e.g., Christy & Sparks, 2015; McCrory et al., 2012; Murray et al., 2017). However, the gap between advanced and school mathematics is still evident. To support prospective teachers in making possible connections, researchers need to further explore the relationship between advanced and school mathematics.

Leikin, Zazkis, and Meller (2018) noted that as mathematicians teach mathematics to prospective teachers, they "act as teacher educators de facto, without explicitly identifying themselves in this role" (p. 452). As such, understanding mathematicians' views is relevant in an effort to facilitate change in undergraduate mathematics teaching that supports prospective teachers. Moreover, Bass (2005) and Hodge et al. (2010) claimed that mathematicians' knowledge, practices, and habits of mind are essential for maintaining the mathematical balance and integrity of the educational process.

Research on mathematicians' views regarding advanced mathematics studies of teachers has not received much attention. The existing empirical research literature is mainly based on interviews with teachers. Only few studies sought the relevance and contribution of academic mathematics studies to secondary school mathematics teaching, taking into account the views of mathematicians in curriculum planning and course design (e.g., Goos, 2013; Hoffmann & Even, 2018; Leikin, Zazkis & Meller, 2018). Nevertheless, many questions remain. For instance, which connections between advanced and school mathematics are important, and how they might lead to the improvement of teachers' practice (Murray, Baldinger, Wasserman, Broderick, & White, 2017).

Theoretical Framing

Ziegler and Loos (2014) identified two dimensions of mathematical knowledge that appear to be critical for teaching: one dimension is knowledge of specific topics, concepts, and procedures, and the other is a more general epistemological knowledge about what mathematics is and what doing mathematics entails. Focusing on the latter dimension, Hoffmann and Even (2018; 2019) identified three aspects of mathematics that the research mathematicians wanted teachers to acquire in their study: (1) the Essence of mathematics, (2) Doing mathematics, and (3) the Worth of mathematics. Each of these aspects includes several sub-themes, a total of nine sub-themes can be found on the left-hand side of Figure 1. These aspects form the Essence-Doing-Worth conceptual framework (henceforth referred to as the EDW framework) intended to serve in studying the relevance and contribution of academic mathematics courses to the teaching of mathematics in secondary schools, and in analyzing teachers' views on what mathematics is.

As we are concerned with potential affordances of a Calculus course to the education of secondary school mathematics teachers, the EDW framework is highly relevant to our study (we additionally expanded this framework, as described in the next section). In particular, we are interested in what mathematicians wish to teach in a Calculus course for prospective teachers. In this paper, we address the following research questions: *How do mathematicians envision a Calculus course designed specifically for teachers? What particular features do they consider important in such a course?*

Method

Participants and Data Collection

24 mathematicians from 10 research universities participated in our study. All the participants were Mathematics Faculty members, representing a variety of specializations within mathematics. All participants have taught a Calculus course in the past or at the time of the data collection.

The mathematicians participated in individual semi-structured interviews aimed to gain insight into how they envisioned a Calculus course designed for teachers. The following guiding questions were posed to the interviewees:

1. What would you like teachers to know and experience about the mathematics taught in university?
2. If you were to design a Calculus course for teachers, how would you adapt an existing Calculus course?

These questions were followed by prompts for expansion and elaboration, as necessary. All the interviews were audio-recorded and transcribed. Additional written artifacts generated by the interviewees were collected for qualitative analysis.

Data Analysis

The interview transcripts were analyzed using iterative and comparative data analysis with the assistance of Nvivo 12. The nine themes from Hoffmann and Even's (2019) conceptual framework were set up as the initial thematic codes in a hierarchical structure (see left-hand side of Figure 1).

In the first round of the data analysis, relevant quotes in the transcripts were coded by theme. As the analysis proceeded, several new themes emerged: *joy, developing creativity, human endeavor,* and *investigating through technology.* In the second round, the connections between initial and emerged themes were explored and identified. The two initial themes *wide and varied* and *lively and developing* were combined for a new theme *human endeavour*, given that two themes were frequently mentioned together. In addition, a third sub-theme *the worth of mathematics as human activity* was added to the main theme *the worth of mathematics*.

In the third round of the data analysis, the quotes assigned to the frequently mentioned themes *investigating* and *using intuition and formalism* were re-examined due to multiple meanings given by the participants in relation to these themes. As a result, we refined the theme *investigating* to three sub-themes: 1) investigating for learning, 2) investigating for teaching, and 3) investigating through technology. Similarly, the theme *using intuition and formalism* was also refined to three sub-themes: 1) use/confront intuition, 2) formalization, and 3) mathematical language and notation. (In the next round of our data analysis we will focus on refining additional recurring themes *thinking and understanding* and *the practical worth of mathematics*)

The right-hand side of Figure 1 shows an overview of our adaption of the EDW conceptual framework. The combined, refined, and new themes are shaded in grey. The numbers in parentheses at the end of each theme indicate the number of the mathematicians, out of 24, who have discussed the corresponding theme during the interviews.

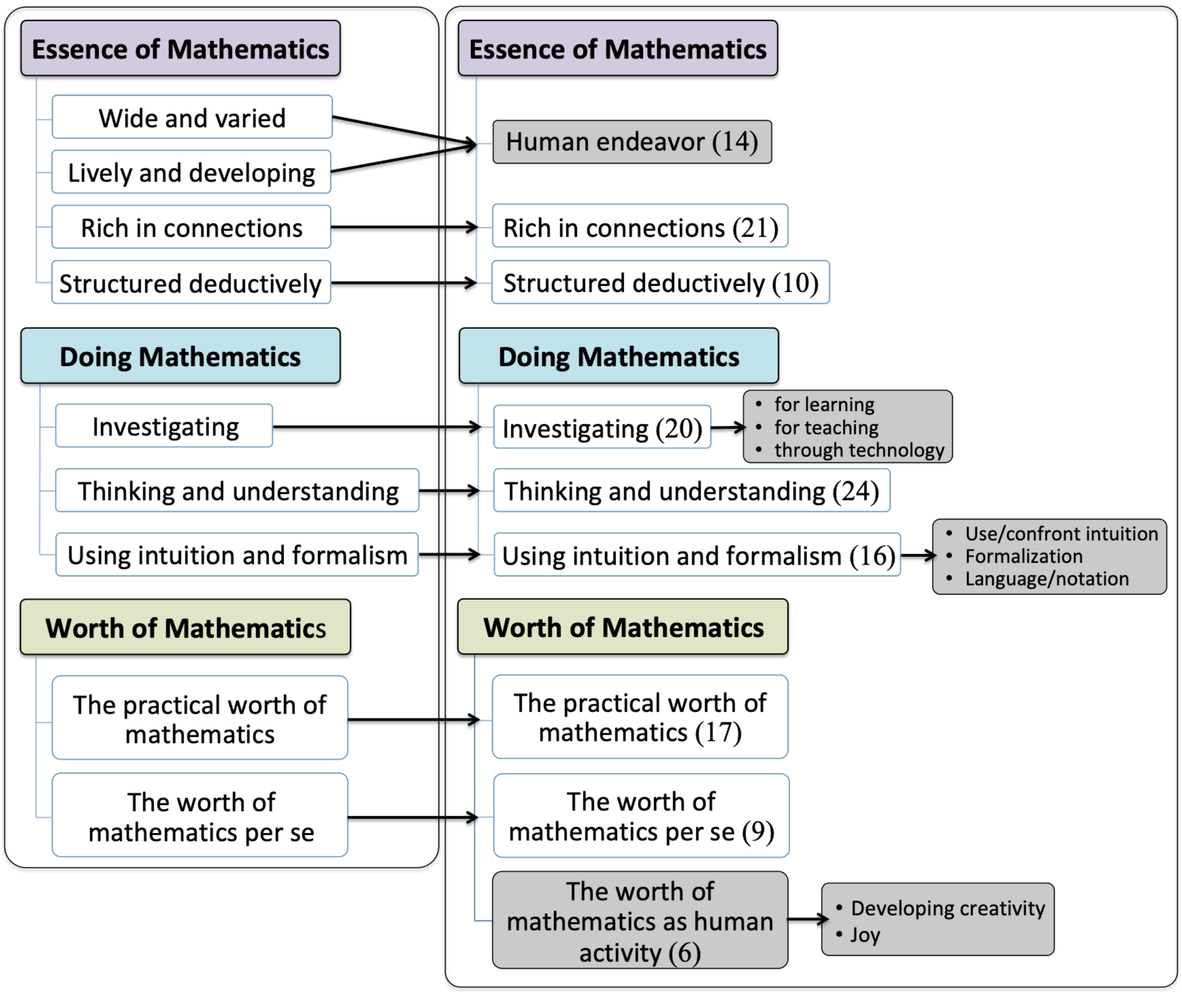

Figure 1. Hoffmann and Even's EDW conceptual framework (on the left) and our adaption (on the right).

Findings

As shown in Figure 1, the mathematicians' responses speak to all the initial themes in Hoffmann and Even's (2019) conceptual framework; however, some themes were mentioned more frequently than others. For example, all 24 participating mathematicians explained how *doing mathematics* enriches the thinking and understanding of the subject. Taking into account the needs of prospective teachers, 20 mathematicians particularly emphasized the important role of investigating in the sense of *doing mathematics*. Additionally, a vast majority of the mathematicians discussed connections between mathematics branches as well as connections to other disciplines.

We present the results of our analysis by focusing on two themes: 1) the essence of Mathematics/Calculus – rich in connections, and 2) doing Mathematics/Calculus – using intuition and formalism. These two themes were chosen based on 1) the limited literature on possible connections between advanced and secondary mathematics, and how these might lead to the improvement of teachers' practice, and 2) an understanding of the key concepts in Calculus, such as derivatives, requires both intuition and formal reasoning. In what follows, we elaborate on these themes. We refer to the participating mathematician as Mi, where i = 1…24.

On the Essence of Calculus – Rich in Connections

Three participants mentioned that regardless of which Calculus course they teach, approximately 70 percent of the course contents overlap (M11), and this is precisely the core – "the most valuable component of a Calculus course" (M9) that they would like to teach prospective teachers. M1 believed that rather than adding another "flavor" to Calculus, generic Calculus would benefit prospective teachers the most. He explained:

> When Calculus is a flavor, there are two things that happen. One is level. You can have more or less proof, more or less theorems, more or less explanation compared to the technical stuff. That is a decision that can be made in a particular course. The other is so-called "word problems in situations that are considered." I am not convinced that the "real" problems that are posted are valuable. Instead of maximizing a function, call that function a revenue function, so suddenly, it is Calculus for business. I think everyone can learn generic Calculus. […] You need to learn the essence.

One critical aspect of the essence of Calculus lies in its richness in connection. As M11 put it, "Calculus is less like a bucket of techniques that you carry around with you but more like a reticulation of ideas that are all at some level connected. To learn mathematics is really to learn about the connections." In the same spirit, M7 argued that the purpose of a Calculus course for teachers, if appropriately tailored, would be to help teachers see the connections between concepts and value these connections.

M6 suggested cutting off about a third of the topics in a regular Calculus course to open up space so that prospective teachers could have time to dig deeper into the essentials – "make it less of a race and more of exploration." M10 also suggested focusing on a handful of central ideas in Calculus rather than cover many topics at a very shallow depth. With interest in making connections between seemingly unrelated concepts, M19 envisioned his Calculus course for teachers with far less content but more time allocated in class for connecting concepts:

> Upon considering the choice of what it is that we ought to teach high school teachers, I think we should pay attention to how deliberate should we be in connecting what we teach high school teachers to what they are going to teach in high school. Giving the teachers really interesting mathematics for an almost purely appreciative goal is maybe insufficient. […] It would be great if teachers think of Calculus as a part of a coherent network of ideas that constitutes mathematics. And I think one way to encourage that view of mathematics is to teach it that way to the high school teachers.

"*Rich in connections*" was also interpreted by the mathematicians in the sense that a mathematical concept can be approached and explained using multiple models and multiple representations. More importantly, developing the flexibility of switching between models and representations should be the goal of such a Calculus course (M4). Taking "π" as an example, M15 argued that while the most usual definition of π is the ratio of the circumference of a circle to its diameter, π could be defined in various ways:

> For example, you can define π analytically: you can define $\frac{\pi}{2}$ to be the place where cosine is 0, having defined cosine as a power series [$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$]. Then say, well, let's define π as twice the smallest positive zero of the power series. And then you can derive all the other properties. […] I think this is closer to where they [teachers] are living because they have known π like since Grade 6... Now that is advanced mathematical knowledge that high school teachers need.

In a Calculus course for teachers, M15 would further challenge teachers by asking why πr^2 is the area when $2\pi r$ is the circumference. "That needs some explanation. And that actually does come up in the first year of Calculus."

On Doing Calculus – Using Intuition and Formalism

Mathematics is not a spectator sport (Phillips, 2005). "Roll up your sleeves and do it" (M10) because "mathematics embodies structure and one has to spend time doing it to understand the structure" (M9). In discussing what it means to "do Calculus", the mathematicians placed value on building and using intuition to understand concepts in Calculus. For example, to introduce the idea of what a limit is, M14 imagined teachers using a computer to plot curves and experiment with the derivatives of those curves:

> I would have them pick a point on the curve. I would have it set up so that they could zoom in repeatedly over and over and over again until eventually they see it looks like a line. […] Where that intuition comes from? To me the intuition comes from making a physical change and seeing what the physical outcome is.

While building and using intuition is an important aspect of doing Calculus, the mathematicians also pointed out situations in which intuition needs to be formalized. For example, M11, M19, and M23 used a typical question on the rate of change to show the necessity of formalization: If a car has traveled 90km in one hour on a road with an 80km/h speed limit, did the car break the speed limit? Intuitively, the answer is obvious. Justifying the intuition, however, requires a careful examination of the global and local behavior of a function, and an understanding of average versus instantaneous speed. The transformation from intuition to formalization is often challenging, yet this is where the fundamental concepts in Calculus come into play (M23).

In addition to formalizing intuition, the mathematicians also discussed situations in which intuition and mathematical formalization need to be reconciled. M11 stated that,

> It is dangerous to believe your intuition is always right. But you should trust your intuition because you want to believe by reasoning through you will have a deeper understanding. […] It is intuition, math, intuition, math… that kind of pattern.

Furthermore, M11 provided two related problems that he would work with students while attending to their intuition:

1. If a full bucket of water is poured into a small cup, then the cup is going to flow over. If this bucket of water is poured into the Lake of Ontario, have the beaches been flooded?

2. One makes a metal piece of track around a basketball tight. If the metal track is increased by a meter, then there will be a gap between the basketball and the track. Now replace the basketball with the earth, would this one-meter make any difference?

In explaining how intuition may support or hinder reasoning, M11 stated:

> Your intuition is a key guide. And in most of the time in a Calculus course, intuition is really something to be developed. But you also want to use the mathematics to verify whether your intuition is correct. And sometimes a simple mathematical tool, the idea of relative sizes, in this case, can actually show you where you have misconceptions.

In M11's view, it is crucial for teachers to learn to confront intuition with mathematical tools and reasoning. This resonates with M14, "I would put much more focus on knowing when to use what tool to evaluate whether you have a sensible answer."

In line with M11 and M14, M18 shared her experience in teaching derivatives, which highlights the role that counter-examples play in challenging one's intuition. As M18 explained, in a first-year Calculus course, many students could get an intuitive understanding that if the derivative is positive at a point then the function is monotonically increasing on the interval around that point. This intuitive understanding can be challenged by exemplifying oscillating functions and can be further corrected as follows: if the derivative is positive on an *interval*, then the function is monotonically increasing on that interval, by the Mean Value Theorem. As stated by M18, "There are these weird functions that have oscillations, which is not obvious. […] However, counter-examples break their [students'] intuition and allow them to appreciate why a theorem is needed."

Conclusion

Acknowledging the significant contribution of mathematicians to the mathematical education of teachers, we set to address how mathematicians envision a Calculus course designed specifically for teachers and which features they consider important in such a course. Utilizing the EDW framework (Hoffmann & Even, 2018, 2019), we analyzed interviews with 24 mathematicians, as a result of which the framework was extended and refined.

Our findings suggest that mathematicians believe the primary purpose of a Calculus course for teachers, if such a course were to be designed, is to communicate the nature of mathematics and to provide prospective teachers with opportunities for mathematical investigations. This resonates with Ziegler and Loos' (2014) discussion on the importance of knowing and understanding the nature of mathematics and what constitutes doing mathematics. This is also in line with Burton's (1998) argument that mathematicians often see connections set within a global image of mathematics as an important part of knowing mathematics. With particular attention to Calculus, the mathematicians provided a variety of examples that can shape and expand the teachers' understanding of mathematics, and in turn, contribute to the teaching of mathematics in school.

The contribution of our findings is twofold: a) we expanded prior research on the role of mathematicians in teacher education by focusing on an envisioned Calculus course for teachers; and b) we extended and refined a theoretical framework which we believe would be useful in future research on how advanced mathematical knowledge may serve secondary mathematics teachers. To conclude, the mathematicians' broad visions of how to teach Calculus to prospective teachers deserve attention among mathematics educators and researchers. The findings of our study call for further research that focuses on mathematics courses designed specifically for teachers and their potential contributions to teacher development.

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Interpreting Undergraduate Student Complaints about Graduate Student Instructors through the Lens of the Instructional Practices Guide

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College and department administrators take undergraduate student complaints about Graduate Student Instructors (GSIs) seriously. However, little research has been done to examine the nature of undergraduate student complaints across multiple mathematics departments from the lens of student-centered instruction. In this study, we compared formal (i.e. documented in writing by the student) undergraduate mathematics student complaints about GSIs at two universities over five years. Complaints were analyzed by coding the contextualized concerns described in the complaints using the Mathematical Association of America's Instructional Practices Guide to align complaints with topics discussed as best-teaching practices. Results demonstrated that concerns about classroom and assessment practices were the most prevalent. Concerns about classroom practices were slightly more abundant and more pervasive throughout the semester than concerns about assessment practices. Additionally, an outside-of-class issue undergraduate students raised was regarding the effectiveness of GSIs communication via emails.

Keywords: Student Complaints, Instructional Practices Guide, Graduate Student Instructors, TAs, GTAs

Within the current trend towards a consumer-driven higher education model with funding diminishing for higher education institutions (Hasan, Ilias, Rahman, & Razak, 2008), there is a great need to consider undergraduate students' concerns with quality teaching in mathematics service courses (i.e. mathematics required for non-mathematics majors; Harris, 2007). This consideration has been shown to be important for the often-clogged Science, Technology, Engineering, and Mathematics (STEM) pipeline where STEM students change majors in part due to their experiences in mathematics classes (Seymour, 2001). Nationally, graduate student instructors (GSIs)[1] in mathematics departments teach these courses and hold significant influence over the STEM pipeline as well as opportunities for other majors who require certain service mathematics courses (Belnap & Allred, 2009).

Within this context, student complaints have become a critical factor for collegiate administrators because they are a means of quality assurance (Hasan et al., 2008). However, our review of the literature found a dearth of research around student complaints. This lack of literature may stem from limited frameworks on best-practices for teaching undergraduate mathematics students, thus limiting structures in which to look at student complaints. Recently, the Mathematical Association of America (MAA) released the Instructional Practices Guide (IPG, 2018), where evidence-based research provided effective methods of student-centered

[1] GSI was used instead of TA (Teaching Assistant) because GSI references graduate students who are full instructors of record.

instruction for undergraduate courses (p. x). With the release of the IPG, administrators and researchers have the opportunity to examine various aspects of student experiences, including complaints, through the lens of research-supported best practices of student engagement to see if such issues are being raised by undergraduate students of GSIs.

Purpose of Study

As the IPG is gaining national popularity and there is a significant push for improving service courses in mathematics departments that are frequently taught by GSIs, our study focuses on analyzing student complaints with a focus on the IPG's instructional practices: *design practices*, *assessment practices*, and *classroom practices*. To do so we collected undergraduate student complaints to mathematics departments from any courses taught by GSIs at two doctoral-granting universities over five years and analyzed their content according to the IPG's instructional practices and what time in the semester the complaints occurred. Complaints from this study were found to include multiple concerns, illustrating apprehensions, worries, or fears expressed by the student. As such, we define a concern as a specific context described by the student that indicated inappropriate actions (from the student's perspective) by the GSI. We define a complaint as a collection of concerns a student chose to directly express to the mathematics department (e.g. department chair, graduate director, or course coordinator) about their GSI. Our research questions used the IPG's instructional practices to code complaints and concerns to determine: (1) What type of undergraduate mathematics complaints and concerns occur with GSIs and (2) When did certain types of concerns occur within the semester?

Related Literature

Higher Education, GSIs, and Complaints

Because GSIs teach hundreds of thousands of undergraduate mathematics students each semester (Belnap & Allred, 2009; Speer & Murphy, 2009) supporting their development has been identified as a key component for successful collegiate mathematics teaching (Bressoud, Mesa, & Rasmussen, 2015, p. 117). Moreover, GSIs' teaching methods, goals, and beliefs greatly influence undergraduate mathematics education (Yee et al., 2019). For these reasons, departments are providing GSIs with robust and regular professional development (PD) around instructional practices (Yee & Rogers, 2017), including multiple day orientations, courses focused on mathematics pedagogy, mentoring programs, and post-observation feedback (Rogers & Yee, 2018). Mathematics departments and researchers continue to focus on supporting and improving GSIs' student-centered instruction (Rogers & Yee, 2018; Speer & Murphy, 2009: Yee & Rogers, 2017). Consequentially, many GSI PD programs are striving to focus on research-based, student-centered teaching strategies as illustrated in the IPG (Speer & Murphy, 2009). To this end, answers to our research questions can illuminate what types of instructional practices GSIs may need more support in implementing based on student feedback.

To understand some background literature about student complaints, we note that students' status within a university depends in part upon a sense of mutuality within their community (Ahier, Beck, & Moore, 2003). Furthermore, one essential practice for good governance of higher education institutions is an opportunity for students to bring about complaints via an effective internal process (NCIHE, 1997) and that this process is 'fundamental to the relationship between students and universities' (Harris, 2007, p. 566). In addition, research has indicated that complaints about GSIs is one measure of the success of a PD program and of good teaching (Ellis, Deshler & Speer, 2016; Friedberg, 2005).

Although the IPG is designed to facilitate effective teaching, we have found limited literature connecting student-centered teaching methods to student complaints in undergraduate mathematics courses. In this work, we begin the process of aggregating complaint data, systematically examining the issues of concern raised by students in undergraduate mathematics courses taught by GSIs, and frame the work using the IPG to consider complaints through research-based teaching practices.

IPG Framework

> We must gather the courage to venture down the path of uncertainty and try new evidence-based strategies that actively engage students in the learning experience. We must gather the courage to advocate beyond our own classroom for student-centered instructional strategies that promote equitable access to mathematics for all students. (IPG, 2018, pp. vii-vi)

For over a decade, many STEM disciplines have also declared this impetus for change, as expressed in this quotation from the IPG's declaration of values, emphasizing active learning (Freeman, 2014). As we think about GSIs' needed courage and values, it is vital to identify issues they may have with evidence-based teaching strategies and methods of actively engaging students so that we may support them. Another layer of stress and complexity for GSIs is that they have learned from primarily lecture-based mathematics courses but are asked to actively engage students in their own classrooms (Yee & Rogers, 2019).

To structure guidance for mathematics instructors, the IPG describes instructional practices in design, assessment, and classroom practices and emphasizes the dynamic interaction between these practices. These instructional practices are an appropriate framework for this study because they provide a breakdown of critical issues of teaching with student learning as the primary driver. Since this study focuses on student complaints, the need to be focused on student understanding and interaction is critical.

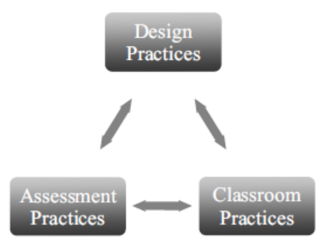

Figure 1. IPG Instructional Practices Model (p. x).

The IPG also recommends considering complaints as an important aspect of learning for mathematics instructors, explaining:

> Student evaluations may suffer during early implementation of a student-centered design. For example, students may complain that an instructor just stands there and makes them do all the work. Untenured and contingent instructors particularly require support from colleagues and chairs as they work toward becoming more effective instructors (IPG, 2018, p. 106).

Our research has indicated student complaints resonate with GSIs (Rogers & Yee, 2017) and thus the results of this study can help inform researchers and administrators of issues students perceive with GSIs.

Methods

Data Collection

Forty-nine complaints about GSIs were collected from two universities in the US over five academic years. The complaints were about GSIs who taught service courses (i.e., Introduction to Statistics & College Algebra). Complaints were obtained by contacting course coordinators, graduate program directors, and department chairs to collect emails that recorded formal complaints about GSIs, not end-of-semester student evaluations. In a few cases, email histories of complaints could not be found but instead the research team collected communications about the complaints by directly talking to the course coordinator (or other faculty members), the student, and/or the GSI. While additional information was gathered about individual complaints (e.g. follow up discussions or meetings), this study focuses on instructional practices that would cause undergraduate students to reach out to administration to discuss concerns. To this end, it is important to note that all results of this study focus on *student perceptions* of instructional practices, not the practices themselves.

Data Analysis

To answer our first research question, the complaints were coded by research assistants at two universities according to the concerns expressed within the complaints. Complaints from multiple students about a single GSI were counted as separate complaints as they were sent separately to administration. A single complaint regularly had multiple concerns that directly aligned with the sections of the IPG (e.g. classroom, design, and assessment practices). To answer our second research question, we analyzed the complaints by looking at when in the semester the complaints occurred. Complaints were anonymized through the coding process using a Google form which omitted any information that would connect the complaint to the GSI. To refine the qualitative analysis of the complaints to a more granular level than the three instructional practices of the IPG, researchers used Rogers and Yee's (2018) topics of concerns for novice GSIs as subcodes within classroom, design, and assessment practices.

Despite the robust design of the IPG, there were a few concerns outside of class that didn't fit clearly into an instructional practice of the IPG. For example complaints included concerns about technology, such as email conversations. The IPG referenced technology as a cross-cutting theme, but we found complaints and concerns focused on how GSIs communicated via email, which is not discussed in the IPG (e.g. a student discussed sending multiple emails, or mentioned not receiving responses, or a GSI sending large amounts of emails in a short period of time). To properly identify these complaints and concerns, we open-coded such concerns as outside-of-class issues with open subcodes. Consider the following concern:

> I emailed [the GSI] right away, and on Tuesday in class I asked her if she got my email, and she said she couldn't remember so I explained the situation and she said that I needed to finish the assignment and to try again and if it didn't work, then I should email her right away. So I tried again and it didn't work, so I emailed her and didn't get a response. So today in class I tried telling her about it again and she said that she forgot I emailed her.

We coded this concern as outside-of-class and subcoded as email discussions.

Student complaints were often explicit about the concerns within the complaint because the complaint was sent to administration. Many complaints were coded to be about multiple concerns within different instructional practices. For example, one complaint contained the following concerns:

[Concern 1] She also seems to struggle to explain the information in a way that is effective for the group as a whole…[Concern 2] My main concern, however, is something that was said about the grading of our tests…[She] explained that she had to be 'picky' about how she graded our test because if she were to give all of us good scores she feared being questioned about making her tests too easy for the class…[Concern 3] The information being taught in the class is moving at a very slow pace, and when it is realized that we are falling behind on time then we have to rush through material.

The first concern was coded as dealing with classroom practices and subcoded as the GSI being unclear in communicating the material. The second concern was coded as dealing with assessment practices and subcoded as a concern about the grading of summative assessments. The third concern was coded to be about design practices and subcoded as the GSI being unorganized because that was the concern as perceived by the student.

Some concerns were coded multiple times within the same instructional practice:

[Concern 1] I was not successful in the course due to the instructor's way of teaching and clarity. I made the effort of going to his office hours where I found no help at all, due to his unwillingness to help… [Concern 2] This specific course is very disorganized and doesn't follow dates according to the syllabus.

The first concern was coded as dealing with classroom practices because the concern discussed the GSI's method of teaching in the classroom. The first concern was then subcoded as the GSI being unclear in communication and presentation of material and as an Outside-of-Class Issue, subcoded for office-hour meetings. The second concern was coded as a design practice concern and subcoded as the GSI being unorganized.

A section of "Other" was also provided within each instructional practice, which allowed the research assistants to create new subcodes as needed. Altogether, the "Other" category was only used once for complaints that did not align with the original subcodes of Rogers and Yee's (2018) topics of concern. For interrater reliability, two additional researchers checked over 50% of the coding. Initial interrater reliability was above 90%. All researchers openly discussed any coding disagreements until there was 100% agreement among all researchers.

Results

With a single complaint receiving multiple concerns within instructional practices, there were a total of 100 concerns raised from the 49 collected complaints about GSIs. Table 1 summarizes descriptive statistics of complaints and associated codes.

Table 1. Descriptive Statistics about GSI Complaints

Complaints by Academic Year	2014-2015 9 Complaints	2015-2016 15 Complaints	2016-2017 17 Complaints	2017-2018 2 Complaints	2018-2019 6 Complaints
Complaints by University	University of South Carolina 22 Complaints		Bowling Green State University 27 Complaints		
Concerns by Instr. Practice (Percentage)	Classroom Practices 40 Concerns (40%)	Design Practices 8 Concerns (8%)	Assessment Practices 39 Concerns (39%)	Outside-of-Class Issues 13 Concerns (13%)	

The precipitous drop off of complaints in the latter years (2017-2019) may be due to a peer-mentor program implemented at both universities, but due to limited space, this analysis was omitted. When looking at instructional practice type, we see student concerns referenced classroom practices (40%), then assessment practices (39%), then outside-of-class issues (13%), and then design practices (8%) in decreasing order.

To answer the second research question we looked at when the concerns occurred within a semester. Figure 2 shows the frequency of concerns with respect to the timing of the semester in which the complaint was received.

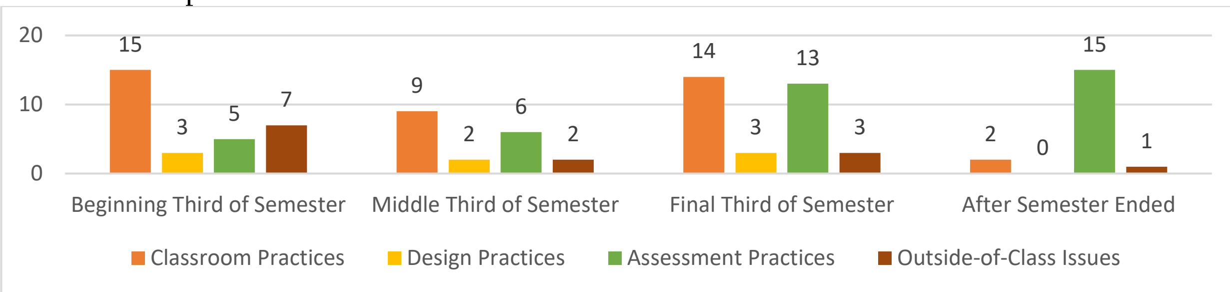

Figure 2. Instructional Practice Types of Concerns Timeline

By focusing on a single instructional practice throughout the semester, we see interesting trends. First there, is a steady increase in concerns about assessment practices (5, 6, 13, then 15). This aligns with the fact that a majority of assessment concerns focused on the grading of summative assessments (Fig. 3) that often accumulate throughout a semester. Second, classroom practices (15, 9, 14) had the highest number of concerns during each third of the semester and then dropped after the end of the semester (2 concerns). Third, design practices had the fewest concerns during every time period in the semester (3, 2, 3, then 0). Fourth, the outside-of-class concerns remained low after the beginning third of the semester (7, 2, 3, then 1). When looking at any single time within the semester, the beginning of the semester had the largest number of concerns about classroom practices while the end of the semester had the largest number of concerns about assessment practices.

Figure 3 provides a further breakdown of concerns by IPG practice as well as outside-of-class issues. Referring to Table 1, recall that different instructional practices had different numbers of complaints, thus each graph in Figure 3 has a different number of concerns.

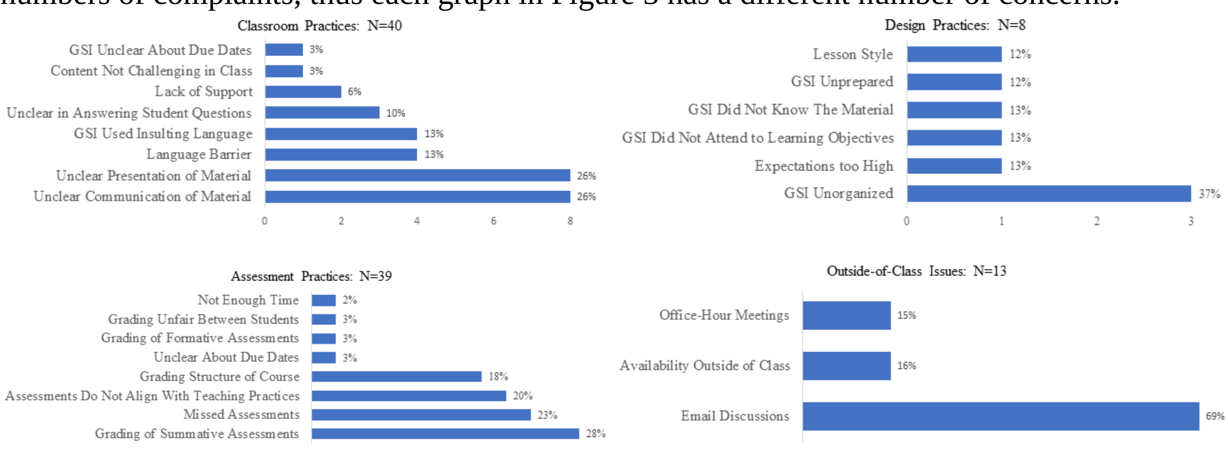

Figure 3. Complaints by Instructional Practice Breakdown.

Classroom practices had the largest number of concerns (N=40). Within classroom practices, unclear communication of material (26%), unclear presentation of material (26%), language barrier (13%), insulting language (13%), and unclear answers to questions (10%) all had at least 10% of the concerns. Within assessment practices (N=39), the four largest concerns were the grading of summative assessments (28%), missed assessments (23%), assessments not aligning with teaching practices (20%), and grading structure of course with (18%). Outside-of-class concerns (N=13) primarily revolved around email discussions (70%) while office hour meeting issues and availability outside of class each had two concerns (15%). It is interesting to note that while summative assessment dominated assessment concerns, formative assessment of GSIs was only mentioned in one concern. Within design practices (N=8), GSIs being unorganized (37%) had three concerns while others, such as expectations are too high (12%) had only one concern.

Looking across all three instructional practices, classroom and assessment practices each held about 40% of all concerns with classroom practices being slightly larger. Often, it is believed students are complaining directly about the grading of summative assessments, but this indicates that students are also aware of the connections between classroom practices and student learning. Additionally, within the classroom practices we see students saw fit to complain to administration often about communication and presentation of the material. Moreover, we see that students do not complain about the need for, lack of, or inclusion of collaborative learning, which is a central piece of the classroom practices within the IPG.

Conclusion

In answering our first research question, Table 1 and Figure 3 show that classroom practices had the largest number of concerns for undergraduate student complaints, with a strong focus on GSIs' ability to make expectations clear, clearly present content, and communicate with students. Assessment practices were also a concern for students, especially summative assessments. Student concerns about design practices indicated that GSIs needed to be organized. A surprising result was that undergraduate students complained 69% of the time about email discussions within the outside-of-class issues. As shown in the example in the data analysis, students complained about punctual responses to emails. This is interesting, and because a focus on GSI email practices is not a common element of GSI PD programs, it may be a critical issue that GSI PD programs need to address for student success within their class.

To answer our second research question, Figure 2 showed that classroom practice concerns were dominant during the semester and assessment practice concerns were the most popular after the semester ended. Comparing this with Figure 3, it is not surprising that assessment practices increased over the semester as most concerns (e.g. grading of summative assessments, missed assessments, assessments aligning with teaching practice) take time for students to have impressed upon them within a semester. Concomitantly, the most popular concerns about classroom practices are issues that do not take weeks to impress upon students (e.g. communication and presentation of material, answering of student questions). It is important to note that classroom practices, not assessment practices, had the most concerns.

This study found that analyzing undergraduate student complaints and concerns can inform the field for GSI PD around student-centered instruction. The results suggest a few important conclusions. First, there is a large need to have discussions about emails in GSI PD and how they have become a critical method of communication with the student. If we are to be student-centered with our instruction that focuses on student thinking, we must also be aware of the many mediums through which students express their understanding and confusion, such as email. Second, classroom practice concerns were more pervasive than assessment concerns and were the most popular concern during the semester, illustrating that students do not only complain to get their grades changed, but understand the connection between teaching and meaningful learning (Cohen, Raudenbush, & Ball, 2003). Third, within design practices, students complained if the GSI seemed unorganized. This suggests organization needs to be a priority for GSIs to take ownership of the structure they are creating from the perception of the student.

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Students' Meanings for the Derivative at a Point

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The purpose of this study is to examine students' meanings for the derivative at a point. While students may associate rate of change with derivative, this does not mean that the meaning they have for derivative is productive. This study explores students' responses to a typical calculus 1 problem that uses derivative to determine a linear approximation. A framework is provided for describing and analyzing students' meanings for the derivative at a point by discussing students' usage of time and meaning for change in the context of a fish growing over time.

Keywords: Derivative at a Point, Calculus, Student Meanings

When students go through a typical calculus course it would not be surprising if they interpreted the derivative at a point ($f'(3) = 6$) as concerning solely one point as opposed to using the value of the derivative over some interval by coordinating changes in input and output values. Students are often introduced to function as a correspondence (Sfard, 1992) and see one input being mapped to one output. It should be natural then that as this notion is rarely challenged, students' conception of function as a mathematical object (Thompson & Sfard, 1994) has the property of being concerned with a single value. This issue is important because derivatives are fundamentally about change, yet how can you talk about change if there is only one instance involved? Students tend to hold multiple disconnected meanings for the derivative at a point such as graphically as the slope of the tangent line, verbally as instantaneous rate of change, physically as speed or velocity, and symbolically as the limit of the difference quotient (Zandieh, 2006). Given this, the research question this paper explores is:

What meanings do students have about the derivative at a point?

Literature Review

Literature on derivatives has been extensive thus far, ranging from but not limited to the limiting process (Oehrtman 2009, Szydlik 2000, Roh 2008), conceptions of rates of change (Byerley & Thompson 2017, Hackworth 1995, Thompson 1994, Confrey & Smith 1994), and student understandings of the derivative as a function (Park 2013, Habre & Abboud 2006, Zandieh 2000). This study aims to contribute to research on student understandings of derivative as a function by investigating student meanings of the derivative at a point and their meanings for change.

Derivatives are about change, and thus a discussion of students' meanings for change and how they come to form images of change is central to students' construction of derivative. In the context of a quantity changing, Castillo-Garsow (2010, 2012) has termed two distinct student images of change as *chunky* and *smooth*. Some students may think about change happening discretely where a quantity changes from one point to another but with no imagery of motion between those two points (chunky image of change). Other students may view change as a change in progress by conceptualizing a quantity as taking on values as time flows continuously and smoothly (smooth image of change). These kinds of thinking about change would likely influence the meaning a student constructs when encountering the limit of the difference quotient.

Theoretical Perspective

The Mathematics of Students

This study utilizes Radical Constructivist theories of learning (Thompson, 2000), taking the stance that it is impossible to know another's thinking. Therefore, investigating student thinking has the goal of building models of students' mathematics (Steffe & Thompson, 2000) that may be used as an explanatory model for why students produce certain responses. Use of this approach to build an explanatory model of students' thinking can be useful for describing the process of developing a productive meaning for derivative at a point. The word 'meaning' is going to be used in the way that Thompson (2013) uses it to describe mathematical meaning. It is the organization of an individual's experiences with an idea that determines how the individual will act. Meanings are personal, and they might be incoherent, procedural, robust, or productive, but these meanings are used by individuals to respond to mathematics tasks and make sense of and/or access mathematical ideas. For example, a person's meaning for derivative might only be associated with calculating limit of the difference quotient, while another's meaning for derivative involves the slope of a tangent line. Since meanings are personal, if one student writes a response similar to another student, it cannot be assumed that they both have the same meaning.

Conceptualizing Derivative at a Point

In a standard Calculus 1 course, students are introduced to derivative through the limit definition; $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{(x+h)-(x)}$. One productive meaning would be having the right-hand side of the equation as representing a convergence of average rates of change. This convergence to a value leads to what one may call an 'instantaneous rate of change' that represents the relationship of two quantities changing with respect to one another's relative size. One needs to instantiate this rate of change by managing output changes with respect to small input changes. This coordination of changing quantities is in line with Carlson's covariation framework Mental Action 4, where someone is coordinating average rate of change over contiguous fixed intervals of the independent variable (Carlson et al, 2002). Finally, one must also demonstrate that using a single derivative value as a rate of change over some interval will usually net an estimate due to the instantaneous rate of change changing as you progress through an interval (Mental Action 5). Mental Action 5 of Carlson's covariation framework consists of someone coordinating the instantaneous rate of change of a quantity with continuous changes in the independent variable i.e.: The rate at which a quantity is changing is also changing with respect to the independent variable. To use a derivative value as a rate of change having some value *m*, one must imagine the input quantity varying and simultaneously also imagine the output quantity varying *m* times as much as the variation in the input quantity. The numerical value of the rate of change is quantifying the multiplicative relationship between the changes in two covarying quantities. This meaning for rate of change and variation is described in greater details in (Thompson 2011, Thompson & Carlson 2017, Thompson & Saldanha 2003).

Zandieh (2006) has shown however that many students do not see derivative as a ratio of two different quantities. Nor do students immediately demonstrate that the tangent line is an approximating function that involves instantaneous rate of change. Zandieh remarks that despite the initial imagery of receding secant lines to a tangent line, students tend to recall the finished product and conflate the tangent line with the slope of the tangent line.

Methodology

The purpose of this study was to investigate student thinking for the purpose of modeling students' understandings (Steffe & Thompson, 2000) of the derivative at a point.

Clinical interviews (Clement, 2000) were conducted with 11 students who were enrolled in a Calculus 2 course at a southwestern university. This study was part of a larger study that involved more tasks, only the first task is presented here. The purpose of the 1-on-1 clinical interviews was to establish each student's initial meanings for the idea of derivative and how that student applied her meaning for derivative to a linear approximation problem.

Data Analysis

This study categorizes different aspects of how someone might consider the derivative at a point. One category describes of what I shall call Static Meanings, and the other describes Dynamic Meanings. In addition to these two meanings, students can also have a *Procedural Meaning*; one in which a student explains the derivative at a point using only mathematical procedures such as applying the Power Rule and plugging in a value without referencing the quantities involved. In a discussion with a student holding a *Procedural Meaning* for derivative, the student does not discuss the quantities in the situation nor mentions quantities changing or what the ratio of two quantities might represent. In additional to classifying students as holding a Static or Dynamic meaning, the derivative at a point framework (Table 1) includes two categories (time and change) that are used to describe how/if a student conceptualizes the passage of time, the meaning that students attribute to the word 'change' in this context, and students' covariational reasoning. Table 1 outlines the categories and the associated behaviors that would evidence each meaning. It may seem that having a Static Meaning for time would imply a Static Meaning for change (and vice-versa), however the data from the interviews indicates otherwise and thus a discussion of both aspects is pertinent in how students interpret the derivative at a point.

Table 1. Student Thinking about Derivative at a Point Framework

Aspect	**Static Meanings**		**Dynamic Meanings**	
	Meanings	**Behavior**	**Meanings**	**Behavior**
Time	1 point of time is involved. No interval of time.	• Absence of a passage of time. • Interpreting the derivative as the value that is computed or indicated (e.g., what the needle on a speedometer points to).	An interval for the input quantity is considered.	• Discussion of an interval or requiring more than 1 point of time.
Change	Change as a difference between two points	• Change has already happened or will happen. • "In 1 month, the fish will gain 6 ounces" • Change happens in discrete chunks (Castillo-Garsow, 2010)	Change as referring to a rate, i.e.: How two quantities vary together	• Indication of a change in progress, or usage to estimate a value using an arbitrary interval. • Association of the derivative value with how two quantities change together.

Results

Task 1 – "Interpreting and Utilizing the Derivative at a Point"

Task 1 employs a typical linear approximation problem that is a type of problem that most Calculus 1 students would have been exposed to. Students worked through both parts of the problem and then were asked about their responses. This was used to both elicit students' meanings for the derivative at a point and how they utilize the derivative at a point to approximate a future value. Figure 1 shows the task that students were presented with.

> Given that *P(t)* represents the weight (in ounces) of a fish when it is *t* months old,
>
> a. Interpret the statement $P'(3) = 6$
>
> b. If $P\ (3) = 15$ (and $P'(3) = 6$) estimate the value of $P\ (3.05)$ and say what this value represents.

Figure 1. Task 1 – Interpreting and Utilizing the Derivative at a Point

Only 4 of 11 students wrote down an answer akin to one that would be deemed as correct; "At 3 months the fish is growing at a rate of 6 ounces per month". However, when each student was pressed for what this meant to them, different meanings emerged. One student, Gemma, mentioned instantaneous rate of change, and her meaning for the derivative at a point became apparent when she gave an example of what she meant (Table 2). Gemma expressed that she was only thinking about one point in time (Line 5), and if you had more than that you would be getting an average rate of change which would not be the instantaneous rate of change (Lines 6-8). Gemma demonstrated a Static interpretation of the derivative at a point due to her belief that the derivative at a point is about a single point in time (Line 5), but also simultaneously reveals her belief that change requires more than one point (Line 10). Despite this though, she was not demonstrating that she was able to associate this rate of change with the covariation of time and weight; instead her utterances suggest that she views the rate of change at an instant as a speedometer that is displaying a speed, but this is not about change in progress.

Table 2: Gemma's Explanation for Instantaneous Rate of Change

Int:	You said something about instantaneous rate of change, can you explain what that means?
Gem:	Instantaneous rate of change… So let's say like a car is driving from point A to point B and you want to find its speed or velocity at a certain point in time like at exactly minute 5 like that instant you want to find its rate of change, it's velocity. I would say because you can't just find its average like saying average rate of change because that would give you an average you know not at that instant.
Int:	And is the car moving at this *point*?
Gem:	Yeah, it's not.

Another of the four students, Cyrus, explained his interpretation of the value of 6 using estimation language, demonstrating his Dynamic Meaning for both time and change (Table 3). Cyrus used the 6 to determine an approximated change in weight with a change in 2 months of time netting him 12 ounces. His consistent usage of "probably" and "just about" (Lines 2,7) indicated his awareness that these changes would be approximate. Cyrus' response suggests that he is instantiating this 6 ounces per month over a period of 2 months to net an estimation of change in weight, additionally he reveals that his choice of 2 months was arbitrary (Lines 10-14), further supporting the claim that Cyrus' meaning for the derivative at a point includes an image of a covarying relationship between the quantities of weight and time.

Table 3: Cyrus' Explanation for the Derivative at a Point

1	Cyr:	So, it's the end of the month right now… between the beginning of this month and the
2		end of the month that's just about to hit, it will grow just about 12 ounces probably.
3	Int:	So just to be clear on the numbers, you're saying from the beginning of the 2nd
4		month to the end of the 3rd month.
5	Cyr:	Yes
6	Int:	It would grow…
7	Cyr:	It would probably grow about 12 ounces, in practicality, if someone didn't know
8		anything about math that's kind of what that means, we're only taking the
9		measurement of this right here.
10	Int:	So, when you decided to say between like this 3rd month is in the middle and it would
11		grow 12, can you expand or shrink this interval?
12	Cyr:	Yes
13	Int:	Does the 3rd month always have to be in the middle of it?
14	Cyr:	No, that's just the sample size, the time scale I used was arbitrary, and the 2 and 3
15		month thing was arbitrary.

In the 2nd part of Task 1, Cyrus further demonstrated his Dynamic Meaning for both time and change as he continued using estimation language throughout the entire discussion (Table 4). In fact, he was the only student among the 11 who verbalized an awareness of the approximation taking place. Other students did not think the value they found was an estimate until questioned about it, even though the question clearly said 'estimate'. Additionally, Cyrus exhibited a strong understanding of why this is an approximation when he explained that "the rate of change probably won't change much between 3 and 3.05" (Lines 7-8), an indication that he was aware that the instantaneous rate of change will change during this interval, but also recognizing that the value of 6 is close enough due to the interval being small. When asked about the multiplication that he did (Line 4), Cyrus conveyed that he interpreted the derivative at the input value 3 as relating the change in weight of the fish and change in time, and used it as a rate of change over the time interval of 0.05 (Lines 5-9).

Table 4: Cyrus' using the Derivative at a Point to estimate a future value

1	Int:	So what did you do here?
2	Cyr:	I did 0.05*6 and got 0.03, so I estimated 15.3. I know it's close, but I know that's not
3		P(3.05).
4	Int:	Why did you do this part over here with 0.05*6?
5	Cyr:	So that was the estimation, it was 3 and 0.05 months, at 3 months it was changing at 6
6		ounces per month, and at 3.05 months it probably is not going to be changing much
7		faster or much less but that's an estimation. The rate of change probably won't change
8		much between 3 and 3.05. So, I multiplied the rate of change which was 6, times the
9		value added on to 3 (0.05).
10	Int:	So what was 15.3 again?
11	Cyr:	15 was the weight of the fish at 3 months, and 15.3 was the estimated weight of the
12		fish at 3.05 months.

Andrew is a student who wrote "After 3 months the fish is growing by six ounces" when he interpreted $P'(3) = 6$. In the discussion that followed his response, Andrew explained that after the third month has progressed the change in the weight is 6 ounces (Table 4). In fact, when asked about the units for the 6, Andrew deliberately said that it is "change in ounces" and not simply ounces (Line 11). Here, Andrew described change as the difference in weight between two points in time, for him this value of 6 did not carry a dynamic relationship between weight and time, rather it demonstrated that Andrew was using chunky reasoning by thinking of an entire discrete chunk of change in weight. Andrew did not articulate time passing when he explained the derivative value of 6 as indicated by his choice in units and explanation, rather he imagined two different points in time and talked about the difference in weight between them; it appeared that Andrew viewed $P'(3) = P(4) - P(3)$. Unlike Gemma, Andrew did not appear to think of the derivative at a point as regarding only the time at 3 months, rather Andrew appears to have a Dynamic Meaning for time since he thought about an entire month and a Static Meaning for change since he was thinking about the difference between two instances of time.

Table 5: Andrew's Explanation for the Derivative at a Point

1	And:	So it's at three months and it's growing by six ounces… it's
2		weight increases by six ounces… that's how much it grew that
3		month so for the third month it grew six ounces.
4	Int:	Like in that entire third month?
5	And:	It gained 6 ounces.
6	Int:	So you're saying that if I have like a calendar month **Draws a*
7		*calendar**, over the course of all this you're saying that the total
8		change in weight is…
9	And:	6 ounces yeah.
10	Int:	Okay, so what are the units for 6?
11	And:	Change in ounces.

In the 2nd part of the task, Andrew wrote down work like Cyrus', but the meaning Andrew attributed to his expression of $6 * (0.05) = 0.3$ was different from Cyrus' (Table 6). Andrew initially set up a proportion of $\frac{x}{6} = \frac{0.05}{1}$, which led him to $6 * (0.05)$. His meaning for the value of 6 explains why Andrew set up a proportion. For Andrew, 6 represented one entire chunk of change in weight (Lines 7-8) rather than the covarying relationship between weight and time. In comparison when Cyrus did 6 * 0.05, Cyrus explained that 0.3 was 6 times as large as the change in time of 0.05, Andrew however used 6 * 0.05 to determine what portion of 6 was in this 0.05 months (Figure 2). Cyrus had indicated that 0.05 was the change in time from 3 months and Andrew articulated that 0.05 represented "time into the month" (Line 5) and his phrasing indicated that he thought about the entire month as his referential unit. His usage of "into" reveals that he thought about the entire chunk of 6 ounces of change and that you are somewhere inside of this chunk. Thus, Andrew utilized a proportion since he was thinking about what portion of the total change of 6 ounces is going to happen 0.05 months into the entire month. Andrew continued to exhibit a Static Meaning for change and Dynamic Meaning for time as revealed in his chunky reasoning of change.

*Table 6: Andrew's Explanation for 6 *(0.05)=0.3*

Int:	So why would 6 *0.05 give you that change in weight? What does the value 0.3 mean?
And:	This would be how much it would have gained in 0.05 months.
Int:	So what does 0.05 represent?
And:	That's time into the month
Int:	What about 6?
And:	6 is the weight that we know that is has to change throughout the entire month.

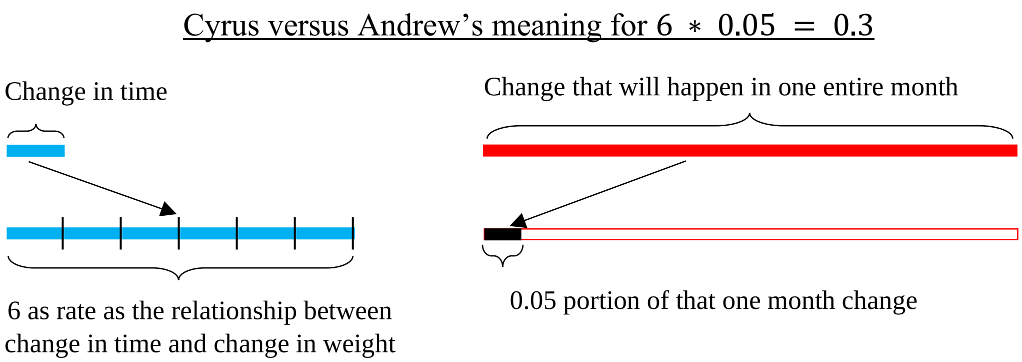

*Figure 2: Cyrus' (Blue) meaning for 6 * 0.05 versus Andrew's (Red)*

Discussion

The results from this study support past research findings that students have diverse meanings for the derivative at a point and the idea of change. The results also revealed that these differences may be a result of how each student considers the passage of time (or lack thereof) and what each students' meaning for change is in the context of the derivative at a point. The framework provided should be used by teachers as part of a conceptual analysis (Thompson, 2008) for anticipating what students need when teaching the derivative at a point by deliberately discussing time and change as students are introduced to the derivative at a point. The results of this study support that students' notion of change significantly impacts the meaning they attribute to the derivative at a point, therefore teachers should consider how they may have their students reason with smooth reasoning about change when it comes to derivative.

Additionally, this study reveals that current Calculus 1 questions about linear approximation do not sufficiently assess what students actually understand about the derivative at a point. As shown by analyzing Cyrus' and Andrews' work, correct written statements and responses do not imply that the student holds the understandings that we would desire them to have. Similarly, Gemma and Cyrus both wrote a correct response for interpreting $P'(3) = 6$, yet only Cyrus' explanation matched what he wrote, whereas Gemma had a different meaning. In fact, despite that only 4 students had written a correct response for the meaning of $P'(3) = 6$, 10 of the 11 students were able to approximate the value of $P(3.05)$ correctly. This suggests a need for better assessments for differentiating between the diverse meanings for derivative at a point that students hold.

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A Comprehensive Hypothetical Learning Trajectory for the Chain Rule, Implicit Differentiation, and Related Rates: Part I, the Development of the HLT

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Students can learn more deeply when conceptual understanding is at the forefront and connections are made between topics. We hypothesize that such understanding and connections can be achieved for the chain rule, implicit differentiation, and related rates through the construct of nested multivariation (NM). In this first paper, we describe the process of creating a hypothetical learning trajectory (HLT) rooted in NM for this sequence of topics. This theoretical paper contains our conceptual analysis, literature review, and construction of the HLT.

Keywords: chain rule, implicit differentiation, related rates, multivariation, covariation

Among the goals of mathematics education are (a) conceptual understanding and (b) connections across topics (Hiebert & Carpenter, 1992; Hiebert et al., 1997; Schoenfeld, 1988). As a key course in undergraduate STEM education, calculus education should especially aspire to meeting these goals. Yet there are topics within calculus that are still often treated as separate, and procedurally, even though they could be taught in a conceptual, connected way. In particular, the topics of the chain rule, implicit differentiation, and related rates contain powerful, interrelated ideas (see Austin, Barry, & Berman, 2000). Although researchers have suggested conceptual connections between them (Clark et al., 1997; Cottrill, 1999; Infante, 2007; Martin, 2000), these topics still tend to be examined in isolation of each other, hampering our ability to create instruction that tightly connects them. Also, while some have suggested removing some of these topics from curriculum (Callahan & Hoffman, 1995; Douglas, 1986), we instead believe calculus would be stronger if they remain in the curriculum but are taught conceptually. In this theoretical paper, which is the first of two papers, we propose to unify these topics into a coherent whole through the construct of nested multivariation (NM) (Jones, 2018). In doing so, we construct a hypothetical learning trajectory (HLT) (Simon, 1995) for teaching this portion of the curriculum in a conceptual, connected way. (NM and HLT are defined subsequently.)

We organize this paper as follows. First we define NM and provide a conceptual analysis of it within these three topics. Second, we examine the research related to these topics, and explain how they informed the creation of the HLT. Third, we describe pilot studies that helped us revise the HLT. We then present our HLT for these three topics. In a separate second paper, we will discuss the results of testing this HLT in a small-scale teaching experiment.

Nested Multivariation and Conceptual Analysis

To begin, previous research states that covariation is important in understanding and solving related rates problems (Infante, 2007). Yet, because multiple quantities are in play, it may be more appropriate to conceptualize these topics through *multivariation* (Jones, 2018), which builds on and extends covariation. In brief, multivariation refers to situations in which *more* than two quantities may change in relation to each other. In particular, Jones (2018) used the term *nested multivariation* for situations involving a function composition structure, $f(g(x))$. That is, if x changes, it induces a change in g, which in turns induces a change in f. If derivatives are interpreted as a rate of change, this structure is inherent to each of the chain rule, implicit differentiation, and related rates. Thus, we decided that an HLT for these topics could be grounded on this construct. While one can certainly use covariation to reason about just x and g, or just g and f, or even just x and f, the full comprehension of the situation requires a step further in the form of nested multivariational reasoning (NMR). We now briefly outline a conceptual analysis for how NM is inherent to these three topics.

NM in the chain rule. The chain rule says that given a function $f(g(x))$, then $df/dx = df/dg \cdot dg/dx$. This expression, read right to left, involves how much g changes as x changes, then how much f changes as g changes, and finally (on the left of the equals), how much f changes as x changes. In other words, it describes how the covariational relationship between x and f is determined by the NM relationship going from changes in x to changes in g to changes in f. For example, suppose g changes twice as fast as x (i.e., $dg/dx = 2$) and f changes three times as fast as g (i.e. $df/dg = 3$). Then for a change in x, the corresponding change in f is $3 \cdot 2 = 6$ times as large (i.e. $df/dx = 6$). The multiplicative nature of the chain rule comes from scaling the change in f relative to g (df/dg) by how much g changed with respect to x (dg/dx).

NM in implicit differentiation. Consider the equation for a circle: $x^2 + y^2 = r^2$. If one conceptualizes one variable, say y, as a function of another explicitly present variable, say x, by restricting the domain or range in an appropriate way, then we say $y(x)$ is an implicit function of x. Implicit differentiation is then finding dy/dx. To do so, one might need to think of each side of the equation as a function, as in $f(x) = x^2 + \left(y(x)\right)^2$ and $g(x) = r^2$ (Mirin & Zazkis, 2019). One can then take the derivative of these two equal functions and set them equal to each other. Some parts of this process do not involve NM. For example, covariation permits the conclusion $dr^2/dx = 0$ for constant r. For $x^2 + y(x)^2$, there is also not NM in dx^2/dx, since the only quantities are x and x^2. NM appears when seeing how much $y(x)^2$ changes as x changes: dy^2/dx. The function composition structure leads to the nested changes in $x \rightarrow y(x) \rightarrow y(x)^2$, indicating NM.

NM in related rates. Consider again the equation $x^2 + y^2 = r^2$. Instead of conceptualizing y as a function of x (or vice versa), suppose x and y were both functions of time (or another variable). In this case, they are what we term *functions of an implicit variable*, because the independent variable is not explicitly present in the equation. Rather, one must insert t, $x(t)^2 + y(t)^2 = r^2$. In such situations, a *related rates problem* is defined as finding the rate of one variable with respect to time (or other implicit variable) when the rate of the other with respect to time is known. Where does NM come into play? Again, by thinking of each side as a function of t, $f(t) = x(t)^2 + y(t)^2$ and $g(t) = r^2$, one can differentiate these equivalent functions. To differentiate f, NM is involved in the nested changes $t \rightarrow x(t) \rightarrow x(t)^2$ and $t \rightarrow y(t) \rightarrow y(t)^2$. As t changes, it induces a change in x (and y), which then induces a change in x^2 (and y^2).

NM in our HLT. We used this conceptual analysis as the basis for our HLT in two ways. First, we used nested multivariational reasoning (NMR) as the way to support a conceptual understanding of these three topics during the learning process. Second, we used the presence of NM in all three topics as the way to attempt to make strong connections across them.

Research on the Chain Rule, Implicit Differentiation, and Related Rates

Research on these topics is fairly scant overall, with the largest conclusion simply that students struggle to make sense of these topics (Jones, 2017; Martin, 2000; Piccolo & Code, 2013; White & Mitchelmore, 1996). This suggests the need for an HLT to incrementally build an understanding of these topics, as it likely would not come easily or naturally. From background research, we also decided it would important to see derivatives as a multiplicative comparison between changes (Thompson, 1994), which connects nicely to NM's basis in changes in nested quantities. Thus, the earliest part of the HLT seeks to create these meanings first. Then, for the chain rule, Clark et al. (1997) explained the importance of basing the chain rule in the students' understanding of function composition. In this way, our reliance on NM for our HLT also fits, because NM explicitly attends to the nested structure of function composition.

Within research on related rates, the most important influence to our HLT was Infante's (2007) detailed dissertation. We used or extended Infante's work in four major ways. First, Infante explained that it was important for her students to see the multiplicative nature of the chain rule, $df/dx = df/dg \cdot dg/dx$. However, because Infante's students had already learned the chain rule, her work was focused on post-hoc sense-making. We realized that it would be important for us to spend an entire portion at the beginning of the HLT just on this idea, especially since our students would be learning it for the first time. As in the conceptual analysis, we believed NMR would be key for developing this understanding. Second, Infante explained that a "delta equation" of the form $\Delta x/\Delta t = \Delta x/\Delta y \cdot \Delta y/\Delta t$ was instrumental in seeing connections between the quantities of a situation and to recognize the form of the chain rule. However, from our use of informal infinitesimals, we revised this to be a "d" equation, $dx/dt = dx/dy \cdot dy/dt$. Infante also wondered at the possibility that creating a delta equation had become a "procedure" for her students, so we wanted to make sure this process had meaning. Third, Infante highlighted the importance of making "time" an explicit quantity in related rates. We used her suggestion by encouraging students to consider all of the changing quantities within a given problem to help make time salient. Fourth, Infante noted that students rarely referred back to the diagram or model for the problem. We built on this by regularly asking them the meaning of each variable, which helped them make frequent connections back to the diagram or model.

In addition, a few other studies lent ideas to the creation of our HLT. For example, research has suggested the importance of using familiar contexts in introducing and developing mathematical content (Boaler, 1993; Bonotto, 2005), which we chose to do. However, while we had planned on using an expanding circle to introduce the topics, we chose not to because we learned that the presence of "π" in an answer can potentially obscure the units of a quantitative result (Dorko & Speer, 2015). Thus, we waited to do a circle/sphere problem later in the HLT.

Pilot Studies

Once we had determined NM to be the basis of the HLT, conducted a conceptual analysis, and reviewed the research literature, at that point we had constructed a draft of the HLT. We then went through a cycle of pilot interviews in which we tested specific parts of the HLT with individual students to refine the HLT. This resulted in the following changes. First, even with the guidance of Clark et al. (1997), we still underestimated the time students needed to dissect function compositions. Thus, we added more time at the beginning of the HLT to do so. Second, the pilot studies showed students were unprepared to think about derivatives quantitatively as a ratio of changes. Thus, in our opening section, we inserted prompts meant to develop quantitative meanings for basic derivatives. Third, as the pilots progressed, realized that there were so many ideas that would come up during an exploration of the chain rule that students struggled to keep track of and organize them. Consequently, as a pedagogical move, we decided it would be important for the instructor to rewrite students' conclusions about the multiplicative nature of the chain rule on a separate sheet of paper so the students would be able to more easily identify the patterns in their own reasoning. For example, the contexts introducing the chain rule dealt with two rates that could multiply to an overall rate, so that they started with df/dg and dg/dx and combined them to get $df/dg \cdot dg/dx = df/dx$. While that may *look* like the chain rule, it is actually the reverse. But by keeping track of these types of results, students could be led to start with df/dx, reverse that thinking, and break df/dx down into $df/dg \cdot dg/dx$.

Finally, we realized that as we tried to help students gain some procedural fluency with symbolic functions, the issue of function composition came up again in terms of students identifying which function was composed "inside" of which. We had used the example $h(x) = \sin(x^2)$, but because of that difficulty we changed this to first be $h(x) = \sin(f(x))$. This was helpful for the last pilot student to see that *f* was a function inserted into the sine function. We then added to the problem to have them consider both $\sin(x^2)$ and $[\sin(x)]^2$ to round out their experience.

Our Hypothetical Learning Trajectory

To put more precision on what an HLT is, Simon (1995) explained that is consists of three components: "the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). Having described the process that led to the creation of the HLT, we now present it. The HLT includes five "stages" meant to be traversed over four 50-minute interview sessions. Each stage has one overarching learning goal, which is accomplished by meeting several subgoals inside of that stage. For ease, we discuss the HLT one stage at a time by presenting the learning goals, select learning activities and questions, and the trajectory of learning. Please note that while we use "we" to imagine ourselves in the role of teaching as we explain the HLT, "we" should be read as "the instructor."

The first stage involves using NMR to develop the multiplicative nature of the chain rule. Table 1 shows the main learning goal and subgoals for this stage.

Table 1. The Learning Goals for Stage 1

Stage 1	**Develop the multiplicative nature of the chain rule.**
1a	Interpret *df/dg* as how many times as large the change in *f* is than the change in *g*.
1b	Interpret *dg/dx* as how many times as large the change in *g* is than the change in *x*.
1c	Interpret *df/dx* as how many times as large the change in *f* is than the change in *x*.
1d	Conceptualize how changes in *x* affect changes in the other two variables simultaneously.
1e	After finding specific values of *dg/dx* and *df/dg,* construct $df/dx = df/dg \cdot dg/dx$ at a point.

We open stage 1 with a simple context involving linear rates of change. The students are shown the prompt: "Let's say you make \$9/hr at your job and you're obsessed with chocolate… You can buy 0.15 lbs of chocolate per dollar." The students are then given the function notation $D(h) = 9h$ and $c(D) = 0.15D$. To scaffold meaning for derivatives as ratios of changes (addressing subgoals 1a and 1b), several key questions are asked including: (1) What is the value of dD/dh? (2) What is the meaning of dD/dh in our context? (3) What are the units of dD/dh? These questions are repeated for dc/dD as well. To address the function composition nature, we then ask: What would be the meaning of $c(D(h))$? To ensure NMR is being used, we also ask: How does hours worked affect the amount of chocolate you can buy? To work toward subgoals 1c and 1d, we prompt: If we could find dc/dh, what would that mean in our context? Then, to begin using their reasoning to progress toward the chain rule, we use the questions: What is the value of dc/dh? How do you know? Through this learning activity, we anticipate students can begin to develop quantitative meaning for basic derivatives and begin to use NMR to think through the nature of the function composition. Building on that NMR, we then expect students to think through the nested changes in h to D to c to begin reasoning about how the rate of change of c with respect to h would be dc/dD times as big as dD/dh.

This stage contains two other activities meant to accomplish similar things to the chocolate context. In the next activity, temperature, expected attendance at a carnival, and revenue are related and similar questions are asked. One difference is that instead of using "suggestive" letters to represent the functions (c, D, h), we represent attendance as "g" and revenue as "f" in order to sometimes use more abstract notations. The last context involves time elapsed, distance run, and calories burned. In the second and third contexts we also use relationships between quantities that do not change at constant rates, to introduce that idea. However, at this point we still focus in only on a single point in time to examine their instantaneous rates.

The second stage of the HLT is meant to generalize the chain rule and help students gain procedural fluency. Table 2 shows the main learning goal and subgoals for this stage.

Table 2. The Learning Goals for Stage 2

Stage 2	**Generalize the chain rule and gain procedural fluency.**
2a	Continue to construct the multiplicative nature of the chain rule.
2b	Generalize the chain rule to any function composition *f(g(x))*.
2c	Gain procedural fluency with the application of the chain rule.

The opening learning activity in stage two is to continue with the running example, but to add symbolic functions into it. The student is given the prompt: "Let's say in a perfect world, you run at a constant rate of 0.1 miles/minute." The student is shown the functions $D(t) = 0.1t$ and $c(D) = 20D^2 + 40D$. Questions at this point include: (1) What is an equation for dD/dt? (2) What is an equation for dc/dD? (3) What are dD/dt and dc/dD at 20 minutes? These questions are intended to help students begin to connect the symbolic derivatives with the meaning for derivatives developed in stage 1. To return to function composition and the chain rule, we ask: What is an equation for $c(D(t))$? What is dc/dt at 20 minutes? This requires explicit attention to symbolic function composition and an interpretation of the chain rule at a specific time value, which relates to subgoal 2a. Then to move toward subgoal 2b, we ask: How can you use what you have found so far to write a general equation that will give you dc/dt at any time t? How do you know? This is followed by an instruction to write a symbolic expression for dc/dt in terms of only t (i.e., without "D" in the expression). After doing so, the instructor highlights that the expression is the derivative of the composition of functions, $c(D(t))$.

To continue to achieve subgoal 2b, the next activity states, "Say that now you are 'Dash' from The Incredibles and you are running at an incredible pace." The functions are then changed to $D(t) = 2^t$ and $c(D) = D^3$. The purpose of this is force functions that will not easily simplify with each other as in the previous activity. Similar questions are used, and the instructor similarly summarizes that the product of the two derivatives is, in fact, the derivative of $c(D(t))$. At this point, we ask: What patterns do you notice? Given any function for $c(D(t))$, what is dc/dt? Here is where keeping track of the student's previous ideas can be used to lead them to realize that what had previously been the combining of two derivatives to get a third can be reversed by thinking of the one derivative that must be broken down in terms of two derivatives. Finally, to achieve subgoal 2c, the students are asked to think about $g(x) = \sin(f(x))$ and to find the derivative dg/dx. This is followed up by taking the derivatives of $h(x) = \sin(x^2)$ and $j(x) = [\sin(x)]^2$.

The third stage focuses on using NMR to identify functions of implicit variables and work through simple related rates problems. Table 3 shows the learning goals for this stage.

Table 3. The Learning Goals for Stage 3

Stage 3	**Develop the idea of variables being functions of the implicit variable time and recognize subsequent existence of compositions of functions.**
3a	If variables change with time, conceptualize them as functions of time and represent them as such
3b	Create compositions of functions structure for functions of implicit variables
3c	Recognize the need for, and correctly use, the chain rule for functions of implicit variables

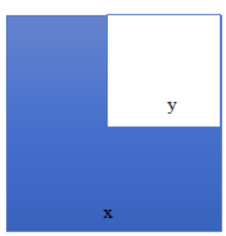
By the end of stage 2, students should be comfortable with the idea of the chain rule, and this is now extended in stage 3 to introduce a related rates problem. Note that in common textbooks (e.g., Stewart, 2015), implicit differentiation comes before related rates. However, we believe that simple contexts involving related rates may be easier to understand and to use NMR for than implicit differentiation. In fact, in stage 4 we capitalize on understandings developed in stage 3 to help extend NMR to implicit differentiation.

This shorter stage contains only a single problem: "The body of a snowman is in the shape of a sphere whose radius is melting at a rate of 0.25 ft/hr. Assuming the body stays spherical, how fast is the volume changing when the radius is equal to 2 ft? Remember that for a sphere, $V = \frac{4}{3}\pi r^3$." At this point, we ask the students to think of all the things that are changing with time. This is to help time become explicit and to help achieve subgoal 3a by determining what changes with time. The instructor then writes these as symbolic functions of time, as in $V(t)$ and $r(t)$, and asks the students whether the equation can be written as $V(t) = \frac{4}{3}\pi r(t)^3$. Time is allowed for using NMR to recognize the nested changes in $t \rightarrow r \rightarrow r^3$. This recognition may be important for connecting this new activity to the previous activities, helping achieve subgoal 3c. Key questions at this point are: (1) What are we trying to find? (2) How can you represent it as a derivative? (3) How can you represent 0.25 ft/hr as a derivative? (4) How can we find *dV/dt*?

The fourth stage of the HLT switches from a related rates problem to using NMR to work through simpler implicit differentiation problems. Later in stage 5, we extend to more complicated related rates and implicit differentiation problems. While this may seem to jump between topics, recall that a major point of this HLT is the connectedness of these topics. Since we do not want them to be compartmentalized, it makes sense for students to build understanding of them simultaneously. Table 4 shows the learning goals of stage 4. Figure 1 then shows the context used to introduce stage 4, which while not "real world," involves familiar ideas.

Table 4. The Learning Goals for Stage 4

Stage 4	**Develop the idea of implicit functions and recognize existence of compositions of functions.**
4a	Given an equation with variables x and y, conceptualize y as an implicit function of x, or x as an implicit function of y. Represent these symbolically as $y(x)$ or $x(y)$.
4b	Recognize the need for the chain rule in taking the derivative of an implicit function



A square with side length x has a corner cut out in the shape of a square of side length y. The blue area must always equal 1. That is, $x^2 - y^2 = 1$. The question we will attempt to answer is: As x changes, by how much does y have to change?

Figure 1. Context used to introduce implicit differentiation

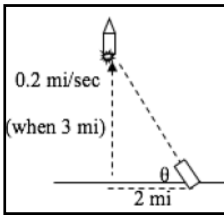
The first action in this activity is to discuss that y is a function of x, because whatever x is determines what y has to be. By drawing on ideas in the previous activity in stage 3, this is written as $y(x)$ to help achieve subgoal 4a. Key questions at this point include: (1) Can you write what we are trying to find as a derivative? (2) How does $[y(x)]^2$ relate to what we have been discussing in our previous sessions? The second question allows NMR to come up again by making connections to previous work. We also ask how much the blue area changes as x changes, and explicitly write down "$d[blue]/dx$" as we ask this. This helps draw on meanings of derivatives as ratio of changes to lead to student to the conclusion that $d[blue]/dx = 0$. We then take the lead to explain that if this derivative is 0 and $[blue] = x^2 + [y(x)]^2$, then $d[x^2 + y(x)^2]/$

$dx = 0$. At this point the student is reminded of the original question and is asked what derivative relates to that. Assuming students know that the derivative of a sum is the sum of the derivative, we ask: What is the difference between dx^2/dx and $d[y(x)]^2/dx$? This hits at subgoal 4b by allowing student to connect to previous thinking that the derivative of y^2 will require the chain rule. Guiding the student through the mechanics of the derivative procedure then allows them to produce an answer. We also reinforce subgoal 4a by reversing what counts as the function and asking: If y changes by some amount, how much will x change? This stage is concluded by doing this process again with a second example of an abstract symbolic equation: $x^2 + [f(x)]^3 = 9$.

The fifth and final stage of the HLT extends the ideas already built up to more complicated related rates and implicit differentiation problems. Recall that we intentionally alternate between related rates and implicit differentiation to support connections. Table 5 shows the learning goals for stage 5 and Figure 2 shows the activities used in this final stage. These activities are meant to help the students continue (a) to develop and solidify their understanding and of related rates and implicit differentiation, (b) to see connections across the two types of problems, and (c) become procedurally fluent with these types of problems.

Table 5. The Learning Goals for Stage 5

Stage 5	**Extend ideas to more complicated implicit differentiation and related rates contexts.**
5a	Continue to gain competence in recognizing compositions of functions in these types of problems
5b	Recognize need to use the chain rule for functions of implicit variables or implicit functions
5c	Gain procedural fluency with more complicated related rates and implicit differentiation problems



A camera is filming a shuttle launch… [see image]… how fast should the angle be increasing when the shuttle is 3 miles above the ground?

(a)

An airplane is flying horizontally at an altitude of 1 mile and speed 500 mi/hr over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles from the station.

(b)

Let $\sqrt{x + y} = x^4 + y^4$.

Find *dy/dx*.

(c)

Figure 2. Context used to extend related rates (a and b) and implicit differentiation (c).

In these activities, we intend for the students to be more independent in their thinking and work, with us acting as guides for when they are stuck or confused. Doing so is important for letting the students develop their own fluency with these types of problems, as in subgoal 5c.

Conclusion

In this paper we have described the development of a hypothetical learning trajectory for the topics of the chain rule, implicit differentiation, and related rates that is meant to teach these concepts conceptually and interconnectedly. We explained how the construct of nested multivariation was used as the foundation, and how a conceptual analysis, the research literature, and pilot studies provided structure for how a learning process might unfold. In a second paper on this topic, we will relate a small-scale teaching experiment with four individual students using the entire HLT. This second paper will help prepare this HLT for a future full-scale classroom teaching experiment. We believe such an HLT and teaching experiment to be crucial for improving calculus curriculum, rather than simply discarding difficult topics from the curriculum. Our HLT has structured a learning sequence that uses NMR to develop strong meanings for derivatives and the chain rule, and to show the close relationships that has to implicit differentiation and related rates. We believe this HLT can help students see these topics as a connected whole, as opposed to isolated individual topics.

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Mathematicians' Proof Repertoires: The Case of Proof by Contradiction

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This paper reports findings from task-based interviews with 6 mathematicians, conducted to document mathematicians' in situ application of their orientations towards proof by contradiction; that is, their reflective dispositions towards proof by contradiction. However, analyses suggest that when engaging in situated analyses mathematicians' behaviors are better characterized as proof repertoires: sets of action-based dispositions tacitly codified into relationally-determined proving routines and best thought of as replicable agent-tool interactions. This theoretical report introduces these constructs (i.e., orientations and repertoires), and illustrates the utility of the latter when interpreting mathematician's ways of reasoning about proofs by contradiction in situ.

Keywords: Proof by Contradiction, Indirect Proof, Proof Repertoires

For a moment, imagine you are a young college student enrolled in your first "proof" course. You have been assigned Chapter 17. You sit down at your desk and read the following:

> This chapter covers proof by contradiction. This is a powerful proof technique that can be extremely useful in the right circumstances. We'll need this method in Chapter 20, when we cover the topic of uncountability. However, contradiction proofs tend to be less convincing and harder to write than direct proofs or proofs by contrapositive. So this is a valuable technique which you should use sparingly.

Figure 1. Chapter 17: Proof by Contradiction - Introduction

What would you think of the method? What impression would you have? The method is extremely useful but to be used sparingly? It's less convincing, yet valuable? Let's consider the mathematician who authored the text. What was the mathematician's intent? The method itself is not described nor aspects of its application. Instead, an attitude towards the method is shared. Perhaps the mathematician hoped to enculturate the student into the discipline, not just its practices but also its attitudes and inclinations? Given the conflicting nature of the remarks it is not clear. What is clear, however, is the commonality of expressions of attitudes towards proof by contradiction in introductory texts. Indeed, chapters on the method often include statements best characterized as *orientations towards proof by contradiction*: reflective dispositions; that is, attitudes and inclinations arising from one's reflection on and interpretation of past and present events, which guide one's behavior when characterizing, describing, valuing or objectifying proofs by contradiction. For instance, consider the two text excerpts in Figure 2. The first describes proof by contradiction as less elegant and less clear than direct proofs. The second states it is, "perhaps the strangest method of proof" (Cunningham, 2012, p. 93). These are not tidbits to guide one's application of the technique, one's recognition of contexts best-suited for its application or at developing one's insights into the method's underlying structure. Instead, these are expressions of mathematicians' *orientations towards proof by contradiction*.

A proof by contradiction is often easier, since more is assumed true; you are able to assume both the hypothesis and the negation of the conclusion. On the other hand, a proof by contradiction is likely to be less elegant than a proof by contrapositive. In any case, for elegance and clarity, it is better to choose a direct proof over an indirect proof whenever possible (Barnier & Feldman, 2000, p. 43). Text: *Introduction to advanced mathematics*
There are times when it is not easy to see how to prove a mathematical statement, say Ψ. When this happens one should try the strategy called proof by contradiction. This strategy is perhaps the strangest method of proof (Cunningham, 2012, p. 93). Text: *A logical introduction to proof.*

Figure 2. Two text excerpts on proof by contradiction.

And, while one might assume enculturative remarks are common in *Introduction to Proof* texts, one would be hard pressed to find similar sentiments in the chapters on other proof techniques.[1] Moreover, outside of these texts, further evidence of mathematicians' *orientations towards proof by contradiction* are easily found. For instance, consider the commentaries shown in Figure 3.

Commentary A
… it would be foolish to repudiate "reductio ad absurdum" as a tool of discovery. It may present itself naturally and bring a decision when all other means seem to be exhausted as the foregoing examples may show. We need some experience to perceive that there is no essential opposition between our two contentions. Experience shows that usually there is little difficulty in converting an indirect proof into a direct proof, or in rearranging a proof found by a long "reductio ad absurdum" into a more pleasant form from which the "reductio ad absurdum" may even completely disappear (or, after preparation, it may be compressed into a few striking sentences). In short, if we wish to make full use of our capacities, we should be familiar both with "reductio ad absurdum" and with indirect proof. When, however, we have succeeded in deriving a result by either of these methods we should not fail to look back at the solution and ask: *Can you derive the result differently*?
Commentary B
Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess play: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.
Commentary C
Without proof by contradiction, a mathematician is a pugilist with his hands tied behind his back.

Figure 3: Commentaries A, B and C

The first, Commentary A, is from Polya's famous 1945 text, *How to solve it*, in a section titled "Objections." In it we find an inclination; namely, should one produce an indirect proof then it is wise to seek an alternative. The second, Commentary B, is from G. H. Hardy's famous 1940 text, *A Mathematician's Apology*. Here we find a somewhat contrary attitude; namely, the method is "one of a mathematician's finest weapons," due to the fact one's whole theory is at risk (i.e., the mathematician offers the game). One cannot read these remarks without thinking of Gauss's efforts to prove Euclid's fifth postulate by contradiction and instead developing hyperbolic geometry. Lastly, in Commentary C, we find sentiments attributed to Hilbert as he reacted to the intuitionists and put forth calls for attention to consistency within mathematical systems. Indeed, taken together these commentaries provide further evidence of mathematicians' *orientations towards proof by contradiction*. Why might these attitudes and inclinations be of interest?

Theoretical Roots and Branches

In the United States, there have been several waves of educational reforms, which sought to change mathematics classroom practices and student achievement through innovative curricular materials. Seeking to understand the impact of these reforms, researchers began to explore and develop theories of mathematics curriculum use (Remillard, Herbel-Eisenmann, & Lloyd, 2009). One outcome of these efforts was the construct of an *orientation towards curriculum*: "a set of perspectives and dispositions about mathematics, teaching, learning and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials and consequently the curriculum enacted in the classroom and the subsequent opportunities for student and teacher learning" (Remillard & Bryans, 2004, p. 364). *Orientations* are of interest, for as Remillard and Bryans show, they impact curriculum use and student-teacher-curriculum interactions. Taking these findings into consideration, it is reasonable to posit the general notion of an orientation might be useful in describing mathematician's perspectives of and dispositions towards proof by contradiction curriculum that together influence how mathematicians engage with such proofs and that impact how mathematicians produce resources and discuss and enact such proofs in the classroom and the subsequent opportunities for student learning; that is mathematicians' *orientations towards proof by contradiction*. Indeed, it seems reasonable to assume such orientations will affect not only the content, tasks, and activities brought into the classroom but also the discourses by which they are communicated, the instructional milieus fostered, and the enculturative practices employed. Hence, they warrant researchers' attention.

Related Research

Researchers have not explored mathematicians' views of indirect proof or the extent to which mathematicians prefer direct rather than indirect arguments in the context of comparative assessments. However, researchers have explored mathematicians' instructional practices related to proof (*cf*. Alcock, 2010; Fukawa-Connelly, 2012; Iannone & Nardi, 2005; Weber, 2012). In particular, Lai and Weber's (2013) explored factors mathematicians consider when modifying a given proof for the purpose of increasing its pedagogical value. They found that mathematicians attended to the audience's background and they considered how to explain results via diagrams or otherwise highlight key ideas when modifying proofs for pedagogical purposes. Inglis and Mejia-Ramos (2009) explored the extent to which mathematicians' sense of persuasiveness is influenced by authority. The authors found that arguments attributed to known mathematicians were more persuasive to the study participants. Together, these studies suggest factors other than validity and rigor influence mathematicians' personal and pedagogical evaluations of proofs. None of the above-mentioned studies, however, explored mathematicians' attitudes towards indirect proof or the influence of such attitudes of mathematicians use or selection of proofs.

The Study

The aim of the study was to explore the mathematicians' views of indirect proof by examining the ways in which mathematician's orientations towards proof by contradiction impact mathematician's comparative assessments of sets of paired direct and indirect proofs and of the commentaries. To carry out this work, 6 mathematicians (M1, M2, M3, M4, M5, and M6) were recruited from a state university in the western United States. The mathematicians were selected on the basis of being active teacher-researchers, that is, research publishing mathematicians responsible for 2-3 courses (8-12 instructional hours per week). During the 30-60 minute interviews, the mathematicians were given tasks used in Brown's (2018) study of university students' reasoning about indirect proofs. Due to space limitations, the discussion is restricted to the Type 1 tasks, which are Task 1 and 3, (see http://bit.ly/2H3ct2p), and the Type 3 tasks. Type 1 tasks were comparative assessment tasks (Jones & Inglis, 2015), which asked participants to select the "best" of two proofs, using whichever criteria they wished, and to provide a selection rationale. These tasks involved a pairing of a direct proof and an indirect proof in the form of: a proof by contradiction (1B) and a proof by contraposition (3A). In the case of Theorem 1, the direct proof (1A) was a longer argument involving more complex content (e.g., series) than the indirect proof (1B), which only required definitions for even and odd. The Theorem 3 task involved two proofs, which were similar in complexity, content, and length. The indirect proof was a proof by contraposition since prior work indicated such proofs are often regarded as proofs by contradiction (e.g., See Figure 4). For the Type 3 tasks, participants were shown the set of commentaries in Figure 3, without any other information and asked, "Which commentary best aligns with your personal views?"

Methods

To examine the impact of mathematician's orientations towards proof by contradiction, a *thematic analysis* (Braun & Clark, 2006) was employed. Specifically, participants' responses were analyzed to identify recurring attitudes or inclinations elicited during the comparative assessment tasks and commentaries. As will been seen, this analysis did not prove fruitful. Consequently, a secondary analysis explored the mathematicians' comparative assessment practices; that is, their situated ways of assessing the affordances and constraints of each method in relation to the theorem, their own knowledge and tendencies.

Findings

A primary finding of the study is that the participants' comparative selections varied by task rather than by proof form. In other words, participants did not uniformly select the direct proof as the "preferred" proof, either personally or instructionally (i.e., for classrooms).

Theorem 1		Theorem 3		Theorem 4	
Argument A (Direct)	Argument B (Indirect)	Argument A (Indirect)	Argument B (Direct)	Argument A (Direct)	Argument B (Indirect)
1	5	0	6	3	3

Table 1: Comparative Selection Task Results

Indeed, assessments of the "preferred proof" focused on which was perceived to be the most elegant, convincing or comprehensible proof rather than on proof form. And, none of the participants uniformly regarded a specific proof form as less favorable. To be sure, across the 18 comparative assessments, there was only one instance in which a participant's remarks were indicative of an *orientation towards proofs by contradiction* (see Figure 4). And, even then, other grounds were provided for the selection; namely, that the proof leveraged a key definition (for set equality) and its' related proof practices (see Figure 4).

> M2: … my personal preference (Theorem 3, direct proof B) … it seems to me more direct (…) this (3A, indirect proof) is proof by contradiction (…) *usually not my favorite kind of proof*. I like this more (the direct proof) because it embodies the idea of how to show two sets are equal … show that one is a subset of the other and the other is a subset of the former and … I sort of prefer that.

Figure 4. A mathematician's expression of an orientation towards proof by contradiction

Looking more closely at the mathematicians' rationales, we find this variation in preference is rooted in the mathematicians' assessments of each proof's affordances and constraints, rather than the mathematicians' orientations towards proof by contradiction. Further evidence of this trend can be seen in M2, M3 and M4's remarks when comparing the Theorem 1 proofs.

M2: I find it (the direct proof) a little more convoluted. This (the indirect proof) I found more … briefer … more straight forward…

M3: 1A (the direct proof) is interesting to me because I hadn't ever thought about that as a way to prove this statement and therefore that makes it more interesting (…) I still probably have a preference for 1B (the indirect proof) because it… it is cleaner, faster, doesn't rely nearly as much on other information, like, you don't need to have knowledge about series, you don't need the result (…) about a^2(…) this (direct proof) is one of those arguments that when my students make this … something like this… I think, oh this is interesting. I'm going to have fun reading through this and checking this all through, while I'm thinking (…) it does feel like they used a sledge hammer to kill an ant.

M4: this one (1A, the direct proof) uses a rather significant fact … in order to like … as the engine to get going … whereas this (1B, the indirect proof) sort of only relies on the basic definitions and structures so it… the sort of amount of theoretical machinery I see involved in this argument (1B) is less than the theoretical machinery involved here (1A) …you have to know more things in order to do this one (1A)… So, I prefer (1B).

In these remarks, we see several considerations. Which proof is "cleaner?" Which is "faster?" Which has a lower cost in terms of the information required? Which will require less time to parse? And, surprisingly, we see a mathematician referring to an indirect proof as "more straightforward." At the same time, we do not see consideration of the underlying method. To be certain, instead of arguing proofs by contradiction are less elegant (or clear), the participants' remarks conflict with those in the texts (*cf*. Figure 2). And, while this is not surprising, given that the Theorem 1 proofs were designed to explore the limits of the attitudes expressed in the texts by providing a lengthier more complex direct proof, it is interesting that the participants did not describe the set of proofs as an exception to prior observations of or trends in such proofs.

Theorem 3: Suppose a set A has the property, for any subset B, $A \subseteq B$. Then, $A = \varnothing$.	
Proof A:	**Proof B:**
Suppose $A \neq \varnothing$. Then there exists an a, such that $a \in A$. Hence, $A \not\subseteq \varnothing$. Thus, there exists a subset B for which $A \not\subseteq B$.	Assume A has the stated property. Recall, that $\varnothing$ is a subset of every set. Thus, $\varnothing \subseteq A$. By the given property, $A \subseteq \varnothing$. Thus, $A = \varnothing$.

Figure 5. Theorem 3 Task

Turning to the Theorem 3 task (Figure 5), one might expect to find expressions of attitude since all of the mathematicians selected the direct proof (See Table 1). Indeed, the proofs were similar in length and complexity – characteristics that might bring proof type to the fore. Yet, outside of the single response in Figure 4, expressions of general inclinations or attitudes were not observed. Instead, what was observed was an expert's assessment of the resources required, the resources employed, and the alignment between the given approach and one's anticipated practices. For example, consider M6's rationale where he speaks of how he would approach the proof – *his own proving tendencies* – and his surprise related the alternative:

M6: And, the thing about this one is (…) I naturally would prove things this way (direct proof), and I would never have thought to do this (indirect proof) ... showing that there's an element and then reaching the conclusion that that set is not a subset of the empty set … who would think to do this?

And M3's rationale, where her own inclinations guide personal preference, while at the same time an affordance of the indirect proof is considered in relation to instructional settings:

M3: I think I have a preference for 3B (the direct proof) but I think that's because that might be how I would have done it. (…) I'm not sure that I strongly have an instructional preference (…) I like this (direct proof) ... (…) I can pick B to be the empty set. I like that notion of it and I think that once they are done with it, they can see they've done the containment both ways and, yay, they're equal! However, I love … here (the indirect proof) the idea of thinking about what does it mean to say something is not the empty set.

To be sure, in the 18 comparative assessments elicited, only one included an explicit expression of general attitudes or inclinations toward proof by contradiction; that is, statements indicative of *orientations towards proof by contradiction*. Instead, the mathematicians tended to engage in detailed, context specific assessments of the affordances and constraints of the given proofs, the resources required and/or utilized, and the mathematicians' own proving practices. Thus, it seems that mathematicians' orientations towards proof by contradiction may not describe or predict mathematicians' *in situ* selective comparisons of proofs direct and indirect proofs.

	M1	**M2**	**M3**	**M4**	**M5**	**M6**
Commentary Selection	Parts of A, B, & C	None, Part of A	Parts of A and C	B and some of A	None; B is the least problematic.	All A, B, & C but not one

Table 2: Type 3 Task Responses

In regard to the Type 3 tasks, we see in Table 2 participants not only held dissimilar views but also did not agree with any single commentary. Instead, the mathematicians tended to hone in on aspects of the statements they felt might be useful and, more generally, to react negatively to the idea there was any utility to these broad sentiments. For instance, consider the remarks of M5, who described the commentaries as detrimental to students' development.

M5: …. points are being made in a level of generality that I just don't think holds … and these blanket statements, I think can be detrimental … because they might insinuate …well, then all that stuff you learned about *reductio ad absurdum* is a waste of time. And, well, actually, it's not a waste of time…it's another perspective that might help a student understand a given problem that they might not be able to understand using the other techniques.

M5 then argued Commentary C implied mathematicians did not have other fighting techniques. He countered by arguing a "better analogy" would be mathematicians "can't throw an upper cut." Then, speaking across the commentaries, M5 noted Commentary B was the least problematic since it asserted the least about the method of proof. M2 expressed a similar perspective. Specifically, M2 noted she disliked all of the commentaries, but Commentary A was the least objectionable since it was "more moderate." M2 went on to note proof by contradiction is both "important and useful" and "it's just another strategy in the tool box, so we don't want to lose it." These remarks highlight two recurrent themes in the mathematicians' responses: (1) the commentaries were too general to be useful; and (2) proof by contradiction is a necessary tool, which is well suited *for certain mathematical situations*, a point explained by M4.

M4: I do not agree with this one (Commentary C) …I don't think that you have your hands tied behind your back if you don't use proof by contradiction […] *it also sort of depends on who you are boxing*.

Discussion

Researchers have argued students find indirect proofs particularly difficult to accept (Antonini & Mariotti, 2008) and that students find these proofs less convincing (Leron, 1985; Harel & Sowder, 1989). Understanding students' ways of knowing do not develop in a vacuum, this study sought to explore mathematicians' *orientations towards proof by contradiction*, arguing these would impact students' instructional milieus and resources. And, while the resources mathematicians develop for students clearly indicate mathematicians' *orientations* impact students' instructional resources, the impact of these orientations during mathematicians' *in situ* assessments is less clear. To be sure, the mathematicians' in situ assessments were not indicative of general attitudes, inclinations, or biases towards or against proof by contradiction. Instead, what was observed was an assessment of an argument's affordances and constraints, it's utilization of resources and their connection to the theorem to be proved. In other words, their selections of preferred arguments were not rooted in *orientations* but rather appear to arise from interactions between the agent's (the mathematician's) ways of reasoning, the tool (the proof technique) and the theorem to be proven (the aim of the agent-tool interaction). Hence, mathematician's *in situ* assessments are not best characterized in terms of mathematicians' orientations towards proof by contradiction but rather are better characterized by mathematicians' *proof repertoires*: action-based dispositions emerging from one's content knowledge, proving skills, and reasoning habits, which are tacitly manifested as situated proving routines; i.e., situated agent-tool interactions.

Concluding Remarks

The reported study aimed to explore mathematicians' *orientations towards proof by contradiction* impact on mathematicians' *in situ* comparative assessments of paired proofs and commentaries. What was found, however, was rather than bring their orientations to bear, mathematicians assessed the aims of the theorem and sought the best *tool* for leveraging available resources and assets, highlighting habitual practices, and communicating ideas; that is, they brought their proof repertoires to bear on the theorem and paired proofs. And, while this finding is not necessarily surprising, given most who have taken advanced courses are familiar with refrains such as, "proof techniques are like tools in a tool belt," this finding is remarkable. Sentiments indicative of *orientations towards proof by contradiction* are readily found in the introductory materials mathematicians produce for students and, are often the first sentiments students encounter. Yet, one would be hard pressed to find remarks descriptive of the behaviors observed during the mathematician's comparative assessments; that is, remarks about the characteristics of theorems, resources, or disciplinary contexts that warrant use of this tool or of the practices (moves or tendencies) that lead to one's selection of the method; that is, remarks that reflect and reveal mathematician's proof repertoires. Thus, there appears to be a misalignment between mathematicians' in situ behaviors and the enculturation practices employed in "Introduction to Proof" texts, at least in the context of *proof by contradiction*.

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[1] The term *enculturation* is used in the sense of Kirshner (2004), who defines enculturation as "the process of acquiring cultural dispositions through enmeshment in a cultural community" (pp. 769).

The Theory of Quantitative Systems: Deconstructing "Symbolic Algebra" to Understand Challenges in Linear Algebra Courses

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I devised the Theory of Quantitative Systems (Sipes, 2019) as a lens for considering the complexity of comprehending linear algebra. The theory resulted from a study of Duval's (1999, 2006, 2017) Theory of Semiotic Representation Registers. With my theory I consider strictly algebraic contexts for systems of linear equations. This report discusses some of the theory's details which may provide insights into challenges encountered in linear algebra courses. Appendix A is a one-page presentation of how I employed Duval's theory in creating my own.

Keywords: linear algebra, systems of linear equations, mathematics comprehension, registers of representation, reversibility, enmeshed communication

Introduction & Rationale

The Theory of Quantitative Systems allows for the exploration of the mystery, to some degree, of the challenges experienced in linear algebra courses by focusing on the ubiquitous topic of systems of linear equations. Linear algebra requires a different approach than adopting a single category of "symbolic algebra" and rejecting its consideration for less abstract areas of research. The theory resulted from taking a more incisive look into Duval's theory than its frequent reduction to a theory of "multiple representations".

Deconstructing "Algebraic"

My theory specifically targets analytic representations of systems, here understood as the algebraic symbolism used to denote them. In mathematics education research, a dichotomous view often emerges: mathematics is either procedural or conceptual (Hiebert & Lefevre, 1986). As a result, focus is often placed on what is considered to be conceptual mathematics and the symbolic is neglected. This may result from not distinguishing *ordinary algebra*, the familiar algebra of real numbers, from other contexts like matrix algebra. While ordinary algebra may be a sufficient partition or grain size in some contexts and at some levels of mathematics, I assert that the complexity of linear algebra requires the adoption of an alternative perspective.

With my theory, I partition "symbolic algebra" into distinct categories, each with its own means of notation and rules of engagement. The more finely-grained categories may provide insight into the seemingly inexplicable wall many students experience in first year linear algebra courses; facets that may go unrecognized by experts in their everyday practice are identified. Awareness may help linear algebra experts support students in their learning. Further, instructors of courses prerequisite to linear algebra may be better able to support students if they attend to mathematics beyond the algebra of real numbers, *ordinary algebra*. Where others acknowledge the variety of mathematical representations (including graphical and tabular) as a source of students' struggles in mathematics, I focus exclusively on analytic representations.

Comprehending Linear Systems: Foundations for a Theory

Reflecting on Duval's Theory of Semiotic Representation Registers (1999, 2006, 2017) and conducting an analysis of students' written exam data led me to further analyze the potential cognitive complexity of comprehending systems of equations. Hereafter, I use "linear systems" or “systems of linear equations” to refer to the common algebraic listing of equations for which simultaneous solutions are sought, if solutions exist. I use the term “quantitative system” or simply “system” to refer to the set of quantitative relationships represented by a linear system. I devised the idea of a *quantitative system* to be consistent with Duval’s notion of a mathematical object; for him a mathematical object is an entity that we only have access to in the physical world through a semiotic system. Duval’s ideology necessarily means that he does not consider a semiotic representation, as something concrete and perceptible, to be a mathematical object. In that case, linear systems like the one shown in Figure 1 are but one way of denoting the quantitative system under consideration. Other representations of the quantitative system are possible; I focus on four possibilities. Theoretically, I do not give the linear systems representation primacy over any other representation for a quantitative system. The quantitative system is *the represented* while the four ways to designate the system are *the representations*.

$$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - z = 0 \end{cases}$$

Figure 1. A linear system, one way to represent a quantitative system.

Of the ways to denote a quantitative system shown in Figure 2, the linear system is often been taken to be the object of study, the represented. The other representations are taken as alternative means of expression in the study of the linear system. That is, the linear system is taken to be *the represented* while the other three are taken to be the representations. Payton (2017) took a slightly different but similar perspective by considering the linear system to be the primary *representation*. He called his perspective “systems-centric”, but ultimately he concluded that the four denotations are “representations of one another” (Payton, 2017, p. 90). Either way, a study of quantitative systems is often initiated with the linear systems representation. Larson and Zandieh (2013) conducted a related study of students’ interpretations of the matrix equation $A\vec{x} = \vec{b}$. They found that one way that students interpret equations of the form $A\vec{x} = \vec{b}$ is to think of them in terms of a linear system. Rather than focus on any particular representation, I find it useful to think in terms of a quantitative system as the object of study and to consider all representations of the quantitative system to be equitable.

Linear System	**Vector Equation**	**Augmented Matrix**	**Matrix Equation**
$\begin{cases} 2x + y - z = 5 \\ 3x - y + 2z = -1 \\ x - y - z = 0 \end{cases}$	$x\begin{bmatrix}2\\3\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\\-1\end{bmatrix} + z\begin{bmatrix}-1\\2\\-1\end{bmatrix} = \begin{bmatrix}5\\-1\\0\end{bmatrix}$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 5 \\ 3 & -1 & 2 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right]$	$\begin{bmatrix}2 & 1 & -1\\3 & -1 & 2\\1 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\-1\\0\end{bmatrix}$

Figure 2. Four ways to represent a quantitative system.

Duval (2006) conjectured that comprehension in mathematics requires the coordination of at least two registers (systems) of semiotic representation; he theorized that at any time, we have at least two registers engaged. (One of these is often natural language, which is not accounted for in my theory.) If no particular representation of a quantitative system is given prevalence, six pairings of representations are possible from the four representations shown in Figure 2. However, research has shown that reversibility is not automatic (Krutetskii, 1976; Pavlopoulou, 1994), so translation between the pairings should be considered in each direction. The 12 resulting types of translations are shown in Figure 3.

Linear System	►	Augmented Matrix
Linear System	►	Vector Equation
Linear System	►	Matrix Equation
Augmented Matrix	►	Vector Equation
Augmented Matrix	►	Matrix Equation
Vector Equation	►	Matrix Equation
Augmented Matrix	►	Linear System
Vector Equation	►	Linear System
Matrix Equation	►	Linear System
Vector Equation	►	Augmented Matrix
Matrix Equation	►	Augmented Matrix
Matrix Equation	►	Vector Equation

Figure 3. Types of Translations of Representations of a Quantitative System.

Visual consideration of Figure 3 alone may make one aware of previously unconsidered intricacies involved in working with linear systems. Numerous translations are possible, and the nature and demands of each translation differ. While the preceding analysis of the variety and combinations of representations reveals a complex situation, much more can be said if Duval's cognitive hypotheses are considered. The preceding analysis was the basis of a new theory, the Theory of Quantitative Systems (TQS). I summarize and rationalize the TQS in what follows.

The Theory of Quantitative Systems (TQS)

The Theory of Quantitative Systems incorporates the perspective that all semiotic representations that may arise in working with a linear system have equal precedence. That is, no particular representation is chosen as the object of study while the other representations are taken to be pointers to that particular representation. Rather, all representations (Figure 2) are taken as pointers. This raises the question: what are the pointers signifying? I assert that the object of study (the signified or the represented) is the set of quantitative relationships that could be regarded as underlying the written representations; I refer to the underlying set of quantitative relationships as a *quantitative system*.

The construct *quantitative system* establishes a particular entity as the object of study. While the idea is abstract, it establishes an invariant object of study that is not a semiotic representation. This characteristic is missing from perspectives that take one of the representations as the object of study (the represented) and assert that another representation signifies the first. Further, perspectives that all representations are representations of one another may leave one to wonder if we are signifying anything beyond collections of notation. Pinpointing a particular object of study allows for new and potentially productive ways to think and speak about linear systems.

I consider the aforementioned analysis surrounding the 12 translations in Figure 3 to be an analysis of the mathematics; I assert that the analysis supports an argument for the complex nature of working with systems. However, much more can be said in terms of cognition. A discussion of additional cognitive complexity requires a keen focus on the definition of *registers of representation* and applying the definition differently and at a new level of granularity. Note that my discussion makes an explicit distinction between the mathematics and the cognition that I do not find in existing literature that appropriates Duval's theory.

New Categories for *Registers of Representation* (and "*Symbolic Algebra"*)

When Duval's theory is described simply as a theory of "multiple representations", the discussion rarely includes specificity about what constitutes a *register of representation*. As a result, what are taken to be categories of registers are quite broad. An example is Pavlopoulou's (1994) dissertation study. For her, categories of registers were: graphical, symbolic, and tabular; the distinctions allowed her to obtain important and convincing results in linear algebra (Artigue, Chartier, & Orier, 2000). Similar categories (graphical, symbolic, and tabular), while not explicitly discussed in terms of "registers", have been productive for analysis in other contexts; the function concept is an example. I assert, however, that a greater level of specificity is possible, and even necessary, to better understand challenges in the learning and teaching of linear algebra.

Duval (1999) used *register of representation* to mean a system of signs and symbols used to communicate along with the ways those signs and symbols are used to represent and process mathematical thinking; *registers* include both representations and a means of processing. This means each of the four representations (Figure 2) for a quantitative system (linear system, vector equation, augmented matrix, and matrix equation) is suggestive of a *register*; each representation resides within a different mathematical system with its own rules of association and means of computation. For example, vector equations involve scalars and vectors along with the rules of vector addition and scalar multiplication, whereas matrix equations are composed of matrices that can be manipulated according to the rules of matrix algebra. Rather than adopting register categories like graphical, symbolic, and tabular, I see declaring each mathematical system to be a *register of representation* as being consistent with Duval's notion. For example, a matrix algebra is a register since it involves specific representations (matrices) and rules for associating them (the operations and properties of matrix algebra). As a result, "symbolic algebra" can be broken down into subcategories required to effectively consider the complexity of linear algebra.

An Additional Level of Complexity

Having established that, for me, each mathematical system is a different register, I am positioned to speak about additional cognitive complexity. The 12 translations that I have identified are translations between differing registers (conversions). The 12 conversions say nothing of translations within a particular register (treatments). That is, the 12 do not address translations like manipulating a vector equation using properties of vector spaces to obtain a result that resides in the same register, a vector space. If we appropriate Duval's (1999, 2006, 2017) cognitive hypothesis that the kind of cognition required for conversions differs from the kind of cognition required for treatments, we can imagine an intensely complex scenario. The complexity goes beyond acknowledging that various semiotic systems and combinations of representations make working with systems challenging. The hypothesis that cognitive mechanisms for working within a register differ from the cognitive mechanisms for working across registers takes the discussion of complexity to a new level.

Implications of the Theory of Quantitative Systems

Theoretically, the Theory of Quantitative Systems accomplishes at least two things. First, the designation of a *quantitative system* provides a mechanism for clearly distinguishing between the represented and the representation in the context of linear systems. Second, the TQS highlights the potential cognitive complexity of working with systems. I have discussed the idea of cognitive complexity; I will say more about decoupling *the represented* and *the representation*. In the following I also discuss additional implications that may be more practical in nature.

Decoupling the Represented and the Representation

Duval described what he called the cognitive paradox by asserting that mathematics comprehension requires that we not confuse *the represented* with *the representation* even though the only (perceptible) access we have to the represented mathematical object is through the symbols we use to denote it. I propose the descriptor *enmeshed* for situations where "the represented" is not distinct from "the representation"; in essence, *enmeshed* is descriptive of the confounding of the representation and the represented. In that case, one may have an *enmeshed conception* where there is no distinction in his/her mind between the represented and the representation. One may also engage in *enmeshed communication* where the communicator may on some level be aware of the distinction between the represented and the representation, but his/her communication does not make the distinction explicit. In other words, one may cognitively separate the represented and the representation while communicating in such a way that there is no clear boundary between the two.

Consider the situation where an instructor, who clearly knows the difference between the represented and the representation, communicates in such a way that the distinction is not clear; students may conclude that they are studying the notation, the linear systems representation itself as shown in Figure 1. (This may be appropriate in a theoretical course; see the A Qualification section below.) Given the idea of a quantitative system, experts can reflect on the cognitive paradox and consider ways his/her communication might promote the confounding of the representation with the represented amongst students. That is, experts can reflect on a potential *expert blind spot* (Nathan, Koedinger, and Alibali, 2001) where their content knowledge prevents them from viewing the content in terms of students' development and learning processes.

The need for such explicit distinctions is documented in literature. Hillel (2000) studied videotaped sessions of lecturers teaching on the topic of eigenvectors and eigenvalues. He noted that the experts moved fluidly between various representations, giving little notice to the nuances of notation and meaning; enmeshed communication is likely what he observed. I also note that Harel (2017) referred to the linear systems representation as both *the represented* and *the representation*, and Trigueros (2018) seemed to use the word "model" interchangeably to refer to the application (traffic flow) and the symbolic representation of the situation. These could be instances of enmeshed communication, though I qualify this observation below.

In summary the construct *quantitative system* is a mechanism which provides an individual (on an intrapersonal level) with a way of thinking which clearly distinguishes between the representation and the represented. As a mechanism, it allows for the identification of an object which is not (nor can be taken to be) notation. On an interpersonal level, the idea of a quantitative system provides a language for communicating which avoids the confounding of the represented and the representation. The construct should allow experts to consider whether their communication is enmeshed in ways that may influence students to fall victim to the cognitive paradox. More broadly, the Theory of Quantitative Systems provides a new way to think and speak about systems of equations.

A Qualification. I see considering the notation of a linear system to be *the represented* as acceptable in a theoretical course. As I understand theory related to systems of equations, we have systems of equations in the algebra of real numbers which we can translate to matrix algebra; working in the matrix algebra informs us about the system of equations. This a common practice in mathematics: reframing a problem in a context with different affordances to gain insight into the original problem. One example is linearizing. In the case of linear systems and matrices, the values of the entries of a solution matrix are precisely the values that solve the system; reframing the linear system in the matrix algebra informs about the system. The notation of the linear system is *the represented*; it is recast as a *representation* in another mathematical system to leverage the features of the new mathematical system. Abstractly moving between mathematical systems to process mathematics is what I refer to as a theoretical approach.

Practical Implications/Considerations

Gauging Flexibility. Analysis of a clinical interview that I conducted with a student revealed that he engaged with four of the 12 translations I identified. I suggest that the experts can use Figure 3 to consider their own flexibility in moving between registers; further, the table can be used as an instrument to assess students' flexibility. Increased awareness by experts will position them to better support students in their learning, especially in the direction of good *structure sense* -- the parsing of mathematical expressions into appropriate units. (Note the way I use *structure sense* differs from its use in other mathematics education literature.)

Side-by-Side Presentations. Representations of systems are often presented side-by-side in instruction and textbooks without acknowledgment that translating differs depending on the direction of translation. This could be because representations are inappropriately given the mathematical characteristic of equality. If we think of representations as occurring within different languages, "equality" is not an appropriate descriptor of language. I see applying a mathematical property like equality to representations as a symptom of an enmeshed conception; the mathematical object and the notation used to communicate it are indistinct. Study of clinical interview data suggestive of a student's confounding of the idea of mathematical equality with the idea of alternative representations of the same entity led me to consider this distinction.

Alternative approaches. I note that the bulk of undergraduates take a single course in linear algebra that is applied in nature. In the case of the top-selling linear algebra textbook in the United States (Lay, Lay, and McDonald, 2015), writing a system as a linear combination of columns (which I call a vector equation) is given a place of prominence that is uncharacteristic in more traditional texts (e.g., Hoffman & Kunze, 1971). Placing the additional representation alongside the others complicates the consideration of what the object of study is. While I assume the authors laid out a plan of study they saw as logical and/or productive in terms of *content*, the effect of the alternative approach on the quality of students' understandings may need further investigation. Such an approach may make the distinctions afforded by the Theory of Quantitative Systems more crucial than more traditional theoretical approaches.

Conclusion

One who argues that only a few translations are needed for solving systems may not recognize the implicit expertise involved in knowing which translations are optimal. The sequence of translations along a solution trajectory may seem quite natural to experts given their experience and proficiency; their expertise may allow them to focus on a small subset of translations in ways they may not realize. Is the same true of the student experience? Is the learner able to focus on the required subset of translations in usual solution trajectories, or does the learner have a murkier experience of multiple representations which are not related in any particular way? I claim that we do not know; a tool like the Theory of Quantitative Systems may help answer such questions.

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Appendix A

The Theory of Quantitative Systems

Duval's Three Sources of Incomprehension in Mathematics
1) Treatments, 2) Conversions, and 3) The need to recognize many representations as indicating the same mathematical entity.
The Cognitive Paradox
Duval's suggestion that although we only have (perceptual) access to mathematical objects through the symbol systems that we use to represent them, mathematics comprehension requires that we not confound the representations with the represented mathematical object.

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Duval's Constructs	My Constructs
A mathematical *object* is an abstract entity that we only have access to through a symbol system.	The set of quantitative relationships represented by a number of linear equations, the *quantitative system*, is the mathematical object.
Transformations are changes to a representation both within a register (symbol system) and between registers (symbol systems) including graphical ones.	Translations are changes to a representation both within and between analytic registers.
Treatments are *transformations* within a register.	Treatments are *translations* within a register.
Conversions are *transformations* between registers.	Conversions are *translations* between registers.
Congruence/Incongruence of a transformation takes into account two criteria: transparency and unit-by-unit translation.	I take transparency to mean visual similarity and units to be sub-pieces of algebraic notation.
Registers of representation are differing symbol systems in which representations can be expressed.	Each mathematical system is designated to be a register of representation. A vector space is an example of one register of representation; representations are formed from scalars and vectors.
Transforming in reverse is a relevant consideration.	*Translating* in reverse is a relevant consideration.
Distinguishing between the representation and the represented is fundamental to mathematics comprehension.	The quantitative system is clearly defined to be the represented mathematical object. All representations, including the linear systems representation, are given equal precedence.

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The Theory of Quantitative Systems designates a set of quantitative relationships as the object of study in the context of systems of equations. The four representations involved are given equal precedence, and acknowledgement is given to the differing nature of translating between two representations depending on the direction of translation. Equitizing the representations and taking reverse translations into account allows the identification of 12 possible translations between the representations.

Working Towards a Unifying Framework for Knowledge for Teaching Mathematics

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In this discussion, we frame an argument to no longer invoke the Mathematical Knowledge for Teaching (MKT) framework (Ball, Thames, & Phelps, 2008) at the post-secondary level of mathematics education research, since it is theoretically based in the work of elementary school teachers. After reviewing the MKT components and related research regarding the extension and modification of the MKT framework to higher mathematics education, we propose a recasting of Shulman's (1986, 1987) initial formulation of content knowledge for teaching as a route for developing a unifying framework for mathematics education. This report uses Precalculus adjunct instructors' interview data to propose a coding framework by applying Shulman's principles and categories of content knowledge.

Keywords: Knowledge for teaching; Mathematics instructor development; Precalculus

Knowledge for teaching mathematics has been widely researched and discussed since Shulman's (1986) initial introduction of *pedagogical content knowledge* as a construct to consider separately from *subject matter content knowledge* and *curricular content knowledge*. The issue of advancing and integrating instructors' knowledge for teaching is vital to supporting students' conceptions of mathematics (Musgrave & Carlson, 2017). Knowledge for teaching directly impacts the ways in which an instructor approaches their teaching including task selection, making sense of student thinking, and posing questions. This paper describes the larger project this work stems from, provides an overview and discussion of research on knowledge for teaching mathematics, and details our argument for a unifying framework by relating our coding schemes to Shulman's original framework.

The AMIRS Project

At our university, Precalculus is taught almost exclusively by adjunct instructors. As a response to the teaching population and concerns for student achievement and retention, we have been providing professional development and support specifically to our Precalculus adjunct instructors through an NSF funded project, *Adjunct Mathematics Instructor Resources and Support: Improving Undergraduate Precalculus Teaching and Learning Experience* (AMIRS Project; NSF Award #1712058). In the AMIRS project, we have developed, and continue to refine, a model for promoting adjunct instructors' learning through course coordination and job supports. As part of a larger effort to improve our Precalculus program, we adopted a research-based Precalculus curriculum and have consistently provided a variety of supports (e.g., course coordination, summer workshop, weekly online meetings, access to trained embedded tutors) to help our instructors feel more integrated into our department and to aid them in implementing the curriculum. By building a model of adjunct instructor resources and support, this work contributes to a deeper understanding of how such efforts impact (1) instructor knowledge and instructional practices, (2) job satisfaction, and (3) student academic success and retention in STEM majors. This understanding may help other departments and institutions that have a similar need for adjunct instructors in courses with multiple sections better support their instructors, thus ultimately improving student outcomes.

This paper uses data from the pilot and first years of our project to present a theoretical discussion of instructor knowledge for teaching Precalculus. The source of data was transcripts of interviews that were conducted at the beginning and end of each semester. The interviewer asked questions related to benefits and challenges of the selected research-based curriculum, their teaching practice, the supports they had been receiving through course coordination, and their job satisfaction. In our course coordination and review of the data, as instructors engaged with the research-based curriculum, we found that the various forms of their content knowledge were developing. We aim to understand the ways in which instructor content knowledge has evolved and how this knowledge was leveraged for instruction and developed over time. As a result of our analysis process and attempts to connect to relevant literature, this paper questions the ways in which researchers have been framing conceptions of knowledge in the secondary and post-secondary spaces. We then conclude by proposing a path forward in understanding knowledge for teaching mathematics at the secondary and post-secondary levels.

A Review of Research on Mathematical Knowledge for Teaching

Shulman's (1986) AERA Presidential Address birthed educational research's current framing of knowledge for teaching by parsing Pedagogical Content Knowledge (PCK) from subject area content knowledge and curricular content knowledge. This framing served as an impetus for mathematics education research to take up work in this area to determine what is needed for teachers to effectively teach mathematics. In particular, much of the work has capitalized on developing the area of PCK, since this is the knowledge that will facilitate student learning through teachers' transformation of content area knowledge in combination with general pedagogical knowledge. The importance of understanding PCK, as well as mathematical and curricular content knowledge, is undeniable as the field works towards the improvement of mathematics teaching and learning.

The *Mathematical Knowledge for Teaching (MKT)* framework (Ball, Thames, & Phelps, 2008) builds on Shulman's (1986) three main categories of content knowledge by theorizing about examples in elementary mathematics teaching. Ball et al. (2008) substantiate their framework using the work of elementary mathematics teachers. Their framework, commonly referred to as 'the egg', subcategorized *Specialized Content Knowledge (SCK)* for teaching as separate from common content knowledge, and knowledge of content and students and knowledge of content and teaching as part of PCK, resulting in a two-part oval figure consisting of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge as the main categories. Ball et al. (2008) also included *curricular knowledg*e within PCK "provisionally" (p. 402) but stated: "We are not yet sure whether this may be a part of our category of knowledge of content and teaching or whether it may run across the several categories or be a category in its own right" (p. 403). Furthermore, Ball et al. (2008) introduced the phrase *horizon knowledge* as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 404). This term was also provisionally placed, this time within SMK. The implications of these unsubstantiated placements will be discussed below in the presentation of our coding scheme.

The MKT framework is widely studied and referenced in all grade bands of mathematics education research, despite its basis in elementary mathematics teaching. It seems that just as PCK as a concept became immensely popular, so has MKT, as its use is pervasive in the field. However, when MKT is invoked by mathematics education researchers, we must be aware that the entire meaning of the framework, including the subcomponents are called upon as well. This implication is of concern because "the expansion of these theoretical constructs beyond the grade levels at which they were developed should be approached cautiously, with attention to the ways in which the ideas can and cannot be applied to other populations of teachers" (Speer, King, & Howell, 2015, p. 112). The following sections of this paper discuss relevant literature on extending and altering the MKT framework to secondary and post-secondary mathematics education research.

Mathematical Knowledge for Teaching Beyond Elementary Education

As researchers at the secondary and post-secondary levels use the MKT framework, they frequently find the need to develop something similar either by forcing the components of MKT to fit their data or by modifying the components or definitions. As Hill, Sleep, Lewis, and Ball (2007) state regarding assessments of teacher knowledge, mathematics education researchers "may miss important elements of the knowledge that makes teachers successful. In addition, using these theoretically impoverished tools also limits the conclusions that can be drawn" (p. 122). As a research community, we should ensure that we are not missing elements that apply to higher levels of mathematics education and appropriately develop the tools and frameworks we use. For example, in their attempt to apply MKT, Speer, King, and Howell (2015) "found the framework insufficient to explain" (p. 113) their data. They describe, "in seeking to explore the robustness and generality of the framework, we pursued disconfirming cases that illustrate ways in which the framework may need to be refined or expanded" (p. 113). These points demonstrate that we require a framework (other than MKT) that is theoretically grounded in the work of secondary and post-secondary teachers.

Howell (2012) argues that the work around what knowledge is needed for secondary teaching is an important and unanswered question. Early studies (e.g., Begle, 1972; Hanushek, 1981; Monk, 1994) have inconsistent findings regarding the relationship between courses taken by teachers and student achievement. What we do know, however, is that teachers should at least know the mathematics they will be teaching: the problem is that we do not have a good understanding of the depth of knowledge needed or what other mathematics teachers should understand (Howell, 2012). Part of the issue is that we still do not know if "the most important types of mathematical knowledge for secondary teaching are similarly distinct [similar to elementary school] from what is learned in traditional mathematics courses" (Howell, 2012, p. 19). In Howell's discussion of Speer and King's (2009) work, she points out that in the MKT framework, "SCK is supposed to be the specialized content knowledge uniquely demanded by teaching" (Howell, 2012, p. 36). However, she makes the connection that "much of the unpacking and connecting of ideas that is required in teaching might also be demanded of professional mathematicians. If we take as given that the goal of a mathematics degree program is to produce potential mathematicians, then these types of knowledge are not specialized" (Howell, 2012, p. 36). Howell (2012) continues: "SCK as defined by the LMT group's model [Ball et al., 2008] does not account for the type of mathematical background most secondary teachers have, and therefore generalizing the model to include secondary teachers may mean modifying its definitions" (p. 36). Howell (2012) concludes that the MKT framework has not been, and most likely cannot be, generalized to secondary mathematics teaching. The frameworks to date that have used MKT as a basis have used expert opinion, which is not a reliable method for understanding teacher knowledge.

Although the MKT framework does serve to unify the various types of knowledge (elementary) teachers need and is useful for all researchers to orient themselves among the varying terms, it does not comprehensively serve the secondary and post-secondary communities. If anything, attempts to extend or rework the MKT framework beyond the elementary level has caused this research area to become fragmented (Speer, King, & Howell, 2015) into various applications or conceptions. Speer et al. (2015) cite, for example, the Knowledge of Algebra for Teaching (KAT) project (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012), which proposes a framework to test MKT for algebra, or frameworks that work on various items such as test item development, curriculum, or validation of instruments. In reference to the underlying theory of these types of instruments, Hill et al. (2007) point out: "Despite claiming to cover roughly the same terrain, these projects [KAT, SII/LMT, DTAMS] have strikingly different approaches to specifying domains for measurement … In fact, they do not have matching conceptual maps, and the authors claim that none match Shulman's PCK either" (p. 131). We agree with Hill et al. (2007) that if we are "interested in building theoretical coherence around mathematical knowledge for teaching, the variety of approaches is distressing" (p. 131).

In a different way, Silverman and Thompson (2008) call for "an orienting framework for the field" (p. 6) in order to improve teacher quality through the development of MKT. Rather than focus on the skills and understandings for teaching, as MKT does, Silverman and Thompson (2008) focus on understandings for instructional sequences and the network of ideas that can support students' reasoning. Additionally, Hauk, Toney, Jackson, Nair, and Tsay (2014) propose alternate definitions for subcomponents of the MKT framework's PCK terms with the purpose of integrating a dynamic rather than static view. In this way, Hauk and colleagues have chosen to include beliefs or orientation with PCK. This makes sense from an equity standpoint and is another component of unpacking teacher knowledge specifically at the secondary and post-secondary levels. These alternate conceptions of MKT may inform our ways of thinking about knowledge for teaching mathematics, but do not build the constructs to create a unifying view of knowledge for teaching mathematics.

It seems that in the attempts to further develop MKT at the secondary level, the work has either "relied on assumptions, most often assuming that secondary MKT is similar to elementary MKT or assuming that expert opinion suffices to construct a theoretical framework" (Speer, King, & Howell, 2015, p. 112). Researchers who have attempted to develop more concise definitions at the post-elementary levels, have incorporated the same terminology proposed by the MKT framework (Ball et al., 2008). This type of work does not support the development of a unifying framework since it alters the intended definition, causing confusion in the field as to which meaning is being referenced. Rather than refining or expanding the framework at the post-elementary levels, we are proposing a recasting of Shulman's (1986) original definitions for all of mathematics education research.

Developing a Framework for Knowledge for Teaching Mathematics

Based on the literature reviewed here, it seems that researchers have held on to the MKT acronym one way or another by using the same terminology for subcomponents but redefining them for higher levels of mathematics education; or, by using the same constructs as Ball et al. (2008) of SMK and PCK but erasing the subcomponents. We are arguing that the MKT constructs should not be used as a unifying framework as we aim to both categorize types of knowledge and integrate the dynamic nature of the development of knowledge and are working to theoretically ground the categorizations and developmental process in our data.

Hill et al. (2007) state, "we are worried about two trends. One is this field's tendency to conflate teachers' knowledge of mathematics for teaching with other types of knowledge or skill" (p. 150). In our work, we are very clear that we are attempting to capture instructors' *content* knowledge; that is, we want to understand the mathematical knowledge behind their teaching, not only their pedagogical knowledge in working with students. There may be interactions among these types of knowledge, but our focus for the present work is on content knowledge.

Principles for Content Knowledge

In the beginning stages of our interview data analysis, we did a first-level coding to locate where instructors were discussing their content knowledge: we were not yet concerned about what type of content knowledge it was, just that it was related in some way to the content of Precalculus. Rather than depend on others' frameworks, we returned to Shulman's initial formulation: "A teacher is a member of a scholarly community. He or she must understand the structures of subject matter, the principles of conceptual organization, and the principles of inquiry" (Shulman, 1987, p. 9) for the field. Shulman (1986) describes the first of these three: "To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the *structures of the subject matter* [emphasis added]" (p. 9). To define these structures, he refers to Schwab (1978): "The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established" (Shulman, 1986, p. 9). The substantive structures and syntactic structures make up what we have termed *Structures of Mathematics (SOM)*. Content knowledge that is classified as SOM, such as rules, procedures, definitions, and axioms, is the subject matter knowledge that mathematics majors would also hold. An example of SOM in Precalculus is that of a teacher referring to the definition of covariational reasoning.

In addition to SOM, "the teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened" (Shulman, 1986, p. 9). To code instances of this type of content knowledge, we use *Principles of Inquiry (POI).* Areas of research we see as related to this principle are the Mathematical Habits of Mind (Cuoco, Goldenberg, & Mark, 1996) and the Mathematical Practices of generalizing and problem solving (National Governors Association, 2010). Examples of POI would include problem solving techniques, use of language (in Precalculus, wording related to quantities and covariation), and checking reasonableness of mathematical work.

Shulman (1986) continues to explain that we also expect teachers to "understand why a given topic is particularly central to a discipline whereas another may be somewhat peripheral" (p. 9), which we termed to be *Principles of Conceptual Organization (PCO).* PCO includes the importance of concepts in relation to prior and subsequent courses (e.g., for Precalculus, how covariation may connect to Algebra or to Calculus), and the connections that exist between concepts. The conceptual web of content knowledge for an instructor directs the time and emphasis placed on particular concepts or procedures in their class, much like a scope and sequence of a course. For Precalculus, an example of PCO is that covariational reasoning can serve as an organizing structure for the subject (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

Through our first-level coding using these three principles of SOM, POI, and PCO, we found that each existed for the instructors' content knowledge, but it was not enough to be able to tell a meaningful story about the instructors' interactions with the curriculum and the subsequent progression of their content knowledge. As a result, we turned back to Shulman (1986) and began to integrate his "three categories of content knowledge: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (p. 9) in order to further code the data.

Content Knowledge Categories as a Second Dimension

As our team coded the first round of interview data, two things were clear: there were pieces of data that were distinctly PCK due to their connection to the classroom; and, the research-based curriculum we utilized was having an impact on the instructors' content knowledge. Subsequently, we re-coded the same interview data using Shulman's (1986) three content knowledge categories to differentiate among *Mathematical Content Knowledge (MCK), Pedagogical Content Knowledge (PCK),* and *Curricular Content Knowledge (CCK)*. We defined MCK to include subject matter content knowledge that did not have the essence of PCK or CCK, and these MCK data points were codable as SOM, POI, or PCO, as previously described.

PCK, the subject matter knowledge for teaching, is further specified by Shulman (1986) into two subcategories. First, he defines *Representations* (coded as PCK-R) as "the most powerful analogies, illustrations, examples, explanations, and demonstrations … the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Instructors choose representations in their work to prepare for lessons and to enact lessons, which calls on particular content knowledge. Second, Shulman (1986) identified *Conceptions* (coded as PCK-C) as "the conceptions and preconceptions that students of different ages and backgrounds bring with them … teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners" (pp. 9-10). The strategies cited here may refer to representations (PCK-R), however, conceptions (PCK-C) are used to respond to student thinking, therefore, we established the conception code as the use or application of a representation. Within this subcategory, we are able to leverage our knowledge of mathematics education research to include teaching strategies such as building classroom discourse or active learning situations.

Curricular knowledge is defined by Shulman (1986) as the "full range of programs designed for teaching, … [and] the variety of instructional materials" (p. 10). Within curricular knowledge, he describes three subcomponents: the *Tools of Teaching; Lateral Curriculum Knowledge;* and, *Vertical Curriculum Knowledge.* First, the tools of teaching (CCK-T) "present or exemplify particular content and remediate or evaluate the adequacy of student accomplishments" (Shulman, 1986, p. 10). We consider our research-based Precalculus textbook to be a tool as it provides a problem-based orientation of Precalculus. Second, lateral curriculum knowledge (CCK-LCK) "... underlies the teacher's ability to relate the content of a given course or lesson to topics or issues being discussed simultaneously in other classes" (Shulman, 1986, p. 10). Examples of CCK-LCK are multidisciplinary connections such as science lab experiments or computer programming. In our pilot and first year interview data, there were few examples of CCK-LCK, but we saw overlap of these examples with POI (principles of inquiry), since the ways of thinking in these contexts allowed application of content knowledge. Third, vertical curriculum knowledge (CCK-VCK) is the "... familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years" (Shulman, 1986, p. 10), much like horizon knowledge. CCK-VCK is categorized as a curricular content knowledge, rather than MCK, since it is related to how the mathematical topics are organized. In conjunction with PCO (principles of conceptual organization), CCK-VCK demonstrates the connections among mathematical concepts, which instructors may build, reorganize, or re-prioritize when interacting with a curriculum.

It is clear that mathematics education researchers have found it useful to differentiate between MCK and PCK. It has not been determined, however, whether CCK is subsumed under PCK, as Ball et al. (2008) proposed, or if it is its own category. Due to the nature of our work and the heavy influence of the interaction between the instructors and the curriculum, we treat CCK as a separate category. While coding, we did notice that there was overlap between PCK and CCK because the curriculum was influencing the decisions made for the classroom (Rahman, 2018). While instructors may reference the curriculum and curricular resources, when the statement is in service of student learning, we decided this would be related to PCK and not CCK. Further connections between PCK and CCK will be considered in subsequent analyses.

Interactions between Categories and Principles of Mathematical Content Knowledge

The principles of SOM, POI, and PCO can easily be seen in the MCK data examples. Within PCK and CCK, we also found instances that could be cross-coded as SOM, POI, or PCO, which provide a second dimension of analysis of the data. If we view the categorical buckets of MCK, PCK, and CCK as content categories, the principles of SOM, POI, and PCO can cut across all three buckets, thereby providing a way to look across the types of knowledge. Not all PCK and CCK data points were able to be classified as a principle, but this tactic allows us to tell more of the story of our instructors' development. These two dimensions incorporate both the static and dynamic nature of instructor content knowledge.

Ball et al.'s (2008) work only directly refers to Shulman's (1986) main categories of content knowledge, they do not explicitly incorporate his principles of SOM, PCO, and POI. We argue that these principles give rise to different conceptualizations of SMK: connections throughout the field of mathematics can be explained using PCO; ways of thinking and knowing can be described using POI; and, syntactical (symbolic, notation) and structural particulars about the subject matter itself can be detailed through the principle of SOM. Moreover, we propose that these principles can also be applied to the categories of PCK and CCK as ways of thinking about the content for pedagogy and in relation to curriculum. Without these cross-cutting principles, it would be difficult to account for the different ways of knowing and thinking in mathematics and in teaching and learning the subject, and therefore, difficult to work towards a unifying framework.

Concluding Thoughts

The categories and principles of content knowledge have allowed our research team to ground our analysis in a framework that classifies the types of content knowledge and also characterizes the type of thinking behind the use of the content knowledge. As a result, we are beginning to develop conjectures regarding the rationale behind instructors' use of particular content knowledge and the ways in which their content knowledge may be developing.

We plan to continue refining this framework through subsequent years of project data and more types of data (e.g., classroom observations, PLC participation) in order to build a more complete picture of each instructor's development. Our goal is to conduct a more comprehensive analysis for telling the story of how our instructors' content knowledge is transforming through their interaction with and implementation of the research-based curriculum. We also hope to further understand how existing constructs from prior research can fit into this framework, and how this framework may provide possibilities for analyzing other sets of data with the intent of having common language to discuss secondary and post-secondary mathematics knowledge for teachers.

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A Meanings-Based Framework for Textbook Analysis

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Textbooks are one of the most important tools teachers and students use in any course. Thus, textbooks can greatly influence what occurs in classrooms. Textbook analysis provides insight into the rolls and effects of textbooks have on the potentially implemented curriculum for any course. This paper proposes a new framework that builds from several existing ones while introducing a focus on analyzing the authors' conveyed meanings by drawing upon Thompson's theory of meanings. While developed to address a need for textbook analysis in introductory statistics, the proposed framework can work well in mathematics and at different levels (i.e., not just introductory courses). Further, the proposed framework is a starting point for developing a more robust framework.

Keywords: textbook analysis, introductory statistics, meanings

There is a need for doing a textbook analysis in introductory statistics courses in today's age. As far as I can find in the literature, there have been four such analyses—one wide ranging analysis and three that looked at specific topics: Cobb's (1987) analysis of sixteen texts across the statistics curriculum, Cai, Lo, and Watanabe's (2002) comparison of US and Asian treatments of the *sample arithmetic mean*, Pickle's (2012) dissertation on statistics content in middle school math texts, and Hoekstra, Morey, and Wagenmaker's (2018) examination on confidence intervals. Since Cobb's article, there have been a number of events that have impacted the mathematical and statistical education landscape: NCTM developed and released standards documents (2000), the Third International Mathematics and Science Study (1995), the Guidelines for Assessment and Instruction in Statistics Education (2005; 2016; 2007), the Common Core State Standards in Mathematics (2010), as well as the rise of open source and online materials. However, there has not been a concerted effort to evaluate textbooks used in statistics courses. My goal with this report is to propose a general framework for evaluating textbooks across the mathematics and statistics curriculum. Due to space limitations, I will focus in this paper on describing existing frameworks and their insufficiencies. Then share my proposed general framework as a work-in-progress. Last, I will give uses and limitations.

Existing Frameworks

Textbook analysis refers to the examination of the content of a textbook with the intent to describe the topics contained, level of content, and/or the use of student-centered language. In the following section I will describe several existing frameworks as well as a framework not originally developed for analyzing textbooks but serves as a useful tool in the present endeavor. I will include critiques and limitations on them to highlight their shortcomings.

Cobb's framework. Cobb (1987) built his framework to address three personal convictions:

> 1) Statistics is fundamentally and primarily concerned with analyzing real data. 2) Data analysis, including inference, is both intellectually challenging and intrinsically interesting. 3) Until recently, most authors of introductory statistics textbooks have managed to do a superb job of concealing from their readers the truth of the first two points. (Cobb, 1987, p. 321)

Cobb selected 11 then new textbooks as well as five older texts to use in his analysis. Cobb's framework looked at three areas: technical level and exposition, the topics covered, and the quality of the exercises.

The technical level and exposition aspect centered on Cobb's assessment of how much mathematics a student needed to understand the text; "I have found it helps to think explicitly about the way in which a book implicitly calls on students to *use* mathematics" (Cobb, 1987, p. 322, emphasis in original). To this end, he categorized texts into nine levels with levels 3-5 being a partial ordering. His levels varied from having no symbols in the text (the least demanding, level 1) to heavy usage of algebra (level 6) to extensive usage of multivariate calculus (the most demanding, level 9). To analyze texts in this dimension, Cobb focused on two questions: to what extent does a formula support a basic concept? and to what extent does the author rely on symbols to connect concepts? While short on details about this aspect of his framework, Cobb does give some examples. In his *sample arithmetic mean* example, he argues that an "author who relies heavily on formulas is likely to slight this connection [between the *arithmetic mean* and a histogram's balance point] even if the right words appear in the text" (p 323). I must point out that from the article, Cobb espoused thinking about the *sample arithmetic mean* in highly procedural terms; "the average of a list a numbers equals their sum, divided by how many there are" (p. 323). The signs of procedural favoritism occur within the examples on the *sample arithmetic standard deviation* and sampling distribution. Within the examples of the normal distribution and the sampling distribution, Cobb demonstrates a treatment of distribution that matches what Moore and Thompson (2015) refer to as static shape thinking; the normal distribution is a shape in the graphing plane to be recognized and the Central Limit Theorem is about a shape and not a way to use a process's long-run behavior to make inferences.

Cobb's topics dimension looked at how textbooks handle six areas: regression, ANOVA, exploratory and resistant methods, computers, probability, and nonparametrics. For the most part, Cobb evaluates these areas on a simple paradigm of how much/far do they cover? In terms of exploratory data analysis, he makes a strong case to differentiate the techniques of exploratory data analysis (i.e., stem-and-leaf plots, box plots) from exploratory attitudes. For the computer dimension, Cobb acknowledges that he supports computers in a statistics course but if there is not active usage of a computer, then why bother including computer output? That is, unless students use the computer, they will struggle to see the need of learning to read output.

The final dimension of Cobb's framework (the exercises) is best summarized in the following quotation: "Judge a book by its exercises, and you cannot go far wrong" (Cobb, 1987, p. 331). Within this portion, Cobb calls for looking at three aspects: the data used, does the problem answer an interesting question, and the ratio of thinking to calculating. In terms of the data used, Cobb looked at whether the text used real data (did the data come from a cited source) or whether the data felt made up. While Cobb counted the number of data sets, he "excluded lists of numbers intended solely for drill" (Cobb, 1987, p. 333); this choice has the effect of biasing the results of Cobb's framework in the data set area. Exercises, to Cobb, should mimic the work of statisticians and thus, the problem should not end with a calculation. Rather the problem should end with the student using the calculated value to answer some question about the underlying context of the problem. Finally, Cobb proposes but does not quantify a ratio of how much the text asks students to perform calculations/carry out procedures to how much the text askes the students to thinking and reason.

Cobb's (1987) framework provides a nice example for analyzing [statistics] textbooks. He proposes several areas to keep in mind (exposition, topic coverage, and exercises) and highlights some of the unique aspects for statistics textbooks (usage of data). However, his framework lacks a rigorous way to evaluate the content and belies a focus on procedural thinking.

TIMSS framework. As part of the Third International Mathematics and Science Study, 48 countries participated in the planned textbook analysis (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). The framework that Valverde et al. developed centered on three aspects (structure, presentation, and performance expectations) as well as physical measurements (e.g., number of pages, total page area, use of graphics, etc.). The dimension of textbook structure centered on how each textbook incorporated content, presentation formats, and expectations throughout the text. This allowed the research team to develop a schematic for each book (see Figure 1). Each vertical hash reflects a block (base unit of analysis) and how researchers coded that block in terms of type (referring to the underlying rhetoric of the block), content (topic), and performance expectations.

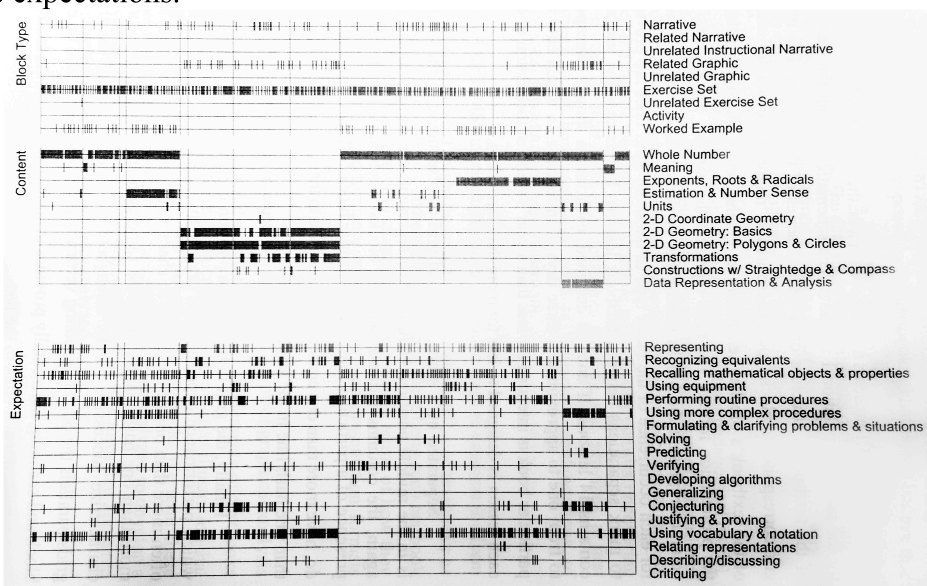

Figure 1. Example of a grade 4 math textbook schematic from TIMSS (Valverde et al., 2002 p. 56).

Valverde et al. (2002) note that when they examined the content of the 400+ books in math and science, they did not examine the specific mental processes involved at any level of the texts in their cross-national setting. Such a task would be sensitive to cultural differences and challenging for any type of inter-rater reliability measurement. Thus, the TIMSS analysis did not dig into the content (i.e., conveyed meanings, coherence) beyond topic and student performance expectations. Performance expectations refer to specific behaviors that the block of text would ask of the student such using vocabulary, performing routine or complex procedures, conjecturing, and relating different representations. While not explicitly stated so, Valverde et al.'s expectation categories are reminiscent of Bloom's taxonomy.

Shield and Dole's framework. Building from the American Association for the Advancement of Science's Project 2061, Shield and Dole (2013) built their framework centered on extant research and standards documents to establish specific curriculum content goals. These goals then serve as the basis to evaluate a textbook. To do this analysis, they established three indicators they felt would contribute to each goal. Focusing on the portions of textbooks dedicated or closely related to their target area (ratio, rate, and proportion), they characterized the amount of evidence for each indicator as High, Medium, Low, or No[ne]. Shield and Dole position their framework as but one part of analysis of a topic. Their framework does not take into consideration how teachers or students use a textbook, only what opportunities for mathematical experiences the textbooks provide students.

Shield and Dole's framework stands in contrast to both Cobb's and TIMSS in that there was a more rigorous focus on the content presented in the texts. While Cobb payed some attention to content, Shield and Dole used a systematic approach to first articulate and then assess content goals. However, Shield and Dole looked for whether a textbook had evidence of their indicators and did not describe the meanings that books' authors conveyed. For example, all of the textbooks that Shield and Dole examined showed medium or high evidence for the indicator of clear links to fraction equivalence, but they do not present what those links are for each text.

Herbel-Eisenmann and Wagner's framework. Herbel-Eisenmann and Wagner (2007) developed a framework centered on how a textbook's language and images might position a student in relation to mathematics, classmates, teachers, and others as well as their experiences. To do so, they propose a series of questions of the form "How might a textbook position students in relation to…" for each of the aforementioned areas. They instantiate this by examining the language choices of the authors (conscious and not) such as the use of personal pronouns and modality. Drawing upon other research, Herbel-Esienmann and Wagner also examined the role of inclusive imperatives (e.g., explain, justify, etc.) and exclusive imperatives (e.g., calculate, graph) tie to mathematicians' activities of thinking and scribbling. This particular framework sets aside macro-structural elements like those in TIMSS, topics like Cobb, and the assessment of goals like Shield and Dole in favor of examining language choices of the authors. The language used by text authors can form an inclusion barrier for students, pointing to potential equity issues missed by the other frameworks thus far discussed.

O'Keeffe and O'Donoghue's framework. Fan, Zhu, and Miao (2013) reported that 63% of 100 published textbook studies focused on analyzing mathematics features, exercises, or comparisons. This helped O'Keeffe and O'Donoghue (2015) highlight that many textbook analyses did not focus on language use. Drawing upon the work of Halliday, they lay out the three core aspects their framework: the ideational function, the interpersonal function, and the textual function. The ideational function focuses on the how the text might elicit some form of activity through processes such as material (doing), mental (thinking), relational, verbal (exchange of meanings), existential, and behavioral. Examining the social relationships between the author and others is the domain of the interpersonal function. Here the researchers looked at how the author conveys his/her position with the reader and subject matter. They achieved this by looking at how the author intrudes into the narrative, uses symbolism, specialist language, and imperatives. The third aspect of the framework, the textual function, examines how the language used relates to the intended purpose of the text.

The interpersonal function has a strong overlap with framework of Herbel-Eisenmann and Wagner's (2007) framework while the ideational function somewhat links with TIMSS performance expectations (Valverde et al., 2002). While O'Keeffe and O'Donoghue's framework provides a lens into the language used, their framework does not provide a mechanism for assessing that language in terms of student thinking. A text which conveys that randomness is another word for haphazard (a problematic meaning), avoids passive sentences (ideational), uses informal sentences (interpersonal), and using a narrative format which encourages reading (textual), would come across just as positively in this framework as one that does the same except conveys randomness as a process's attribute which minimizes sources of bias and enables exploring long-run behaviors (a productive meaning).

Exam characterization framework. The last existing framework is not originally for textbook analysis. Rather the exam characterization framework (ECF; Tallman, Carlson, Bressoud, & Pearson, 2016) provided a team of researchers with a way to characterize Calculus I final exams as part of the MAA's larger investigation of successful college calculus programs. The ECF consists of three aspects. The first is the item orientation which characterizes the cognitive behavior necessary for answering the item. Tallman et al. drew upon Anderson and Krathwohl's (2001) update to Bloom's taxonomy but made two alterations. Tallman et al. added a recall and apply procedure level and an applying understanding level to better describe the required cognitive behaviors elicited by an item. The second dimension is that of item representation, encompassing both how the task appears as well as that of the solicited solution. Representation categories include applied/modeling, symbolic, graphical, and explanation. The final aspect, item format, refers to whether the item is multiple choice, short answer, open-ended, or a word problem. While they referred to the framework as an exam characterization, their focus on items easily transfers to textbook exercises. However, exercises are only one part of a textbook and the ECF does not assess other aspects of textbooks.

Proposed Framework as a Work-in-Progress

The most common limitation to the aforementioned frameworks is their inability to characterize assess or characterize the meanings conveyed and the coherence of those meanings. The framework I'm proposing tackles this explicitly by drawing upon a theory of meanings (Thompson, 2013). I'm treating the term "textbook" expansively to include not only print books, but also PDFs and web-based texts. My framework consists of five main components: the textbook's context, the conveyed meanings and coherence, language used, the structure, and the exercises. Due to space, I will only give a general overview of my framework.

The textbook's context. The context of a textbook refers to those aspects that help position a text within a wider environment. Primary examples here include: what is the author's stated purpose/intent for the textbook is? Is the text meant to be a compendium of procedures or treatise meant to help a novice progress to concept mastery? Is the text presented as being in line with a particular standards or guidelines document (e.g., Common Core, GAISE)? Answering these questions allows us to assess whether the actual text supports the purpose. By examining any preface, letters to…, publisher blurbs, author's descriptions of the text as well as noting the intended audience, and the authors' backgrounds, I propose that you could build an image of the textbook's context.

The conveyed meanings and coherence. This component is the heart of my framework. The others attempt to be indifferent to the meanings conveyed whereas my framework places a primacy on this aspect. Shield and Dole (2013) provide a starting place for such an endeavor. However, while they chose specific curriculum goals and indicators, my framework takes a more descriptive approach. Rather than assessing a text for a particular meaning (e.g., functions are descriptions of co-variational relationships between quantities), this component of my framework asks the researcher to discern the meaning conveyed by the text. For example, does the text convey that a 95% confidence interval of (a, b) means that a person is 95% confident the parameter's true value is between a and b (an example of the fundamental confidence fallacy), or does the text convey that all of the values between a and b are more probable (the likelihood fallacy), both, or something else. To characterize conveyed meanings, the researcher needs to take the text as a whole and not take a single instance as the body of evidence.

Thus, there must be an articulated theory of meanings underpinning such work. I find Thompson's theory to be the most useful here. A meaning refers to the space of implications (actions, images, and other meanings) that results from a person assimilating an experience, forming an understanding of that experience (Thompson, 2013, 2016; Thompson, Carlson, Byerley, & Hatfield, 2014). The set of implications is the meaning that the person ascribes to the experience based upon the inference that accompanies the assimilation. Thus, meanings are idiosyncratic. Given that no one has direct access to another's meanings, how do people learn meanings from one another? Thompson (2013) resolved this through the notion of a conveyed meaning. As two people communicate with one another, each builds a model of other and how the other might interpret his/her words and actions. The meaning that one person gives to the other's communication is the conveyed meaning. The conveyed meaning is still a set of implications but constrained by the receiver's de-centering and belief that the sender made an honest effort to convey his/her thinking. Here, the conveyed meaning of the text is the possible implication space elicited rather that what the author intended.

To assist with this component, a researcher should draw upon the literature, especially that with clearly articulated theories of learning. Wilson's (2005, 2009, 2012) notion of a construct map is of great benefit. The construct map provides an ordering of descriptions of different meanings for the same concept. The ordering could be based upon a hypothetical learning trajectory/progression or might be set by the notion of how productive the meanings are in relation to one another. In either case, the researcher needs to articulate the basis used. Each construct map used helps sensitize the researcher to what meanings the authors could convey through the text.

By attending to meaning, the researcher can also examine the coherence of the text. How the conveyed meanings for different concepts fit and participate with one another is a direct consequence of this approach. This approach, grounded in the notion of conceptual analysis (Thompson, 2008), provides a strong way to assess a focus on conceptual understanding espoused by many standards documents. Additionally, the researcher can make use of other theories for coherence; for example, quantitative reasoning (Thompson, 1993, 2011) has been used in several areas (e.g., Moore, 2012; Oehrtman, Carlson, & Thompson, 2008).

The language used. The third component of the framework draws upon the framework of Herbel-Eisenmann and Wagner (2007) as well as that of O'Keeffe and O'Donoghue (2015). Here, the focus is on the inclusion/exclusion of actors and ownership in the language used. These would link to the ideational and interpersonal functions, as well as how the text might support the reader in positioning herself in relation to the material. The language that authors use not only plays a role in what mathematical/statistical meanings they convey, but also that of the role of the reader. The message that a reader understands about their participation in the mathematical/statistical endeavor of the text is a conveyed meaning.

The structure. The structure component draws upon the TIMSS framework (Valverde et al., 2002) as well as the notion of the textual function (O'Keeffe & O'Donoghue, 2015). Breaking the text down into blocks which the researcher then classifies allows for the development of schematic such as that shown in Figure 1. Such an analysis can allow for the examination of whether the textbook is consistent with structure, thereby speaking to the textual function. While the TIMSS framework provides a solid foundation for block types, there is a 2D limit to these types. Given the advances in online technologies, I feel that new block types including animations and videos (related/unrelated; embedded/linked), applets (related/unrelated; embedded/linked) and student usage of technology are worthy additions.

The exercises. This component is in line with performance expectations of the TIMSS textbook study, but adopts the ECF (Tallman et al., 2016) for analysis. Tallman et al.'s framework provides a solid foundation to assess items included in textbooks that could be linked in later research with exam analysis. For statistics, I must add a fourth dimension to the ECF: the inclusion of data. Cobb's (1987) dichotomy is too course for what statisticians work with and what my own experiences support. Thus, for this fourth dimension, an item might have No Data, Summarized Data (i.e., values of various statistics for an unavailable data set), fake data, sanitized data (cleaned, real data that the student can treat as if fake), and real data where the student will need to engage in data cleaning/wrangling before conducting further analysis.

Uses and Limitations of the Proposed Framework

I must stress that this framework is a work-in-progress. My intent with this paper is twofold: begin crystalizing my own thinking about the framework and to elicit feedback for improvement.

I foresee using this framework not only for the analysis of textbooks and their comparison, but to position such an analysis within the greater context of mathematics and statistics education studies. The focus on conveyed meanings affords an opportunity to draw upon and link research into how student think about ideas to how textbook authors choose to present them. Existing construct maps can help link the textbook analysis to existing research and the analysis could help identify meanings that have not yet identified for inclusion in construct maps. This framework could allow for a more nuanced examination of how textbook authors incorporate the recommendations of various reform efforts in STEM education.

There are several limitations to this framework. First, the framework is quite expansive, and the limited space of this proceeding cannot do justice. Each of the frameworks that came before are involved and my proposed framework does not suggest a way of limiting the workload. Additionally, the focus on conveyed meanings requires the researchers to have a comprehensive knowledge of the literature around various topics and demands careful reading of the text. I suspect that while such a heavy front end might be burdensome, the end results of using the framework will ameliorate this limitation.

A second limitation builds from the first: the researcher's beliefs and understandings will influence the discernment of conveyed meanings as well as the nature of any construct maps used. Take three people who hold three separate views of the sine function: the first views the sine function as being about the sides of a triangle, the second relates sine to circles, and the third views sine as describing periodic co-variation. Each one of these people will prioritize different treatments of the sine function, potentially mischaracterizing treatments that are beyond their own meanings. The first person might not understand something the third writes and potentially classify the text as incoherent or unnecessary for students. Conceptual analysis, articulation of their theory of learning, and construct maps help guard against this issue.

A third limitation with this framework deals specifically with the focus on conveyed meanings. Given that the analysis is on textbooks, I cannot speak to the meanings or intentions of the authors who wrote them. This is where the context dimension serves a mitigating factor to this limitation. My intention for this framework is not to make any claims about the authors' meanings, just what meanings the authors have conveyed to a reader.

This framework shares a fourth limitation with all of the mentioned frameworks: the framework does not attend to how teachers or students use the text, in or out of the classroom. While the implementation of a text is important for investigating what students learn, a first step is identifying what the text could support. This framework fulfills this initial step.

The proposed meanings-focused framework provides one way to draw upon the literature to examine textbooks in a new light while attending to past works. Further, the framework offers a way to describe what meanings the authors might convey to students who use and read the texts.

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A Potential Foundation for Trigonometry and Calculus: The Variable-Parts Perspective on Proportional Relationships and Geometric Similarity

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Until recently, the variable-parts perspective on proportional relationships had been overlooked in mathematics education research. We argue that the variable-parts perspective is implicit in contexts of geometric similarity, including trigonometry and the notion of slope, and that it is well-suited to situations involving varying increments of change, as in calculus. We explain how a variable-parts approach to these topics is related to approaches and findings of research on trigonometry, rate of change, and the Fundamental Theorem of Calculus. Our arguments are theoretical, but lead to the empirically testable hypothesis that an explicit focus on the variable-parts perspective could be productive for learning geometric similarity, trigonometry, slope, and calculus.

Keywords: Proportional relationships, geometric similarity, trigonometry, slope, calculus, variable-parts

Trigonometry and calculus are difficult topics for students (e.g., Thompson, 1994; Weber, 2005). In this paper we explore the idea that the variable-parts perspective on proportional relationships (Beckmann & Izsák, 2015), a perspective that had been largely overlooked in mathematics education research, might be a valuable foundation for trigonometry and calculus.

The variable-parts perspective (Beckmann & Izsák, 2015) is a way to conceptualize how two quantities vary together in a fixed ratio. Key to this view is that quantities are measured with respect to two units: a fixed unit and a variable unit. The fixed unit, called a "base unit," could be a standard measurement unit such as 1 centimeter or 1 Liter. The variable unit, called a "part," is x base units, where x can vary over some set of nonnegative numbers (e.g., all nonnegative real numbers). Thus, if A and B are fixed nonnegative numbers, then two quantities that consist of A parts and B parts, respectively, vary together in the fixed A to B ratio by varying x, the number of base units that make 1 part.

In this paper we discuss why the idea of measuring with respect to two units, one that is fixed, such as 1 cm, and one that can vary with respect to the fixed unit, is worth focusing on in geometric similarity, trigonometry, and calculus. We propose that a variable-parts perspective on proportional relationships might help to make this idea explicit and lay a foundation for trigonometry and calculus.

The Variable-Parts Perspective and Geometric Similarity

In this section we describe how to see situations of geometric similarity from the variable-parts perspective. In Euclidean space, geometric similarity comes from applying distance-preserving transformations and dilations, which scale every distance by the same factor. For simplicity, in what follows, we restrict our discussion to 2-dimensional Euclidean space equipped with Cartesian coordinates, and we focus only on dilations, which we assume are centered at the origin.

The key connection between the variable-parts perspective and geometric similarity lies in seeing dilations in terms of two systems of Cartesian coordinates on the same coordinate axes: one in which 1 unit of distance is a fixed "base unit," such as 1 cm, and a second set of coordinates in which 1 unit of distance is a “variable part” consisting of x base units, where x can be any nonnegative real number. When $x = 1$, the two systems of coordinates are the same. In other words, when $x = 1$, for each point in the plane, its base-unit-coordinates (i.e., its coordinates measured in base units) is the same as its parts-coordinates (i.e., its coordinates measured in parts). A dilation with scale factor s can be interpreted as keeping the parts-coordinates of each point fixed while changing the length of 1 part from 1 base unit to s base units.

Figure 1 illustrates dilations and geometric similarity from the variable-parts perspective. The light (grey) grid lines are 1 base unit (1 cm) apart and they heavy (red) grid lines are 1 part apart. As the scale factor of the dilation changes from 1 to 2 to 2.5 to 3, 1 part—the distance between the heavy (red) grid lines—changes from 1 cm to 2 cm to 2.5 cm to 3 cm. Throughout, the shaded triangle’s vertices have the same parts-coordinates: (0 parts, 0 parts), (4 parts, 0 parts), and (4 parts, 3 parts); the length of the horizontal side, w, is always 4 parts and the length of the vertical side, h, is always 3 parts. But when expressed in terms of the base unit, 1 cm, the coordinates of the triangle’s vertices and lengths of its sides change as the scale factor changes.

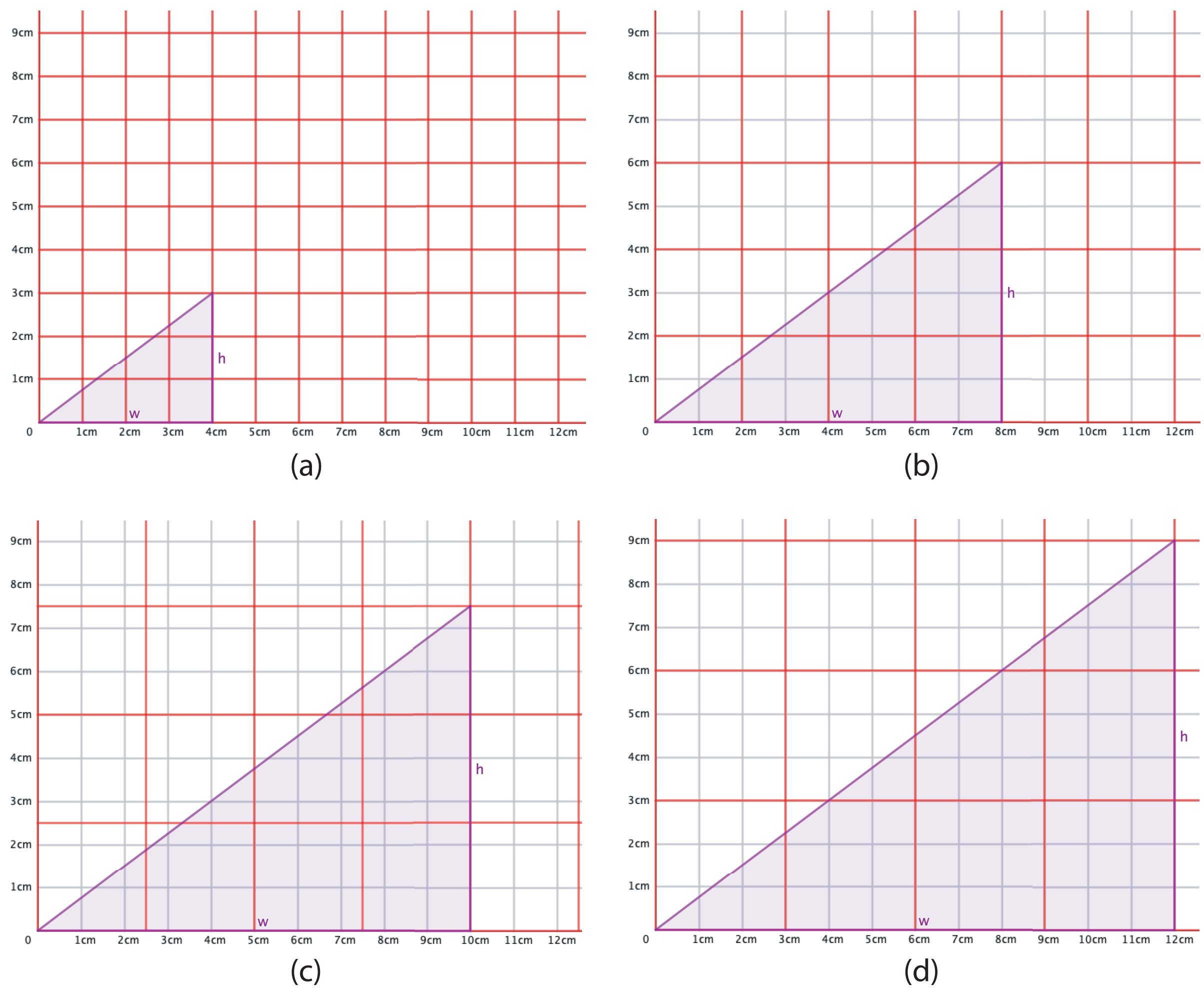

Figure 1. Dilations with scale factor (a) 1, (b) 2, (c) 2.5, and (d) 3. The bold (red) grid-lines remain 1 part apart, but 1 part varies: (a) 1 cm, (b) 2 cm, (c) 2.5 cm, (d) 3 cm.

Because the vertical side-length of the triangle, *h,* is always 3 parts and the horizontal side-length, *w,* is always 4 parts, *h* is always ¾ of *w*, and therefore *h* = ¾ *w*. This equation, which was derived by reasoning about variable parts, expresses the proportional relationship between *w* and *h*, and can also be interpreted as an equation for the line through (0,0) and (4,3). From this perspective, the constant of proportionality or slope, ¾, is the answer to a measurement question: how many (or how much) of the horizontal side-length *w* does it take to make the vertical side-length *h* exactly? As we discuss in the next sections, the idea that one can use a variable-part—a quantity whose size changes when measured in terms of base-units—as a unit of measurement in its own right is implicit in trigonometry and potentially useful in calculus. We therefore conjecture that a variable-parts perspective could be a valuable foundation for those topics.

The Variable-Parts Perspective and Trigonometry

Trigonometry relies on geometric similarity: Trigonometric ratios are well-defined because all triangles that have the same angles are similar. Figure 2 shows the standard trigonometric setup of a right triangle inscribed in a unit circle, but from the variable-parts perspective: a base-unit is 1 cm and 1 part is *r* cm, the radius of the circle and hypotenuse of the right triangle.

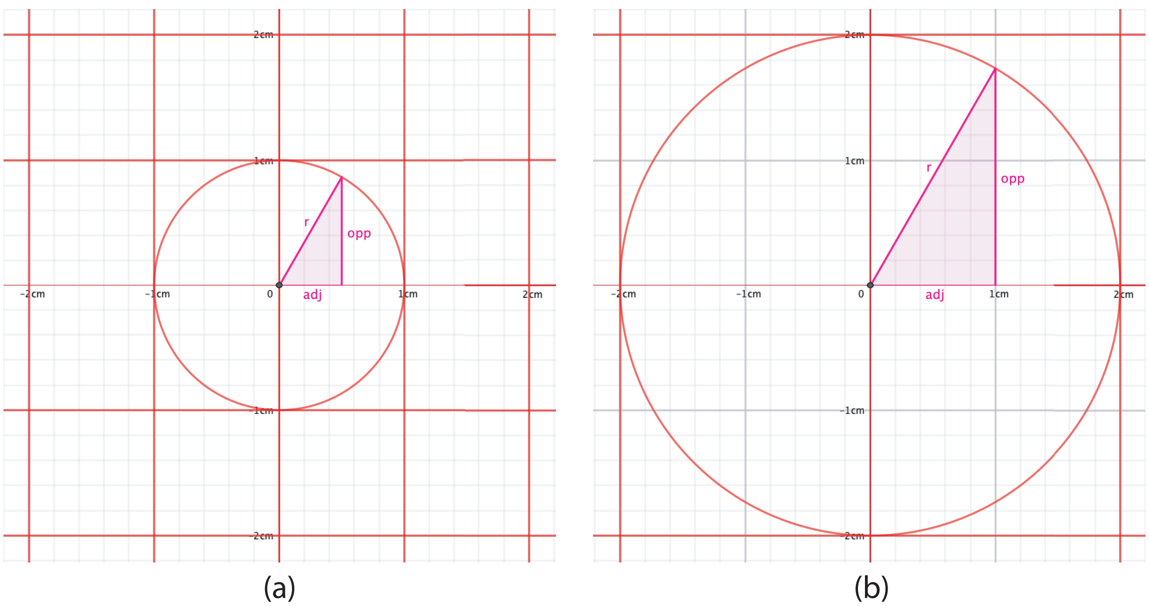

Figure 2. Taking r, the radius of the circle, to be a variable part, the side lengths of the triangle and the length of the arc subtended by the positive x-axis and the hypotenuse of the triangle are a fixed multiple of r, regardless of the size of r in base units (centimeters).

Taking the perspective on geometric similarity discussed above, lengths in the coordinate plane can be measured in base units (centimeters) or in parts (in this case the radius of the circle). When a dilation scales the circle and triangle, length measurements change when viewed in terms of centimeters but remain fixed when measured by the radius. This leads to the notion of radian measure of angles: an angle that is subtended by an arc whose length is that of the radius of the circle is what we call an angle of measure 1 radian. The sine and cosine can also be understood as lengths of the vertical and horizontal sides of the triangle, respectively, when measured in terms of the radius of the circle, which is also the hypotenuse of the triangle.

This perspective on trigonometry, viewed via a variable-parts approach to geometric similarity, fits with the approach Moore (2013, 2014) took in his teaching experiments on angles and trigonometric functions. For example, in one teaching experiment session, Moore (2013) intentionally gave students Judy and Zac different length strings to represent the radius of a circle. Moore asked the students to determine how many string-lengths mark off the circumference of their circle and to create angles that cut off arcs of 1 string length and 1.5 string lengths. Moore gave the students strings of different lengths "with the hope that the students realize that the size of the circle used to create an angle subtending a particular number of radii is inconsequential" (Moore, 2013, pp. 236 – 237). Students Judy and Zac came to understand that arc length can be measured by the radius of a circle and that the measure is independent of the size of the circle. For example, Zac interpreted an arc length of 0.61 as 61% of the radius and also explained that "it's always the same percent of the circle it's cutting out for each different circle" (p. 239).

Moore (2014) found that Zac's ideas of measurement, including reasoning about a radius as a unit of measurement, supported Zac to construct the sine function in terms of radii as well. Zac interpreted the sine and cosine as a percentage of a circle's radius, regardless of the size of the circle, and was able to use these percentages to find coordinates of points on a circle. Zac was also able to extend this interpretation of sine and cosine to a right triangle situation by interpreting the hypotenuse in the same way as a radius of a circle. Thus, Moore's teaching experiment fostered ideas that are consonant with a variable-parts perspective on angles, sine, and cosine.

The Variable-Parts Perspective and Rates of Change in Calculus

Calculus concerns situations involving varying rates of change. To cope with these, there is the notion of average rate of change over smaller and smaller increments, and the notion of instantaneous rate of change. We propose that two aspects of the variable-parts perspective may make it a valuable foundation for calculus. First, the variable-parts perspective foregrounds viewing rates as measurements: how many (or much) of one quantity (such as a change in x) it takes to make another quantity (such as a change in y). Second, it foregrounds measuring with a quantity that varies (such as a change in x over smaller and smaller increments).

One way to think about a rate of change is as how much one quantity changes when another quantity increases by 1 unit. But it may not be clear to students how to interpret a rate of change when the amount of the second quantity changes by something other than 1 unit, especially if the second quantity is shrinking towards 0. In their study of secondary teachers' meanings for measure, slope, and rate of change, Byerley and Thompson (2017) asked a sample of 250 teachers for their meaning of a slope when the slope of a line was calculated as 3.04. The majority of the teachers described the slope as for every change of 1 in x, there is a change of 3.04 in y, or as how to move horizontally and vertically along the graph, such as moving to the right 1 space and up 3.04. When these teachers were asked how to interpret 3.04 if x changes by something other than 1, only 8% were able to convey a multiplicative meaning, such as that x can change by any amount and then y will change by 3.04 times the change in x.

The variable-parts perspective offers an interpretation of the average rate of change that applies even when quantities shrink to 0. The average rate of change of a function over an interval is given by the difference quotient, namely Δy, the change in y (the dependent variable) divided by Δx, the change in x (the independent variable) over the interval. Taking a variable-parts perspective, we can view Δx as 1 part, whose size can be any number of base-units, and therefore can shrink towards 0. We can view 1 part, Δx, as a measurement unit and use it to measure Δy. This interprets the difference quotient as a measurement: how many (or much) of Δx it takes to make Δy exactly.

Figure 3 illustrates this measurement interpretation of the average rate of change over smaller and smaller intervals. For the first interval (Figure 3a), it takes 2.2 of Δx to make Δy exactly. For a slightly smaller interval (Figure 3b), it takes 2.25 of Δx to make Δy exactly. For a yet smaller interval (Figure 3c), it takes 2.253 of Δx to make Δy exactly. The changes Δx and Δy are shrinking: in terms of base-units, their measures are smaller and smaller. Even so, the measurement values 2.2, 2.25, and 2.253 are approximately constant, and in fact would converge to specific number as Δx goes to 0 (keeping the left endpoint of the intervals fixed). This limit is the instantaneous rate of change of the function at the left endpoint of the intervals and also the slope of the tangent line to the curve at that endpoint. In general, differentiable functions are exactly those for which such measurement values converge as the intervals (with a fixed left endpoint or right endpoint) shrink to 0.

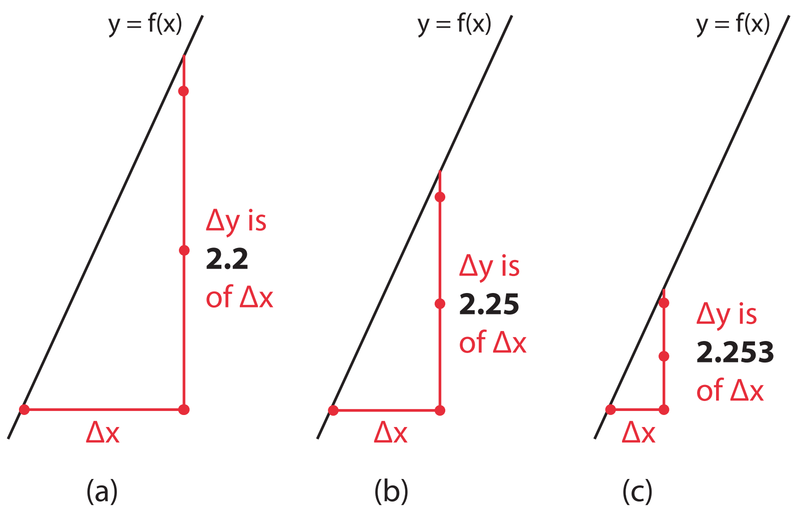

Figure 3. For a differentiable function, as Δx shrinks to zero, how many of Δx it takes to make Δy exactly remains approximately constant: (a) 2.2, (b) 2.25, (c) 2.253.

Johnson's (2015) study suggests that the view of average rate of change discussed above might be accessible and productive for students. The students in Johnson's study considered how the height and volume of liquid in a bottle covaried for bottles of various shapes. Although height and volume were not measured in the same units and were not even the same kind of quantity, all three students compared changes in height with changes in volume, for example by describing amounts of increase in volume as greater than amounts of increase in height over a portion of the bottle with widening width. A next logical step beyond *comparing* a change in y with a change in x (greater than, less than, or equal to) would be to *measure* a change in y in terms of a change in x, which fits with a variable-parts perspective. Johnson noted that for students to reason about instantaneous rates of change, they must have opportunities to shrink the length of an interval but that, even so, they might still make comparisons between amounts of change in quantities. Johnson therefore suggested that "a root for reasoning about instantaneous rate of change is something other than envisioning change as having occurred in completed chunks" (pp. 107 – 108). One root could be the understanding that for suitable functions on a small enough intervals, when the change in y is measured by the change in x, that measurement is very nearly constant, and approaches a fixed number as the changes shrink to 0.

The Variable-Parts Perspective and the Fundamental Theorem of Calculus

The variable-parts perspective offers a natural way to view the Fundamental Theorem of Calculus. Thompson (1994) pointed to the importance of developing a mature image of rate for understanding calculus and identified weak schemes for average rate of change as a source of difficulty, in particular for understanding the Fundamental Theorem of Calculus. As we described above, from a variable-parts perspective, a difference quotient and an average rate of change are most naturally interpreted in terms of measurement, where the unit of measurement is a part that varies in size (a variable part). Thus, a variable-parts perspective may offer a different image of average rate of change than images developed in most treatments of calculus, and this image might be helpful for understanding the Fundamental Theorem.

The approach to the Fundamental Theorem of Calculus presented below relies on interpreting the Riemann sum not in terms of area under a curve but rather in terms of incremental increases of the original function. Figure 4a shows how, for $y = f(x)$, the y values increase as x values increase in increments of Δx. For each such increment, the average rate of change of the function over that increment is the measure of Δy in units of Δx. In other words, the average rate of change over an interval of length Δx is how many of Δx it takes to make Δy exactly. In Figure 4a, we see that these measures range from approximately 1.7 to approximately 2.2.

Figure 4b shows the graph of the derivative $y = f'(x)$, and shows the values of $f'(x)$ increasing from 1.7 to 2.2 between $x = a$ and $x = b$. These values are the instantaneous rates of change of the function f at six points in the interval from a to b. These instantaneous rates of change are approximated by the average rates of change of f over six intervals of size Δx between $x = a$ and $x = b$, as shown in Figure 4a. Notice that in Figure 4a, the (approximate) values 1.7 through 2.2 are results of measurement by the unit Δx, whereas in Figure 4b, the values 1.7 through 2.2 are measured in base units (e.g., inches). Thus, the (approximate) numerical values correspond, but the measurement units do not.

One part of the Fundamental Theorem of Calculus states that the integral from a to b of $f'(x)$ is $f(b) - f(a)$. To see why this should be true, consider a Riemann sum that approximates the integral: $f'(a)\Delta x + f'(a_1)\Delta x + f'(a_2)\Delta x + f'(a_3)\Delta x + f'(a_4)\Delta x + f'(a_5)\Delta x$, where a_i denotes $a + i\Delta x$. In this example, this Riemann sum is equal to $1.7\Delta x + 1.8\Delta x + 1.9\Delta x + 2.0\Delta x + 2.1\Delta x + 2.2\Delta x$. The terms in this sum are approximately equal to the six different values of Δy in Figure 4a, namely the six increments by which y changes between $x = a$ and $x = b$. The sum of these increments is the difference $f(b) - f(a)$. The other part of the Fundamental Theorem of Calculus can be deduced from this part, and we will not explain that here.

Conclusion

The variable-parts perspective on proportional relationships (Beckmann & Izsák, 2015) provides new avenues into persistently difficult topics in mathematics. A fundamental aspect of this perspective is the idea of measuring in two units, one that is fixed, and one that is variable. In particular, we discussed how to view similar triangles from this perspective and used those to examine trigonometry, rate of change, and the Fundamental Theorem of Calculus. We explained how the variable-parts perspective fits with approaches and findings of Moore (2013, 2014) on angle measure and trigonometric functions. We also explained how to interpret average and instantaneous rates of change from a variable-parts perspective in terms of measurement. This interpretation leads to an explanation of the Fundamental Theorem of Calculus that does not rely on an image of the integral as the area under a curve. Our interpretation of research on rates of change (e.g., Byerley & Thompson, 2017; Johnson, 2015; Thompson, 1994) suggests that a variable-parts perspective has the potential to provide students with a productive route into calculus. Future empirical work could explore this hypothesis.

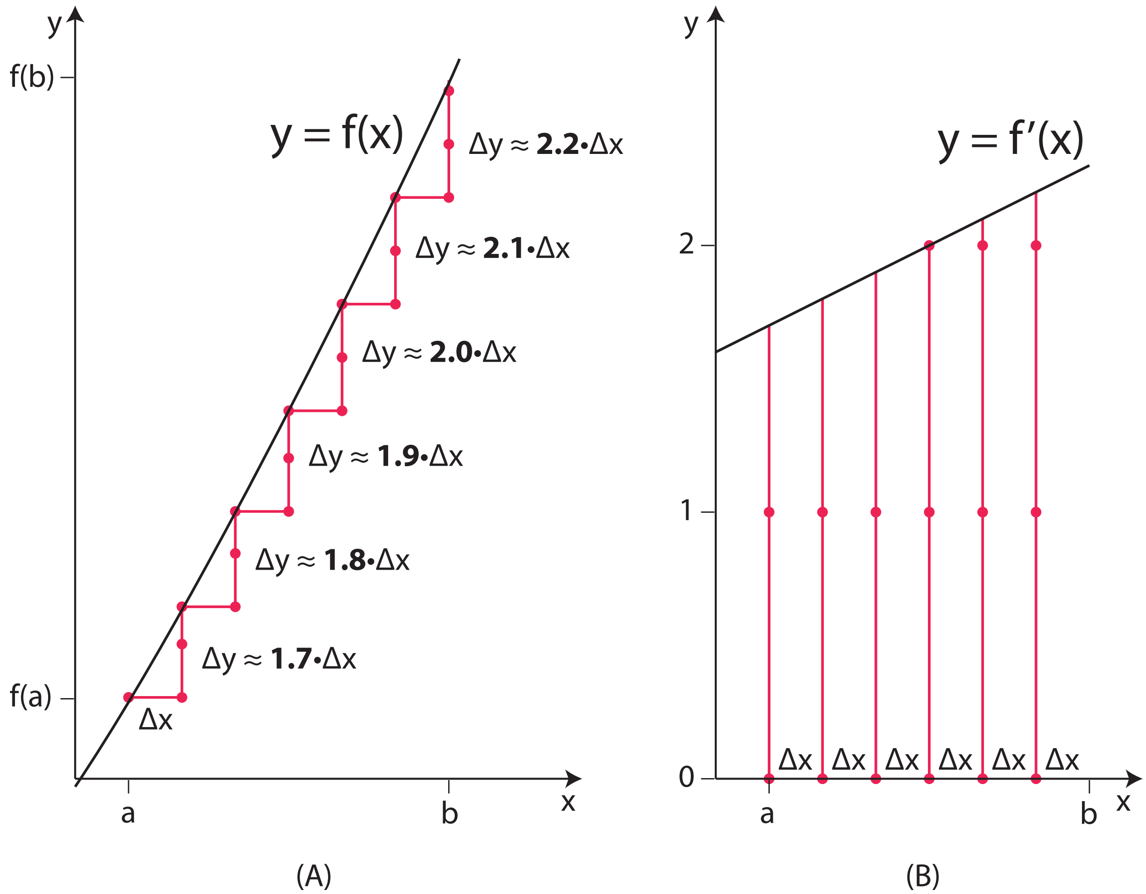

Figure 4. (a) Graph of y = f(x) showing that average rates of change of f over intervals of length Δx are the measures of Δy in terms of Δx and range from about 1.7 to about 2.2. (b) Graph of the derivative y = f′(x).

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Theorizing Teachers' Mathematical Learning in the Context of Student-Teacher Interaction: A Lens of Decentering

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In this paper, I propose a theorization of mental processes involved in teachers' learning of students' mathematics in the context of student-teacher interaction. Specifically, I combine the Piagetian scheme theory with the notions of first- and second-order modeling to characterize types of mathematical learning that may occur when a teacher decenters. I also discuss the affordances and methodological considerations in light of the proposed framework.

Keywords: Decentering, Second-Order Modeling, Piaget, Student-Teacher Interaction, Cognition

Teacher knowledge of students' mathematics has been a growing research area in mathematics education. Authors of the AMTE Standards for Preparing Teachers of Mathematics (2017) have emphasized the importance of teachers having foundational understandings of students' mathematical knowledge and, importantly, being committed to developing such understandings through ongoing teaching practices. Leading mathematics education researchers have also recommended that teacher education programs turn into training teachers as researchers and supporting teachers to *learn through teaching* (Franke, Carpenter, Levi, & Fennema, 2001; Leikin & Zazkis, 2010; Sherin, 2002; A. G. Thompson & Thompson, 1996).

An underlying assumption of these proposals is that teachers' learning of students' mathematics is a constructive, dynamic, and adaptive process. Instead of telling teachers what researchers have known about students' mathematics, it is more effective to advance teachers' sensitivity to students' mathematics and support their construction of and reflection on students' mathematics. Despite a growing number of studies on teacher knowledge of students' mathematics (e.g., Empson & Junk, 2004; Hill, Ball, & Schilling, 2008; Johnson & Larsen, 2012) and teaching practices sensitive to students' mathematics (e.g., Davis, 1997; Jacobs, Lamb, & Philipp, 2010; Lobato, Clarke, & Ellis, 2005; Sleep & Boerst, 2012), there is an absence of research that characterizes teachers' learning processes of such knowledge or practices at a cognitive level. This latter focus contributes to the field in two ways. First, it allows us to shift from theorizing what teachers should know or evaluating what teachers know to characterizing how teachers come to know something. This shift is significant in that it allows us to avoid making claims about teacher knowledge from a deficit perspective (e.g., teachers fail to understand or misinterpret students' mathematics, teachers lack knowledge of students' mathematics) and instead focus on understanding the mechanism by which teachers construct such knowledge. Second, it allows us to shift from understanding how and what teachers *do* to how and what teachers *think*. Gaining insights into the mathematical meanings teachers ascribe to their students is a critical means to providing explanations for teachers' actions during interaction with students (Doerr & Lesh, 2003; Teuscher, Moore, & Carlson, 2016).

Therefore, I consider that providing a fine-grained analysis of the cognitive mechanism of teachers' learning of students' mathematics is a starting point for explaining and supporting teachers' enactment of teaching practices sensitive to student thinking. In this paper, I propose an analytical framework that affords such a line of inquiry. Specifically, I extend the Piagetian scheme theory to outline the types of mental processes that may occur when a teacher engages in understanding students' mathematics during interaction with the students.

Modeling and Decentering

In this section, I elaborate on the constructs of *modeling* and *decentering* that serve as the theoretical foundations of my proposed framework.

Modeling

According to the epistemology of radical constructivism, knowledge, as the product of knowing, is not a representation of objective truth—rather, it functions and organizes viably within a knower's experience and is idiosyncratic to the knower. Therefore, we do not have access to anyone else's knowledge; at best, we can construct hypothetical *models* of others' knowledge that are viable with our observation of their behaviors (Steffe & Thompson, 2000b). As Doerr and Lesh (2003) defined, models are systems of interpretation.

Steffe and colleagues distinguished between two types of models. *First-order models* are "models the observed subject constructs to order, comprehend, and control his or her experience" (Steffe, von Glasersfeld, Richards, & Cobb, 1983, p. xvi), which no one else can have access to. A student's (or a teacher's) first-order mathematics (i.e., *mathematical schemes*) consists of the student's (or the teacher's) mental actions that govern the student's (or teacher's) mathematical perception, activity, and anticipation of results of the activity. Because teachers do not have access to their students' first-order mathematics, what we call teachers' knowledge of students' mathematics is at best the teachers' *second-order models* (Steffe et al., 1983) of the students' mathematics. These models are the teachers' inferences or interpretation of the students' mathematical thinking that can explain the students' observable actions. To be clear, teachers' second-order models of students' mathematics are also mathematical schemes comprising the teachers' first-order mathematics; second-order is referencing to the teacher conceiving the knowledge as from the perspective of the students.

Considering the role of a researcher who is observing a teacher interacting with a student, we need to define *third-order models* (Wilson, Lee, & Hollebrands, 2011), which refer to the researcher's second-order models of the observed teacher's second-order models of the student's mathematics. Such models consist of the researchers' inferences about how the teacher may interpret the student's mathematics that can explain the observable student-teacher interaction (and any other interactions with the teacher). This notion of third-order models aligns with Simon and Tzur's (1999) *accounts of teachers' practice*; that is, explaining teachers' perspectives from researchers' perspective.

Modeling is the cognitive process by which an individual constructs any types of these models. Modeling is an iterative, cyclical process where an individual continually develops, tests, and refines models of a given situation. A modeling perspective on teacher development is significant in that it allows us to account for teachers' continual growth (Doerr & Lesh, 2003). Specific to the context of student-teacher interaction, a teacher's modeling activity involves her or him interacting with students and interpreting the students' thinking either in-the-moment or retrospectively. The teacher's models can continually develop as the teacher observes new features of students' thinking or further reflects on existing models of students' thinking.

Decentering

Modeling necessitates decentration. The notion of *decentration* can be traced back to Piaget's work, in which he describes children's ability to separate themselves from the environment or other objects. It is non-trivial for an infant to model properties of other objects until they can decenter from self and conceive of the objects existing independently (Ginsburg & Opper, 1988). Another example relates to children's construction of space. A child, who is engaging in the

Three Mountains Task (Figure 1) and viewing the mountains at Position A, may consider that another individual, who is viewing the mountains from Position C, can see the same things as what she herself or he himself sees (Piaget, Inhelder, Langdon, & Lunzer, 1967). The child is unable to decenter from the viewer-centered reference frame and describe the perspective of the others. Decentration is also evidenced in children's development of speech and socialization (Piaget, 1955). Children transition from speaking with an assumption that others can see and understand in the same way as they do to being able to simultaneously take into account what they want to express, the listeners' needs, and how the listeners may interpret their talking.

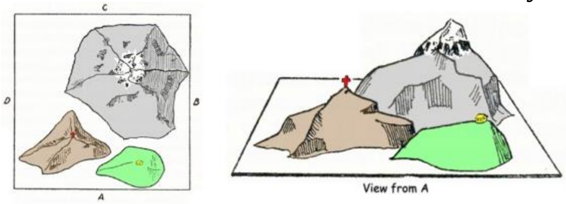

Figure 1. The Three Mountains Task (Piaget et al., 1967, p. 211).

Extending Piaget's work on decentration, Steffe, Thompson, Confrey, and colleagues have used the construct to conceptualize teacher decentering during constructivist teaching (Confrey, 1990; Silverman & Thompson, 2008; Steffe & Thompson, 2000a). They defined *decentering* as a mental action of setting aside one's own thinking and attempting to understand the perspective of the others. Teachers' construction of second-order models of students' mathematics requires the teachers to decenter from their personal mathematical knowledge in an attempt to understand the mathematics of the students that may differ from their own. Carlson, Teuscher, and colleagues further introduced decentering as a lens through which researchers could analyze student-teacher interaction (Bas Ader & Carlson, 2018; Teuscher et al., 2016). Specifically, they inferred from teachers' observable actions to characterize the extent to which the teachers were intentional in engaging in second-order modeling and reflecting on aspects of students' thinking.

Building upon these researchers' frameworks, Mills, Johns, and Ryals (2019) characterized the patterns of mathematics tutors' observable actions that imply decentering (or not decentering) in the moment of tutoring. Other researchers have also applied the construct of decentering to characterize broader educational phenomena such as teachers' listening practices (Arcavi & Isoda, 2007), argumentations (Rasmussen, Apkarian, Dreyfus, & Voigt, 2016), student-student interaction (Hoyles, 1985), and teachers' anticipation of student solutions (Walters, 2017). These researches primarily used decentering as a label for a certain type of process, action, or disposition; their claims foregrounded the interaction patterns or teacher actions that signaled teacher decentering (or not decentering). I have not identified research that aims at providing a fine-grained analysis of mental processes involved in teacher decentering, including the mental processes by which second-order models are constructed or the mathematical consequences of those constructions. In the following, I aim at expanding the extant literature by asking: *What mental processes are involved when a teacher decenters from her or his personal mathematical knowledge and engages in learning students' mathematics*?

A Framework for Teachers' Mathematical Learning Through Decentering

To provide a theoretical tool for the above inquiry, I propose a framework in which I specify the types of mathematical learning that may occur when a teacher decenters. Central to this framework is Piaget's notions of *assimilation* and *accommodation* (von Glasersfeld, 1995). *Assimilation* refers to a mental process by which an individual's existing schemes structure her or his experience and determine what he or she attends to. However, there are experiences where the individual’s enactment of existing schemes results in some conflict due to features not compatible with those schemes. *Accommodation* refers to a mental process by which an individual actively revises or reorganizes her or his schemes to reconcile the conceived conflicts in her or his experience. Accommodation is the mechanism by which *learning* takes place.

A Conceptualization of Student-Teacher Interaction

Steffe and Thompson (Steffe & Thompson, 2000a; P. W. Thompson, 2013) conceptualized social interaction between individuals as each individual reciprocally and continually assimilating the language and observable actions of the other, and constructing second-order models of the other’s knowledge until (ideally) no accommodations are necessary for successful assimilation (i.e., intersubjective construction of knowledge in social interaction). I use Figure 2 (left) to illustrate this process in the context of student-teacher interaction. This model captures the cognitive mechanism of student-teacher interaction by specifying each individual’s enactment and construction of knowledge of the other. Each interaction cycle continues as the teacher and the student construct second-order models of each other’s meanings, monitor and anticipate each other’s responses, and correspondingly adjusting their models of each other.

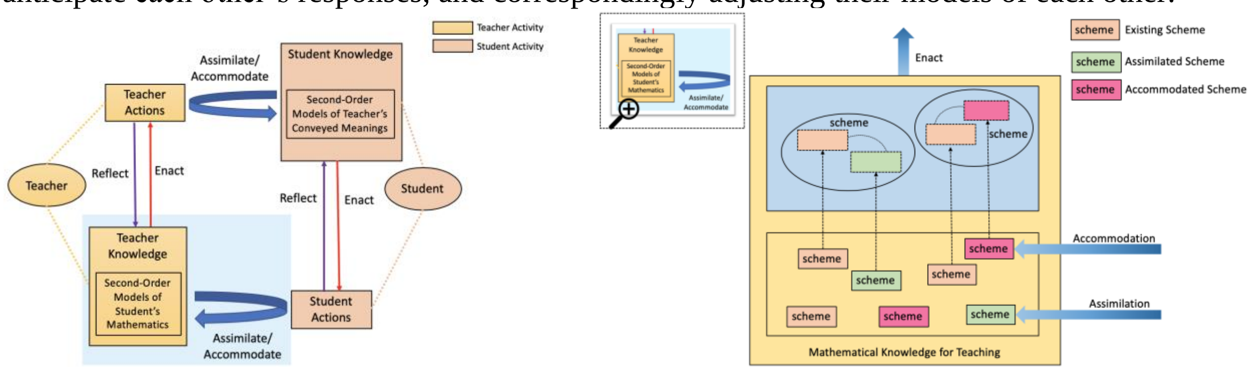

Figure 2. (left) A model of student-teacher interaction and (right) mental processes underlying teachers’ learning of students’ mathematics (Liang, 2019).

Mental Processes Underlying Teachers’ Learning of Students’ Mathematics

To unpack the mental processes underlying a teacher’s learning of a student’s mathematics (blue box in Figure 2, left), I distinguish between three types of schemes in Figure 2 (right). The orange schemes are the teacher’s existing schemes prior to her or his interaction with the student. Constituting the teachers’ first-order knowledge, these schemes consist of the teacher’s personal mathematical knowledge, intended mathematical meanings specific to the interaction, and previously constructed, internalized second-order models of students’ mathematics. They are the source knowledge the teacher leverages to interact with students. The green schemes are the schemes to which the teacher assimilates the student’s actions, hence a subtype of existing schemes. The pink schemes are the schemes the teacher revises or reorganizes to account for novel features of the student’s actions through accommodation, hence updated versions of existing schemes. Further, a teacher can mentally operate on some of these schemes to compare them, identify their relationships, or coordinate them (see the four upward arrows in Figure 2, right). Whether an existing scheme will turn green or pink or coordinate with other schemes is wholly determined by the nature of a teacher’s existing schemes and the teacher’s personal ways of organizing and refining her or his scheme system in response to students’ actions. Thus, which mental process occurs can differ across teachers, even if they observe what a researcher perceives to be similar student actions.

Types of Decentering

In Figure 3, I outline three types of decentering based on the different types of mental processes described above. Type 0 Decentering involves a teacher entirely operating on her or his first-order mathematics (or existing schemes). It occurs when a teacher does not interact with or listen to students. Type 1 Decentering involves a teacher constructing second-order models of a student through assimilation. It occurs when the student's actions are similar to actions that the teacher has abstracted in prior learning experiences or prior interaction with students. The teacher recognizes the student's actions by applying her or his existing schemes and does not make any significant modifications or reorganizations because she or he does not abstract novel mental actions from the current observation of the student's activity.

Type 2 Decentering involves a teacher constructing second-order models through accommodation. When a teacher observes novel student actions and is perturbed by those actions, the teacher revises her or his schemes by either including new compositions into the schemes or restructuring the scheme system. While Type 1 decentering describes how a teacher's first-order knowledge supports her or his second-order modeling, Type 2 Decentering describes how second-order modeling can be generative to a teacher's first-order knowledge.

Type 3 Decentering involves a teacher operating on both second-order models of students' mathematics and first-order mathematics and reflecting on how they are related, similar, or different. It requires the teacher to consciously treats her or his schemes as objects and reflects on the different features and mental actions of the schemes. As a result, the teacher experiences reorganizations in her or his schemes such that they are more coherent to a network of related mental actions and operations. Another important outcome is that, by being aware of how the student's ways of operating differ from the intended goal understandings, the teacher can hypothesize potential construction the student can make (i.e., *zone of potential construction* (Norton & D'Ambrosio, 2008; Steffe & D'Ambrosio, 1995)) and determine how to act in ways that are hypothetically beneficial for the student's development.

Decentering	Scheme Activity	First-/Second-Order Modeling	Scenario
Type 0		- Entirely operate on first-order mathematics	When there is no student-teacher interaction (e.g. pure lecturing)
Type 1	Assimilation	- Construct in-the-moment second-order models via assimilation; - Activate first-order knowledge likely accompanied by modest modifications in first-order knowledge (the first-order knowledge could be internalized second-order knowledge)	When a teacher recognizes student thinking as being similar to her or his own thinking or some former students' thinking
Type 2	Accommodation	- Construct in-the-moment second-order models of students' mathematics via accommodation; - The genesis of modifications in first-order knowledge	When a teacher constructs novel student thinking
Type 3	Accommodation	- Operate on second-order models of students' mathematics and first-order mathematics; - Reorganize first-order mathematics including internalized second-order models	When a teacher reflects on how student thinking is different, similar, or related to her own thinking

Figure 3. A framework for teachers' mathematical learning through decentering.

Affordances and Limitations

I acknowledge that this framework is theoretical, and I encourage future researchers to test and refine it with empirical data. Here, I highlight three potential contributions of the framework.

First, by extending the Piagetian scheme theory (initially used to characterize children's learning) to the context of student-teacher interaction, I specify different types of mental processes involved in teacher's learning of students' mathematics in such context. Piaget's theory of students' mathematical learning can provide the same grounding for understanding teachers' learning of students' mathematics. However, such an extension requires me to operationalize the notions of assimilation and accommodation in slightly different ways, leading to different focuses of my inquiry. For example, understanding teachers' assimilation (as it relates to Type 1 Decentering) in this context requires me to examine how the teachers' mathematical schemes are organized in the teacher's minds and, more importantly, how these schemes govern the teachers' assimilation of students' actions.

Regarding accommodation (or Type 2, Type 3 Decentering), I focus on how the teachers' learning of students' mathematics can be generative to teachers' mathematical knowledge. Shedding light on this constructive and dynamic nature of teacher knowing of students' mathematics provides me an alternative lens to examine student-teacher interaction and teacher knowing. I am positioning teachers as lifelong learners and reflective practitioners (Schön, 2017). I attend to not only how their possessed knowledge influences their understanding of students' mathematics, a common focus in the extant literature, but also ways in which teachers continually refine their mathematical knowledge as a result of interacting with students.

An important question related to teacher accommodation is: what kinds of mathematical schemes allow a teacher to accommodate rather than assimilate? If we compare teachers who are able to engage in Type 2 and Type 3 Decentering to those who engage in Type 1 only, what is different in terms of the nature, content, and structure of their mathematical schemes? One hypothesis is that teachers need to be consciously aware of the mental actions constituting their mathematical schemes (as opposed to the behavioral outcome of applying those schemes) (Tallman, 2016) in order to discern how students' mental actions are different, similar, or related to their own. Otherwise, the teachers may focus on modeling students' behaviors and attempt to engender changes in those behaviors, which does not provide intellectual need (Harel, 2008) for the teachers to model students' mental actions or accommodate mental actions constituting their own mathematical schemes. Future research can test this hypothesis by empirically investigating the relationship between teachers' conscious awareness and decentering.

Second, this framework elaborates on the notion of decentering by describing how teachers' decentering activity may vary by the extent to which they modify their mathematical knowledge in response to their interaction with students. Decentering is not about teachers abandoning or losing sight of their mathematical reality (Hackenberg, 2005) (in fact, they cannot). Instead, it is a mental process of teachers actively applying their mathematical knowledge to discern students' mathematics and, meanwhile, modifying and reorganizing their own mathematical knowledge. Decentering, Type 2 and 3 in particular, can be a source for teachers' reflective abstraction and mathematical learning (Steffe & Thompson, 2000a).

Third, empirical findings relevant to this framework will inform teacher education in terms of advancing teachers' sensitivity to students' mathematics. Mathematics education researchers have made substantial progress in modeling students' mathematics regarding various mathematical domains and created research programs and curricular tailored to those research findings. However, teacher education programs aiming at training teachers to teach in ways sensitive to students' mathematics "have not been grounded in a similarly extensive research based on the nature of teachers' knowledge and its development" (Doerr & Lesh, 2003, p. 128). Fostering teacher development (including knowledge and skills) requires us to understand such developmental processes sufficiently (Simon, 2000). This framework suggests that it is a cognitively demanding task for teachers to develop knowledge of student' mathematics in ways that can support them to act upon students' mathematics; namely, Type 3 Decentering is necessary to do so, and assimilation only is insufficient. Therefore, we have shifted our work to supporting teachers in reflecting on students' ways of thinking and accommodating their own mathematics. Such supports should be more than skill- and pedagogy-based supports. Related to what I have mentioned above, a critical start for teacher educators may be designing content courses aiming to increase teachers' conscious awareness of their own mental actions.

Methodological Considerations

Documenting the evolution of teachers' learning of students' mathematics sensitive to their ongoing interaction with students creates some methodological challenges. Prior researchers have either characterized teachers' knowledge of students' mathematics independent of teachers' interaction with students or investigated student-teacher interaction without providing a fine-grained analysis of the cognitive structure of teachers' second-order models constructed through that interaction. We lack a methodology that simultaneously allows teachers to engage in second-order modeling of students' mathematics and allows researchers to observe, hypothesize, and document the teachers' modeling processes (i.e., third-order modeling).

To address this issue, I propose to expand the constructivist teaching experiment methodology (Lesh & Kelly, 2000; Steffe & Thompson, 2000b), a methodology that involves researchers generating fine-grained second-order models of individual students' (or teachers') mathematics through interaction with them. As illustrated in Figure 4 (each arrow indicates the direction of influence), I expand the roles of a researcher and a teacher. In Phase 1, the researcher engages in cycles of interaction with the teacher and reflectively generates and refines second-order models of the teacher's mathematics.

In Phase 2, at the teacher level, the teacher engages in cycles of interaction with individual students and generates and refines second-order models of the students' mathematics both in the moment of interacting and through retrospective analysis supported by the researcher. At the researcher level, the researcher takes the role of an observer while the teacher is interacting with the students and generates and refines three types of models (indicated in blue boxes) that can explain the observable student-teacher interaction. By comparing these models with the second-order models constructed in Phase 1, the researcher infers whether assimilation or accommodation (or which type of decentering) has occurred in the teacher's mind. Then, the researcher interacts with the teacher with three purposes: (1) stimulating the teacher's retrospective analysis of students' mathematics, (2) testing and refining inferences of the teacher's mental processes and second-order models, and (3) fostering conscious awareness and reorganization of the teacher's mathematical schemes (including schemes constructed through interacting with students). Collectively, this interaction between the researcher and the teacher allows the researcher to promote and characterize the teacher's mathematical development, especially how decentering (or second-order modeling) is generative to such development.

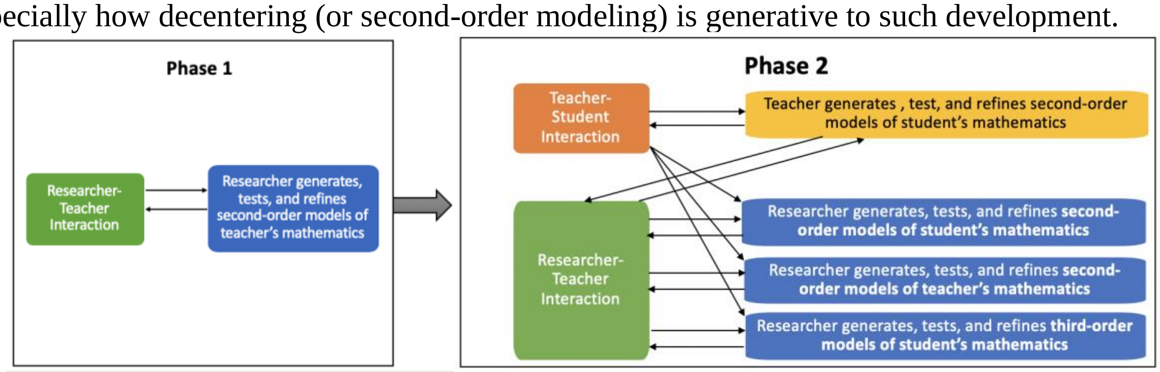

Figure 4. An adapted teaching experiment for modeling teachers' learning of students' mathematics.

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A Quantitative Reasoning Framing of Concept Construction

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Researchers are producing a growing number of studies that illustrate the importance of quantitative and covariational reasoning for students' mathematical development. These researchers' contributions often are in the context of learning of specific topics or developing particular reasoning processes. In both contexts, researchers are detailed in their descriptions of the intended topics or reasoning processes. There is, however, a lack of specificity relative to generalized criteria for the construction of a concept. We address this lack of specificity by introducing the construct of an abstracted quantitative structure. We discuss the construct, ideas informing its development and criteria, and empirical examples of student actions that illustrate its use. We also discuss potential implications for research and teaching.

Keywords: Quantitative reasoning, Covariational reasoning, Abstraction, Concept.

Steffe and Thompson enacted and sustained research programs that have characterized students' (and teachers') mathematical development in terms of their conceiving and reasoning about measurable or countable attributes (see Steffe & Olive, 2010; Thompson & Carlson, 2017). Thompson (1990, 2011) formalized such reasoning into a system of mental operations he termed *quantitative reasoning.* Over the past few decades, other researchers have adopted quantitative reasoning to investigate students' and teachers' meanings in various ways. Some researchers have adopted quantitative reasoning to characterize individuals' meanings within specific topical or representational areas including exponential relationships (Castillo-Garsow, 2010; Ellis, Özgür, Kulow, Williams, & Amidon, 2015), graphs or coordinate systems (Frank, 2017; Lee, 2017; Lee, Moore, & Tasova, 2019), and function (Oehrtman, Carlson, & Thompson, 2008; Paoletti & Moore, 2018). Other researchers have adopted quantitative reasoning to characterize types of individuals' reasoning. A predominant example is reasoning about quantities changing in tandem, or *covarying* (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012, 2015b; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015).

We introduce the construct *abstracted quantitative structure* that marries and extends these two research themes by offering framing criteria for concept construction. Defined generally, an abstracted quantitative structure is a system of quantitative operations a person has interiorized to the extent they can operate *as if* it is independent of specific figurative material. That person can thus re-present this structure in several ways including to accommodate to novel experiences permitting the associated quantitative operations. As we illustrate in this paper, an abstracted quantitative structure is a type of quantitative reasoning that has implications for an individual's meanings within specific topical or representational areas, and her or his engagement in other related forms of reasoning. In what follows, we first discuss background information on quantitative (and covariational) reasoning that underpins the abstracted quantitative structure construct. We then provide a more formal definition for an abstracted quantitative structure and data to illustrate both indications and contraindications of individuals having constructed such a structure. We close with its implications for research and teaching.

Background

Quantitative Reasoning

Thompson (2011) defined quantitative reasoning as the mental operations involved in conceiving a context as entailing measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships). A premise of quantitative reasoning is that quantities and their relationships are idiosyncratic constructions that occur and develop over time (e.g., hours, weeks, or even years). A researcher or a teacher cannot take quantities or their relationships as a given when working with students or teachers (Izsák, 2003; Moore, 2013; Thompson, 2011). Furthermore, and reflecting one implication of the present work, a researcher or teacher should not assume a student has constructed a system of quantities and their relationships based on actions within only one context (e.g., situation, graph, or formula).

Thompson (Smith III & Thompson, 2008; Thompson, 1990) distinguished between *quantitative operations/magnitudes* and *arithmetic operations/measures* to differentiate between the mental actions involved in constructing a quantity via a quantitative relationship and the actions used to determine a quantity's numerical measure. Following Thompson (1990), we illustrate these distinctions using a comparison between two heights. Thompson (1990) described that an additive comparison requires one to construct an image of the measurable attribute that indicates by how much one height exceeds the other height (*Figure 1*). Constructing such a quantity through the *quantitative operation* of comparing two other quantities additively does not depend on having numerical measures, nor does it require executing a calculation; an important aspect of Thompson's quantitative reasoning is that it foregrounds constructing and operating on magnitudes (i.e., amount-ness) of quantities in the context of figurative material (e.g., coordinate systems and phenomena) that permit those operations. *Arithmetic operations*, on the other hand, are those operations between numerical measures such as addition, subtraction, multiplication, etc. that one uses to determine a quantity's measure, and are often in the context of inscriptions or glyphs that signify quantities but do not provide the perceptual material to operate on quantitatively (Moore, Stevens, Paoletti, Hobson, & Liang, online).

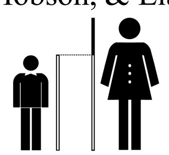

Figure 1. An image of an additive comparison based in magnitudes

Covariational Reasoning

A form of quantitative reasoning involves constructing relationships between two quantities that vary in tandem, or covariational reasoning (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Researchers have conveyed that covariational reasoning is critical for key concepts of K–16 mathematics including function (Carlson, 1998; Oehrtman et al., 2008), modeling dynamic situations (Carlson et al., 2002; Johnson, 2012, 2015b; Paoletti & Moore, 2017), and calculus (Johnson, 2015a; Thompson, 1994; Thompson & Silverman, 2007). Researchers have also illustrated that covariational reasoning is critical to constructing function classes (Ellis, 2007; Lobato & Siebert, 2002; Moore, 2014).

Carlson et al. (2002), Confrey and Smith (1995), Ellis (2011), Johnson (2015a, 2015b), and Thompson and Carlson (2017) are researchers who have detailed covariation frameworks and mental actions. Due to space constraints and the empirical examples we use below, we narrow our focus to a mental action (or operation) identified by Carlson et al. (2002). A critical mental action, especially for differentiating between various function classes, is to compare amounts of change (Figure 2, MA3). MA3 is also important for understanding and justifying that a graph and its curvature appropriately model covarying quantities of a situation (Figure 3) (Stevens & Moore, 2016). Furthermore, and as we illustrate in more detail below, such reasoning enables understanding invariance among different representations of quantities' covariation (Moore, Paoletti, & Musgrave, 2013), which is the foundation for an abstracted quantitative structure.

Mental Action	Descriptions of Mental Actions
MA1	Coordinating the value of one variable with changes in the other
MA2	Coordinating direction of change of one variable with changes in the other variable
MA3	Coordinating amount of change of one variable with changes in the other variable
MA4	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable
MA5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

Figure 2. Carlson et al. (2002, p. 357) covariational reasoning mental actions.

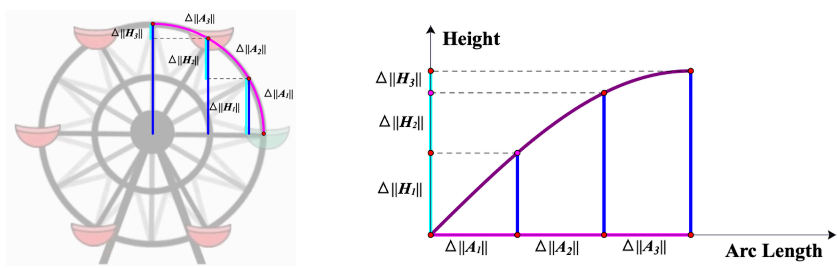

Figure 3. For equal increases in arc length from the 3 o'clock position, height increases by decreasing amounts.

Figurative and Operative Thought

Piagetian notions of *figurative* and *operative* thought (Piaget, 2001; Steffe, 1991; Thompson, 1985) also inform our characterization of an abstracted quantitative structure. These two constructs enable differentiating between thought based in and constrained to figurative material (e.g., perceptual objects and sensorimotor actions)—termed *figurative thought*—and thought in which figurative material is subordinate to logico-mathematical operations, their re-presentation, and possibly their transformations—termed *operative thought*. Quantitative and covariational reasoning are examples of operative thought due to their basis in logico-mathematical operations (Steffe & Olive, 2010). To illustrate the figurative and operative distinction, Steffe (1991) characterized a child's counting scheme as figurative if his counting required re-presenting particular sensorimotor actions and operative if it entailed unitized records of counting that did not require the child to re-present particular perceptual material or sensorimotor experience. As another example, Moore et al. (online) illustrated figurative graphing meanings in which prospective secondary teachers' graphing actions were constrained to particular perceptual features (e.g., drawing a graph solely left-to-right) even when they perceived those features as constraining their graphing of a conceived relationship. In contrast, Moore et al. (online) described that a prospective secondary teacher's graphing meaning is operative in the event that the perceptual and sensorimotor features of their graphing actions are persistently dominated by the mental operations associated with re-presenting quantitative and covariational operations across various attempts to construct graphical re-presentations.

Abstracted Quantitative Structure

Our notion of an abstracted quantitative structure draws on the aforementioned constructs to apply and extend von Glasersfeld's (1982) definition of concept to the area of quantitative and covariational reasoning. von Glaserfeld defined a *concept* as, "any structure that has been abstracted from the process of experiential construction as recurrently usable…must be stable enough to be re-presented in the absence of perceptual 'input'" (p. 194). In the introduction, we defined an abstracted quantitative structure as a system of quantitative (including covariational) operations a person has interiorized to the extent he or she can operate *as if* it is independent of specific figurative material. Using von Glaserfeld's framing, an abstracted quantitative structure is a system of quantitative operations that an individual has interiorized so that it:

1. is recurrently usable beyond its initial experiential construction;
2. can be re-presented in the absence of available perceptual material including that in which it was initially constructed;
3. can be transformed to accommodate to novel contexts permitting the associated quantitative operations, see generalizing assimilation (Steffe & Thompson, 2000);
4. is anticipated as re-presentable in any figurative material that permits the associated quantitative operations.

Clarifying 2., an individual having constructed an abstracted quantitative structure can re-present it in thought and through the regeneration of previous experiences. Clarifying 3., a feature of an abstracted quantitative structure is that it can accommodate novel contexts through additional processes of experiential construction within the context of figurative material in which such construction has not previously occurred. This action is a hallmark of operative thought because it entails an individual transforming and using operations of their quantitative structure to accommodate to novel quantities and associated figurative material, as opposed to having fragments of figurative activity dominate their thought (Thompson, 1985). This action is also a hallmark of quantitative reasoning because it enables conceiving mathematical equivalence in a context differing figuratively from that in which a quantitative structure has been previously constructed (Moore et al., 2013). Clarifying 4., an abstracted quantitative structure's mathematical properties (e.g., quantities' covariation) are anticipated independent of any particular instantiation of them, thus understood as not tied to any particular quantities and associated figurative material. It is in this way that the quantitative operations of an abstracted quantitative structure are abstract; the individual not only understands that the operations are re-presentable in previous experiences, but she also anticipates that the operations *could be* relevant to novel but not yet had experiences (e.g., some coordinate system not yet experienced).

We next use empirical examples to illustrate the extent students have constructed an abstracted quantitative structure. Each example is drawn from a study that either used clinical interview (Ginsburg, 1997) or teaching experiment (Steffe & Thompson, 2000) methodologies to build second-order models of student thinking (Ulrich, Tillema, Hackenberg, & Norton, 2014). It was in our reflecting on these second-order models (and those developed during other studies) that we identified themes in their reasoning, one of which is the notion of an abstracted quantitative structure. We acknowledge the way we have defined abstracted quantitative structure presents an inherent problem in attempts to characterize a student as having or having not constructed such. First, it is impossible to investigate a student's reasoning in every context in which an abstracted quantitative structure could be relevant. Second, to characterize a students' quantitative reasoning necessarily involves focusing on their enactment of operations in the context of particular figurative material. No conceptual structure is truly representation free, as "operations have to operate on something and that something is the figurative material contained in the operations, figurative material that has its origin in the construction of the operations" (L. P. Steffe, personal communication, July 24, 2019). For this reason, we find it necessary to use the criteria above to discuss a student's actions in terms of *indications* and *contraindications* of her or him having constructed an abstracted quantitative structure.

A Contraindication of Re-Presentation

Critical criteria of an abstracted quantitative structure are the ability to re-present that structure in the absence of available perceptual material and the ability to transform its operations to accommodate to novel contexts. As a contraindication of these criteria, consider Lydia's actions during a teaching experiment focused on trigonometric relationships and re-presentation (Liang & Moore, 2018). Prior to the actions presented here, Lydia had constructed incremental changes compatible with those presented in Figure 3 (left). We took her actions to indicate her reasoning quantitatively and subsequently presented her the *Which One?* task. The task (Figure 4, left) presented numerous red segments that varied in tandem as the user varied a horizontal (blue) segment, which represented the rider's arc length traveled along the circle. We asked her to choose the red segment that covaried with the blue segment in a way compatible with the vertical height and arc length of the rider. We conjectured this would help determine the extent she could re-present her previous actions in a similar context with less perceptual material available (i.e., the circle) and novel material (i.e., the red and blue segments).

Lydia became perturbed as to whether or not the horizontal red segment should vary at a changing rate with respect to the horizontal blue segment. After much effort, she abandoned considering the segments in the horizontal orientation and re-oriented them vertically. She chose the correct segment by checking whether the heights matched pointwise within the displayed circle (Figure 4, middle). Both her questioning how the red segments should vary with respect to the blue segment and her requiring re-orienting the red segments were a contraindication of her having constructed an abstracted quantitative structure. We thus returned her to the question of whether the chosen red segment and blue segment entailed the same quantitative relationship as she identified in her previous activity (see Figure 3, left; from Liang and Moore (2018)):

Lydia: Not really…Um, I don't know. [*laughs*] Because that was just like something that I had seen for the first time, so I don't know if that will like show in every other case…Well, for a theory to hold true, it like – it needs to be true in other occasions, um, unless defined to one occasion.
TR.: So is what we're looking at right now different than what we were looking at with the Ferris wheel?
Lydia: No. It's – No…Because I saw what I saw, and I saw that difference in the Ferris wheel, but I don't see it here, and so –
TR.: And by you "don't see it here," you mean you don't see it in that red segment?
Lydia: Yes.

As the interaction continued, Lydia expressed uncertainty as to how to determine if the blue segment and her chosen red segment entailed the same relationship as she had illustrated in her previous activity, although she knew the segments were correct pointwise. As a further contraindication of her having constructed an abstracted quantitative structure, it was only after much subsequent teacher-researcher guiding and their introducing perceptual material using their pens (Figure 4, right) that she was able to conceive the red and blue segments' covariation as compatible with the relationship she had constructed in the Ferris wheel situation.

Liang and Moore (2018) illustrated Lydia's repeated engagement in quantitative and covariational reasoning eventually led to her re-presenting quantitative operations including transforming those operations in novel contexts. This enabled her to conceive mathematical equivalence across numerous contexts including situations, oriented segments, and Cartesian graphs. As indication of having constructed abstracted quantitative structures, Lydia re-presented particular quantitative operations in contexts without given perceptual material (Lee et al., 2019).

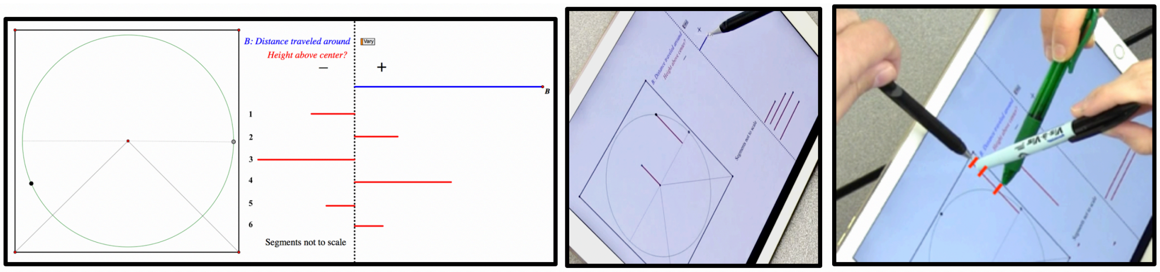

Figure 4. (left) Which One?, (middle) Lydia checking segment pointwise, and (right) Lydia attempting to re-present a quantitative structure with assistance.

An Indication of Re-Presentation and Accommodation

We turn to two prospective secondary teachers'—Kate and Jack—actions when asked to determine a formula for an unnamed polar coordinate system graph (Figure 5, which is $r = \sin(\theta)$; see Moore et al. (2013) for the detailed study). After investigating a few points, Kate and Jack conjectured that $r = \sin(\theta)$ is the appropriate formula and drew a Cartesian sine graph to explore their conjecture (from Moore et al., 2013, p. 468). Important to note, Kate and Jack were not familiar with graphing the sine relationship in the polar coordinate system.

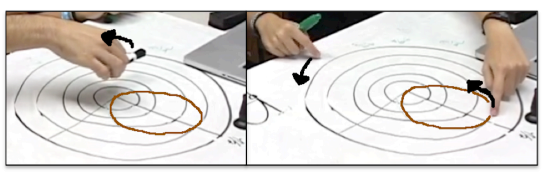

Figure 5. Kate and Jack covary quantities with respect to the given graph (from Moore et al., 2013, p. 467).

Kate: This gets us from zero to right here is zero again [*tracing along Cartesian horizontal axis from 0 to π*]. So, we start here [*pointing to the pole in the polar coordinate system*].
Jack: Ya, and you're sweeping around because [*making circular motion with pen*], theta's increasing, distance from the origin increases and then decreases again [*Jack traces along Cartesian graph from 0 to π as Kate traces along corresponding part of the polar graph*].
TR.: OK, so you're saying as theta increases the distance from the origin does what?
Jack: It increases until pi over 2 [*Kate traces along polar graph*] and then it starts decreasing [*Kate traces along polar graph as Jack traces along Cartesian graph*].
TR.: And then what happens from like pi to two pi.
Kate: It's the same.
Jack: Um, same idea except your, the radius is going to be negative, so it gets more in the negative direction of the angle we're sweeping out [*using marker to sweep out a ray from π to 3π/2 radians – see Figure 5*] until three pi over two where it's negative one away and then it gets closer to zero [*continuing to rotate marker*].
TR.: OK, so from three pi over two to two pi, can you show me where on this graph [*pointing to polar graph*] we would start from and end at?

Kate: This is the biggest in magnitude, so it's the furthest away [*placing a finger on a ray defining 3π/2 and a finger at (1, π/2)*], and then [the distance from the pole] gets smaller in magnitude [*simultaneously tracing one index finger along an arc from 3π/2 to 2π and the other index finger along the graph – see Figure 5*].

Kate and Jack's actions indicate their having constructed (or constructing) an abstracted quantitative structure associated with the sine relationship. They transformed and re-presented the quantitative and covariational operations they associated with a Cartesian graph to accommodate to a polar coordinate system displayed graph. This re-presentation enabled them to conceive two graphs as representing equivalent quantitative structures despite their perceptual differences, which is a contraindication of their reasoning being dominated by figurative aspects of thought. We note that Kate and Jack did not provide evidence related to criteria 4. Such evidence would involve their identifying the potential of not yet experienced coordinate systems and associated graphs that enable re-presenting the same quantitative structure.

Discussion and Implications

We envision the construct of an abstract quantitative structure as useful in several ways. First, it provides criteria to research and distinguish between students' meanings in terms of their foregrounding figurative material and activity and their foregrounding logico-mathematical operations such as a quantitative structure. In our description of Lydia's activity, we underscore that she did not encounter much difficulty assimilating the figurative material; she was able to assimilate the segments and their variation to quantitative operations. Rather, Lydia struggled to accommodate the relationship she constructed with previous figurative material in a way that she could re-present it with novel figurative material. Seeing how difficult it was for Lydia to re-present a relationship within a circular context further demonstrates how powerful Kate and Jack's reasoning was because not only did they re-present a quantitative structure in a novel context, but they abstracted the associated operations such that they could identify the same relationship within a perceptually different representational system. We, therefore, call for researchers and educators to attend not only to students' meanings for various representations (e.g., Cartesian coordinate system, polar coordinate system, formulas, tables, etc.), but also to the quantitative structures students construct and the extent they can re-present (and potentially transform) those structures. In doing so, we can obtain more detailed insights to the extent students construct mental operations in which figurative material is a consequence of those operations including how those operations enable accommodating to novel contexts.

Second, we hypothesize that students' abstracted quantitative structures play an important role in their productive generalization (Ellis, 2007) and transfer (Lobato & Siebert, 2002). Researchers have recently characterized the role of different forms of abstraction in generalization (Ellis, Tillema, Lockwood, & Moore, 2017). Researchers have also recently characterized different forms of transfer including how a student's novel activity can result in cognitive reorganizations regarding their previous activity (Hohensee, 2014; Lobato, Rhodehamel, & Hohensee, 2012). We envision students' construction of abstracted quantitative structures to be a province of each, and we argue that future research should explore these potential relationships as it relates to students' mathematical development.

Acknowledgments

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A Proposed Framework of Student Thinking around Substitution Equivalence: Structural versus Operational Views

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This theoretical paper explores one way in which student conceptions of substitution equivalence might be classified along a spectrum between operational and structural thinking, which we generalize from Sfard's theories of the Genesis of Mathematical Objects and research on student conceptions of the equals sign. We provide some sample student work that illustrates different ways in which students may exhibit structural or operational thinking about substitution equivalence. The aim of this paper is to provide an initial testable framework of student thinking around substitution equivalence that could be used as a basis for future studies that confirm, refute, or refine this framework, or use the framework to explore the relationship of different kinds of student thinking around equivalence to other mathematical conceptions and skills.

Keywords: equivalence, equals sign, relational thinking, operational thinking, structural thinking

One of the very first mathematical skills that children are expected learn in preschool is the ability to recognize certain types of "sameness". They are expected to learn that the number 5 represents any set with five objects, and to recognize and identify special classes of shapes. As children move through elementary school, they are asked to refine and create a more nuanced category scheme for their classification of shapes: instead of just the number of sides, they must classify shapes based on side length, angle size, etc. In elementary school students are first asked to write down formal mathematical expressions of equality, where they indicate which arithmetic expressions are equal to each other. In middle school through college, students solve equations and simplify expressions, using substitution equivalence—often without explicitly being told—or even realizing—that that is what is underlying their work.

In fact, equivalence is central to mathematics at all levels, and across all domains. For example, in algebra, equivalent expressions/equations and equivalence classes of function types play a central role; in calculus, functions that have the same derivative form an equivalence relation; in set theory cardinality is an equivalence relation; and the concept of isomorphism is an equivalence relation that occurs in abstract algebra, differential geometry, topology, etc.

In mathematics education, much research has focused on studying how students think about the equals sign (see e.g., Knuth, Stephens, McNeil, & Alibali, 2006), because students' conceptions of the equals sign have been shown to be related to their ability to perform arithmetic and algebraic calculations. However, equality is just one example of the larger concept of equivalence—other types of equivalence occur extensively throughout the K-16 mathematics curriculum, but are rarely, if ever, taught under one unifying idea called equivalence. For example, if we look at the *Common Core Mathematics Standards*, the word equivalent, equivalently, or equivalence is used 53 times but only two of these times does it refer to something other than equality (Common Core State Standards Initiative, 2017). On the other hand, a number of different kinds of equivalence (e.g. similar or congruent figures, categorization of function types) are contained in the *Common Core Mathematics Standards* but are never explicitly labeled as a type of equivalence, nor are the commonalities among these different types of equivalence highlighted. Because of space constraints, here we focus on just one kind of equivalence—substitution equivalence—and outline a framework for how we might classify student thinking around it, along a spectrum from operational to structural thinking.

Theoretical Lens

We draw on data collected across multiple classes and multiple years at a single community college at the City University of New York, including cognitive interviews, classroom observations, and open-ended questionnaires, analyzed through a framework that emerged through a combination of *field-based hypothesis* (Harel, 2017) —"observations of learners' mathematical behaviors in an authentic learning environment"—and through *conceptual analysis* (Von Glasersfeld, 1995), heavily influenced by Sfard's theories of the *Genesis of Mathematical Objects* (Sfard, 1995) and existing research literature that classifies students' understandings of the equals sign as operational versus relational (e.g., Knuth et al., 2006; Stephens et al., 2013). In a sense, the framework proposed here could be seen as generalizing existing frameworks of student thinking around the concept of the equals sign into a more expansive framework that includes any kind of substitution equivalence (of which equality is just one example).

Sfard defines operational and structural thinking as different ways of *perceiving* mathematics. In operational thinking, a student thinks of mathematical entities as a process of computation; in structural thinking, they think of them as abstract objects in and of themselves which can then been seen as objects for even higher-order processes (Sfard, 1995). In research literature on how students conceptualize the equals sign, a similar notion is often utilized, in which student thinking is classified as operational (students think of the equal sign as a "do something" sign that tells them to calculate whatever is on the left and put the answer on the right) or relational (students see the equals sign as describing the relationship between the two sides) (e.g., Knuth et al., 2006). Some research also adds an additional structural category of student thinking, which is the extent to which students recognize properties (e.g. commutative) that justify why two expressions are equal (e.g., Stephens et al., 2013).

A Proposed Model of Student Thinking Around Substitution Equivalence

In this paper we focus specifically on *substitution equivalence*, or the fact that two expressions, equations, or other mathematical objects are equivalent if each one can be transformed into the other through a sequence of substitutions carried out through a combination of correct interpretation of syntactical structure and appropriate use of mathematical properties. We define *substitution* more broadly than has been done in much existing research and curricula, as the process of replacing *any* mathematical object (or any unified sub-part of an object) with an equivalent object, regardless of complexity. For example: if $x = 2$, replacing the x in the expression $2x^2 - 5$ with 2 is a substitution. But we also see the following as substitution: replacing the equation $2x - 3 = 7$ with the equivalent equation $2x = 4$, and replacing the expression $2x^2 + 5x - 3$ with the equivalent expression $2x^2 + (-x + 6x) - 3$ (where $-x + 6x$ is substituted for $5x$). We do not restrict this definition to algebra only—all calculation in arithmetic can also be seen as a process of substitution. For example, each step in the following calculation depends critically on the notion of substitution equivalence:

$$7 + 5 = 7 + (3 + 2) = (7 + 3) + 2 = 10 + 2 = 12$$

Whether or not this is addressed directly in current curricula or instruction, these calculations are only possible through substitution equivalence—in fact all arithmetic requires at least a basic notion that we can replace a mathematical object only with another equal/equivalent object.

We use the term *syntactical structure* to refer to the structure of a mathematical object *as written*; so, when we say that a student shows evidence of syntactical structure sense, we mean that they appear to be able to accurately describe the *meaning* of the existing structure of the object. For example, if a student can describe that $(2x+1)(x-7)$ represents two sub-expressions $2x+1$ and $x-7$ being multiplied together, or that $4+8\div 2$ represents the sum of 4 and the result of the calculation $8\div 2$, they are showing some aspect of syntactical structure sense. This idea is closely related to *surface structure* as defined by Kieran (1989) and others in linguistics (e.g., Chomsky, 1966), but we choose the term syntactical structure to avoid ambiguity, since in practice "surface structure" has often also been used to denote a *superficial* notion of structure (i.e. noticing symbols or patterns on the page, without correctly ascribing meaning). We note that a student may have syntactical structure sense without necessarily having: 1) a deeper understanding of the operations involved (e.g., they may be able to understand the syntactic meaning of $4+8\div 2$ without having a robust understanding of division); or 2) the ability to identify or generate equivalent objects (e.g., they may understand the syntactic meaning of $(2x+1)(x-7)$ without being able to correctly simplify it).

Operational versus Structural Views of Substitution Equivalence

We now present a framework that we theorize may help to categorize student thinking around substitution equivalence—but first, a few caveats. We cannot actually know what students are thinking, so when we categorize student work as operational versus structural, we do not claim that this category describes the student, but rather that it describes the student's visible work or communicated explanations on a particular problem. For example, a student may have a structural view of equivalence, but provide their explanations from an operational perspective, because they perceive it to be the accepted sociomathematical norm of the classroom (e.g., Yackel & Cobb, 1996). We also note that operational and structural views of equivalence are not necessarily binary—a student may have both (or parts of each in different proportions) when reasoning, and they may use different ways of thinking on different types of problems; there are cases where operational thinking is a legitimate alternate way of reasoning (e.g., Sfard, 1991). We conceptualize student thinking around equivalence as a spectrum that describes the extent to which students are able to use structural thinking appropriately and robustly, with students who exhibit only operational thinking on one end, and students who use structural thinking with robust understating in a wide variety of mathematical situations on the other (see Figure 1).

Operational view of substitution equivalence: Mathematical tasks such as simplifying, rewriting, or solving are about "operating on" objects based on a system of rules. The rules come from an authority and are often based on superficial structure criteria (e.g., "If you see *this*, do *this*"). The goal is to generate an "answer".

Structural view of substitution equivalence: Mathematical tasks such as simplifying, rewriting, or solving are about using syntactical structure, along with specific known properties and appropriately chosen substitution, to generate or identify a new object that is equivalent to the original one. The goal is to generate a string of equivalence relations.

Figure 1. Operational versus structural views of substitution equivalence as a continuum

Operational vs. structural approaches to substitution equivalence: an illustrative example from arithmetic. Before we present the more detailed theorized framework of student thinking around substitution equivalence, we first briefly present a simple illustrative example from arithmetic. In the second-grade curriculum in the U.S., students are taught to find the sum of two numbers by breaking numbers apart strategically, and then adding the results back together. Some different approaches are illustrated in Table 1 below.

Table 1. Various approaches to solving the arithmetic question $7+5=?$		
Approach 1 (operational):	Approach 2 (structural):	Approach 3 (structural):
$7+5=?$ $10-7=3$ $5-3=2$ $10+2=12$	$7+5$ $=7+(3+2)$ $=(7+3)+2$ $=10+2$ $=12$	$7+5$ $=(10-3)+5$ $=10+(5-3)$ $=10+2$ $=12$

Approach 1 is operational and is a commonly used method (Common Core State Standards Initiative, 2017); here there is no obvious substitution equivalence relationship between any of the individual steps (e.g. $10-7=3$ is not substitutionally equivalent to $5-3=2$). While substitution is in fact taking place, none of the written steps make this explicit. We label this method operational because it can be experienced by students as a sequence of steps that must be carried out to produce an answer, while the underlying *structural* nature of the arithmetic expression is not drawn upon *explicitly* in order to produce a sequence of *equivalence relations*. Consider, on the other hand, the same steps of computation, but written in a form so that the substitution equivalence is made explicit, as in Approach 2. Here, the same steps are written so that each expression is equivalent to the one on the previous line, as indicated by equals signs.

Characterizations of operational versus structural thinking about substitution equivalence. We now propose characterizations of operational versus structural views of substitution equivalence. These characterizations include both descriptions of student thinking, as well as behaviors that we theorize are common among students who use each way of thinking. We theorize these connections based on what we have observed in the classroom, in student answers to open-ended questions, and in student explanations during cognitive interviews, but we do not claim to prove this relationship here. The aim of this paper is simply to establish a potential framework which could be proven, disproven, or refined by future empirical work. Table 2 outlines five different categories of mathematical work or thinking that we theorize reveal information about whether the student is thinking operationally or structurally.

Table 2. Characterizations of operational versus structural thinking about substitution equivalence

	Operational thinking	Structural thinking
View of transformation	Students see transformations of expressions and equations (or other mathematical objects) as a process of "operating on" the original object itself.	Students see each step in a transformation as the process of replacing one mathematical object with another equivalent one through substitution, using properties and existing syntactical structure.
Organization of work and use of equality/ equivalence signs	Calculations may seem jumbled, disorganized or unrelated (because students do not "see" the connections between the procedural steps they are taking), and equals signs or equivalence arrows that connect one step to another are often missing.	Student work is typically written as a sequence of equivalence relations, where each line is equal or equivalent to the previous one. Appropriate signs are used to illustrate this relationship (e.g. equals sign, equivalence arrow, etc.).
Justification	Students often cite justification from an authority (instructor, textbook, rule) or vaguely formulated experience ("we did problems like	Students will attempt to justify using arguments about equivalence, including pointing to specific mathematical properties and

	this in class, and in those cases we always multiplied") rather than through logical reasoning.	syntactical structures to justify why equivalence holds.
Structure cited	If structure is cited, it is superficial in nature, and does not tend to reveal actual syntactic structure sense (e.g. "parentheses mean we should multiply").	If structure is cited, the student shows evidence of syntactic structure sense (i.e. a deeper understanding of the actual meaning of the expression/ equation/object, as written).
Goal	The goal of transformations is to generate an "answer".	The goal of transformations is to generate a sequence of equivalence relations. The final object in this sequence may be particularly useful (e.g. desired form, fully simplified).

Examples of Different Aspects of the Framework

In order to illustrate each of these different aspects of the framework, we present some samples of various kinds of student work.

View of transformation. For this aspect, we present some examples from student interviews where they were asked to consider whether the exponent could be distributed in each of the two expressions $(x^2y^3)^2$ and $(x^2+y^3)^2$:

Student 1 (operational view): [in both cases] you distribute the exponent to everything in the parenthesis. [hand motion as though "scooping" exponent back into the parenthesis]… you would add the two exponents. So it would be x^4 and y^5… That's how you kind of get rid of the parenthesis and get rid of the outer exponents by distributing it in the inside.

Student 2 (structural view): [The first expression is] x^2y^3times x^2y^3….. the second one [is] x^2+y^3 times x^2+y^3.

Student 1 uses language suggesting that they see simplification as a process of "doing something" to the expression: e.g., "add exponents", "get rid of parenthesis". Student 2 describes replacing each expression with an equivalent one that reveals the underlying meaning.

Organization of work and use of equality signs. In the following samples of student work, students were asked to partially simplify $(x^2+2x-1)-(3x^2-4x+7)$.

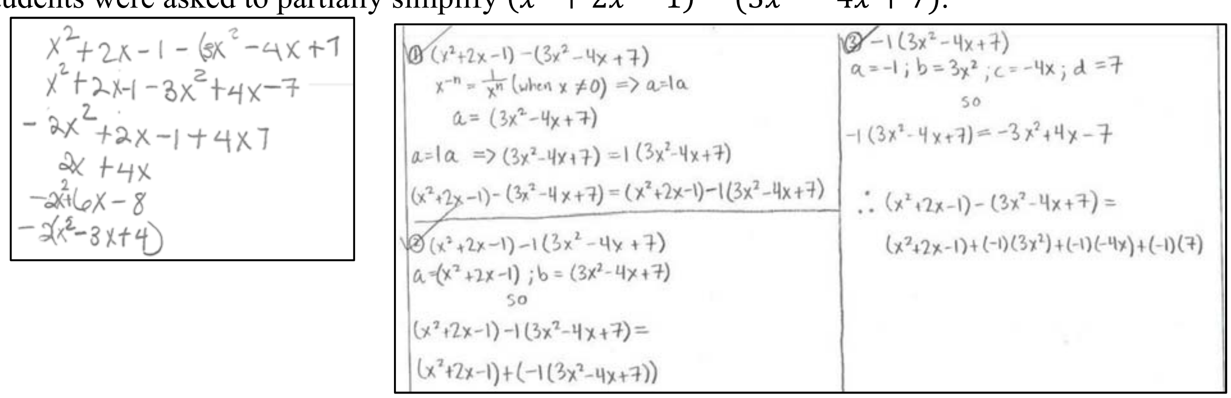

Figure 2: Examples of operational and structural approaches, characterized by work organization & equality signs

In these examples, the student on the left uses no equals signs (or any other indicator of the relationship between steps). They often "lose" important symbols or other features of the problem: in the first step, only the opening parenthesis has been included, and in the third step the subtraction sign before the 7 is missing. While the work on the whole is correct, there is no evidence that the student sees each step through the lens of substitution equivalence rather than just through procedures (e.g. distribute the negative, add the coefficients, etc.). The student on the right directly uses substitution into given properties to establish that one expression is equivalent to another, and they use an equals sign in each step to indicate these relationships[1].

Justification. In the following two examples, students were interviewed about what would happen when you substituted $2y$ in for x into the expression $2x^2 - 7x + 3$ (the options were multiple choice).

Student 1 (operational view): $2y^2 - 7(2y) + 3$ is without the brackets [in the first term] and $2(2y)^2 - 7(2y) + 3$ is with the brackets… [I picked the first one] because … when the teacher was doing it she… didn't put in all of them, the brackets, she only used either the first section or the last section…. Like whenever she's tryin' to solve it in the first section she put the two brackets, but then the second one before she got to the answers, she said that it's 2y instead of the bracket in the answer.

Student 2 (structural view): if you're putting something in for something else, I always usually keep parentheses around it to make sure that I maintain whatever structure the original function had... especially in this one [the first term] because x is being multiplied, and squared, the parentheses really make a difference because if we don't have them, we could get a different, the wrong answer…. you don't need [the parentheses in $-7(2y)$] as long as you make sure you multiply the -7 by 2.

In this example, the first student is relying heavily on the authority of their instructor, describing how she used parentheses sometimes but not others; because the instructor only kept some parentheses but not others, the student felt that an expression with one bracket was better than one with two. This student's reasoning didn't engage with a notion of equivalence or the underlying meanings of the expressions. In contrast, the second student mentions structure explicitly, and explains how removing the parentheses gives a different result that is not equivalent to the original one (although they do not use the word equivalent itself).

Structure cited. In these two examples, students were asked to explain whether or not the two expressions $(2x - 1)(3x - 4)$ and $2x - 1 \cdot 3x - 4$ have the same meaning.

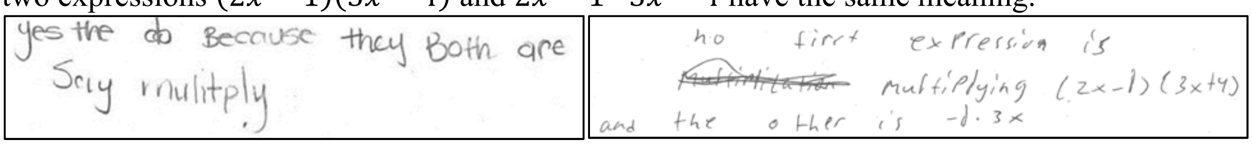

Figure 3. Student work citing different structures related to substitution equivalence

In these two examples, the first student notices that there is multiplication in the expression, but they do not see the underlying meaning of the expression with respect to the multiplication—their sense of structure is only superficial. However, the second student notices that *what* is being multiplied varies in the two expressions, therefore changing the meaning—thus they show some deeper syntactic structure sense.

Goal. In the following two examples, students were interviewed about what would happen if you substituted $(8 + 3)x$ in for $8x + 3x$ into the expression $(8x + 3x)^2$ (the options were

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[1] We note that the framework is not intended to claim that the student on the left has no conception of substitution equivalence, or even that they are not using it to generate the written work—rather, we see the lack of equivalence signs and the missing symbols as something that seems to occur more often with students who are thinking operationally, and which might be one characteristic that, when used in conjunction with other characteristics in the framework, may help us to identify students who are thinking more operationally.

multiple choice).

Student 1 (operational view): [I didn't pick $(8+3)x^2$ because] I've actually never seen a problem that has x out of the parentheses like that… I personally wouldn't know how to simplify $(8+3x)^2$ or $\left((8+3)x\right)^2$, so that's why I picked $8+3x^2$.

Student 2 (structural view): [I didn't choose $(8+3)x^2$] because … the square is supposed to be for the whole $(8+3)x$. And then $8+3x^2$, wouldn't be it because the $8x$ plus the $3x$ are both being squared…[For $(8+3x)^2$], the x isn't being taken out of the 8 here, …so you need to keep the x on the outside of the parentheses. And then $\left((8+3)x\right)^2$ I chose because the x is on the outside of the parentheses, but there is also a parentheses holding everything and then there is the square.

In these examples, the first student never mentions equivalence or equality—instead they focus on which expressions look like ones they have seen before in class, or can be simplified. Their goal seems to be simplification, without attention paid to equivalence. The second student, on the other hand, seems to have as a goal a chain of reasoning that justifies why their chosen answer is equal to the original expression. They consider each possible answer and analyze why it can or cannot be equivalent to the original expression.

Here we have presented just a few illustrative examples of how student thinking around substitution equivalence might be viewed. We hope with this proposed framework to help begin conversations around students' conceptions of equivalence. The framework may allow us to analyze student work (or instruction) in various math classes, including algebra, calculus, and more, in a way that brings student conceptions of substitution equivalence to the fore.

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The Modeling Space: An Analytical Tool for Documenting Students' Modeling Activities

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One of the most challenging aspects of doing research in the mathematical modeling genre has been finding an appropriate characterization for the complex interaction of knowledge and cognitive acts that result in coordination of situational referents and mathematical inscriptions. To this end, we introduce the modeling space and illustrate its descriptive and analytic utility.

Keywords: mathematical modeling, quantities, theory development

To date, the most common characterization of modeling is a "translation" from the real world to the mathematical world. Typically, the "translation" is envisioned as leveraging a one-to-one correspondence between words (or real-world objects) with mathematical counterparts. This characterization is not only empirically inaccurate, but consequently impedes progress in theorizing the teaching and learning of modeling. First, translation between natural languages is typically unbalanced; it is rare that an individual is equally knowledgeable in her first and subsequent languages. Second, the characterization is reductionist since "reliance on translation cues…is more characteristic of students who possess only algorithmic knowledge of the target task and who circumvent the interpretive process of mathematical modeling" (Martin & Bassok, 2005, p. 479). We do not imply that language translation is trivial work, only that referring to mathematization of a real-world situation into a mathematical problem as translation ignores the complexity of modeling which has, in turn, limited the field's ability to systematically investigate it as an idiosyncratic activity. Presently, the field lacks a methodologically cogent approach to studying modeling that reflects the full range of mathematical or real-world meanings individuals might ascribe to a single representation. It also faces an abundance of partially explanatory theories focusing on just one aspect of modeling (e.g., translation). The field is thus ill-equipped to account for how those meanings shift during the process of modeling or to develop means for studying how instructors may impact the process. The broader goal of the research project is a theoretically and methodologically coherent way to trace the evolution of a student's mathematical model, including the representations she produces and her meanings for these representations. Our contribution is twofold: a theoretical construct robust enough to account for students' idiosyncratic and shifting meanings during mathematical modeling and an accompanying descriptive mathematical model capable of tracing the evolving student model.

Relevant Theoretical Constructs

To address the larger question of how to document model evolution, we first acknowledge that the extant theories of mathematical modeling collectively posit both internal (mental) and external (representational) aspects of a mathematical model which must both be accounted for. To date, no one theory does so comprehensively. When multiple theories each offer partial explanations for a phenomenon, such as model evolution, The Networking Theories Group (2014) advocates strategies for coordinating those theories. We adopted the strategies they recommended such as combining and coordinating in order to generate "deeper insights into an empirical phenomenon" (p. 120). We first focused on the aspects of relevant theories that foregrounded conventional mathematical representations (Czocher & Hardison, 2019). However, tracing only the evolution of representations was insufficient for capturing the breadth of ways that a mathematical model could change could change. We next sought to incorporate the theory of quantitative reasoning (Thompson, 2011) by considering the quantities students projected into a situation to be modeled and the interplay between quantities and representations (Czocher & Hardison, under review).

Modeling can be seen as a process of unification among a sign, a referent (the object the sign stands for), and an interpretant (Kehle & Lester, 2003). In mathematics, a sign can be part of an inscription in mathematical notation. In our view, the referent could stand for a real-world object, for a quantity, or for another conceptual entity. Then the interpretant can be characterized as the mathematical conception of how that quantity relates to other quantities. Sherin's (2001) theory of symbolic forms provides one interpretation for how meaning can be read into equations, which are composed of signs. A symbolic form consists of a template and a conceptual meaning (the idea to be expressed in the equation). For example, _ + _ = _ expresses a "parts-of-a-whole" relationship. The blanks can be filled with a single symbol or a group of symbols representing quantities or combinations of quantities (perhaps related via other symbolic forms). Familiarity with symbolic forms helps individuals "know" to use certain operators (e.g., + or ×) or relationships and to know where to place the symbols of quantities in an equation. Symbolic forms are building blocks of equation generation.

Mathematizing a situationinvolves generating mathematical representations and assigning semantic meanings compatible with the modeler's conception of the situation at hand. That is, the modeler must identify relevant quantities and describe how they vary together. Thompson's (2011) theory of quantitative reasoning offers relevant insights. First, Thompson takes the strong position that quantities are mental constructs, not characteristics of objects in the world. It immediately follows that a *quantification* process is carried out by an individual in order to conceive of quantities. Quantification is "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bilinear, or multi-linear) with its unit" (p. 37). One can conceive of various instantiations of the object, with each instantiation manifesting different extents of the relevant attribute, and coordinate these instantiations with a value. We operationalize quantification as the set the of operations an individual can enact on a particular attribute (Hardison, 2019). Observing a phenomenon and conceptualizing that there are quantities and that they can vary (or may be constant) is foundational to formulating a meaningful mathematical model. For example, quantities like distance and velocity may be more readily available for high school students than torque, electrical current, or GDP.

Quantitative reasoning more generally entails conceiving of quantities and relationships among quantities. It involves conceiving of relationships among quantities whose values may vary independently. These constructs allow modeling to viewed as conceiving and representing relationships among the quantities involved. Coordinating quantities and attending to relationships among quantities is *covariational reasoning* (Carlson, et al., 2002). It involves identifying ways to combine quantities through operations and trace their changes, rates of changes, and intensities of changes whether they are directly measurable or not (e.g., Johnson, 2015). Relationships can be identified through observation, a priori reasoning, or through knowledge of principles rooted in physical theory.

When quantities and relationships being modeled are expressed externally in mathematical notation, they become the mathematical representation of a physical model. The mathematical representation is intended to convey an individual's mathematical model (mathematical concepts, objects, and structures) and the relationships among the constituent inscriptions' situational quantitative referents. In this way, the theory of covariation of quantities elaborates an important aspect of how the conceptual counterparts to mathematical models are formalized into mathematical representations and expressed in symbolic forms. In the next section, we introduce the modeling space, a theoretical construct we use to communicate the quantities present in a student's mathematical model, to document changes in this model over time, and to provide some predictive utility for modifications a student might or might not make to her model.

The Modeling Space: Construct and Representational Tool

We preface our discussion of the modeling space by noting that we distinguish between the quantities an individual projects onto a situation, operations (quantitative or numerical) enacted on quantities or their values, and the representations (inscriptions as well as utterances) she uses. We refer to the set of mathematical models a student might generate within a given modeling task as *the modeling space*. We intentionally draw an analogy to a Cartesian product space. Each quantity corresponds to a dimension of the product space. We view the modeling space as the set of mathematical relationships that act via composition on the situationally relevant quantities available to the student. In this section we build to a mathematical description of an individual's modeling space. For example, suppose that in the course of addressing a modeling task about a falling body under the influence of gravity only, we are able to infer that Janet has introduced the quantities *initial height above ground*, *time elapsed*, *current height above ground*, *mass of the object*, and *initial velocity*. We represent her available quantities, organized by type, as the sequence $(\mathrm{TIME}, \mathrm{HT_I}, \mathrm{HT}, \mathrm{MASS}, \mathrm{V_I})$. Her modeling space would be all of the meaningful (to her) mathematical combinations of those quantities. For example, one element in the modeling space Janet may represent is *current height above ground*, $h = h_0 - v_0 \cdot t$ (Eqn 1), where the symbol names correspond to experts' conventions.

We can formalize the modeling space by structuring it with a descriptive mathematical model. Suppose during modeling, a student projects a set of N quantities, $Q = \{q_1, q_2, \dots q_N\}$ onto the referent situation. At interview time τ, we assign one of three values from the set $S = \{-,0,1\}$ to each quantity (– means the potential quantity q_i has not yet been projected by the individual, 0 means q_i has been projected by the individual at some $t < \tau$ but is not referred during a time period of analytic interest, 1 means there is evidence that q_i is referred during the vignette). The set S has a natural order $- < 0 < 1$. We can impose a commutative binary operation, the *tropical addition* defined by $a * b = \max(a, b)$. S is then a monoid[1] under $*$ with identity $-$. We then form a Cartesian product from the set S over the N dimensions supplied by Q:

$$M = \prod_{i=1}^{N} S_i = \underbrace{\{-,0,1\} \times \{-,0,1\} \times \cdots \times \{-,0,1\}}_{N\ times}$$

The product M is again a commutative monoid under the operation $+$, coordinate-wise addition using the operation $*$.

Elements of M, written as $(s_1, s_2, \dots, s_N)$ are assigned to a vignette of student work on a modeling task according to whether *during that vignette* each quantity q_i is used during the vignette (value 1), has been projected prior to the vignette but is not used during the vignette (value 0), or has not yet been projected onto the situation (value $-$). The individual's mathematical model evolves over time and is captured by changing values of s_k for each q_k. Thus, we model the start of the interview with *M* as expressed as $(-, -, \cdots, -)$. As quantities are

[1] A monoid is a set that is closed under an associative binary operation and contains an identity. It is different from a group because its elements lack inverses.

introduced, the $-$'s are replaced with $0'$s and $1'$s. For example, if $N = 4$, the notation $(-,0,1,1,0)$ would indicate that the individual has available four quantities and is using quantities q_2 and q_3, but not q_1 and q_4. The quantity q_1 has not yet been projected. At the end of the interview, M is expressed as a sequence of $0's$ and $1's$. We can use elements of M to represent the accumulated modeling space (all quantities we infer a student has projected up to at a given moment during the interview) or, by acknowledging that q_i is a function of time and treating elements of M as finite states, we could trace change over designated periods of interview time. It is beyond the scope of this paper to fully explore the distinctions; instead we wish to exemplify the construct and its utility to support further theorizing by treating accumulated quantities.

Suppose Janet considered gravity, but she has not yet explicitly introduced it as a quantity, but does later. We can represent her mathematical model as $m_1 = (1,1,1,0,1,-)$, where – indicates that gravity has yet to be introduced. Note, the symbol 1 denotes only that the quantity is "active" for the individual. The symbol does not signify *which* value, unit, or mathematical symbol the individual associates with the quantity at that moment. Because a given attribute of an object can be measured in different ways, expressing gravitational acceleration in feet per second squared or object-lengths per second per second would map to the same coordinate value. In this way, the sequence $m_1 = (1,1,1,0,1,-)$ could represent any number of mathematical inscriptions whose elements are associated with quantitative referents (e.g., a graph and an equation), mathematical relationships among the quantities, or quantitative operations among the quantities. Said differently, elements of M are an equivalence class of mathematical models that compose quantities in Q. The collection of possible mathematical compositions and expressions of those compositions that can be mapped to $m \in M$ we refer to as the *modeling space of m* and indicate as $\widetilde{m}$. The collection of all possible models she might conceive using the quantities available to her during the interview would be her modeling space, $\widetilde{M}$.

Constructing Merik's Modeling Space

Data were collected as part of a larger study of the characteristics of tasks and facilitator interventions that could elicit specific mathematical modeling competencies among undergraduate STEM majors. Pursuant to this goal, we conducted a series of think-aloud task-based interviews. Participants addressed a variety of modeling and applications tasks requiring participants to make simplifying assumptions about the contextual situation. Tasks included opportunities to use arithmetic, algebra, calculus, and differential equations. In this analysis, we present the work of Merik, an engineering student who had completed linear algebra and calculus 3, because he was especially articulate in describing his mathematical thinking and regularly exhibited a variety of modeling competencies as he engaged in modeling tasks. One problem he addressed was the Monkey Problem: *A wildlife veterinarian is trying to hit a monkey in a tree with a tranquilizing dart. The monkey and the veterinarian can change their positions. Create scenarios where the veterinarian aims the tranquilizing dart to shoot the monkey.*

We chose Merik's work on the Monkey Problem because he introduced many different inscriptions, quantities, and mathematical representations indicating that it would be possible to closely examine changes in his mathematical and contextual knowledge about the situation. Merik interpreted the prompt as an invitation to find a model representing the situation. He was given unlimited time and was assured that his responses were not being judged for "correctness." We provisionally accepted all of his work without actively teaching, leading, or removing ambiguity (Goldin, 2000). A key aspect of the interview protocol and subsequent analysis was to assume Merik's interpretations of his own work differed from our own. Follow-up questions and interventions were intended to either clarify his thinking or to test any conjectures the interviewer had about his thinking in the moment. The audio/video recorded interview session lasted 46 minutes and was subsequently transcribed.

We followed the procedure outlined in Czocher and Hardison (2019, under review), which had three stages to analysis: (1a) identify and catalog all mathematical representations by examining the spatial and temporal organization of inscriptions on Merik's paper (1b) determine whether the representations or their meanings may have changed, (2) identify quantities Merik projected onto the Monkey Problem context, and (3) document whether there was sufficient evidence to infer whether the quantitative situational referent of a given inscription changed during the interview. We elaborate on (2) and then show how the modeling space construct enabled (3).

According to our theoretical frame, a quantity is an individual's conception of a measurable attribute of an object in a situation. We analyzed the interview and identified situational attributes to which Merik attended in the Monkey Problem. By situational attributes, we mean we were able to infer a referent within the Monkey context with a quality that Merik might have quantified (e.g., the tree's height). Instances in which Merik mentioned generic attributes—those for which we were unable to infer situational referents (e.g., distance)—were not considered situational attributes. Additionally, we searched for evidence that suggested Merik might have quantified these situational attributes. In particular, we sought evidence of Merik engaging in mental operations necessary for, or suggestive of, a conceived measurement process for each situational attribute. Through iterative cycles of analysis, we stabilized a set of 8 criteria that we took as evidence of quantification during mathematical modeling. Three independent coders systematically applied those criteria to the video and transcript; disagreements were resolved through consensus seeking. A quantity was included as a potential quantity for Merik if it met at least one inclusion criteria (see Table 1). We recorded the times at which we could infer that situational referents actively served as counterparts to inscriptions and symbols within the representations (or not).

Table 1 Potential quantities projected onto the Monkey Problem context, chronological order

Quantity	Type	Time	Description
$\mathrm{ANG}_{\mathrm{STR}}$	Angle	2:08	Measure of angle gun is aimed relative to the horizontal, for straight path
$\mathrm{DIST}_{\mathrm{VET/TREE}}$	Length	2:09	Horizontal distance from vet to the tree/under the monkey.
$\mathrm{HT}_{\mathrm{MKY/GUN}}$	Length	2:10	Height of the monkey relative to the vet's gun.
$\mathrm{VVEL}_{\mathrm{DART\text{-}I}}$	Rate	2:47	Initial vertical velocity of the dart
ACCDART	Rate	3:20	(Vertical) acceleration of dart
$\mathrm{HT}_{\mathrm{GUN/GRD}}$		3:35	Height of gun (or vet) relative to ground.
$\mathrm{HT}_{\mathrm{TREE/GRD}}$	Length	4:13	Height of the tree
$\mathrm{DIST}_{\mathrm{VET/MKY}}$	Length	4:36	Length of the straight path from the vet's gun to the monkey.
$\mathrm{ANG}_{\mathrm{PAR}}$	Angle	6:04	Measure of angle gun is aimed relative to the horizontal, for parabolic path
$\mathrm{IVEL}_{\mathrm{DART}}$	Rate	11:37	Initial linear velocity of the dart.
$\mathrm{HT}_{\mathrm{DART}}$	Length	15:35	Height of the dart
TIME	Time	16:08	Elapsed time
$\mathrm{ANG}_{\mathrm{VET/3D}}$	Angle	24:38	Measure of the plane angle formed by a designated axis and the line through the tree & veterinarian in 3-space.
$\mathrm{HVEL}_{\mathrm{DART\text{-}I}}$	Rate	25.42	Initial horizontal velocity of the dart

In total, we identified 14 potential quantities that Merik cumulatively introduced to structure the Monkey Problem. Thus $Q = (\Big\{ \mathrm{DIST}_{\frac{\mathrm{VET}}{\mathrm{TREE}}}, \mathrm{HT}_{\frac{\mathrm{MKY}}{\mathrm{GUN}}}, \mathrm{HT}_{\frac{\mathrm{GUN}}{\mathrm{GRD}}}, \mathrm{DIST}_{\frac{\mathrm{VET}}{\mathrm{MKY}}}, \mathrm{HT}_{\frac{\mathrm{TREE}}{\mathrm{GRD}}}\ \mathrm{HT}_{\mathrm{DART}}, \Big\},$

$\{\mathrm{ANG}_{\mathrm{STR}}, \mathrm{ANG}_{\mathrm{PAR}}, \mathrm{ANG}_{\mathrm{VET,3D}}\}, ,\{\mathrm{VVEL}_{\mathrm{DART,I}}, \mathrm{IVEL}_{\mathrm{DART}}, \mathrm{HVEL}_{\mathrm{DART}}, \mathrm{ACC}_{\mathrm{DART}}\ \}, \{\mathrm{TIME}\}$)
At interview time τ, we represent the active equivalence class within his modeling space via the tuple Q with appropriate substitutions from S made for each quantity.

Illustrations

Merik initially imposed a right triangle and considered the angle to fire the dart such that the hypotenuse would pass through the vet and the monkey. However, after introducing ACCDART, Merik stated that he was seeking a quadratic equation because "that is the path the bullet is going to follow." At this moment, there were no inscriptions resembling a quadratic equation, so we interpreted his stated goal to produce an equation as indicating an implicit symbolic form relevant to him. To elicit the form from Merik as well as to gain insight into the situationally specific meanings Merik might have for it, the interviewer asked, "What variables and parameters would be present in your equation?" Merik immediately inscribed $f(x) = Ax^2 + Bx + C$. At this point we were unable to infer that Merik had projected meanings specific to the task at hand. Moments later, Merik explained, "I know that my A is negative 10," which indicated he was attending to gravity based on his earlier activities. As Merik continued, he indicated that B "would be whatever the initial velocity is, which I don't have." Merik went on to explain, "the image of 30 feet which is, in this particular case, 40 feet." He also explained that the "image of 0 is 0." These specific values were references to Merik's earlier simplification of the task wherein he considered a specific scenario: the vet was 30 feet from the tree and the monkey was 40 feet high. Although Merik substituted 0 for C, we were unable to infer whether Merik had any situationally specific quantitative referent for C at this point in the interview. For this portion of the interview (roughly 9:30-11:40), the equivalence class for models he could generate was $(\overbrace{1,1,0,0,0,0}^{\text{lengths}}, \overbrace{0,1,-}^{\text{angles}}, \overbrace{0,1,-,1}^{\text{rates}}, \overbrace{-}^{\text{time}})$, which corresponds to his quadratic.

One of Merik's chief difficulties in constructing a model to his satisfaction lay in the fact that there were competing meanings attached to the quadratic template □ = □ · □2 + □ · □ + □. At times the symbol x represented the horizontal position of the dart (implicitly at a given moment in time), while at others it implicitly represented time elapsed in the situation. The three meanings in play were $f(x)$ as predicting vertical position in terms of elapsed time, $f(x)$ as predicting the vertical position in terms of horizontal position, and $f(x)$ as an alternative representation of the mathematical object parabola. The shift itself was not consciously realized by Merik. Indeed, he abandoned the representation at 11:45, as he referred a previously identified quantity, the angle the veterinarian should fire at $\mathrm{ANG}_{\mathrm{PAR}}$ prompting a new inscription. Based on the quantities we could infer were *active* for Merik from 9:30-15:40 The equivalence class for models he could generate was: $(\overbrace{1,1,0,0,1,1}^{\text{lengths}}, \overbrace{1,1,-}^{\text{angles}}, \overbrace{0,1,-,1}^{\text{rates}}, \overbrace{-}^{\text{time}})$ He did not resolve the competing schema until 15:16, when in response to an interviewer prompt to provide explicit meanings for the symbols Merik realized that plugging in 30 for x and 40 for y (distances) was not compatible with the parabola which recycled the symbol x for time. Thus, it was not until after 16 minutes into his work on the problem that Merik referenced time in a way that we could infer he had projected the quantity onto the situation. The representation for quantities available to Merik for composition became $\left(\overbrace{1,1,0,0,1,-}^{\text{lengths}}, \overbrace{1,1,-}^{\text{angles}}, \overbrace{0,1,-,1}^{\text{rates}}\ \overbrace{1}^{\text{time}}\right)$.

Due to space constraints, we provide one more example from later in the interview. At this point, Merik's stated goal was to find ANGPAR and we symbolize the *active* equivalence class within his modeling space as $\left(\overbrace{1,1,0,0,0,0}^{\text{lengths}},\overbrace{1,1,0}^{\text{angles}},\overbrace{0,1,0,1}^{\text{rates}}\overbrace{1}^{\text{time}}\right)$. The interviewer intended to direct Merik's attention to the angle between the straight-line path and the angle that would produce the parabolic path, asking "How do you anticipate the two angles will compare?" Merik responded that ANGPAR would be "larger not by a wide margin but I think that because the way that it's traveling more like [[draws arc'ed curve between the veterinarian and the monkey]] then you have to aim up more to increase the angle." Thus, Merik was able to consider variation in $\mathrm{ANG_{PAR}}$ in relation to $\mathrm{ANG_{STR}}$. However, this was not sufficient for quantifying the difference between the two angles, even after the interviewer prompted him to think about finding an angle measure between two curves and he responded that he could use tangent lines to do so. He introduced the signs $\underline{u}$ and $\underline{v}$ and the inscription $\underline{u} \cdot \underline{v} = \frac{\cos\theta}{|\underline{u}||\underline{v}|}$. We lacked observable evidence to support the claim that θ corresponded to a situational referent. We suggest that Merik did not (in that moment) apply his formula because, for him, the angle between $\underline{u}$ and $\underline{v}$ was not salient as a quantity. His reasonable, sensible options for modeling the situation were constrained by the quantities that he had available. This perspective explains not only why the prompt or his further mathematical conceptual work did not help him to make progress, but also why it could not help him – without a quantity, the formula had no situational meaning.

Value and Future Work

Our theoretical and methodological considerations have resulted in an examination of Merik's mathematical modeling activity as a process of composition of quantities via mathematical operations. Our approach separated the acts of quantification from the acts of introducing variables from the acts of generating inscriptions from the acts of ascribing meaning to mathematical inscriptions. We found evidence that each of these acts can be carried out independently or to varying degrees of alignment. These aspects are overlooked when viewing modeling as translation. The modeling space enables a precise description of this finding. As a theoretical tool, (1) it predicts at any given moment, a student's modeling process will be constrained by the elements in her modeling space at that time and (2) we can trace how the modeling space expands and supports (or excludes) formation of mathematical relations over time. The modeling space (at least partially) predicts, and simultaneously constrains, the mathematical models the student might produce. Ultimately, the research community's goal is to articulate opportunities for effective pedagogical intervention. In contrast to the majority of research in the modeling genre, which tends to be representations-forward, our theoretical and methodological approach put quantities in the fore. Because the modeling space is focused on documenting and tracing meanings as well as inscriptions, it may be able to support models of pedagogy as well. Our analyses offered explanations for two key moments for pedagogical intervention: one successful intervention (asking the student to be explicit about his meanings for symbols) and one failed intervention (introducing a strategy based in quantities Merik had not projected) and the outcomes of the interviewers' moves are reflected as amendments to the modeling space. As a representational tool, the modeling space approach affords overviews of an individual's work as a time series, could facilitate comparison of individuals' productions, be used to evaluate potential task prompts and indicate potential sites for interventions as well as predict whether those interventions are likely to be taken up by the student. Finally, the methodology moves the field one step closer to being able to trace changes in a mathematical model: how they are precipitated, ways they change, and how students respond to interventions.

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What is Encompassed by Responsiveness to Student Thinking?

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The way instructors respond to student thinking is an important topic of education research, yet a lack of clarity and consistent use of terminology prevents clear communication among researchers on this topic. Researchers use a wide variety of terminology, including teacher follow up, sensitivity to students, uptake, and responsiveness to refer to leveraging student thinking during various instructional practices. In this paper, I use thematic analysis to analyze 34 articles that draw on constructs related to responsiveness to student thinking from within the science and mathematics education literature. Results from this analysis shed light on a distinction between responsiveness as a disposition and as enacted responsiveness. This work has implications for professional development providers interested in supporting instructors in implementing active learning and student-thinking centered practices.

Key words: Responsiveness, Responsive teaching, Student thinking, Literature analysis

STEM education researchers would benefit from a cohesive definition and understanding of the terminology used to describe student-thinking centered instruction that uses student ideas and contributions to inform classroom interactions. These instructional practices that are responsive to students' thinking have been shown to enhance students' conceptual understandings (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), promote student agency and voice (Coffey, Hammer, Levin, & Grant, 2011), and promote equitable participation (Empson, 2003; Robertson, Scherr, & Hammer, 2016). Despite these benefits, there is not a consensus about what responsiveness to student thinking entails (e.g., Lineback, 2015; Empson, 2014).

Robertson, Atkins, Levin, and Richards (2016) summarized responsive teaching, drawing primarily on the K-12 science education literature, in three themes: (a) *a foregrounding of the substance of students' ideas*, instantiating "intellectual empathy" by listening to students with the goal of understanding their thinking rather than to simply evaluate or correct it, (b) *recognizing the disciplinary connections within students' ideas*, identifying "disciplinary progenitors" (Harrer, Flood, & Wittmann, 2013) or "seeds of science" (Hammer & van Zee, 2006), and (c) *taking up and pursuing the substance of student thinking*, where the classroom direction emerges and is moved along by the students' contributions. Although Robertson, Atkins, Levin, and Richards (2016) have begun this work focusing on the K-12 level, there is still a need to synthesize and make explicit a definition rooted in researchers' conceptualizations of instructor responsiveness to student thinking that extends to the literature and terminology used at the post-secondary level. Further, there remains a lack of precision in the terms used to describe responsiveness and responsive teacher moves (like "take up"). Additionally, there are a number of instances where researchers use the terms responsive or responsiveness without clearly articulating the intended meaning, indicating that there is a shared understanding even though they are used to describe a variety of forms and teacher actions. An explicit definition will allow the education research community to better leverage existing research that encapsulates responsiveness to student thinking and responsive instruction, and to further our understanding of responsiveness and ways to support instructors in leveraging student thinking.

In this paper, I develop a definition for *responsiveness to student thinking* rooted in the various terms, definitions, and characteristics of responsive instruction that are currently used in the math and science education literature. Specifically, my research questions is: How can responsiveness to student thinking be defined so that it encapsulates existing and related terms/constructs and provides math and science education researchers with a common language to discuss responsiveness to student thinking moving forward?

Selection of Literature

To arrive at a collection of literature related to instructor responsiveness to student thinking, I did a thorough search using the following selection criteria to identify articles and books relevant for this analysis: (1) literature that defines a term or construct indicating responsiveness to and the leveraging of student thinking, or (2) literature that explicitly discusses teacher activity that is responsive to students or their thinking (beyond evaluating the correctness of student responses). I began with Robertson, Scherr, and Hammer's (2016) book *Responsive Teaching in Science and Mathematics*, examining all references to identify those that met at least one of the selection criteria. I then conducted a search of the databases Academic Search Premier, PsycINFO, ERIC, and Psychology and Behavioral Sciences using the key words "responsive*" and "math*", and excluding "cultural*" since Culturally Responsive Teaching is beyond the scope of the analysis for this paper. This resulted in a corpus of 17 articles. I looked through all the citations in these articles and added six that met the selection criteria. To this corpus, I included 11 articles that met the selection criteria and were recommended by other mathematics education researchers.

The goal of this analysis is to identify and understand the terms and constructs math and science education researchers use to describe responsiveness and responsive instruction, working towards a definition that clearly articulates what is encapsulated by these terms. Here, I exclude literature related to factors that affect instructors' ability to be responsive to students' thinking, such as their mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008) and generative listening (e.g. Yackel, Stephan, Rasmussen, & Underwood, 2003) because they are outside of the scope of the research question and goal for this work. However, I acknowledge that these are important features to consider in order to fully understand instructor responsiveness.

Thematic Analysis: What is Encompassed by Responsiveness?

To develop a definition for responsiveness to student thinking that encapsulates related terms and constructs, I conducted a thematic analysis on the corpus of literature selected using the criteria discussed above. In a first level of coding, the data were categorized by the following themes: constructs related to responsiveness (satisfying selection criteria 1), teacher activity related to responsive instruction (satisfying selection criteria 2), and curriculum and resources that support responsive instruction.

Forms of responsiveness

After this initial categorizing, I coded the data to determine the forms that responsiveness to student thinking can take on based on the descriptions in the literature. The results from the thematic analysis indicate that responsiveness has been described in a variety of forms (see Table 1), which highlights the poorly defined and multi-faceted nature of responsiveness to student thinking. Several constructs take the form of discourse moves in student-teacher interactions (e.g., Collins, 1982; Brodie, 2011; O'Connor & Michaels, 1993; van Zee & Minstrell, 1997; Adhami, 2001), with some focusing on teachers' comments/questions and how they use, expand, or clarify student contributions (e.g., Collins, 1982; O'Connor & Michaels, 1993; Adhami, 2001). A prominent characteristic of these constructs is the cohesion between the student contribution and teacher's response, indicated by an explicit reference to the student's contribution.

In addition to being referenced as discourse moves, responsiveness has been described as characteristic of interactions, mental actions and knowledge, an orientation, and a skill set. Namely, Bishop, Hardison, and Przybyla-Kuchek (2016) refer to responsiveness as a characteristic of an interaction that is evidenced by how student ideas are "present, valued, attended to, and taken up as the basis of instruction" (p. 1173). Teuscher, Moore, and Carlson's (2015) and Ader and Carlson's (2019) use of decentering takes into consideration a teacher's attempt to understand student reasoning when considering how teachers respond to student contributions, shedding light on their mental actions. Jaworski's (2002) defines sensitivity to students as a component of a teacher's knowledge of student thinking, in addition to an attention to students' needs and interactions with students, indicating an orientation. In Hackenberg's (2010) conceptualization of mathematical caring relations, she considers these relations to require an orientation to monitor and respond to student thinking and energy, and Jacobs, Lamb, and Philipp (2010) describe professional noticing as a set of interrelated skills.

Table 1. Aspects of Responsiveness from Data Corpus

Aspect of Responsiveness	Constructs from Data Corpus (Citation)
Discourse Moves	Uptake (Collins, 1992); Follow-up (Brodie, 2011); Revoicing (O'Connor & Michaels, 1993); Reflective toss (van Zee & Minstrell, 1997); Responsive questioning (Adhami, 2001)
Characteristics of Interactions	Responsiveness (Pierson, 2008; Bishop, Hardison, & Przybyla-Kuchek, 2016)
Mental Actions & Knowledge	Sensitivity to students (Jaworski, 2002; Potari & Jaworski, 2002); Decentering (Teuscher, Moore, & Carlson, 2015; Ader & Carlson, 2019)
Orientations	Mathematical caring relations (Hackenberg, 2005; 2010)
Skill Set	Professional noticing (Jacobs, Lamb, & Philipp, 2010)

Responsiveness as a Disposition or Enacted

This second level analysis of the form of responsiveness to student thinking highlights a prevalent distinction between two aspects of responsiveness to student thinking. Multiple researchers allude to the difference between a tendency to be responsive and the enactment of responsiveness. This distinction indicates that it is productive to consider responsiveness as a disposition (i.e. tendency to be responsive) and responsiveness in action (i.e. enacted responsiveness). While dispositions themselves can be difficult to identify and understand, researchers have studied dispositions in relation to how they impact behavior. Thornton defines *dispositions in action* as

> habits of mind including both cognitive and affective attributes that filter one's knowledge, skills, and beliefs and impact the action one takes in classroom or professional setting. They are manifested within relationships as meaning-making occurs with others and they are evidenced through interactions in the form of discourse (2006, p. 62).

Drawing on Thornton's (2006) definition, the constructs, terms, and conceptualizations of responsiveness were then coded as evidencing a responsive disposition, enacted responsiveness, or both. I began by highlighting all excerpts pointing to aspects or "habits of mind" that impact action (indicating responsiveness as a disposition), and all excerpts demonstrating teacher activity (indicating an enacted responsiveness). The following sections include themes and a description for responsiveness as a disposition and enacted responsiveness. I then used open coding to determine emergent subthemes. Table 2 provides the emergent codes, and an example coded excerpt.

Table 2. Codes and Example Coded Excerpts.

	Code	Coded Excerpt
Responsiveness as a Disposition	Attention to student thinking	"Responsiveness to students' mathematical thinking is a characteristic of interactions wherein students' mathematical ideas are present, valued, attended to" (Bishop, Hardison, & Przybyla-Kuchek, 2016)
	Attention to other student needs	"[Mathematics teachers] start to act as mathematical carers when they work to harmonize themselves with and open new possibilities for students' mathematical thinking, while maintaining a focus on students' feelings of depletion and stimulation" (Hackenberg, 2005)
	Valuing of student thinking	"Responsive questioning can be identified [by]…genuine acceptance and engagement with pupils' ideas" (Adhami, 2001)
Enacted Responsiveness	Eliciting student thinking	"Of particular interest was …whether or not the teacher elicited students' thinking beyond what was required to solve the particular problems posed" (Franke et al., 2001)
	Responding to other student needs	"[Mathematics teachers] start to act as mathematical carers when they hold their work of orchestrating mathematical learning for their students together with a… respon[se] to fluctuations in positive energy" (Hackenberg, 2010)
	Responding to/on the basis of student thinking	"We conceptualize *responsive teaching* as a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continuously adjusted during instruction in response to children's content-specific thinking" (Jacobs & Empson, 2015)

Responsiveness as a disposition. A number of the responsiveness descriptions in the literature point to the impact of an underlying disposition on an instructor's demonstration of responsiveness. In Hackenberg's (2005; 2010) description of mathematical caring relations, she discusses instructors' orientation to monitor and respond to energetic fluctuations, indicating an awareness of the need to provide appropriate follow-up to students that is rooted in an accurate interpretation of student understandings. Similarly, Ader and Carlson (2019) discuss teachers' propensity to decenter, which aligns with one's orientation to engage in decentering acts, and professional noticing (Jacobs, Lamb, & Philipp, 2010) specifically points to an intentionality in recognizing opportunities to connect mathematics with students' current understandings. This indicates that there is an underlying attribution of value to student contributions in professional noticing since the instructor aims to leverage productive student thinking and understandings in making these connections (also see Potari and Jaworski, 2002).

Together these constructs illustrate what is encompassed by a responsive disposition towards student thinking with themes including the valuing of student thinking and ability to do mathematics, and an awareness of the need to provide appropriate follow-up to facilitate the students' mathematical progress. This valuing and awareness of appropriate follow-up aligns with Thornton's (2006) definition of disposition in action since these are the "habits of mind" or orientations that impact the actions that a teacher takes. Further, "they are manifested within relationships as meaning-making occurs" in how a teacher attends, engages, and attempts to understand student thinking, looking for opportunities to make connections to disciplinary conventions. In addition to shedding light on responsive dispositions, these constructs also demonstrate an enactment of this disposition, "evidenced through interactions" (Thornton, 2006, p. 62) or teacher activity.

Enacted responsiveness. All constructs and descriptions overviewed in this paper highlight instructors' activity that exhibits responsiveness, or in other words, their enacted responsiveness. Instructors demonstrate responsiveness in the ways that they attempt to understand student thinking, engage with student reasoning, and in how they facilitate opportunities for student contributions to arise, drawing out student content-related thinking. Further, the phrase "elicit[ing] student thinking" is mentioned explicitly in descriptions of five constructs (see Table 3). Nine constructs do not explicitly reference eliciting student thinking, but they do refer to creating space for contributions or opportunities for students to (further) reveal their understandings (see Table 3). Seven constructs rely on a demonstration of student thinking in order to leverage it during in-the-moment instruction (see Table 3). These teacher actions illustrate how an instructor draws out student contributions or engages with student reasoning, which is critical to developing the understanding of student thinking necessary to effectively leverage it during instruction.

Table 3. Aspects of Enacted Responsiveness

Enacted Responsiveness	Construct from the Data Corpus (Citation)
Eliciting student thinking	Fennema et al., 1996; van Zee & Minstrell, 1997; Adhami, 2001; Abdulhamid & Venkat, 2018; Pierson, 2008
Creating space for student contributions	Johnson, 2013; Kuster, Johnson, Keene, & Andrews-Larson, 2017; Laursen & Rasmussen, 2019; O'Connor & Michaels, 1993; Collins, 1982; Brodie, 2011; Saxe, Gearhart, & Seltzer, 1999; Franke, Turrou, & Webb, 2011; Lineback, 2015
Leveraging student thinking	Jaworski, 2002; Hackenberg, 2005; Jacobs, Lamb, & Philipp, 2010; Teuscher, Moore, & Carlson, 2015; Empson, 2014; Koole & Elbers, 2014; Jacobs & Empson, 2015

Another similarity between these descriptions and definitions, shedding light on an important component of enacted responsiveness, is that it requires an actual response from the instructor indicating they are attending to the students' thinking or contributions *and* are responding based on this. In particular, it is important for an instructor to respond to students in ways that are consistent with student demonstrations of understanding. Instructors can respond by leveraging, building on, or incorporating student thinking/ideas into the discourse of the class, and by using student contributions to inform instruction. Additionally, instructors often respond in order to progress students' content-related thinking, supporting the development of rich meanings, or to facilitating and connecting student ideas with other student contributions or with established disciplinary conventions.

Discussion: Defining Responsiveness to Student Thinking

After considering these various constructs and the definitions or descriptions provided by the authors, I propose the following definition for *responsiveness to student thinking*, highlighting the two distinct dimensions of responsiveness as a disposition and enacted responsiveness:

Responsiveness to student thinking as a disposition is attending to student thinking, disciplinary meanings, and needs (including both cognitive and affective needs), and an awareness of the need to provide an appropriate response based on student understandings or contributions. A responsive disposition relies on an instructor's valuing of the substance of student thinking.

Enacted responsiveness to student thinking is responding to student thinking by focusing on the substance of student contributions (including those that are incorrect or incomplete) when determining the direction of the lesson, through leveraging and building on student contributions, and by making connections between student ideas and what is accepted as disciplinary conventions.

These components of responsiveness as a disposition and enacted responsiveness are specifically exhibited by various conceptualizations of responsiveness through the constructs analyzed in this paper. Figure 1 illustrates the relationship between responsiveness as a disposition and enacted responsiveness, highlighting how they complement one another to support a more complete understanding of responsiveness to student thinking. Responsiveness as a disposition is an underlying component that influences how an instructor makes meaning from their students' thinking and contributions by impacting how they create space for student ideas to emerge, in how they engage with student ideas, attend to student thinking, interpret demonstrations of student understandings, and in how they elicit, build, clarify, and reflect on student contributions. These actions, influenced by responsiveness as a disposition, are further evidenced by how an instructor responds based on student thinking, leveraging student ideas in instruction and supporting disciplinary connections rooted in the substance of student contributions. Although not explicitly documented in the diagram, it is important to note that responsiveness can be demonstrated both during in-the-moment instruction and in how an instructor uses student thinking in other instructional practices, such as in planning or assessing.

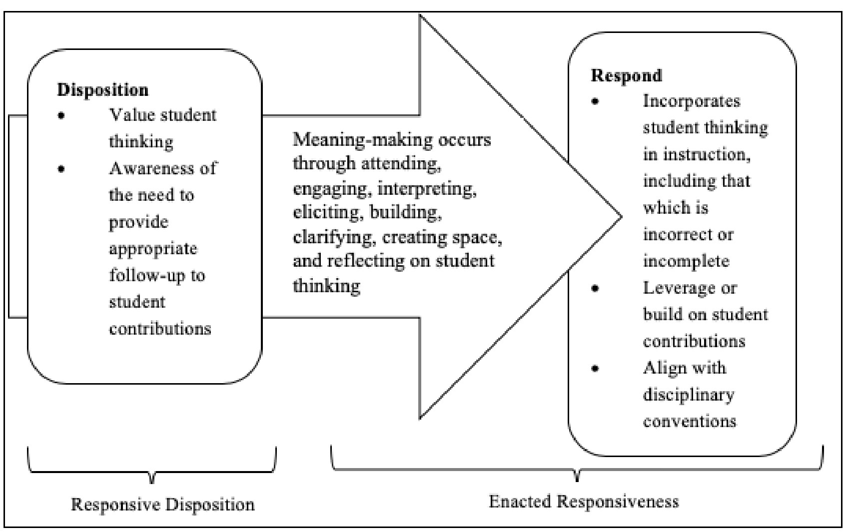

Figure 1. Responsiveness to student thinking as a disposition and as enacted responsiveness.

I propose that these two components, responsive disposition and enacted responsiveness, are each necessary but not individually sufficient to constitute responsiveness to student thinking. That is, actions that respond to student thinking (enacted responsiveness) that are not driven by underlying values and disposition that support these actions (responsive disposition) are not responsive to student thinking. For example, if an instructor only engages with students' thinking on a surface level through observable behavior, such as restating a student's contribution without attempting to understand the substance and meaning within the student's statement, then this interaction would not be considered responsiveness to student thinking. Similarly, if an instructor values student thinking and is aware of the need to provide appropriate follow-up to student thinking (responsive disposition), but does not act on this (enacted responsiveness), this would also not be considered responsiveness to student thinking, as there is no evidence of a response.

Considering responsiveness as a disposition and as enacted responsiveness together allows for a more complete picture of what is encapsulated by responsiveness to student thinking, and highlights the importance of both an instructors' underlying interest and valuing of student thinking and response on the basis of student thinking. Responsiveness to student thinking, as I have defined it in this paper, includes both an underlying and enacted disposition, and includes an assumption that instructors demonstrating responsiveness to student thinking are attending and responding on the basis of (underlying) student thinking rather than simply responding to student contributions without using the contribution as a window into student conceptions.

Conclusion and Implications

In collecting a selection of literature describing constructs and activity related to responsiveness for the thematic analysis, I sought to include a broad yet representative sample of literature, primarily focusing on better understanding how responsiveness was exhibited in responsive instruction. In the presentation of this work, I will also address the theoretical perspectives that underlie these conceptualizations of responsiveness. These conceptualizations all included descriptions of instructor activity, but the thematic analysis also illustrated an underlying aspect of responsive instruction – the presence and impact of responsiveness as a disposition. Although this underlying component of responsiveness was not explicitly discussed in conjunction with the evidence of an enacted responsiveness in all of the constructs, it is a critical component of responsiveness to student thinking, enabling effective student-thinking centered instruction.

There is an increased desire by departmental leaders (department chairs and course coordinators) to include more active learning at the post-secondary level, but this runs counter to the perspective of many individual instructors' who believe in the "strength and utility of lecture" (Laursen, 2019, p. 112). Thus, there is a need to identify ways to support attending to and the leveraging of student thinking within the current instructional environment. Since responsiveness to student thinking is not limited to in-the-moment instruction, it allows instructors to be responsive without completely abandoning the instructional practices that they are most comfortable employing, while also shifting their instruction to be more student-thinking centered. Further, by distinguishing between responsiveness as a disposition and an enacted responsiveness as dimensions of responsiveness to student thinking, professional development providers can leverage both of these aspects in supporting responsive student-thinking centered instruction. It is important to note, however, that there is still work that needs to be done to more fully understand responsiveness as a disposition, enacted responsiveness, and ways to support instructors in their endeavors to teach responsively. Specifically, there is a need to investigate demonstrations of responsiveness that go beyond in-the-moment inquiry-based instruction, particularly at the post-secondary level. Further, since instructors are likely to enact responsiveness differently, it is important to better understand these differences and their impact on instruction and student learning (Lineback, 2015).

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A Theorization of Learning Environments to Support the Design of Intellectual Need-Provoking Tasks in Introductory Calculus

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Intellectual need is the need that students feel to understand how and why a particular mathematical idea came to be. We are interested in creating tasks that calculus instructors can use to provoke intellectual need. However, the current suggestions for designing such tasks lack detail and don't account for several issues specific to undergraduate introductory calculus. In this theoretical paper, we discuss the idea of intellectual need, explore three issues related to the teaching of calculus, and present a theoretical model that task-designers can use to frame important factors that affect the development and use of these tasks.

Keywords: Calculus, Intellectual Need, Task Design

As instructors who regularly teach introductory calculus at the undergraduate level, we routinely look for ways to support our students' learning. In particular, we want our students to "engage with mathematical content in ways that go beyond using known facts in standard procedural ways" (Mesa, Burn, & White, 2015, p. 84). One idea that we have found particularly promising is Harel's (e.g., 1998) idea of intellectual necessity (which we describe in detail below). In particular, we are interested in designing tasks that calculus instructors can use to provoke intellectual need in their students.

In this theoretical paper, we describe the idea of intellectual need, discuss three issues specific to designing intellectual need-provoking tasks for introductory calculus, and then present a theoretical framework for describing features of learning environments that will help task designers identify important factors in creating effective need-provoking tasks.

Intellectual Need

Harel and Tall (1991) first alluded to the idea of intellectual need, proposing that "if students do not see the rationale for an idea... the idea would seem to them as being evoked arbitrarily; it does not become a concept of the students" (p. 41). This was later formalized by Harel (1998) as the *necessity principle*: "For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need" (p. 501).

The idea of intellectual need is framed similarly to Piaget's notion of disequilibrium. Specifically, Harel (2013b) describes it as the perceived need to resolve "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (p. 122). As Meyer (2015) put it, "if math is the aspirin" then intellectual need is "the headache." This perturbation is rooted in the individual's experience within the discipline—in this case, of mathematics—and is based on the learner's epistemological justification for the mathematical concept, where an epistemological justification is "the learner's discernment of how and why a particular piece of knowledge came to be" (Harel, 2013a, p. 8).

For example, suppose students are given data for the fuel economy of cars driving between 45 and 75 mph at 5-mph intervals, and asked to determine the speed that would maximize the fuel economy. The students might initially guess that the largest reported fuel economy value would be the maximum, but then realize that the maximum could occur at a speed that wasn't reported in the table. The students might find a function that could model the data, but, without a technique or tool for finding the maximal value of this function, would be limited to guessing and checking. This inability to find the exact value may cause a perturbational state for the students, which is the intellectual need for how to find maximal values of functions, and lead students to see the epistemological justification for such a tool or technique.

Intellectual need is related to other theories of teaching and learning. For example, in Dewey's (1938) theory of inquiry, individuals experience indeterminate situations and then, through the process of inquiry, frame these as problematic situations which they feel the need to resolve. Similarly, in Brousseau's theory of *problematique*, mathematical problems "constitute means to challenge the pupils' initial conceptions and to initiate their evolution" (Balacheff, 1990, p. 260) and the students' learning "depends on their recognition and re-construction of problems as being their own" (Balacheff. 1990, p. 259).

The theory of intellectual need has been widely used to design instructional tasks. For example, Harel (e.g., 2013b) has conducted numerous teaching experiments for various mathematical topics based on the necessity principle. Leatham, Peterson, Stockero, and Van Zoest (2015) used the idea to design and analyze effective "openings" for lessons, and Abrahamson, Trninic, Gutiérrez, Huth, and Lee (2011) used the idea to design "hooks" for their curriculum of mediated discovery. Other teachers and researchers have used the idea to conduct professional development workshops (e.g., Meyer, 2015), design instructional tasks (e.g., Koichu, 2012; Caglayan, 2015; Foster & de Villers, 2015) and analyze classroom instruction (e.g., Rabin, Fuller, & Harel, 2013; Zazkis & Kontorovich, 2016).

There are currently a few suggestions for developing intellectual need-provoking tasks, but these generally lack detail. For example, Harel (2013b, p. 149) proposed that instructors:

1. Recognize what constitutes an intellectual need for a particular population of students, relative to a particular subject
2. Translate this need into a set of general questions….
3. Structure the subject around a sequence of problems whose solutions contribute to the investigation of these questions….
4. Help students elicit the concepts from solutions to these problems.

However, we believe that these recommendations aren't situated in a way that lets us account for some important factors that might influence their design, implementation, and success of the tasks. In addition, in the case of undergraduate introductory calculus classes, there are several issues that either act as a barrier to designing intellectual need-provoking tasks or introduce aspects that aren't well-theorized by the current literature on intellectual need. We describe three of these issues below.

Issue 1: What can Students Experience Intellectual Need For?

The first issue we explore is the limits of intellectual need—that is, if we want to design tasks that provoke students to experience an intellectual need for X, what sorts of things can X be?

Harel (2013a) defined mathematical knowledge as "all the ways of understanding [i.e., products such as definitions, conjectures, theorems, proofs, problems, and solutions] and ways of thinking [i.e., mathematical practices used to create the products, heuristics, and beliefs about the nature of math knowledge] that have been institutionalized throughout history" (p. 5). In the examples in the literature, the mathematical knowledge for which a student might experience intellectual need is generalized, abstracted, or broadly applicable, and is typically a mathematical product rather than a practice. For example, the literature includes intellectual need-provoking tasks for “complex numbers” (Harel, 2013a); “diagonalization” and “algebraic reasoning” (Harel, 2007); “function notation” (Thompson, 2013); “eigenvalues” and “eigenvectors” (Caglayan), and “scalar products” (Foster & de Villers, 2015).

In addition to traditional procedures and concepts that might be taught in an introductory calculus class, instructors might include learning goals such as mathematical practices (e.g., like those described in the Common Core State Standards (National Governor’s Association, 2010). Similarly, Schoenfeld (2012) advocated for students to learn sense-making and problem solving as part of mathematics and, following Brousseau (translated by Warfield, 2007, pp. 20-21): “some mathematical knowledge in its full richness—not merely a statement of a mathematical concept, but its meaning, its uses, its connections to prior knowledge, the context in which it is likely to be encountered, the language commonly used to express it.” In terms of Harel’s framing of mathematical knowledge, we want the theory of intellectual necessity—in its current form—to allow students to experience an intellectual need for a way of thinking.

In the current framing—and in the examples in the literature—mathematical knowledge for which one might experience intellectual need appears to necessarily apply beyond a single context. For example, in the fuel efficiency problem described above, the specific numeric answer to this question is not “mathematical” knowledge because it is not generalized or broadly applicable. However, we question how general the resolution of a perturbational state needs to be. For example, could “using derivatives to find optimal fuel efficiency” be the target of an intellectual need-provoking task? Must the mathematical knowledge relevant to such a question be a general solution method for optimization that deals with general principles and works across various problem contexts?

We also question whether the resolution to a perturbational state needs to be free of (non-mathematical) context. For example, if we ask students to think about relationships between distance, velocity, and acceleration functions, we might identify “derivative functions” as the target for an intellectual need-provoking task. However, students might view “kinematic equations” as the resolution to their perturbational state. Would this count as “mathematical” knowledge? In particular, since 98% of students enrolled in introductory calculus classes are not planning on a career in mathematics (Bressoud, Mesa & Rasmussen, 2015), it might be valuable for mathematics tasks to attempt to provoke intellectual need for concepts in specific non-mathematical contexts.

Issue 2: The Role of Affective Need and Context

The second issue we explore is the ramifications of the prior experience of students in introductory calculus classes. Bressoud (2015) found that roughly 60% of students in introductory undergraduate calculus classes have already taken a “calculus” course. In terms of intellectual need, this raises the question of whether a particular situation *can* engender a perturbational state if the students already (believe they) possess the knowledge that addresses the situation. For example, if the instructor poses an optimization problem to students who already know to “set a derivative equal to zero,” then the students are unlikely to find the task problematic because they already have the knowledge that will resolve any of the optimization problems (whether they have epistemological justification or not).

In these cases, we see additional aspects of “need” coming into play that can help steer the students toward intellectual need and connecting that need to the knowledge they already have. In particular, we hypothesize that *affective need*, which “has to do with people’s desire, volition, interest, self-determination, and the like” (Harel, 2013b, p. 146), plays an important role in getting calculus students to engage with tasks that can then provoke intellectual need. Harel suggests that affective need is often linked to students' perceived obligation to participate in school and respond in particular ways to teachers, to increase social or economic status, or to advance societal goals. Thus, we hypothesize that a student's "real-world" interests as well as their beliefs about mathematics, learning, and their role in a math classroom have a significant impact on their experience of affective need. Although Harel acknowledges the importance of affective need, noting that it "is the drive to initially engage in a problem and pursue its solution" (2013b, p. 136), he discusses affective need "only indirectly" (Harel, 2013a, p. 8).

We hypothesize that experiencing affective need is a prerequisite to intellectual need and that this is particularly critical for students who already "know" what they are being asked to learn. Specifically, even if students experience a perturbation from a task, we hypothesize that they might not take steps to resolve this state without experiencing affective need (see Balacheff, 1990). In cases where students believe that their role in a math class is simply to memorize and replicate the procedures demonstrated by their instructor, they may only "learn" the mathematics in order to do the similar problem on the test rather than constructing new mathematical knowledge to resolve a perturbational state.

To clarify the role of affective need, we distinguish between what we call the *problem context*, which is the specific situation and a question about that situation, and the *problematic mathematical situation*, which is the broader mathematical question that is encountered as one digs into the problem context. In our example, the problem context is a situation about fuel consumption. The problematic mathematical situation *inside* that problem context is the issue of not knowing how to identify maxima (or minima). We note that this distinction does not only apply to "real-world" problems. Even abstract mathematical problems have both a problem context and a (possibly) problematic mathematical situation.

If a student views the problem context as interesting and worth pursuing, it can stimulate the affective need that creates engagement with the problem. This engagement can then lead students to encounter the problematic mathematical situation. If the students are engaged in solving a problem due to their interest in the context and don't know beforehand that the lesson is on a particular topic, they may experience some of that disequilibrium before realizing that the "set the derivative equal to zero" method is its solution. When the connection is made to the previous knowledge, it *does* become a resolution to the problematic situation, and epistemological justification can be created. For example, if students are interested in fuel efficiency, then they might be curious about its answer. This affective need causes them to engage in the problem and persist in trying to answer it. As they do, they may have guesses about its approximate value, but experience the issue of not knowing precisely what speed maximizes the fuel efficiency. If the connection is eventually made to the optimization process, then the student is able to connect possibly pre-existing knowledge to the situation.

Issue 3: Local vs. Global Need and Relationship with Curriculum

The third issue we explore is the instructional constraints faced by teachers in introductory calculus classes. As Rasmussen and Ellis (2015) note, successful calculus programs at large universities typically have systems for coordinating aspects of their calculus classes, including having common syllabi, textbooks, and exams, as well as having coordinators who decide on the emphasis of various topics and try to ensure uniform coverage and pacing of topics. The current literature on instruction that focuses on intellectual need either analyzes stand-alone tasks or a self-contained sequence of tasks. Thus, there are not clear, theoretically grounded ways to account for instructional constraints in the design and implementation of intellectual need-provoking tasks.

Harel (2013b) distinguishes between *global intellectual necessity*, which refers to a "major problem" (p. 147), and *local intellectual need*, which "emerges along the way to solve a major problem" (p. 147). In an instructionally-constrained classroom, we suspect that it would be challenging to create or implement a full curriculum—a "selected sequence of activities, situations, contexts, and so on, from which students will, it is hoped, construct a particular way of thinking" (Thompson, 1985, p. 191) that would provoke global intellectual need. However, with sufficient knowledge of the curricular constraints, it may be possible to design tasks that provoke local intellectual need for targeted concepts and, if there is consistency among the tasks, enable instructors to help their students see connections between these moments.

A Theorization of Learning Environments for Intellectual Need-Provoking Tasks

In order to support the design of tasks that instructors can use to provoke intellectual need for topics in introductory calculus, we need a theoretically grounded model of the various factors in a calculus class that would influence the use of these tasks. To do this, we build on prior theorizations of learning environments to incorporate the aspects of Harel's theory and additional aspects that we described above.

There are numerous theorizations of learning environments. The most widely cited are those of Chevallard (1982) and Brousseau (1997) who described didactical situations in terms of three components, as represented in *Figure 1a*:

- Students, including their background knowledge and reasoning
- Mathematical knowledge (in its "full richness" as described above)
- The teacher, including their choice of pedagogy and learning materials

This theorization was largely concerned with mathematical practice—particularly, how practice outside the classroom was transformed into practice within the classroom.

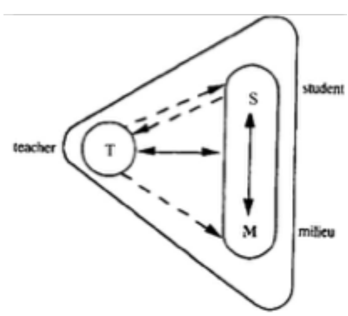

Figure 1a: Brousseau's (1997) model of a didactical situation

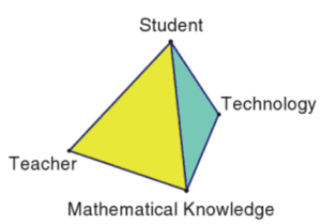

Figure 1b: Olive et al.'s (2010) model of a mathematical learning environment

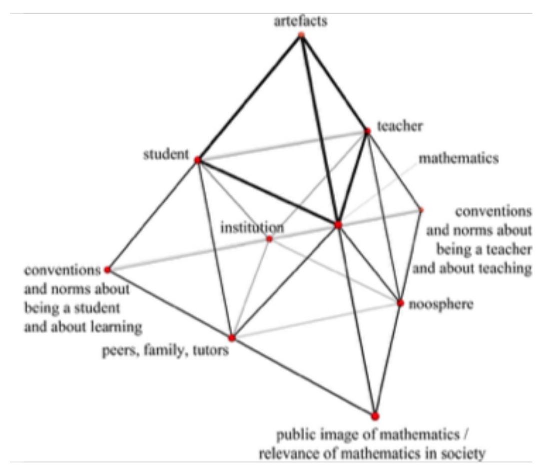

Figure 1c: Rezat & Sträßer's (2012) socio-cultural model of a mathematical learning environment

As Schoenfeld (2012) noted, subsequent models of learning environments tended to start with Brousseau's model, but framed its components in terms of epistemological and psychological relationships. For example, Olive et al. (2010) investigated the way "technology" mediated students' construction of mathematical ideas in a classroom setting and added this as a node in a didactical tetrahedron (as shown in *Figure 1b*). Similarly, Rezat (2006) built on Vygotsky's activity theory and added the "textbook" as a fourth node, and Johnson, Coles, and Clarke (2017) added "task" as a fourth node. Rezat and Sträßer (2012) attempted to incorporate many of the socio-cultural elements that were part of Brousseau's original theorization, for a total of 10 nodes in their model (as shown in *Figure 1c*).

Harel's (e.g. 2013b) original theory suggests that an instructional model needs to attend to three aspects: the students' current ways of thinking and their "background and knowledge" (p. 145); the target mathematical concept and the related " epistemology of a discipline" (p. 146); and the teacher's attention to these two aspects in order for "students' mathematical behaviors to become oriented within and driven by these needs" (p. 145). However, we believe that such a theorization wouldn't capture some of the important aspects of a learning environment germane to the teaching of introductory calculus. In addition to the issues discussed in the previous sections, we believe that a theoretical model of a learning environment that supports the design and analysis of intellectual need-provoking tasks needs to address some of the aspects present in these other models and enable instructors to address the three issues described above.

We propose a new model of a learning environment designed for the purpose of supporting the development of intellectual need-provoking tasks, as shown in *Figure 2*. In this model, the task is represented at the center of the student-teacher-math triangle, but we intend it to have components at the center of each of the seven embedded triangles. Although we don't have enough space to deeply explore the meaning of each vertex and triangle, we describe below some of the key ways this model is both connected to and extends previous models.

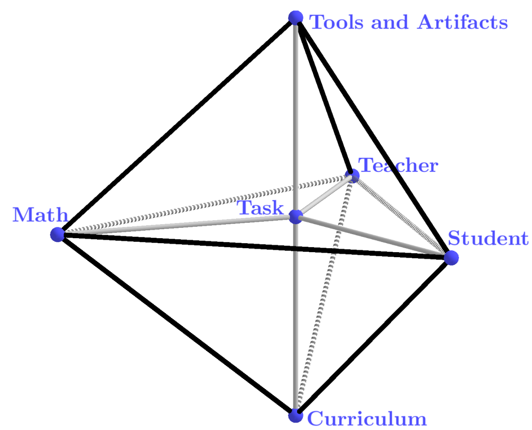

Figure 2: The model of a learning environment

At the center of the model is the task that is intended to provoke intellectual need. Following Johnson, Coles, and Clarke's (2017) theorization of tasks in the context of learning environments, this task node includes the written/verbally-expressed task itself, the designer's intended instructional purpose—both mathematical and curricular—and anticipated student responses, the teacher's intentions, and the students' activity. Thus, the task necessarily interacts with the teacher, students, mathematics, and the curriculum. In addition, Johnson, Coles, and Clarke (2017) note that the task is viewed "as a collective social performance" (p. 814) and, thus, necessitates the incorporation of the social dimensions of Brousseau's original model.

In the curriculum node we include the written, intended, and enacted curriculum (Remillard, 1999), the course syllabus, and learning goals; we also include the timing and design of assessments, since these may be created on a department level, particularly for courses with multiple sections. In many cases, instructors have little control over the written and intended curricula, and the enacted curriculum is dependent on the learning goals and assessments.

Several of the previous models of learning environments (e.g., Olive & Makar, 2009; Rezat & Sträßer, 2012) describe the ways in which artifacts mediate students' and teachers' interaction with mathematics. In contrast to Johnson, Coles, and Clarke (2017), we believe that these tools and artifacts—the reference materials, computational devices, symbolic notation, etc.—play a substantially different role than the tasks themselves: the task provides a context and purpose for the students' work, while the artifacts and tools mediate the work itself. The idea of tools is particularly germane to intellectual need-provoking tasks: As Lim (2009) points out, the necessity principle implies that "students should be allowed to use whatever tools are available to them" (p. 94) and, if they are artificially constrained, will not experience intellectual need. For example, if a student has access to mathematical software that automatically computes maxima and minima of functions, basic optimization tasks might fail to put them in a perturbational state.

The mathematics node is broadly defined to incorporate both mathematical processes and the contexts used in the tasks that might be—from the students' and instructors' perspectives—inseparable from the more abstractly-defined mathematical concepts. Following Brousseau's theorization, this view of mathematics incorporates the ways the mathematical concepts are embedded in and influenced by social practices and values.

The node for the teachers includes the ways they implement instructional activities, their sequencing and timing of these activities, the ways they model mathematical practices, invoke and interpret mathematical concepts, and relate to the students. In addition, this node includes the teachers', students', and task/curriculum designers' conceptions of the roles that a teacher plays in the classroom.

The students are described in terms of their ways of thinking, their background knowledge, their beliefs about mathematics and role in the classroom, as well as the contexts, actions, and practices that influence their affective need and might let us generate "hooks" for prompting them to engage in mathematical activity.

In this model, the primary triad of student-teacher-math is meant to represent Brousseau's original model, including the ways each of these aspects are socially and culturally situated. The tools/artifacts and curriculum nodes create collections of triads with the student, teacher, and math nodes; each of these triads is meant to be analyzed as suggested by Rezat and Sträßer, focusing on the ways the tools, artifacts, and curriculum mediate the interactions of the other pairs of nodes while, following their Vygotskian perspective, still being socially situated.

The task node can be viewed either as within a triad with another pair of nodes or as part of the entire model. In this way, it provides a guide for task designers by focusing on all of the ways the task relates to and influences the other nodes, yet simultaneously allows the designer to focus on particular triads for analysis. For example, the task designer needs to think about mathematical concepts and how they are used, presented, and sequenced in a particular curriculum; similarly, the designer needs to think about things that might stimulate students' affective needs and the tools and artifacts that are available to the students in order to select tasks that will motivate and, potentially, put the students in a state of perturbation.

By providing detail about each node and explicitly accounting for the various connections between the nodes, this theoretical model can support the work of task designers and instructors to create and effectively implement tasks that provoke intellectual need for undergraduate calculus students and, thereby, promote student learning.

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Knowledge for Teaching at the Undergraduate Level: Insights from a STEM-wide Literature Review

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Research on Mathematical Knowledge for Teaching has helped the education community understand the complex, knowledge-related factors that shape instructors' practices and the learning opportunities they create for students. Much of this work has occurred in the context of K-12 teaching. Although expanding, research on knowledge for teaching undergraduate mathematics is not extensive. A similar situation exists in science education. To help support these research efforts and theory development, we analyzed literature on knowledge for teaching STEM content at K-12 and undergraduate levels. Findings take the form of cross-disciplinary themes and descriptions of how components of knowledge for teaching are defined in different disciplines. This cross-disciplinary view into research on knowledge for teaching provides assistance to researchers who wish to leverage work from outside of mathematics and the analysis also reveals some under-examined areas of knowledge for teaching that might be productive foci for future research in RUME.

Keywords: knowledge for teaching, pedagogical content knowledge, pedagogical knowledge, STEM disciplines

Introduction and Rationale: The Value of a Cross-Disciplinary View into Research on Knowledge for Teaching

Evidence-based instructional strategies can improve outcomes for all students and the retention of students from underrepresented groups in undergraduate STEM degrees (Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, 2014; Laursen, Hassi, Kogan, & Weston, 2014). As a result of this potential, there have been repeated high-profile calls for substantial reform in teaching practices in undergraduate STEM. Achieving widespread adoption and effective use of evidence-based teaching strategies demands attention to the role of college instructors, including what instructors know and are able to do. Although work in this area has increased in recent years, undergraduate mathematics instructors' knowledge and teaching practices have not been extensively researched (Speer, Smith III, & Horvath, 2010). Examining the role of teaching knowledge in evidence-based instruction and how to support its development are crucial to progress in reforming undergraduate instruction. In settings where more extensive research on teachers' knowledge and practices has occurred (e.g., K-12 grade levels), products of that work have informed development of theory and have shaped professional learning opportunities provided for teachers to support their development of teaching knowledge and practices (see, e.g., Fennema, Franke, Carpenter, & Carey, 1993; Hill, Rowan, & Ball, 2005). As in mathematics, for science, much of the theory development and use of findings in teacher professional development has occurred at K-12 grade levels. Thus, a similar situation exists in the context of undergraduate instructor knowledge and practices in science disciplines as we see in mathematics.

We contend that there will be benefits to both educational theory and practice from efforts to broaden the scope of research on knowledge for to undergraduate levels and by having such efforts be informed by the existing literature base across science, technology, engineering and mathematics (STEM) disciplines. Some researchers have been conducting research into knowledge for teaching at the undergraduate level but the amount of activity in this area has been relatively small. In an effort to encourage and support additional research in this area, an inter-disciplinary team of researchers has conducted a review and analysis of literature about studies of knowledge for teaching across STEM disciplines.

In this Theoretical Report, we present outcomes from our work to compare and contrast what is known about knowledge for teaching in STEM disciplines. Identifying similarities across disciplines can provide additional evidence for the strength of those aspects of theory and findings. Noting where contrasts exist in how knowledge for teaching is defined or theorized can also help advance efforts by bringing to light areas that have been under-examined in particular disciplines. These efforts also reveal how particular terms are defined and used in multiple disciplines which can enable researchers to more productively learn from research done on knowledge for teaching outside of their own discipline and across grade levels.

Methods Used to Identify and Analyze Relevant Literature

Identifying Articles

Our goal was to identify peer-reviewed articles (both journal and conference proceedings) about teaching knowledge of undergraduate STEM instructors. We were seeking articles based on empirical investigations using data specifically about instructor knowledge. In the course of our search, we also located theory-focused articles. Identifying such articles required a two-stage approach because our initial, broad search returned many articles where teacher knowledge was related to the topic of the study but was not the actual focus of the research investigation.

In the first stage of the search we did the following:

1. Created a preliminary set of keywords to use in our journal archive, conference proceedings archive and database searches.
2. Searched peer-reviewed discipline-specific education journals, science and STEM education journals, and specific conference proceedings (e.g., RUME, PERC).
3. Further expanded our list by searching the reference section of each paper that met our criteria. We also adopted search terms from these papers and conducted additional searches to find other relevant work by the same authors.
4. Searched databases such as Google Scholar.

Our resulting preliminary criteria for inclusion of papers were:

   a. Level has to focus on undergraduate instructors teaching STEM courses (not courses for pre-service STEM teachers in a college setting)
   b. Focus is a STEM discipline
   c. Articles match keywords (at least one term from each of three categories): pedagogical content knowledge (PCK), mathematical knowledge for teaching (MKT), pedagogical knowledge, teaching knowledge, AND discipline, AND college or undergraduate level.

A second stage was needed to exclude from consideration studies that included discussion of teacher knowledge but where the actual investigation was focused on something else. Often in these cases, the introduction/motivation or the implications for practice section of the paper included interesting discussion of how findings could relate to issues of teacher knowledge and/or teacher professional development but the analysis was not on knowledge for teaching. Our review also did not include studies where the focus was on curriculum implementation with data only from student performance. We read each article found during the first phase of our search to determine which should be included in our final set. Note: Each researcher carried out this phase of the work for articles in their own discipline and for one additional discipline. In cases where the researcher was unsure of the fit of the work with our criteria, an additional researcher read the paper and a decision was reached via discussion.

In the end, our criteria for inclusion are items a – c listed above, plus this criterion:

d. Study is an empirical investigation that utilizes data on teacher knowledge relevant for teaching undergraduate STEM content. Or article addresses theory-related issues related to our focal topic.

Analysis of Articles

As part of our search process, for each article identified, we made note of several feature of the work: the research questions addressed, the theoretical perspective and/or framework used, the categories and sub-categories of teacher knowledge addressed, data collection and analysis methods and the primary findings. In a vein similar to cross-case analysis done using various types of data (e.g., Yin, 1989), we then looked for themes within disciplines as well as themes or patterns that were apparent when we examined these features across all the STEM disciplines.

Findings from Our Review

Here we report findings related to the prevalence of this work as well as the categories and sub-categories of teacher knowledge addressed in the articles we identified in our search.

Research on Knowledge for Teaching Undergraduate STEM Content is (Relatively) Scarce

Impressions that we had from our experiences conducting research on knowledge for teaching in our individual disciplines (i.e., mathematics, biology and chemistry) were confirmed by our systematic search across STEM. Research on knowledge for teaching undergraduate STEM content is not plentiful (especially as compared to the literature base that exists for teachers of K-12 grades). We found 23 journal articles that fit our criteria (and an almost equal number of conference proceedings papers). We found nine in chemistry, two in engineering, six in physics, three in biology, and three in mathematics. The conference proceedings articles were almost equally divided between physics and mathematics with 14 found in RUME conference proceedings. By comparison, a Google Scholar search on the phrase "mathematical knowledge for teaching" without restricting results to the undergraduate level returns over 8,000 results.

Knowledge for Teaching is Described and Categorized Differently in Different Disciplines

We focus the rest of our discussion of results on what was revealed as we looked at how knowledge was described and categorized among the different disciplines. Here we draw on work done in K-12 contexts as well as that done in undergraduate contexts to more fully describe the particular ways that knowledge for teaching is addressed in the different STEM disciplines. Then we look at which categories have been the object of investigation in the different disciplines at the undergraduate level.

It is not surprising that researchers in different disciplines have developed different ways of describing and sorting teacher knowledge. What we found, however, is that the ways knowledge is categorized may also be shaping the nature of the work done in those disciplines. The leading framework in science education is PCK (Gess-Newsome, 2015; Hume, A., Cooper, R., & Borowski, 2019). The leading framework in mathematics is MKT (Ball, Hoover Thames, & Phelps, 2008). There is considerable overlap between the PCK frameworks used in science education research and MKT. MKT separates teaching knowledge into two broad areas: subject matter knowledge and PCK. Both frameworks divide PCK into constituent components. However, the MKT framework characterizes PCK as just one of two categories of content-specific knowledge used in teaching. In contrast, PCK is used in science education as an umbrella term for all content-specific knowledge used in teaching. In science, empirical research and theory development has focused (mostly) on testing, refining and seeking community consensus on sub-categories of PCK. In mathematics, similar work has occurred, with theory-development in the form of refining definitions of sub-categories. However, perhaps because of the existence of categories outside of PCK, the work has also included explorations of additional types of knowledge beyond those found in the MKT model (e.g., knowledge for discourse).

Another difference among disciplines is how general, non-content-specific knowledge for teaching is addressed. This Pedagogical Knowledge (PK) is characterized as more generalizable (i.e., relevant across many topics one teaches) than sub-categories of PCK or MKT. It has been the focus of investigations in some STEM contexts (and consider as a component of knowledge for teaching that discipline) but in other disciplines, it is not part of the knowledge considered.

Below we summarize the ways these sub-categories are described across STEM disciplines. This summary is intended to aid those who work in one discipline to read and understand work in other disciplines. In addition, the summary is preparation for the cross-discipline analysis of the prevalence of attention to these sub-categories given in a subsequent section of this report.

Components of PCK. PCK is knowledge specific to a topic and to a grade level. Research has often aimed to identify and describe its components, but differences persist in what researchers include within PCK (Hume, A., Cooper, R., & Borowski, 2019). Most studies of PCK in K-12 mathematics and science education examine at least two components: knowledge of student understanding and knowledge of instructional strategies and representations (Chan & Yung, 2015; Depaepe, Verschaffel, & Kelchtermans, 2013; Park & Oliver, 2008). Knowledge of student understanding includes awareness of students' prior knowledge of a topic, common difficulties students encounter as they learn a topic, and variation in student thinking about a topic (Ball et al., 2008; Grossman, 1990; Magnusson, Krajcik, & Borko, 1999; L. Shulman, 1986). Knowledge of student understanding is referred to by different names, including knowledge of content and students (in the MKT model), knowledge of students' understanding of science (Magnusson et al., 1999; Park & Oliver, 2008), knowledge of learners (Sickel & Friedrichsen, 2018), knowledge of student thinking (Ziadie & Andrews, 2018), knowledge of students (Chan & Yung, 2015), and knowledge of student ideas (Robertson et al., 2017). These differences in terminology may sometimes represent subtle differences in meaning, but we understand these researchers to be referring to some or all of the definition we provided above.

PCK also includes knowledge of instructional strategies and representations. This encompasses knowledge used in identifying and designing examples, analogies, visual representations, activities, and other approaches to facilitate student learning of a topic (e.g., Magnusson et al. 1999, Ball et al. 2008). It may also include sequencing of particular content. For instance, when introducing a rule for differentiation in a calculus class, an instructor draws on her knowledge of instructional strategies and representations to select which examples to use first. This component of PCK also includes knowledge of which features of examples should be avoided because they may not highlight the key ideas. As is true for knowledge of student understanding, this component of PCK goes by multiple names. Knowledge of instructional strategies and representations is known as knowledge of content and teaching in the MKT framework (Ball et al. 2008).

Knowledge of student understanding and knowledge of instructional strategies are the most commonly described and investigated components of PCK, but some researchers make further distinctions. Some frameworks also include knowledge of assessment and knowledge of curriculum (e.g., Magnusson et al. 1999, Park & Oliver 2008, Ball et al. 2008). Knowledge of curriculum, which is called knowledge of content and curriculum in MKT, includes awareness of standards related to a topic, curricular programs and resources for teaching a topic, etc. Teachers rely on this knowledge to differentiate between big ideas and less important or more peripheral ideas related to a topic (e.g., Magnusson et al. 1999, Park & Chen 2012). Knowledge of assessment includes knowledge of the dimensions of learning to assess related to a topic and methods that can be used to assess that learning (e.g., Magnusson et al. 1999).

The other half of MKT. In the MKT model, PCK represents one of the two main categories. The other category captures knowledge of mathematics that is used to do the work of teaching but is not blended with knowledge of students or teaching (Ball et al, 2008). For example, figuring out how to generalize a procedure that works in one context (e.g., whole numbers) so that it is also mathematically valid in another context (e.g., rational numbers) may be part of the work a teacher does. However, making that determination draws just on content knowledge and not on elements of PCK. Additionally, this type of mathematics task is NOT likely to present itself in the context of other types of mathematics-intensive work such as engineering, accounting, etc. Thus, the type of knowledge used to do this kind of mathematical work of teaching is referred to as specialized content knowledge. It is mathematical knowledge that is "specialized" to the work of mathematics teaching. Specialized content knowledge is also what teachers utilize to determine whether a nonstandard approach a student has taken to a problem is mathematically valid and to do other validity-checking in real-time while teaching and while examining students' work.

In addition to specialized content knowledge, the non-PCK components of the MKT model also encompass common content knowledge and horizon content knowledge. Common content knowledge represents knowledge that is used by teachers and by others, is learned via courses at K-12 and higher levels, and is not specific to teaching. Horizon content knowledge represents knowledge of particular mathematical ideas that appear in different mathematical areas or grade levels. Horizon content knowledge comes into play when, for example, an instructor is discussing a topic that students will encounter again at a later grade level. This work of teaching is referred to as "trimming" (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). An example of trimming that indicates a lack of horizon content knowledge is when an elementary school teacher says that "multiplication makes things bigger." Students will eventually encounter multiplication involving values less than one, which does not make things bigger. Therefore, this idea may make it harder for students to learn content in later years.

Pedagogical knowledge. Other domains of teaching knowledge are applicable across topics rather than specific to particular content. Such knowledge has received less attention than topic-specific knowledge, and so is not as clearly defined. Pedagogical knowledge may include knowledge of how people learn, general principles and approaches to instruction and assessment, classroom management and organization, and motivation (e.g., Auerbach & Andrews, 2018; J. König, S. Blömeke, P. Klein, U. Suhl, A. Busse, 2014; Morine-Dershimer & Kent, 1999; L. S. Shulman, 1987). Knowledge of how people learn includes knowledge of learning theories and principles about learning. For example, instructors may rely on deep understandings of constructivism to plan lessons that engage students in problems and tasks that require them to generate their own explanations and reasoning (e.g., Auerbach & Andrews 2018). Knowledge of student motivation includes awareness of strategies to motivate individual students and the class as a whole (e.g., Konig et al. 2014). Knowledge of classroom management and organization includes approaches to setting up expectations for students and structures to support those expectations, in order to maximize class time spent on content (Morine-Dershimer et al. 1999).

The Sub-Categories of Knowledge Examined Differ Among Disciplines

In addition to being described and categorized in different ways, components of knowledge for teaching have received differing amounts of attention in research among STEM disciplines. In an effort to convey this finding, we first created a representation (Figure 1) that includes the MKT diagram and other sub-categories of knowledge that are found in work on knowledge for teaching done outside of mathematics. The color-coding indicates which discipline(s) use the particular sub-categories in their descriptions of knowledge for teaching. Note: This figure represents sub-categories that are used in work at K-16 levels.

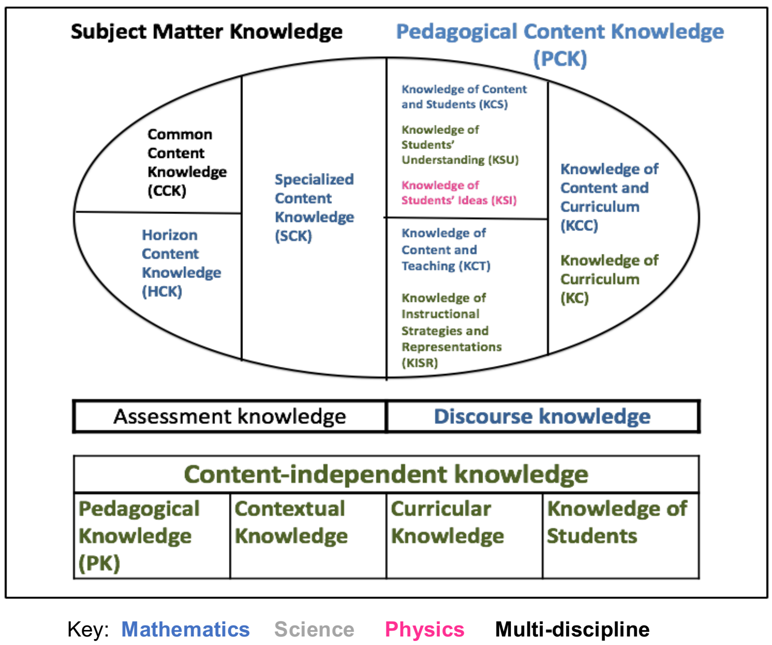

Figure 1. Sub-categories of knowledge for teaching from across STEM disciplines

Using the representation in Figure 1, the sub-categories that have been the object of investigation in the context of undergraduate teaching are highlighted (in yellow, in Figure 2) for several STEM disciplines.

Discussion and Conclusions

Based on our review, knowledge for teaching undergraduate STEM content is not an area that has yet been the object of extensive research. However, there are some signs that this may be changing (e.g., the number of conference proceedings (which can represent early-stage work) has increased in recent years) and our finding suggest various potential ways to enhance work in this area.

First, as Figures 1 and 2 illustrate, there are certain types of knowledge that have been the object of productive studies in some disciplines that have not yet been examined in others. For example, Specialized Content Knowledge has received quite a bit of attention from the mathematics community and it seems plausible that such knowledge is also relevant to teaching science and therefore studies of it might be of value. Pedagogical Knowledge has received attention in biology but has yet to be the object of much investigation in undergraduate mathematics. We have a pressing need to understand how to support instructors in developing evidence-based teaching practices and doing so has been challenging. It may be useful for those in RUME to expand investigation of knowledge for teaching to include some of the more general, discipline-independent types of knowledge that appear to be helping to leverage change in disciplines such as biology.

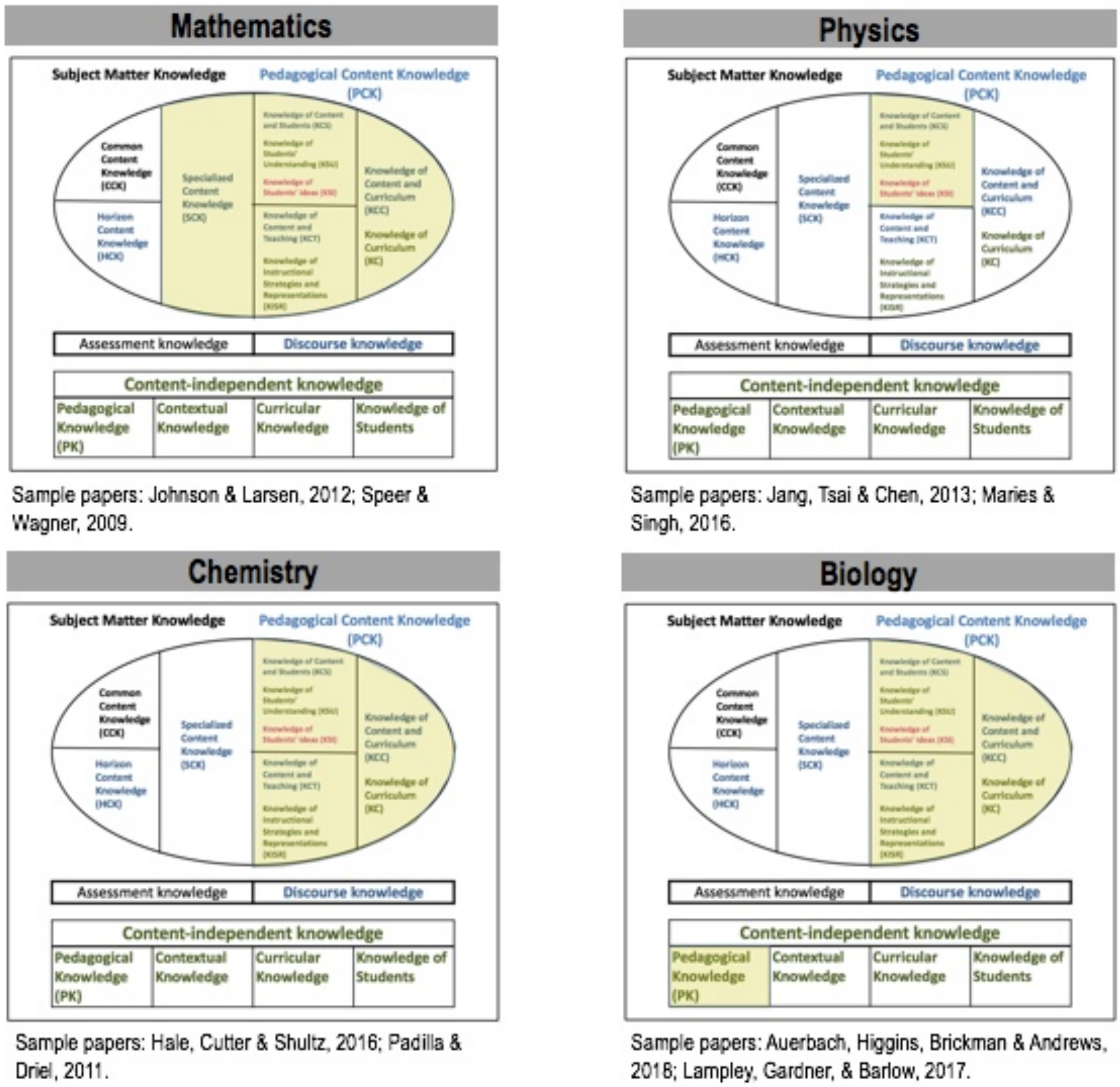

Figure 2. Sub-categories of knowledge for teaching that are the focus of research at the undergraduate level across STEM disciplines

We hope findings from this review of literature aid researchers in learning from and utilizing theory and findings from all STEM disciplines. By describing how the (often similar but slightly different) terminology is used across disciplines, we aim to help researchers locate and make use of the work of our STEM colleagues so that additional advances can occur in our efforts to understand and support the development of the types of teaching practices that provide rich learning opportunities for all students.

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Operational Meanings for the Equals Sign

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This paper examines the various characterizations of the operational (non-normative) meanings of the equals sign discussed in math education literature. It provides both an exposition and a critique of the various classifications of students' misunderstandings of the equal sign and of equations. This meta-analysis provides valuable starting points for future research that deepens this active area of exploration.

Keywords: equals sign, identity, equation, operational

The study of student conceptions of the equals sign is a longstanding theme in math education research (Byrd, McNeil, Chesney, & Matthews, 2015; Renwick, 1932). The literature tends to categorize student understanding as either *operational* or *relational*. The *relational* understanding is the normative understanding that we, as math educators, want students to construct, whereas *operational* understandings are the unproductive views that many students unfortunately hold. In Mirin (2019), I analyze the ways that the math education literature characterizes relational understandings through the lens of intellectual history regarding the equals sign. I present here the operational counterpart, providing a review of the various subcategories within the operational umbrella. This review includes both exposition and critique of various authors' categorizations of students' operational meanings.

The importance of the equals sign in mathematics cannot be understated. Four of Euclid's five common notions deal with equality, and the fifth deals with inequality (Euclid, 2013). The identity relation is important between not only numbers, but also other mathematical objects, such as sets and functions. For example, the fundamental theorem of calculus asserts identity of functions, and the axiom of extensionality provides an identity criterion for sets. The prevalence of the equals sign in all levels of mathematics underscores its importance for college students as well. Moreover, some authors have provided evidence to suggest that a robust understanding of the equals sign is important for learning early algebra (Byrd et al., 2015; Knuth, Alibali, Hattikudur, McNeil, & Stephens, 2008). Since many college students take early algebra (college algebra), this discussion is relevant for undergraduate mathematics education. However, the math education literature on the equals sign focuses on K-12 students, so I address it. I discuss some of that literature as a way of describing ways of thinking about the equals sign. The extent to which these ways of thinking extend to college students is an open question to be investigated in the future.

Before discussing details of the subdivisions of the operational umbrella, I provide a more general description of student (mis)understanding of the equals sign and the data that accompany it. This will provide a basis for the more in-depth characterizations in subsequent sections.

Many students struggle with accepting equations of the form (i) "5=2+3", (ii) "5=5", and (iii) "3+2=4+1", preferring equations like (i') "2+3=5", (ii') "5+0=5", and (iii') "2+3=5+1=6", respectively (Behr, Erlwanger, & Nichols, 1980; Byrd et al., 2015; Denmark, Barco, & Voran, 1976; McNeil et al., 2006; Oksuz, 2007; Sáenz-Ludlow & Walgamuth, 1998). The equations (i), (ii), and (iii) can be described as "rule violations" and characterize operational understandings of the equals sign (Oksuz, 2007). The idea behind this terminology is that students with operational understandings are accustomed to seeing the equals sign in contexts like "2+3=5", where "2+3" is an arithmetic problem to which "5" is the answer. Thus, the equations violate the rule that to the left of the equals sign is an arithmetic problem on the right of which is a single numeral as an answer. A common explanation posited for such understandings is that students view the equals sign as a command to perform an operation. These understandings of the equals sign that involve arithmetic, problems, answers, and calculations are characterized as "operational".

Table 1. Equations that students frequently reject accompanied by preferred alternatives.

	Rule Violation	Preferred Equation(s)	
(i)	5=2+3	2+3=5	
(ii)	2+3=4+1	2+3=5+1=6	2+3=5 / 4+1=5
(iii)	5=5	5+0=5	

Having laid out the fundamentals of the topic of discussion, the following discusses several different characterizations of the operational approach to the equals sign. This includes the characterization of the equals sign as expressing equality of magnitude, as a relation between problem and answer, as punctuation, and as an instruction. These classifications of understanding often fail to account for context, despite the fact that student understanding of the equals sign seems to be context-dependent (McNeil & Alibali, 2005).

The Meaning of the Equals Sign

Frege and Wittgenstein argue that one should never ask for the meaning of a word in isolation (Frege, 1980; Wittgenstein, 2014). Instead, one should consider the meaning of a word in the context of a sentence. The equals sign is essentially a word, and an equation in which it appears is a sentence. This means that what is written on either side of the equals sign is inextricably tied to the meaning of the equals sign. Further, the meaning of a sentence is tied to the usage of the utterer of the sentence. So, the meaning of an equation is tied to its context of usage. Hence, we must consider the context and usage of the equals sign when examining student understanding.

The modern mathematical consensus is that the equals sign represents the identity relation between objects; that is, A=B if and only if A and B are identical (Hodges, 1997; Klement, 2019). Equations involve both sameness and difference. "A=B" asserts that A and B are the same (identical objects), yet both "A" and "B" are different names. Thus, we use informative equations to assert that two names or representations actually refer to the same thing. The sameness (in referent) is what makes the equation true, and the difference (in name or representation) is what makes it informative. For example, "2+3=4+1" is true because "2+3" and "4+1" both name the *same* number, and it is informative because "2+3" and "4+1" are *different* representations of that number (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

Following other authors who have used the historical development of a concept to contextualizes student learning of it (e.g., Sfard (1992), Thompson & Carlson (2017), and Harel, Gold, & Simons (2009)), I do the same thing for the equals sign. Attempts to characterize the meaning of the equals sign and the identity relation is a longstanding theme in intellectual history (Mirin, 2019). In the late 1800's, there was a widespread debate about whether in mathematics,

"=" represents true identity. Some mathematicians and philosophers "posited some weaker form of 'equality' such that the numbers 4×2 and 11-3 would be said to be equal in number or equal in magnitude without thereby constituting one and the same thing" (Klement, 2019). We can see that the idea of = being a sameness relation weaker than true identity might also exist in the minds of students; Behr, Erlwanger, & Nichols (1976) claim that one particular student, D (a high IQ 6th grader who had no issues with the three rule violations), views "equals" as something different from "the same". When presented with the equation "6/2=3", D claimed that 3 is a whole number, but 6/2 is not. When asked to clarify how 6/2 can be equal to 3 yet have different properties, D clarified "It just says that they are equal to each other, they have the same value, but that doesn't mean that they are the same number" (p.13). To D, 6/2 is a fraction, not a whole number, whereas 3 is a whole number. The authors explain that, in this view, "=" means "has the same value" rather than "is the same number". Observe that this is the same view that existed amongst intellectuals in the 1800s. This interpretation of the anecdote suggests that situating a student's understanding in historical context might help us understand the student's understanding of equality. However, we might be too quick to claim that D views equality to mean something other than true identity. The authors do not seem to consider that the issue with D is not his conception of equality but that of "number". Perhaps D views a number as a numeral (its name), which would account for him not viewing 6/2 and 3 to be the same number. The idea that the equals sign represents a relation between names was once endorsed by Frege, so again, this idea exists in intellectual history (Mirin, 2019). Further, use-vs-mention ambiguity is plausible when we see how rampant it is even within the math education literature (Mirin, 2019). This interpretation of the anecdote suggests that we ought to consider not just how students understand the equals sign in isolation, but instead consider how students understand what is written on either side of it.

The Equals Sign as a Relation, Just not the Right One or Between the Right Things

Several authors seem to characterize the operational view as not viewing the equals sign as a relation (Byrd et al., 2015; Denmark et al., 1976; Kieran, 1981). I argue against this as a dichotomy. When we look carefully at how students understand what is written on either side of the equals sign, rather than looking at how they understand the equals sign devoid of context, we can see that a student with an operational understanding might indeed view the equals sign as a relation. Denmark et al. (1976), in their 4-level characterization of various operational conceptions, describe two of these viewpoints, Levels 2 and 3. The Level 3 understanding is almost the same as the Level 2 understanding, except it treats the equals sign as a symmetric relation, so that students accept Rule Violation (i). Here we focus on Level 2. Denmark et al. (1976) characterize a Level 2 understanding as one in which the student has the meaning that the equals sign is a "one-directional operator separating a problem from its answer" (p. 31). The authors use this classification to describe a student who would view "3+4=7" to mean that "7" is the answer to the problem "3+4". That expresses a relation. Let us call the relation R. So, aRb ("a=b") if and only if "b" is the answer to the problem "a". The idea that students with operational views think of the equals sign as expressing some sort of relation is corroborated in Behr et al. (1980), who found that most children view sentences "2+5=5" and "2+3=7" as either true or false. The fact that these children ascribed a truth value to such statements suggests that they view the equal sign as expressing a relation, at least in these contexts. From the Level 2 and 3 perspectives, it is false that "5" is the answer to the problem "2+5".

Situations in which students do not ascribe truth-values to rule-violating equations also confirm the idea that they might view the equals sign as expressing a *relation* between problem and answer. Students' responses to rule violations can be seen as students attempting to make sense of the equals sign under their understanding of equality as a non-symmetric relation between problem and answer. This becomes most apparent when we consider how students with these perspectives view not just the equals sign in isolation, but in the context of what (from the students' perspective) is written on either side of it.

Students do not always view each side of an equation (e.g. "2+4") as representing a unified number but instead a process or a problem (Behr et al., 1976; Renwick, 1932; Zazkis & Liljedahl, 2004). It is not as if students with an operational conception necessarily view equality as some *other* relation (other than identity) between numbers. Instead, students with a Level 2 conception view the equals sign as expressing a relation between different sorts of objects (problems and answers). To a student with this Level 2 conception of the equals sign, the equations might not make sense: "3=3" reads "3 is the answer to 3", "5=2+3" reads "2+3 is the answer to 5" and "2+3=4+1" reads "4+1 is the answer to 2+3." It is not that under this conception these assertions are necessarily false, but they are incoherent in a similar way to which we might find the following infamous assertions incoherent:

Colorless green ideas sleep furiously (Chomsky, 2002)
Quadruplicity drinks procrastination (Russell, 2007)

Noam Chomsky used "colorless green ideas sleep furiously" as an example of a sentence that is syntactically correct but semantically incoherent, and Bertrand Russell used "quadruplicity drinks procrastination" as an example of a sentence that is meaningless and hence lacking in truth value. In other words, these sentences sound meaningless because they each express a relation between things that are not potentially capable of having that relation (category mistakes). Both of these sentences are *wrong*, but not necessarily false. The words individually are not meaningless, but the context surrounding them makes the sentences incoherent. *Drinks* is a relation between beings and liquids, and a relation that does not involve quadruplicity or procrastination. Similarly, to a student with a Level 2 conception, = is a relation between problems and answers, and something like "2+3" is not an answer to such students (it's a problem). Denmark et al. (1976) describe a student who views a statement like "2+3=4+1"(Rule Violation (ii), Table 1) as statements of *two* problems, changing it to "2+3=5 / 4+1=5. They, as well as other researchers, also describe children who explain that it would be "wrong" to write "2+3=4+1" with the rationale that the "answer" should be written after (to the right of) the problem (Renwick, 1932; Sáenz-Ludlow & Walgamuth, 1998). Similarly, students tended to call equations like "3=5" and "3=3" (Rule Violation (iii)) "wrong", rather than "false", and modified them to equations like "3+5=8" and "3+0=3" (Behr et al., 1976). Some students, when asked for the meaning of equations of the form "a=a", explained that they could *mean* something like "a+0=a" (Behr et al., 1976). This suggests that these students viewed equations of the form "a=b" not as *false,* but instead representing some sort of category mistake (since "a" isn't a math problem).

In multiple studies, students demonstrate a propensity to correct equations of the form "5=2+3" to "2+3=5" (e.g., Behr et al., 1980; Denmark et al., 1976). For example, in Behr et al.'s (1976) study, one student, when presented with "$\square$=1+2" wrote "3" in the box but read the sentence aloud as "Two plus one equals 3". Another student rewrote "$\square$=2+3" as "2+3 = $\square$" before putting a "5" in the box, explaining that the original prompt is "backwards" and asking the interviewer "do you read backwards?". It is plausible that these students viewed the equals sign as expressing a non-symmetric relation between problem and answer (Level 2), and that, in an effort to make sense of rule violations, corrected the equations to be of a form that made sense to them and their conception of the relation expressed by the equals sign. Such students might attempt to correct such sentences in the same way that I would want to correct "Harvard studies at Sylvia" to "Sylvia studies at Harvard". *Studies at* is a nonsymmetric relation between individuals and educational institutions. Hence, when presented with such a sentence, we would correct it so that it expressed a relation between individuals and educational institutions. Similarly, for a student with a Level 2 view, the equals sign expresses a nonsymmetric relation between problem and answer, and they would therefore want to correct "5= 2+3" to "2+3=5" as a way of correcting or making sense of what they see as a category mistake.

The existence of the Level 2 and 3 conceptions, as described in Denmark et al. (1976), suggests that some students with nonnormative understandings of the equals sign still view the equals sign as expressing a particular relation. A closer look at context – specifically, how students understand what is written on either side of the equals sign – elucidates how and why students might respond to the rule violations the way that they do.

Punctuation

Several authors draw the analogy between operational views of the equal sign and the way that one might view a punctuation mark (Denmark et. al, 1976; Renwick, 1932; Saenz-Ludlow and Walgamuth, 1998). These characterizations succeed at explaining students rejections of Rule Violation (ii) in favor of the preferred equations. However, they fail to thoroughly consider context and usage of the equals sign.

Jones, Inglis, & Gilmore (2011) characterize a *punctuative (operational) conception* as being when a student views the equals sign as "a place-indicator for writing down a number"(p.81). They describe this punctuative conception as a use of the equals sign in a way that is "analogous to the full stop in written language" (p.81). A student with this conception might think of the equals sign as a way to mark where the problem and the answer are, similarly to how we might think of the full stop as a way to mark where one sentence begins and another ends. They explain that when a student writes "2+3=5+1=6" and rejects equations like "2+3=4+1" (Rule Violation ii), he is using the equals sign as a punctuation mark. This again begs the question, how would such a student be using the equals sign in a singular equation, such as "2+3=5", when there is only one step to the computation and hence nothing to keep track of? Hence, we can see that this punctuative classification of student understanding of the equals sign fails to account for even the most basic contexts (single equals sign).

Jones et al. (2011) seem to be motivated by the descriptions provided by Renwick (1932). Renwick (1932) describes this usage of the equals sign in running equations as a symbol of distinction: "to the child mind, would appear to be to separate rather than bridge" (p.173). Renwick does not explain *what* the equals sign separates. The first equals sign in "1+2=3+4=7" could be seen to separate the problem "1+2" from the answer "3", or it could be seen as a way of separating the equation "1+2=3" from the equation "3+4=7". The former interpretation seems more likely, but both interpretations are problematic. The latter does not account for how a student would understand equations with one equal sign, such as "1+2=3". Consistent with the former interpretation, Renwick (1932) is attempting to contrast the student conception (to separate) with a normative conception to bridge. Viewed this way, since the equals sign *bridges* "1+2" and "3" (in the sense that 1+2 and 3 are the same number), students would view the equals sign as *separating* "1+2" (the problem) from "3" (the answer). However, this contrast is highly questionable: one could argue, from the relational/normative perspective, in the same vein that the equals sign *separates* "1+2" from "3". While, normatively, equality is a sameness relation (and we use the equals sign to express that relation), the equals *sign* as a linguistic object really does *separate* two names of numbers. The very nature of the equals sign being a (binary) relation symbol means that part of its role is to distinguish the object(s) that hold that particular relation. Recall the earlier discussion: an equation involves both sameness (of object) and difference (of representation). The equals sign really does visually separate "1+2" from "3". Not only do "4+1" and "5" look different, but they bring different things to mind; "4+1" brings addition to mind, and "5" doesn't. Thus, this characterization of student use of the equals sign is limited in scope and fails to carefully consider the actual context of *usage* of the equals sign as separating *different* representations.

Do Something! The Equals Sign as an Instruction

Jones et al. (2011) introduce the punctuative view as a way of contrasting it with what they call the *imperative* view. A student with this conception of the equals sign might characterize the equals sign to mean *calculate the answer to the problem*. In other words, an imperative view of the equals sign involves viewing the equals sign as a command to *do something*. Jones et al. (2011) are responding to a conflation of imperative versus non-imperative views in the literature. The literature seems to allude to both "command to calculate" and "relation between problem and answer" conceptions without being explicit about this distinction, just as I did in the introduction to this paper.

McNeil et al. (2006) characterize an operational conception both as a sign announcing a result of an arithmetic operation and as a command to add the numbers without discussing any differences between these conceptions. Knuth et al. (2008) make a similar conflation; they refer to an operational conception both as interpreting the equals sign as an announcement of result and as a *do something* symbol. These authors were not distinguishing between *types* of operational conceptions, nor did they explicitly refer to the conception of announcement of a result as distinct from a do something symbol conception. Instead, the *do something* symbol conception and the *announcing a result* conception are treated synonymously as operational conceptions. This provides a label for the *command to calculate* operational understanding alluded to in the literature. I adopt the "imperative" terminology to refer to the view of the equals sign as a command to calculate. I am not specifically referring to the way that Jones et al. characterize it, but I am adopting their useful terminology for classifying this operational viewpoint described in the literature by using it as a short-hand for "command-to-calculate".

Treating the imperative view of "=" as non-normative seems to fail to consider the roll of context. As discussed earlier, the meaning of a symbol cannot be divorced from its context. This lack of consideration can lead to a particularly rigid view of student understanding and meaning in general. In some contexts, it seems that "=" really *does* mean "calculate" – for example, when using certain kinds of common calculators (Figure 1). When students are given a worksheet with "2+3=" followed by a blank space, they are being told to calculate in the same way that pressing "=" on such a calculator prompts the calculator to compute. In these situations, the equals sign really does *instruct* something (the student, or a calculator) to calculate something. So, if a student explains that '=' means calculate, that student is correct in these contexts, that is, in reference to "1+2=". Arguably, this is a correct understanding of the meaning of "=" in these contexts.

Figure 1. A TI-108 calculator

Authors who characterize students as viewing "=" as a command to calculate seem to infer that, because students interpret "1+2=" to mean "calculate 1+2," the students view the equals sign itself as meaning "calculate". Why should we conclude that the equals sign, specifically, is what the student interprets as a command to calculate? What about the rest of the expression? After all, "+" actually denotes an operation (notice the "operation" in "operational"). It seems possible that students might interpret "1+2" to mean "calculate 1+2", despite there being no equals sign. I hypothesize that, if we were to give one group of students a worksheet with inscriptions like "1+2" followed by a blank, and another group of students a worksheet with inscriptions like "1+2=" followed by a blank, the two groups would respond similarly. If this is the case, can we really say that the meaning of command to calculate resides in the equals sign itself?

When we look carefully at students' equal sign *usage*, we can see a relationship between an imperative (e.g. Level 2) and a non-imperative operational conception. Picture a student who thinks of things like "1+2" as problems and things like "3" as answers to such problems. Such a student might see "1+2=" followed by a blank in the context of a math worksheet and (rightfully) assume that he should write the answer "3" to the right of the equals sign. The student might view "1+2=3" as a record of an already completed calculation. Suppose he interprets "a=b" to mean that b is the answer to a. Then "1+2=" followed by a blank is an incomplete sentence, and he (rightfully) interprets that he is to fill in the blank by completing the sentence. Arguably, this is a normative imperative viewpoint. When we ask students to fill in the blank in "2+3= blank+1", we are asking them to complete a sentence so that the sentence becomes true. We expect them to use the context to realize that they are to do something (fill in the blank); we don't expect them to read "2+3=blank+1" as an already true sentence.

This discussion begs the question – which comes first, thinking of the equals sign as a command to calculate, or thinking of the equals sign as expressing some sort of relation (so that equations are sentences capable of being true or false)? The previous paragraph explains how a student might use context to switch between imperative and non-imperative interpretations; if a student thinks of equations as sentences, then a blank prompts him to complete the sentence. It is still an open question whether students, developmentally/chronologically, understand the equals sign as a word in a sentence or as an imperative. Perhaps this is a question more generally about language acquisition – do children first learn the command "close the door," or do they learn the proposition "the door is closed"?

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Mathematical Limitations as Opportunities for Creativity: An Anti-deficit Perspective

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This theoretical report connects an anti-deficit perspective on students' mathematical sense making with students' mathematical creativity. The report specifically examines the role of mathematical limitations during the different stages of the creative process. We discuss mathematical creativity in the specific context of constructing everyday examples to explain basis in linear algebra. We consider this construction as a creative activity. We argue that mathematical limitations in students' initial examples are not only reasonable but can also provide opportunities for more mathematical creativity. While identifying the mathematical limitations led some students to dismiss their examples, thereby ending the creative process, for others it led to further engagement in the creative process through deeper exploration of the everyday example and the mathematics. This report shows that not only does flexibility with full mathematical precision with the examples allow for an anti-deficit interpretation of the students and their sensemaking of the mathematics, such flexibility also sustains the creative process.

Keywords: creativity, anti-deficit perspective, basis, linear algebra

Mathematical learning, particularly at the undergraduate level, is still largely a process of acquiring, engaging, or rediscovering a set of predetermined set of ideas. There is an ongoing tension between treating mathematical learning as the acquisition of knowledge and meaningfully participation in the creative process of mathematics (cf. Sfard, 1998). Discussions about students' creativity at the undergraduate level are often situated in the context of proving theorems and solving traditional mathematical tasks (Karakok, Savic, Tang, & El Turkey, 2015), and in this way constraining within these bounds. We argue that these distinctions and contexts also reflect a tension between two values of the mathematical community: mathematical precision and mathematical creativity. This is the context of the discussion of this theoretical report.

Theoretical Frameworks

This report is also motivated by the discussion that was started in Adiredja (2019) about deficit perspectives on students' mathematical thinking in undergraduate mathematics. The author defines deficit perspective as "the perspective that focuses on inherent problems in students' knowledge and attributes those problems to students' shortcomings" (Adiredja, 2019, p. 413). The author also argued that important values and practices in mathematics may inadvertently support deficit interpretation of students' mathematical works. In particular the author identified the overreliance on the use of formal knowledge and coherent formal mathematical language as indicator of students' understanding, and the expectation for full consistency in that understanding. Adiredja (2019) highlighted how the deficit interpretation is often aided by existing deficit societal and research narratives when interpreting the mathematical work of students from marginalized groups.

In this paper we examine the connection between the tenets supporting deficit perspectives and mathematical creativity. In particular we focus our theoretical report on the role of mathematical limitations, which can be interpreted as mathematical "deficiencies" of the students from a deficit perspective, in the creative process. We argue that not only are mathematical limitations during a creative process reasonable, they can in fact be interpreted from an anti-deficit perspective as opportunities for creativity.

In this report, we discuss mathematical creativity in the context of constructing a conceptual metaphor (e.g., Lakoff & Johnson, 1980; Lakoff & Núñez, 2000) for the concept of basis. Work on conceptual metaphor focuses on the use of a metaphor to unpack the structure of a concept and ways that entities in the concept interact. In the undergraduate mathematics education literature conceptual metaphor has been used to examine student understanding of a number of concepts (e.g., Oehrtman, 2009; Zandieh & Knapp, 2006; Zandieh, Ellis, Rasmussen, 2017). In the data illustration section, we focus on students' metaphors, as expressed in everyday examples they constructed to explain basis. Our determination of creativity focuses on the originality and the divergent thinking of students' responses and examples (Torrance, 1966; Kwon, Park, & Park, 2006), and the new insights these examples reveal about the mathematics and students' understandings (Liljedahl & Sriraman, 2006). We select illustrative examples that are "imaginative, clever, elegant, or surprising, beyond analytical thinking" (Milgram & Hong, 2009, p. 152), which further showcased students' creativity.

Data Illustration

We draw our illustration from the data we collected in previous studies where we interviewed students about their understanding of basis in linear algebra (Adiredja & Zandieh, in press; Zandieh, Adiredja, & Knapp, 2019). The context of the two studies are quite different. The first study specifically recruited a group of undergraduate women of color in the United States. The second study participants happened to be a group of advanced (senior undergraduate or graduate) male students in Germany. Our previous analyses of the data focused on analyzing the different everyday examples students generated, and the kind of sensemaking about basis the examples revealed.

For purposes of this paper we reanalyzed the data and focused on instances when students came across a previously unaccounted mathematical idea while they were constructing their example. We found that students responded to a mathematical issue in their examples in one of three ways. Some students closed down the creative process by dismissing their example. A group of students continued the creative process by revising their examples to account for the mathematical idea. Other students further engaged in the creative process by generating a new example to account for the mathematical idea. We discuss these responses as they relate to the creative acts that students were engaging in.

Eliana Dismissing Her Storage Room Example

Eliana was one of the American students who ended up dismissing her example after finding a mathematical issue with it. The first example Eliana constructed during the interview was using her arms and legs as a basis to cover the room. She then switched to another example where she talked about the dimensions of a storage room as its basis. She explained,

> You can fill it based on how much you have. How much room you have. Sort of… I don't think that really applies because a basis is the least amount in order to cover an entire space. That doesn't really work because the space would be limited. And with a basis it's just a representation, like a representation of an unlimited area.

Eliana recognized one mathematical limitation of the room example: the room represents a finite dimensional space, which a vector space is not. Many student examples often involved a similar constraint. For example, Simon one of the German students, offered an example that initially used a finite dimensional space. He came up with the example of building a house (vector space) using wood, stones, and glass (basis vectors).

Simon highlighted the way that his example captured the notion of linear independence, but he noted “maybe can’t get the vector space of the house. It’s maybe a bit difficult to see that.” Consistent with the finiteness issue in Eliana’s storage room example, Simon shifted the vector space. He said, “I guess it would be everything you can build up from stone, glass, and wood. It could be a house; it could be something else.” Simon revised his example to address the vector space issue.

Mathematically, Eliana initially focused on accounting for the *minimal* characteristic of the basis vectors in constructing her example. Simon initially focused on the linear independence of his example. It is common (and reasonable) for students to initially attend to a particular idea about basis in constructing their example. Students would typically work through their example and attempted to account for other ideas (e.g., infinite vector space).

Presenting these two cases together also allowed us to see that recognizing a mathematical limitation can lead to the end of the creative process. As one develops expertise in a particular topic, one becomes more aware of the different aspects and nuances of the topic. Such critical awareness when applied at the beginning of a creative process might close up opportunities for more creativity. Moreover, the two cases together also show that not only are students able to identify such mathematical issue in the example, such mathematical issue can actually be addressed without dismissing the initial creative work, using resources from students.

Lilianne Revising Her Religion Example

As Simon demonstrated, students can further engage in the creative process by revising their initial example to account for additional mathematical ideas. Lilianne was one of the American students who similarly engaged in this process. During the interview, we asked students the extent that their example fit the formal definition of a basis. This was Lilianne’s analysis of her example that uses religious doctrines (basis vectors) to make moral decisions (vector space).

> It doesn’t convey as much for, for linear independence because you, you can have a particular situation that happens and you can apply different doctrines to that one point. /…/ I feel bad for what I did to my sister and I want to say I’m sorry. The reason why I do that and the reason why I decided that, um, could be because Christ said it when he, when he was talking to, something, love your neighbor. Okay, that’s why. But you could also, there’s another commandment where it’s like “love God and God says love your neighbor” so that's why I decided that. Or there’s like, in the scriptures I read this story that he was forgiving, and I was like, want to be like that. So that’s why I decided to do this. So, you have different ways to get to that. So it doesn't as much convey that independence.

This excerpt suggests that Lilianne might have conceptualized linear independence as the uniqueness of the solution to $c_1v_1+c_2v_2+\ldots+c_nv_n=b$, where v_i’s were doctrines and b was feeling bad and apologizing to her sister. In other words, there were different basis vectors that she could have chosen to arrive at the same decision. We asked her if there was a way to modify the example so that it captured the independence idea. She modified her basis vectors.

> So then my first thought would be for the basis it would be “God loves us,” “God speaks to prophets” and “God speaks to us.” Those are the three that and from those truths there’re a lot of things that come out of that. And a lot of things we can also use. Um if you have the Ten Commandments, that comes from those three. /…/ So that would be

dependence off of those. So then if a thing happens and we want to apply, we use those main points. Indirectly we're using those main points cause we're like, we're all bases but it would be based off of those main ones.

Lilianne revised her example by modifying the basis vectors in her example. In this explanation she recognized the potential for more than one set of basis vectors ("A lot of things we can also use"). She also explained that other potential basis vectors (e.g., Ten Commandments) can be expressed as a linear combination of her new basis vectors.

This case shows that recognizing a mathematical issue (e.g., not addressing linear independence) led Lilianne to further develop her initial creative example, thereby continuing the creative process. At the same time, Lilianne's case also shows that students' creativity can serve as a space for mathematical sensemaking. We can interpret her examples as providing her opportunities to connect different aspects of her understanding of basis and linear independence. The next case also illustrates this point as Jocelyn, one of the American women, made sense of a theorem about basis that she had not understood previously.

Jocelyn Generating a New Set of Examples for Spanning Set Theorem

In Zandieh et al. (2019) we included cases with the students in Germany when the construction of a new example was specifically motivated by the student's dissatisfaction with the ability of their current example to address some aspect of basis. Here we add to those examples a case where a student constructed a new example to account for a theorem about basis that she had not previously understood. Nadia offered two contrasting examples in response to a theorem that described two ways to construct a basis. Mathematically, it sounds like an application of the Spanning Set Theorem. She explained the first way to construct a basis and how it fit with her understanding of basis:

If you have some set and it's not a basis, then you take out dependent ones. It has the same span. So the way I think of basis is, I'm just going to collect all of these vectors so that I know for sure I span the entire space. And then I'm going to do a bunch of tests and checks so that I can throw out all of the ones that are combinations of the rest.

The first way was to remove any vectors that would make the spanning set linearly dependent. The second way was to start with a linearly independent set and keep adding vectors until they span the desired vector space. She noted that up until this moment in the interview, she had not understood the theorem.

You can start with something like small… or like start with one or two vectors. And you know they're linearly independent. So then every time you try to add something in, so then you can, you know, like span more. /.../ So I guess you would have to define, like, what you want to span. The goal is to span that. So then you pick one thing in there. Then okay, so the span's this, but I still have the rest of [the space]. So then I can put in this [other] vector check if they're linearly independent and see if I span that thing I was trying to span. If I don't, then I pick another vector. So then I would stop whenever I like I've covered the whole space I was trying to span.

We asked her if there is anything in the real world that operates like these two ways of constructing basis:

That's an interesting question. I guess I immediately thought [of] art. I don't know if that's a good example. So you know how you can start with a giant block of something and carve at it, chip out of it whatever through things I don't know. Like you take pieces away from it to reveal product. And in other types you, like in, I don't know. What's a good example? I'm making a collage or something and I see putting things together.

Jocelyn did not specify the basis vectors and the vector space in her examples. However, she was able to capture both processes of basis construction in a set of new examples. What was interesting about this example is that we asked her if she could also apply these two ways of constructing basis with her fashion example she provided earlier in the interview, where she used tops, bottoms, and shoes as basis for a wardrobe. This was her revision of her example:

Oooh. K. So I guess I could if I start off—The easiest way would be if I start off with a giant wardrobe and I have like five heels and three pairs of running shoes and like, twenty skirts, then I could reduce that and, like, one pair of heels and like, one pair of tennis shoes and one skirt. I still have all these other things but the basis would be the, like one from each one. I'm not sure about how you would, like, build. I mean, I guess I could start off with like here's a pair of shoes and then, ok, so then I need a bottom so then I could have like a skirt, pants. And then I could add like a shirt...I guess it kind of works. If I'm shopping, and I don't have a jean jacket. I'll buy a jean jacket. So kinda.

In addition to constructing new examples with the art sculpture and the collage, Jocelyn also revised her fashion example and developed the two ways of constructing basis in the context of that example. This results in a much richer everyday example that account for even more aspects of basis. In other words, an unaccounted mathematical issue (the theorem) ended up generating new examples (new creative product) and enriching a previous example (continuing of a creative process).

Discussion

The three different cases we presented illustrate the different ways that mathematical limitations in students' examples serve as opportunities for creativity. An anti-deficit approach to these examples is not to deny the existence of the mathematical limitations. It is to focus on changing the way we interpret them, not as deficiencies attributed to the students, but instead as potential opportunities for further creativity. In all three cases students attended to the limitations, whether in not capturing notions of linear independence or infinite vector space, or in thinking about other mathematical ideas like the Spanning Set Theorem. In the first case we saw how attending to the mathematical imprecision or a mathematical inconsistency in the example led to an end of the creative process. In contrast, the other two cases showcased how a mathematical limitation led to a creative product of a new example, or a further engagement with the creative process through revision of an existing example. All three cases also emphasize that enriching the mathematics behind a creative product is as productive as the creation of a new product.

We would be remiss if we did not highlight the significance of these creative acts for students' mathematical understanding. The act of constructing everyday examples engages students with mathematical creativity outside of the context of traditional problem solving and proving. Moreover, it provides students an opportunity for abstracting linear algebra concepts beyond the real vector spaces. Students examples already use general vector spaces such that a shift to consider the vector spaces of polynomials or matrices would not be too far. Moreover, without access to calculations, the activity also gives students opportunities to discuss the notions of span and linear independence in a conceptual way. In other words, this activity of constructing everyday examples provide opportunities for students to engage with both mathematical creativity and mathematical learning. However, this theoretical report highlights the importance of separating mathematical analysis of the example from analysis of students' understanding and creativity. Deficit perspectives would focus on mathematical limitations in the examples as deficiencies in understanding. An anti-deficit perspective would instead highlight their role in showcasing students' understanding and creativity.

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A Tour of Cognitive Transformations of Semiotic Representations in Advanced Mathematical Thinking

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The purpose of this paper is to expand on Duval's (2017) notion of Cognitive Transformations of Semiotic Representations (CTSRs) and search for CTSRs that go beyond treatments and conversions at a school and entry undergraduate level. Examples of CTSRs that illustrate a progression of mathematical thinking through calculus and real-analysis to topology; and examples from representation theory of finite groups that blends abstract algebra and linear algebra will be explored.

Keywords: Advanced mathematical thinking, Conversions, Semiotic representations, Registers

Introduction

To represent a mathematical object we use signs: symbols, words, geometric figures, diagrams, gestures, etc. Representations for a mathematical object, regardless of whether they are internally held or externally expressed, for which a learner actively forms and binds with semantic significations are called "semiotic representations" (Duval, 2017, p. 25; Arzarello, 2006). For a mathematics learner, switching between a variety of semiotic representations for many mathematical objects is a crucial skill; a mental move from one semiotic representation to another is called a cognitive transformation of semiotic representations (CTSRs) (Duval, 2000a, 2006). For a mathematics educator, "Semiotic representations are not only useful for working with or about the objects. If we want to describe, from a cognitive point of view, the mathematical way of working in mathematics, we must focus on the transformations of semiotic representations and analyze the different kinds of transformations (Duval, 2017, p. 31)." Duval's (2006) paper distinguished between two types of CTSRs: treatments and conversions with a focus on school and entry undergraduate level mathematics. Motivated by Duval's work and questions it inspired, the theme of this paper is to expand upon his perspectives by taking a tour of CTSRs at an advanced undergraduate and graduate level; we also search for a theoretical language to describe and analyze these CTSRs.

As an extension of Duval's (2006) paper we explore the following questions: (a) What is a type of CTSR that goes beyond treatments and conversion? (b) What are a few examples of CTSRs that learners are expected to perform within advanced undergraduate and graduate level coursework? (c) What are examples of CTSRs that occur within or depend on other CTSRs? We begin this paper with an overview of Duval's (2000a, 2006, and 2017) work on CTSRs for mathematical objects and two types of CTSRs: treatments and conversions. After a review of treatments, conversions, and the learning milestone of "conversion fluency" the above questions will be explored (Duval, 2017, p. 110-111). The goal of this paper is to obtain a model that summarizes a progression of advanced mathematical thinking (-thinking) [Dreyfus et al., 1990; Harel & Sowder, 2005] restricted to CTSRs on a mathematical object.

Duval's Perspective of Semiotic Representations and CTSRs

When working with a mathematical object it is important to consider the "content" of the semiotic representation (SR), the "system" and "register" the SR is produced in, and the "represented object" (Duval, 2017, p. 27). The content of a SR refers to the qualities of an object that the SR makes conceivable. For example, let the object be a normal linear operator on a vector space V, where V is a finite dimensional inner product space over the complex field. Since the operator is normal, the operator can be expressed as a diagonal matrix with respect to some basis. A SR for the operator that is a diagonal matrix makes the eigenvalues, as content, clear-cut to see. A SR that is a matrix and is not diagonal, lower or upper-triangular hides the eigenvalues from view. In general, different semiotic representations may dazzle certain or suppress certain qualities of a mathematical object. Cognitive systems used to generate SRs are: the automatic (non-conscious) neural system and the intentional system (consciously controlled). In the automatic system "The representation IS THE OUTCOME of a direct access to object" as in the case of intuition. The intentional system brings, "…into play the semiotic system (mentally or materially)" to produce a semiotic representation that "DENOTES the represented object" (Duval, 2000a, p. 66). Within the intentional system are sub-systems called registers; types of SRs have been theoretically classified according to the register(s) used to produce them (Duval, 2000a, Duval, 2017). Figure 1 shows four main registers: discursive, non-discursive, multifunctional and monofunctional. The discursive register is used to represent mathematical objects using spoken words or written symbols of a language. The non-discursive register is used to represent mathematical objects through a non-spoken language such as visual images, geometric shapes, graphs, and diagrams. A multifunctional register is used to produce SRs consisting of vernacular language, writings, and drawings that are free from the strict confines of formal mathematical language. A SR produced in this register by a learner is predominantly a reflection of that learner's intimate and unique view of the mathematical object. In contrast to the multifunction register, a monofunctional register is used to produce SRs that are restricted to writings in formal mathematical language such as logical quantifiers, algebraic symbols, axioms, and formal definitions. A SR produced in this register is predominantly a reflection of standard mathematical practices (Duval, 2006, 2017). Figure 1 is a theoretical cross product: (discursive, non-discursive) x (multifunctional, monofunctional) registers.

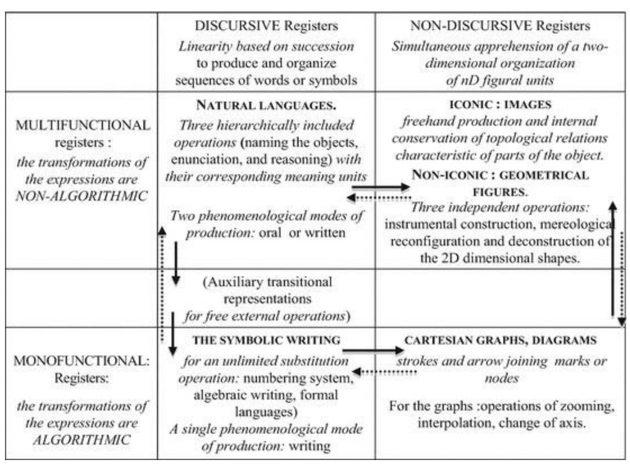


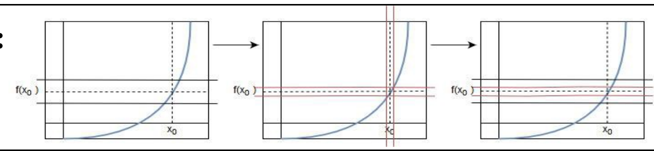
Figure 1: Registers and semiotic transformations given by arrows between registers (Duval, 2017, p. 85).

In order to survive in a mathematical environment, the learner must at least be able to perform cognitive processes termed transformations of semiotic representations (CTSRs) (Duval, 2017). A CTSR is a mental mapping from one semiotic representation to another. "In any semiotic transformation, it is necessary to distinguish between the starting representation and new representation produced, i.e. the arrival representation (Duval, 2017, p.43)." In some cases, the starting representation and target representation are produced using the same register. In other cases, the starting representation and target representation are produced through different registers. To theoretically separate these two cases, Duval (2006) made the distinction between two types of CTSRs: treatments and conversions. A treatment is a change in SR without a change in register. An example of a treatment is solving algebraic equations while staying in a monofunctional x discursive register. The more involved mental process of keeping the object the same and switching from "starting" to "target" SRs through different registers is called a conversion (Duval, 2006, p. 112). The arrows in Figure 1 depict the moves a learner could make in going from one register to another (i.e., natural language, diagrammatic drawings, symbolic writing, etc.). A conversion from one SR produced in one register to a SR produced in a different register may be algorithmic, step by step procedures, for example there are step by step procedures that are used to go from a symbolic SR of a function to drawing a graph of a function. Procedures can be acquired through "instrumental" (memorizing the procedure)" or "relational learning" (creation of the procedure after "relational understanding") (Skemp, 1978, 1979, p. 285-286). In contrast with an algorithmic move, a non-algorithmic from one register to another has no formulaic or standard mapping (Duval, 2017). To practice a conversion, that can be made algorithmic, and to prime the reader for more advanced CTSRs that will be presented later in this paper, we now explore continuity in calculus.

An Example of a Conversion in Calculus

This example in Table 1 is a conversion for the mathematical object: continuity of a real function at a point. The initial SR of this object is produced through a discursive x monofunctional register and the target through a non-discursive x monofunctional register (or vice-versa). This conversion is scaffolded with an "auxiliary transitional representation", SR B, as an intermediate step to go from the initial SR to target SR (Duval, 2017, p. 91). Some aspects in going from SR A to SR C can be made algorithmic using steps [1], [2], [3], and [4] in SR B.

Table 1. Example of a conversion in calculus.

Object: continuity of a real function at a point	
Register	**Semiotic Representation (SR)**
Discursive x Monofunctional	**A**: Let $X, Y \subseteq \mathbb{R}$. The function $f: X \to Y$ is continuous at a point $x_0 \in X$ if for any $\epsilon > 0$ $\exists \delta > 0$ such that for $x \in X$ we have: $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$.
Discursive x Multifunctional (bits of natural language) x Monofunctional (Auxiliary transitional representation)	**B**: [1] Choose any real number ϵ bigger than zero and examine $\|f(x) - f(x_0)\| < \epsilon$ as a horizontal tube of width 2ϵ centered around the horizontal line $y = f(x_0)$. [2] Now imagine $\|x - x_0\| < \delta$ as a vertical tube with a width 2δ centered around the vertical line that runs through x_0. [3] Try to find a number δ so that the new horizontal tube made by horizontal lines running through $f(x_0 - \delta)$ and $f(x_0 + \delta)$ sits inside the initial horizontal tube in [1]. If you can find a δ that gives you this inner horizontal tube, then it is the same as writing $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$. [4] Can you repeat [1]-[3] for any ϵ no matter how small? If the answer is yes, we say f is continuous at x_0.
Non-discursive x Monofunctional	**C**: 

As learners progress from a calculus to an undergraduate real-analysis course there is a rising expectation for them to produce auxiliary representations and make conversions on their own. However, the treatments and conversions undergraduate learners are expected to do and what they are cognitively able to do at a point in time may be antipodal (Thomas et al., 2010). In a recent study, Sandoval and Possani (2016) found that a majority of undergraduate learners in an introductory linear algebra course could not perform geometric treatments adding vectors nor conversions between verbal, algebraic, and geometric registers for: vectors and planes in R^3. They observed that, "One important result shows that, once students choose a register to solve a task, they seldom make transformations between different registers, even though this facilitates solving the task at hand (p. 109)." At some point the learner must overcome such difficulties to reach the learning milestone of "conversion fluency" for a variety of mathematical objects (Dreyfus, 1991; Duval, 2017, p. 110-111). Duval (2017) affirmed that, "without any explicit training of the conversion fluency, there is no possible mathematics learning for most students … (p. 110)."

The Milestone of Conversion Fluency

The construct of conversion fluency is a form of representational fluency (RF). RF is "the ability to create, interpret, translate between, and connect multiple representations" of a mathematical object(s) (Fonger, 2019, p. 1). Conversion fluency of a mathematical object is achieved if the learner can: (a) recognize and generate multiple semiotic representations of a mathematical object in a variety of registers (b) realizes that the semiotic representation is used to represent an object, but is not the object itself, (c) can begin with an initial semiotic representation and arrive at a target for a variety of initial and target choices, and (d) integrate the semantic bindings attached to multiple representations. "As a result of this process [integration], one has available what is best described as multiple-linked representations, a state that allows one to use several of them simultaneously, and efficiently switch between them at appropriate moments as required by the problem or situation one thinks about (Dreyfus, 1991, p. 32; Duval, 2017)." Drawing from others perspectives [Dreyfus, 1991; Fonger, 2019], we non-discursively view and discursively state a learner's 'fluency digraph' $F_o$of a specified mathematical object o to be a directed graph built up over past experiences: where the vertex set is a collection of semiotic representations S_o, a relation $R \subseteq S_o \times S_o$ which is a set of pairs (initial, target) semiotic representations, and arrow set $A(G_o) \subseteq R$. Graduate level learners and mathematicians are known to have fluency for numerous mathematical objects (Duval, 2017, p. 110-111). The size, $|V(F_0)|$, of their fluency digraphs may be quite large, but they don't semiotically denote all vertices in the digraph during mathematical activity; rather they make a choice that is situation dependent. To account for this, we call a subgraph of a learner's fluency digraph F_0 that is semiotically denoted and explicitly used during mathematical activity a 'semiotic container' c^o. With the theoretical background and language of SRs, registers, CTSRs, fluency digraphs, and containers in hand we now move on to an additional classification of CTSRs. This additional classification of CTSRs makes the distinction between a class of CTSRs that includes conversions and a second class that does not include conversions to address the research question: (a) What is a type of CTSR that goes beyond treatments and conversions?

An Additional Classification of CTSRs Beyond Treatments and Conversions

Based on Duval's (2000a, 2006, 2017) theoretical perspectives the first classification, classification 1 transformations (C1Ts), is to: move from an initial to target semiotic representation(s) for the same mathematical object. In mathematics some notions of "sameness" with respect to mathematical objects are: definitional equivalence, one-to-one correspondences, bi-directional set containment, isomorphism, and homeomorphism (Bourbaki, 1968). This first classification may or may not involve change in register; this class includes conversions. We also consider a second classification (C2Ts): varying a mathematical object by manipulating its structure or properties and necessarily varying the semiotic representation because the object has changed. In addition to having a change from an initial to a target semiotic representation(s) there is a change from an initial object(s) to a related target object(s). Some mathematical notions of mappings between two related but different mathematical objects are: non-bijective homomorphisms, free functors, and forgetful functors. It follows that this second classification includes: adding structure and/or properties to obtain more specialized objects; and forgetting structure and/or reducing properties to obtain a more general object. Within the classification of C2Ts we include generalization of a mathematical object(s). There are many forms of generalization in the mathematics education literature (Tall, 2002a; Mason, 2012). For this paper we remain within the scope of generalization of a mathematical object by induction from several initial example objects or as the idea of loosening the initial object(s) by preserving some but not all its structure and properties to obtain a target object(s). We now explore examples of C1Ts in the context of representation theory of finite groups over the complex field. Next, examples of C2Ts that are related to continuity across calculus, real-analysis and topology will also be explored to address the research question: (b) What are a few examples of CTSRs that learners are expected to perform within advanced undergraduate and graduate level coursework?

Examples of Class 1 CTSRs in Representation Theory of Finite Groups at a Graduate Level

Let the object be the inner product space of class functions $H = \{f: G \to \mathbb{C}: f(x) = f(gxg^{-1})\ \forall g\ \in\ G\}$ for a finite group G, these are functions that are constant on conjugacy classes of G. What is the integrated multi-representation that you use to semiotically denote this object? What registers are you using? What is a Hermitian inner product for this space? Let this be the initial semiotic container. Two target semiotic representations of the inner product space H can be constructed by finding two different basis B_1 and B_2. For the first C1T take your initial semiotic container. Let the target semiotic representation be the span of B_1 where B_1 is the list of functions $f_1, f_2, \ldots f_k$ defined as:

$$f_{i(x)} = \begin{cases} 1, & K_i \\ 0, & K_i, \quad i \neq j \end{cases} \quad where\ \{K_i\}\ is\ the\ set\ of\ congjuacy\ classes\ on\ G.$$

For the second C1T take your initial container for the space of class functions H and a different target semiotic representation will now be constructed using the irreducible representations of G over $\mathbb{C}$. Let $p_1, p_2, \ldots, p_m$ be representatives for the irreducible representations of G over $\mathbb{C}$ up to isomorphism. Define the character X_i: $G \to \mathbb{C}$ such that $X_i(g) := Tr(p_i(g))$. By properties of the trace function each X_i is a class function. In fact, the list of these class functions forms a basis B_2 of H. The first C1T is a move from initial container for H to $span(B_1)$. The second C1T is a move from initial container for H to $span(B_2)$. In both cases there was a change from an initial to target semiotic representation of the object H; and the mathematical object H remained the same $H = span(B_1) = span(B_2)$. Combining these two C1Ts leads to a major mathematical result [Theorem 7, Serre, 1977]: the number of irreducible representations of a finite group G up to isomorphism over $\mathbb{C}$ is equal to the number of conjugacy classes. This follows directly from a result in undergraduate linear algebra. It takes a great deal of mathematical knowledge developed throughout undergraduate and graduate level experiences that merges linear algebra and abstract algebra to make these higher-level C1Ts. These C1Ts require conversion fluency for a variety of mathematical objects and proof. In addition, many CTSRs including C1Ts and C2Ts occur within these two C1Ts. Due to the complexity of the CTSRs that occur within these two C1Ts and this papers length constraint, we switch back to the topic of continuity to explore the research question: (c) What are examples of CTSRs that occur within or depend on other CTSRs?

Examples of Class 2 CTSRs and CTSRs that Occur Within Other CTSRs in Real-Analysis towards Topology

In many cases one CTSR may be involved in another. It is important for theoretical coherency to initialize the mathematical object, specify the initial and target of the total CTSR, and specify sub-CTSRs that occur within the total transformation. For example, generalization as the total C2T from the initial object 'continuity of a real function at a point' to the target object 'continuity of a function between two metric spaces at a point' may involve more localized cognitive transformations such as the conversions in Table 1. Or other CTSRs but now restricted to sub-objects. For example, the learner may link multiple semiotic representations, such as those in Table 1, to gather information about the sub-object 'open interval in ℝ' in order to generalize this sub-object to the target sub-object 'an n-dimensional open ball in a metric space' with target semiotic representation: $B_n = \{x \in X\colon d(x_0, x) < \delta\}$ where (X, d) is a metric space. Within this sub-C2T is the sub-sub-C2T in which the learner must focus on the object 'the Euclidean distance function' to reach a more general target object a distance function on a metric space (Reed, 2018); To make the C2T from 'continuity of a real function at a point' to the target object 'continuity of a function between two metric spaces at a point' at least requires that the learner is able to parse through the initial object in multiple registers to detect pertinent sub-objects for sub-CTSRs. A second C2T can be made to go from the initial object 'continuity of a function between two metric spaces at a point' to the target object 'continuity of a function between two topological spaces at a point'. A sub-C2T from a metric space to a metrizable topological space and examples of topological spaces that are not induced from a metric space are needed to make this second C2T. One such example is the Zariski topology. It can be accessed by passing into the mathematical domain of Commutative Algebra. We now end with a model that summarizes a progression of advanced mathematical thinking (-thinking) [Dreyfus et al., 1990; Harel & Sowder, 2005] restricted to CTSRs on a mathematical object.

A Model for the Progression of CTSRs in Advanced Mathematical Thinking (-Thinking)

Overtime through various mathematical experiences a dramatic shift takes place for the mathematics learner. Before the first encounter with a mathematical object it does not seem to exist in their mind. At the first encounter the semiotic representation of the object feels foreign. They get acquainted and become familiar with the object by performing treatments, conversions, and C1Ts. Eventually one gains control of the object. At times an object with its semiotic representation comes to mind as an involuntarily reaction; other times one intentionally recovers the object from the mind and becomes choosey with how to semiotically represent or manipulate/change the object using C2Ts. This cycle of getting acquainted, gaining familiarity, and gaining control repeats as new objects are encountered. This progression is summarized in

Figure 2 below. From the examples of C1Ts and C2Ts at an advanced undergraduate to graduate level we took note of many supplementary processes that were involved.

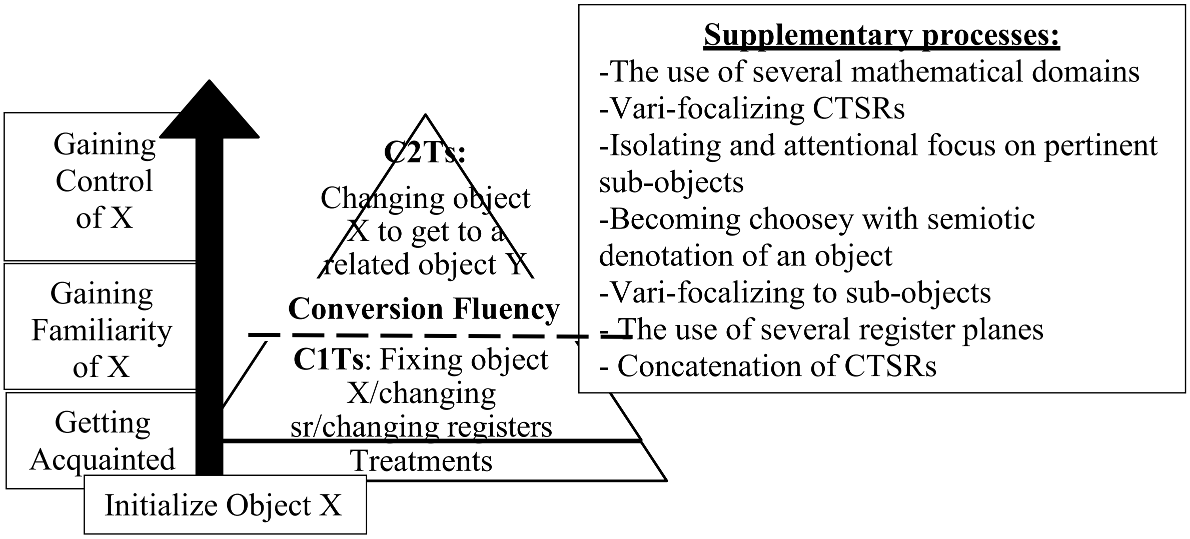


Figure 2. A progression of advanced mathematical thinking (-thinking) with respect to mathematical objects.

Some supplementary processes are: "vari-focalizing" many objects, zooming in on the object(s) to drop into a sub-object(s) level and zooming back out; and "vari-focalizing" CTSRs, zooming in to perform sub-transformations, concatenating them together, and zooming back out to see a total move (Skemp, 1979, p. 115). To transition from the Figure 2 summary model to a CTSR analysis tool, it is important to choose a theoretical designation of registers $r_1, \ldots r_n$ (Ely, 2017; Duval, 2017). To account for both the multi-modal (multiple registers used in tandem) and uni-modal (a single predominant register) case (Arzarello, 2006), we let Σ to be the power set of $r_1, r_2, \ldots r_n$; we call elements of Σ register letters. We visualize register letters as planes. C1Ts are used to travel from SR to SR of the same object among the register planes. These movements occur within the digraph of that object. Even after conversion fluency is obtained for a mathematical object, C1Ts may be revisited to increase the size of the fluency digraph. C2Ts are used to travel from an object to a related non-equivalent object. CTSRs, even C1Ts, can be quite involved. Both C1Ts and C2Ts can occur within or across mathematical domains.

Concluding Remarks

In this paper we focused on CTSRs at an advanced undergraduate and graduate level. However, the literature suggests that instructional materials and techniques that train these CTSRs, in level appropriate contexts, should be presented at all stages of mathematical development from school to a graduate level (Harel, 2006). This paper is a piece of a more expansive project in the works on mathematical intuition in proof. We hope to attain a more developed theoretical lens of CTSRs that builds from the ideas presented in this paper. This lens will be applied to analyze CTSRs and investigate the interplay between CTSRs and intuition. We plan to investigate how graduate learners classify irreducible linear representations (irreps) of D_6, D_8, D_{10} and D_{12} and how they use CTSRs in these example cases to gain intuition that leads to a final semiotic denotation for a general target object 'irreps of D_{2n}'. Within this total C2T, we hope to find CTSRs that may be accessible to learners taking introductory abstract algebra and advanced linear algebra coursework.

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Metacognition: An Overlooked Dimension in Connecting Undergraduate Mathematics to Secondary Teaching

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This theoretical report discusses metacognition as a tool for connecting teachers' experiences as learners of undergraduate mathematics to their teaching practice. Academic mathematics has long been considered an essential component of secondary teacher preparation. Current initiatives to make academic mathematics relevant to teacher knowledge and practice focus on course design and the role of instructors. This limits the potential for undergraduate mathematics to impact teacher practice, by restricting attention to select courses. To maximally leverage the impact of academic mathematics on teaching, teachers should be active participants in forming connections. In this report I discuss a metacognitive practice in which teacher are guided to reflect on their experiences as learners of undergraduate mathematics and use these experiences to inform their teaching practice. The report is presented from a theoretical perspective, but it emerged from empirical data. Examples are presented and discussed.

Keywords: Secondary teacher preparation; Undergraduate mathematics; Key Memorable Events; Metacognition

Introduction

Secondary mathematics teachers are generally required to learn academic mathematics, by which I refer to the content typically studied in the course of an undergraduate mathematics degree. A significant portion of coursework in teacher preparation programs content courses in undergraduate mathematics (Tatto et al., 2012).

In the early 20th century, Felix Klein recognized that academic mathematics and secondary mathematics were essentially different disciplines (Klein, 1932). Despite this, the "necessity" for secondary mathematics teachers to know academic mathematics was taken as a given; the assumption underlying its contribution to the work of teaching unexamined, "expecting the Intellectual Trickle-Down Theory to work overtime to give these teachers the mathematical content knowledge they need in the school classroom" (Wu, 2011, p. 372). The necessity assumption was challenged by Begle (1972) who found no empirical evidence to support a connection between teachers' knowledge of academic mathematics and student achievement in algebra. A later meta-analysis confirmed this finding (Crosswhite & Begle, 1979).

Despite requiring secondary mathematics teachers to study academic mathematics, teacher preparation programs often fail to make explicit how opportunities to learn academic mathematics should be structured to positively impact teaching (CBMS, 2012). Beyond this, studies show that mathematics teachers courses in academic mathematics as neither relevant nor necessary (Wasserman, Weber, Villanueva, & Mejia-Ramos, 2018; Zazkis & Leikin, 2010). Teachers view school mathematics as different from academic mathematics (Beswick, 2011). The structure and implementation of undergraduate mathematics courses is suggested as a possible culprit (Dreyfus, 2002; Lai, 2018; Weber, 2004).

Purpose

This theoretical report addresses a factor largely overlooked by recent efforts to make academic mathematics relevant to teaching secondary mathematics: the teachers themselves. Existing initiatives to connect undergraduate and academic mathematics focus externally on course design and the role of instructors; they present a collection of curated experiences for teachers to apply their knowledge of academic mathematics in pedagogical contexts. Actively involving teachers in the process of finding and leveraging connections can increase the potential for academic mathematics to impact teaching practice.

In this report I discuss the potential for teachers to incorporate metacognitive practices to connect their experiences as learners of undergraduate mathematics to their teaching practice.

Conceptual Grounding

In recent years increasing attention has been focused on making undergraduate courses for mathematics teachers relevant to the work of teaching (CBMS, 2001, 2012). Current thinking focuses on: connecting the contents of school and academic mathematics (Zazkis & Mamolo, 2011); purposefully designing academic mathematics courses to influence secondary teaching (Wasserman, Weber, Fukawa-Connelly, & McGuffey, 2019); courses co-taught by mathematicians and mathematics teacher-educators (Zaslavsky & Cooper, 2017) and capstone courses taken at the end of undergraduate studies connecting academic and school mathematics (Lai, 2018).

These initiatives are limited in scope, scale and impact: Custom-designed courses embedding academic mathematics in pedagogical contexts are applied on a small scale (Lai et al., 2019), and many prospective teachers study academic mathematics courses with a general population of undergraduates (Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos, & Weber, 2017). Capstone courses, typically taken at the end of undergraduate studies, are often taught by mathematicians who lack knowledge and confidence in their ability to create connections that are relevant to secondary teachers (Lai, 2018).

Beyond these logistical limitations, custom-designed courses *curate* situations in which academic mathematics is embedded in pedagogical contexts. That is, pedagogical applications are pre-selected by instructors. In addition, capstone courses may be "too little, too late," as teachers may have already formed (negative) opinions of the utility of academic mathematics before reaching their final year of study.

These limitations raise the question of how academic mathematics courses across the spectrum can be maximally leveraged to positively impact teaching practice. One answer may lie with teachers themselves. Research shows that reflective metacognitive practices positively impact teaching and learning. Below I define metacognition as used in this report and discuss how metacognitive practice have been used previously to positively impact teaching.

Metacognition

Metacognition refers to one's thinking about thinking or thinking about knowing (Flavell, 1979). It includes awareness and control over one's thinking and learning (Swanson, 1990). Reflection is a component of metacognition (Von Wright, 1992).

Teacher reflection is a metacognitive practice through which teachers: analyze pedagogical situations; plan and set goals; monitor actions; evaluate results and review their professional thinking (Colton & Sparks-Langer, 1993). Successful university mathematicians and mathematics educators use metacognitive practices to reflect on their teaching (McAlpine &

Weston, 2002). Providing university instructors with reflective to scaffold their thinking has been shown to improve teaching (McAlpine & Weston, 2002).

Integrating Metacognition with Conceptual Frameworks for Improving Teaching

Research frameworks for understanding teacher knowledge and productive practices have, on occasion, been coopted to serve as tools for metacognitive reflection. I give some examples below:

The Knowledge Quartet is a four-component framework describing how different dimensions of teachers' knowledge of mathematics and pedagogy impact classroom practice (Rowland, Huckstep, & Thwaites, 2005). Recently it has been applied as a metacognitive lens through which university instructors of mathematics can reflect on their teaching practice (Breen, Meehan, Rowland, & Breen, 2018).

Mason (2002) introduced the Teacher Noticing construct as means for teachers to discipline their in-the-moment responses to student thinking. In a 2011 study Breen et al (2011) used Teacher Noticing as a metacognitive prompt for university mathematics instructors to identify and reflect on the critical incidents that occurred in their classes such as particular interactors with students.

The examples above describe metacognitive applications of conceptual constructs for teacher knowledge and practice. In the first example, teachers reflect on the knowledge they bring to teaching; in the second they reflect on how attention to student learning can inform teaching. Below I propose a similar integration in which teachers reflect on their experiences as learners of undergraduate mathematics to inform their teaching.

Learning in Undergraduate Mathematics Courses

Marmur (2018) developed the construct of Key Memorable Events (KME)—classifying events which students identify as memorable and meaningful in support of their learning. KMEs are typically accompanied by strong positive or negative emotions. Marmur and Koichu (2018) identify eight key types of KME relating to: surprise (at the direction of the lesson), discourse (on how to approach a challenging problem); suspense; bridging (intuitive and analytical discourse), questions (which engage student attention); symbols theoretical loose ends; and connections.

Marmur suggests a practical application of the construct: Instructors of academic mathematics leverage *students'* KMEs as pedagogical tool to guide lesson design. Below I discuss a metacognitive application of KMEs, in which structured reflective questioning is used to guide teachers' experiences as learners of undergraduate mathematics to identify and understand their *personal* KMEs. They are further guided to use personal KMEs to inform lesson design and teaching practice.

Context:

A large private University in the US offers a Masters' degree in Teaching Mathematics. Students (generally in-service teachers) are required to enroll in a pedagogical seminar taught in parallel to Calculus 1 or 2. The seminar and calculus course are taught by different instructors. The teachers enrolled in the course and seminar have previously studied calculus, but for many it is a distant memory. The purpose of the seminar is to use the "happenings" in calculus to motivate discussions on teaching mathematics. The pedagogy seminar is loosely defined with instructor autonomy to structure the class and define its goals.

In a pedagogy seminar in Summer 2019 I used KMEs as a “hook” to connect the events in calculus lessons to teaching practice. All documentation of teachers’ activities and responses in the seminar are presented anecdotally to comply with strict IRB requirements for reporting on data that was initially unblinded.

Teachers were introduced to KMEs in the first seminar meeting. They were invited to reflect on the previous days’ calculus lesson and identify a KME. In this session KMEs were general in nature, describing, for example, frustration in being unable to “keep up” and appreciation for the instructor’s willingness to “go back.” In subsequent weeks, students were asked to identify and explain a KME from at least two of the three weekly calculus lessons, by submitting written responses to the following prompts[1]:

1. Does anything memorable come to mind about your calculus class yesterday? (positive or negative—and not necessarily in a mathematical sense.)
2. If yes, why was the event memorable? If no, what might have made the class more memorable?

The written responses were used to structure class discussion. The initial KME prompts were focused on KME-*noticing,* however in class discussion, students were asked to identify the narrow mathematical landscape in which the KME occurred and to discuss the role of learning and instruction played in triggering the KME. After two weeks reflective prompts were expanded to include these elements, allowing time for each student to independently narrow the focus of their KME:

3. What, specifically, was happening mathematically?
4. What type of teaching/learning was going on in at that moment?

At this point class discussion shifted to what these KMEs might “look like” in a secondary mathematics classroom. This idea was first modeled; subsequently teaching and learning scenarios incorporating these KMEs were created collaboratively. the reflective prompts were again expanded:

5. What would this memorable event “look like” in a class that you teach?

In the final weeks of the seminar, teachers found their own “pedagogical contexts.” Working in pairs they designed learning activities based on insights developed from personal KMEs and the mathematical content/learning landscape in which they occurred. Examples are shown in Table 1.

[1] Many thanks to Ofer Marmur for developing these prompts with me

Table 1. From personal KME to potential for student KME to Learning activity

Personal KME	Calculus context	Potential for student KME	Learning activity
Wow/bridging	Calculus instructor illustrated the connection between the accuracy of approximation by Taylor polynomials using dynamic graphic software	Dynamic graphic software can be used to dramatically present an algebra concept	Used dynamic graphic software to discover principles about the behavior of polynomials
Noticing the frustration/ knowledge gap of others	Undergraduate students in calculus class did not understand the ratio test	If students had the necessary background, they could experience a positive KME connecting prior and new knowledge	Designed an activity to illustrate exponential growth in the discrete case to motivate geometric series and partial sums

Theoretical and Practical Utility

At the conclusion of the seminar participants indicated that the reflective practice of using KMEs to connect academic mathematics to the content, teaching and learning of secondary mathematics increased the relevance of the academic course to their teaching practice. They planned to continue the practice in other academic courses and anticipated that KME reflection would similarly increase the relevance of these courses.

Reflective KME prompts, guiding teachers to identify a personal KME, imagine a similar KME for students and design an instructional activity to elicit this KME closely parallels the stages of teacher knowledge described by Silverman & Thompson (2008). Indeed, KME reflective practice offers a metacognitive perspective on the development of this knowledge.

Ball and Bass (2009) describe a type of knowledge for teaching mathematics, labeled horizon content knowledge, as an awareness of a large mathematical landscape, engaging aspects of mathematics that are useful to students. Through reflective KMEs, the teachers in the pedagogy seminar exhibited several elements identified by Ball and Bass as benefits of a mathematical horizon. Through reflective KMEs, teachers: anticipated and made connections between undergraduate mathematics and secondary mathematics; identified precursors to future mathematical confusion; and noticed and evaluated mathematical opportunities

Conclusion

Secondary mathematics teachers are required to study a significant amount of undergraduate mathematics. Recent initiatives to make this mathematics relevant to the work of teaching have primarily focused on course design and instruction. This theoretical report discusses a metacognitive dimension to the work of connecting school and undergraduate mathematics. Through guided reflective practice teachers can be enlisted as active partners, connecting their experiences as learners of undergraduate mathematics to the content, teaching and learning of secondary mathematics.

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How Mathematicians Attend to Learning Goals for Teachers

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Many mathematicians face the unique challenge of acting as teacher educators without necessarily having experience working in secondary education. We propose that examining the learning goals mathematics faculty attend to, as well as how they attend to these goals, provides promising insight into how mathematicians lead content courses for teachers, despite this challenge. In a study which interviewed six mathematics faculty, we were able to categorize the types of learning goals mathematicians seek to address in these content courses. To understand how mathematicians attend to learning goals, we developed an elaboration of the instructional triangle model which reconceptualizes the interactions between mathematicians and students, as well as the environmental influences of these instructional interactions.

Keywords: Content Courses for Secondary Teachers, Mathematicians, Mathematical Knowledge for Teaching

Multiple studies have sought to examine the importance of a teacher's individual mathematical knowledge in developing mathematical knowledge for teaching and the impacts on student achievement (e.g., Baumert et al., 2010; Blunk, 2007; Hill, Rowan, & Ball, 2005). As these relationships are better understood, it is necessary to examine applications to the undergraduate experiences of mathematics teachers. It is well-documented that many secondary teachers find the content of their mathematical undergraduate courses inconsequential to future teaching practice (Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Wasserman, Weber, Villanueva, & Mejia-Ramos, 2018; Zazkis & Leikin 2010). It follows that changes to the structure and instruction of these courses should be considered to better highlight connections between undergraduate mathematics courses and teaching practice.

Mathematics courses for prospective and in-service teachers are principally taught by mathematics faculty. Many mathematicians face the unique challenge of educating teachers without necessarily having worked in secondary level classrooms. However, few studies have examined the practices of mathematics faculty (Speer, Smith, & Horvath, 2010). As modeled by Sleep (2009), understanding instructors' learning goals is one means to understanding teaching practice. Consequently, we aim to understand the selection and enactment of learning goals by mathematics faculty when leading courses for mathematics teachers.

We investigated the two research questions:

1. What learning goals do mathematics faculty attend to when planning content courses for prospective and in-service secondary mathematics teachers?
2. How do mathematics faculty attend to these learning goals?

Conceptual Perspective, Terminology Used, and Prior Literature

In this paper, we define *mathematician* as a faculty member in a university department of mathematics who engages in teaching and mathematics scholarship, while holding a mathematics doctoral degree. *Learning goals* are conceptualized as statements of intended impact on a student's learning. This conceptualization is intentionally broad; as Sleep (2009) has noted, there is limited vocabulary to describe "learning goals" due to under-conceptualization (p.1). Despite its lack of specificity, however, the term "learning goals" holds significance for practitioners.

Following Lampert (2001) and Cohen, Raudenbush, and Ball (2003), we take *instruction* to mean the "interactions among teachers and students around content, in environments", where instructional interactions signifies "no particular form of discourse but to teachers' and students' connected work, extending through, days, weeks, and months" (Cohen, Raudenbush, & Ball, 2003, p.122). These interactions can be represented by an "instructional triangle" (Figure 1).

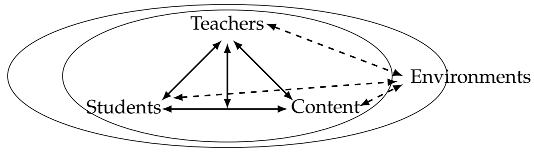

Figure 1. Instructional Triangle

In any particular instance of instructional interaction, teachers and students work collaboratively in actions that enact learning; meanwhile, teachers simultaneously interact with curriculum content. Ultimately, learning cannot ultimately take place without interactions between students and content; consequently, teachers facilitate this interaction through activities such as investigating problems or discussion to support the success of learning. It must be noted that instruction occurs "with and in *environments*" (Cohen, Raudenbush, & Ball, 2003, p.127). This domain of instruction entails managing elements such as "other teachers, school leaders, parents, district policies, state requirements, and more", which act not only as external influences but are entwined with the ongoing interactions between teachers, students, and content (Cohen, Raudenbush, & Ball, 2003, p.127).

In the undergraduate context, an undergraduate instructor is a teacher, and secondary teachers are students. Henceforth, we use *instructor* to designate undergraduate faculty, *teacher* for secondary teacher, and *student* for secondary student.

Following Schoenfeld (2010), we take instructors' decision making to be modeled by an instructor's resources, goals and orientations. Schoenfeld defines

- Resources as an individual's knowledge in the context of available material and other resources;
- Goals as an individual's conscious or unconscious aims for instruction; and
- Orientations to encompass an individual's beliefs, values, biases, and dispositions.

As indicated by previous work, the conceptual perspective given by the instructional triangle provides a reasonable way to understand someone's teaching practice; Lampert (2001) utilized these ideas in examining and exposing her own teaching practices. We propose that combining this perspective with a particular focus on the instructors' goals for teachers and the orientations demonstrated toward those goals provides a promising examination of teaching practice. However, existing research examining learning goals has largely focused on the K-12 level rather than the collegiate level (Sleep, 2009).

Data & Methodology

Participants

The participants of this study were six mathematics faculty at a variety of undergraduate institutions. The participants agreed to participate in the study after receiving email invitations distributed to previous participants in teacher education workshops for mathematicians. Some, but not all, of these workshops were co-organized by one of the authors of this study.

Protocol

Data was collected during structured online interviews via Skype. Before the interview, participants were asked to send the investigators a description of a favorite problem used in a mathematics course for in-service or pre-service secondary teachers which exemplified their philosophy of teaching such courses. The associated research question is noted in parentheses.

To begin the interview, participants were Skype-chatted the following general questions:

1. What are the most important goals you have had for (pre- or in-service) teachers in mathematics courses for teachers? (RQ1)
2. What kind of experiences or knowledge have you drawn upon to carry these out in your teaching, especially the times that you felt like your teaching particularly supported those goals? (RQ2)
3. How does your use of your favorite problem exemplify teaching toward these goals?

Participants were then asked these questions regarding the provided favorite problem:

1. In your view, if you had to say it in one sentence, what specifically are the most important content or practice goals you have in mind when using this task? (RQ1)
2. How does working on this task get at these content and practice goals? (RQ2)
3. What are some ways you use to tell whether the teachers learned what you intended them to learn through working on this task? (RQ2)
4. Has the way you've used this task changed over time? How did these ways support the goal? (RQ2)

Analysis

To answer our first research question, we used open coding and constant comparative coding to capture and categorize the variation of responses from the participants (Strauss & Corbin, 1994). To answer our second research question, we coded data according to an elaborated model based on an instructional triangle (Lampert, 1990; Cohen, Raudenbush, and Ball, 2003), as well as constant comparative coding to interpret and modify this model based on our results (Miles, Huberman & Saldana, 2012). The use of open coding and constant comparative coding was appropriate for this study given the limited work in this area; in addition, open coding can mitigate bias in classifying intended learning goals (Strauss & Corbin, 1994).

Preliminary Results

Recall that our first research question asked to determine *learning goals that mathematicians attend to* in content courses for secondary mathematics teachers. Table 1 shows the four categories we developed from our analysis of the participant interviews, with illustrative excerpts. We note that although the interview protocol specified "content or practice goals", the mathematicians described goals outside of these categories. Table 2 shows the how often these categories appeared in the initial round of coding. The last three categories may contribute a possible elaboration of the "meta-mathematical" aims found by Leikin, Zazkis, and Meller (2017) in their interview of mathematicians teaching practicing secondary teachers.

Table 1. Categorization of Learning Goals

Code	Description	Illustrative Interview Excerpt
Mathematical Content	Responses involving mathematical concepts, skills, or procedures.	"And so one of the goals … is going back and visiting some of those topics that they might have skipped."

Habits of Mind	Responses involving practices applied across mathematics.	“The primary goal is to give the students an experience with problem solving.”
Pedagogical	Responses involving specific methods and practices that could be employed in future teaching scenarios.	“Thinking about when this would or wouldn’t be appropriate with students, connecting it back to the classroom…So making that switch from being a student to being a teacher.”
Affective	Responses involving an individual’s perspective of their mathematical capabilities or of mathematics as a whole.	“One of my main goals is really recognizing that they will continue to grow and they need to put themselves in situations in where they continue to grow.”

Table 2. Frequency of Codes

Learning Goals			
Mathematical Content	Habits of Mind	Pedagogy	Affective
17	17	7	4

Our second question asked *how mathematicians attend to their intended learning goals in instruction*. We interpreted “instruction” in terms of interactions in an instructional triangle. We coded mathematicians’ descriptions of their enactments by component of this triangle. Arguably, the “environments” component of this triangle are not well-understood (Neal & Neal, 2013). To make sense of this component, then, we consulted theories and frameworks in teacher education. We found it useful to (1) *split bi-directional arrows into uni-directional arrows*, and (2) interpret mathematicians’ descriptions of environments in terms of *obligations* in the sense of Herbst and Chazan (2011, 2012). In the presentation, we will discuss how we used this theory, as well as specific codes associated to each uni-directional arrow. In the space allowed in this brief report, we present only a summary of our result for the second question, in the form of an elaborated instructional triangle (Figure 2).

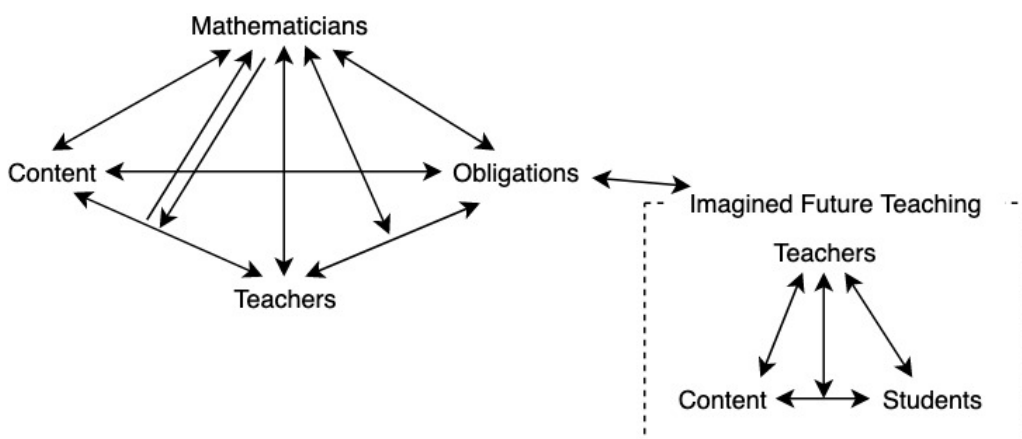

Figure 2. Elaborated Model of Instruction in Content Courses for Secondary Teachers

Discussion

This study set out to understand the learning goals mathematicians seek to address in content courses for in-service and prospective teachers, as well as how these goals are enacted. By asking mathematicians to provide and examine a favorite problem used in these courses, we were able to retroactively “observe” a concrete example the participant’s practice. Moreover, we were able to interpret the instructional enactment of these favorite problems via a theory of instruction, thereby understanding not only how the instructional choices of mathematicians create opportunities to reach these learning goals, but how mathematics faculty attempt to ensure and measure the success in actualizing these goals.

The elaborated model of the instructional triangle we have developed provides a novel framework to capture mathematicians’ teaching practice. In particular, as we work to decompose the relationship between mathematicians, here acting as instructors, and the ongoing interaction of teachers, here acting as the students, with content. In particular, we can distinguish those actions that originate with the mathematicians, such as distributing a task or problem, and those actions that describe interactions of teachers and content. In doing so, we may better see not only the intentions of the practice of mathematicians, but also how the intentions unfold in instruction.

Moreover, our elaborated model proposes a conceptualization of the “environments” domain of instruction in terms of *obligations*. In particular, this contribution seeks to add context to this domain of instruction, which hereto lacks specificity to encapsulate the many external forces which are interwoven into other instructional interactions. As introduced by Herbst and Chazan (2011, 2012), *professional obligations* refer to the requirements that the profession of mathematics gives to those acting as a mathematics teacher, and thereby shape their instructional decisions. In the situation of teacher education, mathematics faculty and students both take on the role of mathematics teachers. Our conceptualization allows us to theorize the influence of *imagined future teaching* on the instruction of these university content courses. The obligation to future teaching provides a simultaneously external and internal influence on mathematicians, teachers, and content; these future interactions may not physically occur in this classroom, but provides the uniting motivation for all of the participants of these content courses.

We note two limitations of the study. First is that this study relies on a reporting of practice of mathematics faculty, rather than actual practice. Consequently, our recorded data may be obscured by factors such as personal expectations and the passing of time, as opposed to independent, immediate observations. Second is the limited sample size of this study. Nonetheless, our study was able to document the intentions and instructional decisions of mathematics faculty when preparing courses for such a specific, and critically important, audience.

There is still much work to do to understand the intended learning goals of mathematics content courses for secondary mathematics teachers, as well as how these learning goals may be enacted through the instruction of mathematics faculty. Nonetheless, this study provides insight for future work, with practical applications for the preparation of mathematics teachers.

Questions for Audience

Our preliminary results give way to several questions we wish to discuss at our RUME presentation.

1. Does our proposed elaboration of the instructional triangle overlook any crucial elements of instructional interactions in mathematics courses for secondary teachers?
2. Are there ways to highlight the role of mathematical knowledge for teaching in our elaborated model?
3. Elementary teaching may be different in nature from secondary teaching. What potential differences may there be between goals or instructional interactions for mathematics courses for elementary teachers compared to secondary teachers?

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Linking Terms to Physical Significance as an Evaluation Strategy

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One expected outcome of physics instruction is that students develop quantitative reasoning skills, including evaluation of problem solutions. To investigate students' use of evaluation strategies, we developed and administered tasks prompting students to check the validity of a given expression. We collected and analyzed written and interview data at the introductory, sophomore, and junior levels. Tasks were administered in three different physics contexts: the electric field due to three point charges of equal magnitude, the velocity of a block at the bottom of an incline with friction, and the final velocities of two masses in an elastic collision. An unexpected strategy used by some students was to associate physical significance with terms in an expression (e.g., associating a term in an electric field expression with a specific point charge). We explore the significance of these responses and propose explanations for the frequency of this phenomenon in different contexts.

Keywords: Evaluation, physical significance, physics, problem solving

I. Introduction

The preeminent goal of a science education is the ability to think critically. Consequently, the National Research Council identified the development of critical thinking as a general learning goal for undergraduate science, engineering, and mathematics education (National Research Council, 2011). In physics, critical thinking is fostered through problem solving. One aspect of critical thinking is evaluation: making a judgement about the value, nature, character, or quality of something or someone. Many models of problem solving in physics explicitly include evaluation of problem solutions (Polya, 1945; Heller, Foster, & Heller, 1997; Tuminaro & Redish, 2007).

Problem solving in physics also involves mathematical reasoning. This constitutes the ability to describe the physical world using mathematical representations (sometimes referred to as "mathematization" (Uhden, Karam, Pietrocola, & Pospiech, 2012)) and to ascertain the physical consequences of mathematical manipulations of these laws. One facet of mathematical reasoning is a set of quantitative reasoning skills that includes performing dimensional analysis, considering limiting cases, using approximations, and identifying errors in solutions. These skills are collectively known as validity checks and are some of the unspoken examples of what it means to "think like a physicist" (Van Heuvelen, 1991). All these expert physicist skills involve integrating physics and mathematics in a way that makes sense physically (Loverude, 2015).

Thus, as an important element of critical thinking and a crucial component of mathematical reasoning in physics, evaluation is indispensable to physics. There are many models of using mathematical reasoning in physics (Bing & Redish, 2009; Redish & Kuo, 2015; Uhden et al., 2012; Wilcox, Caballero, Rehn, & Pollock, 2013). While these studies point out that evaluation is important, they are focused on other aspects of the problem-solving process, e.g., correctness of computation and activation of correct mathematical tools. Only a few studies have explicitly focused on evaluation skills. (Lenz, Emigh, & Gire, 2019; Loverude, 2015; Sikorski, White, & Landay, 2018a). Sikorski ,White, and Landay (2018) showed that the validity check most taken up by students after instruction is unit analysis. Warren (2010) found that students' problem solving performance improved when they were explicitly taught evaluation strategies. However, there are no studies focused on how or when students develop the skill of validity checking.

Consequently, the broad goal of our study is to understand how physics students develop mathematical reasoning by examining how validity checks can help these students consolidate their physics and mathematics knowledge.

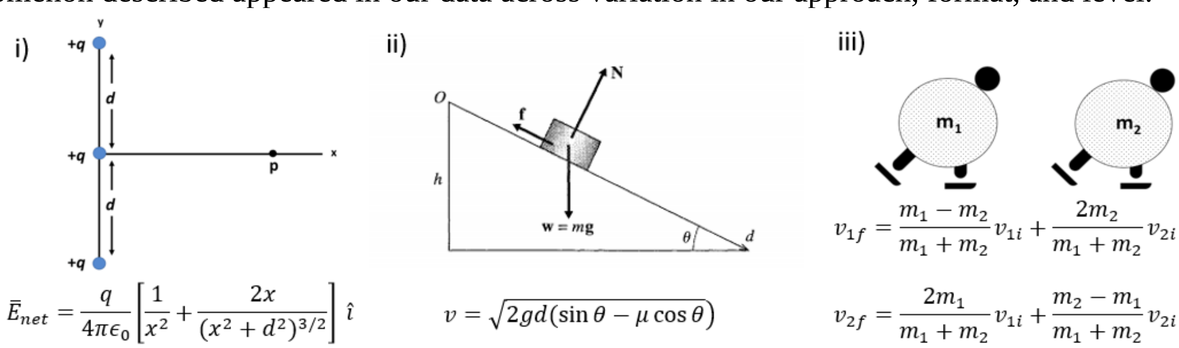
To this end, our research questions are:

1. To what extent do students check the validity of derived expressions when prompted?
2. How do students make sense of an expression that when they check its validity?
3. What determines the strategies used in checking the validity of a given expression?

II. Study Design and Methods

To answer these questions, we designed tasks asking students to check the validity of given expressions relating quantities in a physical scenario. These tasks were given in both interview and written form and administered at different levels of the curriculum as well as with different problem contexts. In this paper, we focus on three introductory-level tasks (see Fig. 1). In each of these tasks, students were given a correct expression for one or more quantities: the velocity of a block at the bottom of an incline with friction; the electric field at a point near three point charges of equal magnitude; the final velocities of two masses in an elastic collision. The students were first prompted to describe how they would go about checking if the expression was reasonable and then asked to use their suggested approaches to determine if the expression is likely to be correct.

The tasks were administered in various physics courses at a public research university in New England. Interview subjects were volunteers, solicited in the class of interest. The interviews were conducted in pairs. The written data collection depended on the way that the course instructor thought would optimize participation, including short in-class quizzes with or without an offer of extra credit, and extra credit items on an exam. While it is not possible to eliminate all potential variables, the phenomenon described appeared in our data across variation in our approach, format, and level.



$$\bar{E}_{net} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{x^2} + \frac{2x}{(x^2+d^2)^{3/2}}\right]\hat{\imath}$$

$$v = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2}v_{1i} + \frac{2m_2}{m_1 + m_2}v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2}v_{1i} + \frac{m_2 - m_1}{m_1 + m_2}v_{2i}$$

Figure 1: Figures and given expressions for the assigned tasks: (i) the electric field at a point some distance from three point charges of equal magnitude; (ii) the velocity of a block at the bottom of an incline with friction; (iii) the final velocities of two masses in an elastic collision. For each task, students were asked (a)"Without knowing the correct answer, how would you go about checking if your solution is reasonable?" and (b)"Using the approach(es) you described in part (a), determine whether or not this is likely the correct result.

III. Data Analysis

Data analysis in our study used modified grounded theory, as the analysis was in part based on previous literature and there were some expectations of certain categories. Our research design and data analysis have focused on emergent patterns in the data. Written data were open-coded, with phrases in a response categorized based on an overall theme. For instance, on the inclined plane task, responses where students suggested plugging in numbers to check a velocity value were coded as "plug in numbers." To analyze interview data, we transcribed the videos and coded for approaches that were also present in the written data, then for new ones that emerged in the interview. The interview codes were not based on the presence or absence of certain words or phrases but in the overall approach with which the student seemed to tackle the prompt.

Theoretical framework

One challenge of our project is that there is no framework for analyzing validity checks in the PER literature. However, we examined our results through the lens of *epistemological framing*. Principally, we used the work of Bing and Redish on epistemic framing via warrants (Bing & Redish, 2009). In their study, upper level physics students were asked to explain why they thought their solution to physics problems were correct. Their responses were categorized into 4 groups (frames): *invoking authority* (mathematical results used without explicit justification), *calculation* (computation leads to trustable result), *physical mapping* (quality of fit between mathematics and a physical situation), and *mathematics consistency* (similarity between mathematical structure underlies two superficially different physics situations). We used this framework because our task also deals with epistemology as it focused on students justifying why they thought a given solution was reasonable.

IV. Results

General Results

On each of the three tasks, our analysis showed a myriad of student approaches, with most students suggesting and/or attempting more than one approach for a given task. Overall, most introductory students did not perform validity checks when prompted. On the inclined plane task for instance, only 4% of introductory students considered limiting cases or checking units.

We categorized the evaluation tools used by students into three groups based on their underlying warrant: *consistency of computation*, *consistency with authority*, and *consistency of physics with mathematics*. One class of responses in *consistency of computation* was solving (again) for the given expression. This was the most popular response on all three tasks. Other computation-based approaches included integrating or differentiating the given expression and rearranging the expression to solve for a constant or given variable. The consistency with authority category included responses that base correctness on checking with an external authority like the professor, textbook or teaching assistant.

The *consistency of physics with mathematics* category includes attempting to check the units of the expression, considering limiting cases, and citing an expected covariation (e.g., “the velocity increases as the angle increases”). The last subset of responses in this category are those that make sense of an expression by pointing out the physical significance of a multi-parameter term in a given expression. We label this phenomenon “grouping,” since it resembles a grouping “procedural resource” in the literature in which students grouped parameters before conducting algebraic manipulation while solving a separation of variables problem in an intermediate mechanics context (Wittmann & Black, 2015).

For the purposes of discussion in this paper, we define grouping as identifying a subset of a mathematical expression that is bigger than one mathematical symbol and that is associated with some physical significance. It is not uncommon for groups to be separated by mathematical operators or the equal sign. Further, grouping involves making sense of terms in an expression and explaining its significance using the physics at play in the physical scenario described by the expression. We believe that for this process, the association is driven by physics as the students do not just focus on mathematical operations in the expressions but on their significance of the situation at hand.

To demonstrate how we apply this definition of grouping, we identify and describe specific occurrences in our data. We will present examples from the point charge task.

Grouping

To evaluate the magnitude of the electric field (E) due to a point charge (q), one uses the Coulomb expression $\left(\vec{E} = {}^{kq}/_{r^2}\,\hat{r}\right)$ where k is Coulomb's constant, r is the distance between the source charge and the point where the electric field is being calculated, and $\hat{r}$ is the direction radially outward from the point charge to that point. To evaluate the electric field due to multiple point charges, the electric fields due to the individual charges are added together using vector superposition.

In five pair interviews using this task, we observed three types of grouping: *grouping for distance* (4 pairs), *grouping for projection* (2 pairs), and *grouping for electric field* (4 pairs).

1. **Grouping for distance**

This category involved associating segments of the expression with distance. In the following excerpt, Evan and Emily try to make sense of and check the validity of the given expression.

Ev: So $1/x^2$, that's the distance for the charge in the middle.

Em: Charge in the middle, yep. So, we are just going to make sure that this [line earlier drawn from the bottom charge to point p] is the right one.

Ev: So, that [$1/x^2$] makes sense. And then $\frac{2x}{(x^2+d^2)}$ …

Em: So that's the distance, If this [x-axis] is what we're calling x, that's $x^2 + d^2$ is r so that would make sense. And then there's two of them, that is why it is raised to the three halves.

In the above excerpt, Evan and Emily are grouping terms by connecting a group of symbols to a distance on the diagram. First, they decide that $\frac{1}{x^2}$ is present/correct/warranted because "that's the distance for the charge in the middle." They also conclude (incorrectly) that the $(x^2 + d^2)^{3/2}$ makes sense because $x^2 + d^2$ is the distance from the off-axis to point *p*. The students call this distance "*r*." They also rationalize that the three-halves power is because there are two off-axis charges.

2. **Grouping for projection**

This category involved associating segments of the expression with the projections of the electric field in the *x* and *y* directions. In the excerpt below, two first-year students, Martin and Nate try to make sense of and check the validity of the given expression.

Martin: Off the top of my head I don't know why we add it [*2x* in numerator] on but if it's because of the components of the electric field caused by these [circles top and bottom charges] point charges and the formula says you have to add it on then I would agree with that.[…]

Nate responded to this, drawing *x* and *y* components of the electric field due to the top and bottom charges. He then continued:

Nate: This [$\frac{2x}{(x^2+d^2)^{3/2}}$] term is, umm, due to this [*x* component of top charge E-field] and this [*x*-component of bottom charge E-field] combined, so that is where you get the *2x* in the numerator.

In the above conversation, Martin expresses uncertainty about the physical meaning of the *2x* in the expression and Nate proceeds to explain. He breaks down the electric field vector into *x* and *y* components, showing that the *y* components of the E-field cancel so that the *x* components of the electric field of the top and bottom charges add up to yield the *2x* in the given expression. In the rest of the interview, Nate and Martin do not explicitly connect the terms in the expression (as a whole) to the electric fields due to the charges. However, another set of first-year students (Dom and Jake) do so.

3. **Grouping for electric field**

This category involved associating segments of the expression with the expression for electric field due to a point charge. In the following excerpt, In the excerpt below, Jake tries to make sense of and check the validity of the given expression.

J: … if you are to multiply this [$\frac{q}{4\pi\epsilon_0}$] in here [$\frac{1}{x^2} + \frac{2x}{(x^2+d^2)^{3/2}}$] you will get that [$\frac{q}{4\pi\epsilon_0 x^2}$] back so we know at least this [$\frac{q}{4\pi\epsilon_0}\left(\frac{1}{x^2}\right)$] part of the equation is accounting for the first [middle] charge, and without checking this [$\frac{2x}{(x^2+d^2)^{3/2}}$], we know it looks like we are going to be using this same this [draws a box around $\frac{q}{4\pi\epsilon_0}$ in $\frac{q}{4\pi\epsilon_0 x^2}$] bit right here. which would be assuming the q's are the same which they are according to this diagram. Then this [$\frac{q}{4\pi\epsilon_0}$] would just be constant.

In distributing $\frac{q}{4\pi\epsilon_0}$ into the rest of the expression, Jake compares the given expression to the iconic equation for electric field ($E = \frac{q}{4\pi\epsilon_0 r^2}$). He recognizes that the $\frac{q}{4\pi\epsilon_0 x^2}$ term accounts for the electric field of the "first" (middle) charge, while $\frac{2x}{(x^2+d^2)^{3/2}}$ is connected to the other two charges, and accounts for the distance to each charge using trigonometry.

4. Instances of grouping in written data

We first noticed the phenomenon of grouping during the interviews of first-year students on the point charge task. This prompted us to reexamine the written data in search of similar responses. This was challenging because written responses do not give the level of insight into in-the-moment thinking as interviews. It is possible that a student might have also grouped terms implicitly while working on the task. Nonetheless, we did observe a few occurrences of grouping in the written responses. Figure 2 shows the work of a sophomore who connects segments of the expression to the physical scenario it describes, which is the core motivation for grouping. At the introductory level (n=157), grouping prevalence was 7% for distance, 6% for projection, and 6% for electric field. At the sophomore level (n=20), 30% of the students grouped for distance, 15% for projection, and 15% for electric field. At the junior level (n=17), 29% grouped for distance, 12% for projection, and 47% for electric field.

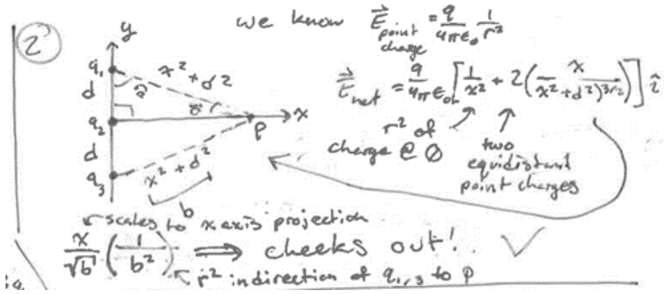


Figure 2: A sophomore's written work showing (successful) grouping.

V. Discussion and conclusion

In analyzing tasks that probe student use of evaluation strategies in problem solving, we documented students making sense of an expression by grouping terms into sub-expressions that have physical significance (e.g., Coulomb's Law). Because of the coupling of mathematical and physical meaning, grouping should be considered an expert-like skill, consistent with frameworks that model mathematical reasoning in physics (Bing & Redish, 2007; Brahmia, 2014; Wilcox et al., 2013). We distinguish this version of grouping from the procedural resource of the same name (Wittmann & Black, 2015), for which students were grouping terms to then perform mathematical operations in a separation of variables problem. The grouping identified here, while symbolically similar, is motivated by physical or geometric significance rather than algebraic convenience.

Moving forward, we will evaluate the similarity of grouping to deliberate or automatic chunking in the psychology literature (Gobet et al., 2001). We will also explore relevant theories or frameworks in the mathematics education research literature, including structural reasoning and justification (Harel & Soto, 2017; Harel & Sowder, 1998). Finally, it seems the prevalence of grouping as a strategy in validity checking varies with context; additional work will explore factors responsible for this variation including salience and the affordance of the equation (Heckler, 2011).

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Adapting the Norm for Instruction: How Novice Instructors of Introductory Mathematics Courses Align an Active Learning Approach with the Demands of Teaching

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As undergraduate mathematics instructors continue to implement active learning strategies and practices, researchers investigate the factors that contribute to classroom environments that are conducive to this approach. In this preliminary report, four "novice" instructors share their reflections, challenges, dilemmas, and personal growth from teaching introductory mathematics courses via an active learning approach. Instructors navigated institutional demands and innovative, "flexible" learning spaces to make reasonable pedagogical decisions. By examining the practicality of their decisions with respect to teaching norms and obligations, this study emphasizes the many resources and supports that instructors utilize to improve their teaching.

Keywords: Introductory Mathematics, Active Learning, Learning Spaces, Teaching Decisions

Introduction and Literature Review

Recently, there has been a general effort in science, technology, engineering, and mathematics (STEM) departments across the nation to implement active learning strategies at the university level (Association of American Universities, 2017; Conference Board of the Mathematical Sciences, 2016). While some researchers claim that active learning is clearly the best way to help students learn mathematics (Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt, & Wenderoth, 2014; Michael, 2006), others (including faculty and instructors) doubt the effectiveness and feasibility of these strategies (Le, Janssen, & Wubbel, 2018; Michael, 2007). Instructors' hesitancy or resistance to implement evidence-based teaching practices can be due to their personal perceptions of these practices, lack of support, institutional factors such as course coverage pressure and classroom layouts, or qualities of the innovative approach (Keller et al., 2017; Lund & Stains, 2015; Shadle, Marker, & Earl, 2017).

While many recent studies report on active learning practices and outcomes in upper-level mathematics courses (e.g. Larsen, Johnson, & Bartlo, 2013; Rasmussen & Wawro, 2017), fewer studies investigate active learning practices in introductory courses, which non-STEM majors are often required to take (Speer, Smith, & Horvath, 2010). The focus on upper-division courses is apparent in research on instructor preparation and professional development as well. Studies that do examine the preparation of instructors of introductory courses (e.g. pre-calculus) tend to focus on graduate teaching assistants (GTAs) rather than instructional faculty (e.g. Beisiegel, 2017; Milbourne & Nickerson, 2016).

Most empirical studies at the post-secondary level highlight programs and techniques that work for experienced instructors but not for novice instructors (Speer et al., 2010). Speer and colleagues claimed that these studies were not beneficial to novice instructors who wished to learn about the *practice* of teaching (e.g. planning lessons and making pedagogical decisions during class). This leads to limited research-based support for novice instructors, especially those who wish to change or improve their teaching practices.

Due to an increased interest in non-lecture pedagogies among faculty at the undergraduate level, researchers are trying to better understand what factors influence pedagogical decisions (Biza et al., 2016). Adding to the limited research in undergraduate settings, the K-12 literature contributes significant insights about the factors that influence teachers' instructional decisions, listing two of the main factors as experiential knowledge and institutional factors. Researchers are currently exploring whether factors related to the physical learning space may also influence instruction.

There is a popular notion that post-secondary instructors simply teach mathematics the same way that they learned it, since they generally have limited *formal* teaching preparation (Oleson & Hora, 2014). Similar claims had been made about K-12 teachers but were later countered as researchers came to recognize teachers' important experiential knowledge (Shulman, 1986). Indeed, the practices and pedagogical knowledge of post-secondary instructors are influenced by a multitude of factors. Instructional decision making is a complex, non-linear process and the oversimplification that instructors "teach the way they were taught" is a deficit view, because this view ignores the efforts that instructors make to improve their teaching (Oleson & Hora, 2014). While time spent in the classroom can have positive effects on instructors' experiential learning of teaching, it can be influenced by the demands and culture of their institution (Keller et al., 2017; Oleson & Hora, 2014). As undergraduate faculty are encouraged to employ evidence-based, non-lecture practices, researchers should pay attention to how this support and encouragement is carried out (Henderson & Dancy, 2008).

Instructors make pedagogical decisions based on demands imposed on them by external people and structures. Community college instructors in Mesa and colleagues' (2014) study stated that the course syllabus and student placements often influenced their pedagogical decisions. Course syllabi felt constraining because there was so much pressure to "cover" a vast amount of content. This encouraged instructors to use a traditional teaching approach (i.e., lecture) in order to fulfill their course obligations. Researchers lament that mathematics educators have long known of the effectiveness of active learning methods, but course designers and administrators are still hesitant to design courses that follow this structure (Reinholz, 2017).

The ecology of the learning environment includes factors that are a part of the physical space before teachers or students even enter the classroom (Doyle, 2014). Physical features of classrooms, such as layout, furniture arrangements, size, and technology, can affect student engagement and attitudes (Weinstein, 1979). For instance, the proximity of students to the front of the room can influence how often they interact with the teacher (Doyle, 2014), and, in post-secondary settings, greater distances between the instructor and students may imply that the instructor has more authority (Lim, O'Halloran, & Podlasov, 2012). The relationship between space and learning is still not well understood, but researchers are addressing this topic by first examining the relationship between space and pedagogy (McNeil & Borg, 2018).

Internationally, there is a persistent interest in the physical design of learning spaces in undergraduate courses (McNeil & Borg, 2018). Some researchers claim that the physical space does not influence instruction and that the relationship between space and pedagogy is more complex than a cause-and-effect explanation (Mulcahy, Cleveland, & Aberton, 2015). However, other studies found that the type of classroom played a role in the instructors' teaching moves (King et al., 2014; Stains et al., 2018). Stains et al. (2018) observed over 2000 STEM undergraduate classes across 25 U.S. institutions to better understand what factors correlate with different teaching approaches. Lecture dominated the instructor practices across all observations, taking up an average of 75% of the total class time. Stains and colleagues (2018) also examined features such as classroom layout and class size. They found that classrooms with non-traditional layouts or "flexible seating" were more likely to be associated with a student-centered instructional approach. However, they emphasized that a reformed layout did not necessarily affect or improve the pedagogy in that space. Thus, it is important to continue to explore the relationship between undergraduate pedagogy and space.

Theoretical Framework

The practical rationality of mathematics teaching framework, established by Herbst and Chazan (2011), helps researchers understand why some innovations are taken up while others are viewed as not feasible or not in alignment with the practice of teaching mathematics. Relying on Bourdieu's (1998) notion of *habitus*, or practical reason, practical rationality posits that there are demands and constraints in teaching, related to people, objects, and perceptions, that influence pedagogical decisions and actions.

Similar to many other professions, teaching places various obligations on those who teach. The demands of teaching are regulated by four professional obligations for instructors of mathematics: disciplinary, individual, interpersonal, and institutional (Herbst & Chazan, 2011). Disciplinary obligations require the mathematics teacher to be a steward of the discipline; that is, they must accurately convey and represent mathematics content, practices, and applications. Individual obligations hold the instructor accountable for individual students' behavioral, cognitive, emotional, and social needs. Interpersonal obligations involve the interactions between instructors and students and include the use of shared resources, such as discursive, physical, and social spaces and times. Institutional obligations observe the various components of the schooling administration, such as curriculum, schedules, and policies.

The four obligations acknowledge that instructors are members of a common practice, rather than individuals who disregard external factors of teaching. Instructors, as members of the practice of teaching undergraduate mathematics courses, are assumed to consider similar factors that are reasonable and relevant to their practice (Herbst, Nachlieli, & Chazan, 2011). Within the practice of teaching undergraduate mathematics, there are norms, or routines and actions that are common and expected. Breaching or rejecting an established norm is an active decision that instructors must make, since the norm is the customary and often comfortable way of teaching. Rejecting or adapting a norm is synonymous with challenging the customs of the common practice, which is why this action is often met with resistance.

The practical rationality framework guided the analysis of data. In particular, I focused on how instructors acknowledged and responded to classroom norms, especially while analyzing the obligations connected to their practice of teaching.

Research Questions

I address the following research questions in this portion of the study:

1. How did undergraduate mathematics instructors analyze, navigate, and react to classroom norms that they encountered while teaching via an active learning approach?
2. How did these instructors perceive that the demands of teaching introductory-level courses aligned with an active learning approach?

Methods

The data presented here are part of a larger qualitative, dissertation study. This study took place at a large, public, research-oriented university that recently invested in re-engineering classroom spaces as part of the Association of American Universities (AAU) initiative to improve undergraduate STEM teaching (AAU, 2017). As of Fall 2018, this university boasted a total of 30 innovative learning spaces distributed across campus, ranging in size from 24 to 264 students. This included 25 collaborative learning spaces (CLSs) and five flexible learning spaces, which are CLSs with adaptive furniture that allow for individual as well as group workspaces.

Participants of this study include four instructional faculty members of the mathematics department at this institution. Together, they taught three introductory courses: college algebra, pre-calculus, and a data-based course intended for social science majors and other non-STEM majors. Each instructor taught multiple sections of one or more of these three courses. These sections took place in a variety of classroom spaces, including CLSs, flexible learning spaces, large lecture halls, and smaller, traditional classrooms.

In the larger study, I observed each instructor's teaching at least four times using the Observation Protocol for Active Learning tool (Frey et al., 2016) and taking field notes. Observations informed lesson debrief conversations and subsequent interviews, all audio recorded. Lesson debriefs were casual, 10-minute discussions immediately following observations that resembled informal interviews (Hatch, 2002). Semi-structured, or formal, in-depth, interviews (totaling about 70 minutes per participant) built on prior conversations and the observations to understand the teaching perceptions and rationale of instructors.

This paper focuses on the analysis of lesson debriefs and interviews using multiple qualitative coding techniques. In particular, I used Saldaña's (2016) description of *concept coding* to highlight teaching norms and "instructional situations" (Herbst & Chazan, 2011) that instructors mentioned. These norms were guided by relevant research in undergraduate mathematics classrooms, as well as Herbst and Chazan's (2011) description of norms and their four professional obligations: disciplinary, individual, interpersonal, institutional.

Additionally, I utilized *versus coding* (Saldaña, 2016) to illustrate the conflicts, inconsistencies, dilemmas, and stressful situations that instructors described related to teaching college mathematics courses via an active learning approach. Occasionally, an instructor used the word "versus" to explain a decision, dilemma, or contrast that they observed. Sometimes, an instructor phrased a contrast in a more general way, such as "this or that", "on the one hand…on the other hand", "I chose X instead of Y", so the versus code was implicit from their discussion of the situation. I compiled all versus codes across instructors and sorted into emergent themes.

Preliminary Findings

The findings shared in this preliminary report illustrate the teaching practices of four undergraduate instructors who attempted to reject the norm of lecturing and instead teach introductory mathematics courses via an active learning approach. Although they shared a wide variety of experiences, they all considered themselves a "novice" in some way. For Charles, the fact that he had been teaching mathematics at the post-secondary level for over 40 years was irrelevant because this "new way" of teaching was so different and challenging for him. Maggie's 20 years teaching high school mathematics helped her to create her own engaging activities and lesson plans, but she continuously compared the teaching environment of high school to her current position as a first-year college instructor. Amber and Chelsea were both novices in a traditional sense, acknowledging that this was their first-year teaching college classes post-graduate school, but emphasizing that they focused on their "different experiences" and "better education" rather than their "lack of experience."

All four instructors took a different approach to teaching via active learning, but all expected a high level of participation and collaboration from students. They expressed that active learning was "easier" in some courses due to the structure of the course and amount of content coverage expected. For instance, Amber felt that her data-based course was conducive to daily classroom collaboration and introduced groupwork norms on the first day. Meanwhile, Charles gradually implemented more group-based activities in his college algebra sections but struggled to "break the mode" of lecturing in his pre-calculus sections. Although he claimed that the content coverage and student personalities in pre-calculus were not conducive to active learning, he later reconsidered when his students "requested" more opportunities for in-class collaboration.

One pedagogical factor that some instructors did not anticipate was the design and type of classroom that they taught in. Maggie taught two sections of pre-calculus back-to-back, one in a CLS and the other in a traditional "rows and columns" classroom. As she walked from the first classroom to the next, she reflected on the lesson and adapted it in her head for the second space, acknowledging that the physical layout of desks, tables, and resources greatly influenced how she approached her lesson. Similarly, Chelsea critiqued the design of her classroom for not being more aware of students, referring to practical features such as room size, table shape, and position of the projector screens. She claimed that students who sat in certain parts of her CLS were hard to interact with and these "weird issues that no one saw coming" were "implications of the physical space that affect mental learning."

All four instructors were reflective and critical of their teaching, carefully considering and comparing different pedagogical decisions and practices. This was apparent from the *versus coding*, which revealed a total of 168 instances of decisions, dilemmas, and misalignments. Major themes emerged from the versus instances, including considerations of students, dilemmas about specific teaching practices, and conflicts between planned lessons and the classroom space.

A more thorough discussion of these findings will be shared during the presentation. Specifically, I will provide an analysis across instructors to compare how they analyzed norms and teaching demands with respect to external factors and obligations.

Discussion Questions

This study emphasizes how researchers' view of an "experienced" or "novice" instructor may be too narrow at times; a new approach to teaching can make even a veteran instructor feel like a novice. The instructors in this study were extremely reflective and adaptive when planning lessons and teaching mathematics and brought considerable expertise to their practice; however, they still acknowledged the challenges of an innovative approach. While Amber was a novice instructor, she showed no vacillation in rejecting lecture-based norms and adopting practices for collaboration, weighing the disciplinary obligation of teaching mathematics well more heavily than external factors. This contrasted with Charles' hesitancy to implement collaborative activities in all of his classes, interpreting his students' learning preferences and behavior as inconsistent with groupwork during class (individual obligation), until they told him otherwise.

An active learning approach paired well with CLSs overall, but sometimes conflicted with institutional demands, such as content coverage and common exams (institutional obligations) and student attitudes and shared resources (interpersonal obligations).

I propose the following questions and prompts for discussion with the audience:

1. In addition to the examples I provided in my analysis, what are other typical or atypical norms for teaching undergraduate mathematics?
2. Are "typical" norms changing in the undergraduate mathematics education literature?
3. Do your institutions view physical classroom characteristics as pedagogical factors?
4. What are implications of this work for mathematics instructors who want to teach via active learning and possibly are new to this approach?
5. What are your suggestions for extending and furthering this work?

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Examining the Qualities of Schema in Topology

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According to Skemp (1979), a schema is a structure of connected concepts that determines the effectiveness of our director systems. He defines many qualities of a schema, including the strength of connections and the existence of high-order schemas. We use undergraduate-level Topology tasks to investigate two examples of rich schemas to learn more about these two qualities and to see what they may look like practically.

Keywords: Topology, schema

Background

In his book, Skemp (1979) lays out details for how a director system operates and influences our actions, both physical and mental. The effectiveness of our director systems in reaching our goal states depends on our schemas. "[O]ne requirement [of problem solving] is the ability to set aside existing pre-formed habits and make equally available for use the whole of an appropriate schema." (p. 174) Schema, however, is a term that can be elusive and difficult to define. We will use Skemp's (1979) description of schema.

> A schema is a structure of connected concepts. The idea of a cognitive map is a useful introduction, a simple particular example of a schema at one level of abstraction only, having concepts with little or no interiority, and representing actuality as it has been experienced. A schema in its general form contains many levels of abstraction, concepts with interiority, and represents possible states (conceivable states) as well as actual states. (p. 190)

This study is part of the first author's dissertation work where we investigate knowledge and learning in the area of Topology in order to understand the construct of schema. There is little research in the area of Topology before recent years (Cheshire, 2017). Some may argue that this is because Topology has no place in research on undergraduate mathematics education, but we disagree. Topology is a course that is offered at the undergraduate level, and even if it is not a requirement for graduation, it is usually a requirement if students are to continue on into graduate school. Students usually view Topology as highly abstract and difficult and we believe that research into why this is the case can be beneficial for education in topics in addition to Topology.

In a preliminary study, Berger and Stewart (2018) found evidence that most undergraduate students were still in a beginning stage of schema development for a basis by the end of their first semester of Topology. In order to try and understand why this is, they turned to Skemp (1979) and his list of schema qualities because "The qualities of our available schemas are crucial determinants of our success in action." (p. 191). Berger and Stewart (2019) investigated some of these qualities and discussed why they can be difficult for learners in a Topology context. The main objective for the current project is to understand Skemp's qualities of a schema so that they may be less abstract and more useful for future work. We work towards this goal by seeking out examples of "rich" schemas from Topology in order to gain more insight about what these qualities might look like practically.

Method

In this case study, we interviewed three graduate students whose research field is in or closely related to Topology. They were given four tasks that are considered to be introductory point-set Topology tasks for undergraduate or beginning graduate students. For each task, we asked them to think about the problem, explain it to us as if they were explaining to students of theirs, and then reflect on what prerequisite concepts, skills, or techniques they thought were necessary for undergraduates to possess when working through the task. Because the participants were experienced with conducting research related to Topology, we interviewed them with the perspective that they are experts in the field. The interviews were video recorded and later transcribed.

We then began an initial thematic analysis of the transcripts with open coding; searching for common themes among the responses (Maguire & Delahunt, 2017).

For this paper, we look at two of the participants, Jordan and Luke, and some of their ideas related to the first task (see Figure 1). We identify a few themes for each of them and a related schema quality for each. The two schema qualities that we focus on here are strong connections within a schema and a schema being of a high-order, making it more general.

1. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \left\{x \mid \tfrac{1}{2} < |x| < 1\right\}$$

$$B = \left\{x \mid \tfrac{1}{2} < |x| \leq 1\right\}$$

$$C = \left\{x \mid \tfrac{1}{2} \leq |x| < 1\right\}$$

$$D = \left\{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbb{Z}_+\right\}$$

Figure 1. Task 1.

The Case of Jordan

Jordan worked through the first task rather efficiently. He did consider set C longer than the other sets and eventually decided to skip the details of it and maybe would come back to it, which is referenced below. His work is found in Figure 2.

Jordan: Here is like a visual presentation of it and here's like this open-interval presentation. They have to be able to go back and forth between these things. So, that's one thing. You need to know the definition of the topology and the subspace topology and then you need to...so, like there's this notion of like an interior point to a set. So, a point's in the interior if there's like some open set around it. So like an alternative definition for openness is that every point's interior. So, you might find that helpful to make some of these arguments like this one. And I think that's what you need to do here [set C] although it doesn't seem as easy to me.

Interviewer: Yeah.

Jordan: Oh, I can finish that then [set C]. Yeah. I think that's exactly what you need. So, set theory, interior points, and then, of course you need to know the subspace topology. You need to know stuff about bases.

[Writing on board.]

Anything else I used?

Interviewer: Where did you use a basis?

Jordan: You need to know what the basic open sets in R look like to maybe argue stuff about interior points.

Interviewer: Okay. Yeah, that's true.

Jordan: Maybe something about sequences. I don't think this [set D] is like a kind of a normal set.

…

Jordan: Really what was going on here is using my set theory to go back and forth between different pictures. I don't know if people...okay, I don't do this. Maybe some people do. I'm not going to be able to tell you whether that's open or not unless I have some sort of visualization of that.

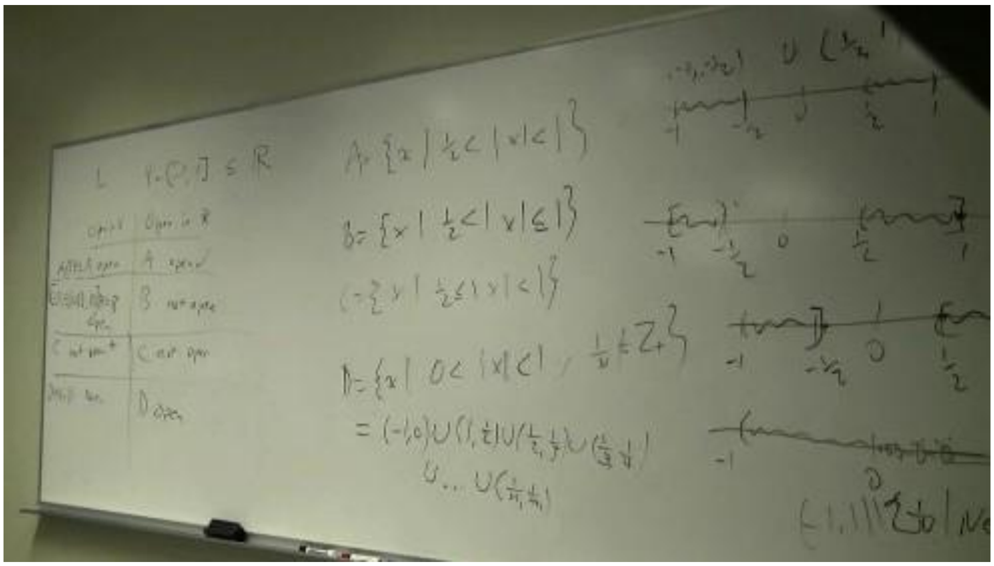

Figure 2. Jordan's board work.

Jordan shows that his Topology schema contains several connections with concepts like set theory, sequences, and interior points. For example, he talks about interior points and bases as a tool for deciding whether or not a set is open in a topological space, whereas a typical definition uses set inclusion (a set U is open in X if U is an element of the topology on X). We can see that these connections are strong, not just because he was able to complete the problems easily, but also because of his comfort with switching representations and utilizing alternative definitions. From this, it seems that one way of identifying strong connections within a student's schema is to notice the fluidity and frequency with which a student changes representations in a problem. Utilizing alternative definitions usually implies a movement between concepts, again showing a strength in the connections between the concepts.

The Case of Luke

Luke somewhat over complicated the problem when working. He did not say or do anything incorrect, but rather he tried to use higher-level concepts in thinking about set C and even thought about questions that weren't necessarily asked for in the given task.

Luke: Okay. So, and then you want, C can be, C says I can be at least a ½ away and then here. Okay. Then this one… There may be a better way to talk about and get a basis for the subspace topology. … I want to say if you want a basis for the subspace topology, you can just intersect all the basis elements of the regular topology [with Y] and if you know that, then it's clear that this is not going to be open because it's...anything around here contains an open set. ... I want to say, it's not open and not closed.

Interviewer: In Y or R?

Luke: In Y.

Interviewer: Okay.

Luke: You could also say like...You could put limit points. So, like, it's not closed because it doesn't contain this limit point and it's not open because this is in my set and anything around that also contains stuff that's...I'm getting too much stuff. I always have to get at least an open interval around there. Okay. Yeah. If you had a closed bracket then it would be closed in this thing. Because you have closed sets that intersect your subspace. … It doesn't tell you it's open though. Okay. I want not-open as opposed to not-closed.

...

Interviewer: So, what kinds of things do you feel like they need at the level? What kind of background knowledge do they need to have to be able to think about this problem?

Luke: Okay. I think you want to have different definitions of closed and open on hand so you want to know the limit point definition and think about your complements and you want, if you have a basis, you should use it. That's the main things. Maybe you don't need to know what a sub-basis is. Yeah.

Interviewer: Okay. Cool.

Luke: Clever things like a set is closed if it's equal to its closure. It's like, it'll come up occasionally.

Luke tried to think about a basis for the subspace topology to help with this problem, which is not necessary to actually complete the problem. He also spent a good amount of time trying to answer if the set C was closed in addition to deciding if it was open, which was more than the task asked for. Both of these things are typical characteristics of *doing* mathematics: using lemmas and 'clever things' to simplify a proof and posing additional questions. A schema quality that seems apparent with this participant is that he possesses a high-order schema. "By a high-order schema, we mean one containing high-order concepts...This determines its generality, and thus the extent of its domain." (Skemp, 1979, p. 190) In this example, we consider a basis to be one high-order concept that the participant has in his Topology schema. He views a basis as a useful tool: "If you have a basis, you should use it." This possession of a high-order schema seems to be backed by this participant's habits when doing mathematics.

Discussion Questions

The next steps are for us to continue coding through the data for mathematical themes and rich schema qualities. Our objective is to learn more about Skemp's schema qualities through any patterns that emerge and to incorporate them in a useful theoretical framework.

1. What other themes can serve as evidence of Skemp's (1979) qualities of a schema?
2. Are all of these themes observable?

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A Conceptual Blend Analysis of Physics Quantitative Literacy Reasoning Inventory Items

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Mathematical reasoning flexibility across physics contexts is a desirable learning outcome of introductory physics, where the "math world" and "physical world" meet. Physics Quantitative Literacy (PQL) is a set of interconnected skills and habits of mind that support quantitative reasoning about the physical world. The Physics Inventory of Quantitative Literacy (PIQL), which we are currently refining and validating, assesses students' proportional reasoning, co-variational reasoning, and reasoning with signed quantities in physics contexts. In this paper, we apply a Conceptual Blending Theory analysis of two exemplar PIQL items to demonstrate how we are using this theory to help develop an instrument that represents the kind of blended reasoning that characterizes expertise in physics. A Conceptual Blending Theory analysis allows for assessment of hierarchical partially correct reasoning patterns, and thereby holds potential to map the emergence of mathematical reasoning flexibility throughout the introductory physics sequence.

Keywords: physics, quantitative reasoning, mathematization, conceptual blend, assessment

Introductory physics is characterized by using simple mathematics in sophisticated ways. Physics equations tell stories. For example, to an expert, $a = -9.8\ m/s^2 + -b(2m/s)$, the equation is recognized as describing the acceleration of an object that is in free fall and experiencing air resistance. The two forces causing the acceleration in this case are the gravitational force and the force of air resistance. The coordinate system is set by the choice of sign for the acceleration due to gravity, "downwards" in this case is chosen to be in the negative direction. At this instant, the object is moving upwards at 2 m/s since the one-dimensional velocity vector is in the positive direction. The negative signs in front of each of the two terms on the right-hand sign of the equation carry different meanings. The first negative sign is an arbitrary choice that determines the coordinate system, while the negative sign on the second term indicates that whatever the sign of the velocity is, this contribution to the (vector) acceleration will be in the direction that is opposite to the direction of motion. So, the sign of the velocity, and the sign of the contribution from gravity must agree with the choice of coordinate system, but the sign in front of the second term does not because it indicates opposition, which is independent of the coordinate system used. Black and Wittman provide evidence that many of these nuances are lost on physics majors at the junior level, even though the mathematics involved is at the precaclulus level, and they are well beyond that stage in their mathematics course taking (Black & Wittmann, 2009).

Even as students move beyond the introductory sequence to using newly learned mathematics (calculus, linear algebra, differential equations), there is mounting evidence that although they don't struggle to execute the mathematics, they don't really understand why they do what they do, and they'd like to (Caballero, Wilcox, Doughty, & Pollock, 2015).

In the work described in this paper, our intention is to understand and assess the nature of student reasoning about the quantitative models that make up introductory physics, as the air resistance equation above exemplifies. Sherin's Symbolic Forms provides a framework of the kind of reasoning most physics instructors would like to see in their students as the outcome of having taken an introductory physics course (Sherin, 2001). While Symbolic Forms emerge from observations of students engaged in successful problem solving, the students in Sherin's study are high achieving students in their last semester of introductory physics at an elite institution. Most students who take introductory physics are less sophisticated mathematically and come from less educationally privileged backgrounds. We consider Symbolic Forms as a *learning objective* of the introductory physics course.

In the next sections, we describe the Physics Inventory of Quantitative Literacy (PIQL), which is designed to help assess this learning objective, and Conceptual Blending Theory (CBT), which provides a framework for understanding the integration of mathematical and physical reasoning (Fauconnier & Turner, 2008). The purpose of this paper is to examine items from the PIQL through the lens of CBT. We argue that CBT analysis lends itself to the development of an inventory that promotes a growth mindset associated with mathematical reasoning in physics.

Physics Inventory of Quantitative Literacy (PIQL)

Physics, as perhaps the most fundamental and the most transparently quantitative science discipline, should play a central role in helping students develop quantitative literacy (Steen, 2004). We coin *Physics Quantitative Literacy (PQL)* to refer to the rich ways that physics experts blend conceptual and procedural mathematics to formulate and apply quantitative models. Quantification is a foundation for PQL, using established mathematics to invent and apply novel quantities to describe natural phenomena (Thompson, 2010; Brahmia, 2019). These quantities then allow for the investigation of patterns and relationships, which in turn anchor the quantitative models that are the hallmark of physics as a discipline. PQL, involving sophisticated use of elementary mathematics at least as much as elementary applications of advanced mathematics, is more challenging for students than many instructors may realize (Rebello, Cui, Bennett, Zollman, & Ozimek, 2007; Brahmia & Boudreaux, 2016).

Although the mathematics involved in introductory physics quantification is typically algebra or arithmetic, a conceptual understanding of this mathematics is fundamental to the sophisticated task of reasoning in the context of strange new physics quantities. Despite its importance, little work had been done to measure progress of PQL in introductory physics students as a result of instruction. The Physics Inventory of Quantitative Literacy (PIQL) is a valid and reliable reasoning inventory that is under development by the authors with the intention of being used to track changes in student quantitative reasoning over the course of the introductory physics sequence. Carefully validated forced-response questions probe student reasoning about quantification in introductory-level physics contexts. Analysis of PIQL results allows us to track progress in students' PQL and determine features of PQL that are particularly challenging.

We've identified three facets as the basis of quantification in introductory physics: proportional reasoning, reasoning about signed quantities, and co-variational reasoning. The choice of these three facets as foundational for physics quantitative literacy was supported by work done in both physics education research and mathematics education research. Much of our thinking about the domain of co-variational reasoning originates in mathematics (Carlson, Oehrtman, and Engelke, 2010). The development of the Pre-calculus Concept Assessment served as a foundation for our thinking about covariation in physics.

Concept and reasoning inventories, by nature, use expert-like reasoning and understanding as a metric with which to assess student reasoning. In the process of developing items for the PIQL, we developed an organizational framework by which to characterize the different uses of the negative sign in contexts of introductory physics (Brahmia, Olsho, Smith & Boudreaux, 2018). This framework proved useful not only in the development of assessment items for the PIQL, but also, potentially, for instructors and researchers in characterizing student understanding of the different meanings of the negative sign in physics contexts. We continued work on a framework for the natures of proportional reasoning in introductory physics (Boudreaux, Kanim & Brahmia, 2015) and have begun work for an analogous framework for co-variational reasoning.

In its current state, the PIQL items (including the distractors) are representative of expert natures of proportional reasoning, co-variational reasoning and reasoning with signed quantities. The item distractors emerge from open-ended version analysis of student responses, which are then refined to align with validated expert natures. We conduct student think-aloud interviews to refine or reject items. We use CBT (Fauconnier & Turner, 2008) as a framing for the analysis of student responses to help inform both item refinement and implications for instruction.

Results from the Physics Inventory of Quantitative Literacy (PIQL) administered in large-enrollment calculus-base courses - when the items are scored dichotomously (either all correct or incorrect) - indicates only a very modest improvement in students' quantitative reasoning as a result of introductory-level physics instruction. By incorporating patterns that emerge from a CBT analysis, we are encouraged at the potential to recognize hierarchy in students "incorrect" answering, and thereby be able to better understand, and assess, the development of PQL over the course of the introductory physics sequence.

Conceptual Blending Theory

Conceptual Blending theory (Fauconnier & Turner, 2008) describes a cognitive process in which a unique mental space is formed from two (or more) separate mental spaces. The blended space can be thought of as a product of the input spaces, rather than a separable sum. We view the development of expert mathematization in physics occurring not through a simple addition of new elements (physics quantities) to an existing cognitive structure (arithmetic), but rather through the creation of a new and independent cognitive space. This space, in which creative, quantitative analysis of physical phenomena can occur, involves a continuous interdependence of thinking, most of which is subconscious, about the mathematical and physical worlds.

The following are elements found in any blend (see Fig. 1a). This static diagram represents connections that activate and deactivate. It is not depicting actual stages in a temporal way.

- Input spaces: contains the concepts involved
- Generic Space: structure that the inputs share, and maps onto input spaces
- Blended Space: related to generic space but contain more structure in which the inputs are indistinguishable
- Projections: represented by lines connecting the rectangles; project in either direction

As an example of CBT analysis, we present an abridged version of their Buddhist Monk problem. The reasoning is abstract and cognitively complex; its analysis is shown in Fig. 1(b).

> A monk begins at dawn one day walking up the mountain, reaches the top sunset, stays several days then he begins at dawn to walk back and reaches the bottom at sunset. Make no assumptions about his starting or stopping or about the pace of the trips. Is there a place on the path that the monk occupies at the same hour of the day on the two separate journeys?

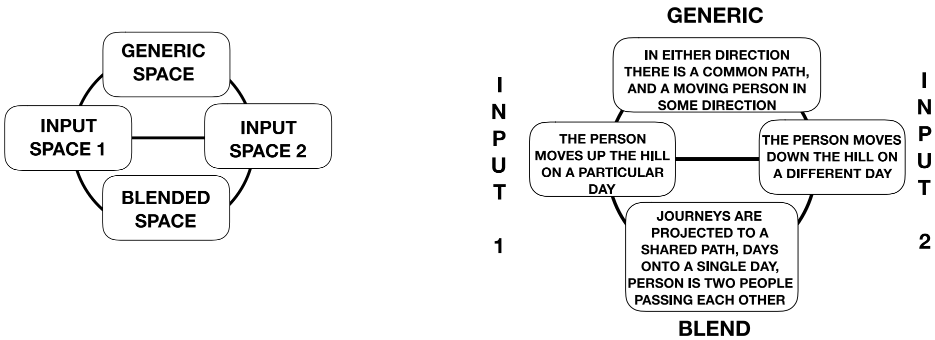


Figure 1. (a) Elements of a typical conceptual blend, (b) CBT analysis of Buddhist Monk problem

CBT Analysis of PIQL Items

In what follows, we present a CBT analysis of Alex and Jessie solving two PIQL proportional reasoning items to illustrate our method. We use dotted lines in the CBT analysis to indicate unstable knowledge.

PIQL Item 1

A miner is trading steel for lead. According to the current exchange rate, each kilogram of lead is worth 1.6 kilograms of steel. The miner trades M kilograms of steel. Which of the following expressions helps figure out how many of kilograms of lead the miner will get?

a. $M \cdot 1.6$ b. $M/1.6$ c. $1.6/M$ d. $1/(1.6 \cdot M)$ e. None of these are helpful.

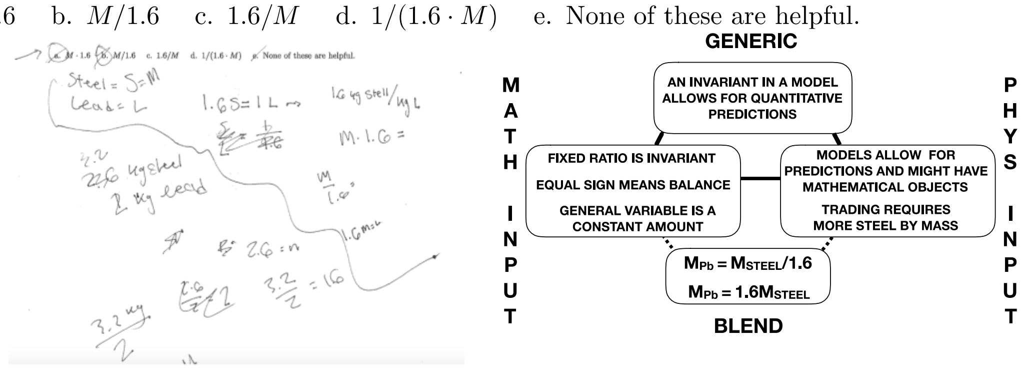


Figure 2. (a) Alex's work showing multiple attempts to make sense of the general variable "M", (b) CBT analysis

Although Alex's reasoning, when considered dichotomously as correct or incorrect, is in fact incorrect, there are resources used that are productive here. Alex recognized that they were looking for an invariant quantity, but their reasoning broke down when trying to reason with a general variable. In physics, we often pose questions in which the value of a quantity is represented by a general variable. Its value doesn't change during the problem, but it can take on any value. Many students, like Alex, struggle to activate mathematical reasoning that is familiar to them when they are asked to reason with quantities represented as general variables. There is weakness in the projection between the blended space and the input spaces. The generic space is stable though, and can be considered a resource.

PIQL Item 2

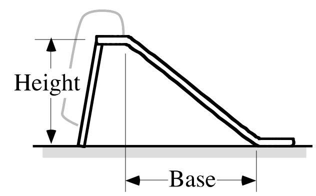
Interviewer: Describe your thinking as you find a solution to this problem.

Jessie: Okay, so in this problem, we essentially have a triangle and we know two sides of the triangle and can find the third. And if the third side of the triangle, like the hypotenuse is longer, then it'll be like less steep. It's like the taller it is and then narrower the base, the steeper the actual slope of the slide. And you can just Pythagorean theorem this.

You are purchasing a slide for a playground and would like to get the steepest one you can find. For four different slides, you have the measurements of the length of the base of the slide (measured along the ground), and the height of the slide.
You decide to use this information to rank the slides from **most steep** to **least steep.** Which of the following choices is the best ranking?

Slide	Base	Height
A	8 ft	12 ft
B	5 ft	9 ft
C	6 ft	9 ft
D	12 ft	8 ft

a. $A > B = C > D$
b. $B > C > A > D$
c. $A = B > C > D$
d. $B > A = C > D$
e. $A = D > C > B$



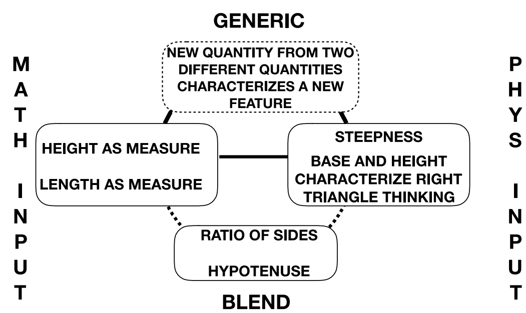


Figure 3. CBT analysis of Jesse's reasoning

The most common student responses to this question are shown in the blend in Fig. 3. Jessie activates the generic space shown in Fig. 3, but it is unstable; they aren't creating an independent quantity. Jessie shows weakness in the projection process between the blend and each of the two input spaces. They never reconcile that the hypotenuse is appropriate only if the height is held constant, in which case it is also unnecessary since the length alone would represent the steepness. In addition, they fall back on a common heuristic of a physics classroom, where a productive resource for Jessie when working with vector components is to "Pythagorean theorem this" to find a measurable quantity (e.g. speed from the components of a velocity vector).

Discussion

This CBT analysis facilitates parsing out specifically where student reasoning is productive, and where problems lie. Invariant reasoning is a resource for Alex, and Jesse recognized the need to create a single new quantity from two others. These resources can seed future interventions. In both cases, we see the projection between the blend and the input spaces is unstable, and for Alex (and many students) quantity expressed as a general variable destabilizes reasoning. Neither student was completely wrong. The CBT analysis provides seeds on which reasoning can grow, providing a potential pathway to help students strengthen their PQL.

Acknowledgments

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Would You Take Another Inquiry-Based Learning Mathematics Course? Links to Students' Final Exam Grades and Reported Learning Gains

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Despite active learning's promise for student gains across cognitive, affective, and social domains (Laursen et al., 2011), lecture-based teaching dominates instruction in college mathematics courses in the United States (Stains et al., 2018). In this study, college students enrolled in a Calculus I course responded to an inquiry-based learning (IBL) pedagogical approach via a survey that included Likert-scale items and open-ended questions. We sought to determine if there was a relationship among student performance in the class, students' reported learning gains, and whether or not they would take another IBL mathematics course. Those who responded positively to IBL (49%) performed better on the final exam than those who responded negatively (35%). Additionally, over 55% of those who responded positively reported gains in their ability to work on mathematics collaboratively and communicate mathematical ideas whereas less than 30% of those who responded negatively to IBL reported similar gains.

Keywords: active learning, inquiry-based learning, calculus, student surveys

In traditional, lecture-based courses, communication typically involves the teacher alone posing questions, evaluating student responses, and conveying meaning "while students are the passive recipients of the teacher's 'unalterable' messages" (Knuth & Peressini, 2001, p. 9). Though this practice is familiar and comfortable for students, research indicates that students learn more when the class structure provides active learning opportunities in lieu of passively listening to lectures (Freeman et al., 2014; Laursen, Hassi, Kogan, Hunter, & Weston, 2011). Active learning encompasses a broad range of classroom practices, such as group work, student presentations, and whole-class discussions. By exploring mathematical concepts and discussing their thinking with others, students can enhance their conceptual understanding of mathematics, develop positive mathematical identities, and improve their communication and collaboration skills (Laursen et al, 2011; Stein, Engle, Smith, & Hughes, 2015; Walshaw & Anthony, 2008).

In this paper we report results of a study focused on inquiry-based learning (IBL), which is "a form of active learning in which students are given a carefully scaffolded sequence of mathematical tasks and are asked to solve and make sense of them, working individually or in groups" (www.inquirybasedlearning.org). This study investigated how college students enrolled in a Calculus I course responded to an IBL class structure. In particular, we ask: What reasons do students provide for why they would or would not take another IBL mathematics course? What relationship exists among student performance in the class, students' reported learning gains, and whether or not they would take another IBL mathematics course?

Literature Review

Despite active learning's promise for student gains across cognitive, affective, and social domains (Laursen et al., 2011), traditional lecture-based teaching is the dominant form of instruction in college mathematics courses in the United States (Stains et al., 2018). Research has found that students develop deeper conceptual understanding of mathematics when they work collaboratively and communicate with others, and furthermore, cooperative learning seems to promote positive interpersonal relations, motivation to learn, and self-esteem among students. (Bossert, 1988; Knuth & Peressini, 2001). In IBL math classrooms, students explore mathematical concepts, share mathematical authority, and use communication as a thinking and learning tool to generate mathematical knowledge (Knuth & Peressini, 2001).

Researchers at the University of Colorado Boulder (Laursen et al., 2011) found that students in IBL mathematics courses reported their own active participation and in-class interactions as the most helpful features of the IBL classroom. Students described developing a better understanding of mathematics through figuring out mathematics on their own rather than being told the punchline. Additionally, working with their classmates promoted improvement in students' collaboration skills, enjoyment doing mathematics, and use of communication to develop deeper understanding. However, some students in IBL courses noted that working in groups, the workload, and the frustration they felt when they could not solve a problem hindered their learning (Laursen et al., 2011). Our study sought to explore the connections among the features of an IBL Calculus course that students identified as influencing their learning, affective gains students reported, and students' final exam scores.

Method of Inquiry

Context and Participants

This research was set within two sections (n = 50 and 44) of an IBL Calculus I course in a large, public, midwestern university, whose mathematics department is considering a transition to less lecture-based introductory mathematics courses. Every semester since Spring 2018, the Mathematics Department at XX University has piloted 2–3 sections of "active calculus" in which, consistent with the structure and philosophy of IBL, students spend most of their class time working in small groups facilitated by the instructor and two learning assistants (LAs) to solve problems that emphasize key concepts and relationships of calculus. While groups worked on tasks, the instructor and LAs circulated to troubleshoot student difficulties, pose questions, check student understanding, encourage group discussions, and generally encourage student thinking and autonomy in problem solving rather than directly answer student questions.

Each class period, the students completed tasks designed to build conceptual understanding of a particular concept. For example, early semester tasks focused on a position function for an object in motion, $s(t) = 64 - 16(t - 1)^2$, and required students to graph the function, describe how its motion changed, calculate average velocity over several intervals, connect velocity to the position graph, and ultimately, connect average and instantaneous velocity to the slopes of lines, limits, and the derivative. The ideas and key concepts were developed and articulated by the students before the instructor added formal definitions and notation to their ideas.

The two sections combined had 94 students and, of those, 84 agreed to participate in this study. Of the participants, 49 were females and 35 were males; 55 were first-year students and 29 were upper classmen; and 61 majored in mathematics or science, 19 in engineering or computer science, 3 in education, and 1 was undecided.

Data Collection and Analysis

Two weeks prior to the semester's conclusion, we administered a survey to gather student perceptions of the course. The survey consisted of multiple-choice Likert scale questions and open-ended questions, all of which explored various aspects of students' Calculus I experiences: their perceptions of the structure of the course, their interactions with the instructor and the learning assistants, and their understanding of specific concepts and procedures. We collected final exam scores. The exam questions were built from a pre-determined list of required topics used every semester and included a combination of open response, fill-in-the-blank, and multiple-choice questions that assessed both conceptual understanding and procedural fluency.

To analyze the survey data, we open coded open-ended questions and performed descriptive statistics on the Likert scale items and final exam data. We further looked for trends in the Likert-scale survey items, the open-ended survey items, and final exam data across participants who would take another IBL math course and those who would not. For this paper, we analyzed the following open-ended survey question: Explain why you would or would not take another math course structured this way.

Results

First, to determine participants' responses to the IBL approach to calculus instruction, we examined their 5-point Likert-scale responses to the question, "Would you take another math class structured like this (using group work on problems with professor and LAs available to assist followed by class discussion) again rather than lecture?" Based on participant responses to this question, we identified three categories of participants: those who would "definitely" or "probably" take another IBL math course (Likert scale response of 4 or 5), those who would "definitely not" or "probably not" take another IBL math course (Likert scale response of 1 or 2), and those who "might" take another IBL math course (Likert scale response of 3). We hereafter refer to these groups as IBL-repeaters, IBL-non-repeaters, and IBL-maybe-repeaters, respectively. The average Likert scale response across all participants was 3.29 (a slightly better than a neutral response) with 30 participants (35%) identifying as IBL-non-repeaters and 41 participants (49%) identifying as IBL-repeaters. The main reason IBL-non-repeaters were not interested in taking another IBL math course was their desire for lecture. IBL non-repeaters made 34 references to needing more lecture and 11 participants specifically stated they learn better though lecture; as one described, "I learn better through lecture so group work just made me confused never knowing the material before the worksheet other than teaching ourselves the preview the day before." Others indicated they wanted more lecture, but mainly "to establish the new topic." The two most common reasons IBL-repeaters cited for wanting to take another IBL math class were the interactive nature of the class and their opportunities for learning. One participant described how the course structure supported her understanding of mathematics:

> Group work really helps me understand what we are doing. In lecture it is easy to zone out and never do practice problems so when it comes time for the test I feel very unprepared. Doing problems every day and getting help from others makes me feel a lot more prepared.

Another participant's response highlights the way interacting with classmates was helpful:

> Because I was in a group, I had more fun learning this complicated Math. Sitting in a lecture every single day with no interaction does not help me learn at all. Being in a group also makes me feel comfortable asking questions, because I don't feel like I am stopping an entire lecture just for my, probably simple, question.

Of the 84 participants, 13 (16%) said they "might" take another IBL math course. Their comments displayed a mix of positive and negative reactions to the structure of the course. The IBL-maybe-repeaters wanted more lecture for new material and felt more comfortable with lecture even though they enjoyed the interaction in the IBL class. They also noted the important influence of the quality of their group – when group members worked well together, IBL-maybe-repeaters tended to have positive comments about the course structure, and they had more negative comments when they identified their group as not cohesive.

Second, to determine participants' reported gains in learning and affective domains, we examined their responses to Likert scale questions in which they rated their areas of improvement. Table 1 shows what gains each of the three groups reported making over the semester. IBL-repeaters reported gains in working on math with others, willingness to seek help from others, and communicating math with others – all areas in which the course was designed to support gains. Over 50% of the IBL-repeaters reported gains in these areas, whereas in all categories but one, less than 30% of the non-IBL-repeaters reported gains.

Table 1. Areas in which participants reported making gains organized by likelihood of taking IBL course again.

In what areas did participants report making gains?						
Areas of Participants' Self-Reported Gains	Take IBL Again (n = 41)		Not Take IBL Again (n = 30)		'Maybe' Take IBL Again (n= 13)	
	n	**%**	***n***	**%**	***n***	**%**
Working on math with others	31	75.6	8	26.7	4	30.8
Willingness to seek help from others	30	73.2	10	33.3	4	30.8
Understanding of math concepts	28	68.3	3	10	4	30.8
Working on math on my own	27	65.9	7	23.3	3	23.1
Communicating math ideas or processes	26	63.4	8	26.7	4	30.8
Persisting to solve challenging problems	22	53.6	5	16.7	5	38.5
Confidence I can do math	24	58.5	4	13.3	6	46.2

Third, to determine participants' mastery of course content we examined the final exam data for each of the groups (Table 2). The average exam score for IBL-repeaters was 71.9% as compared to the 62.8% exam average of non-IBL-repeaters. Of the 41 IBL-repeaters 16 scored above a 78% and 9 scored above a 68% on the final exam. Though 61% of the IBL-repeaters earned a score above the mean for all Calculus I students from that semester, only 35% of non-IBL-repeaters scored at the same level and only 10% of those scored in the highest quartile. Exam scores suggest that those who were successful in the course would take an IBL course again and those who were not successful would not take an IBL course again.

Table 2. Quartiles of participants' exam scores organized by likelihood of taking IBL course again.

How did participants who would or would not take an IBL course again score on the final exam?						
Exam Score Range (Quartiles)	Take IBL Again (n = 41)		Not Take IBL Again (n = 30)		'Maybe' Take IBL Again (n = 13)	
	n	**%**	***n***	**%**	***n***	**%**
[0 – 59.6)	8	19.5	13	43.3	0	30.8
[59.6 – 68.75)	8	19.5	6	20.0	6	46.2
[68.75 – 78.25)	9	22.0	8	26.7	3	23.1
[78.25 – 100]	16	39.0	3	10	4	30.8
Exam Score Mean	**71.9%**		**62.8%**		**72.8%**	

Discussion

The small size of the IBL-maybe-repeaters group (n = 13) relative to the other two groups limits the claims we can make about their perceptions of course, and hence, we will focus our discussion on IBL-repeaters and IBL-non-repeaters. The results paint a coherent picture of how well this course structure worked for IBL-repeaters and how poorly it worked for IBL-non-repeaters. Of 84 participants, 49% identified as IBL-repeaters. Based on their responses to open-ended questions, it was important to them that this course structure encouraged and supported interaction with their classmates, which in turn helped them improve their understanding of calculus. Furthermore, the majority of IBL-repeaters reported gains in mathematical communication and cooperative learning skills – factors that this course structure was intended to support. Additionally, 68% of IBL-repeaters reported gains in understanding or making sense of mathematical concepts, which was corroborated by their relatively high average final exam score (71.9%). Thus, the students who were successful in this course structure appreciated the interaction with and knowledge offered by their classmates. They parlayed that into valuable gains in working and communicating with others and they demonstrated proficiency with calculus content via their final exam scores.

On the flip side, the 30 (35%) IBL-non-repeaters fixated on the lack of lecture in the course. Their comments indicated they thought they learned better from lecture or needed lecture for new material. These students did not seem to appreciate the opportunity to work with and learn from their classmates. Very few of the IBL-non-repeaters reported gains in cooperative learning or understanding mathematics. Additionally, their performance on the final exam (average 62.8%) indicated a lack of proficiency with calculus content and their average final exam score was nearly 10 percentage points less than the IBL-repeaters who embraced the course structure. Thus, the IBL-non-repeaters did not capitalize on the collaborative nature of the course structure or demonstrate understanding of the course material at the same level as the IBL-repeaters. We wonder what additional supports the IBL-non-repeaters needed to be successful.

There was a clear distinction in responses from students in terms of their appreciation of collaborative work and the opportunity to figure things out on their own versus their desire for lecture and need to be explicitly shown how to do problems. We posit four potential explanations for our findings. First, students may have felt lecture was needed due to insufficient in-class support. However, LAs established an instructor-to-group ratio no worse than 1-to-5 which suggests support was available when needed. Second, IBL-non-repeaters' lack of comfort and familiarity with the course structure may have prevented them from using resources effectively. Students did not recognize classmates' knowledge nor the benefit of communicating their mathematical ideas. These students might need more support in learning to work effectively with a group and in understanding how the course structure relies on student autonomy. Third, their idea of learning math (e.g., copying a procedure first modeled by an instructor) was incongruous with IBL. Survey comments showed that IBL-non-repeaters did not value instructor questioning to determine the root of their struggles in lieu of lecturing with directive answers to questions. In addition to teaching mathematics, instructors may need to teach students what it means to learn mathematics (autonomous problem solving, making sense of and communicating mathematical ideas, not memorizing and executing procedures). Fourth, students' difficulties may have been exacerbated by this course structure in ways that a traditional lecture course would have masked. In an IBL course, students are challenged daily to do mathematical work in ways they are not challenged in a lecture-based course. Without that challenge students are not put in the uncomfortable position of recognizing what they do not understand or cannot do independently.

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Active Learning Approaches and Student Self-Confidence in Calculus: A Preliminary Report

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In this report we present the first collection of results of a study of student attitudes towards mathematics during the first year of implementation of a fully active learning in mathematics (ALM) approach in Calculus I at a large, urban, research intensive (R1) institution. Students were randomly assigned to either a treatment section or control section with students in the treatment group participating in the new curriculum. The Attitudes Towards Mathematics Inventory (ATMI) was used to measure student attitudes at the beginning and end of the course and the results compared. The ALM curriculum was found to increase student confidence while the traditional approach decreased it.

Keywords: active learning, calculus, student attitudes and beliefs

Calculus acts as a transition point for students in almost every science, technology, engineering or mathematics (STEM) degree program. Studies such as the *Characteristics of Successful Programs of College Calculus* (CSPCC) have shown that students' experiences in Calculus I have significant impacts on their pursuit of STEM majors and on their attitudes toward mathematics (Bressoud, Carlson, Mesa & Rasmussen, 2013). At the same time, recent work has noted that classes with components that involve active learning in mathematics (ALM) tend to improve student success outcomes (Freeman, et al, 2014) and mathematics education researchers are making a case for increased active learning in mathematics (Laursen & Rasmussen, 2019) and in particular noting success in introductory courses (Rasmussen, et al, 2019). In this work, we present a discussion of results from the first year of implementation of an ALM approach in Calculus I at a large, urban, research university. This approach integrates the Modeling Practices in Calculus (MPC) curriculum, a student-centric curriculum where students emulate the practices of mathematicians in the classroom to learn Calculus I. These practices include but are not limited to: students working in groups to facilitate sense making; building a classroom community; use of multiple representations; fostering constructive perseverance; developing problem-solving skills; and building proficiency with terminology, language constructs and symbols.

Literature Review

Student attitudes towards mathematics play an important role in their engagement with the subject. There have been several studies of students' attitudes toward mathematics. One of the most common instruments used in research is the Fennema and Sherman (1976) instrument which consists of nine subscales and assesses many aspects of student's attitudes and feelings towards mathematics and the learning of mathematics. Unfortunately, this instrument had 108 items, which made it logistically time-consuming to administer to students. Adediwura (2011) made use of only four of the Fennema-Sherman scales (1976) believed to be the most relevant to students' learning. They designed and validated a 17-item survey to measure students' attitudes toward mathematics in four domains: confidence, perceived value of mathematics, interest in mathematics, and negative feelings towards mathematics. Adediwura (2011) confirmed this four-factor structure using a sample of secondary school students. Tapia and Marsh (2004) reported the development of a new instrument, called the Attitudes Toward Mathematics Inventory (ATMI), to measure high school students' attitudes toward mathematics. Their instrument was originally composed of 49 items that measured six domains: confidence, anxiety, value, enjoyment, motivation, and parent-teacher expectations. Items were scored using a five-point Likert Scale, with responses ranging from "strongly disagree" to "strongly agree." An exploratory factor analysis was run and four factors were identified: value, self-confidence, enjoyment, and motivation (Tapia, 1996). To determine whether the four-factor model would hold for a population of college students, a confirmatory factor analysis was then conducted by Tapia and Marsh (2002). Results from this analysis on 134 undergraduate mathematics students indicated that the four-factor model holds. High Cronbach alpha coefficients were also reported for each subscale ($\alpha_{\text{self-confidence}} = 0.96$, $\alpha_{\text{value}} = 0.93$, $\alpha_{\text{enjoyment}} = 0.88$, $\alpha_{\text{motivation}} = 0.87$). These scores indicate a high level of repeated measure reliability for both the total instrument and the four subscales.

Faculty and instructional practices play an important role in shaping student attitudes towards mathematics and in particular student experiences in calculus have a substantial impact on their attitudes. The Characteristics of Successful Programs in College Calculus (CSPCC) project, Sonnert et al. (2015) found that faculty who employed 'good teaching' practices (e.g., allowed time to understand difficult ideas, help outside of class, clarity in presentation and answering questions) had the most positive impact on student attitudes, particularly with those students who began with a weaker initial attitude. Bressoud et al. (2013) found that 'good teaching' practices are most strongly correlated with maintaining student enjoyment of and confidence in mathematics. This work found that faculty who employed 'ambitious teaching' practices (e.g., group work, student explanation of thinking, word problems, whole-class discussion) had a small negative impact on student attitude but that this 'ambitious teaching' had a more positive effect on students who already enjoyed a positive attitude toward mathematics. Moreover, without good instructor to student relationships, 'ambitious teaching' practices could be counterproductive (Bressoud et al., 2013). Sonnert et al. (2015) suggest more probing into the impact of 'ambitious teaching' on students' attitudes, particularly because teaching characteristics were found to interact with class size. In this study, we implemented a fully active learning in mathematics (ALM) approach in calculus that incorporates all of the 'ambitious teaching' characteristics (as identified in the CSPCC study). The approach integrates the Modeling Practices in Calculus (MPC) curriculum and an instructional approach that is student-centered and studio-based. It is intentionally designed as a culturally appropriate model, as it allows students to try out their ideas in a low-stakes environment (i.e., make mistakes), receive ongoing formative assessment, and participate in a community of learners. In this way, we integrate a number of these ambitious teaching practices into a coherent classroom environment that attempts to empower students in their mathematical learning. We present here one portion of the outcomes from that implementation.

Background on ATMI

The ATMI comprises four scales with a total of 40 items which measure self-confidence, enjoyment, motivation, and value. Eleven items were reversed-coded and given the appropriate value for data analysis. The self-confidence scale includes 15 items, and measures students' "confidence and self-concept of (their) performance in mathematics" (Tapia & Marsh, 2004, p. 17). The enjoyment scale includes ten items, and measures "the degree to which students enjoy working (on) mathematics" (Tapia & Marsh, 2004, p. 17). The motivation scale includes five items, and measures students' "interest in mathematics and (their) desire to pursue further studies in mathematics" (Tapia & Marsh, 2004, p. 17). The value scale includes ten items, and measures "students' beliefs on the usefulness, relevance and worth of mathematics to their lives" (Tapia & Marsh, 2004, p. 17).

Methods

In order to obtain a random sample of the student population and measure their response to the curriculum, students were allowed to register in sections that were capped at twice the normal size for calculus courses. The week prior to the first day of class, students were randomly assigned to either a treatment or control section at the same time of the original, larger course. In the spring of 2019 semester, a total of N=261 students enrolled in ten sections within the study, with N=130 of these in five treatment sections and N=131 in five control sections. Students were administered the ATMI instrument at the beginning and end of the course in all treatment sections, and in three of the control sections in this semester.

In this report, we examine the responses of three control sections responding and the matching treatment sections. For this group, N=59 students responded to both pre- and post-survey instances in these three treatment sections, and N=47 responded to both in the three matching control sections.

Generating an ATMI Factor Score

ATMI factor scores were generated by combining the scores for each of the 40 items on a Likert scale of 1-5 grouped by factors labeled value (10 items), enjoyment (10 items), motivation (5 items), and self-confidence (15 items). Each factor score is the sum of the item scores for that factor. The factor ranges and total factor difference ranges possible for each factor are shown in Table 1.

Table 1. ATMI Factor Properties

Factor	Items	Factor Range	Difference Range
Motivation	5	5 to 25	-20 to 20
Confidence	15	15 to 75	-60 to 60
Enjoyment	10	10 to 50	-40 to 40
Value	10	10 to 50	-40 to 40

Results

We observed decreases in Motivation, Enjoyment and Value for treatment group students, and an increase in Confidence. The decreases were observed to be approximately 1-3% of the response total with a largest decrease in Value, while the increase in Self-confidence was approximately 1%. Of the control sections, three administered the ATMI survey during the spring term. We observed decreases in all four factors of Motivation, Confidence, Enjoyment and Value among control group students. The decreases for each factor were larger than for the treatment group ranging from 4.5% for Value to 8.1% for Motivation. The resulting pre- and post-survey item average scores for each factor as well as factor total and item average differences for survey respondents in the corresponding treatment and control groups for these sections are shown in Table 2.

Table 2. Factor totals and item averages for Treatment and Control Groups

	Treatment Group (N=59)					Control Group (N=47)				
Factor	Pre	Post	Difference			Pre	Post	Difference		
	Item Avg	Item Avg	Factor Total	Item Avg	Gain	Item Avg	Item Avg	Factor Total	Item Avg	Gain
Motivation	3.25	3.16	-0.44	-0.09	-2.70%	3.16	2.9	-1.28	-0.26	-8.10%
Confidence	3.55	3.58	0.44	0.03	0.80%	3.49	3.25	-3.58	-0.24	-6.80%
Enjoyment	3.64	3.58	-0.58	-0.06	-1.60%	3.39	3.19	-2.02	-0.2	-6.00%
Value	4.14	4.02	-1.2	-0.12	-2.90%	3.92	3.75	-1.75	-0.17	-4.50%

Analysis of Population Differences

In order to analyze the differences between groups using parametric tests (e.g., independent-samples t-test), a common assumption is that the dependent variable is approximately normally distributed for each group of the independent variable. For this study, an ATMI Difference Score for each of the four factors was calculated to serve as the dependent variable and the independent variable is the two groups.

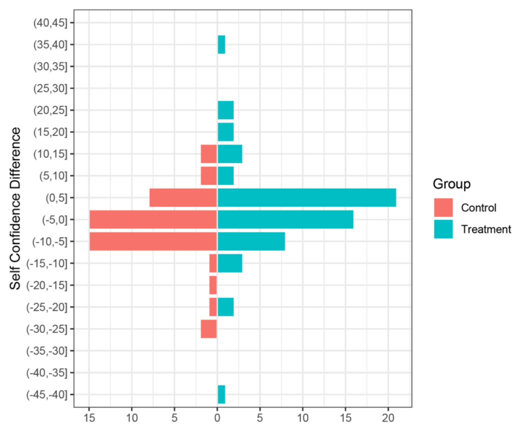

Figure 1. Population distributions for Self-confidence factor.

We analyzed the ATMI Difference score for each factor using Shapiro-Wilk's test and found that the scores were normally distributed for control sections ($p = 0.486$) but not MPC sections ($p < 0.001$). The ATMI Difference Scores data obtained from MPC sections failed to pass the assumptions necessary for an independent-samples t-test for Spring 2019. Figure 1 shows the distributions of differences for the self-confidence factor.

A rank-based nonparametric Mann-Whitney-Wilcoxon U-Test was used to determine if there is a statistical significance between the control (N=47) and MPC (N=59) groups on ATMI Difference Scores for the four subscales: self-confidence, value, enjoyment, and motivation. Differences for value, enjoyment and motivations were all found to be insignificant. We found the increase in the mean difference in self-confidence for the treatment group to be significantly different ($U=1000$, $Z=-2.69046$, $p = 0.007202$, $r=0.25889$) from the decrease in the factor level for the control group. With $r=0.25889$ we note a medium effect size related to the treatment and suggests that most of the effective change in the total ATMI response is related to this reported self-confidence.

Conclusions

Student attitudes interact with course structures and instructional practice in complicated ways that differ dramatically from student to student. In this project we are attempting to capture a large cross section of that variability using randomly assigned populations in treatment and control sections of a Calculus 1 course where an active learning implementation has been adopted for all classroom contact time in the treatment sections while traditional lecture-based methods are the primary modality for control sections.

For one of our assessment protocols, we implemented the ATMI survey of student attitudes. We did not find statistically significant differences in gain scores between the treatment and control groups for Motivation, Enjoyment, and Value. Student attitudes became more negative in the control sections after completing their calculus course. We observed statistically significant differences between the treatment and control groups for their overall response levels in the sense that positive sentiment towards mathematics decreased less for treatment students than for control students. Within this scale, for the Confidence factor of the survey, students reported an increase in Confidence for the treatment sections while this factor decreased for the control students. We conclude in this preliminary stage that active learning as implemented in our protocol benefits student self-confidence.

Our intent in designing the curriculum and classroom practices used to implement it focused on making the calculus content culturally relevant and the classroom experience empowering. These initial outcomes are encouraging indicators of that experience for students. In subsequent work, we will attempt to determine if this in turn is related to observed increases in student success. Similar to the study conducted by Sonnert et al. (2015), future work should examine these negative differences in attitudes to explore how instructor pedagogy influences students' attitudes towards mathematics. As noted by Bressoud, Mesa, and Rasmussen (2015), 'good teaching' practices improve students' attitudes. It is reasonable that students in the treatment sections show increases in self-confidence because they experience 'good teaching' practices (e.g., providing explanations that students understand, listening carefully to student comments and questions, and allowing time for students to understand difficult ideas).

Future work should also examine the small negative differences in gain scores for the Motivation, Enjoyment, and Value factors. Sonnert et al.'s (2015) work that found that faculty who employed 'ambitious teaching' practices, such as group work, class discussions and student presentations, had a small negative impact on student attitudes. Current research is mixed regarding connections between ambitious teaching and affective outcomes, such as student attitudes and beliefs (Bressoud, Mesa, & Rasmussen, 2015). Future research should establish the link between ambitious teaching practices and student attitudes towards mathematics.

Although the randomized control design enhances this study's generalizability, the low response rate in contrast serves as a limitation to the ability to generalize the results. While response rates are relatively low, it is less clear that this development is a major source of nonresponse bias (Czaja & Blair, 2005). Other concerns include a number of outliers in the populations as well as the non-normal distribution of the treatment group data. Further work in the upcoming academic year will collect substantial amounts of additional data and will allow us to analyze these outcomes for underlying patterns. Our goal is to collect one more semester of data to include 16 sections total, including 8 treatment and 8 control sections. These additional data points will hopefully allow for analysis of item responses and subpopulations.

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Defining Key Developmental Understandings in Congruence Proofs from a Transformation Approach

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Previous work by the authors (2019) identified two potential key developmental understandings (KDUs) (Simon, 2006) in the construction of congruence proofs from a transformation perspective for pre-service secondary teachers in an undergraduate geometry course. We hypothesized the independence of the potential KDUs in previous work, meaning that students may have one potential KDU but not the other, and vice versa. We tested this hypothesis with analysis of an expanded data set and found that this hypothesis did not hold in general. We report on the expanded analysis and discuss the implications for the scope and limitation of the potential KDUs. Such work can lead to a more precise understanding of how these potential KDUs may be addressed by instructors of undergraduate geometry courses.

Keywords: geometry from a transformation approach, pre-service teacher training, secondary education

A change has come to K-12 geometry instruction, and as a result changes to preparation of future teachers must follow. Many guidelines (NCTM, Catalyzing Change in High School Mathematics: Initiating Critical Conversations, 2018) and standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) now recommend or require the teaching of geometry from a transformation perspective instead of the more traditional approach originating from Euclid's *Elements* (Sinclair, 2008). The concepts and proofs involving congruence and similarity now appeal to rigid motions: reflections, rotations, and translations. That is, two figures are said to be congruent if and only if there exists a sequence of rigid motions carrying one figure onto another. This definition is notably different from those in Euclid's *Elements,* where the criteria for congruence differs for each type of shape. Thus, the reader will note that the differences in mathematical structure between the transformation and *Elements* contexts are substantial.

The resulting danger is that some future teachers may lack the content knowledge to handle the new approach. Without sufficient content knowledge, they may struggle to know what can be proved in this new context and how these proofs may be structured. This lack may affect how they write lesson plans and course materials, adapt or modify materials for the context of their class, and evaluate student thinking and alternate approaches. Future teachers may need support in the transformation context to allow them to thrive in the teaching of geometry.

Relationship to Prior Literature

To answer the call, some undergraduate instructors are beginning to incorporate transformation geometry into their geometry courses for future teachers. Because transformation geometry is becoming a more prominent feature of geometry in post-secondary contexts, research on how pre-service teachers learn these topics are particularly salient. However, at this point research on how pre-service teachers learn transformation geometry is just beginning. Jones and Tzekaki (2016) noted the "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" (p. 139).

Some key results informing our work are the following. Edwards explained that students in middle school through undergraduate contexts tend to view transformations from a motion view (2003), as opposed to a map view of transformations. A motion view is characterized by conceptualizing transformations as physical movements, such as picking up a figure and shoving it to where it needs to go. A map view is characterized in terms of inputs and outputs of transformation, and distinguishes the preimage from the image. For instance, a person with a motion view may think of an image and preimage of a figure as being the same object, simply with a different location. But a person with a map view can hold the idea that the image and preimage as different objects, and hence compare them. Research conducted after Edwards' (2003) study with middle school students corroborate her results, even for high school teachers (Portnoy et al., 2006; Hegg et al., 2018; Yanik, 2011). These results also note the difficulty that a motion view may present to generating congruence proofs from a transformation approach.

Based on analysis of future teachers' work on two congruence proofs on a midterm examination, we previously highlighted the importance of supporting pre-service teachers in understanding *both* directions of the "if and only if" in the definition of congruence. Further, we identified two potential key developmental understandings (KDUs; Simon, 2006), stated below:

"Potential KDU 1: Understanding that applying the definition of congruence to prove congruence of two figures means establishing a sequence of rigid motions mapping one <u>entire</u> figure to the other <u>entire</u> figure" (St. Goar & Lai, 2019, p. 251).

"Potential KDU 2: Understanding that using a sequence of transformations to prove that two figures are congruent means justifying deductively that the image of one figure under the sequence of transformations is exactly the other figure" (St. Goar & Lai, 2019, p. 252).

As the results by the authors (2019) were based on analysis of teachers' work from a single, timed assessment, more work is needed to interrogate the accuracy of these potential KDUs.

Further, *we previously hypothesized the independence of these potential KDUs, meaning that teachers might hold KDU 1 but not KDU 2, or hold KDU 2 but not KDU 1*. We generated this hypothesis empirically from examples of teachers' work in our previous analysis. In considering the literature, we might also support and refine this hypothesis as follows. First, potential KDU 1 pertains to constructing a sequence of rigid motions, and not explicit deductive reasoning about images and preimages, which is the scope of potential KDU 2. Second, constructing a sequence of rigid motions can be consistent with either a motion view or a map view. However, deductive reasoning as needed for congruence proofs might require distinguishing between overlapping figures. Although this could be done under a motion view, it seemed plausible to us that conceiving transformations as maps was more likely to support a teacher in careful work with images and preimages – particularly if the figure is disconnected. It seemed plausible that it is more difficult to conceive of "moving" a disconnected figure than "moving" a connected one. In lieu of the literature, although it is possible for these potential KDUs to be independently held, *the following is a better hypothesis: Teachers hold neither potential KDU (if neither motion or map view is developed), potential KDU 1 but not potential KDU 2 (representing a motion view), or both potential KDUs (representing a map view).*

Research Questions

Hence, we proceeded with the following research questions, with the same teachers' work on different congruence proofs than previously analyzed: (1) Do we continue to see evidence of the previously identified potential KDUs? (2) What are the scope and limitations, including the independence, of these potential KDUs?

Conceptual Perspective

Based on Usiskin and Coxford (1972), a *transformation approach* assumes without proof that rigid motions (e.g., reflections, rotations, and translations) are bijections of the plane that preserve both distance and angle measure. Additionally, under such an approach, two subsets of the plane are considered to be congruent if and only if there exist a sequence of rigid motions mapping one subset to the other. Similarity is treated analogously, incorporating dilations.

We consider an *Elements approach* to be one where definitions of congruence are based on categories of geometric figure. For example, two triangles are considered to be congruent under an *Elements approach* if each angle and side of one triangle are congruent to corresponding components in a second triangle. This definition of congruence could not, as stated, apply to an object such as a circle. This contrasts with the definition in the transformation approach, which applies to any geometric figure in the plane. Additionally, in an *Elements approach*, one of the triangle congruence criteria, such as SAS, must be taken as axiomatic, whereas they are all theorems in a transformation approach (Venema, 2012).

Key developmental understandings (KDUs) are described by Simon (2006). A key developmental understanding has two primary aspects: (1) Achieving a KDU represents a conceptual advance by the student. A conceptual advance is "a change in a students' ability to think about and/or perceive particular mathematical relationships" (Simon, 2006, p. 362) and (2) Acquiring KDUs does not tend to happen "as the result of an explanation or demonstration. That is, the transition requires a building up of the understanding through students' activity and reflection and usually comes about over multiple experiences" (Simon, 2006, p. 362).

As Simon noted, KDUs generally cannot be found by a mathematician examining their own understanding of a topic, but rather through observing students' mathematical work. As a result, our first steps in identifying these potential KDUs have been through the analysis of future teachers' work. Simon noted also that KDUs may be identified with varying amounts of detail and that "the level of detail specified for a key developmental understanding is adequate if it serves to guide the effort for which it is needed (e.g. curriculum design, further research)" (Simon, 2006 p. 364). Hence our analysis here is meant to achieve this necessary detail so that the potential KDUs can be used to improve undergraduate geometry curricula and research.

We use the term "potential KDU" rather than "KDU" because we see our understanding of teachers' understanding as a work in progress that is only based on analysis of written work as opposed to cognitive interviews, which would be ideal and needed to substantiate a claim of being a KDU. We return to this critical piece in the discussion and questions to the audience.

Data Collection and Analysis

We collected the coursework of twenty teachers in an undergraduate geometry course taught by the second author. We examined homework assignments and midterm exams from throughout the semester for tasks where teachers specifically worked on congruence proofs. Here we report analysis of four tasks. This resulted in 69 total proof submissions included in the analysis.

We coded teachers' work on tasks based on evidence of potential KDU 1 and KDU 2. During the course of this analysis, if some criteria had to be changed, then codes were reworked to reflect these updated criteria, consistent with constant comparison (Strauss & Corbin, 1994).

Results

Addressing the first research question, the basic statements of the potential KDUs remained intact after analysis of teachers' work on additional tasks. Addressing the second research question, this analysis provided possible disconfirming evidence for the independence of the potential KDUs. We begin this section by reviewing the scope and limitations of the potential KDUs, and then compare evidence of each potential KDU.

Scope and Limitations of Potential KDUs

Potential KDU 1 is primarily focused on the construction of the sequences of rigid motions. That is, in order to have this potential KDU, teachers must construct a sequence of rigid motions from one entire figure to another entire figure. This means that aside from the creation of the rigid motions themselves, the rest of the deductive logic in a transformation proof is not a part of this potential KDU.

Potential KDU 2 focuses on the deductive reasoning used in the proof. Specifically, teachers need to attempt to deductively show that their transformation extends to the entire figure. Note that a teachers' work need not show entirely correct logic in order to show evidence of this potential KDU so long as they are attempting to extend arguments about the image of a transformation to entire figures and are using deductive logic to do so.

(Non) Independence of Potential KDUs

We hypothesized previously the independence of potential KDU 1 and potential KDU 2, meaning that, teachers' capacity to engage in deductive reasoning about the correctness of a proof may not depend on their capacity to construct sequences of rigid motions. We refined our view in lieu of the literature to hypothesize that it is most likely that teachers may hold neither potential KDU 1 nor potential KDU 2, hold potential KDU 1 and not potential KDU 2, or hold both. Our analysis suggests that our initial hypothesis is not well-supported, but our new hypothesis is. For brevity, we limit discussion of this to a visual summary of the results of this analysis, shown in Figure 1.

Homework Task 1	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	12	2
No Evidence of KDU 2	2	0

Homework Task 2	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	9	0
No Evidence of KDU 2	1	6

Midterm Task 1	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	4	0
No Evidence of KDU 2	12	1

Midterm Task 2	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	7	3
No Evidence of KDU 2	5	5

Across all Tasks	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	32	5
No Evidence of KDU 2	20	12

Figure 1: The above is a summary of evidence of potential KDU 1 and potential KDU 2 across two homework tasks and two midterm examination tasks.

Discussion and Conclusion

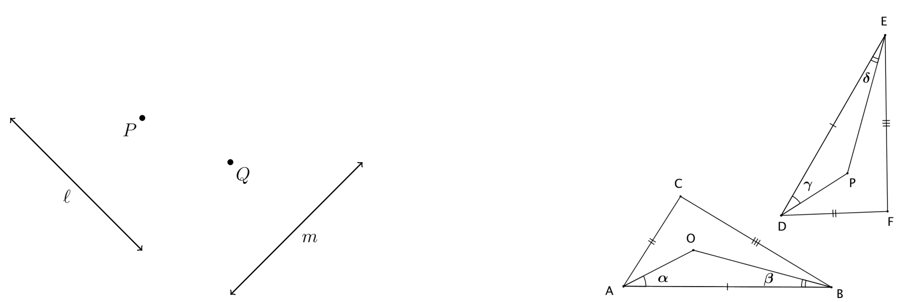
In this report, we expanded on the research by the authors (2019) by analyzing future teachers' work on transformation congruence from an undergraduate geometry course. The results confirm the viability of potential KDU 1 and potential KDU 2 as codes for teachers' written work on congruence proofs from a transformation approach. Moreover, the results do corroborate the authors' revised hypothesis that that teachers may hold neither potential KDU 1 nor potential KDU 2, hold potential KDU 1 and not potential KDU 2, or hold both. In other words, the *least likely scenario* is that teachers hold potential KDU 2 but not potential KDU 1. Indeed, across the tasks, there are only 5 out of 69 instances (7%) where teachers' work shows evidence of potential KDU 2 but not potential KDU 1. It is encouraging to see that in almost half of the cases, 32 out of 69, teachers' work showed evidence of both potential KDUs. In roughly one third of the cases, teachers' work showed evidence of potential KDU 1 but not potential KDU 2. About 12 out of 69 cases showed evidence of neither potential KDU.

However, the task where most cases of the work showed evidence of both potential KDUs (Homework Task 1) was a homework task asking teachers to prove the veracity of the ASA triangle congruence criterion. This is an instance where the figure has only one component, rather than having multiple components or disconnected components (cf. St. Goar & Lai, 2019). This result corroborates the result of Hegg et al. (2018), who also found that teachers could perform triangle congruence proofs using a transformation approach.

While our work was able to corroborate part of our revised hypothesis described above, the revised hypothesis was based on the construct of map view and motion view. We were not able to deduce from the available written work which type of view a teacher might hold, and as a result further research is needed to investigate this possible role of motion view and map view.

Questions

After analysis of teachers' work on congruence proofs in a second course, our next research step will be to confirm the existence of these two potential KDUs. Given that Simon explains that KDUs can be specified by observing "students engaged in mathematical tasks (Simon p. 363)", it follows that observing teachers in action, for instance in the context of interviews, may be a reasonable way to proceed. Moreover, because KDUs are intended to help researchers shed light on learners' conceptions, we are also interested in examining the hypothesis that the potential KDUs are linked to the conceptions of map view or motion view as introduced by Edwards (2003). Hence our primary questions for the RUME audience are: (1) What are ways to detect map view or motion view in the context of cognitive interviews? (2) What selection of tasks shown in Figure 2 might be most fruitful for these interviews? For what reason? (3) How do we balance our agenda of investigating potential KDUs for proofs and the aim of examining conceptions of congruence such as map view or motion view?



Midterm Task 1 (above):
Note that point P is one unit from line ℓ, and point Q is one unit from line m. Show that $P \cup \ell \cong Q \cup m$.

Midterm Task 2 (above):
Show that the two figures above are congruent as marked.

Figure 2: The above shows the two midterm tasks we analyzed. Homework Task 1 asked students to verify ASA. Homework Task 2 asked students to show that two rectangles with the same dimensions are congruent.

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Attending Mathematics Conferences as a Means for Professional Development: A Preservice Teacher's Evolving Identity

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In this preliminary report, we present data and initial findings on a preservice teacher's developing identity of becoming a teacher of mathematics after attending a regional mathematics conference. We are particularly interested in reporting on one case, Krista's, experiences leading up to the mathematics conference, her experience during the mathematics conference, and how these experiences led to the evolvement of her identity as a preservice teacher of mathematics. Reflections on the conference, focus group data, and autoethnographies written by the students were collected, and we present the preliminary analysis with the overall goals of: discussing the issues/strengths of using autoethnographies as a research methodology in mathematics education, using Krista's experiences to help recreate similar conference experiences for other preservice teachers, and to discuss how Krista's identity developed as she became a novice teacher of mathematics.

Keywords: Teacher identity, Preservice Teachers, Professional Development

Introduction

Imagine a successful secondary mathematics teacher. Highly motivated and focused on student learning, this person is devoted to their profession. However, they are isolated geographically and they do not feel part of a larger community of educators since they talk with just a few people about their lessons and pedagogical strategies. Now picture that same teacher at a regional math education conference, energized to be surrounded by hundreds of like-minded individuals, all focused on teaching mathematics better. Once an island, now the teacher is awash in colleagues and community. These experiences engage teachers and preservice teachers alike in a social, collegial atmosphere. We aim to illustrate how these experiences can have a positive impact on one's professional identity as a mathematics teacher, which has been linked to lower turnover (Ingersoll, 2001; Moore & Hofman, 1988).

In this preliminary paper, we discuss the initial findings from our case-study of a preservice mathematics teacher as they attended a regional mathematics conference. Research on preservice teachers' attendance at conferences is minimal, especially when coupled with how attendance at conferences impacts their identity of being a teacher of mathematics. While many studies discuss teacher identity in general (e.g. Beijaard, Meijer, & Verloop, 2004; Gee, 2000), we were specifically interested in how an early professional development opportunity could influence teachers' identity before entering the profession.

In our study, we analyzed autoethnographies, reflections, and focus groups to capture the preservice teachers' views on teaching as well as their views on attending the regional mathematics conference. Specifically, we are seeking to answer the following research question: How did the participant's identity of being a teacher of mathematics evolve after partaking in a mathematics conference?

Background Literature & Theoretical Perspective

We are interested in how attending a mathematics education conference influenced a preservice teacher's sense of professional identity as a teacher of mathematics. It is therefore critical that we illustrate our meanings for the rather nebulous concept of one's identity as well as how to frame that identity within how we view ourselves as members of a community, in this case, the community of teachers of mathematics. These illustrations help support our choice of a case study methodology and the use of autoethnographies, a rather novel research tool in mathematics education.

There is no consensus amongst researchers on a clear definition for professional identity. In their meta-analysis of research concerning teachers' professional identity, Beijaard, Meijer, & Verloop (2004) note that some studies define it contrarily to one another while others fail to claim any kind of definition at all. Their synthesis of more than twenty studies conducted in the 1980s and 1990s resulted in their identification of four essential features of teachers' professional identity. The first feature is that professional identity is not static; it is dynamic, shifting throughout a teacher's career. One's professional identity answers not only "Who am I at this moment?" but also "Who do I want to become?" (p. 122). Second, professional identity is co-developed via the self and interactions with the professional community, there is no objectively defined identity all teachers should aspire to as all have different experiences to build upon. A teacher's sense of professional identity is intimately related to their sense of belonging to their professional community, both locally and more broadly. Third, the authors claim that a teacher's identity is not a singular construct but is composed of sub-identities that "harmonize" with one another (p. 122). Agency is the final feature. Teachers must assume an active role (and be given space for such an assumption) in their profession to develop a sense of professional identity. Collectively, we take these features as illustrative of our working definition of teachers' professional identity.

One's sense of professional identity has been linked to teacher turnover (Gaziel, 1995; Moore & Hofman, 1988). For example, Israeli elementary teachers who took a sabbatical to engage in professional development had lower desire to leave their jobs (Gaziel, 1995). Teacher evaluation programs can have a negative impact on teachers' professional identity when asserting what professionalism looks like for teachers without appropriate support and reflection opportunities (Bradford & Braaten, 2018). Therefore, engaging preservice teachers in activities that support the development of their professional identities may provide a way to combat high turnover in the profession (Ingersoll, 2001).

Gee (2000) says identity is "being recognized as a certain 'kind of person' in a given context" (p. 99). While this may not capture all the subtlety of Beijaard, Meijer, & Verloop's work, it speaks to the intimate nature of the relationship between a teacher's professional identity and the acceptance of their peers and mentors who are already considered members of the community of mathematics teachers. To that end, we framed professional identity with Allport's (1954) notions of *in-groups* and *reference groups*. An in-group is a group or community you feel you belong to; Allport claims his best definition is "to say that members of an in-group all use the term *we* with the same essential significance" (p. 173). A reference group "is an in-group that is warmly accepted, or a group in which the individual wishes to be included" (p. 178).

Putting these elements together, we have a structure with which to frame our study. We view preservice teachers as having some sense of professional identity but that this identity is in flux and sensitive to external influences, such as being welcomed by professional educators as equals at an education conference. We see them as interacting with the community of math teachers as a reference group at first, since they aspire to be teachers but may not see themselves as members of that community just yet and therefore do not see the community of teachers as an in-group to which they belong. Our research goal is to explore how one student, Krista's, professional identity developed after attending a regional mathematics education conference. In the next section, we describe our research methods and how we are analyzing the data.

Methods

Data from this study was collected at a large northwestern university in a secondary mathematics methods course. One of the authors was the instructor for the course and collected the data through a variety of assignments throughout the semester. As part of the course, students attended a regional mathematics education conference aimed at mathematics teachers at the K-12 and undergraduate level. The instructor provided transportation for all students attending and students stayed overnight at the conference location. They were instructed to attend sessions during each available time throughout the day, although no requirements for which sessions they must attend were given. After attending the conference, students were asked to write a reflection of each session they attended as well as an overall takeaway from the conference. Students also partook in an informal discussion, or focus group, that asked a series of questions related to their experience at the mathematics conference. Near the end of the course, students were asked to write an autoethnography to narrate and analyze their experiences and membership within the in-group of teachers of mathematics. Autoethnography is “a qualitative research method and genre in which researchers analyze their personal experiences (‘auto’) to explain and understand cultural and societal context of those experiences (ethnography)” (Hall, p. 3, 2016). It therefore can serve as both a research methodology for the individual completing it and/or as a product for further analysis, which is how it is being used in this study. The autoethnography allowed us a window into our how our participants viewed themselves in relation to the community of mathematics teachers, whether as an in-group or just as a reference group.

The data collected consisted of six students who completed the autoethnography, conference reflection, and focus group. The autoethnography and reflection were text documents that students submitted as part of their coursework; however, the focus group was filmed and then transcribed for data analysis. Once all data was collected, the two authors of this study read through each of the three artifacts for all participants. One student participant, Krista (pseudonym) was selected as an individual case study because she provided the most detail in her written work as well as during the focus group.

Our goal in the preliminary results is to report on Krista’s experiences leading up to the mathematics conference, her experience during the mathematics conference, and how these experiences led to the evolvement of her identity as a preservice teacher of mathematics. Furthermore, we plan to describe our process of using autoethnography as a research methodology as well as our framework for attending the regional mathematics conference. It is our hope that other mathematics teacher educators use these tools to provide quality professional development to preservice teachers.

Preliminary Results: The Case of Krista

Autoethnography: Identity as a K-16 Student

Krista’s identity as a future teacher of mathematics was shaped by experiences throughout her own K-16 schooling. It was in fourth grade that Krista first recalled wanting to be a teacher. Her teacher, Mrs. Andrews (pseudonym), “had an impact on every student that she taught”, and Krista was no exception. On a particularly frightening day, Mrs. Andrews had a diabetic episode during class and had to be rushed to the hospital. The students were worried and wanted to show they cared as much for their teacher as she did for them. Krista made a card and had the students sign it for Mrs. Andrews. Krista reflected that "if it wasn't for that day, where the love of her students shone through, I probably wouldn't want to be a teacher as strongly as I do now".

Once in middle school, Krista took on the role of taking care of her younger siblings, which "really took a toll" on her. Her teachers sat her down and asked her what was going on in her home life; she opened up to them about the struggles she was experiencing. A couple days later the teachers signed a card for her and told her "they knew life was hard right now, but that it would get better." Krista reflected on this story by saying "...up until this point, I didn't realize that teachers could also be your friend… for these teachers to be there for me like they were, I knew that I wanted to be that person for other people in the future".

During her time in her teacher preparation program, Krista's feelings about teaching were reinforced. She commented on how "whenever I was making a lesson plan, or I was surrounding myself with math teachers or students, I felt like I was in my element". Once during a field trip with middle-school students, Krista recalled "wanting to impact every student… I wanted to show every student that math could be fun to do and learn if you had the right teacher." Krista also described having a "life goal of winning Teacher of the Year nationally." She knows that this is a lofty goal, but she wants to "make an impact on the students' lives" just as her former teachers did for her. Krista continued by paraphrasing Scholze (2013) by saying that "for most teachers, being a teacher is their identity and they know that is what they were meant to do". She believes this is something she identifies with and wants to use her "drive to help people in everything that I do in life".

Regional Mathematics Conference Attendee

During the fall of her junior year, Krista attended a regional mathematics conference as part of a requirement in her mathematics methods course. The students travelled to the conference as a cohort, along with their professor (one of the authors) and spent an overnight to attend both days of the conference. Throughout the conference Krista did her best to submerse herself in the mathematics teacher community. She raised her hand to ask questions during the talks she attended, she approached teachers after a talk to discuss their ideas, and she made connections between the content in the sessions she attended and the material she had learned in her teacher preparation program. For example, Krista talked about how she heard "mention of the Five Practices [for Mathematics Discourse]... and Growth Mindset a lot" throughout the sessions she attended. She also discussed how the conference "was the first time I actually interacted with high school teachers besides those who taught me… it was really cool to be in that environment where they treated you as equal." She went on to say that the interactions she had with the inservice teachers helped reaffirm her love of math by seeing "just how much everyone else loved math… you could be yourself and love math… we just fit right into this group."

In addition to immersing herself in the teacher community, Krista's beliefs about the value of attending conferences grew. She commented that she figured the conference would be fun; however, she found that "the talks were a lot more engaging than I was expecting… there wasn't one talk that I went to where I was bored or didn't find the information useful." She went on to explain how the experience of attending a conference, "is something that I think everybody who is going to be a teacher should do. It really gets your feet wet with what it looks like to be a teacher… I hope to go again next year and gain even more experience and learn more than I did this time around." She even went on to say that she would be "super down" to present at conferences down the road. "All of the talks were about stuff they [the presenters] were passionate about, and I could talk about something I am passionate about forever."

However, to Krista and her fellow classmates, the most influential piece of the conference was more than spending time with the inservice teachers and attending sessions. In her reflection of the conference Krista wrote,

> I know that I am supposed to talk about which talk had the most impact, but I don't think that is what stuck with me the most. The talks were extremely interesting, and I learned so much about being a teacher, but that isn't what had the most impact. The time that I had with the six other students with me is something that I will never forget. I gained a bond with each of them… those 20 hours in the car brought us together… I have lifelong teacher buddies and that is something that I don't think I could have gained from any other experience except this one.

Krista also described that by each student attending a variety of sessions, they were able to discuss what they had each learned and "instead of going to just 6 sessions, it was like you went to 15 or 20". Further, by attending the sessions as a cohort they were able to influence each other to break out of their comfort zone and attend sessions they might have otherwise overlooked.

Impact of the Mathematics Conference Experience on Krista's Developing Identity

In order to understand how Krista's identity as a mathematics teacher evolved after the conference experience, we first had to make sense of her identity as a K-16 student and as a conference attendee. As a K-16 student, Krista felt her sense of duty as a teacher was to make a difference in her students' lives. She wants to be able to show her students the same empathy her teachers showed her by caring and taking an interest in their lives outside of her classroom. During her time attending the mathematics conference, Krista was able to integrate herself as part of the teacher community, even though she was still a preservice teacher. We view this as evidence she is moving from seeing the community of teachers just as a reference group and more as an in-group, she felt welcomed by the presenters and teachers she met. She also found herself as part of an in-group with her own peers from her methods course, as she now felt she had people to call upon as she began her career as a novice teacher.

Our view of professional identity includes four key elements: professional identity is (1) dynamic, (2) co-developed via the self and interactions with the professional community, (3) composed of sub-identities, and (4) requires teacher agency. Attending a mathematics education conference and completing the autoethnography allowed Krista to develop within each of these four elements. Here, we highlight the first two elements. Her sense of belonging increased with being welcomed by the community of mathematics teachers, indicating this aspect of her identity was and is subject to change. Additionally, her interactions with teachers was a driving force behind the shift in her beliefs about whether and how she belonged to the community, indicating these changes are functions of both Krista's beliefs and her interactions with her community.

As part of our session, we hope to have a brief discussion with the audience about our findings and what our results mean for mathematics teacher educators. Specifically, 1) what issues/strengths exist concerning the use of autoethnographies as a research methodology in mathematics education?; 2) how does Krista's experience attending the mathematics conference allow us to help other mathematics teacher educators to recreate a similar experience for their preservice teachers?; and 3) how do the preliminary results point to the developing identity of Krista as she becomes a novice teacher of mathematics?

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Multivariate Functions, Physical Representations, and Real World Connections

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Understanding functions of two variables is difficult even for students who have studied multivariate calculus. The eight participants in this study, prospective middle and high school mathematics teachers, constructed physical models of functions of two variables by creating and assembling sets of transparencies representing cross-sectional planes. The models provided "chunky" representations of surfaces defined by function rules of two variables. This study reports ways in which students with the experience of constructing these models interpreted and operated on a real-world example of multivariate functions (wind chill index) that was reported in a national newspaper. Participants were confronted with potential cognitive conflicts in that the article contained typographical errors that eliminated a variable from the function rule and omitted essential graphical information. First rounds of analysis suggest the importance of focusing on what is being represented rather than solely on the form of the representation.

Keywords: Multivariate functions, concrete representation, function, fundamental planes

Students intending to engage in a mathematics-intensive major are typically required to complete several calculus courses, including a course focused on multivariate functions. In spite of extensive work with $\Re$-to-$\Re$ functions and significant exposure to functions of two variables, many of these students operate at only the action or process levels (Arnon et al., 2014) when working with functions of two variables. Research by Trigueros Gaisman and Martinez-Planell (2010, 2012) has suggested that those who have a deeper understanding of planar slices of surfaces representing functions of two variables have an overall better understanding of multivariate functions. This raises the question of how to help students develop this deeper understanding of the planar slices. Given the well known premise that concrete experiences with mathematical ideas can help in developing younger students' mathematical understanding, we asked how college students' understanding of multivariate functions might be affected if they had a physical experience constructing a representation of functions of two variables. We developed an activity that gave students a hands-on experience of constructing a physical model of a function of two variables.

To investigate the effects of this activity, after engaging students in constructing physical models of functions of two variables, we observed the ways in which they interacted with and connected various multivariate representations of a function of two variables in a real-world setting. In this manuscript, we discuss students' interpretations of various representations of functions of two variables drawn from a real-world situation after they constructed a physical model of a function of two variables.

Background

In the United States, students often begin their study of multivariate functions during their third course in undergraduate calculus. Although college students in mathematics-intensive majors are typically the only ones who have the opportunity to study this topic, investigations of how students without prior multivariate function experience deal with functions of two variables raise important questions about developing an understanding of multivariate functions. Yerushalmy (1997), for example, introduced functions of two variables to seventh grade students by asking them to consider a real-world example involving the cost of a rental car, given the distance driven and the number of days for which the car is rented. She found that the students had to reconsider notions with which they were familiar in the context of univariate functions, needing to extend their understanding of dependency of variables in order to develop a graphical or tabular representation of a function of two variables. Understanding of multivariate functions by students who had recently completed a multivariable calculus course was the focus of an interview study conducted by Trigueros Gaisman and Martinez-Planell (2012). Only four of the thirteen participants in their study gave evidence that they had achieved an object-level understanding of multivariate functions and their different representations. Kabael (2011) further documented the difficulties that college students might face when working with functions of two variables, particularly regarding graphing and identifying domain and range.

The question remains–what strategies can be used to develop that understanding? Weber & Thompson (2014) introduced a method of visualizing functions of two variables, particularly for those of the form $f(x,y) = g(x) * h(y) (e.g., f(x,y) = x^2 * y^3)$. Their approach makes use of students' understanding the simultaneous covariation of three variables by having students imagine a univariate function *sweeping out* into a third dimension to create a graph of a function of two variables. Covariational reasoning can occur continuously, referred to as a smooth image of change, or as occurring in intervals, referred to as a chunky image of change (Castillo-Garsow, Johnson, & Moore, 2013). Sweeping out relies upon students' ability to smoothly employ covariational reasoning. The physical model used in this study (described in the Methodology section) employs a chunky approach, in which each planar subset can be viewed as the next edge of the interval. Trigueros and Martinez-Planell's 2010 article lends support to this idea. Based on their data, students that were able to visualize planar subsets of the graph of a function of two variables (*fundamental planes*) were more successful in performing tasks involving functions of two variables. Our study examined how construction of physical (chunky) models of functions of two variables can affect students' interpretations of functions of two variables presented in real world settings

Methodology

This study examines the use of planar slices of the graph of a function of two variables to enhance students' understanding of multivariate functions. The study occurred in the context of a course on functions designed for prospective secondary and middle school mathematics teachers. The primary strategy for the development of understanding of multivariate functions was to have students construct a physical representation (we called this an Office Max Model[1]) of the graph of a function of two variables.

The functions course was designed to increase the mathematical understanding of function of prospective middle school and secondary teachers. When the study occurred, students had already engaged in studying different conceptions and definitions of function, different families of function, and properties of those families. Prior to the study, the course had focused entirely

on $\Re$-to-$\Re$ functions, with no involvement with functions of two variables ($\Re^2$-to-$\Re$ functions). This study focused on students' interpretation of various representations of functions of two variables in a real world setting after an in-class experience with physical representations of functions of two variables.

[1] The phrase "Office Max Model," suggested by Walter Seaman, was used to describe a model that could be constructed using only supplies available at office supply stores.

The Office Max Model

The physical representation that students used in our class activity was what we called the Office Max model. Over the past decade, we have used the Office Max Model to engage students with representations of functions of two variables. The Model consisted of physical − configurations of sets of "slices" of the surface defined by a function rule of the form $z = f(x,y)$, with each slice recorded as a two- dimensional graph on a heavy transparency that could be stood on its edge. One set of transparencies consisted of representations of the intersection of the surface with a plane parallel to the y–z plane and for which the value of x was held constant for each slice. The transparencies were positioned on their edges and lined up corresponding to the value of x used for that transparency. One set of transparencies consisted of representations of the intersection of the surface with a plane parallel to the x–z plane and for which the value of y was held constant for each slice. These slices were lined up corresponding to the value of y. Participants in this study constructed two models for their assigned function rule (e.g., See Fig 1 for $z = y^2 + \sqrt{x}$), giving them access to two different angles from which to view the surface.

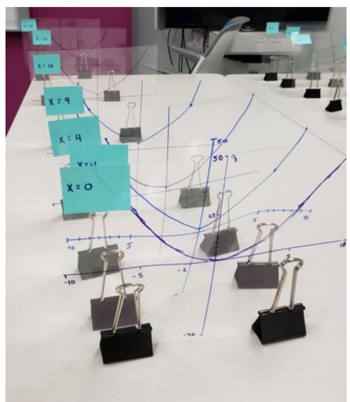

Figure 1. Office Max Model $z = y^2 + \sqrt{x}$ shows the transparencies used for the graphs with fixed x-values

Participants

The set of participants consisted of the eight volunteers of the eleven students enrolled in the functions course. Three of the participants were enrolled in a secondary (grades 7–12) mathematics certification program, that required a multivariate calculus course for entry into the program. Five were enrolled in a middle level (grades 4–8) mathematics certification program that required only first semester calculus for entry into the program. Four of the participants (three in the secondary mathematics program and one in the middle level program) had completed a course in multivariate calculus.

Data and data collection

Data consisted of: video-recordings of (1–1.5 hour) pre-activity task-based interviews that occurred prior to any class activities that involved multivariate functions, written observation notes on students' constructions of the Office-Max Model, video-recordings of (1–1.5 hour) post-activity task-based interviews, and written reflections on the activity completed by the participants. For each interview, one of the authors (usually the first) served as interviewer, and the other served as videographer.

The pre-activity interview was designed to provide information on interviewees' conceptions of univariate function and contained questions about function and related concepts. Information gleaned from these interviews was used to situate the effects of the physical representation on students' understanding of function. One of the questions asked participants to interpret function notation and another presented a 3D graph and asked participants to explain the relationship between the function rule and the graph. After the pre-activity interview, in the only other classroom activity involving functions of two variables (modeled after a question in Yerushalmy's (1997) study), students were asked to suggest ways to use a table to represent the relationships among three variables. In the following class session pairs of students constructed Office-Max Models for a function of two variables whose rule they were given. Mathematics Education doctoral students recorded notes on the work of each of the participant-pairs. The portion of the post-activity interview reported in this manuscript centered on analysis of various representations of the wind-chill index that were presented in a national newspaper article in addition to analysis of the graphs of functions of two variables.

The wind-chill article appeared in the national newspaper almost three decades ago. The article revealed and recounted the development of a new formula for wind chill. Of particular interest was the set of representations for the wind chill index presented in the article. As it appeared in its published form, the article drew the first author's attention because its inclusion of several typographical errors presented the potential for introducing cognitive conflict related to functions of two variables.

Participants were asked to read the article and interpret the graphs. The symbolic, graphical, and tabular representations of the wind chill index included several errors or omissions. The new function rule in the text of the article contained only one variable (air temperature) (This was a typographical error. A correct presentation of the rule appeared in the tabular representation.). A graph associated with the old and new wind chill functions depicted wind chill temperature as a function of wind speed without mentioning that the graph represented only the wind chill temperature associated with an air temperature of 5 degrees. A spreadsheet chart recorded the wind chill for pairs of wind speed and air temperature. Because the article was replete with opportunities for cognitive conflicts in interpreting and connecting a range of representations of functions of two variables, it was an ideal context for observing participants' understanding of multivariate functions.

Data analysis

The authors are in the process of analyzing student understanding of functions of two variables principally as revealed in responses to the interview questions about the wind chill index function. A first level analysis consisted of relistening to the pre-activity and post-activity interviews for one participant from each of the pairs, with each author proposing observations of responses that revealed some understanding of $\mathfrak{R}$-to-$\mathfrak{R}$ or $\mathfrak{R}^2$-to-$\mathfrak{R}$ functions. Before recording an observation, the two authors discussed and revised the claim until both agreed. As these observations were negotiated and recorded, the authors consulted the written documents for clarification and corroboration. A second round of analysis focused on evidence of the connections or lack of connections that the participants made across representations.

Observations Related to Results

At this point in the analysis, we are not ready to make claims about the results of our study. After the second round of analysis, however, our attention has been drawn to three aspects of the data. We noticed distinct differences in the ways in which participants drew on various representations of the function of two variables that they were studying (the Wind Chill Index).

(1) *Focus on inscriptions*. Some participants focused only on the inscriptions themselves, for example observing only the shape of the graph without recognizing what that shape indicated about the relationships among variables or how the graph represented a subset of the values in the spreadsheet. (2) *Treatment of absence of needed variable*. Every student eventually recognized the absence of the wind speed in the uncorrected new version of the Wind Chill Index. Some students recognized the illogic of having only one independent variable in the new function. Others created explanations of why it might make sense for the Wind Chill Index to have only one variable (e.g., Maybe this is an index for only warm days when wind doesn't make that much difference). (3) *Connections to a physical representation of function of two variables.* Only one of the participants, without prompting, referenced the Office Max Model as she was addressing the wind chill representations, but most of them were able to make some substantive comment about the relationship between their Office Max work and their work on the wind chill problem. As we continue our analysis, we would appreciate input regarding the following questions.

Questions for Discussion

1. What can be learned from other studies that focused on the impact of college students' use of physical representations on their understanding of other mathematical concepts?
2. For what theories of representation (in addition to Duval's) should our study account?
3. Our post-activity interview situated participants in potential cognitive conflicts. Are there caveats about what one might learn from such interview strategies?

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Novice and Expert Evaluation of Generic Proofs

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Mathematics education researchers do not yet agree on a shared characterization of proof, or generic proof; neither do mathematicians. Weber's "clustered concept" (2014) isolates six properties that, to personally varying degrees, comprise a valid proof. A generic proof consists of a sufficiently complicated example that illustrates the underlying structures of a generalized argument. Such arguments can be utilized to support student learning of proof. We examine students' and experts' perceptions about proof, explicitly generic proofs. Our findings support Weber's "clustered concept," and extend its utility by showing commensurate properties are also valued by students. In short, we observed that all participants in our study valued an array of psychological and social factors while determining a proof's validity. In addition, our findings suggest that experts do not consider generic arguments to be a proof.

Keywords: Proof, Generic Proofs, Student Thinking,

The transition from computationally focused courses such as calculus to those involving the evaluation and creation of written arguments has long been noted as difficult for students (e.g., Moore, 1994; Selden & Selden, 1987). Proof and proving however, make up a large portion of the upper-level courses that are required of undergraduate mathematics majors. In an effort to support this transition, the teaching and learning of mathematical proof has garnered increasing interest from the mathematics education community. This topic however is not solely important for undergraduate students; as pointed out by Stylianides (2007) many researchers have also stressed the importance of proof and proving across all content areas and K-12 grade levels. As many undergraduate mathematics students go on to teach, the education community has a responsibility for supporting undergraduates in both successfully engaging in proof and proving, and in developing a broad enough conceptualization of proof that will in turn allow them to support others in engaging in appropriate proof activity as well. This is currently problematic as mathematics education researchers have yet to agree on a shared characterization of proof (for a discussion of this see Weber, 2014) and even expert mathematicians disagree (and express awareness of the lack of a shared definition) when it comes to what exactly constitutes a proof; Weber and Czocher (in press) provide a recent example of this. This means a significant portion of the work we expect our students and teachers to engage in and support others in learning, is not well defined or understood.

Background and motivation

In response to the ambiguity, mathematics education researchers have introduced a few theoretical frameworks characterizing proof. For instance, Weber (2014) proposed a characterization in the form of a clustered concept. Namely, proof was proposed as being an argument that: (a) persuades a knowledgeable expert a particular claim is true, (b) provides an understanding of why a statement is true, (c) presents, though deduction, a theorem as a consequence of already established claims, (d) is sufficiently transparent to the point that a mathematician can be reasonably expected to fill in potential gaps, (e) is made utilizing a representation system conforming to communal norms, and (f) has been approved by the larger mathematical community. Within this framework, one might consider an argument to be a proof if some combination of these properties were present. Weber (2016) later provided empirical evidence supporting the claim that different mathematicians will value each of properties differently. This means proof cannot be thought of solely as a convincing argument containing elements of logical deduction.

Of particular importance with regard to the teaching and learning of proof, is the notion of a *generic proof.* Here we use the term generic proof to mean an argument that makes particular characteristics or underlying structures of a particular class explicit by using mathematical operations or illustrations on a single representative from that class. Some researchers have stressed the productivity of using generic proofs as a pedagogical tool that can help support student learning. For instance, Leron and Zaslavsky (2013) noted that the use of generic proofs is appropriate as it allows students to "engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism" (p. 27).

Looking across the research regarding generic proofs, we find much of the same issue; researchers have yet to agree on how exactly a generic proof should be characterized. One characterization specific to generic proofs that fits well with the framework provided by Weber (2014) was proposed by Reid and Vallejo (2018); they argued the existence of psychological and social factors that determine whether or not someone will consider a generic argument to be a proof, stating:

"Psychologically, for a generic argument to be a proof it must result in a general deductive reasoning process occurring in the mind of the reader, that convinces the reader that there exists a fully deductive inference structure behind the argument. Socially, for a generic argument to be a proof it must conform to the social conventions of the context" (p.250).

Additionally, it is not yet understood how mathematicians evaluate generic proofs. Miller, Infante and Weber (2018) examined how mathematicians grade students' proofs and their findings suggested, at least in the context of grading student generated proofs, mathematicians find generic proofs acceptable. Noting "most mathematicians did not take off points for the students writing generic proofs or not listing the scope of variables"(p. 33). Due to this observation they explicitly called for more research aimed at better understanding how experts evaluate generic proofs. Given the important role of generic proofs with regard to student learning, this call for more research is even more relevant. In this paper we report the findings from a small study aimed at understanding how experts and novices evaluate generic proofs.

Methods

This pilot study had 12 participants consisting of 7 undergraduates recruited from upper level, proof-intensive mathematics courses (e.g. real analysis, coding and information theory, and graph theory and combinatorics) and 5 mathematicians. The students were interviewed during the end of the semester, and experts were interviewed a few weeks after classes had ended. Each of the participants were presented with a generic proof for each of five different mathematical statements. The generic proofs ranged in quality; some left large gaps, were missing cases or were unnecessarily difficult while others were complete and even generalized. The arguments were presented one at a time and the participant was given time to familiarize themselves with the statement and its argument. They were then asked a variety of questions concerning how convincing the argument was, whether it was a proof, what changes could be made that would make the argument more proof like, etc. The interviews were video recorded and transcribed.

The transcripts were analyzed using methods consistent with thematic analysis (Braun and Clarke, 2006). In this report we focus on two of the five arguments presented, Argument 1 one concerning numbers that have an odd number of factors (see figure 1) and Argument 5 concerning divisibility by 3 (see Zaslavsky et al., 2016).

Statement: Every square number has an odd number of factors.

Proof: Consider the square number 36. Its factors are 1,2,3,4,6,9,12,18 and 36. Indeed, it has 9 factors and 9 is an odd number. Observe that we can pair up eight of these factors so that the pair of each product is 36:

$$1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 36.$$

The one leftover number, 6 is the square root of 36. We see then that if we start with any square number n^2, it's factors occur in pairs (each pair having one number greater than and one number smaller than the square root, n), except for the square root, n, itself. Because we have pairs of factors with one left over, there must be an odd number of factors.

Figure 1: Argument 1

Results

Generally speaking the students and experts tended to agree that the arguments presented to them were not proofs. Looking across arguments 1 and 5, only one student considered Argument 1 to be a proof, and two considered Argument 5 to be a proof. Similarly one expert considered Argument 1 to be a proof — with the caveat that the first half of the argument, "the example", be removed — and none of the experts considered Argument 5 to be a proof. Despite characterizing the arguments as examples, the students in our study tended to be much more convinced by the arguments compared to the experts. When asked how convincing they found the arguments on a scale of 1 to 5, student responses typically ranged from 2 to 5, whereas expert responses usually ranged from 1 to 2, though there were some noted exceptions. When asked to explain their scores, the participants expressed multiple criteria by which they were judging the argument. We discuss some of this in more detail in the following paragraphs.

After analyzing the data it was determined that the participants, holistically speaking, evaluated the arguments based on the following criteria: 1) *clarity* — being easy to read and understand, 2) *generality* — whether or not the generality of the statement matched the generality of the argument, 3) *logical validity* — that each step followed logically and correctly from the previous step, 4) *destination* — the argument ends with the statement it is trying to prove and, 5) *communal formatting* — being laid out in a particular fashion utilizing attributes that have been determined to be necessary by others. It was the presence of these properties (or lack thereof) that was cited as the reasoning behind how convinced they were or why they did or did not feel an argument was a proof. It should be` noted that participants often cited the same reason for being convinced (or not) and why they considered the argument to be a proof (or not). In addition, the criteria were not mutually exclusive.

Argument 5 – Divisibility by 3

Considering Argument 5, Student 2 said they were very convinced giving a score of 5. They also considered it to be a proof noting:

> "It definitely goes through a step-by-step process of showing us how this can be solved, and then it tells us reasons why this is the way it is. Like, it says, because both 99 and 9 are divisible by 3, 8 plus 5 plus 2 must be divisible by 3 for 852 to be divisible by three".

A different student, Student 1 was also convinced, assigning a score of 4 but they did not consider it to be a proof. When asked why they were convinced they said "it doesn't consider a general case, but it's really close" when asked to explain further they said "it does only consider one case, but in that case it doesn't rely on the actual values of the digits." They later stated it was not a proof because "the statement says a number, so *any* number, in this case there's only *one* number," noting "it doesn't extend to the general numbers in ℝ." They were then asked why that would make it more proof-like and they replied "honestly it's because that's what I've been doing for like the past three years… that's what was taught to me. In my previous classes, you need to do it generally." This seemed to indicate that despite seeing the properties at play in the proof, and being convinced by the argument, they did not consider it to be a proof because they've been taught examples are not sufficient. Student 7 made a similar statement noting how the argument could be made for any number, but then stating "so there seems to be a pattern, but I don't know if this pattern holds for all numbers." While Student 2 is explicit about valuing clarity and logical validity of the argument, Student 1 was making sure the generality in the statement matched that of the argument, which may be a rule imposed by others in the community, and Student 7 needed to make sure argument actually ended with the statement.

The experts seemed to weigh the generality and alignment between the statement and argument much more heavily. When asked how convinced by Argument 5 they were, Expert 4 replied "a one or a two." The reason provided was the same for why they felt the argument was not a proof, "so it has the crux of the idea, but it's still an example," then noting "if the statement is for any number of any number of digits, so there's considerably more work to be done." Expert 1 had a similar opinion saying they really liked it but that they "would want to see that extra step of take the specific example now recognize that this can be done for any number and then generalize it." Expert 2 scored this as a 1, citing "again, just an example. This appears at least to be a universal statement, so, to prove a universal statement, an example is never enough." Expert 3 was not convinced but said "it's motivating the beginning of a real proof." Here we can see many of the same notions expressed by the students.

Argument 1 – Factors for n^2

When discussing Argument 1, Expert 3 gave it a score of "four or five… ignoring the numerical part." When asked why they needed to ignore that part they replied:

"examples tend to be misleading in proofs. They tend to… be useful to help motivate a proof, but they tend to push people towards thinking that the example is representative of everything, and so it's harder for someone to decide if that's really true or not."

Interestingly when asked what they would change to make the argument more proof like they replied "I guess that question really comes down to who was the target audience," indicating proofs would contain different elements if they were written for different audiences. Expert 4 assigned this a score of 1 noting:

> "It is purely an example. There is absolutely no generality in it. And so it could very well have been a total fluke. On the other hand, it does, I wouldn't give it a zero because the, the person has picked up on the right idea of pairing different factors together and that the square root will be the odd man out."

Here we can see that the experts do value in some sense the properties being presented in the argument as Expert 3 was fairly convinced and Expert 4 noted the person "picked up the right idea," but they seem to have a general sense of skepticism as well. For the experts, it seems that examples are inherently misleading, or are at least potentially so.

Many students characterized the arguments as examples as well, but their reasons for this being a problem were different from the experts; Student 1 rated this a 3 on the convincing scale noting "it does the example of 36 then it says this is true for all numbers and, but doesn't do any reasoning." Student 5 assigned this a score of 2, saying it "only gives the example of 36 and so there's a lot more square numbers that we don't necessarily know about." Student 2, after giving it a score of 3, said "I feel it is more an explanation of an occurrence, not an actual proof of something."

Discussion & Questions

The criteria by which we found our participants to be evaluating proof align with those proposed by Weber (2014) and Reid and Vallejo (2018). The properties of clarity and logical validity were often cited by students and directly align with the importance of Reid and Vallejo's logical inference, and Weber's emphasis on transparency and deductive consequence. In addition, it seems many of the students are aware that examples are not acceptable as proof but their reasons are often indicative of them following a communal norm. That is, our students tended to not consider the arguments as a proof citing expectations that were set in previous course work. In particular, despite the students being able to use the same logic presented in the arguments to generalize the generic proofs, students found the arguments insufficient because they view the presence of examples as taboo or as something others don't value.

Most experts immediately insisted that an example was insufficient for proving the statements associated with both Arguments 1 and 5, despite often times noting the argument got at the right idea. This exhibits a need for logical validity that encapsulates the jump from the insight of an example to the final generalized statement, that is, the destination. In the eyes of an expert, our concepts of logical validity and destination correspond to Weber's third concept: a proof presents, though deduction, a theorem as a consequence of already established claims. To be clear it is not the logical deduction within the generic argument itself they question, but rather if the seemingly salient properties of the example are actually germane to a process through which the generalized conclusion can be drawn. While the near universal objection of generic proofs by our experts may appear to be evidence that the presence of an example violates a communal norm, comments such as how "examples tend to be misleading in proofs" or "might just be a fluke" demonstrate that the experts' understanding of logical deduction is deep. As opposed to the students, experts are not criticizing the proofs based solely on an externally imposed requirement that proofs not include examples. That is, the experts were wary of generic proofs as they felt reasoning with the examples may have introduced additional assumptions that mistakenly lead to the desired conclusion.

Given our findings point to experts' propensity for considering generic proofs as "not real proofs" we challenge the notion that experts accept generic proofs as suggested by the findings of Miller, Infante, Weber, 2018. This difference in findings suggests that more research must be done in this area. In addition, our findings highlight an interesting issue; mathematics education researchers value using generic proofs as productive ways for supporting student learning, while mathematicians may not find them to be valid arguments.

With these findings in mind, we have the following questions:

- Would we observe the same results with different generic proofs? In particular, would generic proofs with more detail be more acceptable to experts?
- How do we best incorporate generic proofs into teaching proof but still support a healthy skepticism of examples?
- Why are experts more skeptical of examples than students?

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Proving Activities of Abstract Algebra Students in a Group Task-based Interview

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This proposal presents data from a session of a design-based research study in which three undergraduate mathematics students compare two attempts to prove that group isomorphism preserves the abelian property. We offer examples of students' proof construction, comprehension, and validation behaviors in a classroom-like setting. We observe that these three processes alternate and interact in this kind of teaching/learning environment. We argue this space thus provides new opportunities for investigating the relationships among the three kinds of activities to complement previous studies that isolate any one such activity.

Keywords: Student proving activities, abstract algebra, group interview

The field of research on proof and proving in mathematics education has received increasing attention and focus over the past several decades. As an essential and indisputable aspect of mathematical practice, engaging productively with proofs is an important, yet difficult, part of a mathematics students' experience (Stylianides, Stylianides, & Weber; 2017)

Despite the prevalence of proving in advanced undergraduate classrooms, majority of researchers have leveraged clinical settings with individual student interviews. Moreover, the types of student proving activities investigated in research have often fallen into one of the three following categories: proof comprehension, proof construction, and proof validation (e.g. Mejia-Ramos et al, 2012; Weber, 2001; Selden & Selden, 2003). Investigating different proof-related activities separately has been appropriate inasmuch as proving is a highly complex activity that engages a number of component competencies. We aim to build on this foundation of literature via studying students proof activity more generally as they interact with each other and proofs.

In this proposal, we discuss the some of the various proving activities of a group of three undergraduate students in a task-based interview study. In this group setting, students interacted with a teacher-researcher and the other students in the study modeling a classroom setting. In this study, we are investigating the nature of these proof activities as students engage with proof tasks and with each other. In particular, we illustrate how such activity plays out in this setting and posit that the activities often occur concurrently.

Literature Review & Theoretical Orientation

Our underlying assumption is that proof is a social activity. Thus, students' interactions with the proof, the teacher-researcher, and each other will shape their proof comprehending, validating, and construction activities. In this section, we briefly unpack each of these activities.

Proof Comprehension

A significant part of a mathematics student's experience is to study and read proofs both during lecture and in textbooks. From these proofs, students are expected to learn the norms of the fields and how to construct their proofs. Yet, research has shown the extent to which students understand and comprehend these proofs varies considerably (Mejía Ramos et al, under review).

In order to empirically explore this understanding, Mejia-Ramos and colleagues (2011; 2017) have developed an assessment framework for proof comprehension building on the work of

Yang and Lin (2008). This framework identified a number of related activities that capture both a learner's understanding of local aspects of a proof, but also a learner's holistic understanding of the proof. In particular, assessing local comprehension of a proof includes focus on (1) meanings of terms and statements, (2) logical status of statements and proof frameworks, or (3) justification of claims. Assessing a holistic comprehension of a proof includes focus on the ability to (1) summarize the proof, (2) identify the modular structure, (3) transfer the ideas of the proof to different concepts, or (4) illustrate the ideas of the proof using examples.

This existing literature provides a useful lens with which we explore how to extend the discussion of proof comprehension beyond that of assessment. We leverage this framework to inform the development of the tasks used in this project and to help us identify aspects of students proof comprehension activity in a classroom-like setting.

Proof Validation

The validation of proofs is an essential part of mathematical practice. While it is less common (but perhaps increasing in popularity) for students to be asked to validate a proof explicitly in their classes, proof validation is an important aspect of proof reading and writing. In Selden & Selden's (2003) seminal study on proof validation, the eight undergraduate students were able to assess the logically validity of proof in less than 50% of their evaluations of student produced proofs. Such results are consistent with Fagan's (2019) large-scale study of validation where undergraduate students largely struggled to identify *common validity issues* in proof.

Unlike comprehension, there is not a framework that captures validating activity. However, we can glean insight from Weber' (2008) and Weber and Mejia-Ramos' (2011) studies of mathematicians validating activity. In particular, validating appears to occur in two phases: determining the structure of the argument and then checking each line of the argument. Weber and Alcock (2005) have suggested validating a proof is exploring whether "If (a subset of the previous assertions in the proof), then (new assertion)" is warranted at each line of proof.

From this literature, we can identify validating activity through the lens of evaluation of warrants of claims and appropriateness of proof structure. In a classroom-like setting, students will be positioned to validate one another's' proof contributions.

Proof Construction

The majority of proof research has focused on the activity of proof construction. Despite this prevalence, we still know little about how to best support students in proving. Research has shown the difficulty in proof construction in advanced mathematics, such as Weber (2001) who identified differences in strategic knowledge in graduate and undergraduate students. The undergraduate students lacked knowledge about proof techniques, which theorems are important and when they will be useful, and when and when not to use syntactic strategies.

While most literature documents difficulties, we can gain insight into some aspects of construction activity that may be fruitful. Coarsely, proof construction can be divided amongst work in the formal representations and work in informal representations. From the formal side, Selden, Selden, and Benkhalti (2018) have suggested an impactful part of proving is anticipating the overall structure of a proof (proof framework) based on the structure of the claim to be proved. Recognizing what assumptions can be made at the outset and identifying the necessary final inference can help a student get started with proof construction.

From the perspective of informal representations, Weber and Alcock (2004) documented that students may find insight and proving success by first developing semantic arguments (such as through the exploration of diagrams and examples). However, moving between informal arguments and formal proofs is particularly challenging and i large scale studies was observed as highly infrequent (Zazkis, Weber, & Mejia-Ramos; 2016). Meanwhile, Zazkis, Weber, and Mejía-Ramos also identified elaborating, syntactifying, and rewarranting as activities that supported students who successfully translated between informal arguments and formal proofs.

The activities involved in constructing proof range from developing informal intuition, analyzing the formal structure of statements, and converting between informal and formal arguments. Within a classroom-like setting we may consider how these activities unfold between students and how an instructor may help support students develop strategic knowledge. The present research focuses on the proving activities of three students interacting in a group interview, designed to simulate an interactive classroom. Moreover, we present a preliminary investigation of the interaction between proof comprehension, validation, and construction.

Methods

This report considers data from one session of the first implementation of a design-based research project developing tasks to orchestrate discussions around proofs in abstract algebra[1]. The participants in this iteration were three undergraduate students, each of whom had completed abstract algebra. The sessions were led by a researcher with experience teaching abstract algebra and supported by another researcher with experience teaching introductory to proof courses. In the session to support students' proof validation activity, students compared two attempts to prove the claim: *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

In the session, students reviewed the relevant terms, attempted to prove the claim themselves, and shared their approaches. After this, students were provided the two student proof attempts shown in Figures 1 and 2. We note that in the G-first attempt, the arbitrary elements are selected from G rather than H. However, this proof fails to leverage surjectivity to argue that all elements in H are necessarily the image of elements in G. The H-first attempt is a valid proof.

Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof:

Let $a, b \in G \ni \varphi(a \circ b) = \varphi(a) * \varphi(b)$

$\iff \varphi(b \circ a) = \varphi(a) * \varphi(b)$ since $a \circ b = b \circ a$ (G is abelian)

but we know $\varphi(b \circ a) = \varphi(b) * \varphi(a)$ b/c φ is a homomorphism. So this implies

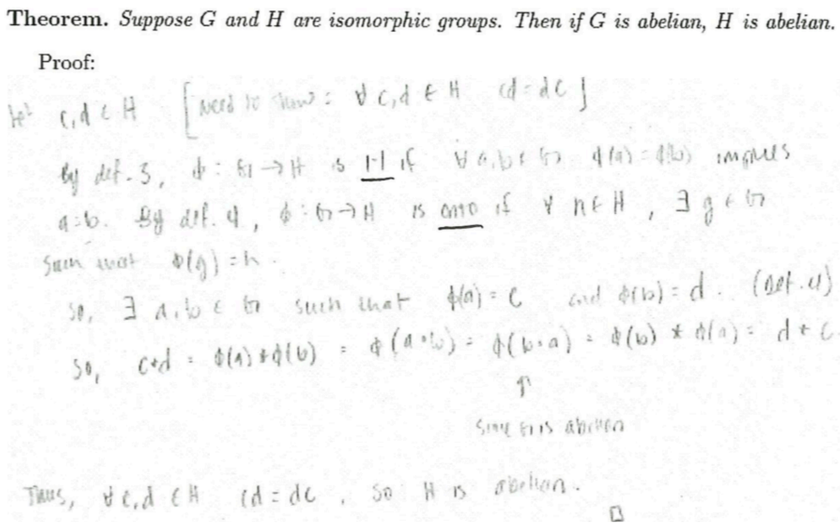
$\varphi(a) * \varphi(b) = \varphi(b) * \varphi(a)$ b/c $\varphi(a), \varphi(b) \in H$

Figure 1. G-first proof attempt.

Participants first read the G-first approach, illustrated its argument in a function diagram, and discussed the approach. They then read, illustrated, and discussed the H-first proof. They then compared and assessed the validity of the approaches. Finally, students investigated the need of the mapping's surjectivity and attempted to modify the claim and the proofs as necessary.

The session was videotaped, transcribed, and independently coded by the first two members of the research team for instances of student proving activity. While this session was developed to encourage proof validation, we quickly saw construction and comprehension activities were also vital to the students' progress through the tasks. These instances were categorized by types of proving activities discussed in the literature, while allowing for additional or new categories.

[1] For the scope of this paper, we focus on one session from one implementation. Further analysis will continue considering data from additional sessions and implementations of the study.


Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof:

let $c, d \in H$ [need to show: $\forall c, d \in H \;\; cd = dc$]

By def. 3, $\phi: G \to H$ is 1-1 if $\forall a, b \in G$ $\phi(a) = \phi(b)$ implies $a = b$. By def. 4, $\phi: G \to H$ is onto if $\forall h \in H$, $\exists g \in G$ such that $\phi(g) = h$.

So, $\exists a, b \in G$ such that $\phi(a) = c$ and $\phi(b) = d$. (Def. 4)

So, $c * d = \phi(a) * \phi(b) = \phi(a \cdot b) = \phi(b \cdot a) = \phi(b) * \phi(a) = d * c$

since G is abelian

Thus, $\forall c, d \in H$ $cd = dc$. So H is abelian. □

Figure 2. H-first proof attempt.

Preliminary Results

We first provide examples of student behavior for each of the three types of proof activities. We then provide an episode in which these occur in quick succession to show how they are not independent in a complex teaching-learning setting, even with a modest number of students.

Students Proof Comprehension Behavior

When presented with a modified claim (asking if a homomorphism preserves the abelian property) to explore the necessity of the surjectivity of the homomorphism, Student A offered:

> I think they work. I was looking at [...] the second proof that we looked at. Their main point that actually proves what the statement is saying, you can still use all of the stuff that they used based on that statement because they didn't really need to use the fact that they're one-to-one and onto. So it being bijective isn't necessary.

This quote suggests that Student A's activity is in the category of proof comprehension since they were considering the *key ideas* of the proof and *justifications* of claims made in the proof.

Students' Proof Validation Behavior

Further along in the discussion around the necessity of the homomorphism's surjectivity, Student A successfully identifies the flaw in the G-first proof attempt. Student A indicated that the proof does not prove the claim (all elements in H commute is not *warranted*) "because if you happen to pick an element in your proof that doesn't map then your proof won't work." Student B later continued, "Right, because we're just picking the ones where a g will match the map to an h." Finally, Student A concludes the G-first proof attempt is "only proving the parts that G maps to are abelian." Here, we see Students A and B working together to establish that the G-first proof is not a valid proof as presented. This behavior is indicative of proof validation because the students are identifying an invalid aspect of the given argument.

Students' Proof Construction Behavior

As the students debriefed at the end of the session, we asked why the H-first proof avoided the pitfalls of the G-first proof.

Student C: I mean, I guess I learned it in proof classes that you like, start with elements of what you're trying to prove and you kind of work back from what you know.

Interviewer: So, kind of the structure of the statement. It's like, if G is abelian then H is abelian, so H is abelian is a piece you're trying to prove.

Student C: Trying to prove, so you take elements in that and you show that those are what you need based on what you're given earlier.

In this exchange, we see Student C considering the *proof framework* of the claim to be proven. In this case, Student C indicated the proof framework stems from the claim to be proven, which is a universal claim about the group H. As such, the proof should consider arbitrary elements of H.

The Interaction of Proof Construction, Comprehension, and Validation Behaviors

At the outset, the students were asked to attempt to prove the claim and then to share their approaches with the other students. Though this first task prompt focused on proof construction, students spontaneously engaged with proof comprehension, construction, and validation. When asked to discuss their individual proof attempts, Student B volunteered:

> ϕ is an isomorphism from G to H and so by definition if you have two elements of group G then, like, x and y so ... So then you have $\phi(xy) = \phi(x)\phi(y)$. And then I mentioned that G was an abelian group and so again, $ab = ba$ for all a, b, and G. [...] then I established if G is a group and x, y are in G then [...] xy is also in G. And since is G is abelian, $\phi(xy)$ would equal $\phi(yx)$ and then I kind of ... That's all I have. So far.

In this quote, we see Student B beginning her proof construction by *stating assumptions* and *leveraging the relevant definitions* of those assumptions (Moore, 1994).

In response to Student B's approach, Student A said, "I think it made sense, using the definition of [...] group *G* was abelian, she was using that to have the ϕ acting on xy the same as ϕ acting on yx and I thought that that worked." Student A revoiced part of Student B's approach and *summarized the main idea*, suggesting that Student A was engaging in proof comprehension. Meanwhile, Student C expressed doubt about the approach, "I like that you unpacked the definitions, just walking down. But I'm not sure that that is in H." This doubt about the elements' membership in H suggests that Student C was engaging in proof validation of Student B's proof.

In this exchange, we see students are capable of organically engaging in these activities in a classroom-like setting in which discussion around proof is encouraged. In fact, a productive student discussion can involve and require switching between types of proof activities.

Discussion

This preliminary report presents evidence that students are capable of authentically engaging in proof construction, comprehension, and validation. In addition, we offer evidence that in classroom-like settings proof comprehension, validation, and construction may occur in tandem as various students produce proofs, others seek to understand them, and then validate them by building on those interpretations. This suggests these activities may need to be investigated holistically to understand the interactions in this type of setting, rather than segmented as they are traditionally investigated. In fact, as the session culminated in modifications of the G-first proof, we conjecture these activities are important components of proof analysis (Lakatos, 1976; Larsen & Zandieh, 2008).

In ongoing analysis of this data and data from our second implementation of these tasks, we are considering the alternation and interplay among these categories of proving activities, proving activities beyond the three reported here (particularly proof analysis), as well as instructor prompts that may encourage students to engage in these behaviors.

Acknowledgments

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Student Responses to an Unfamiliar Graphical Representation of Motion

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Student use of mathematics in physics is an area of current interest in RUME and physics education research (PER). In particular, the function concept has been widely studied in RUME but has received less attention in PER. This study probes the ability of introductory physics students to (1) interpret graphical representations of position vs. time functions and their corresponding derivatives, to (2) translate the graphical representation into a meaningful symbolic representation, and to (3) interpret a novel graphical representation. Data were collected through think-aloud interviews and analyzed using a conceptual blending framework (Fauconnier and Turner 1998). The novel representation was challenging for students but in some cases prompted generative reasoning and re-invention of 'known' rules and relationships.

Keywords: Physics, Kinematics, Function, Covariation, Blending

This work is part of a collaboration to investigate student learning and application of mathematics in the context of physics courses. Our project seeks to study student conceptual understanding of the mathematics encountered in physics courses, to investigate models of transfer, and to develop instructional interventions to assist student learning.

Introduction

The mathematical function is a concept that pervades much of introductory mathematics and physics instruction. Whereas there is extensive research on student understanding of the function concept in the Research in Undergraduate Math Education (RUME) community (Asiala et al 1997, Carlson 1998, Carlson et al 2002, Oehrtman et al, 2008, Moore and Paoletti 2015, Park 2015), there has been relatively little research in Physics Education Research (PER) explicitly attending to the function concept. The PER community has extensively studied student learning of one-dimensional kinematics (the description of motion) in the context of graphs of position, velocity, and acceleration (McDermott et al 1987, Beichner 1994), but very little of this work reflects understanding of related work in the math education community. A goal of this study was to unite RUME and PER work on student understanding of the function in kinematics.

Theoretical Perspective

This work has been influenced by two major theoretical lenses, those of conceptual blending and covariation. As this is preliminary work, we are still exploring whether these lenses are compatible and which might best help explain our results. We briefly describe each.

Conceptual blending is the theory that all human knowledge is pervaded by metaphors constructed from bodily experiences (Fauconnier & Turner 2002). Knowledge is grouped into resources that are activated together at appropriate moments. In the blending framework, these resource groups are termed *mental spaces*. When confronted with a new concept, the mind blends mental spaces to make sense of the notion at hand (Fauconnier & Turner 2008).

Several researchers have used blending to examine how students use mathematics in physics contexts. Two types of blends will be considered in this paper. Single-scope blends use one of the input spaces as the framework for the blended space. Bing and Redish illustrate a single-scope blend between a math space and a physics space in which a problem is solved using math machinery, and units are tacked onto the final answer without any attempt to make sense of the result (Bing and Redish 2008). The math input space is the framework for the blend, and the physics is used superficially. Double-scope blending combines input spaces to create a new mental space with elements non-native to either input space. Bing and Redish also show how students used a double-scope blend of math and physics in order to conduct a physically meaningful calculation. In this example, students exemplified a bi-directional flow of thought, starting in the physics space and using the math space to calculate, or beginning in the math space and using physical intuition to reason about a result.

This study was additionally influenced by research on covariation and in particular the notion of bidirectional reasoning (Moore and Paoletti 2015). Covariation describes the way that changes in one quantity are reflected in a second quantity and has been the basis of numerous RUME studies (Carlson 1998, Carlson et al 2002, Oehrtman et al, 2008). Moore and colleagues have studied student covariational thinking in the context of graphs, contrasting static shape thinking, (in which a graph is considered as an object, much like a wire) with emergent shape thinking (in which a graph is an emergent relationship between covarying quantities). They have often chosen non-canonical graphing tasks as means of eliciting student thinking rather than rote repetitive results. In particular, Moore and Paoletti (2008) state that "denoting the horizontal axis as the independent/input quantity is a convention common to the teaching of mathematics," and that "students understand it as a necessary part of graphs." They suggest switching axes to determine whether students are thinking bidirectionally.

Methods and analysis

We designed a research protocol using three graphs. In each context students were asked a series of questions relating to the function and its derivative, or the position, velocity and acceleration of two objects. The first task used a graph (see Fig. 1a) drawn from a RUME publication on student understanding of a function and its derivative (Asiala, *et al.* 1997). The second was a physics kinematics graph (see Fig. 1b) influenced by PER results on student difficulties with position and velocity (McDermott, *et al.* 1998). The third task reflected the recommendation of Moore and Paoletti to switch axes; we chose a graphical representation that would be novel to students. This task contained the same tasks as the second but used what physicists sometimes describe as the spacetime convention (see Fig. 1c). Spacetime graphs (or Minkowski diagrams) are frequently used in special and general relativity. They contain the same information as traditional position vs. time graphs, but the orientation of the axes is switched, with position on the horizontal axis and time on the vertical. From the blending perspective, we believed the spacetime diagram (see Fig. 1c) presented an opportunity for students to create or revise a blend "on the fly". Because of its orientation, students would have to blend spaces differently to accomplish tasks such as finding velocity and acceleration.

This work has taken place in the context of a two semester introductory calculus-based physics sequence. The courses from which students are drawn are taught in a large lecture format at a large public comprehensive university serving a diverse student population. The course sequence is required for physical science, engineering, and computer science majors, and sections are typically traditionally taught by a variety of instructors, none of whom were

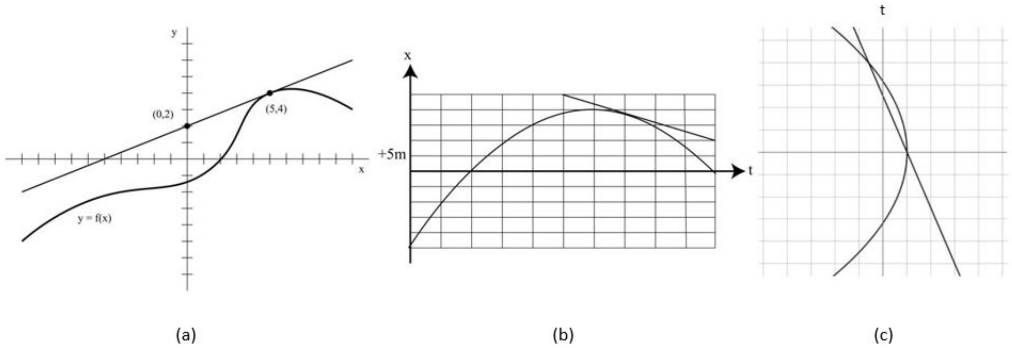

Figure 1. Graphs for interview tasks. Figure 1a is adapted from [4].

affiliated with this research study. The course uses the most common introductory physics text (Halliday, *et al.* 2018) and covers a fairly standard list of topics.

We performed semi-structured think-aloud interviews with volunteer students [N=7]. Interview volunteers were chosen from among students who completed the first semester course with grades of A or B; several were enrolled in the second-semester course. All students had completed relevant instruction on kinematics graphs as well as at least two semesters of calculus. Students came from academic majors including physics, engineering, and mathematics. Data were collected through audio and video recordings and transcribed for analysis.

While reading the transcripts, we began qualitatively identifying the reasoning that students were using and assigning words or phrases as elements of spaces. We identified three mental spaces from student reasoning: a mathematical formalism space, a graphical space, and a physical space. For example, *slope*, *tangent line*, or *rise over run* were categorized as elements of the graphical space. The analogous math space contained elements such as *f '(x)*, *f '(a)*, or *derivative* while the physics space consisted of expressions like *average velocity*, *instantaneous velocity*, or *change in position over change in time*. Based on preliminary coding, we generated what we believe to be an expert blend incorporating these spaces and the correspondence between them. Although prior literature did not distinguish between graphical and mathematical spaces, we chose to treat them as distinct spaces for the sake of our initial analysis. The question of how student reasoning with graphs relates to mathematical formalism is not simple and so this assignment should be regarded as tentative; recent analyses have suggested alternative approaches including 'graphical forms' (Rodriguez, *et al.,* 2018).

After developing the expert blend, we looked in the interviews for specific words or phrases associated with the corresponding mental spaces. We color-coded responses to examine patterns in student thinking. In many cases, extended sequences of a single color were grouped together, suggesting reasoning in a single mental space. Other situations contained multiple colors grouped together, suggesting blending. Examples are provided below.

Results

Analysis of student responses is ongoing. Below we present several results that illustrate the richness of student reasoning prompted by the spacetime diagram. All students expressed discomfort with the spacetime representation, and nearly all struggled to use the graph in Figure 1c in making predictions. For many of the students, the first response was to physically rotate or flip the paper in order to orient the coordinate axes in a more familiar way.

Inventing the Horizontal Line Test

The first example is chosen to show what we interpret as the revision of an existing blend. Student S1 was asked whether a position vs. time function was appropriate for Figure 1c.

I: Would it be appropriate to describe the position of these objects with a function, x of t?

S1: Umm... If you could like rewrite it so that like the x is, you know, there [vertical], and then t is there [horizontal], then you could write it as a function, I think.

I: I will note for the audio listeners that... one of the first things you did was to actually rotate the piece of paper.

S1: Yeah... Because it can't be a function if it doesn't pass the vertical line test... if it's f of x, the f would have to be up here and then the x would be here. So if it's x of t, then the x would have to be up here, the t would have to be here.

The student showed multiple signs of blending: she connected *function* and *vertical line test*, and *x of t* and *f of x*. In other portions of the interview, her responses reflected a strong double-scope blend. The new representation gave her pause; as noted, she rotated the paper to put the axes in familiar positions, and stated that the graph did not pass the vertical line test in the original orientation, and so it would have to be rotated. She was asked to return to the original orientation, and pressed on the original question:

I: ...if I want x of t, I can't draw the graph like that?

S1: Um... I don't think so, no. unless... Cuz if you can't move the axes or like rotate...

S1: Let me think. I think you can.

I: You can? You changed your mind again?!

S1: ...you would draw the vertical line tests this way [draws horizontal lines], and then it would pass...the vertical line test just shows...there's two inputs for the same output.

The vertical line test was mentioned by almost all of the interviewed students as a reason that it would be inappropriate to describe x as a function of t. S1 articulated how the graph failed the vertical line test in its original orientation, but then decided that the test was no longer applicable to this situation. She considered the meaning of a function, namely that every input has a single output, and thus invented a test appropriate for the spacetime convention.

Incorrect vs. Correct Blending Example

For this example, we compare responses to a question about the speed of two objects shown in the spacetime convention. With spacetime graphs, certain automatic tendencies are now incorrect. On a traditional x vs. t graph, a steeper slope translates to a greater speed, but in spacetime convention, this is no longer true. At $t = 4$, where the lines cross in Fig. 1c, the object depicted by the curved line has a greater speed. One student, S2, had constructed a single-scope blend of the physics space and the graphical space, and makes an incorrect conclusion.

I: At t=4, how do the speeds of the two objects compare?

S2: Object 2 is going faster than object 1.

I: And how did you know?

S2: Cuz the slope is steeper... Yeah, the linear line is going faster than object 1.

This student stated that the slope is steeper. His association of *greater speed* and *steeper slope* was no longer correct. In contrast, the student from above, S1, again seemed to revise a functional blend:

I: Which object is faster?

S1: Umm... I think that object 1's speed would be faster.

I: Why is that?
S1: Because it looks like the slope is steeper.

The two answers above contain almost identical language for the speed comparison and follow-up reasoning, however they arrived at exactly opposite answers. Student S1 interpreted her graphical reasoning through the physical knowledge provided by the axis labels. She stated the slope is "steeper", meaning that the object was moving a greater distance in the same amount of time, even though the line is not actually steeper in the standard usage of the term. The student had, in essence, revised her blend such that "steeper" now actually denoted "more gradual".

Initial Velocity Units Example

In this example, a student was asked to determine the velocity of an object on the spacetime graph. She expressed discomfort while considering the units:

S3: So, the velocity of this would be the slope of this line... so -10/4? ...Rise over run...I have a feeling that this is actually supposed to be 4/10 seconds per meters, whatever that means.

The student stated that the velocity is the slope of the line, displaying an incorrect blend of the graphical and physical spaces for this problem. She was reminded of the axes:

I: Which way is seconds and which way is meters?
S3: Oh, wait hold on... this is meters, this is seconds. ...So yeah, I think it's -10/4 seconds per meters. I think that's what it is, or it's the other one, meters per seconds.
I: And then, so that's velocity?
S3: I don't know what that is.

After considering the axes, the student switched the units *seconds per meters* to match the numerical value of the slope. When prompted further to check consistency with the original task of finding velocity, she claimed to not know what her answer was. In our small data set, this was characteristic with students who had constructed strict single-scope blends, which is consistent with the results of Bing and Redish (2007) who described 'tacked on' units. Because the graphical space served as the organizing frame of the blend, the change to the conventional axes made it difficult to use the physics space to interpret the answer.

Conclusions

We have analyzed student responses to introductory kinematics tasks in which we used a novel representation, the spacetime diagram task, using an analysis of conceptual blending. This work is preliminary and involved a small number of students, but results are suggestive. Most students, who had constructed single-scope blends that used the graphical space as their organizing frame, seemed to struggle with the spacetime graph because it disrupted the existing blend. At least one student was able to revise an existing blend and invent new rules an interpretations, possibly due to the strength of her double-scope blend. We believe this supports the suggestion of Moore and Paoletti (2015): "We hypothesize that students who are supported in thinking bidirectionally will construct more productive ways of thinking about function than those who are not."

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Adapting K-12 Teaching Routines to the Advanced Mathematics Classroom

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In recent years, there has a been a push for undergraduate mathematics classrooms to move away from purely lecture to a model where students are more actively engaged in their own learning. Such a transition is hardly a trivial task and requires robust instructional supports. Our recent work endeavors to adapt research-based supports from the K-12 level to the undergraduate abstract algebra classroom. In this report, we share preliminary results from a design-based research project directly aimed at adapting best practices to this new setting. We share several illustrations of how particular teaching routines (Melhuish & Thanheiser, 2017; Teachers Development Group, 2013) can productively unfold in a proof-based setting.

Keywords: instructional practices, proof

The teaching of undergraduate mathematics is a largely understudied practice (e.g., Speer, Smith, & Horvath, 2010). This is even more so the case for advanced mathematics classrooms with proof as their focal practice (see Rasmussen & Wawro, 2017.) From a practical standpoint, we have little knowledge of what may be productive instructional supports for this level classroom. This is not the case in the K-12 setting where there is a set of research-based instructional strategies (colloquially referred to as *best practices*) that include focusing on student thinking, facilitating student discourse, and creating a classroom where students engage authentically with mathematics (e.g., Jacobs & Spangler, 2017). Many instructional supports have been developed and studied in this setting in order to provide instructors with the tools to keep a classroom both student-centered and mathematically productive (e.g., Melhuish & Thanheiser, 2017; Stein, Engle, Smith, & Hughes, 2008).

As we move towards undergraduate classrooms being more student-centered (as recommended by Saxe & Braddy, 2015), we are in a position to develop similar instructional supports for the undergraduate level. In this preliminary report, we share some initial analysis of a design-based research experiment aimed at adapting K-12 supports to the needs of a proof-based abstract algebra classroom.

Background and Framing

If we want students to engage in authentic mathematical activity during classroom lessons, the first step is providing opportunities for them to do so (Cai, et al., 2017). As a mathematics education field, we have a general consensus that we value student-centered classrooms and that noticing student thinking and facilitating meaningful discussion are key instructional practices that support such classrooms (Jacobs & Spangler, 2017). At the K-12 level, there are a number of instructional routines that can serve to support students in engaging in authentic mathematics while continuing to move the mathematics forward. These include teaching routines such as selecting and sequencing student ideas (Stein, et al., 2008) comparing across student strategies (e.g., Durkin, Star, & Rittle-Johnson, 2017), leveraging visual representations (e.g., Arcavi, 2003), and providing students the time and space to make sense of tasks (e.g., Kelemanick, Lucenta, & Creighton, 2016). Through these and other best practices, classrooms can become a place where students do not just engage with open tasks, but their thinking can move the mathematical agenda forward.

Pedagogy at the Undergraduate Level

At the undergraduate level, we know substantially less about how this type of instruction may play out. We have some evidence that active learning (Freeman, et al., 2014) and inquiry-based learning (Laursen, et al., 2014) may support student learning gains, confidence, and more equitable outcomes. However, such analysis has been large-scale, and gives little insight into the nature of these classrooms. In fact, we can find other narratives countering that more student-centered approaches are positively aligned with student outcomes (Sonnert, Sadler, Sadler, & Bressoud, 2015) and are more equitable (Johnson, et al., 2019). Yet, another research-line, research on inquiry-oriented curricula, provides a series of results indicating with careful pedagogy, this type of classroom can be associated with student learning (e.g., Rasmussen & Kwon, 2007). Furthermore, beyond course outcomes, if we value students engaging in mathematical activity, it is propitious to develop classroom settings that are student-focused. This literature raises questions about not whether student-centered classrooms are better, but rather under what conditions they work best.

Answering that question can serve a substantial need for instructors wishing to move away from a traditional lecture model. Johnson, Keller, and Fukawa-Connelly (2018) recently surveyed instructors of abstract algebra classes to discover the nature of their instruction and why they choose to orchestrate their classrooms in particular ways. One important result is that many instructors do not think lecture is better for students; however, they identified a lack of resources and support in implementing ambitious pedagogy.

The Math Habits Framework

Mathematically Productive Teaching Routines	**Catalytic Teaching Habits**	
Generates student engagement in mathematically productive discourse by: (**Structure**) Structuring mathematically worthwhile talk (**Sequence**) Working with selected & sequenced student math ideas (**Public Record**) Working with public records of student mathematical thinking (**Confer**) Conferring to understand students' mathematical thinking & reasoning (**Represent**) Eliciting reasoning about visual representations (**Meaning**) Making meaning of tasks, contexts, and/or language	*Uses questions/actions to elicit students':* • Private reasoning time • Perceptions of the meanings of specific math concepts or properties • Mathematical noticings, wonderings, or conjectures • Metacognition or reflection • Mathematical reasoning on a problem or argument	*Orients students to ideas by:* • Prompting students to analyze contradictions or misconceptions • Prompting students to analyze a strategy or argument • Prompting students to compare or connect across students' reasoning • Exposing mathematical content in a non-verbal mode • Revoicing or recapping student ideas

The underlying hypothesis of our work is that best instructional practices at the K-12 level can be adapted to the abstract algebra setting. As such, we developed a series of tasks and instructional supports focused on key aspects of mathematically productive classrooms. In this study, we are focusing on the Math Habits framework (Melhuish & Thanheiser, 2017), a research-informed framework that captures teaching routines (extended routines that can support student engagement in authentic mathematical activity), catalytic teaching habits (the individual moves that teachers make to engage students in mathematical activity), and the resulting student activity. This framework operationalizes the triangle that occurs between math content, teachers, and student interactions (Cohen, Raudenbush, & Ball, 2003). For the scope of this report, we focus on the teaching portion to illustrate how particular routines may play out in the abstract algebra setting.

Methods

The data presented in this report stems from a design-based research project focused on developing instructional supports for the abstract algebra classroom. We developed three central tasks related to fundamental proof activities: validating, constructing, and comprehending. In this report we focus on the validating task (which also includes proof analysis to modify and test proofs and statements, c.f. Lakatos, 2015). The task centers on the proof that the Abelian property of a group is structural (preserved by isomorphism). We conducted two iterations with undergraduates who had recently completed introductory abstract algebra (three and four undergraduates, respectively). In the spirit of design-based research, we developed a set of conjectures of how students would respond to various instructional moves, then refined and altered tasks based on a close analysis of transcript and video data between implementations. We share data from the second implementation that incorporates substantial analysis-informed changes from the first. This lesson was roughly two hours in duration.

The focal lesson we share in this preliminary report was videoed and transcribed. This data was then analyzed using the Math Habits framework to identify the nature of the instructional moves and resulting student activity. Two researchers, who were not involved in the planning or implementing the lesson, coded the data independently and reconciled differences through discussion.

Sample Results

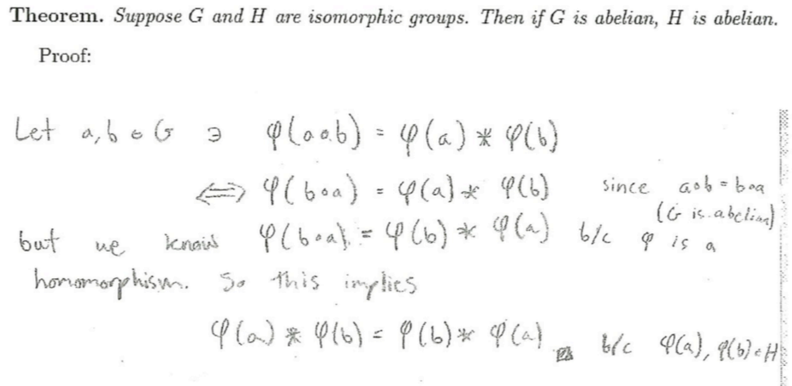
During the course of our focal lesson, 55% of the time was spent with the instructor(s) engaged in at least one teaching routine with all six appearing. Additionally, 62 catalytic teaching habits occurred covering all ten habits at some point in the lesson.

Teaching Routine Vignette 1: Making meaning of tasks, contexts, and/or language.

The first teaching routine that occurred was focused on making meaning of the theorem statement. Students were prompted to read the theorem and “sketch out how you might go about proving it.” This incorporated the CTH: private reasoning time, as the students were intentionally given time to write out their thoughts before engaging with each other. The instructor-researcher then asked, “So what are the types of things that we think about when we're going to prove something?” (CTH: metacognition/reflection). From this prompt, the students suggested the “givens” and “what we want to prove.” The instructor-researcher then created a public record on the board of the students’ suggestions (CTH: exposing content in a non-verbal mode) [Record: G and H are isomorphic. G is abelian. G, H groups; What we want to prove: H abelian.]

At this point we unpacked various vocabulary terms asking students to provide their definitions for abelian and isomorphic (CTH: press for perceptions of the meaning of mathematical ideas). This discussion served both to remind the students of the meaning of terms, but also provide a scaffold for the eventual proof by creating a record that included the definition of isomorphism (existence of an alpha that’s 1-1, onto, and a homomorphism) and abelian (for every $a,b \in H$, $ab=ba$.)

Teaching Routine Vignette 2: Working with public records of students' thinking.


Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof:

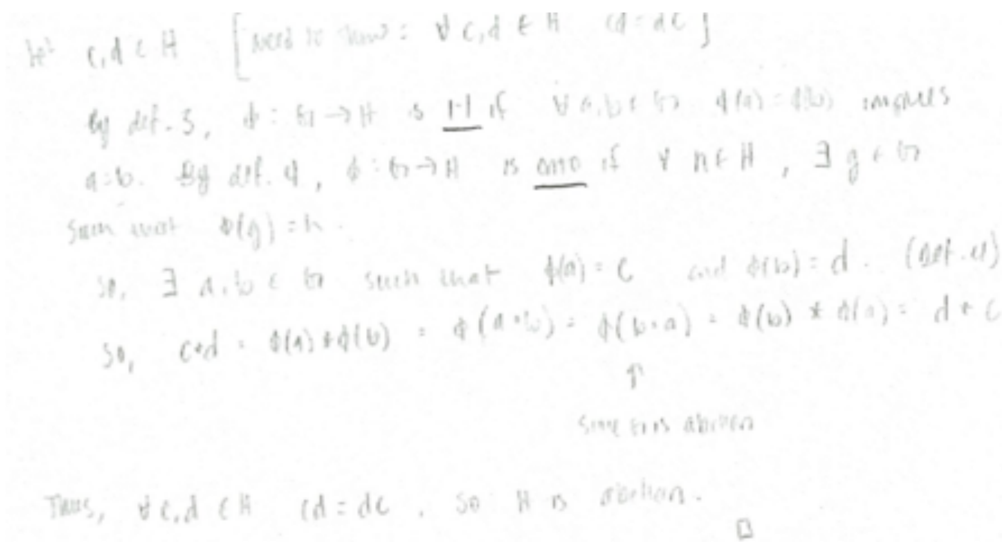
Let $a, b \in G \ni \varphi(a \circ b) = \varphi(a) * \varphi(b)$

$\iff \varphi(b \circ a) = \varphi(a) * \varphi(b)$ since $a \circ b = b \circ a$ (G is abelian)

but we know $\varphi(b \circ a) = \varphi(b) * \varphi(a)$ b/c φ is a homomorphism. So this implies

$\varphi(a) * \varphi(b) = \varphi(b) * \varphi(a)$ b/c $\varphi(a), \varphi(b) \in H$

Figure 1. An invalid (or incomplete) G-first approach.


let $c, d \in H$ [need to show: $\forall c, d \in H$ $cd = dc$]

By def. 5, $\phi: G \to H$ is 1-1 if $\forall a, b \in G$ $\phi(a) = \phi(b)$ implies $a = b$. By def. 4, $\phi: G \to H$ is onto if $\forall h \in H$, $\exists g \in G$ such that $\phi(g) = h$.

So, $\exists a, b \in G$ such that $\phi(a) = c$ and $\phi(b) = d$. (def. 4)

So, $c * d = \phi(a) * \phi(b) = \phi(a \circ b) = \phi(b \circ a) = \phi(b) * \phi(a) = d * c$

since G is abelian

Thus, $\forall c, d \in H$ $cd = dc$. So H is abelian. □

Figure 2. A valid H-first approach

The next vignette we share is focused on the teaching routine of working with public records of student thinking. We note the routine is not focused only on students sharing thinking, but also having students engage with each other's mathematical ideas. In this iteration, we provided the students with two common approaches[1] to this proof (one that argued that the images of arbitrary elements of the domain group, *G*, commute, and one that argued that arbitrary elements of the co-domain group, *H*, commute) as shown in Figures 1 and 2. The two sets of partners were provided one of each of the approaches and the following directions as presenter and listener roles:

Be prepared to explain this proof approach to your classmates. This explanation should include a function diagram that connects to the proof approaches. (CTH: non-verbal representation)

What is one thing about this proof approach that makes sense to you? What is something that you have a question about? (CTH: make sense of a strategy or argument)

[1] See Melhuish, Larsen, & Cook, 2019 for the frequency of these approaches and some of the underlying proof issues.

These prompts positioned the students to make sense of a proof approach, including through using a visual representation, and to engage meaningfully with each other's approaches. This task was particularly productive as the students successfully outlined the proofs including identifying important warrants, asked robust questions (such as identifying when elements are in the domain and codomain), and provided revoicings of each others' ideas.

The next prompts focused on connecting across ideas: "…the next thing that we're going to have you think about... kind of like a series of activities going through... is thinking about what's the same and what's different about these approaches, and you're welcome to chat again with your partner about this" (CTH: compare and connect across strategies). Through this prompt the students identified a number of commonalities including shared warrants (homomorphism property and abelian) and differences (beginning with elements in G rather than H, the use of 1-1 and onto). The instructors again kept a record of the similarities and differences on the board. By comparing and contrasting, the students (who found both arguments compelling) were positioned to notice the important differences across the proofs. Exploring these differences eventually lead to determining which warrants were essential (onto), which were not needed (1-1), and a discussion on how one approach better aligned with the statement via attending to the givens and what we want to prove.

Discussion

The early iterations of this experiment illustrated substantial promise in adapting teaching routines from the K-12 setting to the proof-based setting. First, we provide an existence proof that the types of teaching moves and routines from this setting can be adapted to the formal proof setting. Furthermore, these moves and routines seemed to be productive in the sense that students engaged in authentic mathematical activity (which we defined as proof analysis, construction, and comprehension) as a result of the teaching prompts and tasks. We also found that this iteration was productive in terms of meeting the underlying instructional goals which included: developing an appreciation the role of proof frameworks (Selden & Selden, 1995), deeply exploring the impact of a function being 1-1, onto, and a homomorphism, and arriving at important statement modifications. Comparing student strategies seemed particularly crucial to identifying the important differences between approaches which otherwise may have remained hidden. Furthermore, unpacking the statement in terms of givens and want-to-proves placed focus on the alignment of proof and the statement to be proven. In particular, the students arrived at noting the importance of starting with arbitrary elements from the codomain group, an approach rarely taken by students in introductory abstract algebra classes (Melhuish, et al., 2019).

Currently, this project is in early stages of analysis and implementation. As such, we acknowledge limitations of the generalizability of this work. Currently, we see a great deal of promising in adapting instructional supports from the K-12 level. If we want undergraduate student-centered classrooms to be productive, we need to develop such supports at this level, and then study their impact. Future research will include scaling from small-group lab settings to full classroom implementations to further adapt and refine this work.

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When Covariational Reasoning Does Not "Work": Applying Coordination Class Theory to Model Students' Reasoning Related to the Varied Population Schema and Distribution Graphs

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Covariational reasoning is a fundamental concept that is necessary to interpret graphical representations of dynamic processes; however, some graphical representations, such as distribution graphs (histograms), are intended to be "read" differently and require an alternative set of strategies for eliciting relevant information and drawing inferences. In this study, semi-structured interviews were conducted with twelve general chemistry students to investigate their reasoning related to the varied population schema, the idea that for a given system, molecules vary with respect to different parameters. Students were prompted to discuss distribution graphs that highlight the variation in a system and their reasoning was analyzed using coordination class theory, a framework that builds on the knowledge-in-pieces perspective to define "concepts" and conceptual change. Preliminary analysis indicates that although students that have productive ideas for sensemaking, they may not necessarily use them appropriately, in some cases unproductively using covariational reasoning to interpret the graphs.

Keywords: Coordination Class Theory, Graphical Reasoning, Covariation, Chemistry

In the physical sciences, there is a well-documented challenge associated with students' ability to combine mathematical reasoning and scientific principles to discuss the meaning encoded in graphical representations (Carpenter & Shah, 1998; Glazer, 2011; Ivanjek, Susac, Planinic, Andrasevic, & Milin-Sipus, 2016; Kozma & Russell, 1997; Planinic, Ivanjek, Susac, & Milin-Sipus, 2013; Potgieter, Harding, & Engelbrecht, 2008), with covariation serving a primary role in this process (Rodriguez, Bain, Towns, Elmgren, & Ho, 2019). Covariational reasoning reflects a necessary skill for interpreting graphical representations, drawing connections in advanced mathematics topics, and sensemaking related to dynamic processes (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Habre, 2017; Thompson, 1994). In most cases, approaching a graph by coordinating how x changes with y is a productive strategy for drawing inferences, but there are cases when graphs are intended to be "read" differently. As an example, reaction coordinate diagrams share surface features with graphs (energy on the "y-axis" and reaction progress on the "x-axis"), but should be understood as one-dimensional, with the width of the peak (i.e., horizontal distance) not having any physical meaning. This is often not explicit to students, given they tend to impose time on the x-axis (Popova & Bretz, 2018). Distribution graphs reflect a similar class of representations, in the sense that walking along the graph and describing the general trend is less useful for eliciting key information; instead, students need to be focusing on probabilistic ideas, central tendency, and consider the spread of values. Moreover, research on graphical representations of distributions (histograms) indicates students tend to interpret them in non-normative ways, and as with reaction coordinate diagrams, often impose time on the x-axis (delMas, Garfield, & Ooms, 2005; Lem, Onghena, Verschaffel, & Van Dooren, 2014).

In the context of chemistry, at the core of interpreting a distribution graph (e.g., number of molecules vs. energy) is the *varied population schema*, the ontological view that variation exists within a system, as opposed to viewing a system as being composed of uniform components

(Talanquer, 2015). Acknowledging the dynamic nature and variability of molecules within an ensemble builds on the idea that observable patterns (e.g., macroscopic measurements such as temperature and pressure) are the result of random, stochastic interactions. This is a key idea reflected in the Next Generation Science Standards' cross-cutting concept *systems and system models* (National Research Council, 2012), which iscritical for understanding emergent processes and properties (Chi, 2005; Chi, Roscoe, Slotta, Roy, & Chase, 2012). In this work, we are interested in investigating how students reason about the varied population schema in different contexts, including distribution graphs: *How do students coordinate knowledge to draw inferences related to the varied population schema?*

Theoretical Underpinnings

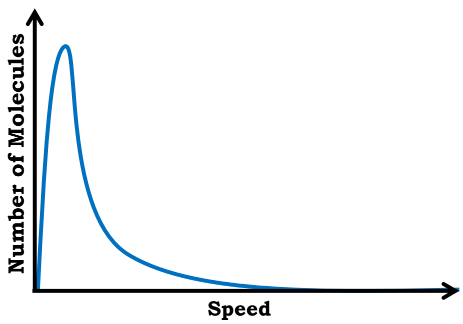
This work is rooted in the underlying assumptions outlined by *knowledge-in-pieces*, which characterizes knowledge as an ensemble of local knowledge structures comprised of fine-grained knowledge elements (diSessa, 1993). Knowledge-in-pieces is closely related to the resources perspective (Hammer, Elby, Scherr, & Redish, 2005), and in this work, *knowledge elements* and *resources* will be used interchangeably to describe the cognitive units associated with knowledge structures. Researchers across a range of contexts have characterized different types of knowledge elements and knowledge structures used for sensemaking, including mathematical resources (Rodriguez et al., 2019; Sherin, 2001), epistemological resources (Hammer & Elby, 2003; Redish, 2004; Tuminaro & Redish, 2007), ideological resources (Philip, 2011), and diSessa's (1993) phenomenological primitives, which reflect intuitive ideas constructed based on experiences.

In this work we used coordination class theory, a framework that builds on the knowledge-in-pieces perspective to define one type of a "concept", a coordination class. In short a coordination class consists of (1) readout strategies (approaches for obtaining and filtering information) and (2) a causal net (associated knowledge elements used to draw conclusions) (diSessa & Sherin, 1998). Within coordination class theory, conceptual change is posited to be the result of the gradual restructuring of the causal net and expansion of the set of readout strategies. Moreover, although not every concept can be described as a coordination class, physical entities involving measurable quantities are good candidates for coordination classes (diSessa & Sherin, 1998), with coordination class theory used previously to describe students' reasoning related to ideas in both physics (Buteler & Coleoni, 2016; Levrini & diSessa, 2008; Ozdemir, 2013; Parnafes, 2007; Thaden-Koch, Dufresne, & Mestre, 2006; Wittmann, 2002; Yerdelen-Damar, 2015) and mathematics contexts (Jacbson & Izak, 2014; Kapon, Ron, Hershkowitz, & Dreyfus, 2015; Wagner, 2006). In this work, we focus on the varied population schema as a coordination class, acknowledging that when students use a coordination class, they might only be using a concept projection (subset of their knowledge elements and readout strategies) and in practice concept projections from multiple coordination classes interact in dynamic ways to form a coordination system (Wagner, 2006). Therefore, in order to provide more information about a students' coordination class, multiple contexts and prompts must be used, in which each different question provides another "piece" (concept projection) of a student's coordination class.

Methods

Data Collection & Analysis

Participants for this study were recruited from a general chemistry course intended for science and engineering majors (n = 12). During semi-structured interviews students were provided different contexts used to probe students' reasoning related to the varied population schema: (1) sealed flask containing neon at room temperature; (2) distribution graph, number of molecules vs. speed; (3) distribution graph, number of molecules vs. kinetic energy. The graphs used in Prompt 2 and Prompt 3 are provided in Figure 1. The prompt involving the flask revolved around students' conception of the variation associated with the particles in the flask (i.e., system) and the prompts involving graphs centered around getting students to discuss the meaning encoded in the graphical representations (including surface features such as peak, area under the curve, axes, etc.), along with the extent in which students made inferences by using ideas related to the variation observed in a system.



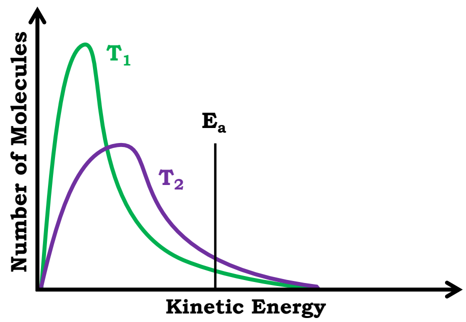


Figure 1. Graphs used in Prompt 2 (left) and Prompt 3 (right).

Data analysis utilized a case study approach involving a line-by-line analysis of each student's interview (Thomas, 2011), with an emphasis on characterizing the features attended to by the students and the knowledge elements they used to discuss the prompts. This process resulted in the construction of a resource graph for each student that represents the ideas the students discussed over the course of the interview (Wittmann, 2006). Thus, the resource graphs represent the knowledge elements of the causal net as opposed to explicitly outlining students' readout strategies; however, information is provided in the resource graph regarding the specific features students attended to (i.e., *readouts*) that resulted in the activation of knowledge elements and we posit that based on the relationship between knowledge elements and readouts, students general approaches for obtaining information, readout strategies, can be inferred.

The goal of the resource graphs is not to indicate every knowledge element discussed by the student; instead, the aim is to use evidence from the interview to posit a potential candidate for each student's causal net, or to be more explicit, a knowledge structure indicating the interaction between elements that are part of causal nets from different coordination classes (i.e., interacting concept projections). A single resource graph was constructed for each student because the stability and relationship between knowledge elements across the different contexts was a deciding factor if a knowledge element should be included in the resource graph (Wittmann, 2006). For example, although a student may have been prompted to discuss the units for the x-axis (e.g., *m/s as units for speed*), if the student did not subsequently connect this to other ideas or bring it up again in a different context, then it is less likely this knowledge element is a central piece to a larger knowledge structure; in contrast, if multiple features in the prompt cause students to bring up a particular idea (e.g., *values vary for different molecules*) or other ideas are consistently related back to this idea, it is more likely to be part of a stable knowledge structure. Thus, in order for a knowledge element to be considered part of a stable knowledge structure, we limited analysis to resource graphs containing three or more connected knowledge elements. Examples of resource graphs are provided in the next section.

Preliminary Results

Initial coding indicates that although students tended to focus on the same features in the prompt, there was a large range of knowledge elements activated as a result of this perceptual cuing. Here, we focus on two students' knowledge structures using resource graphs as a visual tool to illustrate the variation in reasoning exhibited by students (see Figure 2).

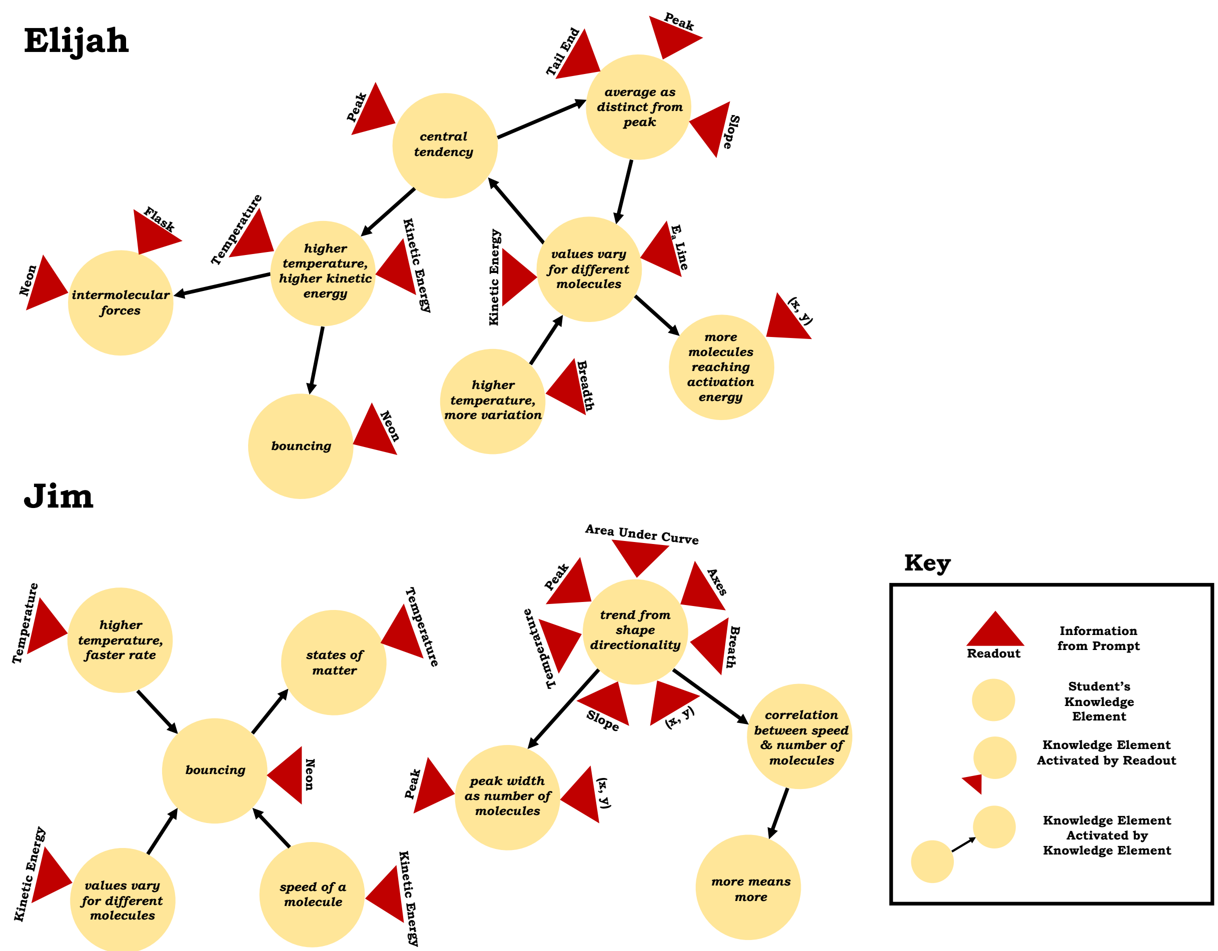


Figure 2. Elijah and Jim's resource graphs.

For Elijah, it seems he has a central node in his knowledge network, *values vary for different molecules*, which was activated by different features in the prompts (E_a Line, Kinetic Energy), and discussing other ideas led back to this knowledge element (*average as distinct from peak*, *higher temperature, more variation*). Looking at Elijah's knowledge structure, we can trace his line of logic that indicates he is viewing the provided graphs as a distribution, in which he recognized that the peak in the graphs provide a measure of central tendency, which is distinct from the average, given the non-symmetrical shape of the distribution. For Prompt 2, during the interview Elijah drew a line to the right of the curve, stating that the average would be shifted because of the curved tail end of the graph:

> "… the highest peak, it's at that value of speed that we're getting the most molecules in the neon vessel. … we have the highest number of molecules moving at a specific speed, the average is likely to be close to that … it would be fairly close towards the right on the peak. … Yeah, curves down. But I feel like that has more effect on the graph there by shifting the actual average speed of the molecules towards the right."

In the case of Jim, an interesting feature is the presence of two large, stable knowledge structures, one revolving around the phenomenological primitive *bouncing* (diSessa, 1993), a reference to the behavior of the particles in the flask (Prompt 1), and the other revolving around the graphical form *trend from shape directionality* (Rodriguez et al., 2019), used to describe the distribution graphs provided in Prompt 2 and Prompt 3. Based on the presence of two knowledge structures, we can infer that Jim did not connect the non-graphical prompt with the graphical prompts, suggesting he was attending to the surface features, rather than the underlying principles. It is also worth noting, the large number of readouts that led Jim to focus on *trend from shape directionality*, an inference regarding the graph's general tendency to increase or decrease based on the shape. At its core, this idea is closely related to what Carlson et al. (2002) describe as "coordinating the direction of change of one variable with changes in the other", a foundational concept associated with covariational reasoning. Jim's description of the graph depicting number of molecules vs. speed (Prompt 2) is provided below:

> "If number of molecules go higher, the speed would decrease, and if number of molecules go lower, the speed would increase. … Because as number of molecules goes higher the, they would bond together more, which would make them go faster. While, if there's not a lot of molecules then, there would be nothing pushing against each other to make them go faster."

As shown in Jim's discussion, it is clear he is interpreting the graph in the same way you would "read" other graphs in general chemistry (e.g., concentration vs. time). Moreover, the context leads Jim to suggest causality associated with the relationship between the number of molecules and their speed, reflecting a "centralized causal process" perspective of the phenomena (Talanquer, 2015).

Conclusion and Questions

Analysis is still ongoing and we are interested in how the variation of knowledge elements and readouts observed across the sample relates to students' readout strategies. It is also worth considering how instruction can use this knowledge to help students draw connections and attend to relevant features. Although Jim and other students in the sample seemed to have reasoning centered around inappropriately applying covariational reasoning, it should be noted that covariational reasoning is generally productive, with students simply needing more support regarding how to use these ideas effectively. These interests and others are provided below as potential questions for tentative directions for this project:

1. Based on the relationship between knowledge elements and readouts, what are students' readout strategies?
2. How can we promote meaningful connections between knowledge elements and expand the productive readout strategies available to students?
3. How can coordination class theory be applied to other contexts?
4. To what extent are the knowledge structures used by students stable across contexts, including distribution graphs depicting varied populations?

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Undergraduate Learning Assistants and Mathematical Discourse in an Active-Learning Precalculus Setting

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Undergraduate learning assistants (ULAs) are becoming more popular in mathematics classrooms. With this growth, there is a need to understand ULAs' roles in the classroom. Using a four-level distinction from univocal (ULA was the only voice in the conversation) to dialogical (ULA was fostering group discussion), we investigated the discourse between ULAs and students while in an active-learning setting. Our conjecture was that much of the discourse would fall in either univocal or dialogical; however, the ULAs' discourse was an array of all four ratings. We attribute this to the ULAs recognizing the "intellectual need" of the students in the moment.

Keywords: undergraduate learning assistants, mathematical classroom discourse, precalculus

Background Literature

Undergraduate Learning Assistants

Using undergraduate learning assistants (ULAs) is becoming a more common practice in STEM departments across the US. Over 100 institutions have implemented the model created by the University of Colorado Boulder Physics program (Learning Assistant Alliance, 2016; as cited by Sellami et al., 2017). In the research on ULAs, there have been increased student performance in content assessments (e.g., Otero, Pollock, & Finkelstein, 2011), including when ULAs were employed after active learning had been fully implemented (Sellami et al., 2017). Some universities investigated the ULA model for the effects on ULAs themselves, including changes in the ULAs' physics (and pedagogical) identities (Close, Conn, & Close, 2013; Nadelson & Finnegan, 2014), as well as the effects of the model as a K-12 teaching recruitment tool (Otero, Pollock, & Finkelstein, 2011).

Use of ULAs may be on the rise because of the chance to have more faculty-student interaction. According to Pavlacic et al., (2018) "faculty-student interaction is crucial for the student and the institution" (p. 3). The authors go on to cite literature to support positive effects of faculty-student interaction (e.g., personal development and academic achievement) and negative effects of not having these interactions (e.g., increased student withdrawal rates and students as passive recipients of information). In the undergraduate mathematics education literature, Webb, Stade, and Grover (2014) described the ULA model at the University of Colorado Boulder, addressing the curricular challenges as well as implementation. In the article, the authors discussed how ULAs perceived their effects on students. The overarching theme was that the ULA acted as an intermediary: "It gives [the students] someone to ask questions to, someone they can relate to…LAs were students just like them…a class with LAs also allows students to be able to communicate with teachers who are their own age" (Webb, Stade, & Grover, 2014, p. 45). We aim to add to the growing ULA literature by investigating the ULA-student interaction in more detail, positing some explanations of student performance and relationships with ULAs. First, we discuss what we mean by interaction in the classroom, so we turn to the discourse literature.

Classroom Discourse in Mathematics

Founded in the socio-cultural learning theory of Vygotsky (1978, 1986), the study of discourse, "or, more commonly, talk," (Imm & Stylianou, 2012, p. 130) in a classroom is an important aspect of mathematics education. For example, Wachira, Pourdavood, and Skitzki (2013) called discourse "a crucial aspect of a [actively engaged] classroom" (p. 2), and Truxaw and Defranco (2008) cited six articles that mentioned "the *quality* and *type* of discourse [being] crucial to helping students think conceptually" (p. 489, emphasis in original).

Imm and Stylianou (2012) found that there were three classifications of classrooms: *high discourse*, which is defined as, "teachers valu[ing] an exchange of ideas with students and where rich, inclusive, and purposeful mathematical conversation happened" (p. 131); *low discourse*, defined as, "when a teacher's talk rarely invited students into mathematical conversation or where one-directional 'telling' was the norm" (p. 131); and *hybrid discourse*, a mixture of the two. Imm and Stylianou (2012) utilized the notions of *univocal* (one-sided) and *dialogic* ("give-and-take communication that uses dialogue as a process for thinking" (p. 489)) discourse from Truxaw and Defranco (2008). In our study, we investigated the discourse between ULAs and groups of students, categorizing those interactions on a continuum developed by Martin (2018) influenced by Truxaw and Defranco's notions of univocal and dialogic.

Research Question

Initially, we set out to research the effectiveness of the ULAs in an active learning classroom. One way to gauge this effectiveness is by observing what the ULA did in the class. Therefore, our research question is: *With what kinds and levels of discourse do learning assistants engage with students in an undergraduate precalculus course?*

Methods

Participants

Three ULAs participated at a large Midwest research-intensive university in Spring 2019. The mathematics department recently began a change in its precalculus program (see Authors, submitted for details) adding ULAs and an emphasis on active learning to its precalculus courses. Jack was a ULA for all six semesters that the new course structure has been used. He was with the same instructor throughout that time as well as two other instructors while working as a ULA. Phoenix has been a ULA for four semesters and worked with three different instructors. Phoenix started out as a student in the new course format and was recruited to be a ULA by her instructor. Art has been an ULA for two semesters but was a ULA for a college algebra class the first semester. Spring 2019 was the first semester for which Art served as a precalculus ULA. Like Phoenix, Art was previously a student under the new course format.

Data Collection

There were two forms of data collection for this study: observations and interviews. There were two rounds of observations performed on each section of precalculus by the first author to maintain consistency with the ratings. An initial observation was done during the 4th and 5th weeks of the semester and another during the 14th and 15th weeks (out of 16 total weeks).

The rating system used during observations was inspired by the levels of mathematical discourse from Martin (2018). These levels of discourse were for a more traditional classroom atmosphere, so adjustments were made from Martin's definitions to fit the ULA and active-learning environment. These adjustments were mostly directed at viewing each individual group as the whole "classroom" and the ULA as the "teacher." The system was laid out in an observation protocol and interactions between the ULA and students were rated 1-4 as they fit the criteria. Many interactions were given a combination of ratings as the interaction would transform or meet multiple rating criteria.

The rating 1 (*univocal* level) was used when the ULA was the only voice during an interaction or part of an interaction. For example, this rating was used when the ULA was explaining a problem, giving an answer, guiding a student through work, or giving formulas.

The rating 2 (*partial univocal* level) was used when the ULA was asking closed-ended questions or prompting specific answers from the students. This also included when the interaction had some input from the student, but the ULA was the primary guide of the interaction.

The rating 3 (*emerging dialogical* level) was used when the ULA was asking open-ended questions or engaging in direct discussion with *one* of the students in the group. Either the student was the main voice of the interaction, or the discussion was split evenly between the ULA and the student.

The rating 4 (*dialogical* level) was used when the ULA-student interaction resulted in group discussion between students. This does not account for all student discussions, but just the discussions that were a product of ULA intervention.

There were three different types of semi-structured interviews conducted with a) precalculus instructors, b) ULAs, and c) students in the course. There were six instructors interviewed mid-semester; four student interviews and three ULA interviews were conducted towards the end of the semester. The second author used a common semi-structured interview protocol for all three types, adjusting questions based on the interviewee's role in the classroom. For example, an important question asked was, "When a student asks a question, how do you (or how does the ULA) typically answer that question?"

Coding

For this study, the program Nvivo™ was used to organize the observations and interviews. The second author gave a number rating to each interaction in real time. Some interactions were given multiple ratings when the interaction reflected multiple levels of discourse. There were two types of coding used on the interviews. Each interview was coded by types of questions, and open-thematic coding was used on the responses of the interviewees.

Results and Discussion

Table 1: Total amounts of each rating by section

Rating	*Sections*						
	A (Phoenix)	*B*	*C*	*D*	*E*	*F (Jack)*	*G (Art)*
1 (univocal level)	6	8	16	14	7	24	18
2 (partial univocal level)	2	6	13	10	13	15	15
3 (emerging dialogical level)	2	5	7	11	6	19	21
4 (dialogical level)	3	3	10	7	10	12	12

Examples for each rating from Art's second observation in Section G follow from a single interaction with a group:

- Student asks if they are doing things right, and ULA notices they aren't. LA starts to explain (1-*univocal*) and then poses questions to that student about the process (2-*partial univocal*).
- Student asks another question, and ULA lets the student talk her way through her question to understand how to move forward (3-*emerging dialogical*).
- ULA uses phrase "you could also think of this as y=5" when talking about the limit of the function 5 to help the student understand, and student says "ohhh". (This is a different method than how the instructor explained it). LA then uses the white board to explain (1-*univocal*) and poses a question to the group and all members get involved (4-*dialogical*).

When starting the research, we had a conjecture in mind: the ULAs would skew toward either *univocal*, because the ULAs were new to the classroom, or *dialogical* because the whole class was to engage in active learning. Instead, all three ULAs demonstrated a balance of all four ratings, as shown in Table 1. We believe this is due to the ULAs addressing the intellectual needs of the students. Harel (2013) defined an *intellectual need* as a "problematic situation prior to the construction of knowledge" (p. 122). It is reflected in one of our student interviews when asked about how her ULA answered her questions:

> If it's like a longer problem-I feel like if it's just a straight forward answer you know you can get without going through a lot of steps, he'll kind of give me the answer, and be like "oh, it's this instead" or he'll kind of hint at it like "you basically gave me the answer", but if it's like a lot of steps, he'll be like "okay, try it this way, and if you don't get the right answer, I'll come back".

A reason we believe the ULAs are attending to intellectual needs is because instead of relying on a uniform approach to answering students' questions, their techniques seem to vary for each situation. When Phoenix was asked about whether she directly answers students' questions, leads students to their own understanding, or facilitates discussion, she replied:

> Usually a mix of all three. Because if they are super struggling, I'll at least try to get them to think about the problem in a different way. Because if the instructor teaches it one way it's like "this is the way to do it", but I'm like "but there's multiple ways to do a problem" … I want them to think independently on their own unless if they're just completely lost. And then I'll sit down and just break it down piece by piece.

We note that Phoenix's total number of observed interactions were lower than the other ULAs because each interaction she had with students was longer than each interaction Jack, Art, and others had with their students. Jack, the experienced ULA, and Art, the first-time pre-calculus ULA, both also mentioned adjusting their questioning technique based on their perception of students' approaches.

The second reason we believe in the intellectual need motivation is that the ULAs show care for the students and were concerned about student success in the class. Jack stated that he is "paid to do something [he] would naturally volunteer for." When talking about what he hopes students take from the class, Jack talked about students' affect and well-being:

We want them to be comfortable with learning through failure… It's much less important if something is originally correct, but we want to check the understanding.

Preliminary Conclusion

By investigating the discourse of ULAs in an active-learning setting, we have introduced a new method of determining engagement. It seems as though the ULAs were balanced in their engagement overall, and that seemed to be best for students' intellectual needs. Finally, the ULAs' affect contributed to discourse; their care was influential for students and TAs alike.

Questions for the Audience

- What are other methods or approaches that could further discern dialogue?
- How much can ULA dialogue contribute to the environment of the classroom?

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Exploring and supporting physics students' understanding of basis and projection

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Linear algebra plays an important role in describing quantum mechanical systems and representing different quantum states. In particular, the ideas of basis and the process of changing basis are fundamental to understanding the nature of quantum states. However, quantum mechanics is abstract and can be very challenging for students. In this paper we describe the development and implementation of an activity that connects a quantum state to a Cartesian coordinate system as an analogy for understanding basis. We further describe changes made from the first implementation as we develop the activity for further use.

Keywords: Linear algebra, Basis, Physics, Quantum mechanics, Vectors

Introduction

Many quantum mechanics systems, specifically those that relate to measuring a particles' inherent angular momentum (referred to as spin), are associated with Hilbert spaces and are well defined using the language of linear algebra. States of particles are identified as superpositions of basis- or eigen-states, which are defined to be orthonormal in physics. Physical quantities that can be measured are described by operators and often represented as Hermitian matrices. Research has shown that the use of geometric and graphical representations can support mathematics students learning of abstract concepts in linear algebra including span and linear independence (Plaxco & Wawro, 2015; Wawro et al., 2012) and physics students in understanding the eigenvalue-eigenvector equation (Karakok, 2019).

More recently there had been a focus on more mathematical topics including student understanding of different quantum mechanics notations and representation of states (Gire & Price, 2015; Wawro, Watson, & Christensen, 2017), phase (Wan, Emigh, & Shaffer, 2019), methods of calculating expectation values (Schermerhorn et al., 2019; Sadaghiani et al., 2018), and the process of normalization (Watson, 2017). However, little work has addressed student understanding of basis. In studying understanding of phase, Wan and colleagues (2019) noted many students do not recognize that a relative phase in a coefficient for a basis state affects the measurement outcomes in another basis. Much of the research addressing student understanding of basis has focused on the two properties required for a basis: span and linear independence (Hannah, Stewart, & Thomas, 2013; Plaxco & Wawro, 2015; Stewart & Thomas, 2010).

Observations in our classrooms and in research interviews, conducted as part of a larger study, have revealed that students think a state vector is changed when represented in a different basis. Given the success of geometric and graphical representations in supporting linear algebra concepts, we seek to extend this to the process of changing basis in quantum mechanical systems. Since bases in quantum are defined to be orthonormal, a direct analogy can be made to the commonly used Cartesian coordinate system. We believe that leveraging this connection could provide students with a concrete way to think about state vectors and basis in quantum physics. As such, we seek to answer the following question.

- To what extent does a graphical representation support student understanding of basis and representation of vectors in quantum mechanics?

This paper describes the chronological development of an activity initially designed to help students understand basis and give a conceptual meaning for change of basis. Preliminary implementation revealed students struggled with the process of changing basis. Based on student feedback and response to the activity, we made modifications and plan further implementation and testing in the coming Fall semester.

Brief Overview of Quantum States and Change of Basis

In quantum mechanics, all information about a system is encoded in a quantum state represented as a vector in a complex Hilbert space. An example of a quantum state for a two-dimensional system is given below using a specialized physics notation, called Dirac notation.

$$|\psi\rangle = \frac{2}{\sqrt{5}}|+\rangle - \frac{1}{\sqrt{5}}|-\rangle \doteq \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Dirac notation defines a column vector for a generic vector as $|\psi\rangle$ and a row vector as $\langle\psi|$ to represent the complex conjugate transpose of $|\psi\rangle$. The symbol, $\doteq$, means "is represented by."

The states $|+\rangle$ and $|-\rangle$ form a basis, and are represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in this choice of basis. Bases are chosen to be eigenstates of Hermitian operators. When making a measurement of a physical property, the possible outcomes are the eigenvalues of the corresponding operator. The probability of a given outcome is the complex square of the coefficient when written in the appropriate eigenbasis. In quantum mechanics we often change basis when discussing different physical properties. For example, the state above is given in the basis of the S_z operator, referring to spin angular momentum in the z-direction. So it is possible to read off the probabilities of measurements. One method to solve for probabilities for measurement along the x direction (for the S_x operator) is to change the basis of the state. Later in the course, students are required to write states in specific bases to describe time evolution.

Change of basis is often performed by writing the state in the new basis with unknown coefficients. Those coefficients are then solved for using a projection, which is mathematically equivalent to a inner product. The new generic state is given as

$$|\psi\rangle = a|+\rangle_x + b|-\rangle_x$$

where $|+\rangle_x$ and $|-\rangle_x$ are the eigenstates of the S_x operator and $a = {}_x\langle+|\psi\rangle$ and $b = {}_x\langle-|\psi\rangle$. The information encoded in the quantum state is not dependent on the basis.

Building an Understanding of Basis

This section describes the design and implementation of an activity that builds on students' prior experience working with two dimensional vectors in their early undergraduate courses. The goal of the activity is to utilize this prior understanding and graphical representations to help students make sense of basis for quantum mechanical states and what it means to change the basis a state is written in.

Overview of the Basis Activity

Part one of the activity rewrites Cartesian unit vectors in Dirac notation, so that they are now represented by $|i\rangle$ and $|j\rangle$ in place of $\hat{\imath}$ and $\hat{\jmath}$. Students first answer whether these unit vectors could be used as a basis, and then are given a state vector, $|u\rangle = |i\rangle + 2\,|j\rangle$ which they are asked to normalize and graph using a set of Cartesian axes (Figure 1). Students then represent $|u\rangle$ as a column vector and are asked to represent the coefficients as an inner product. Lastly, they are

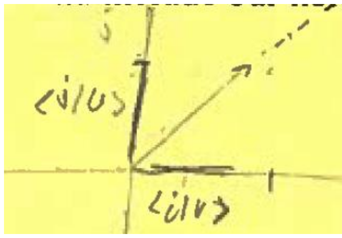
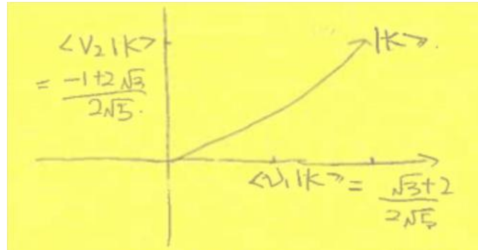
(a)  (b) 

Figure 1.(a) *A student's graph of the vector* $|u\rangle$ *using* $|i\rangle$ *and* $|j\rangle$ *as coordinate axes. They first sketch the vector and then label the inner products on the axes.* (b) *A plot of* $|k\rangle$ *in the basis of* $|v_1\rangle$ *and* $|v_2\rangle$ *from part two of the activity.*

asked to identify this inner product on the earlier graphical representation, as is represented on Figure 1, and discuss the conceptual meaning of the inner product based on the graph. This portion of the activity is designed to help connect the idea of an inner product in quantum mechanics to solidify the idea of a component in a specific coordinate direction.

Part two of the activity introduces a new basis of $|v_1\rangle$ and $|v_2\rangle$, given as column vectors using the Cartesian basis. Students are asked to write $|u\rangle$ in this new basis and call the resulting state $|\kappa\rangle$. Students are explicitly given the expression $|\kappa\rangle = a|v_1\rangle + b|v_2\rangle$ and asked to find the coefficients a and b. While $|u\rangle$ and $|\kappa\rangle$ are the same vector, the goal of the activity is to lead students to this realization. They are asked to graph the vector and label the inner products, now using $|v_1\rangle$ and $|v_2\rangle$ as their axes. When drawing $|v_1\rangle$ and $|v_2\rangle$ as horizontal and vertical axes, the location of the vector appears to be in a different place.

Part three of the activity asks students to find the relationship between $|u\rangle$ and $|\kappa\rangle$ by drawing all vectors on the same set of axes from the beginning of the activity. Careful plotting of these vectors would lead students to see that $|u\rangle$ and $|\kappa\rangle$ are overlapping The activity asks them to state the relationship between $|u\rangle$ and $|\kappa\rangle$, what the inner product would be, and to synthesize a conceptual meaning for change of basis. Lastly, students are asked to connect change of basis in the Cartesian coordinate system to a physics situation involving a quantum state.

Implementation

The activity was first implemented late in the semester, since it was developed following our observation of students' struggling with basis. It was administered at Cal Poly Pomona by two of the authors in a required recitation section for an upper-division spins-first quantum mechanics course. There were 37 students enrolled in the course over two recitation sections. Students worked in groups of 3-4. We note that the university where the activity was administered is a large, public, Hispanic-serving, primarily undergraduate institution. Linear algebra is not a pre-requisite for the quantum mechanics course, but some students had taken a course from the mathematics department that covers topics in both linear algebra and differential equations.

The activity was collected following the session and scanned before being returned to students. Additionally, students were asked to write a couple sentences of feedback following the activity. The remainder of this section presents observations made during the recitation session, the collected artifacts of student work, and student feedback. These results have been used to develop the activity further for the upcoming semester.

Preliminary findings

Observations during the recitation session showed that students were successful with part one of the activity. They were able to easily recognize $|i\rangle$ and $|j\rangle$ as basis vectors and subsequently graph the vector $|u\rangle$ in the first part of the activity (Figure 1). Further, students were able to translate from the Dirac representation of the state to a column vector and write the appropriate inner products. We expect this question to be more difficult when given earlier in the semester when students have had less practice with Dirac notation. Almost all students went on to define the inner products using the language of "projection" as shown in the following student response.

"The inner product is the projection of $|u\rangle$ onto either the $|i\rangle$ or $|j\rangle$ axes. (like a dot product)."

Part two of the activity, where students were expected to use the idea of projection to convert the state to the basis of $|v_1\rangle$ and $|v_2\rangle$, was more difficult. Instructor intervention was needed to cue the calculation of the coefficients. Once the state was written, students were able to graph the state and label the inner products as in part one.

In part three, the majority of students drew $|v_1\rangle$ and $|v_2\rangle$ correctly when graphing all the vectors using $|i\rangle$ and $|j\rangle$ as the basis and were then able to see that vectors $|u\rangle$ and $|k\rangle$ were indeed the same (Figure 2a). Because the components ended up involving square roots, some students plotted the vectors close but not identical. This is fixed after students are encouraged to plot carefully. A few students did not recognize $|v_1\rangle$ and $|v_2\rangle$ as vectors in the $|i\rangle$ and $|j\rangle$ basis, and therefore didn't plot them (Figure 2b).

Also, students acknowledged the inner product between $|u\rangle$ and $|k\rangle$ was 1. The few students who were not able to graph the vectors were able to calculate the inner product and recognize they were identical. Many students felt surprised and excited by this discovery. When synthesizing a definition of change of basis, most students wrote something explicitly related to reference frames and acknowledged the vectors, $|u\rangle$ and $|k\rangle$, were the same, as below.

"It's the same vector but with a new reference frame."

"Changing the basis is just another way of writing the vector. It doesn't physically change the state."

Responses of this nature indicate that the activity was successful in getting students to recognize that change of basis does not physically change the state but is a choice of representation.

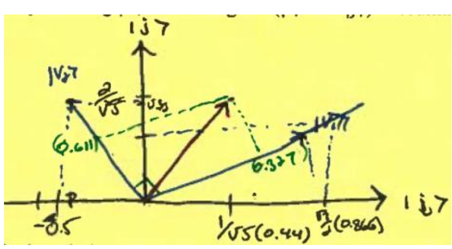
Student Feedback

Students generally found the activity helpful and felt it allowed them to relate the mathematical concepts to the physics concepts.

"The [activity] was great. It generalizes the procedure for changing basis and highlights that changing basis doesn't affect the overall value of the vector, only it's components."

"Made me remember classical mechanics, connecting like a new reference frame."

"I liked it. It was nice to relate the math to the concepts."

(a) 

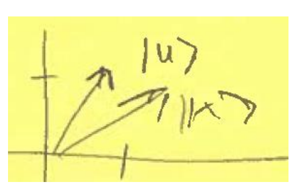
(b) 

Figure 2. Students' work from the last part of the activity showing (a) a correct plot of the vectors where $|u\rangle$ and $|k\rangle$ align and (b)an incorrect plot that doesn't account for the second set of basis vectors.

The feedback received was overwhelmingly positive and suggests that the activity was helpful to students, even those who had already learned change of basis earlier in the semester. Several students mentioned that they felt the activity would have been useful when first learning about basis, which is when the activity is intended to be used.

Discussion and Further Development

Quantum mechanics deals with very abstract and conceptually demanding ideas for students. Observations during the recitation session suggest that utilizing graphical representations that are already well understood by students can help support students' understanding of basis in quantum mechanics. The activity also highlights several areas where we can further support students. Students struggled in part two to associate inner products and projection as useful to carrying out a change of basis, despite being able to recognize that graphical meaning of the inner product as a projection to a specific axis in part one. This further supports the need for the activity, since students don't already have a strong understanding of how different bases relate to each other.

Analysis of a common exam question given at our three institutions further shows that not all students recognize projection as a viable method to change basis. Students were given a state in the basis of S_z and asked to write the state in the basis of S_n for spin in the $\hat{n}$-direction. The conversion between basis states was provided, as was the generic representation $|\psi\rangle\ = a|+\rangle_n + b|-\rangle_n$. The percentages of students correctly finding the state in the new basis using projection varied across the three universities with an average of 45%. Students answering incorrectly either attempted a solution involving excessive algebraic calculation or swapped the basis labels.

Accounting for the difficulty with projection, we plan to incorporate an additional first page for the activity to assist students connecting the inner product for a different basis state to the coefficient of the different basis state. We will also add an additional page at the end of the activity to help students understand the reason why we change bases in quantum mechanics. This uses an analogy to an inclined plane problem from classical mechanics, where a different set of coordinate axes simplifies the vector representation.

The refined activity will be administered in Fall 2019 as part of regular instruction earlier in the course. We'll further plan pre- and post-testing and interviews to study whether students are using projection to carry out a change of basis and several survey questions to probe individual student understanding of basis following the survey. Future work will seek to explore how students connect this understanding to other contexts in quantum mechanics.

Questions for the Audience

1. How far removed is our physics math from math math? Are there mathematical concepts or principles we could add to strengthen this activity?
2. Not all students have taken linear algebra. For those who have, what would/should they be drawing on when they come to our classes?
3. Are there other resources related to linear algebra that we may find useful?

Acknowledgments

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Getting Back to Our Cognitive Roots: Calculus Students' Understandings of Graphical Representations of Functions

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The derivative is a cornerstone of the first-semester calculus course curriculum. The concepts of function, ratio, slope, variable, covariation, and rate of change are considered to be cognitive roots of the derivative (Larsen, Marrongelle, Bressoud, & Graham, 2017)*, and this report argues for explicit attention to student understandings of output of a function particularly when considering the derivative in the graphical context. I draw on the location-thinking and value-thinking constructs of David, Roh, and Sellers* (2018) *for describing students' thinking about outputs of functions to connect student's understanding of output to student's conceptions of differences of outputs. Through a theoretical thematic analysis and a subsequent inductive thematic analysis of clinical interview data with calculus students* (Braun & Clarke, 2006)*, I code the mathematical objects refer to when considering outputs of functions presented in graphical contexts. Codes and potential sources of variation in codes are presented.*

Keywords: Calculus, Graph, Location and Value Thinking, Derivative, Outputs of Functions

Introduction

Numerous research studies report calculus students' difficulty with the concept of the derivative of a function (Larsen et al., 2017; Orton, 1983). Some research has sought to categorize student errors (e.g. Orton, 1983) while other research has worked to identify foundations of the derivative, or rate of change function, that are sources of difficulty for students when considering the derivative (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Monk, 1994; Nemirovsky & Rubin, 1992). Examples of the cognitive roots cited in the literature include ratio and slope (Byerley, 2019; Nemirovsky & Rubin, 1992), rate of change (Thompson, 1994; Thompson & Carlson, 2017; Zandieh, 2000), covariation (Thompson & Carlson, 2017), and variable (White & Mitchelmore, 1996). Research has shown that Calculus students' conceptions of derivative is related to the concepts of function, quantitative reasoning, covariation, and rate of change (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994; Thompson & Carlson, 2017). This work aims to identify how students' understanding of output of a function given a graph of a function relates to the development of the concepts of differences of outputs, difference quotient, and rate of change.

Research points to students' proficiency in computing derivatives of functions when given their algebraic representation difficulty in considering rate of change of a function and the related foundational concepts when given a graphical representation of a function (Larsen et al., 2017; Orton, 1983). Nemirovsky and Rubin (1992) investigated students' ways of reasoning through a series of tasks where students were matching the graph of a function with the corresponding graph of its derivative. They characterized two approaches students used to completing these tasks: resemblance-based and variation-based approaches. Resemblance-based approaches are based on the assumption that the graph of a function and the graph of its derivative will share perceptual attributes, such as increasing over an interval, being below the horizontal axis, and being straight or curved. Variation-based approaches focus instead on local variation of quantities on a graph and the relationship between a function and its derivative. Both approaches rely on students' understanding of the graphical representation of a function. Students focusing on the rate a function is changing are relying on their understanding of the graph of a function as a representation of the covariation of two covarying quantities (Thompson & Carlson, 2017).

Carlson (1998) and Monk (1994) also found that many students' meanings for the graph of a function were being derived from visual attributes of the graph rather than measured values. Monk (1994) found that students would confuse a graph of velocity with a graph of position, and that students interpret two cars to collide when the graphs of their velocities with respect to time intersect (Monk, 1992). These sorts of reasoning seem based on a function's graph as a picture, where points moving along the graph represent the scenario instead of the covarying quantities.

For these reasons, it is my hypothesis that students' understandings of a graphical representation of a function will strongly impact the cognitive development of the concept of derivative and instantaneous rate of change of a function in a graphical context. As a first step towards understanding this relationship, this analysis examines how students understand output of a function and differences of outputs in the graphical context, two concepts necessary for understanding the average rate of change of a function.

David, Roh, and Sellers (2018) suggest that students' meanings for output, graph, and points on a graph can largely impact students' reasoning about mathematical statements. I used David, Roh, and Sellers' location-thinking and value-thinking constructs (defined shortly) to characterize calculus students' understandings of function outputs. I believe these meanings for output impact both the relationships students focus on in their mathematical activity related to derivatives and their graphs and the conceptions that students develop based on these meanings.

In this report, I use this framework to investigate calculus students' understandings of output of a function and differences of outputs of functions. The research questions are:

1. How do first-semester calculus students understand output of a function in the graphical context?

2. How do first-semester calculus students understand differences of outputs of a function in the graphical context?

Theoretical Perspective

Implicit in understanding the derivative of a function in the graphical context is an understanding the graph of a function. Multiple researchers have investigated students' understanding of graph of a function (David et al., 2018; Moore & Thompson, 2015) and points on a graph (David et al., 2018; Thompson & Carlson, 2017). David, Roh, and Sellers's (2018) concepts of location- and value-thinking describe students' understanding of outputs of functions, graph of a function, and points on a graph of a function. They describe the form of reasoning, namely value-thinking, as being characterized by thinking of the output of a function for a given input value to be a value, a point on a graph to be a pair of values (input, output), and a graph of a function as a set of input/output value pairs. Students using location-thinking, however, are characterized as viewing an output of a function as lying in the Cartesian plane along the graph of the function. Further a point on a graph is the resulting output of a given input value, and a graph of a function is a collection of spatial locations in the Cartesian plane.

David, Roh, and Sellers's (2018) classification of student reasoning adopts the radical constructivist perspective (von Glasersfeld, 1995), which I also adopt. Students' meanings for outputs of function impact their meanings for differences of outputs of functions, average rate of change, instantaneous rate of change, and derivative.

To understand how conceptions of output for a graphical representation of function relate to the derivative, I draw on Zandieh's (2000) framework for derivative, where she deconstructs the concept of derivative as a function into three layers: function, limit, and ratio. These layers are connected in the framework through process-object pairs, where a process conception of one layer can draw on the object (or pseudo-object) conception of another layer, forming a chain of process-object pairs (ratio to limit to function). Zandieh also describes contexts where the conception of derivative can be described: the graphical, verbal, symbolic, and physical contexts.

To explore additional cognitive roots of the derivative for a fixed input, I extend Zandieh's (2000) framework to include outputs of functions and differences of outputs of functions. These layers extend the chain of process-object pairs, where process conceptions of differences of outputs can draw on object conceptions of outputs of functions, and process conceptions of ratio can draw on object conceptions of differences of outputs. This gives an explicit link to connect outputs of functions to the derivative for a fixed input. Additionally, I focus on the graphical context in this research. While the details for this framework are beyond the scope of this report, this motivates the notion that students' understandings of the derivative of a function for a fixed input value is related to students' understandings of the outputs of a function. In this report, I use David and colleagues' (2018) notion of location- and value-thinking to describe the layer of outputs of functions. Future work will investigate how students' understandings of outputs of functions impact the meanings they develop for the other layers of the framework.

Methodology

Data for this study was collected as part of a larger study aiming to understand how students' understanding of a graph of a function impacts the development of the derivative of a function for a fixed input. Data was collected from first-semester calculus students enrolled in a calculus course tailored to biological scientists in the summer of 2019 at a large western public institution. Two 90-minute semi-structured interviews were conducted in the first two weeks of the semester one week apart before students learned about instantaneous rate of change. Five students volunteered to participate in at least one interview for the study.

One task was chosen to be the focus of this paper. This task used Cartesian axes oriented in the conventional manner and asked students to represent both the outputs of two inputs and the difference between those outputs. The task administered is in the figure below (Figure 1).

Consider the graph of the function *f(x)* above. The lengths of colored line segments above represent the quantities a and h. Both the *x*- and *y*-axes have the same scale length.

Represent the following quantities to the right.

I. $f(a)$
II. $f(a + h)$
III. $f(a + h) - f(a)$

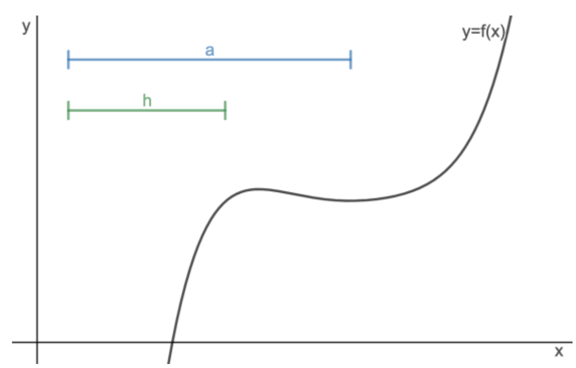

Figure 1: The Interview Task

The first two questions asking students to represent $f(a)$ and $f(a + h)$ were designed to help characterize of students' thinking of output as either value-thinking or location-thinking. The task further gives students no numerical nor labeled values to base their mathematical work on. This choice was made to afford students' understandings of output in the graphical context to be at the forefront of their mathematical reasoning, instead of computation. Analysis of student responses to earlier versions of these items in Spring 2019 revealed that some students computed outputs and differences of outputs of functions numerically and only then matched the computed numerical value to a labeling represented on the graph. For this reason, input values were represented as lengths of line segments. To further investigate the relationship between students' understanding of output and differences of outputs, students were asked to represent $f(a + h) - f(a)$. Students were then asked to estimate the average rate of change of $f(x)$ from input a to $a + h$, though this item was not considered in this initial analysis.

Results

In the theoretical thematic analysis (Braun & Clarke, 2006), three students had instances of reasoning during this task which were categorized as location-thinking. Instances of student's thinking coded as location-thinking involved students labeling and referring to $f(a)$ and $f(a + h)$ as points on the graph of $f(x)$. For example, one student responded this way when discussing $f(a)$:

Student A: So a would be here [gesturing to the mark on the x-axis of distance a from the origin], so then I guess $f(a)$ would be right here [labeling a point on the graph of $f(x)$].

Interviewer: So it is that point, right? The point on the graph, with that x-coordinate of a? [gesturing to the label '$f(a)$' that the student wrote down]

Student A: Yeah. Exactly.

Interestingly, students differed in their methods for how to identify these points along the graph. While two students (A and B) measured a distance of length a from the origin along the horizontal axis, Student C instead measured a distance of length a along the graph, basing the measurement from where the curve intersects the x-axis to the point on the curve that forms an arc length of length a. The student discussed the process of identifying the point through a measurement process along the curve akin to measuring the "length of a trail" on a map, stating:

Student C: So then this [length] a, if you're doing like the maneuvering like I would do if I was trying to, like, figure out the length of a trail, this would be the length of a from here to here... by, you know. Do you kind of see what I'm.... like I took like the whole length of a here and then all I do is, like, on the actual line itself, like, just continue to, like, align this, like, measurement that I'm using with the actual line, and just continuously turn it until I get to [length] a.

Student B was the only student who exhibited moments of value-thinking, referring to $f(a)$ as value or length. When talking about the process of representing different output, Student B began speaking about $f(a)$ and the axes not being labelled. This student made comments about $f(a)$ and whether he should respond to the item in terms of the input's length: "This is a [gesturing to the distance a along the x-axis], and the y-value is whatever fraction of a [gesturing to the distance between the point labeled $f(a)$ and the marking labeled a along the x-axis]".

Other forms of student reasoning about output did not seem to align with the notions of location- or shape-thinking. I inductively coded these instances of student thinking with regards to what mathematical object the student seemed to be referring to. One of the themes I used for describing students' thinking involved one student referring to outputs $f(a)$ and $f(a + h)$ as arcs along the graph of $f(x)$. This student (Student D) seemed to consider $f(a)$ as arc along the graph of $f(x)$ over a finite interval. When asked about what $f(a)$ referred to, she said "it's that whole thing" while gesturing along the graph from one location on the graph to another. It seemed that she was referring to the entire portion of the graph that connected the two points.

The last code used to describe students' understandings of output was $f(a)$ as a distance along the graph of the function. To produce $f(a)$, the student (Student E) measured an arc length of a along the curve similarly to Student C yet referred to the distance along the curve of length a as $f(a)$. When the student was prompted about imaging a string of length a, the interviewer asked:

Interviewer: "If you started the one end here, would the point here [gesturing to a point the student identified], would this point here being the end [of the string], would that be ... that, that distance be the, um, *f(a)*? Would it be just that, sort of, point here? The line segment itself?"

Student E: Yeah, it would be the whole distance from here to here.

Discussion & Future Work

This preliminary analysis suggests that calculus students bring a variety of understandings of output of a function to the classroom. Only Student B utilized value-thinking during this task, and for that student, their reasoning seemed to vary between value-thinking and location-thinking. Three students utilized location-thinking, and they drew on this reasoning in two distinct ways: by measuring length *a* along the horizontal axis and along the graph.

In this analysis, location- and value-thinking alone could not explain the variation in the sample's thinking. For instance, one of the five students (Student D) referred to *f(a)* as an arc along the graph of the function, and another student (Student E) referred to *f(a)* as a measured distance along the graph. One possible explanation is that these students viewed the input *a* as something distinct from a value. David and colleagues' (2018) constructs describe student's thinking about outputs of a function for a given input *value.* Since this task provided students axes with no a priori markings to designate particular values along the axes, students may have understood the input *a* differently from a value. For example, Student D may have understood the length *a* as an interval of input values, in which case, David and colleagues' (2018) constructs would not be appropriate. From this perspective, the arc along the graph may be the graph of the function over this finite interval. Considering this input, Student D's reasoning could be location thinking when considering *f(a)* as the image of an interval of length *a*, where *f(a)* is the set of points on the graph resulting from the output of each input value in the interval. Further analysis into each student's thinking of input may help explain students' responses.

This analysis points to an interesting potential relationship between the items' prompt language and students' inclination to demonstrate location or value thinking. Both students A and B represented *f(a)* (symbolic language) as a point, suggesting reasoning rooted in location thinking. However, both students responded to a follow up question asking to represent the output of *a* (verbal language) with a marking on the vertical axis, suggesting value-thinking. It seems that the students view *f(a)* and the output of *a* distinctly, and favor representing *f(a)* as a point and the output of *a* as a value, though normatively they are considered to be different representations of the same concept. In this sense, the graphical context can provide a setting to illuminate subtle differences in student's thinking. Given this distinction between students' graphical conceptions of *f(a)* and output of *a,* it worth noting that the items in David and colleagues' (2018) study and the current study refer to quantities with symbolic language. Understanding student thinking about output when questions are phrased in these two contexts would help elucidate this apparent distinction.

Lastly consider Student B's response to the interviewer asking about the meaning of *f(a)* and *f(a + h)* in the difference *f(a + h) – f(a)*: "If you're subtracting this [the point on the graph labeled f(a + h)] minus this [the point on the graph labeled f(a)], you're subtracting two coordinates [referring to coordinate pairs]…". Student B's subsequent confusion suggests that considering students' understandings of differences of outputs can help categorize student's reasoning about outputs. This also hints that certain forms of reasoning about outputs may afford or constrain the development of other concepts in later layers of the derivative. Future work aims to help illuminate these affordances and constraints for other layers of the framework.

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Physics Students' Implicit Connections Between Mathematical Ideas

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The Physics Inventory of Quantitative Literacy (PIQL) aims to assess students' physics quantitative literacy at the introductory level. PIQL's design presents the challenge of isolating types of mathematical reasoning that are independent of each other in physics questions. In its current form, PIQL spans three principle reasoning subdomains previously identified in the research literature: ratios and proportions, covariation, and signed (negative) quantities. An important psychometric objective is to test the orthogonality of these three reasoning subdomains. We present results that suggest that students' responses to PIQL questions do not fit this structure. Groupings of correct responses identified in the data provide insight into the ways in which students' knowledge may be structured. Moreover, questions with multiple correct responses may have different responses in different data-driven groups, suggesting that the both the answer choice and the context of the question may impact how students (implicitly) relate various ideas.

Keywords: Physics, Quantitative Literacy, Psychometric Analyses

One major goal of university-level physics courses is the development of mathematical reasoning skills, but despite decades of research on the complex interplay between physics conceptual understanding and mathematical reasoning (Boudreaux, Kanim, & Brahmia, 2015; Brahmia, Boudreaux, & Kanim, 2016a, 2016b; Rebello, Cui, Bennett, Zollman, & Ozimek, 2007; Sherin, 2001; Thompson, 2010), measuring these skills has not gained as much popularity as strictly conceptual assessments such as the Force Concept Inventory or Force and Motion Conceptual Evaluation (Madsen, McKagan, & Sayre, 2017; Madsen, McKagan, Sayre, & Paul, 2019).

Assessing conceptual understanding is inherently easier than assessing generalized mathematical reasoning. The former is tied to specific physics contexts taught over a finite period, while the latter is ubiquitous across contexts and time. Physics education researchers have conducted qualitative case studies to probe students' mathematical reasoning and their transitions to expert-like reasoning (c.f., Hayes & Wittmann, 2010; Hu & Rebello, 2013; Smith, Thompson, & Mountcastle, 2013). While this method of research provides a rich view into how specific students reason in a particular context, little has been published that characterizes the process of emerging expert-like mathematical reasoning across multiple topics and for a large group of students.

We have developed the Physics Inventory of Quantitative Literacy (PIQL) to meet the need fora robust and easily administered multiple-choice assessment to measure students' mathematical reasoning in the context of physics (a.k.a., physics quantitative literacy, PQL) (Olsho, White Brahmia, Boudreaux, & Smith, 2019).The PIQL is intended to test three key components of PQL: ratio and proportion (Cohen & Kanim, 2005), covariation (Carlson, Oehrtman, & Engelke, 2010; Hobson & Moore, 2017; Moore, Paoletti, & Musgrave, 2013;

Paoletti & Moore, 2017), and negativity (Brahmia & Boudreaux, 2017; Vlassis, 2004; White Brahmia, Olsho, Smith, & Boudreaux, 2018, 2019, in press).

We consider PQL to be a conceptual blend between physics concepts and mathematical reasoning (Fauconnier & Turner, 2002). In order to measure the complexity of ideas that students bring from both of these input spaces, we have chosen to include some multiple-choice multiple-response (MCMR) questions in which students are instructed to "select all statements that **must be true**" from a given list, and to "***choose all that apply***" (emphasis in the original text). The MCMR question format has the potential to reveal more information about students' thinking than standard single-response (SR) questions, but it also poses problems with data analysis, as typical analyses of multiple-choice tests assume SR questions. We have previously compared two different methods that could be used to identify groups of questions evident in students' responses to PIQL questions and compare them to the groups defined by our three PQL constructs (Smith et al., in press). In this paper we briefly describe these methods and present our preliminary results from using module analysis for multiple-choice responses, which allows each correct response to MCMR questions to be included separately (Brewe, Bruun, & Bearden, 2016).

Data for this study were collected in two different terms at a comprehensive public university in the Northwestern United States. The PIQL was given as a pretest during the first week of the term in three different calculus-based introductory physics classes: Mechanics ($N = 821$), Electricity and Magnetism ($N = 701$), and Thermodynamics and Waves ($N = 585$). These data do not form a matched set, but we take them as three snapshots in time, which may be representative of a progression through the introductory course sequence: before mechanics (PreMech), after mechanics and before electricity & magnetism (PostMech), and after electricity& magnetism (PostEM). Previous analyses of these data have shown that students' overall scores on the PIQL increase over time, so we can consider students progressing toward expertise through our data (Smith et al., 2018, in press).

We have previously reported preliminary results from applying module analysis to identify coherent groups of PIQL questions and responses (Smith et al., in press). In the current study we interpret these results in new ways that focus specifically on the questions related to covariation. Our work is guided by the research question: In what ways do patterns of students' responses to PIQL questions reveal insights into the ways in which they think about and use covariation in physics contexts?

Previous Results: Coherent Modules Identified from Students' Responses

We have previously used both confirmatory and exploratory factor analysis to identify groups of questions evident in students' response patterns to PIQL questions and compare those groups with our pre-defined PQL constructs (Smith et al., in press). These results showed a poor alignment between response patterns and our three primary constructs of PQL: ratio and proportion (questions 1–6), covariation (questions 7–14), and negativity (questions 15–20); however, a major limitation of factor analysis is that questions must be scored dichotomously as either completely correct or incorrect. This is problematic for MCMR questions because a student who chose one correct response to a question with two correct responses would be coded the same way as a student who chose multiple incorrect responses; therefore, we choose not to emphasize these results here.

To preserve the nuance and complexity of students' response patterns within (and between) questions we used module analysis for multiple-choice responses to examine the network of student responses to PIQL questions (Brewe et al., 2016). Module analysis uses community detection algorithms to identify modules (a.k.a. communities, clusters, etc.) within networks of responses to multiple-choice questions. We have chosen to analyze a network of only correct responses to PIQL questions. The benefit of this method is that we can examine the patterns that arise from students' selections of each individual correct response, which preserves some of the complexity of module analysis questions by recognizing any time a student chooses a correct response. A limitation of the way that we have used module analysis is that we are ignoring whether or not a student chooses incorrect responses in addition to correct responses. Expanding the network to include correct and incorrect responses could address this limitation, but is beyond the scope of the current study.

In network analysis studies, the choice of community detection algorithm seems to depend a lot on personal preference of the researchers. Unfortunately, the InfoMap algorithm used by Brewe et al. (2016) did not yield useful results.[1] In the absence of clear guidelines regarding which community detection algorithm would be most relevant, we chose to compare the modules identified by six different algorithms. We feel confident that modules that are identified by multiple community detection algorithms are representative of the data.

A major result from our analyses was that the modules were not consistent in our three time-dependent data sets. The results went from four modules in PreMech (two with only two responses each) to six modules in PostEM (most with only 2–3 responses). Contrast this to what might be expected for a hypothetical group of experts: true experts would answer all questions correctly, resulting in strong links between all correct responses, and all responses being in one coherent module. Our data show that as students progress toward expertise during the introductory sequence, modules become less coherent, not more. Additional data from upper-division students are needed to examine the continuation of this progression.

The changes in module definitions over time led us to look for consistent patterns across the results, which may represent stable elements of student reasoning. Figure 1 shows the average likelihood that each question pair occurs in the same module as well as the "submodules" that we have identified as being consistent across our analyses. Each of these submodules may be seen as a bright yellow/orange square along the diagonal in Figure 1, with submodule i (in the upper right corner) being the least cohesive (least bright). Some submodules are subsets of our PQL constructs: ratio and proportion (iii), covariation (ii and viii), and negativity (ix). Others include questions from two or three of these constructs (i, iv, v, vi, and vii), emphasizing the connections between these constructs.

The MCMR questions with more than one correct response show some particularly interesting trends. Question 17 has two correct answers (17D, 17G) that group very strongly together. Question 16 also has two correct responses (16C, 16D), but they do not group into the same submodule. Question 9 has three correct responses: 9C and 9D are in submodule i (which is the least coherent submodule mentioned above), and 9A groups equally well with two different submodules, neither of which connects with 9C or 9D.

New Insights: Covariation Questions Group Together and Split Apart

To answer our research question we consider the questions intended to assess students' PQL regarding covariation (7–14). Some of these items were taken directly (with permission) from the Precalculus Concept Assessment (PCA) (Carlson et al., 2010): question 8 asks students to interpret the slope of a graph of a function (modified from the PCA by graphing speed as a function of time), questions 11 and 12 are bottle questions for which students need to either

[1] Wells et al. (2019) report similar difficulty using the InfoMap algorithm.

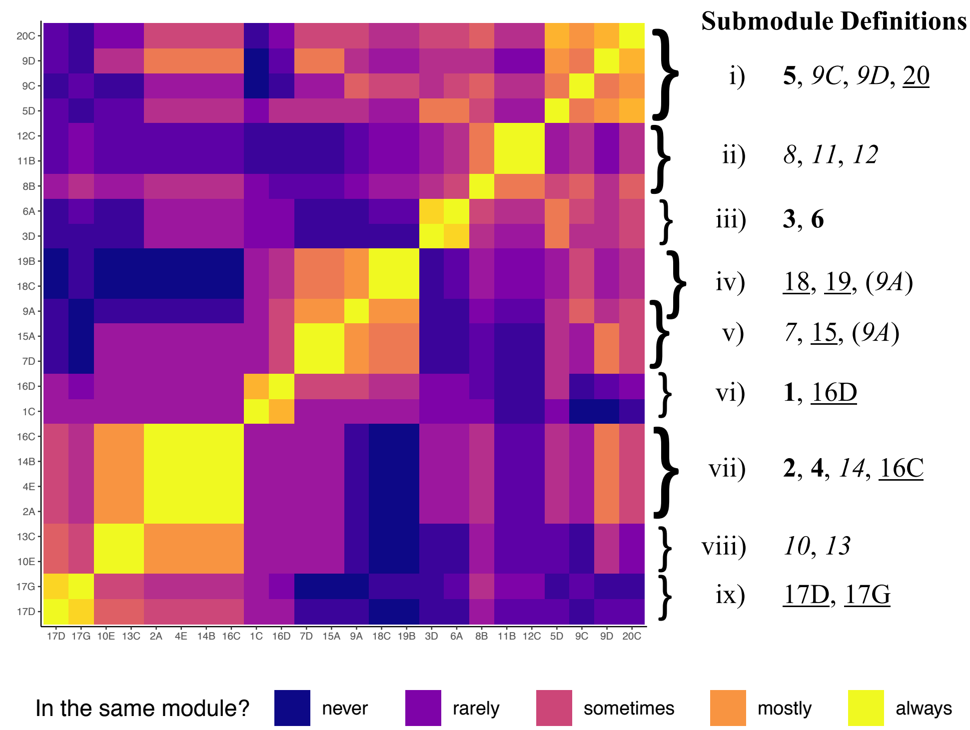

Figure 1. Previous results from applying module analysis to PIQL data (Smith et al., in press). The heat map shows the average cooccurrence matrix from the three data sets (yellow indicating responses that are always in the same module, dark blue indicating responses that are never in the same module). The bright squares along the diagonal are used to identify the consistent "submodules" across data sets. Questions intended to probe ratio and proportion are shown in bold in the definition list, questions for covariation are in italics, and questions for negativity are underlined. Response 9A fits equally with two submodules.

select a bottle to match a given graph (11) or select a graph to match a given bottle (12), and question 14 asks students to interpret a ratio expression with a variable in the numerator and denominator to make claims about both the rate of change of the ratio and its limiting value (in the context of fish in a lake). Question 7 is a variation on the students-professors question in which students must select an equation to represent the statement "There are three times as many quarks as nucleons." Question 9 tells students that the length and width of a flag both increase by a factor and asks which, if any, of the following quantities also increase by the same factor: perimeter, area, length of diagonal, and/or length of curve superimposed on the diagonal. For question 10 students must compare the distances traveled by two joggers who run at different speeds for different amounts of time, and for question 13 students are given the equation $m = k\frac{p}{3n^2}$ and asked what happens to m if both n and p double (with k held fixed).

As seen in Figure 1, these questions show up in five out of the nine submodules, but only two submodules include only covariation questions (ii and viii). We identify commonalities between the questions that appear in covariation-only submodules. Questions 8, 11, and 12 (submodule ii) all require students to interpret graphs, and these are the only such questions on the PIQL. Questions 10 and 13 (submodule viii) both require students to determine the output of a known function with two input variables. The questions in each of these submodules test students' abilities to use a unique and somewhat sophisticated type of reasoning.

It is more difficult to identify why other questions/responses group together (or don't). It makes sense that students would choose 9C (the curve along the diagonal of the flag) at similar rates as they choose 9D (the diagonal itself). But why is 9A (the perimeter of the flag) more strongly connected to questions 7 and 15 (both of which require students to select an equation based on a description) or to questions 18 and 19 (which involve interpreting negative vector quantities)? Questions 2, 4, and 14 all involve ratios, but 16C requires students to correctly compare a positively charged sphere and a negatively charged sphere, which does not (on the surface) seem to be related. More work is needed to reveal why students' responses group in these particular ways.

Summary and Future Directions

As mentioned above, the submodules identified by module analysis do not correspond with the groups defined by our PQL constructs. This suggests that either a) students' PQL cannot be separated into skills regarding ratio and proportion, covariation, and negativity, or b) their skills in these areas have developed similarly such that they are functionally equivalent. Regardless of the interpretation, module analysis reveals complexity and structure that changes over time as students progress through the introductory physics course sequence.

Several questions still remain:

- How do the modules identified by students' responses to PIQL questions change over time throughout the undergraduate curriculum?
- How sensitive are these modules to different forms of instruction? Are they the same at different institutions or in different courses?
- How do students' choices of incorrect responses relate to their choices of correct responses (especially for MCMR questions)? How often do students contradict themselves?
- Are there underlying commonalities that we can identify in each module? How do these relate to previous literature on quantification and quantitative reasoning or students' conceptual understanding of physics?

Module analysis opens the possibilities for future work that goes beyond analysis of only correct responses by identifying modules of incorrect responses as well (Brewe et al., 2016). Including incorrect responses in the module analysis could provide evidence to explain why some questions and responses group together in unexpected ways. This is particularly important for interpreting responses to MCMR questions, as different (in)correct responses can reveal insights into different aspects of a student's understanding regarding both physical concepts and mathematical reasoning. This interplay is essential to measuring PQL.

We plan to look more closely at the dynamics of the defined modules over time by using matched sets of responses collected from the same students at different times, and by expanding data collection beyond the introductory sequence. These longitudinal data will allow us greater confidence in claims regarding how students' response patterns change over time. Future work will also include expanding data collection beyond a single university. The coupling of PIQL MCMR questions with module analysis shows promise for finding patterns of emergent expertise in mathematical reasoning in introductory physics, and beyond, on a scale that cannot be achieved using qualitative research methods.

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Supporting Underrepresented Students in an Undergraduate Mathematics Program

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Increasing diversity in STEM fields is a moral and economic imperative. However, it is unclear what support systems are most helpful to retaining low-income underrepresented students in undergraduate mathematics programs. This paper and presentation is a preliminary report describing and assessing four support systems developed to increase retention in a four-year mathematics program.

Keywords: Equity, Policy

Increasing diversity across Science, Technology, Engineering, and Mathematics (STEM) fields is a goal that research communities and the public agree upon (i.e. PCAST, 2012). Despite agreement, the question of **how** to support underrepresented students to remain in STEM majors is unanswered (Karp, 2011). The purpose of this proposal is to report on the first semester of a 5-year longitudinal study, which was designed to provide support to low-income underrepresented students in an undergraduate mathematics major. In particular, I describe the supports we designed, why they were chosen, as well as qualitative data assessing those supports.

Literature Review and Background

The difficulty in increasing the amount of diverse students in STEM fields is two-fold: students choosing to major in STEM (Snyder & Dillow, 2011), and then students remaining in a STEM major through graduation (Chen 2009; Higher Education Research Institute 2010). This study focuses on the latter issue. Although retention in STEM majors is the same for non-STEM majors (Chen, 2013), there is evidence that women, underrepresented minorities, first-generation students, and those from low-income backgrounds leave STEM at higher rates compared to other groups (e.g. Anderson & Kim 2006; Hill, Corbett, & Rose, 2010; Griffith, 2010). Seymour and Hewitt (1997) reported that many students leaving STEM were high performers who left due to feeling unwelcome or other non-academic factors. While 100% retention is unreasonable, we owe it to STEM students to examine what support programs are most useful to students, and in what ways.

Karp (2011), in a review of non-academic supports designed to help university students, noted that there were four non-academic supports that were helpful to students: building *social relationships*, educating students about *career options*, illuminating the *college structure*, and supporting students through *life issues*. I have added the italicized emphasis to indicate the shortened names that I use to talk about these four areas of support that are necessary to helping students. *Social relationships* means helping students find study groups, and generally providing them with opportunities to meet and bond with other students. Students who built communities at college were more likely to remain in college (Crisp, 2010). Karp (2011) indicated that educating students about *career options* helped students to see why college was useful to them and their future goals. Illuminating the *college structure* is particularly important at institutions where many of the students are first generation college students because the entire structure of how to register for classes, where to go for help, and even how to apply for financial aid can be mystifying. Lastly, Karp (2011) notes that many students leave college because of *life issues*, without realizing that there are resources they make use of, or alternate ways to make up work. However, Karp (2011) notes that most of these supports have been studied without control groups, and further, the studies have not asked students why and how the supports are helpful. There have been other calls for further data collection and documentation of what works to help support underrepresented students in STEM (and what does not) (Estrada, et al., 2017).

I worked with a group of mathematicians to create a program that would address these four supports. The program was created alongside a grant funded scholarship program, where students with financial need would apply to the program, and would be introduced to the tailored supports in their first year. These students serve as the intervention group, which I will call the Math Scholars. The purpose of this proposal is two-fold: (1) to describe the support program we have created, and (2) to report on the first round of interviews conducted with the Math Scholars, and a control group of mathematics majors.

The Support Program

The mid-sized western university where the study was conducted has a student population made up 67% first generation students and 61% of the students are Pell-eligible, indicating a financial need. The overall student population is approximately: 3% African American, 14% Asian (mainly Southeast Asian: Hmong and Cambodian), 49% Hispanic, 6% non-resident students, 3% two or more races, 5% unknown, and 20% white.

Prior to creation of the program, two focus groups were held with groups of current mathematics majors. We conducted the focus group as a way of identifying the specific needs of the students at our university, and in particular, identify places where students hit obstacles in their academic career. In these focus groups, the students were asked what obstacles they had faced, what they wished they had known when they were beginning the program, and other similar questions. They were conducted early in Spring 2017 semester. The themes that came out of these focus groups were: (1) students held (sometimes multiple) off-campus jobs and lived with family members that they were supporting, (2) students were unaware of support systems in the university, (3) students struggled finding other students to work with, (4) students did not feel comfortable coming to faculty members with their academic or life troubles, and (5) students did not know what they could do with a mathematics majors besides teach. In particular, with the comfort with faculty members theme, students voiced that they felt faculty members had led very different lives from their own, and as such, would not be able to understand or provide the support they needed.

Taken together, these reinforce the four supports suggested by Karp (2011): *social relationships* (1), (3), (4); *career options* (5); *college structure* (2), (4); *life issues* (1), (3). With this grounding, I worked with a team of three mathematicians, who I will call mentors, to create supports to help address these four categories. The program, which began in Fall 2018, consisted of four main supports: scholarships, advising, workshops, and problem solving challenges.

Addressing Social Relationships

There were two supports that were created to address *social relationships*: scholarships and problem solving challenges. The scholarships were funded through an NSF S-STEM grant, and each student received up to $10,000 per year depending on their financial aid. The scholarships address social relationships because, for many of our students, they expressed that working several jobs to pay for college made creating an on-campus community nearly impossible. Two students, in particular, noted that they made no connections on campus until their last year because they only attended classes, and went to their employment. Thus, Math Scholars were less likely to need to work, and so could focus more on building relationships on campus.

Secondly, the Math Scholars met with the three mentors every Friday of the semester, and the meetings alternated between workshops and problem solving challenges. During the problem solving challenges, the mentors designed problems that were relevant to the courses the students were taking. The students were encouraged to work in groups, to take charge by writing assertions on the white boards, and the mentors provided help when requested.

Career Options

For career options, we designed intensive advising and focused some of the Friday workshops on career options. The advising took place during Summer 2018, where students were asked what their career aspirations were and what their life goals were. In response to that, the mentors helped them choose the courses to take, with an emphasis on getting them to 18 total STEM credits in their first year, which has been linked to overall persistence in STEM majors (Chen, 2013). Several of the Friday workshops included discussions with alumni who were currently working in industry, or other mathematicians who worked in a combination of research and industry. Most of the speakers were people of color. In future years, the plan is to scale up the focus on career options as part of the workshops, including providing information on internships, workshops on resumes and interviews, and so on.

College Structure

The workshops, particularly in the first year, were mainly designed to address the overall college structure. Addressing this ranged from sharing advice from the initial focus groups, to bringing in representatives from the Student Health Center, and other campus programs, to holding workshops on filling out the FAFSA. The mentors also took requests from the Math Scholars if they wanted to know more about a particular program. For example, the Math Scholars were interested in knowing how they could volunteer to help local high school students, or help in some other way in the community. In response, we helped to connect the interested students to volunteer opportunities.

Life Issues

The life issue category is difficult to plan for in that we do not know what might happen in the students' lives. However, the mentors made it clear that life issues will happen to everyone, and that they would be ready to help, and that the Math Scholars should also help one another.

Methods

The participants in the study were 13 undergraduate freshmen who have chosen a major in mathematics at a mid-sized western public university. Out of the 13 students, 7 of them are Math Scholars, and 6 of them are part of the control group. The control group was chosen out of undergraduate freshmen math majors, and an effort was made to match entry level courses between control group and scholar group. In both groups, the majority of the participants are people of color, and the Math Scholar group has a higher ratio of women to men (4:3 compared to 2:4).

Between November 2018 and January 2019, I conducted semi-structured interviews with all participants. They were paid $25 for their time. They could choose to interview with another participant, so two of the interviews were conducted in pairs. This choice was made to ensure the participants felt as comfortable as they could when speaking with me, since I was expecting they might feel uncomfortable speaking with a professor alone. The interviews lasted between 30 minutes and 65 minutes. The students were asked questions on which supports they had used on campus, if they felt like they had a mentor on campus, if they had a peer group to study with, how college mathematics expectations differed from high school, if they had hit any obstacles yet, and if they had any successes. The interviews were video recorded, and subsequently transcribed verbatim. Transcripts were separated by question, summarized, and from the summaries, overall themes were created. These themes were compared between the Math Scholars, and the control group for differences.

Results, Discussions, and Conclusions

Comfort Level with Faculty Members

Six out of the seven Math Scholars mentioned that they would feel comfortable going to one of the faculty mentors with either academic or life issues. In the control group, one of the participants said they had a faculty member they would feel comfortable going to for help academically. This theme indicates that the support program we designed is helping the participants build *social relationships* with faculty members. However, all of the students mentioned that they were still anxious about going to office hours with any professor. However, the Math Scholars were more likely to make use of office hours (5 of 7 versus 2 of 6). Many of the students in both groups stated they were more likely to go to office hours if they could go with someone else, or they knew other students tended to also be in office hours.

Additionally, one of the Math Scholars shared that they had not filled out their financial aid documents correctly resulting in less financial aid than the student needed to live off campus. This Math Scholar went to one of the mentors about the financial difficulty, and the mentor was able to find alternate sources of funding. Without the program, the student reasoned that they would no longer be enrolled at the university. This indicates that building *social relationships* with faculty members may lead to a better management of *life issues* and *college structure*.

Study Groups

Six of the Math Scholars indicated that they studied with other Math Scholars for their STEM classes, indicating the supports are leading to *social relationships*. They mentioned that the program allowed them to create friendships before beginning classes. One of the Math Scholars described, “Before school started, we like walked around campus together and found where all of our classrooms were.” The Math Scholars were also more likely to feel comfortable going to office hours even if their professors were not the mentors. The Math Scholars also described how the problem solving challenges altered the way that they saw mathematics. “It was like I didn’t know how to start any of the problems. But then I watched other people, and they didn’t know either, but we talked to each other. And we figured it out slowly. It made me realize it’s okay...to talk about what I don’t know.” In contrast, one of the control group students articulated, “I should go [to office hours]. It’s just…I’ll say something that’s not right.” Several of the control group participants study with a couple other students, however, most of them explained that they had not found the right group of people yet.

Time Management

All of the students, control group or Math Scholar group, indicated that they struggled with time management in their first semester indicating that *college structure* is still an issue. Many of them articulated that the homework systems were different when compared to high school with their current homework being turned in before class began and often through an online system.

In high school, all of them turned in paper and pencil homework with a few problems. They also described the difference in the time structure from high school to college in that high school was about 8am-3pm every day, but in college, they might have various gaps at different times. They were used to being able to work on homework at the same time every day rather than having to plan out which times on which days to work on which assignments. Lastly, all participants mentioned that the level of difficulty was significantly harder in college. The gap between high school and college was larger than they anticipated, requiring more time to complete assignments and to study for tests. When asked if there was anything they could have known at the beginning that would have helped them navigate this difference, they articulated it was a difference they had to experience in the moment and react to individually.

Conclusions and Future Directions

In summary, in the first year, the supports have had positive effects on the Math Scholars' *social relationships*, mainly. The Math Scholars also seem to be more connected on campus with mentors, and each other. The Math Scholars were more likely to feel comfortable going to office hours for any mathematics or science professor than the control group, even outside of the three project mentors. The project recruited a second cohort of 10 students that have started in Fall 2019.

We have addressed the issue of time management by requiring the students to attend one hour each week of either tutoring or office hours. Initial results from the second round of data collection are showing that one theme is the students appreciating this extra time to study and work with other people. As the program continues, I will also be comparing overall mathematics course grades of the Math Scholars versus non-Math Scholars in the mathematics department to see the effect on overall mathematics achievement.

Acknowledgments

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Comparison of a Pre-requisite to Co-requisite Model of Remedial Mathematics

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There has been a movement away from the pre-requisite model of requiring a year of remedial algebra towards the co-requisite model of providing support in relevant major math courses. The purpose of this project was to examine students' mathematical beliefs and quantitative reasoning before and after a switch to a co-requisite model.

Keywords: Pre-Calculus/College Algebra/Developmental Math, Curriculum, Policy Change

For years, undergraduate mathematics departments required students to take a placement exam, and then depending on the outcome, students below a particular threshold were placed into a year of remedial algebra. This year of remedial algebra did not count as college credit and if the students did not pass the remedial course, they could not move on to mathematics courses that would count towards their major. Thus, the remedial algebra courses could effectively bar students from obtaining a college degree. We will refer to this system of one year of remedial algebra as the **pre-requisite model**. In recent years, scholars have noted that the pre-requisite system is unfair in two main ways: (1) Students who begin in the remedial program are unlikely to succeed in passing to move on in their degree program, and (2) Students from minority groups (African American, Hispanic, first generation, etc.) were more likely to be placed into the remedial system (Cuellar Mejia, Rodriguez, & Johnson, 2016). As a result of this research, many community colleges and universities are shifting away from the pre-requisite model.

One popular reform is the **co-requisite model** (Rodriguez, Cuellar Mejia, & Johnson, 2018). In this model, rather than taking one year of remedial math, the students are placed into mathematics courses they would need to take for their major, and are provided just-in-time mathematics instruction to help them in those courses. There are some potential benefits to the co-requisite model over the pre-requisite model if the co-requisite model takes into account the following three facets of student learning:

1. Arranging the curriculum in a logical way can introduce epistemological obstacles if content is used differently at the end of one course than at the beginning of the next in the sequence. (Raman, 2004; Czocher, Tague, & Baker, 2013). That is, in designing entry level mathematics courses, we cannot rely solely on the logical sequencing of content to automatically provide cognitive basis for future learning (Tague & Czocher, 2016). Here, we adopt Thompson's (2008) construct of coherence which we leverage *curricular coherence* to describe, "the extent to which mathematics content is logically, cognitively, and epistemologically sequenced" (Tague & Czocher, 2016). The goal is to create curricular coherence across courses, not just within.
2. Initial placement in the mathematics sequence has "profound effects on students' trajectories and on the likelihood that they will achieve their academic goals" (Rodriguez, Cuellar Mejia, & Johnson, 2018, p. 5). Rodriguez, Cuellar Mejia, and Johnson (2018) provided evidence that students who entered in remedial mathematics in the pre-requisite model tended to never exit the remedial program. When the students did succeed in the remedial courses, they were less likely to succeed in the sequential mathematics courses, and as such, less likely to gain a degree. A successful co-requisite model would provide natural exit points from the remedial material.

3. One of the underlying reasons that students may struggle in remedial mathematics has more to do with adapting the structure of college than the mathematics content (Kane, Boatman, Kozakowski, Bennett, Hitch, & Weisenfeld, 2019). A successful co-requisite model would have academic and social supports built in.

Vestal, Brandenburger, and Furth (2015) studied a co-requisite for Calculus, in which students received a letter grade of a C or D in their Pre-Calculus class or received a score of 40-54 on their ACT Compass Trigonometry. The Calculus class proceeded as usual, but a lab component was added as the co-requisite portion. The lab began with questions from the students, followed by a quiz from the previous week's material and reading with plenty of opportunity for peer dialogue. Lab concluded with group work on worksheets. Final grade data was collected for four consecutive semesters and the authors used a Pearson chi-square test to check for a significance in passing and failing rates for students (Vestal, Brandenburger, & Furth; 2015). The passing rate increased from 61% (before labs were introduced) to 66% (after labs were introduced). Although the difference was not statistically significant, the authors argued that any increase in the passing rate is promising (Vestal, Brandenburger, & Furth; 2015). We tend to agree, given the large number of students who get C's or D's in pre-calculus across universities.

The arguments for the co-requisite model are supported with previous research, however, they remain hypotheses because little research has been conducted on how the switch from pre-requisite to co-requisite is affecting students. The purpose of the current study is to examine how the move from pre- to co-requisite shifts students' self-reported attitudes towards mathematics and the extent of impact on aspects of their quantitative reasoning.

Context and Course Design

The main struggles with the co-requisite model are ensuring the three points above are met: (1) curricular coherence, (2) appropriate placement, (3) addressing the college structure, and an additional struggle of instructional coherence. Instructional coherence is the counter part to curricular coherence in that instead of examining the overall coherence of the content, it examines the coherence within a particular course, or the "alignment among in-class materials, out-of-class materials, and target content" (Tague & Czocher, 2016, p. 227). If the students are reminded of the mathematics they need to use, and also able to examine how they might use it differently, they might be more likely to apply that knowledge (Tague & Czocher, 2016). For example, in statistics, it is useful for students to examine and acknowledge the differences between fractions, percentages, and proportions. Many of the co-requisite models intend to improve upon quantitative reasoning, and number sense, rather than focus on exclusively algebra (for example, The California State University Quantitative Reasoning Task Force Final Report, 2016).

Co-requisite models can overcome these struggles by addressing them, and applying current research results. For curricular and instructional coherence, the curriculum developers must address where students are likely to have cognitive obstacles (Herscovics, 1989), which is a way of thinking that might be appropriate in one context, but is inappropriate in another. For example, when adding integers, we enumerate the total number of objects, but when adding fractions, we cannot enumerate the total number of objects in the numerator and divide by the total number of objects in the denominator. For these reasons, and similar, the curriculum for the co-requisite model must be design to match with the content of the original course (instructional coherence), but also address likely cognitive obstacles around quantitative reasoning that were created during the students' previous mathematics career (curricular coherence). Appropriate placement is addressed by using the students previous mathematics career as a guide for where they will likely be successful. Examining multiple measures including overall high school GPA, high school mathematics GPA, course taking patterns, standardized tests scores, and so on can allow for a better prediction of where the student is likely to be successful. Lastly, any co-requisite model should explicitly address obstacles created by the college structure – study skills, mathematical beliefs, creating study groups, etc.

Our study was conducted on a co-requisite model at a comprehensive public university in the Western United States. The university is an Hispanic-serving institution and an Asian American Native American Pacific Islander-Serving Institution. In Fall 2017, a policy change required the department to switch from a pre-requisite model to a co-requisite model, effective Fall 2018. Instead of taking a year-long remedial algebra course, students were placed into four possible parent course options: statistics, liberal arts math, math for future elementary school teachers, or a STEM-track course based in algebra. In addition each student enrolled in a co-requisite lab section.

The co-requisite labs were designed to promote instructional coherence and curricular coherence around quantitative reasoning (Tague & Czocher, 2016) and heavily involved the master's-level Graduate Teaching Associates (GTAs) in planning and execution. The GTA's met every week to review upcoming materials and critically reflect on the effectiveness of previous materials. The content in the labs was aligned with the relevant course material. For example, in statistics, when the students were learning about means, the GTA's led discussions of fractions and led the students to examine how a data set might have to change to achieve a particular mean. Whenever possible, the GTA's explicitly acknowledged how fractions might be used in the context of finding a mean to address instructional coherence by matching the content in the parent course, but also to provide curricular coherence in recognizing that previous math content would be used in a different way. In this way, we hoped the students would be more likely to develop and connect their fraction knowledge to the concept of measures of central tendency.

In all of the courses, we also acknowledged that one of the underlying reasons students might be struggling, could be due to adjusting to college. That is, the students might not understand how college mathematics might be different from high school mathematics, and they might also need help knowing how to think about mathematics, and how to study for mathematics. We addressed these issues throughout the semester, by having the students watch youcubed videos about mathematical beliefs, study during class and explicitly discuss useful study habits, and lastly, explicitly address differences from high school structure to college structure (i.e. homework might be online and due at midnight the day before class occurs, work and studying might take longer). Overall, we anticipated that the switch from pre-requisite to co-requisite models would improve the students' quantitative reasoning, and also their beliefs around mathematics, particularly related to the utility of mathematics.

Methods

We wished to document any impact the co-requisite model might have on the students' success. It was not possible to directly compare students' course grades since each of the courses operated independently. Instead, we decided to measure students' attitudes towards mathematics and students' performance on several quantitative reasoning tasks. The first instrument measured Beliefs about Mathematics and Problem Solving (MAA, 1994). The second instrument measured quantitative reasoning through the Minimal Competency Tasks (MAA, 1994). We designed both instruments to be short enough that students would be likely to finish it, while also allowing us to measure self-reported changes in relevant attitudes and quantitative reasoning content. In particular, we hypothesized that the change in curriculum might affect students' beliefs about mathematics (i.e their perceived usefulness of mathematics, and their efficacy towards the subject).The survey was administered via Qualtrics in Spring 2018 (pre-requisite model) and again in Fall 2018 (co-requisite model). In total, we surveyed 11 classes in Spring 2018, and 25 classes in Fall 2018 for a total of 136 and 219 student responses, respectively.

Results

The beliefs results are shown in Table 1, and the quantitative reasoning results are shown in Table 2. To examine differences in beliefs, we collapsed the two disagree categories into disagree, and similarly collapsed the two agree categories into agree. We conducted a 2-tailed Mann-Whitney test to determine if the differences in beliefs were significant from pre- to post-curriculum switch. The results indicate that students in the co-requisite model were significantly less likely to think that (1) one can do math or one can't ($p=0.008<0.05$), and (2) that mathematics was about memorizing ($p=0.047<0.05$). Although not significant at the $\alpha=0.05$ level, students in the co-requisite model seemed more likely to think that the mathematics they were learning was related to their life ($p=0.093$). Indeed, thinking that the mathematics related to their lives explained 90% of the variance in the difference between the two groups.

Table 1. Math beliefs results – the top percentage is from Spring 2018 (pre-requisite) and the bottom is from Fall 2018 (co-requisite).

Math Beliefs Questions (MAA, 1994)	strongly disagree	disagree	neutral	agree	strongly agree
The math classes I have taken at Fresno State are preparing me to use math in my future career.	3% 4%	9% 7%	28% 31%	41% 39%	18% 19%
The math classes I have taken at Fresno State are preparing me to use math in my everyday life.	7% 2%	15% 11%	37% 37%	26% 41%	15% 9%
There is only one correct answer to any mathematics problem.	14% 14%	36% 29%	18% 27%	24% 23%	8% 7%
If a mathematical problem can be solved, it can usually be done in five minutes or less.	17% 13%	26% 31%	36% 32%	17% 20%	3% 4%
Either people can do mathematics or people cannot do mathematics.	5% 14%	32% 31%	33% 35%	20% 13%	11% 7%
Math is only useful for people who do very specialized work.	11% 10%	35% 39%	26% 27%	23% 16%	5% 7%
Learning mathematics involves mostly memorizing.	1% 6%	11% 12%	32% 35%	38% 33%	18% 14%
Solving some mathematical problems involves knowing different strategies to try.	1% 3%	2% 1%	8% 17%	59% 55%	30% 24%

Among the quantitative reasoning questions, the only question for which the percent correct significantly differed was question 4. On Question 4, the co-requisite students answered correctly more often ($p<0.001$) We are encouraged by improvement on the fraction question because many of the co-requisite labs included intense review of fractions content and its relation to algebraic concepts.

Table 2. The quantitative reasoning results with the pre-requisite percent shown above the co-requisite percent

<table>
<tr><th>Quantitative reasoning questions (MAA, 1994)</th><th>Percent Correct</th></tr>
<tr><td>1. Gloria noted that in June of 1989 her tuition to Super-U increased by 6% and in June of 1990 it increased by 3.5%. Over the two year span what was the percentage increase in tuition?</td><td>0%
0%</td></tr>
<tr><td>2. Harry raises tropical fish and noticed his guppy population increasing according to the following table:
<table>
<tr><td>Time in weeks</td><td>0</td><td>4</td><td>8</td></tr>
<tr><td>No. of guppies</td><td>3</td><td>33</td><td>363</td></tr>
</table>
If the guppies continue to increase at this rate, how many should he expect to have in twelve weeks? Explain your answer.</td><td>53%
61%</td></tr>
<tr><td>3. How many dollars will x pens cost if 5 such pens cost y dollars?</td><td>7%
6%</td></tr>
<tr><td>4. Leni spends ⅓ of her income on rent and ¼ of her income on car expenses. What fraction of her income is left for other expenses?</td><td>24%
41%</td></tr>
<tr><td>5. Allessandria bought 4 peaches and 1 apricot at the farmer's market for $6. Carlos, not knowing what Allessandria had done, went to the same market and bought 5 peaches and 1 apricot for $7.50. How much does one peach cost?</td><td>48%
53%</td></tr>
<tr><td>6. A rectangular garden has a perimeter of 28 yards. The width of the garden is 6 yards less than its length. What is the area of the garden, in square yards?</td><td>19%
24%</td></tr>
<tr><td>7. Jose has a van with a rectangular-shaped space for hauling boxes. The space measures 7 feet long, 4 feet wide, and 4 feet tall. If the boxes that Jose hauls are 2 feet by 2 feet by 2 feet, how many boxes can he fit in his van at once? Briefly explain your answer.</td><td>16%
19%</td></tr>
</table>

Discussion and Conclusion

Even after a full year of remedial mathematics, the Spring 2018 cohort did not match the Fall 2018 cohort's development of positive mathematics beliefs and quantitative reasoning skills. The co-requisite model seemed to have some positive impact on student attitudes towards mathematics, particularly on their beliefs about whether they are capable of learning mathematics. Since positive self-efficacy predicts both learning and persistence (Betz & Hackett, 1983; Bandura, 2006), we view this as a positive result. Students seemed to learn about fractions better in the co-requisite model, which is predictive of overall mathematics achievement in algebra and other upper level mathematics (Siegler, et al., 2012).

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Shifting Pedagogical Beliefs into Action through Teaching for Mathematical Creativity

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Through participation in a research project on fostering creativity in calculus, two instructors showed shifts in their beliefs on teaching. Participation in the project entailed creating mathematical tasks designed to elicit creative responses from students. Support for task development included participation in weekly online professional development sessions. In this paper, we share one instructor's shifts in beliefs as well as alignment of her pre-existing beliefs with pedagogical actions. Preliminary analysis of her entrance tickets to the professional development sessions and her exit interview indicates that this instructor a) shifted her previous beliefs about a perceived time pressure and b) manifested her existing beliefs into actions regarding multiple-approach tasks.

Keywords: mathematical creativity, calculus, pedagogical change, beliefs

Due to the ever-changing landscape of economies and jobs, creativity is a skill that is sought after by employers in STEM fields (Wilson, Lennox, Hughes, & Brown, 2017). As mathematical researchers and instructors, our roles in preparing students depend in our interactions with them, as they comprise the future STEM task-force. Since "the process [of promoting creativity] is influenced by teacher beliefs and biases" (Hershkovitz, Peled, & Littler, 2009, p. 265), positive instructor beliefs regarding the role of creativity in a course are vital to students' experiences of creativity. Thus, instructors should be encouraged to adopt the belief that fostering creativity in mathematics is a vital part of the curriculum. However, one's beliefs may not always align with their actions (Hutner & Markman, 2016) given that beliefs exist within complex social and political systems and in some cases acting on those beliefs may be perceived as a threat to job security. Supported by the model of constructive alignment (Biggs, 1996), we further claim that if instructors believe that fostering creativity is an instructional goal, the next step is alignment of that belief with their own pedagogical actions in the classroom.

Background Literature

One of the most cited definitions of belief is Richardson's (1996): "Psychologically held understandings, premises, or propositions about the world that are felt to be true" (p. 103). A belief is individualistic, meaning that a person with a belief can respect another person's contradicting belief if they find that the opposing belief can be explained in a reasonable and intelligent manner (Philipp, 2007).

Though accepting another's belief is possible, changing one's beliefs has been shown to be a difficult process (Dunbar, Fugelsang, & Stein, 2007; Schoenfeld, 2011; Shtulman & Valcarcel, 2012) because "for an individual to change their beliefs, they need to desert premises that they hold to be true, and often this is difficult and challenging" (Grootenboer & Marshman, 2016, pp. 16-17). One possible way to induce changes in instructors' beliefs is through professional development (PD) since those experiences are critical in influencing thinking about change and enacting change (Capps et al., 2012; Enderle et al., 2014; Woodbury & Gess-Newsome, 2002).

This is largely due to the opportunities in PDs to engage instructors in reflecting on elements of teaching such as assessment, implementation, collaboration, and problem-solving.

In this paper, we share preliminary results demonstrating that providing instructors PD support to engage in the process of fostering creativity in the classroom can both help instructors shift their beliefs and also begin to transform their beliefs into pedagogical actions.

Methods

In Spring 2019, two instructors from a South-Midwest regional university participated in a study on fostering creativity in calculus. In this paper, we focus on Jo Parker, a white female with eleven years of teaching experience.

Instructors attended an initial 2-day, 4-hour (total) PD session in December 2018. During this initial PD, the authors attempted to provide context for discussing mathematical creativity and discuss their requirements for participation, which included uploading previous semesters' calculus materials, attending weekly online PD meetings, filling out entrance tickets prior to the online meeting, and consenting to be interviewed after grades were submitted. Perhaps most importantly, instructors were to implement two creativity-based tasks created by the researchers and create at least four more creativity-based tasks to use in their classroom. The term *creativity-based tasks* describes tasks that could allow for multiple solutions (Leikin, 2014), provide opportunities for students to pose questions/problems then solve their own problems (Haylock, 1997; Silver 1997), or are ill-defined, or open-ended, such that posing questions is necessitated (Kwon, J.H. Park, J.S. Park, 2006). Task development was supported through the PD by providing a list of task features instructors could use (El Turkey et al., submitted). Each time they implemented a task, they were to video-record the class. Also, they were encouraged to incorporate the Creativity-in-Progress Rubric (CPR) on Problem Solving (modeled after the CPR on Proving; Savić, Karakök, Tang, El Turkey, & Naccarato, 2017) with each task.

The first two authors used holistic coding (Saldaña, 2015) while watching the exit interviews. They separately looked for perceived belief shifts or evidence of enacting pre-existing beliefs and met together to discuss the shifts. After agreement, the first author used narrative coding (Saldaña, 2015) to analyze the entrance tickets and end-of-semester interviews.

Results and Discussion

The three most prominent themes extracted from Jo's entrance tickets, PD sessions, and exit interviews were beliefs related to *Time Pressure*, *Multiple Approaches*, and *Posing Questions*. In this section, we show Jo's belief shift in *Time* and also her alignment of a pre-existing *Multiple Approaches* belief to her classroom practices. A cursory mention of Posing Questions is addressed in the Conclusion.

Belief Shift: Time Pressure

Prior to participating in this project, Jo perceived a lack of time that impacted her capacity to incorporate creativity into her Calculus 1 course: "**[d]uring a 'normal'[1] semester, I typically feel time pressure**, so I didn't make as much of an effort to incorporate creativity activities within the classroom" (exit interview, May 2019). During the semester of her participation in this project, she initially assigned the first few creativity-based tasks outside of class, but then changed the last couple of tasks to be done in class because:

[1] Jo refers to the semesters prior to her participation in this study as "normal".

> **I realized that I have time** to do them in class. And I wanted them to do them in class…they worked well together so why not let them utilize their time doing these together? **I have time.** It's really the time dictated and how I did things and our Calc 1 is very packed and since we only meet for three hours a week. **I was very nervous in the beginning** (exit interview, May 2019).

By the end of participating in the grant, she had shifted her thinking about the time pressure. She explained her shift in the exit interview:

> **[a]t the beginning of the semester, I was very nervous about time. Very packed but, who cares—in some sense—I can spend 15 minutes once a week** just saying, "hey do this problem in class" or I can give it to them outside of class and then say, "hey let's come to the board and you all put up your solutions and we can talk about them." I'm much more aware of that.

Alignment of Belief to Pedagogical Practices: Multiple Approaches

The PD entrance ticket on 12/11/18 had the question: *What are some "ways" to foster students' mathematical creativity? Please be as specific as you can.* In Jo's answer, we coded four themes, which are given in bold below.

> I think failure can foster creativity. **[Failure]** You attempt a problem using "standard" procedures and it doesn't lead to anything fruitful. So, you think about other ways to approach the problem**. [Multiple Approaches]** What additional information do you have? Is that information useful? etc. **[Evaluation]** I think making the students the teachers can also foster creativity. Outside of the classroom, they tend to teach one another in alternative ways to how they were taught in class. Why not foster that inside the classroom as well? **[Students as Teachers]**

Her answer shows that prior to participating, she already held the belief that Multiple Approaches is important to foster creativity. However, when asked, *Looking at the ways you listed above, can they be implemented/used in Calculus I course? Why or why not?,* Jo only referred to the ideas related to the Students as Teachers theme and did not mention Multiple Approaches as a pedagogical action that can be implemented: "I think it would be difficult, but not impossible, to get students to "teach" during class. (It is a matter of creating a safe environment.) I think well thought out assignments can work but creating the assignments can be tricky." It is worth mentioning that she acknowledges intentional assignment development is needed to help her enact this pedagogical method of teaching.

Though Jo believed having students approach problems in more than one way can foster creativity, she acknowledged that she does not explicitly model this behavior in her class. In her 2/5/19 entrance ticket, when asked to *Describe a moment in the classroom where a student surprised you with their work or other discourse. Reflecting on that same moment, what aspects of the course or your teaching do you think contributed to that moment?*

> I was actually surprised with…the Limit Task [Evaluate $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ in as many ways as possible]. It appears that most found the limit one way and didn't bother to find it another way. It was like the question just read "Evaluate (blah)." **I think a lot of it has to do with the problems we do in class. We usually only approach them one way and we call it good.** Sometimes I approach a problem [in] multiple ways, but I rarely show my work. (I talk through the alternative method(s).)

Here, Jo referred to a misalignment between instructor's practices in the classroom (i.e. approaching problems only one way) and how students are assessed, in this case through HW.

Constructive alignment between the instructional methods and assessment methods is important because it supports students in reaching instructional goals (Biggs & Tang, 2011), in this case mathematical creativity.

Referring to the Limit Task again during the exit interview (May 2019), the interviewer asked Jo why she called it "cute":

> I love the fact that you can **approach it in many different** ways…I think the traditional way that students think of it is multiply by the conjugate, but I mean, they can factor it… It's just a little trick. And they don't think to factor linear terms. And so, I like that. I think it's cute. **It's a little bit outside the box but it's still within their realm of knowledge.**

Jo created a review assignment for her students that included a question similar to this Limit Task. When she said it didn't go as well as she had hoped, the interviewer asked her to describe what "going well" would mean to her. She responded focusing on multiple approaches:

> **It'd be nice if just another student popped up and say "hey I approached it by factoring or I used the T chart" which I know is not the best method but it gives me intuition as to what's going on.** So…in my mind that's what's going well [means].

What is demonstrated in these two quotes is that not only did Jo include more creativity-based tasks on her final review, she is now using multiple approaches as a metric of success in terms of a class session. Additionally, when she says the T chart method may not be the best, but gives her insight into her students' thinking, she is valuing a student-centered approach rather than a content-centered one.

El Turkey et al. (submitted) coded this review assignment and found that there were 5 creativity-based tasks with several features that have been reported to foster creativity, one of the features being "different approaches." This is in stark contrast to final exam review created prior to her participation in the grant where no questions were coded with creativity fostering features.

Though Jo has put into practice her belief that assigning multiple-approach problems can foster creativity, she extended this belief into her teaching practices. Beliefs of mathematical creativity held by mathematicians have been reported to align with actions in their research (Borwein, Liljedahl, & Zhai, 2014), but Jo mentions that those beliefs may not necessarily align with actions in their teaching:

> Obviously, I think [creativity is] beneficial for teaching, but I mean we do it all the time in mathematics when we're conducting research…But I mean **it's encouraging me to think in different ways** necessarily than I normally do, yeah, and also ideally, hopefully **teach in different ways too that are more beneficial**.

Jo uses creativity while conducting mathematics research, but she expressed a desire to incorporate creativity into her teaching more often, particularly with respect to teaching in multiple ways. This intention also aligns with research showing that differentiated (multiple) ways of teaching are more beneficial for student engagement, as well as social and academic inclusion (Katz, 2013).

Conclusion

Consistent with Enderle et al. (2014), this paper shows that teaching beliefs and practices have a reciprocal nature; not only can beliefs influence the promotion of creativity in the classroom, but the converse can be true: promoting creativity can shift teaching beliefs, biases, and practices. In the previous section, we showed that Jo shifted her belief in not having enough time to cover everything in Calculus 1. Through analysis of her final exam review, we showed that her pre-existing belief (i.e. multiple-approach tasks can foster creativity) was transformed into action. While this provides preliminary evidence of shifting of beliefs and alignment into actions, further analysis of the classroom videos is needed to triangulate this finding.

This quote from Jo on Week 10 of the professional development encapsulates all three themes: Time Pressure, Multiple Approaches, and Posing Questions.

> Admittedly on Monday, my students—**I'm now behind—but they were asking phenomenal questions.** We were talking about increasing and decreasing and then all of a sudden one of the students goes "well doesn't that show you that you have an absolute min at this point?" "Heck yes, it does." …**So, they're saying and thinking great things that I don't normally get out of my students…So they're thinking about things a different way.**"

The quotes above show that the students' reactions are influencing Jo's thinking on pedagogical shifts. Additionally, Jo commented in her exit interview that she had fewer students come to her office and complain about non-routine tasks when compared to a "normal" semester. In other words, "[t]he ways in which teachers perceive, interpret, and act on students' reactions to their attempts to make change affects their sense of what is 'working' in their classrooms" (Woodbury & Gess-Newsome, 2002, p. 768). We believe that the reactions and successes of the classroom were reflected on, discussed, and reinforced through the professional development of fostering mathematical creativity.

As we observed these shifts in beliefs and enacting beliefs, we examined our PD actions using nine critical features of effective PDs (Capps et al. 2012), which are italicized in the next few sentences. The initial PD (*Total Time*) followed by semester-long participation in the PD (*Extended Support*) provided opportunities for instructors to reflect (*Reflection*) on their teaching practices through developing creativity-based tasks (*Developed Lessons*). The research group provided two tasks as models *(Modelling*). Once the tasks were developed with the content in mind (*Content Knowledge*), instructors presented them to the rest of the participants in PD and received feedback on the task itself as well as implementation (*Transference*). Going through this process of taking risks and creating new ideas is parallel to the creative process mathematicians engage in during their own problem-solving processes (*Authentic Experiences*). The participants were presented with the importance of mathematical creativity as posited by mathematicians (*Coherency*). Although the PD sessions touched on all nine of these features, the degrees of engagement in these features varied. For example, our PD was strong in the areas of Total Time, Extended Support, Reflection, Developed Lessons, Content Knowledge, Transference, and Authentic Experiences. Coherence and Modelling are two areas in which our future PDs will seek improvement. For example, the importance of mathematical creativity will be aligned to goals of NCTM, MAA IP guide, and research results (e.g. Omar, Karakök, Savić, El Turkey, & Tang, 2019)

Shifting beliefs is the first step to change, but not the final. Beliefs do not always translate to action, but non-action does not imply that the belief is not held (Hutner & Markman, 2016); there are complex systemic or cultural issues that may be preventing the instructor from acting on their beliefs. In our future research, we will be examining more belief-in-action shifts (Philipp, 2007; Bishop et al., 2003), as we believe "Reflection without action is…[an] armchair revolution" (Friere, 1970, p.149).

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Students' Understanding of Infinite Iterative Processes

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Philosophers, mathematicians, and students struggle with the notion of infinity. This study investigates one undergraduate student's response to tasks involving infinite iterative processes in different contexts. A semi-structured clinical interview was conducted to describe the types of mental structures the participant had constructed to solve the proposed tasks. APOS theory was then successfully able to predict later student's responses. This study also suggests that how students depend on time to understand infinity and infinite processes may be an area for further investigation.

Keywords: Infinity, APOS, Undergraduate-Level Mathematics

Aristotle perceived two different types of infinity: potential infinity and actual infinity. Potential infinity is a continuing process that never ends. Actual infinity is a "definite entity encompassing what was potential." (Dubinsky, Weller, McDonald, and Brown, 2005). The continuing process can be described as an infinite iterative process. This idea can be seen in mathematical concepts such as series, sequences, and limits. The purpose of this study is to use APOS theory to identify what mental constructions a student has available to them and with those identified mental constructs, predict the participant's response to tasks about infinite iterative processes.

Theoretical Framework

According to APOS theory, Action, Process, Object, and Schema (APOS) are types of mental structures an individual could construct in response to different problems or scenarios. A new mathematical concept is formed as one transforms existing mental objects. The transformation is initially conceived as an *action*. This structure is identified by the explicit performance of step-by-step transformations of an existing object or objects. As one repeats and reflects on an action, it then becomes interiorized into a *process*. One is able to apply the transformation without going through each individual step like an action. The process is encapsulated into an *object* as one views the process as complete and a totality that transformations can act upon. An individual's understanding of mathematical topics can consist of different mental actions, processes and objects. A *schema* for a topic is the coherent organization of these mental structures.

Brown, McDonald, and Weller (2008) developed a theoretical description of what mental constructions are involved with infinite iterative processes using APOS theory. They state that infinite iterative processes are the coordination between iteration through $\mathbb{N}$ and repeatable transformations. An action, in this case, would be a small number of iterations of the transformation. This action can be interiorized into a mental process of iterating through any finite segment of $\mathbb{N}$. Coordinating the processes for multiple finite segments can lead to the construction of an infinite iterative process. "The individual understands that an object is obtained for each natural number in order, and that objects are obtained only for natural numbers" (Brown et al., 2008). In response to the attempted action of deducing what is "next," one might encapsulate the infinite iterative process into an object. This object is a state at infinity.

Literature Review

There are two notable studies that apply this APOS framework to identify the mechanisms of encapsulation and interiorization in constructing infinite iterative processes and states at infinity. First, Brown et al. (2008) conducted a study of students' attempts of determining whether or not two sets are equal (see Figure 1, 1.1). One should note that the equality does not hold because each P({1,…k}) contains only finite sets, and thus the infinite union also only contains finite sets. Conversely, P(ℕ) contains infinite sets as well as finite ones. They proposed the initial framework for infinite iterative processes used to construct different power sets and unions.

The study of Brown et al. (2008) was with one infinite process while Dubinsky, Weller, Stinger, Vidakovic (2008) wanted to analyze the coordination of three infinite iterative processes: "between iteration through ℕ and time, between iteration through ℕ and the movements of tennis balls, and between the process resulting from the first two coordinations" (Dubinsky et al., 2008). This problem involved the infinite sequence of times and the process of moving balls in different bins infinitely many times (see Figure 1, 1.2).

Radu and Weber (2011) wanted to see what types of strategies students used when solving infinite iterative processes. Students were given a wide variety of problems in different contexts. The responses for the tennis ball problem aligned with what Dubinsky et al. (2008) found. Radu and Weber identified that students will often generalize certain properties of the finite states as properties if the state at infinity.

Radu (2009) compared APOS theory with and embodied cognition theory called Basic Metaphor of Infinity (BMI) created by Lakoff and Nùñez (2000). BMI states that actual infinity is the "metaphorical result of a process with no end." They are not quite able to describe what this result will look like, only that it exists and is unique. Dubinsky et al. (2008) note that different actions on a completed infinite iteration can change how one encapsulates the process into an object. Radu points out that both theories rely on intermediate states to determine a state at infinity.

1.1) Prove or disprove: $\bigcup_{k=1}^{\infty} P(\{1,2,\dots,k\}) = P(\mathbb{N})$ where P(X) is the power set, i.e. the set of all subsets of X and ℕ is the natural numbers.
1.2) Suppose that we have three bins of unlimited capacity, labeled Holding Bin, bin A, and bin B, with a dispenser button that when pushed, moves balls from the holding bin to bin A. The holding bin contains an infinite set of numbered tennis balls (1, 2, 3,...). Step 1: 1/2th of a minute before 12:00, the dispenser is pressed and balls #1 and #2 drop into bin A, and ball #1 is moved instantaneously from A to B. Step 2: 1/4th of a minute before 12:00, the dispenser is pressed and balls #3 and #4 drop into bin A, with the smallest numbered ball in bin A immediately moved into B. Step 3: 1/8th of a minute before 12:00, the dispenser is pressed and balls #5 and #6 drop from the holding bin into bin A, with the smallest numbered ball in bin A immediately moved into B. If the pattern just mentioned continues, what will be the contents of bin A and bin B at 12:00?

Figure 1. Interview 1 Tasks

2.1) Suppose you are given an infinite set of numbered tennis balls (1,2,3,...) and two bins of unlimited capacity, labeled A and B. At step 1, you place balls 1-10 in bin A and then move ball 1 to bin B. At step 2, you place balls 11-20 in bin A and then move ball 2 to bin B. At step 3, you place balls 21-30 in bin A and then move ball 3 to bin B. If the pattern described continues infinitely many times, what will be the contents of bin A and bin B at the end of this process?
2.2) Suppose you have an empty jar and outside of it you have an infinite number of marbles labeled with 1+1/n where $n \in \mathbb{N}$ (so the first one is labeled 2, the second one 3/2, third one 4/3, etc). At step 1, the marble labeled 2 is put in the jar. For any n>1, at step n you remove the marble labeled 1+1/(n-1) from the jar and put marble 1+1/n in the jar. Assume ALL steps have been performed. What are the contents of the jar at this point?
2.3) Distribute countably many balls into two bins such that after n steps, one bin contained only the ball labeled 1, while the other contained the balls with labels 2,3,...n. The contents of the bins are swapped every odd step. What is in each bin at the end?
2.4) Let v=(1,0,0,...) ∈ ℕ^ℕ. You are going to "edit" this vector step by step. Step 1: v=(0,1,2,0,0,...) Step 2: v=(0,0,1,2,3,0,0,...) Step 3: v=(0,0,0,1,2,3,4,0,0,...) This process is continued ad infinitum. Now assume ALL steps have been completed. Describe v at this point.

Figure 2. Interview 2 Tasks

Methods and Procedures

The study involved one undergraduate student from a large public research university in the southeastern United States. The student, Margaret, was a graduating senior majoring in mathematics that has taken undergraduate and graduate level courses in the past year. This participant was chosen because of their exposure to infinity in different contexts, as well as, their understanding of higher-level mathematics.

We conducted two semi-structured interviews a few days apart. They consisted of tasks that were chosen to highlight the student's understanding of infinite iterative processes in different contexts (see Figures 1 and 2). Follow up questions were asked after the tasks were solved to provide clarification on the student's reasoning. The interviews were videotaped and transcribed for further analysis. Tasks for the first interview are those used in Brown et al. (2010) and Dubinsky et al. (2008). The data obtained in interview one and the APOS framework described previously were used to predict student responses during the following interview. Tasks for the second interview were from Radu (2009).

I decided to present the tasks from the first interview slightly differently than Brown et al. (2010) and Dubinsky et al (2008) due to the participant's high-level math background. Brown's study asked questions about power sets and unions of a finite number of sets how one would show the equality between one set and a union of sets before 1.1 (Figure 1). Dubinsky's study introduced a finite version of problem 1.2 without a correlation to time prior to 1.2.

Results

Interview one

Margaret immediately started to prove containment in both directions for the first task. Initially she misread the question and thought she was trying to prove $P(\{1,2,\ldots,k\}) = P(\mathbb{N})$. She proved one containment and then crossed it out. When asked why she did that, she replied "Let me just try proving containment this way and that way. And I got to the other way [⊇] and I was like oh this isn't true. I don't think." She then went on to give a counterexample of A=2ℕ. This shows that she understood for any fixed k, P(ℕ) is not a subset of P({1,2,…k}). However, for the original problem 1.1, she gave a proof for equality and crossed out her counterexample. She focused on individual elements of an arbitrary subset of ℕ, as shown in Figure 3. When asked to explain why the earlier counterexample, 2ℕ, is not a counterexample anymore she rewrote it as the union of {2}, {4}, {6}, etc. Margaret failed to recognize that in order to prove $\bigcup_{k=1}^{\infty} P(\{1,2,\ldots,k\}) \supseteq P(\mathbb{N})$, the whole union of singletons must be contained in at least one $P(\{1,2,\ldots,k\})$ for some fixed value of k. From her response on this question alone it is difficult to determine what mental constructions she had available to solve this problem.

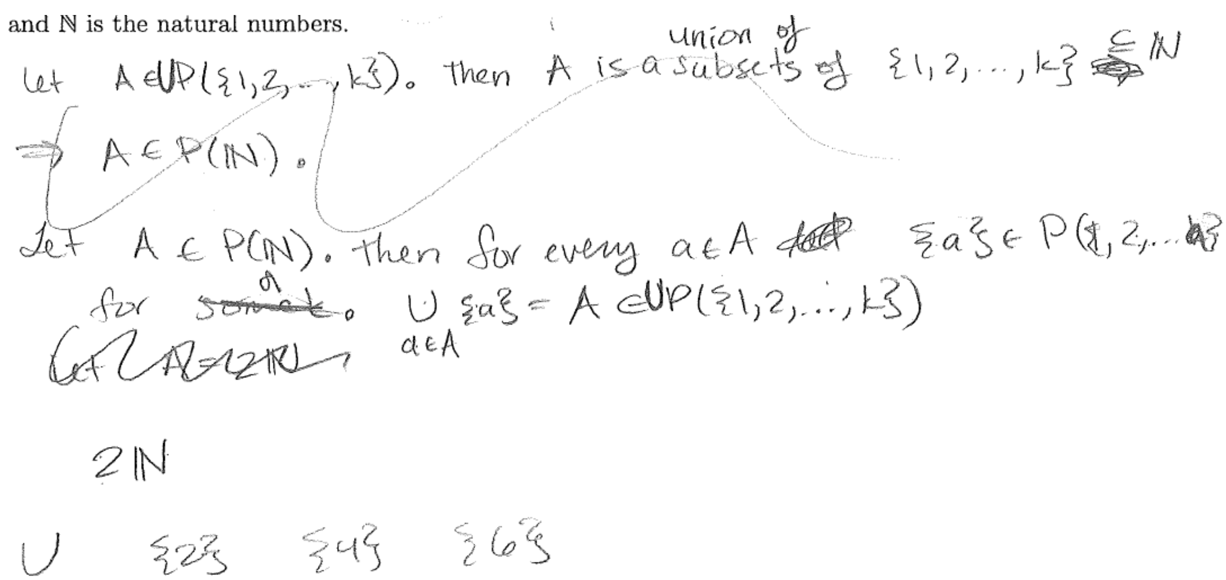

Figure 3. Part of student's work on task 1.1

For the second task, she acknowledged the fact that the number of balls in each bin was increasing for each finite step. This demonstrated her coordination between iteration through ℕ and the movements of tennis balls.

> Obviously everything ends up in here [bin B]…but it's odd because like the number is increasing. So like here [in bin A] you have one ball, two balls, three balls, four balls. So theoretically as this goes to infinity you would have infinitely many balls. But at the same time they're all in bin B. So that doesn't make much sense.

After much internal conflict, she decided bin A was the empty and bin B contained all the natural numbers. At this point it seemed like the timestamps 1/2, 1/4, and 1/8 were just being used as indices for each step. To determine if time played a big role in her encapsulation of this infinite process, I asked Margaret how she would think about the problem if there were no connection to time. "Well it's like you can't ever get there…So it's like it's changing infinitely fast as you get closer and closer. So you can't just like say what's in the bin." Margaret can conceptualize what happens at the end, but she has difficulty describing the process if there is no time limit.

Margaret's response to task 1.2 led us to believe she could encapsulate infinite iterative processes into an object. However, the time-dependent nature of the task might have enabled her to think of this process as completed. The tasks for the second interview were made to be independent of time to confirm whether or not she could conceive the final state as a complete object (see Figure 2). Task 2.1 is very similar to task 1.2 but now without time and slightly modified. Because of Margaret's object understanding of infinite iterative processes, I predicted that she would be able to answer all the tasks in interview two.

Interview two

The absence of a relationship between time and the tasks did not affect Margaret's ability to solve all of the tasks successfully. In fact, she completed each task in less time and with more confidence than those in the first interview. She quickly concluded that bin A was empty and bin B contained all the natural numbers for task 2.1. In task 2.2 she stated that the jar remained empty at the end. After I inquired why a marble labeled 1 wouldn't be in the jar she responded by saying "Well one isn't actually a ball. You only have 1+1/n and [1/n] cannot equal zero for any n." Margaret wrote down two sets, {1} and {2,…,∞}, to describe the ending bins in 2.3. Although "1 won't be in one or the other [bins]," the final state existed as the partition of balls represented by the two sets.

Conclusion and discussion

Margaret misinterpreted both questions during the first interview. This may be due to the typesetting or wording of the problem. It would have been nice to get a baseline of the student's understanding of power sets and unions in the first interview. I do not believe that asking general questions about those topics like in the Brown et al. (2005) study would promote scaffolding.

Initially I conjectured after interview one that Margaret may be dependent on the infinite iterative process' relationship to time. Since she was able to solve all the tasks for interview two, this dependency on time to comprehend infinite processes was not there for Margaret. Retrospectively looking at her response to 1.2, she determined the outcome of the infinite process and then used phrases such as "at this time interval" to explain her reasoning. I could say that she has encapsulated the iteration through $\mathbb{N}$ and time because she used the time stamps as an index rather than a separate process. Although this dependency may not be applicable to Margaret, perhaps there is some relationship between time and infinite iterative processes. Sam from the Dubinsky et al. (2008) study is an example of this. He was able to iterate through $\mathbb{N}$, but was unable to see the movement of tennis balls as complete without the time connection. The concrete stopping time 12:00 enabled Sam to encapsulate the process in totality. This relationship between time and infinite processes would be something to explore in future studies.

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"f(x) Means y": Students' Meanings for Function Notation

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When using function notation, some students describe f(x) as meaning the same thing as the variable y to represent the output value of the function. This paper describes interviews with two students with vastly different interpretations of function notation. The interviews were designed to explore how students interpret function notation when it is used to represent the output for a given input.

Keywords: function notation, student meanings, using function notation representationally

This preliminary report describes research into how students understand using function notation representationally. What we mean by this is using function notation to represent the dependent variable using a given independent variable (e.g. $f(a)$ represents the y-value when x is a for a function f). This is an important area of study for mathematics education because many advanced mathematical definitions rely on function notation. Yet, many mathematics textbooks introduce function notation by writing "*f(x)*" in places where they had previously used "*y*".

Research Questions

The focus of this research was investigating what students understand about the convention of function notation. We were interested in students' interpretations of statements using function notation representationally. For example, function notation could be used representationally by using $f(x)$ to represent the output of a function f in the absence of a calculation. Since many students have only had experiences with function notation like those in most textbooks, we were unsure of how they would interpret the notation.

Our guiding research question was: How do students interpret function notation used representationally?

Literature Review

Function notation is often introduced by writing "*f(x)*" in places where students had previously used "*y*". When used this way, function notation becomes unnecessary and unimportant to the student (Thompson, 2013). Understanding function notation as representing the output value for a given input value is crucial to understanding many topics in advanced mathematics.

Sajka wrote about a student's difficulty with the many parts of function notation; this includes the f, x, $f(x)$, and the meaning of the word function itself (Sajka, 2003). In the interviews for this project, it is clear that one of the students struggled to make meaning from all of these components of the function notation.

We take the theoretical perspective of constructivism by acknowledging that a student's thoughts are personal and private. Hence, we construct models of the students' thoughts that are consistent with the students' behaviors. We obtain data for these models by performing clinical interviews (Clement, 2000). Each student participated in a 40-60 minute recorded clinical interview in which they completed six tasks, three of which are presented here (Clement, 2000). The students were asked to think out loud and describe their reasoning.

Three students were interviewed for this study. All three students were enrolled in the same community college during the time of the interview. Excerpts from two of the student interviews will be reported here. Students' names are pseudonyms in this report. Alex had completed the calculus sequence and a lower division class of differential equations. Molly was enrolled in calculus 1. Neither student had participated in any reform or conceptual mathematics courses.

Results

The first task was adapted from an ASPIRE item (Thompson & Milner, 2018). The students were given a paper with the function definition in Figure 1, and they were asked to circle the part of the function definition, which represents the output of the function.

Circle the part of the definition which represents the output.
$f(x) = x(11-2x)(8.5-2x)$

Figure 1. Students were asked to circle the part of the function definition which represents the output.

Both students circled *f(x)*; however, the students gave very different justifications for this answer. Alex described the symbol *f(x)* as "what is 'outputted' after you put your input, which is the *x*". Molly circled *f(x)* and said "this (referring to *f(x)*) is *y*". She continued by saying, "According to my previous classes, *f(x)* is basically y. I don't really know what it means but it's like *x* is in the parentheses... But, I know this is *y*". Molly's words suggested that she viewed "*f(x)*" as another name for "*y*" yet struggled to understand the meaning behind "*f(x)*".

Throughout the interview, both students' understandings and interpretations of function notation remained consistent with these excerpts. Alex remained consistent with his meaning for *f(x)* as the output for *f* given an input of *x*. Molly tried to make meaning of the parenthesis and x in *f(x)*, but her interpretation of this part of function notation remained unclear.

The second task asked students to explain what the following definitions of an even function and an odd function mean to them. Alex's interpretation is best seen in his explanation of even, and Molly's is best expressed in her explanation of odd.

Explain the following:
*A function f is **even** if f(-x)=f(x).*
*A function f is **odd** if f(-x)=-f(x).*

Figure 2. Students were asked to describe what the definitions of even and odd functions meant to them.

Alex's response for the definition of even is consistent with his answer to task 1. He stated that, "When you have an input (*x*) and you try the opposite input (-*x*), you end up getting the same output either way". Alex continued to interpret function notation as representing the output for a function for a given input.

Molly's interpretation of the definition of odd function was, "if x was negative, then it's y will be negative as well". There are many questions to be raised from this statement that remain unanswered by the interview. How does Molly interpret equal signs? What does x in $f(x)$ mean to her? Does Molly think that $-x$ is always a negative number, by virtue of the negative sign? However, it is clear that Molly was still reading the notation $f(x)$ as representing y because she said that "y will be negative" when asked about $f(x)$.

The third task asked students to explain what the following definition for the sum of two functions means to them. The purpose of this task was to examine how students interpret unconventional uses of function notation (e.g. $(f+g)(x)$).

Explain the following:
The sum of two functions, f and g, is defined to be
$$(f+g)(x) = f(x)+g(x).$$

Figure 3. Students were asked to describe what the definition of the sum of two functions meant to them.

Alex continued to use the same meanings he had already described.

Alex: So, if we have two functions and we're trying to add up the functions according to the same input... if they both have the same input, we are able to get the value of the first function and then the value of the second function and add up those values together. It would be the same as adding up the functions mathematically, getting one big function and then plugging in the value.

Interviewer: Okay. So, you... you described adding two functions and then you described that that's the same as...

Alex: So... in terms of inputs, adding two functions, combining them into one bigger function, and then putting in an input to get an output, will yield the same result...

Interviewer: *interrupts* So what... Okay, before you... sorry. Before you do the "yield the same result", where is what you just said represented in the definition?

Alex: The "$f+g$"... Whereas the "$f(x) + g(x)$", you find your output for the $f(x)$, you find your output for the $g(x)$ and then you add those values, and they end up being the same as if you had added up the functions, then put your input to find one output at the end.

With unconventional uses of function notation, Alex was still able to describe function notation using his previously determined meanings.

Molly, however, interpreted this notation as multiplication. When reading the task out loud, she read the left hand side as "f plus g times x" and the right hand side was read with conventional terminology.

Interviewer: So, what does the "$f+g$" mean in the definition on the LHS?

Molly: Mmmm. Well, it means that these two different functions are being added to each other. I don't know if they're being combined to like… like if they were two different lines on a graph, like one of... Like if one was right here and one was right here, then they're being added to combine in the middle. And then, times x... I don't... I'm not sure what that does. $f(x)$ plus $g(x)$... I get where that comes from. Just from the algebra part of it. You uh... factor this (the right hand side) and it becomes $f(x)$ plus $g(x)$.

Further Research

This project demonstrated that there are many ways in which students can interpret function notation. There is other notation embedded in $f(x)$ to be interpreted which was not addressed. These tasks, and more importantly the follow up questions to these tasks, did not address the student's interpretations of the name of the function, *f*, nor student interpretations of the input to the function.

These tasks do not address student interpretations of the name of the function when reading function notation. This is a significant question because the function name is part of what the student is reading when interpreting function notation. There is evidence that some students think of the function name as a sign post announcing a function or a "beginning of a thought" for a new problem (Sajka, 2003). The implications of these interpretations of function name in these tasks are unclear. Interpretations of the f in $f(x)$ could easily be addressed by including a task where the student is simply asked to explain what they think the f means or represents. Similarly, the task in Figure 2 could be expanded to ask students the meaning of $f+g$.

Student interpretations of the input to the function are also unclear from these tasks. Some of the tasks could address this issue by including more follow up questions asking students to clarify their meaning for the input. But, a new task is needed solely focused on student interpretations of the input to the function. For example, asking students to interpret the meaning of $f(x)=f(x+5)$ could lead to more of a focus on the input to the function. The new task would include follow up questions specifically targeted at the student's interpretations of the x and $x+5$.

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Identifying Covariational Reasoning Behaviors in Expert Physicists in Graphing Tasks

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Covariational reasoning — how one thinks about the way changes in one quantity affect another quantity — is essential to calculus and physics instruction alike. As physics is often centered on understanding and predicting changes in quantities, it is an excellent discipline to develop covariational reasoning. However, while significant work has been done on covariational reasoning in mathematics education research, it is only beginning to be studied in physics contexts. This work presents preliminary results from an investigation into expert physicists' covariational reasoning in a replication study of Hobson and Moore's 2017 investigation of covariational reasoning modes in mathematics graduate students. Additionally, we expand on this work to include results from a study that uses slightly more complex physics-context questions. Two behavioral modes were identified across contexts that appear distinct from those articulated in the Hobson and Moore study: the use of compiled relationships and neighborhood analysis.

Keywords: covariational reasoning, quantitative literacy, physics, graphing

Introduction

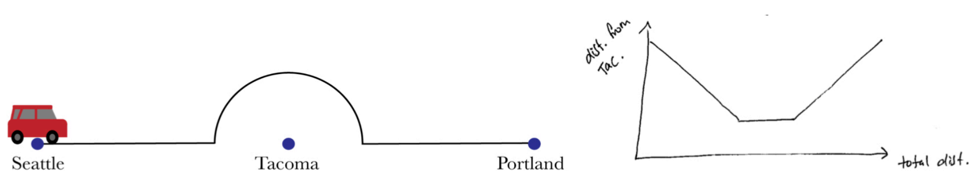
The ability to understand quantities and how they relate to each other is an important skill in day-to-day life. We expect, as educators in mathematics-based disciplines, that our students leave our classrooms with this *quantitative literacy* and carry it forward into the world. Covariational reasoning – how small changes in one quantity affect another quantity – is an essential piece of quantitative literacy. Covariational reasoning has been studied significantly in mathematics education research, but is only beginning to be examined in the physics education realm. Physics, where the focus is often on understanding and quantifying change in a physical system, is an important and useful setting for developing students' covariational reasoning. Therefore, it is integral that we better understand how covariational reasoning manifests in physics so as to more directly enhance our students' quantitative literacy across contexts.

In prior research, covariational reasoning has been shown to be an essential part of student understanding in precalculus and calculus courses (Thompson, 1994; Saldanha and Thompson, 1998; Oehrtman, Carlson, & Thompson, 2008), and plays a key role in forming conceptual understanding of derivatives, integration, and graphing functions (Johnson, 2015; Moore, Paoletti, & Musgrave, 2013; Castillo-Garsow, Johnson, & Moore, 2013; Paoletti & Moore, 2017; Carlson, 1998). Carlson, Jacobs, Coe, Larsen, and Hsu (2002) developed a framework to describe the mental actions of students engaged in covariational reasoning. Here, *mental action* is used to describe a specific behavior in students' reasoning. In 2017, Hobson and Moore reported the mental actions and associated reasoning modes of expert mathematicians (defined in their paper as mathematics graduate students with teaching experience at the introductory level) when engaged in graphing tasks designed to elicit covariational reasoning. After reviewing these works, we were interested how covariational reasoning manifests in expert physicists' reasoning.

We seek to investigate two questions: (a) how do expert physicist behaviors compare to those found by Hobson and Moore (2017) in expert mathematicians, and (b) are these behaviors consistent outside motion-based graphing tasks? The first follows from our literature review, and the second arises because the original graphing tasks are focused on motion-based systems. Producing motion graphs is a heavily practiced skill for physics graduate students engaged in teaching introductory physics, and we were curious if the behaviors we saw were simply due to the high level of familiarity with the types of questions asked. Therefore, we performed two rounds of think-aloud interviews with physics graduate students, following the experimental model provided by Hobson and Moore's work. We claim that physics experts employ two behaviors that are notably different from those articulated in Hobson and Moore's 2017 paper; we call these *using compiled relationships* and *neighborhood analysis*.

Methods

We interviewed two rounds of ten physics graduate students, five of whom participated in both interview rounds. We chose to study graduate students since they are considered to have expert-level knowledge for our introductory-level questions. Each question was designed with an accompanying animation designed to mimic the method used by Hobson and Moore (2017). For both rounds of interviews, participants were given a computer with animations that represented the problems and a paper sheet with the question prompts. They were able to play the animations at will and move between questions until they felt satisfied with their answers.



(A) **(B)**

Figure 1: (a) An animation still, and (b) an example of participant work from "Going Around Tacoma". Participants were given the following prompt: "A few students are going on a road trip, and decide to drive from Seattle to Portland. They want to avoid the traffic around Tacoma, and take the path shown in the animation. Draw a graph of the students' distance from Tacoma vs. their total distance traveled in the space below."

The animations and text from the first set of interviews were adapted from Hobson and Moore (2017) in an effort to reproduce their study with physics graduate students. The items are based around relating two distances in a situation regarding a moving object. For example, our version of Hobson and Moore's *Going Around Gainesville* task, which we call *Going Around Tacoma*, asks the participant to relate the distance from Tacoma to the total distance traveled as a car drives from Seattle to Portland (Figure 1). The second round of interviews was designed in a similar format, but with non-motion based physics contexts. For example, in *Rigamarole*, the participant is asked to relate the gravitational potential to the total distance traveled as a spaceship moves around a planet (Figure 2).

Audio recordings of the interviews were collected, along with the participants' written work. Initial transcripts were produced using the computer program Otter, and subsequently hand corrected (Otter, 2019). The transcripts were coded by one researcher using a modified approach to grounded theory, as the coding was informed by the researcher's literature review. Inter-reliability was confirmed by the other authors in the study.

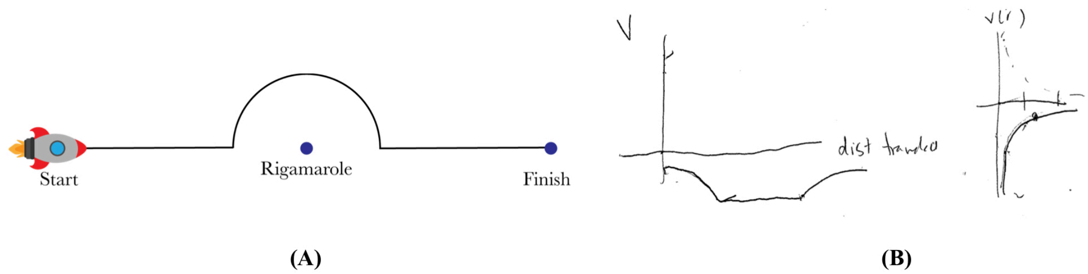


Figure 2: (a) An animation still, and (b) an example of participant work from "Rigamarole". Participants were given the following prompt: "Spencer is piloting a spaceship near the planet Rigamarole. Not wanting to crash into the surface, he carefully directs the ship around the planet, and continues on his journey. The animation represents a simplification of his journey. Create a graph that relates the total distance traveled by Spencer's ship and the gravitational potential energy of the entire system (Spencer's ship and the planet Rigamarole)."

Results

The transcripts from both rounds of interviews yielded rich, detailed reasoning patterns about covariation and therefore analysis is ongoing. In preliminary results, we have identified two reasoning patterns to be discussed here: using compiled relationships and neighborhood analysis. *Using compiled relationships* describes when participants use a previously known relationship between quantities to draw the graph rather than considering the problem at hand directly. *Neighborhood analysis* refers to when participants draw their graphs by identifying physically significant points, considering the slope of the graph around those points (in the points' "neighborhood"), and then connecting those slopes to form a smooth curve.

Using Compiled Relationships

In introductory physics, there exists a small, finite number of functions that are used to model physical situations. We speculate these are drawn upon by expert physicists and employed to reduce cognitive load. Here we discuss two examples of compiled relationships being used: a constant circular motion model employed to assign trigonometric functions in *Ferris Wheel* (Figure 3) and using the inverse relationship between potential and radius from the source object ($V \propto 1/R$) to generate the graph for *Rigamarole*.

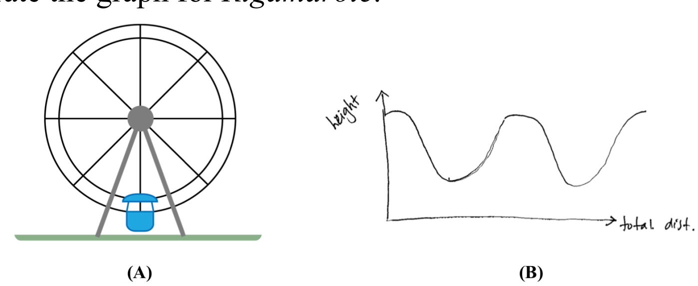


Figure 3: (a) An animation still, and (b) an example of participant work from "Ferris Wheel". Participants were given the following prompt: "As shown in the animation, a cart moves around a Ferris Wheel. Draw the graph of the height of the cart from the ground vs its total distance travelled in the space below."

Ferris Wheel was given in the first round of interviews. In our analysis we saw compiled relationships used when participants applied a uniform circular motion physics model, ubiquitous in introductory physics. Immediately, one participant stated: "I feel like this is where, like, my understanding of trig functions really comes in handy. Because I know this is a circle. And so, the height goes like a trig function." They quickly identified that circles indicate a trigonometric system, and spent their time instead determining the initial conditions rather than the form of the graph itself: "I know it's like, basically a sine wave. But then there's a choice of how to phase it." After deciding that the animation starts with the cart at the top of the curve, and therefore the highest point, they arrived at the correct graph as shown in Figure 3B. Only after significant probing from the interviewer did the student produce reasoning that directly compared the height of the cart from the ground and the distance the cart traveled. In comparison, Hobson and Moore report their participants dividing the Ferris wheel into equal sections and considering the curvature of the graph for each section (2017).

Similarly, in the second round of interviews, *Rigamarole* seemed to spark the use of compiled relationships (Figure 2). For example, one participant used the established physics relationship between potential and position from the source, $V \propto 1/R$, to arrive at their graph (Figure 2B). They began by noticing that position from the source and distance traveled are not the same quantity. They drew a separate graph of the potential with respect to the position, R, from Rigamarole off to the side of the page, and then compared points to determine the shape of the curve:

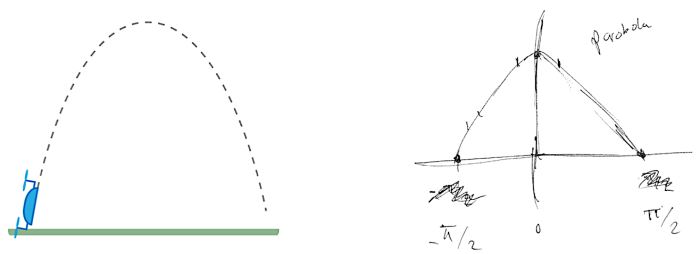
> "So as I go from distance traveled, I'm basically starting at a fixed R_o and I'm going to a smaller R there [gestures to the start and end points of the first section of Rigamarole]. So V is going to go down… so I can copy and paste this section [referring to the potential plot] to either end of the semi-circle."

Here, the student uses their prior knowledge of how potential behaves with respect to position from a source, and then compares distance traveled and position to arrive at the answer rather than directly considering how the potential changes with respect to distance traveled.

Neighborhood Analysis

Neighborhood analysis refers to choosing notable points in the problem, analyzing the rate of change between two quantities in the "neighborhood" near each point, and connecting those rates to form a smooth curve that represents the entire scenario. We found that when unable to relate a compiled relationship to the quantities in question, expert physicists often turned to neighborhood analysis as a way of solving the problem. We present two examples here, one from an expert analyzing *Ferris Wheel* that did not immediately see the uniform circular motion relationship, and another solving *Drone* from the second round of interviews (Figure 4).

While the majority of the physics experts interviewed did apply a uniform circular motion model to *Ferris Wheel*, there were a few that instead performed neighborhood analysis. One representative participant began by identifying significant points on the trajectory (at the top, bottom, and sides) and discussing how the height changes near those points. For example, when analyzing the point at the left side of the Ferris wheel, they stated: "The component of the velocity vector that's pointing towards the ground is going faster, which means its height from the ground will be changing faster [than at the top and bottom] at this point." The graduate student used vectors to assist their understanding of how the cart is moving, and arrived at the conclusion that the cart quickly changes height on the sides and slowly changes height at the top and bottom of the Ferris wheel: "So if I want it to be slow, fast, slow… I think it looks like… [draws a sinusoidal-shaped curve]." They arrived at the same-shaped curve as their peers that applied a compiled relationship reasoning pattern, but did so by only examining the covariational relationship at a few points during the cart's journey. In contrast, Hobson and Moore described the students they interviewed as dividing up the entire journey into equal parts and comparing the change between sections. We speculate that by only analyzing the covariation at points that appear *physically* significant, physics experts are reducing their cognitive load once more, perhaps towards finding a compiled relationship.



(A) **(B)**

Figure 4: (a) An animation still, and (b) an example of participant work from "Drone". Participants were given the following prompt: "A few physics students are playing with a drone, specially designed to reduce air resistance by aligning its axis along the direction of velocity. As the drone moves through the air, the students record the angle of the drone's axis relative to the ground. The animation represents a simplification of the drone's journey. Create a graph that relates the height of the drone from the ground and the angle of the drone's axis with respect to the ground. You may assume the drone experiences negligible air resistance."

In the second round of interviews, we continued to see participants exhibit neighborhood analysis. One participant in particular wrestled with their interpretation of *Drone* for some time. When they first approached the problem, they had difficulty thinking through the relationship and quickly drew the solution shown, articulating little supporting reasoning (Figure 4B). Unprompted, the student later reflected on their answer, remarking, "I guess it makes sense… because the angle is changing really quickly near the middle… if I look at two points in the middle, like, it's changing faster than, say, two points here [gestures to the start of the trajectory]." While they have difficulty conceptualizing how the two quantities were related upon reading the problem, the student is ultimately able to develop an explanation for how the quantities are changing with respect to each other by comparing two representative neighborhoods rather than equal sections of change.

Conclusion

Using compiled relationships and neighborhood analysis are distinct behaviors we identified expert physicists exhibit during interviews. This suggests that physics experts are thinking about covariation differently at times compared to mathematics experts. Physics has recently been discussed in physics education research as a conceptual blend of physics concepts and mathematical understanding. It is valuable to understand how instructors of mathematics and those of physics are using and modeling these behaviors to their students (Eichenlaub and Redish, 2019; Brahmia, Boudreaux and Kanim, 2016). The difference in covariational reasoning we observed suggests that further work is required to better understand how covariational reasoning is used in physics as it likely has an impact on student understanding across disciplines.

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Self-Efficacy in a Flipped Calculus II Classroom

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Self-efficacy (Bandura, 1977), an individual's belief in his or her ability to succeed at a specific task, is a predictor of student performance and persistence in mathematics (Pajares & Miller, 1994; Zeldin & Pajares, 2000). Thus, it is important to understand how students' self-efficacy changes in different settings. When designed carefully, certain mathematics learning environments are more conducive to students' development of self-efficacy as they allow for multiple self-efficacy opportunities (Sawtelle, Brewe, & Kramer, 2012). Flipped classrooms (Lage, Platt, & Treglia, 2000) reverse classroom lecture and out-of-class assignments and may increase self-efficacy, as students have opportunities for collaborative work and instructor feedback during class. The purpose of our study was to investigate changes in students' self-efficacy in a flipped Calculus II course. Quantitative findings included significant increases in students' self-efficacy in Calculus. Qualitative findings revealed that students believe their experiences in a flipped classroom setting increase their mathematics self-efficacy.

Keywords: self-efficacy, flipped classroom, active learning classroom, calculus

Courses traditionally viewed as "gateway" courses to advanced study in mathematics and engineering, such as Physics and Calculus, have been historically perceived by students to be the most difficult in which to succeed (Kalman & Rohar, 2010). Research has shown this perception is rooted in equity and access issues that occur before students even take these early post-secondary courses (Cross, Woods, & Schweingruber, 2009; Leonard & Martin, 2013). As a result, many students have trouble persevering in their studies in mathematics and STEM-related subjects (Hsu, Murphy, & Treisman, 2008). These experiences have an impact on students' self-efficacy, or their beliefs about succeeding at a particular behavior (Bandura, 1977). Studies have shown that self-efficacy is predictive of mathematics performance and persistence (Pajares & Miller, 1994; Pajares & Miller, 1995). These results suggest that designing classroom environments that offer students self-efficacy opportunities is essential in helping students become successful in mathematics. Self-efficacy opportunities are events that influence self-efficacy development (Sawtelle, Brewe, Goertzen, & Kramer, 2012).

Relatedly, learning environments that empower students to explore content with support from peers and the instructor, such as the "flipped classroom," have been shown to improve learning outcomes in STEM courses (Gross, Pietri, Anderson, Moyano-Camihort, & Graham, 2015; Maciejewski, 2016). Previous work on flipped classrooms in undergraduate mathematics education, specifically in the Calculus sequence, has largely focused on performance differences on posttests between students in a flipped classroom and their counterparts in traditional lectures (Adams & Dove, 2018; Albalawi, 2018). Some studies have also focused on students' and instructors' perceptions of the flipped classroom approach (Wasserman, Quint, Norris, & Carr, 2017) and design principles to guide the development of flipped classrooms (Lo, Hew, & Chen, 2017; Song & Kapur, 2017). Research in other subject areas has shown positive results with respect to how the flipped classroom approach impacts students' self-efficacy. In fact, Ibrahim and Callaway (2014) have found that students' mean self-efficacy scores in a technology integration course were higher after using a flipped classroom approach than they were before the approach was used ($t(36) = -4.652$, $p = 0.001$). With newer studies reporting positive results on students' self-efficacy in undergraduate flipped mathematics classrooms (Sun, Xie, & Anderman, 2018), the research team decided to use the flipped classroom setting available in Calculus II to take a close look at students' self-efficacy over the course a semester.

The studies on self-efficacy and flipped classroom design have formed the basis for our study, where we examined students' self-efficacy in a Calculus II flipped course, along with their perceptions of learning in such a setting. This study sought to address two research questions: (a) What is the perception of math/physical science students regarding the use of flipped teaching strategy in a Calculus II course? and (b) How does the self-efficacy of math/physical science students in a flipped Calculus II course change over the course of one semester? Preliminary results from students' self-efficacy surveys and interview data from students in the course were used together in order to better understand STEM students' self-efficacy in mathematics.

Research Setting and Demographics

This study took place in an eight-week "Calculus II for Math and Physical Science Majors" summer course at a large university. The course was taught in a designated "active learning classroom," whose layout and tools allowed for students to complete collaborative learning activities and think about what they were doing (Prince, 2004). The content and difficulty of the course were the same as that of a regular semester implementation of the course. There were 23 students in the class, all of whom had STEM-related majors including engineering, statistics, computer science, and biochemistry. Approximately two-thirds of the students identified as male, whereas the other one-third identified as female. Students reported their race as White (32%), Black (8%), Asian (56%), and Pacific Islander (4%). The research team consisted of one Part-Time Lecturer (instructor for the course), an Assistant Professor and two of her undergraduate researchers, and an Assistant Dean from the School of Engineering.

Pre-Class Activities

Before each class session, students were asked to watch approximately eight to ten instructional videos on the topics of the next class meeting. In most cases, students had two or three days to complete this work during the condensed summer session. Each video was approximately eight minutes long or less, adhering to research-based guidelines for flipped classroom video length (Guo, Kim, & Rubin, 2014). The videos used were chosen from the MathisPower4u (n.d.) YouTube channel primarily for their approach; they outlined theory, demonstrated theory and concepts with visual applets, then gave concrete examples for students to have as references. Students were encouraged to take notes and think aloud as they watched the videos.

In-Class Activities

When students arrived at each in-person class meeting, the first ten minutes were devoted to a student-generated summary of the main points of the videos. During this time, students also had the opportunity to ask clarifying questions if they did not already do so via e-mail or in office hours. Following the first ten minutes, students took a brief quiz, which consisted of two very basic questions that assessed surface-level understanding of the videos due for that class meeting. The purpose of this quiz was to hold students accountable for watching each video set.

Following the quiz, students either went through a two-station rotation or three-station rotation, depending on the activities planned for the class meeting.

Two-station rotations typically consisted of one station with selected WebAssign problems (Cengage, 2019) for students to complete, and another with applications and remediation, depending on what the students needed. The WebAssign platform offered students instant feedback on their answer submissions, along with a suite of scaffolds to support their learning, including hints and video tutorials. Each WebAssign set consisted of approximately ten questions and allowed students five attempts to answer each question correctly. The difficulty of the problems was on par with those given during a normal semester of Calculus II and followed the department's syllabus for all sections of the course. For the application and remediation station, the instructor provided more challenging problems for students who were looking to explore STEM-related applications and also supplied small group instruction and further scaffolding for students who needed more intensive support. Students had an hour to work in each station.

Three-station rotations included a "workshop" station as the third station. The workshops consisted of challenging problems where students had to connect multiple concepts together and explain their reasoning using graphs, tables, and sentences. For example, one workshop problem early on in the course was to test a claim about "average temperature" reported by a news station. Students used concepts from average value and integration to disprove the claim. Their write-ups included graphs, average value calculations by hand, and a discussion of the Fundamental Theorem of Calculus. Students had 40 minutes to work in each station due to the added workshop station. Approximately half of the class sessions had three-station rotations.

During the two- and three-station rotations, one of the stations was run by an undergraduate Learning Assistant (LA), and the other(s) were run by the instructor. Based on the successful model implemented at Colorado University - Boulder in 2001 (Otero, 2015), the University in which the study was conducted offers undergraduate students the opportunity to become LAs to gain experience working with students in active learning settings. Students who qualify to become LAs have earned an A or B+ in the course they are an LA for, successfully completed a course in pedagogy, and have the recommendation of their former instructor. LAs use their pedagogic training to help the instructor facilitate discourse in the classroom to promote active engagement. The LA for this course met with the instructor once a week for each week of the eight-week course to plan, review main ideas of content, and strategize ways to help particular students during class time. The LA's primary role in this study was to facilitate the WebAssign online homework station and encourage students to actively work with one another to solve the problems.

All class sessions ended with whole-group closure and an exit quiz, which consisted of one or two problems that resembled the station work for the class session. Our goal when designing the course was to go beyond the mere inversion of homework and lecture associated with most flipped classrooms and instead offer students more opportunities to work with one another and the instructional team to develop a deep understanding of Calculus II. Having two quizzes per class ensured students' regular attendance and upkeep with out-of-class assignments.

Methods and Analysis

Our mixed-methods approach allowed us to gather quantitative and qualitative data on students' self-efficacy in the flipped classroom. Each instrument and its corresponding analysis is described in the upcoming paragraphs.

Survey

The research team administered the 34-item “Mathematics Self-Efficacy Scale” (MSES) developed by Betz and Hackett (1993) to individual students during the first class meeting and last class meeting before the final examination. This scale was designed to measure students’ beliefs regarding ability to perform various math-related tasks and behaviors (Betz & Hackett, 1993). The MSES had two subscales, the “Mathematics Task Self-Efficacy” subscale and the “Math-Related School Subjects” subscale, which lead to an overall “Total Mathematics Self-Efficacy Score.” Items on the task subscale asked students to rate their confidence on performing mathematics tasks such as, “How much confidence do you have that you could successfully: Add two large numbers (e.g., 5379 + 62543) in your head?” (Betz & Hackett, 1993). Items on the subjects subscale asked students to rate their confidence on earning grades in certain courses, such as, “How much confidence do you have that you could complete the course with a final grade of "A" or "B" in: Basic College Math” (Betz & Hackett, 1993). Each survey item had 10 possible response options, that is 0 to 9, corresponding to “No Confidence at All” to “Complete Confidence” on the scale. Betz and Hackett (1993) reported strong evidence for the reliability of items within the MSES, including a Cronbach’s alpha value of .96 for the total scale, and .92 for both the tasks and courses subscales of the MSES. The pre- and post- surveys were analyzed using a paired t-test, with each survey treated as a separate population.

Interviews

Fifteen semi-structured interviews (Rubin & Rubin, 2011) were conducted during the last week of class. The research team adapted questions from a previous study on students’ self-efficacy in Calculus (Monterrosa, 2015). Students did not provide their names and students’ names were unknown to the interviewer, author two. Each interview was 10-20 minutes long and allowed students to reflect on their self-efficacy in mathematics. Examples of interview questions include, “How do you rate your confidence in math now? Why?” and “What could make you feel more comfortable about math?” (Monterrosa, 2015). Follow-up questions allowed for students to expand on either the survey they took or experiences in the course. All interviews were audio-recorded using a digital recorder, transcribed, and coded using open coding. In this paper, we use the qualitative interviews to supplement the findings from the quantitative analysis by including “rich, thick descriptions” form the interviews (Merriam, 2002).

Preliminary Results

Quantitative data from the pre- and post- surveys indicated changes in students’ self-reported self-efficacy from the start of the term to the end of the term. The results showed a statistically significant increase in students’ math-related school subjects self-efficacy from pre-survey (M = 5.69, SD = 1.44) to post-survey (M = 6.25, SD = 1.30) conditions; $t(22) = 2.52$, $p = .019$. This result implies that at the conclusion of the flipped course, students felt more confident that they could earn an A or B in college courses that involve mathematics. This is consistent with students’ commentary during interviews about their ability to succeed in mathematics. For example, many students described how their level of confidence increased over the course of the term and how continuous practice helped them earn high grades. One student even described, “seeing someone in the same group or maybe level as me doing (math) just builds confidence,” which implies vicarious learning as a source of self-efficacy. Several students also described feelings of validation and satisfaction from their work in the flipped course and how that has changed their beliefs in how they can perform in mathematics courses.

There was also a statistically significant increase in students' mathematics task self-efficacy from pre-survey ($M = 6.62$, $SD = 1.36$) to post-survey ($M = 6.99$, $SD = 1.56$) conditions; $t(22) = 1.7648$, $p = .0915$, using $\alpha = .10$. This result means that at the end of the flipped course, students felt more confident about math-related content and their ability to complete mathematics tasks. These findings were also supported by the interviews. Many students cited the ability to work with groups inside and outside of class, as well as having access to the instructional team to ask questions, as important to their confidence in completing mathematics tasks. For example, one student described his/her experience in the flipped classroom by saying, "In lecture [I] ... talked to my professor, and he answered all the questions that I had, and ... just having more practice with it is what helped me … we're actually coming to class and we're not just listening to him, we're actually asking questions." Another student, who took Calculus II for the first time, added, " I really couldn't guess how my questions would help me but now. . . my grades are so much better, I understand the material so much better, able to ask the questions, like I feel comfortable asking the questions because we're interacting with the professor, and the LA the entire time, like it's not just for like two minutes if you approach them at the end of class."

Overall, there was a statistically significant increase in students' total mathematics self-efficacy from pre-survey ($M = 6.18$, $SD = 1.29$) to post-survey ($M = 6.64$, $SD = 1.36$); $t(22) = 2.38$, $p = .026$, as measured on the MSES. Upon closer examination of the categories within the "Math-Related School Subjects" portion of the survey, we found that 22 of the 23 students rated their Calculus self-efficacy as higher at the end of the term compared to the start of the term, with a statistically significant difference from pre-survey ($M = 5.30$, $SD = 2.38$) to post-survey ($M = 7.04$, $SD = 1.33$) conditions; $t(22) = 4.729$, $p < 0.001$. These findings were also supported by the interviews when students described their ability to succeed in understanding Calculus at a more conceptual level. For example, one student commented, " ... I think this is the most I've actually understood Calculus II and before then I just didn't even, I just followed everything like the formula, but now I know the reasoning of everything." Another commented, "I think I understood the concepts a lot more and I knew when to use one formula versus the next. I understood even everything's formulaic but I know the reason why I'm using this formula." Several other students also discussed their prior experiences in math courses and how that has shaped their success in Calculus II.

Discussion

Self-efficacy is an important dimension to consider when designing an undergraduate mathematics classroom learning environment. The kinds of classroom activities in this study seemed to have an impact on students' self-efficacy; multimedia learning, followed by collaborative in-class work with instructor feedback, proved to significantly increase students' self-efficacy, particularly in Calculus. As a result, increasing different kinds of self-efficacy opportunities is important for students who enter a course with low self-efficacy, particularly early undergraduate STEM courses like Physics and Calculus. Not surprisingly, we also found students were aware of their confidence in math. Discussing and addressing that directly in class may help students with their feelings toward math. Future work in this area will include a closer look at students' learning outcomes and how they correlate with self-efficacy measures and feedback from interviews.

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Implementing an Open Educational Platform in Blended Learning

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Adopting open educational resources (OER) with an organized educational frame has the potential to not only reduce student cost for learning materials but also improve student learning, instructional methods, and educational environments. The aims of this study are to explore student perspectives on the use of an OER platform in blended learning, and to examine student achievement, engagement, and opportunity for learning mathematics. As a mixed project, the data was collected from 423 students in 15 sections of Elementary Statistics during four semesters. The results of this study showed that the use of an OER learning platform in blended learning promoted student engagement and significantly increased student opportunity for learning. There were not significant differences in student achievement between adopting an OER learning platform in blended learning and adopting commercial resources in regular classes. This study will contribute to the knowledge of the open educational use of technology.

Keywords: Open educational resources, blended learning, and learning platforms

Introduction

In the past decade, mobile technology has revolutionized educational environments and instruction. Students have more opportunities to learn their materials because they can instantly access e-books, download educational materials, or do their online homework with feedback. Responding to the educational changes and the rapid use of mobile technology in education, instructors have been interested in improving online recourses and pedagogical methods. For example, students can watch online lectures for each topic or work on problems or projects using Mathematics or Maple, mathematics software, in order to improve their understanding of new concepts. As time goes by, more instructors and students can easily access e-books and online educational resources because of advanced mobile technology (Brent, Gibbs, & Gruszczynska, 2012; Wiley, 2013). Innovative technology allows educators to not only upload their educational resources but also raise communication for sharing valuable resources. As a result, there are numerous qualified open educational resources (OER) online. Instructors at colleges and universities have been interested in adopting qualified open e-textbooks because of rising costs of commercial textbooks. Researchers have reported that qualified OER have promoted instructors' pedagogical practices and redesigned educational materials because of flexibility of OER (Fink, 2003). Researchers have studied adopting OER regarding reductions in student cost of learning materials (UMass Amherst Libraries, 2013; Bliss, Robinson, Hilton, & Wiley, 2012, Wiley, 2013). In addition, there are active studies regarding to improvement on student performance and learning through adopting OER or redesigned instructions with OER. However, there are limitations to student perspectives on OER and effectiveness of the use of OER in terms of student achievement, engagement, and a well-organized pedagogical frame. The aims of this study are to explore student perspectives on the use of an open educational resources (OER) platform in blended learning and to examine student achievement, engagement, and opportunity for learning mathematics in a blended learning environment. Pedagogical approaches of this study were developed by Bloom's taxonomy and blended learning, and used an OER learning platform.

There are two research questions with the following one sub-question to the second question: 1) what are students' perspectives on the use of an OER platform and learning mathematics in blended learning? and 2) how does an OER platform in a blended learning environment affect student achievement, engagement, and opportunity for learning? 2.1) are there significant differences in student achievement between the control and treatment groups?

To answer the research question, there were two groups which were regular Elementary Statistics using commercial educational resources as control groups, and OER Elementary Statistics adopting an OER learning platform as treatment groups. The instructors in the control and treatment groups used a departmental common syllabus. Instructors in treatment groups assigned their students to watch videos before class and gave additional homework from Introductory Statistics published by OpenStax after class. In class, the instructors provided short lectures about 20 minutes and then did activities. Unlike the treatment groups, instructors in the control groups used the same commercial textbook and spent most of the time lecturing.

Theoretical Perspective

The theoretical rationale of this study was built from Mayer's (2001) cognitive theory of multimedia learning (CTML) and the learning process of Bloom's Taxonomy as post-positivism perspective. The three assumptions of Mayer's CTML are 1) "Humans possess two separate channels for processing visual and auditory information"; 2) "Humans are limited in the amount of information that they can process in each channel at one time"; and 3) "Humans engage in active learning by attending to relevant incoming information, organizing selected information into coherent mental representations, and integrating mental representations with other knowledge" (Mayer, 2001, p. 44). With the three assumptions of the CTML, instructors help students improve their learning abilities in three areas: 1) selecting relevant words and pictures during instruction; 2) organizing vital information into working memory; and 3) integrating new knowledge with prior knowledge (Mayer, 2001). Mayer's CTML formed the length and the components of short-format video lectures before class. In addition, activities and learning materials in blended learning were developed on Bloom's Taxonomy. For example, students were assigned to watch mini video lectures before class for Remembering level and then completed interactive web-based exercises through learning platforms for Understanding level. For Creating level, students could post their solutions, notes, or videos on an educational blog. Instead of simply adopting innovative pedagogy, the theoretical frames provided a clear direction for students to adopt appropriate OER and innovative educational technology based on the educational framework. In addition, this study helps readers understand the finds as knowledge and to anticipate the next phenomena.

Literature Review

Researchers have become interested in the effectiveness of open educational resources (OER) since more learning content is available online according to increasing technological advances. Researchers reported the significant impacts of OER on student outcomes. OER refer to any electronic educational resources such as digital textbooks, learning applications, or virtual materials with little or no cost to students (Wiley, 2013). For example, OpenStax, a nonprofit based at Rice University, has provided high-quality peer-reviewed open textbooks since 2012. The benefits of OER are to save money and improve student access to core instructional materials (Wiley, 2013). According to Rice University news and media (August, 2016), 2,026 college systems/schools have adopted OpenStax textbooks and have saved students over $42 million in the 2015 academic year. In addition, the success and retention rates of students who used OER were better than the rates of students who used traditional textbooks (Hilton & Laman, 2012; Feldstein et al., 2012).

In the early 2000s, learning platforms were considered a potential tool as next-generation technology in education (Hill, 2012). A learning platform is defined as "an integrated set of interactive online services that provide teachers, learners, parents and others involved in education with information, tools and resources to support and enhance educational delivery and management" (Hill, 2012). There are several advantages of learning platforms, such as 1) supporting multiple learning social applications instead of just extensions to the enterprise system, 2) creating connections between learners and customization of content based on learner needs, 3) allowing for the discovery of instructional content and user-generated content (Hill, 2012). Therefore, learning platforms provide a single, unified learning destination for students by integrating with any internal or external content source.

Blended learning is an alternative solution to promote students to be active learners and to reduce limits of face-to-face learning environments because of the accelerated technological advances. Blended learning models are more effective pedagogy than purely face-to-face instruction or purely online instruction for students, who are accustomed to technology (Gecer & Dag, 2012). In addition, students in blended instruction enhance their understanding of material and engagement (Garrison & Kanuka, 2004; Alexander, 2001). Thus, a number of instructors from K-12 to higher education have adopted blended learning strategies into their classes to engage students towards meaningful learning outcomes (Garrison & Kanuka, 2004).

Methods

The number of participants was 423 students in 15 sections of Elementary Statistics with three instructors during four semesters at a public university with about 20,000 students in the Southeast. The minimum and maximum age of the participants was nineteen and sixty five years old. The participants were both genders and registered in daytime classes to fit their schedule and other preferences.

This study employed three different data sources: the pre and posttests, semi-structured interviews, and questionnaires. The qualitative data explored student perspectives on the use of an OER platform and learning in blended learning. The quantitative data was collected by three categories: engagement, achievement, and opportunity for learning and supported the findings of the qualitative data. There were 45 volunteers for 15-minute interviews conducted after the questionnaires during the four semesters. All participants answered the 20-minute questionnaires at the end of semesters. Interviews were semi-structured with five open-ended questions regarding student perspectives on an OER learning platform and learning in blended learning. After interviews, the interviewees checked their transcriptions, and peer researchers reviewed the data and results, which were questionnaires, transcripts of interviews, critical themes, categories, and code books based on the strategy of Miles & Huberman (1994). In order to measure student achievement, the pre and posttests were the same question formats with different numerical values between control and treatment groups. The tests were provided at the beginning and the end of semesters respectively. To analyze the quantitative data, two sample t-tests with a significance level of 0.05 were conducted using Minitab, software for statistics. The dependent variables were mathematical achievement by the posttest, engagement, and opportunity for learning.

Findings

As a mixed project, this section describes the results of the qualitative and then the quantitative data. Interviews were analyzed for student perspectives on a OER learning platform and a blended learning environment. From analyzing interviews, there are two main themes: open videos on an OER learning platform and having more time for class activities. As the first theme, qualified open videos through the learning platform strongly helped students understand concepts and solving problems. Some students responded:

> *Nicholas*: I really love the fact that you made a page with instructional videos on Kahn Academy. I feel like the extra online resources have helped me personally work through the problems if I did not grasp a certain concept during lecture.
>
> *Kirsten*: The Khan Academy videos really helped me understand the material because it allowed me to stop and rewind the video when needed and there were practice problems as well.

In addition, students had more time to work on problems as class activities because of the benefits of blended learning environment:

> *Riley*: I also like that you make one person perform a problem on the board. Even though I am a shy individual, it forces me to really think and work through problems on my own to get some initial practice before I work problems on my own.
>
> *Josh*: Working through examples of problems on the board. It helps me greatly to see the step by step processes needed to solve different problems.

The findings of questionnaires were broken into three categories: engagement, opportunity to learn, and an effective educational tool for understanding concepts. Sixty percent of students with an OER learning platform in blended learning believed that they actively participated in class activities. Contrary to students in treatment groups, only 53.8% of students in regular Elementary Statistics positively engaged in class activities. In the category for "opportunity to learn," 61.3% of students in treatment groups believed that they had more opportunities to learn and were motivated to access educational materials through an OER learning platform. However, only 49% of students in control groups had positive answers for "opportunity to learn" even though they could access commercial educational materials, for example, physical or e-books, videos, and problems. In the section for "an effective educational tool for understating concepts," 53.6% students thought that an OER learning platform was a vital tool to understand new concepts. In addition, 81.6% of students in treatment groups were satisfied with overall open educational resources through a learning platform in blended learning.

To measure student achievement, two sample t-tests were employed. The means of pretest of students in treatment groups and control groups were 80.3 and 75.5 respectively. There was significant difference between two groups in terms of the pretest means (p-value =0.00074 with alpha =0.05). In addition, the posttest mean of students in blended learning was 72.7. On the other hand, the posttest mean of students in regular classes was 75.9. Students in the two groups were not significantly different in the means of posttest (p-value=0.05991 with alpha=0.05). Although student achievements were different in the means of pretests, students in the two groups had similar achievement at the end of semesters.

Thus, the findings of questionnaires and interviews supported that an OER learning platform improved student learning and engagement and was an important educational tool for learning. In addition, students were interested in and had positive overviews on blended learning environments because students had not experienced blended learning approach in mathematics. Students in treatment groups should have better achievements according to the findings of questionnaires and interviews. However, the achievements of two groups were not significantly different. The results indicate that adopting OER learning platforms in blended learning played a role as an effective learning approach with no-cost-to-students learning materials. In addition, the innovative pedagogical approach has potential for student learning in mathematics and improvement on pedagogy because of flexibility of OER and reserved time for class activity in blended learning. This study encourages math instructors to be interested in OER learning platforms and improve their innovative pedagogical methods in blended learning. In addition, it contributes to the knowledge of the educational use of technology with no-cost-to-students learning materials.

Lists

1. To analyze the quantitative data for student achievement, what is the appropriate significance level?
2. Are there affordable methods for measuring student engagement?

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Singular and combined effects of learning approaches, self-efficacy and prior knowledge on university students' performance in mathematics

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This article describes a quantitative research project with the sole aim of exploring the factors affecting learning outcomes of students in higher education mathematics. The chosen factors are mainly from students' personal constructs which are approaches to learning, self-efficacy and prior knowledge. Results of analysed two sets of data consisting of 234 engineering and economics students as well as 253 engineering students that offered a first-year mathematics course are reported. The preliminary findings established a differential classification of the students' approaches to learning into deep and surface. It has also exposed the relationship between calculus self-efficacy and approaches to learning with higher self-efficacy students identified with deep approaches and lower self-efficacy students identified with surface approaches.

Keywords: approaches to learning, self-efficacy, factor analysis, structural equation modeling

Introduction

Attrition in mathematics performance of undergraduate students has gained increased global attention among educationists (Eng, Li, & Julaihi, 2010) in which Norway is not an exception. Gynnild, Tyssedal, and Lorentzen's (2005) report of 21.5% to 39.2% failure rate over the span of five years in a first-year introductory calculus course at a Norwegian university is a typical example. This prevalence of mathematics underachievement among university students has among other reasons been a focus of MatRIC, Centre for Research, Innovation and Coordination of Mathematics Teaching. The centre is situated at the University of Agder while extending her research across universities in Norway with a focus on teaching and learning mathematics within other subject programmes such as economics, engineering, natural science and teacher education. On this basis, it supports several research projects both qualitative and quantitative research to improve students' learning experience in mathematics. This study is part of a larger quantitative research project supported by MatRIC to provide empirical evidence on the major factors affecting students' learning outcomes in undergraduate mathematics within economics, engineering, natural science and teacher education disciplines.

Diverse studies abound in literature on empirical evidence of various factors affecting students' performance in mathematics (e.g., Zakariya, 2016). Prominent among these factors are prior knowledge, learning approaches and self-efficacy. These factors have been reported to have high correlations with students' performance in mathematics coupled with strong predictive powers (e.g., Roick & Ringeisen, 2018; Sharp, Hemmings, Kay, & Atkin, 2018). Deep approach is generally associated with increased learning outcomes while surface approach is generally associated decreased learning outcomes. For instance, Maciejewski and Merchant (2016) found in their study that deep approach to learning has positive correlation with student mathematics grades in the first-year while surface approach to learning has negative correlation with student mathematics grades in year two, year three and year four. Also, employing multi-level analysis, Peters (2013) found that mathematics self-efficacy differs across gender with boys taking the lead and positively correlated with achievement. More recently, Roick and Ringeisen (2018) found in their longitudinal study that mathematics self-efficacy exerts a great influence on performance and played a mediating role between learning strategies and mathematics achievement.

However, quite a number of these studies were conducted outside Norway with a few within the country but conducted in other disciplines (e.g., Opstad, Bonesrønning, & Fallan, 2017). Also, given that some of these factors are either context or task specific, findings from other studies may not be applicable to the population of current study. Further, the interplay of these factors and the combined effects on performance have not been investigated within Norway to the best of our knowledge. In a bid to fill this gap and move the frontier of knowledge forward in this direction, the current study emanated with the aim of investigating the correlational effects of students' prior knowledge, learning approaches and self-efficacy on performance in first-year calculus course among university students. Further, the interrelatedness of these factors and their combined effects on performance are also being investigated.

Conceptual Framework

Two well-established psychological theories are important for this current study. These theories are student approaches to learning (SAL) theory (Marton & Säljö, 1976, 2005) and self-efficacy theory (Bandura, 1977, 2012). The SAL theory of Marton and Säljö using phenomenography methodology coupled with some constructivist perspectives of John Biggs and colleagues has provided theoretical foundations for conceptualizing students' approaches to learning in the current study. According to SAL theory, approaches (deep or surface) are not fixed traits, they are context-specific, they are intrinsically and extrinsically motivated and predict performance (Entwistle, 2005; Marton & Säljö, 2005). These basic ideas of the SAL theory have been used to formulate the research questions and the hypotheses of the current study. Moreover, their classification of students' approaches to learning into 'surface' and 'deep' approaches has guided this study towards the selection of an instrument for measuring the construct. On the other hand, the conceptual understanding of perceived self-efficacy has been provided a theoretical structure from Bandura's self-efficacy theory. His self-efficacy theory posits that all psychological and behavioral changes occur as a result of modifications in the sense of efficacy – the conviction to perform a task geared towards a desirable outcome, or personal mastery of an individual (Zakariya, Goodchild, Bjørkestøl, & Nilsen, 2019). Perceived self-efficacy is also not a fixed trait but predicts performance (Bandura, 2012). These ideas have guided the current study in justifying the relations between the constructs.

It is important to remark that these two theories are complementary to each other with a sole aim of making learning activity a meaningful process. They model student's learning experience especially at higher education institutions which fits the aim of the current study. Another shared characteristic of these theories is that they consolidate 'presage', 'process' and 'product' features of the 3P model (Bandura, 1993; Dunkin & Biddle, 1974). Such that prior knowledge can be embedded in the 'presage', perceived self-efficacy and learning approaches in the 'process' and performance of students in the 'product' component of the model. We argue from these common traits that despite the variances in the proponents of these theories there is considerable coherence in putting it together in the current study. Finally, apart from the provision of structures for conceptualization and designing this research, this conceptual framework also helps in the formulation of research questions, selecting research instruments, and justifying the research methods.

Research questions

The following research questions are raised and are being addressed while Figure 1 elicits hypothesized relationships between research variables:

1. How do mathematics learning approaches differ among first-year undergraduate students with respect to the prevalence of deep and surface approaches?
2. What is the singular contribution of prior mathematics knowledge, students' learning approaches and self-efficacy on academic performance in first-year calculus?
3. Do prior mathematics knowledge, students' learning approaches and self-efficacy have inter-correlational effects within each other and between performance in first-year calculus?

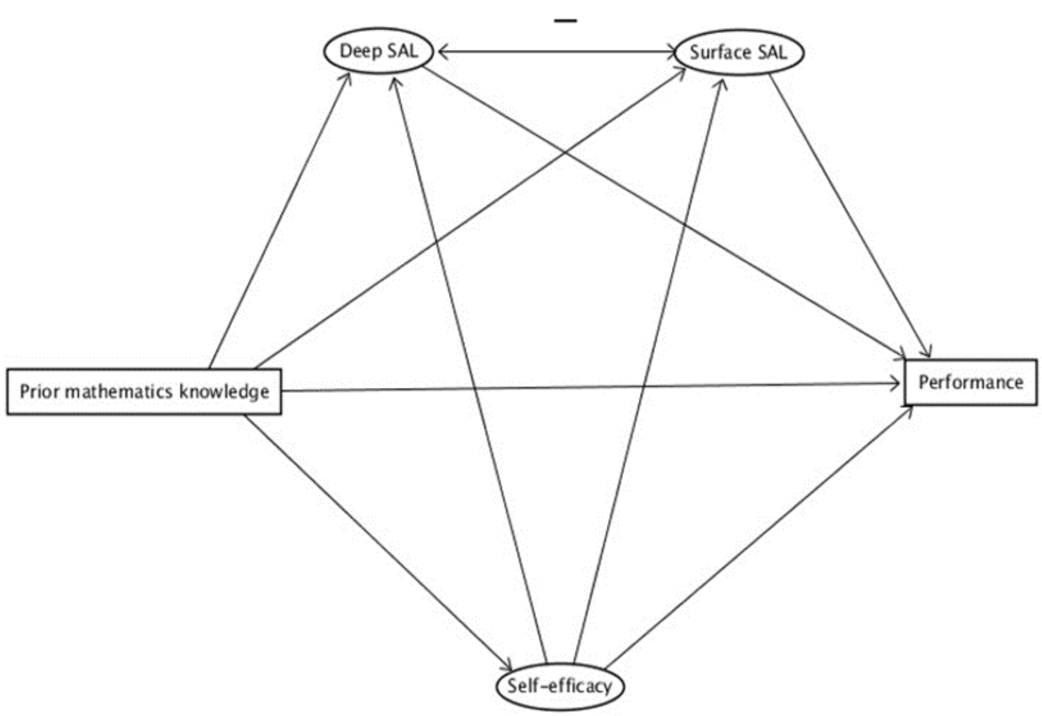

Figure 1. Hypothesized relationships between the research variables.

The oval shapes in Figure 1 represent unobserved (latent) variables while the rectangles represent observed variables. Single-headed arrows indicate the directions of effects and the double-headed arrow indicates a correlation. The figure shows that prior mathematics knowledge, deep approach, surface approach, and calculus self-efficacy have direct effects on student performance in the first-year calculus course. These are hypothesized relations based the two theories adopted in the current study. Also, prior mathematics knowledge was also hypothesized to have indirect effect on performance in the first-year calculus course via calculus self-efficacy and approaches to learning. Further, calculus self-efficacy was hypothesized to have indirect effect on student performance in the first-year calculus course via approaches to learning and there is a negative correlation between deep and surface approaches. The correlation between deep and surface is negative because each student who adopt deep approach to learning is expected to have low rating on surface approach subscale and vice-versa.

Data sources and methods of analysis

Our focus in this study encompassed the first-year engineering undergraduate students in a Norwegian university. These students were chosen because their prior mathematics knowledge before university education can effectively be assessed at the beginning of their first year in the university as compared to students in other years of study. They appear to be more prone to performance attrition in mathematics, lack of perceived self-efficacy, unstable learning approaches, etc, due to their newness in higher education (HE) and early detection of these will aid remedial interventions. Data were collected in two phases. The phase one data collection was completed in the Spring 2019 in which all students who consented to take part in the project filled-out questionnaires used for this study. It involved two data sets consisting of 234 engineering and economics students as well as 253 engineering students that offered a first-year mathematics course. The 234 data set was used to develop a calculus self-efficacy inventory (CSEI) using exploratory factor analysis (EFA). It was also used to establish a relationship between calculus self-efficacy and approaches to learning. Further, the 253 data set was used to validate Norwegian adaptation of the revised study process questionnaire (R-SPQ-2F) – a widely used instrument for measuring approaches to learning. The method of analysis of R-SPQ-2F followed a series of confirmatory factor analyses using Mplus program. The ensuing findings from the analyses of these data are presented in this article.

Phase two data collection is taking place at the time writing, during Autumn 2019 and it involves CSEI, R-SPQ-2F and an instrument for measuring prior mathematics knowledge of the students. Structural equation modeling will be used to analyze the data such that some of the research questions as well as the hypotheses will be addressed accordingly. The structural equation modeling approach was chosen instead of multiple regression analysis as the former is robust enough to account for measurement errors in the instruments and can accurately expose the mediating effects within the constructs of investigation. Some of the findings from the analyses of the second phase data collection will be presented at the conference.

Findings

Our preliminary findings revolve around addressing the first research question and provision of empirical evidence for validity and reliability of the instruments used in the current study. The result of the EFA established a unidimensional 13-item CSEI with all eigenvalues greater than .42, an average communality of .74, and a 62.55% variance of the items being accounted for by the latent factor, i.e., calculus self-efficacy (Zakariya et al., 2019). These were interpreted to be evidence of construct validity in measuring students' internal conviction in successfully solving some calculus tasks. The concise length, task specificity, higher factor loadings, and higher average communality gave CSEI advantages over mathematics self-efficacy scale (MSES) developed by Betz and Hackett (1983) and its revisions (e.g., Kranzler & Pajares, 1997). The inventory was also found to be reliable with an ordinal coefficient alpha of .90. This coefficient portrays evidence of high internal consistency of items in the inventory (Baglin, 2014).

Further, this reliability coefficient is higher than the coefficient of the mathematics task subscale of the MSES reported in (Betz & Hackett, 1983; Hackett & Betz, 1989), and it is within the ranges of the revised MSES reported in the literature (e.g., Kranzler & Pajares, 1997). Out of a total of 234 engineering and economics students it was only possible to identify the scores of 95 students on both the CSEI and R-SPQ-2F due to some technical constraints. Using Spearman' rank correlation coefficient, a significant moderate positive correlation $\rho(95) = .27, p < .05$ (2-tailed) was found between the deep approach to learning and calculus self-efficacy, and a moderate negative correlation $\rho(95) = -.26, p < .05$ (2-tailed) was found between the surface approach to learning and calculus self-efficacy. These suggest that students who adopt a deep approach to learning are more confident in dealing with calculus exam problems than those students who adopt a surface approach to learning. Further, these results also provided evidence of discriminant validity of the CSEI through correctly separation between external latent constructs (Zakariya et al., 2019).

The results of the analysis of the 256 data set (χ^2-value (151, N = 253) = 377.68, $p < 0.05$, standardized root mean squared residual (SRMR) = .072, comparative fit index (CFI) = .844, Tucker-Lewis index (TLI)= 0.824, and root mean square error of approximation (RMSEA) = .077) established construct validity of the Norwegian adaptation of R-SPQ-2F (Zakariya, Bjørkestøl, Nilsen, Goodchild, & Lorås, 2020). The series of CFA confirmed a first-order two-factor solution – deep and surface approaches – as the underlying constructs of the instrument. However, appropriate fits of the tested hypothesized models were found after deleting one item from the original 20-item questionnaire. This finding corroborates most of the reported results on the transcultural validation of R-SPQ-2F in the literature (e.g., López-Aguado & Gutiérrez-Provecho, 2018). Consistent with the literature (e.g., Biggs, Kember, & Leung, 2001), a negative correlation ($r = -.52$, $p < .05$) was also found between items on deep and surface approach subscales which is suggestive of a discriminant validity between the subscales. Moderate reliability coefficients of .81, .72 and .63 were found for deep subscale, surface subscale and the whole instrument respectively using Raykov and Marcoulides' (2016) formula (Zakariya, 2019; Zakariya et al., 2020).

Conclusion

The preliminary findings presented in the previous section have addressed to some extent our first research question by establishing a differential classification of the students' approaches to learning into deep and surface. Psychometry evidence of construct, content and discriminant validity as well as reliability of our measures have been provided. More so, the findings have also exposed an inversely proportional relationship between calculus self-efficacy and approaches to learning i.e. high sense of perceived self-efficacy implying deep approach to learning and low sense of perceived self-efficacy implying surface approach to learning. More results will be explained in detail during the presentation at the conference.

Questions for Audience

1. What are your recommendations towards an improvement of this current project?
2. Are there other ways to measure approaches to learning and calculus self-efficacy within quantitative research methodology and without the use of survey instruments?
3. How can the validity of the current study be improved to foster generalization of results?

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Student Mathematical Activity During Analogical Reasoning in Abstract Algebra

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This paper examines the mathematical activity of two students as they engage in reasoning about topics in ring theory by analogy with topics in group theory. Analogies are represented by mappings between a source and target domain (Gentner, 1983). Participants were given task-based interviews, each focused upon a particular structure from ring theory: rings, subrings, ring homomorphisms, and quotient rings. Techniques from grounded theory were utilized to analyze the interview transcripts with the goal of interpreting and describing the mathematical activity of the students. Preliminary results indicate that there are three main categories of activity: foregrounding the source or target domain, focusing on similarities or differences, and which mapped aspects of the domains are the focus of the reasoning. "Pathways" are identified as a way to capture the dynamic nature of students' reasoning by analogy in abstract algebra.

Keywords: Abstract Algebra, Analogical Reasoning, Group Theory, Ring Theory

Reasoning by analogy is an important component of mathematical reasoning (Polya, 1954). However, research indicates that teachers produce the majority of analogies in classrooms and that students rarely get the chance to engage productively with analogical reasoning themselves (Richland, Holyoak & Stigler, 2004). In addition, research on analogical reasoning has been dominated by an expert/novice paradigm wherein the focus is on whether individuals can generate analogies that meet some predetermined criteria set by an expert (e.g. Holyoak & Thagard, 1989) rather than focusing on the process of analogical reasoning.

The initial aim of this study was to investigate the ways in which students might leverage their knowledge of group theory when learning ring theory as well as study what kinds of connections students might generate between these subjects. Although research exists on students engaging with group theory (e.g., Larsen, 2013) and ring theory (e.g., Cook, 2017), there is virtually nothing in the literature about the potentially viable interplay between the two for assisting students in understanding ring theory after having learned group theory. One way in which this can occur is through analogical reasoning about the various structures in group theory and ring theory. This study will aid in providing a more complete picture of how students reason analogically about mathematical structures in group theory and ring theory.

Review of Literature

Gentner (1983) introduced the concept of representing analogies by mapping between domains. Domains are defined as collections of networks of nodes and propositions comprising knowledge about a structure or situation. In particular, Gentner identified the base domain, also called the source domain, as the familiar domain that acts as a source of knowledge. The target domain is the domain to which knowledge about the source domain is applied. A set of mappings between the source and target domain is an analogy. Different aspects of domains may be mapped: objects, attributes, and relations. Objects are representative of concepts within a domain, whereas attributes are descriptions of those objects. Relations describe ways in which two objects interact. Gentner argues that analogies are defined as those mappings between domains that do not map attributes, but rather map relations.

Stehlíková and Jirotková (2002) described how students might create of the structure of the set of integers mod 99, Z_{99}, by analogy with the structure of the integers, Z. Through engaging students with tasks designed to meet this goal and analyzing their work, Stehlíková and Jirotková identified five phenomena related to the creation of one structure by analogy with another: identifying regularities, identifying anomalies, broadening intuition, adaptation, and obstacles. Identifying regularities and anomalies refers to the observation of similarities and differences, respectively. Broadening intuition is the phenomena of coming to a greater understanding of a common concept between the analogous structure as a result of reasoning by analogy. Adaptation refers to the phenomenon of making suitable changes to the target in an effort to reconcile the differences between two domains. Finally, obstacles refer to the phenomena of understanding a structure being hindered because of a reliance on an analogy to a previously understood structure.

The goal of this study is to further explore the ways in which students might reason by analogy in mathematics. This study aims to answer the following research question: *What is the nature of the mathematical activities that students engage in during analogical reasoning tasks about rings, subrings, ring homomorphisms and ideals?*

Theoretical Framework

Following Gentner (1983), this study assumes that analogies are represented by mappings between a source and target domain. A visual of this representation is seen in Figure 1. However, I modify aspects of Gentner's theory in order to better suit the needs of studying analogical reasoning within the context of mathematics. In particular, I provide definitions of object, attribute, and relation, and adopt a more relaxed position of what counts as an analogy. I discuss these modifications here.

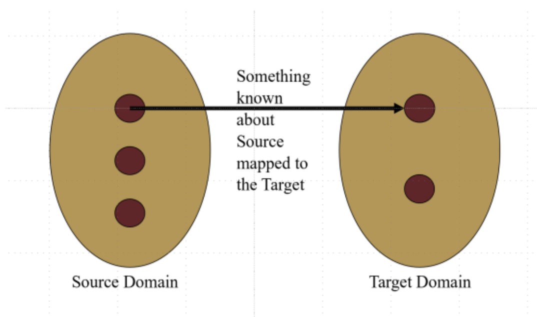

Figure 1. Representation of Analogy as Mapping

I define objects as being mathematical objects or structures within a particular domain. For example, a quotient group or a binary operation could be an object in the domain of group theory. Attributes are then definitional or descriptive properties of objects. For example, an attribute of a group would be that the identity is unique. Finally, relations are connections made between two or more objects within a domain. An example of a relation would be that kernels of ring homomorphisms are ideals. While Gentner argues that an analogy is a mapping that consists primarily of relations, I contend that attributes may act as an important aspect to attend to during analogical reasoning in the context of abstract algebra. For example, a student may map the attribute that the additive identity of a group is unique to the multiplicative identity of a ring being unique.

Research Methodology

Data Collection

A series of task-based interviews were individually administered to two participants recruited from a large public university in the South-Central United States. The interviews were audio-recorded and subsequently transcribed. The first participant, Timmy, was an undergraduate student pursuing a Bachelor's in mathematics. Timmy had previously taken one undergraduate course in abstract algebra covering group theory and a brief introduction to rings. The second participant, Trixie, was a graduate student pursuing a PhD in mathematics education. Trixie had taken one undergraduate course and one graduate course in abstract algebra, both emphasizing group theory. Trixie had no prior exposure to ring theory.

The initial interview focused on prior knowledge of group theory and was used to assess the participants' content knowledge of group theory before beginning to explore topics in ring theory. The interviews that followed then each focused upon a different topic in ring theory, including rings, subrings, ring homomorphisms, quotient rings and ideals. The interview tasks were constructed around three basic types: (1) Explicit elicitation of analogy generation, (2) example generation and checking, and (3) proof-writing. The goal of the first type of question was to explicitly engage the participant in analogical reasoning, whereas the second and third types gave the students a chance to engage and acclimate to the new material while allowing for the opportunity to spontaneously reason by analogy. Examples of these types of questions are seen below:

> 1a. Make a conjecture for a structure in ring theory that is analogous to subgroups in group theory.
> 1b. In what ways is your structure similar? In what ways is your structure different?
> 2. What might be an example of a subring for the ring Z?
> 3. Is the homomorphic image of a commutative ring a commutative ring?

Data Analysis

Instances of analogical reasoning were identified in each of the interviews. These instances were established based on evidence that the participant was making analogical connections between two or more topics. For example, a series of turns of talk focused on an aspect of ring homomorphisms in which connections were made to group homomorphisms could comprise an instance. Multiple instances that were focused on a single topic formed a segment.

Open coding was performed in order to generate concepts and ideas about the students' activity related to analogical reasoning (Corbin & Strauss, 2015). This process allowed for a close examination of the different ways in which the participants were engaging in reasoning by analogy. Categories of student activity were then identified. The coding process resulted in the identification of three key categories: (1) foregrounding either the source domain or the target domain, (2) focusing on either similarities or differences between the source and target domain while engaging with analogical reasoning, and (3) the aspects mapped between the source and target domain. Each instance of analogical reasoning was assigned one code from each category. These codes are briefly described in Table 1.

Table 1. Description of Codes

	Description of Category	*Codes*	*Example*
Foregrounding a Domain	Placing emphasis on a domain when generating an analogy.	Source Foregrounded Target Foregrounded	A student is foregrounding the source domain when mapping the concept of Sylow subgroups to ring theory without providing rationale for the mapping.
Similarities and Differences	Identifying or generating a similarity (or difference) between the source and target domain.	Similarity Difference	A student is identifying a similarity if they recognize or conjecture that there are special types of rings in the same way that there are special types of groups (such as Abelian groups.)
Mapping Aspects of Domains	The aspects of the domains mapped between the source and target.	Object Attribute Relation	Mapping the object of normal subgroups in group theory to the object of ideals in ring theory.

Results

Analogical reasoning was observed as being a dynamic process for the students. In other words, there were various shifts between which domain was foregrounded, whether the focus was on similarities or differences, and whether they were focused on objects, attributes or relations. To capture this dynamic process, *pathways of analogical reasoning* are considered to describe the ways in which reasoning by analogy might evolve while focusing on analogies for a single task or topic. These pathways can be productive (resulting in new conjectures or adaptations made by the student) or unproductive (resulting in obstacles for student reasoning.) I present two examples of productive pathways here.

Observing Differences to Facilitate Adaptation

One observed pathway was seen when the student would begin the process of analogical reasoning by observing anomalies. Following this observation, the student could then begin to make conjectures about the target. During a segment in which Timmy was reflecting on the definition of ring which was presented to him, Timmy noted that the lack of a multiplicative identity was strange. When asked why, he provided the following explanation:

> Because everything in group theory ... there's a very big part of it, is that there needs to be identity and there's also, even clear in this, an identity for addition. It just seems interesting that when we throw in another binary operation it doesn't necessarily need to have an identity as well. Which is just interesting to think about.

In this instance, we see that Timmy is focused upon identifying the differences between the definition of group and ring. He is foregrounding the source domain of group theory in this case because of the emphasis being placed on the definition of group. In addition, Timmy is seen mapping the object of identity in this instance.

Continuing with his explanation, Timmy begins to shift gears into a more generic conversation about the implications of rings not possessing a multiplicative identity:

> Which also, I would say, opens up a lot more possibilities for what your ring is allowed for… like if you do multiplication of positives or something like that when you're talking about integers to have a group and stuff. Or something like that, I don't remember exactly what it was, because the problem of the identity or something like that. But this'll probably open up a lot more options since you won't necessarily need the inverse part to work out or something like that, you know?

Unlike the previous instance, Timmy is now foregrounding the target domain of ring theory because the emphasis is now upon considering what is possible in ring theory given the lack of multiplicative identity. In addition, while his focus is still on differences, his focus has switched from mapping objects to mapping the attribute of elements requiring an inverse under the operation of a group. In particular, Timmy is attempting to adapt to the context of ring theory by arguing that the lack of a need for multiplicative inverses will allow for “more possibilities” for rings compared to groups.

Conjecturing Similarities to Facilitate Adaptation

In the previous example, Timmy was seen to pick out a difference before engaging in adaptation. Another way in which this can occur is by first conjecturing about what could be the same between domains before attempting to make modifications. In the following excerpt, Trixie has been asked what she might expect to see in the study of rings after her second interview in which rings were introduced:

> Oh gosh. Is it like Abelian rings? Or like giving them that type of thing where you give them special names. Special types of rings, "This is the golden ring." So these, you gave me these properties on the last page. But I'm sure if you have all these properties, it's probably a special type of ring.

In this excerpt, Trixie begins by mapping the attribute of being Abelian from group theory into ring theory. During this instance, Trixie is foregrounding the source domain of group theory.

Following this instance, however, Trixie broadens the scope of her conjecture to include “special types of rings.” This marks a transition from relying on her knowledge of group theory to recognizing that rings may differ in substantial ways. This transition is made apparent by noting the shift from foregrounding the source domain of group theory to foregrounding the target domain of ring theory.

In this excerpt, Trixie is seen to make an initial conjecture about how group theory and ring theory could be similar. Afterward, Trixie leverages this conjectured similarity in order to think about what kinds of rings could exist if one were to continue studying rings in detail.

Discussion

This study has identified the concept of “pathways” of analogical reasoning in order to capture and describe the dynamic nature of reasoning by analogy. Two examples were provided to showcase ways in which students might leverage reasoning by analogy to jumpstart mathematical reasoning about topics in ring theory provided that they have learned about group theory. More work is needed in order to fully describe and categorize the different types of pathways that might exist for students as they engage in analogical reasoning. Future work in this area could include a closer investigation of the comparison of analogies generated by the students to determine when and how pathways can be more productive and how productive reasoning by analogy can be facilitated by an instructor.

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How Different is Different? Examining Institutional Differences Prior to Scaling Up a Graduate Teacher Training Program to Improve Undergraduate Mathematics Outcomes

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A comprehensive graduate teaching assistant (GTA) training program in mathematical sciences designed at one institution is being replicated at two peer institutions. This paper presents the findings of a baseline comparison of the three universities undertaken at the start of the project to inform its adaptation and implementation at each institution and the evaluation of its impact. Program components include a first-year teaching seminar, peer mentoring and support from a peer TA coach, a Critical Issues in STEM Education seminar, and K-12 outreach to inform understanding of the pipeline. Differences in undergraduate demographics and performance in introductory mathematics courses, GTA responsibilities, prior departmental GTA training elements, and GTAs attitudes towards teaching mathematics/statistics are presented. Implications for program implementation and assessment of study goals related to institution differences are presented.

Keywords: professional development, teaching assistants, instruction, graduate education

Recent national efforts have focused on graduate teaching assistant (GTA) professional development (PD) (Ellis, Deshler, & Speer, 2016a,b; Olson, Ferrara, Jacobson, Manzanares, 2016; Speer, Ellis, & Deshler, 2017; Speer & Murphy, 2009) and mentoring (Rogers & Yee, 2018; Reinholz, 2017; Yee & Rogers, 2017) to improve academic outcomes for undergraduate students and build both perspective and a pedagogical skill base for graduate student instructors in preparation for future roles as mentors and faculty in the mathematical sciences. Although many programs have been developed for and by specific institutions (Childs, 2008; Childs & Milbourne, 2019; Griffith, O'Loughlin, Kearns, Braun, & Heacock, 2010; Kaplan & Roland, 2018), few GTA PD programs have been translated across multiple institutions.

The goal of our *NSF*-funded project, Promoting Success in Undergraduate Mathematics through Graduate Teaching Assistant Training (PSUM-GTT), is to study how the structure and training components of a program developed at one large university can be adapted for implementation at two peer institutions with different student demographics and distinct geographical locations. In this preliminary paper, we focus on the development of a shared understanding on how to best adapt a model across institutions while embracing the complexities and differences between schools, GTAs and student population. Hence, this paper provides an example of some preliminary baseline comparisons across the three schools conducted prior to the start of the 2019-2020 academic year.

The GTA Training Program

Desired Program Outcomes

The program goal is to strengthen the teaching capabilities of mathematical sciences GTAs in order to improve the academic outcomes of the undergraduates that they teach. Intended outcomes include GTAs' increased preference for student-focused instruction, satisfaction with their teaching training and mentoring, increased attention to equity and inclusive pedagogy in the classroom, and decreased rates of their undergraduate students earning grades of D or F or withdrawing.

The Program Components

The first component is a course that GTAs take about how to teach effectively to support student learning. This GTA training seminar focuses on inclusive pedagogy and best practices in classroom instruction. Common themes in the seminar will provide continuity across universities and opportunities for cross-institution discussion, feedback, comparison and training. Specifically, common articles will be used to guide discussion, reflection and seminar topics. Through a combination of both practitioner and research articles, GTAs will be asked to critically reflect on classroom issues such as assessment, equity, inclusive practices, and classroom culture. Seminar instructors will model specific pedagogical techniques throughout the seminar to highlight how one can facilitate active learning strategies and student-centered pedagogy to promote engagement and inclusivity in classroom practices. GTAs will also work toward the development of an initial teaching philosophy as a final product for the seminar.

Based on research indicating that individualized mentoring and coaching can increase teaching effectiveness, the second component involves one-to-one peer mentoring and instructional support provided by a TA Coach. Each GTA in the program will receive one-to-one peer mentoring on a consistent basis from a GTA who has taught at the university the prior year. The mentoring is designed around a series of "Office Talks" topics to facilitate their discussions, together with at least two classroom observations and post-observation feedback each semester. In addition, an experienced GTA designated as TA Coach at each university will serve as a peer leader for new GTAs and provide in-class instructional support during the facilitation of new instructional practices and activities in a supportive and non-evaluative manner.

In order to help GTAs become reflective practitioners, the third component is the Critical Issues in STEM Education seminar, which is held four times throughout the academic year for all GTAs, to provide the opportunity to interact with invited researchers and practitioners about current issues surrounding undergraduate student instruction.

The final component provides students with outreach opportunities in local K-12 schools and after-school programs to help them attain an understanding of the mathematics pipeline that their students take to college.

Initial Results for College Algebra DFW Rates at Founding Institution

Initial results from the institution that developed this training program indicate that the model has a positive impact on the performance of the undergraduates whom the GTAs teach. GTAs at this institution are typically tasked with teaching two sections of college algebra recitation during each semester of their first year in the program. Rates of students who receive a grade of D or F or withdraw from the course after the census date (DFW rates) are detailed below, including the three years prior to this program and the three years since its inception. Overall, DFW rates have decreased by approximately 11 percentage points since the implementation of the training model.

Table 1. DFW Rates for College Algebra Before and After Implementation

	2013/2014/2015	2016/2017/2018
Fall Semester Aggregate	43.1%	32.6%
Spring Semester Aggregate	42.3%	30.8%
Overall Aggregate	42.8%	31.9%

Exploring Baseline Institutional Differences

In order to accurately frame the results when evaluating the implementation and impact of the training program at the end of the project, the differences among the institutions prior to implementation need to be acknowledged, as the program will be modified to fit within the current structures of the programs at the two new project sites and as some differences among the universities would be expected to influence program results.

Undergraduate Student Demographics

University A, where the graduate teacher training program was developed, is a public institution located in an urban area in the Midwest. Approximately 57% of first-time students and 47% of all undergraduates identify as people of color, approximately half of whom identify as Hispanic and a quarter of whom identify as Asian American. Approximately 50% of all students are first generation. Approximately 50% are transfer students. Approximately 43% of first-time students and 26% of all undergraduates receive Pell Grants. Approximately 80% of the first-time students entering in Fall 2018 took the SAT. The 1st and 3rd quartiles for SAT Math scores were 510 and 600, respectively.

University B is a public institution located in an urban area in the Mid-South. Approximately 40% of first-time students and 41% of undergraduate students identify as Black or Hispanic. The majority of these students identify as Black and the institution has a Predominantly Black (PBI) designation. Approximately 38.5% are first generation students, and 44% are Pell Grant eligible. Approximately 96% of the first-time students entering in Fall 2018 took the ACT. The 1st and 3rd quartiles for ACT Math scores were 19 and 26, respectively.

University C is a public, land-grant institution located in a college town in the South. Approximately 76% of first-time students and 79% of all undergraduate students identify as White. Approximately 15% of all undergraduates receive Pell Grants, and 13% of first-time entering freshmen identify as first-generation college students. Approximately 84% of the first-time students entering in Fall 2018 took the ACT. The 1st and 3rd quartiles for ACT Math scores were 23 and 28, respectively.

Undergraduate Student Outcomes in Mathematics

Table 2 summarizes the DFW rate for each institution for a variety of developmental and introductory mathematics courses. The rates are similar for College Algebra and Calc 1, but have observable differences for PreCalc, Trig, and Calc 2. The current DFW rates for College Algebra for University B and C are lower than University A's DFW rate prior to their implementation of the training program. It should be noted that at University A, prior to Fall 2019, there was no math placement test for courses other than Calc I, while University B used an online placement test and University C used students' ACT or SAT scores to place them into specific math courses.

Table 2. Three-Year Average of DFW Rates for GTA-Taught Sections Prior to Fall 2019

Course Name	University A	University B	University C
College Algebra	29.3%	29.1%	30.0%
PreCalc	32.4%	30.0%	23.8%
Trig	37.3%	18.4%	-
Calc 1	27.1%	26.1%	30.1%
Calc 2	31.8%	31.1%	23.0%

Note: For University A, the three-year average includes years after pilot implementation. For Universities B and C, these are the three-year rates prior to implementation in Fall 2019.

First-Year Graduate Student Teaching Responsibilities

At University A, first-year graduate students are generally assigned as instructors of record for recitation sections in College Algebra. First-year students at University B and C are generally assigned as graders or mathematics tutors due to accreditation constraints (i.e. students must have completed 18 graduate hours in their content area before being able to teach).

Previous GTA Training Programs

Similar to what University A offered prior to the development of the new training program, University B offered a one-semester seminar focusing on topics such as quiz creation, grading, classroom management, instructor demeanor, and effective office hours. First-year students at University C went through a two-semester training program. The first semester covered classroom skills, including effective dialogue with students, grading and partial credit, and included two or three guest speakers on a variety of pedagogical issues. The second semester training course included micro-teaching experiences for students, wherein they were able to put lessons from the first semester into practice.

Current GTA Beliefs about Teaching Mathematics/Statistics

During the first weeks of the Fall 2019 semester (the first semester of implementation at Universities B and C), all GTAs completed an online survey, which asked about prior teaching experiences before and after entering the graduate program, previous preparation to teach (if any), and attitudes towards teaching mathematical sciences using the Approaches to Teaching Inventory (ATI; Trigwell, Prosser, & Ginns, 2005). Separate two-way ANOVAs for the ATI Student Focused and Teacher Focused scores indicated no significant main effects for university or graduate student experience (first year and second year graduate students versus students in year 3 or more in their program) and no significant interaction effect (See Tables 3 and 4).

Table 3. Means and Standard Deviations for Approaches to Teaching Inventory

ATI Subscale	University A		University B		University C	
	1st/2nd Yr. Students *(n =10)*	More Exp. Students *(n =12)*	1st/2nd Yr. Students *(n =9)*	More Exp. Students *(n =14)*	1st/2nd Yr. Students *(n = 8)*	More Exp. Students *(n = 11)*
Student Focused	35.80	38.58	41.22	39.43	39.63	36.36
	(6.16)	(3.99)	(5.04)	(3.89)	(6.70)	(5.38)
Teacher Focused	38.9	36.92	39.44	37.21	40.13	41.36
	(7.19)	(4.81)	(8.40)	(6.90)	(6.15)	(4.63)

Table 4. Two-Way ANOVA Results for University and Student Experience on ATI Scores

ATI Subscale	University Main Effect			Experience Main Effect			University x Exp. Interaction Effect		
	Test Stat	DF	p	Test Stat	DF	p	Test Stat	DF	p
Student Focused	2.20	2, 58	.12	0.34	1, 58	.56	1.97	2, 58	.15
Teacher Focused	1.12	2, 58	.34	0.37	1, 58	.55	0.45	2, 58	.64

Differences with Implications for Implementation and Impact

Several differences among the universities have implications for implementation of the program components at each site. Most importantly, University A and B had similar "prior versions" of training. University C had a training program that was somewhere in between those prior versions and the new comprehensive training program. Hence, Universities A and B will have a one-semester first-year training seminar and University C will keep their two-semester sequence. Secondly, due to the differences in accreditation bodies, University A's first-year graduate students gain teaching experience earlier in their graduate program, starting with their first semester in the program, as compared to later starts for University B and C. Lastly, baseline survey results, which might differ by site, will influence the first-year training seminar instruction at each university. For instance, the seminar instructors could spend time introducing the types of active learning activities that the GTAs indicate they have never heard of. Similarly, if the baseline survey indicates that GTAs are coming in with substantial outside teaching experience or training, this may influence the sequencing and pacing in the content presented in the first-year teaching seminar.

Due to differences in implementation, evaluation of the program components also needs to account for what might be considered a "dosage" effect. There are differences in length of first-year training seminar (University C's is longer) and amount of time possible to gain teaching experiences (University A students could teach more semesters). Additionally, there is a difference due to concurrent training with first semester teaching (University A) versus those with training prior to teaching (University B and C). Additionally, the diversity in the students served and the starting DFW rates (University B and C) also influence how much change could possibly be seen in the DFW rates at each institution.

Although it is the point of the funded project to identify the elements of the components that contribute to GTAs' development as instructional faculty, all of these differences in context need to be considered when exploring the efficacy of the training program and when making suggestions for implementation of the program at other institutions.

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An Example of Computational Thinking in Mathematics

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The term computational thinking has engaged multiple disciplines in discussions about how to prepare students for careers in a technological society. However, a wide variety of definitions and settings has made it difficult for researchers to clarify what is meant by computational thinking. In this paper, we present a case study that explores one setting in which computational thinking and mathematics interact, through an interview with a computational mathematics PhD student. We consider the influence that computational thinking and mathematical knowledge have on each other through three key moments in the student's dissertation process. Additionally, we consider the role of affect in computational thinking and the potential benefits that computational thinking carries for the future of mathematicians.

Keywords: Computational thinking, Computational mathematics, Affect

Mathematics education researchers have suggested that there could be a meaningful relationship between computational thinking and the teaching and learning of mathematics, and some have called for examples of computational thinking and activity within mathematics (e.g., Lockwood, DeJarnette, & Thomas, 2019). In this paper, we examine an hour-long interview with a mathematics PhD student on computational thinking in mathematics. The main theorem of her dissertation was motivated by a need to make a computer program that could list certain mathematical elements efficiently. Thus, we provide a case study of a student that gives insight into ways in which mathematicians interact with and leverage computers in their research. Given the variety of characterizations of computational thinking, this case study helps to shed light on the multifaceted interaction between mathematics and computational thinking, particularly with regards to Wing's (2014) definition. This paper addresses the following research question: *In what ways did the interaction between computational thinking and mathematical knowledge inform a mathematician's work toward solving an open mathematics problem?*

Literature and Theoretical Perspectives

The construct of computational thinking (CT) is gaining attention in several disciplinary areas, including undergraduate STEM departments, K-12 school districts, and the technological industry. A range of philosophical definitions (e.g., Wing, 2006; Aho, 2012), taxonomy frameworks (e.g., Brennan & Resnick, 2012; Weintrop et al., 2015), and similarities to existing concepts like as procedural understanding (Star, 2007) necessitate a pragmatic choice of definition when studying CT. We chose to use Wing's (2014) definition for this interview study because it is widely used in the CT community, is concise, and uses general language that is easy to parse quickly in an interview setting. Wing defines computational thinking as "the thought processes involved in formulating a problem and expressing its solution in such a way that a computer - human or machine - can effectively carry out". It is notable that Wing's work has been recognized as a main initiator of the current movement of CT, and her definition comes from a background of teaching computer science and working in industry. An informal way to describe CT involves the classic "peanut butter and jelly sandwich" thought experiment. Suppose you would like your robot friend to make you a PBJ sandwich. The robot has never made a PBJ sandwich before and they will follow your exact instructions; which you must give from a different room. You might say, “put the peanut butter on the bread”, and the robot puts a glob right in the middle without spreading it around. The point here is that CT invokes a certain algorithmic frame of mind that is potentially unique to working with computers.

For the purposes of this case study, when we refer to mathematical knowledge, we mean the students’ internal construction of mathematical concepts, ideas, and practices (Cobb, Yackel, & Wood, 1992). By affect, we mean the collection of one’s attitudes, beliefs, and emotions (McLeod, 1994). As interviewers, we came to this study with a qualitative, constructivist lens. We understood the interview itself as a meaning-creating experience in which ideas about the interactions between computational thinking and mathematical knowledge were formed jointly through questions, responses, and interpretations.

Methods: Data Collection and Analysis

The data for this preliminary report consists of a single hour-long interview with a mathematics PhD student, Sam[1]. Sam was selected opportunely as a case-study participant, as we had knowledge of her dissertation work in computational mathematics. The interview was conducted by the first author, audio recorded and transcribed for analysis. For analysis, we coded the interview with respect to three key aspects of Wing’s definition: formulating a problem computationally, expressing the solution computationally, and carrying out the solution computationally. This partitioned the interview into sections, where each section consisted of one of Sam’s responses. From here, we coded whether each section involved a moment of computational thinking affecting mathematical knowledge, of mathematical knowledge affecting computational thinking, or a reflexive interaction in both directions. In addition, we coded the interview for affect by coding sections in which Sam mentioned her attitudes, beliefs, or emotions in regard to computational thinking.

Results

We begin by describing Sam’s research and an overview of her dissertation experience. We then examine how she related to Wing’s definition of CT and provide evidence that Sam connected to the definition and thus believed that she had engaged in computational thinking. We then consider three examples of the ways in which computational thinking interacted with mathematics during Sam’s work. Finally, we consider how affect appeared in relation to computational thinking during Sam’s interview, and we discuss the importance of computational thinking moving forward. Sam completed her PhD in geometry and algebraic topology, where she studied ways of tiling the hyperbolic plane with polygons. While reading a computational algebra article in her area, Sam decided that the authors did not list the computational results in a way that was helpful for her own interests, which inspired her to try writing her own program. Stemming from this moment, Sam’s research question became whether she could write a computer program that would list all the groups that satisfied certain properties related to the hyperbolic tilings. However, once she began the process of writing code to store and list these groups, Sam ran into a problem of memory storage. In the interview, Sam remarked on how she could easily conceive of twenty million objects as a pure mathematician, but as a programmer, the number was far too large for her computer’s storage abilities.

[1] Sam requested that her real name be used because she is happy to share her experience in computational thinking.

This motivated Sam to go back to the mathematics and prove what would become the main theorem of her dissertation: a theorem that showed isomorphisms between these groups, which allowed Sam to take the twenty million original groups and condensed them down to a mere 70,000 groups. When she went to write this new isomorphism requirement into her code, Sam struggled to do so – she soon discovered that writing a mathematical theorem about isomorphisms and coding a computer program that takes isomorphisms into account were very different tasks. To solve this quandary, Sam created a physical interface between the mathematics and computer program via a card game that visually demonstrated the algorithm she was trying to write. This physical representation succeeded at helping Sam sort out her understandings, and, ultimately, she created a working program based off this game. Sam used a program called Groups, Algorithms, and Programming (GAP) in her work. Notably, Sam had taken a couple of introductory computer programming classes in her undergraduate program and an hour-long GAP workshop, but she considered herself to be largely self-taught and engaged in programming only as it was needed for her mathematical exploration. Her computer skills were largely self-taught, and as we will see, they benefited largely from her mathematical background when it came to debugging and understanding the results of her code.

A Computational Mathematician

Here we consider how Sam viewed computational thinking in relation to her work and her identity as a mathematician. In the interview, we had asked Sam to respond to Wing's definition. In the statement below, Sam considered how much the definition reminded her of her own work.

> Sam: I think that this matches my expectation when I hear the words computational thinking, and I feel like that was a huge part of what I did for the thesis actually. In some sense I think that the biggest barrier when I first got started was this idea of translating the notation of mathematics and the thought processes of mathematics to something that could be turned into a computer program.

Sam identified with the idea of figuring out how to think computationally and frame her problem in a way that a computer could understand. Furthermore, Sam believed that completing this dissertation changed the way she viewed herself as a mathematician.

> Sam: So, it definitely did change my opinion of myself as a mathematician. I thought of myself just as pure mathematics, but I started to think about myself more as a computational mathematician as a result of this project, 'cause I liked it so much.

We thus interpret that Sam began to consider herself more of a computational mathematician as a result of this project, which suggests that she found her experience to be meaningful. We now turn to the ways computational thinking appeared in Sam's dissertation process.

Computational Thinking Influencing Mathematical Knowledge

Sam's initial foray into computation had immediate difficulties, due to the sheer number of groups she had. This led her to formulate a mathematical problem: reduce the number of groups that she needed to study. From here, Sam worked on developing the main theorem of her dissertation, the proof of which showed such an isomorphism among the groups existed and allowed her to examine fewer of these groups. In this way, engaging in computation inspired Sam to create new mathematics, one of the potential results from interactions between CT and mathematics. Regarding coding the isomorphism theorem into her computer program, Sam talked about how she had to think about isomorphisms in a completely different way in order to formulate the mathematics in a way that the computer could program.

Sam: I had to sort of remove myself from the way that I usually thought about geometry and topology. I had to completely break away from the way I was used to thinking about group isomorphisms. Um, and try to encode that information differently in the software.

Thus, we have evidence that the use of a computer as a frame for thinking led Sam to think about mathematics in a different way. Sam also talked about how this way of thinking would not have occurred to her without the computer program and that it wasn't an entirely useful way of thinking about groups outside of programming. This gives us further reason to believe that CT is something that may intersect with, but is still distinct from, doing mathematics.

Connecting Computational Thinking and Mathematics

Once she had a working theorem, Sam needed a way to connect the areas of the pure mathematics and the computer programming. This connection was ultimately formed through a concrete puzzle game that she created.

Sam: I kind of came up at some point with a card game of sorts which was the equivalent puzzle, which was the equivalent of this problem that I was trying to solve with computers, as sort of an intermediate step between the mathematical way of thinking about it and the computational way of thinking about it.

This card game consisted of using a set of 25 colored domino tiles and finding different ways to make five groups of five dominos each that satisfied the certain properties Sam was looking for in her lists of groups. Sam believed that if she could solve this problem, it would help her to understand exactly what the groups looked like, and thus how to find them with her computer code. Indeed, this game gave Sam the insight she needed to write her computer program. Further, this game demonstrates one way that computational thinking and mathematical knowledge can manifest together as observable activity.

Mathematical Knowledge Influencing Computational Thinking

While Sam understood the solution to her problem through the card game, getting the computer program to run properly was a much longer endeavor. One important piece of this process was debugging computer code, a facet of computational thinking that is mentioned explicitly in many definitions or taxonomies (e.g., Weintrop, et al., 2016). Combining mathematics and computational thinking is extremely powerful because the mathematics offers a unique type of justification for the program output. In the quote below, Sam talks about a mathematical concept called abelianization, and how the output of her program was not matching her mathematical intuition of this concept.

Sam: So, I was subcategorizing all of my groups based on the abelian invariants, and I was getting ones that were a little bit strange. I was getting some, which didn't uh, which the only abelianization was the trivial one. You cannot map into any other abelian group, and I said to myself actually, I thought I'm not sure about this. This feels wrong. So based on that sort of mathematical computation, I went back to the program and it turned out I had put a 9 instead of a 10 somewhere, and I said wait a minute oh I see, so I reran everything with the correction and I began to believe what came out.

This section highlights an instance of mathematical investigation having an impact on how a mathematician thought about a computational situation. We argue that we see ways in which Sam's mathematical knowledge informed her reasoning about the computational thinking and activity in which she was engaged. Sam's mathematical knowledge gave her a powerful tool to validate her computer code and the output it was creating. It is worth noting that this worked well because of Sam's extensive mathematical knowledge and trust in her mathematical ability. For a student with less mathematical confidence, it is potential that erroneous computer code could convince them of incorrect mathematics or shake their mathematical confidence.

Computational Thinking and Affect

There were several moments in the interview when Sam mentioned ideas like confidence, approachability, fun, and motivation. The thesis process fundamentally changed the way she viewed computational mathematics and her own identity as a mathematician. Below, we see Sam's response to a prompt about how she felt about combining computation with pure mathematics.

> Sam: To me, it seems a lot more manageable. I know that there are some people who really don't like programming because of how precise you have to be when you're talking to the computer. But for me, there was something reassuring about I can sit down and I can write this program and I can be done, and when it finishes running I will have something in my hands that I obtained, as opposed to a proof which may or may not go anywhere… It just made the subject feel a lot more approachable to me.

The tangibility of having something functional in her hands at the end of the programming process was influential in Sam's attitude towards computation. She also found this aspect of programming to be motivating towards finding results, as seen in the below quote.

> Sam: And somehow, I felt more motivated to get a particular result. Like I felt really motivated to get something really nice that was really conducive to a program that I could write and could finish quickly, so I went out and searched for that thing.

She went on to describe this as the most influential and satisfying part of her thesis work, which speaks towards the nature of computational thinking being a worthwhile and important way to engage students in mathematics. Thus, we see an area for future research that explicitly studies the affective nature of computational thinking in mathematics and how it changes student beliefs about what constitutes mathematics, and whether they consider themselves a mathematician.

Conclusion

This preliminary report explored one interview with a PhD student and the interactions between CT and mathematical knowledge that helped them solve an open mathematics problem. The multiple facets of this relation between CT and mathematics are exciting and leave many opportunities for future research. In particular, we believe there is more research to be done on ways of thinking about mathematics that are primarily for helpful for computer programming, the activities that individuals engage in to translate between mathematical knowledge and a computer program, and the use of mathematical knowledge to verify code output. Additionally, there is an open avenue of research for the role of affect in computational thinking, and the unique benefits it provides mathematics students. Due to the wide variety of mathematical subfields, types of computational programs, and definitions of CT, more work is needed to develop a full picture of this relationship, which foreshadows a bright future for mathematics education researchers.

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The Motivations and Perceived Success of Different Calculus Course Variations

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Precalculus to single-variable calculus (P2C2) courses are a critical first educational step for students pursuing any science, technology, engineering, and math (STEM) degree. As students come into university courses with a wide range of preparation and skill, institutions need to be ready to meet their needs. One way this issue is being addressed is by the development of different course variations in the P2C2 courses. In this study we focus specifically on what motivates the creation of such course variations and how their success is perceived. Using open-ended survey responses from key faculty and instructors at ten institutions we identify themes relating to both the motivation and perceived success of course variations.

Keywords: Course Variation, Calculus, Precalculus

Calculus is a branch of mathematics related to the study of continuous change, but how one learns the subject of calculus is often embedded within a system of multifaceted and complex educational structures. For instance, nearly 14% of high school seniors will take a class referred to as Calculus while approximately 25% of entering college students will need some form of developmental math (US Department of Education, 2009). This spread in students' exposure and opportunities to learn mathematics creates challenges for university systems and faculty seeking to support student success in introductory mathematics and encourage their interest in pursuing a STEM major. Supporting students through the transition from high school to university mathematics has also been greatly impacted by recent pressures to reduce offering developmental courses since they have been shown to disproportionately impact students of color from obtaining a degree (Attewell, Lavin, Domina & Levey, 2006). One structural resource used to address this issue at universities is the development of *calculus course variations* (Bressoud, Mesa, & Rasmussen, 2015). Calculus course variations as they have been conceived are structural or curricular modifications to the traditional course sequencing of precalculus, differential calculus, and integral calculus (P2C2). For instance, some course variations include reconfiguring the standard-length of an integral calculus course to two semesters, increasing the contact hours through a co-requisite course, offering a version of calculus specifically for engineering, or shifting the order and focus of the calculus content.

In one of the only large-scale studies to examine calculus course variations, Voigt, Apkarian, Rasmussen and the Progress through Calculus Team (2019) documented that of 223 PhD And Master's granting institutions, calculus course variations were relatively common occurring at 78% of institutions. Such course variations are often targeted to support different student populations, these were categorized as variations for students with less preparation, for students with more preparation, and designed for students majoring in specific disciplines. Using student success data, Voigt et al. (2019) demonstrated that for students with less preparation "who are typically at the highest risk of failing, course variations resulted in similar passing rates, essentially levelling the playing field for these students" (p. 1). Although such findings are a promising resource to support student success, less is known about the motivations behind implementing such course variations, and whether key stakeholders at a university regard them as successful. As such this study seeks to address: *How do key* *stakeholders at an institution characterize the motivations and goals of a given calculus course variation? And how do they perceive the success and attainment of their stated goals?*

Methods

The data for this study comes from a larger, multi-phase national project (Progress through Calculus, PtC) that conducted 12 case studies of mathematics departments, and specifically their P2C2 programs. The project utilized census survey responses from graduate degree granting institutions to select a variety of institutions; which varied in size, type, populations that they serve, etc. Each institutional case study consisted of three site visits (Fall 2017, Spring 2018, and Spring 2019) consisting of individual and group interviews with various members of the department and university. Additionally, instructors and students were surveyed about their course experiences (Apkarian et al., 2019). After the Spring 2018 visit, we identified 14 course variations offered by 10 of the 12 departments. A more targeted visit was conducted in Spring 2019 at the departments that needed further investigation. We refer to the 10 schools with the following pseudonyms: Alpine University (AU), Canyon Crest University (CCU), Desert Bloom University (DBU, Dunshire University (DU), Fog Mountain University (FMU), Maple State University (MSU), River Rock University (RRU), Rolling Hill University (RHU), Sandpiper University (SU), and Treeline University (TU).

The data presented in this report draws on instructors' and key faculty open-ended survey responses on two surveys (see Table 1 for the survey items). The aim of both surveys was to get information about the variations and its perceived success. We administered the Instructor Survey to all instructors teaching a P2C2 course during Spring 2019 and if the participant indicated that they had experience teaching a variation, then they were shown the items in Table 1. We also administered the Key Faculty Survey to faculty that the project team identified as course historians (e.g., a department member that played a role in creating the variation) allowing the team to gain insight from faculty who may not have taught the course during Spring 2019. In total, our data consists of responses from 55 Instructor Surveys and 26 Key Faculty Surveys.

To analyze the survey responses, we first focused on the Key Faculty Survey. We read each response multiple times, open coded the responses, and compared notes to form closed formed codes. Then we individually tagged the responses with the closed form codes and compared tags. There was a high level of agreement between coders and when there was disagreement, we discussed until we reached an agreement. Discussion of the tags generated new codes, which we then returned to the data and refined the tags. We are currently in the process of repeating our analysis with the Instructor Survey responses. Here we present the findings from analysis of the Key Faculty Survey response and will provide additional insight using the instructors' responses during the presentation.

Table 1. Course variation survey items

Key Faculty Survey Items	Instructor Survey Items
1. What was the original intention or motivation when creating [course variation]?	1. In your opinion, what is the purpose or goal of [course variation] compared to [comparison traditional option]?
2. In your opinion, do you think [course variation] is meeting its goal? [Yes, No]	2. In your opinion, is [course variation] achieving its purpose? [Yes, No, I don't know]
3. [If 2=yes] In what ways is [course variation] meeting its goal?	3. [If 2=yes] What is it about [course variation] that is helping it achieve its purpose?

4. [If 2=yes]What explains the success in your opinion?
5. [If 2=no] In what ways is [course variation]not meeting its goal?
6. [If 2=no]What explains the lack of success in your opinion?

4. [If 2=no] How do you think [course variation] is coming up short and why do you think that is happening?

Sample Results

Below we present our findings from the Key Faculty survey in two sections. First, we describe the motivations that arose for creating the variation as well as what the different resulting variations were. We then investigated how the surveyed key players identified the success of the variation (or lack thereof). Our presentation will include results from analysis of the full data set which includes responses from the Instructor Survey.

Motivation for the Variation

In response to the question: "What was the original motivation and intention when creating [course variation]?" we found that three main themes arose that motivated the creation of the course variations: (1) students need more support with pre-req material, (2) content was being presented too procedurally making students miss fundamental ideas, and (3) students had specific needs related to their majors. Only two responses fell into more than one category, for some responses additional motivations were identified and coded as either "other" or "N/A".

Responses identified as "needs prerequisite support" addressed issues for students who needed additional support to succeed in a first semester calculus course. This theme covered responses identifying students which (1) have a lack of prerequisite knowledge, (2) had borderline placement, (3) come from a different prerequisite structure than at the current university, or (4) needed more just-in-time support. For example, one response pointed students with a "weaker background" in math as well as misalignment of the P2C2 sequence as the motivation, saying:

> "[I]t was to service students who (A) may have a weaker background in mathematics and needed more review of elementary concepts as they went along (algebra, functions, etc.), and/or (B) were out of sequence with the traditional Calculus sequence (had not had Trig, for example) and wanted to complete Calculus by year's end."

Another response identified the need for just-in-time support as well as borderline placement:

> "[course variation] was created for students that passed but scored low on our placement test. We noticed that these students typically struggled with Calculus I. [...] The motivation for the other half semester courses was to be able to help struggling students sooner by letting them repeat the half semester course."

Whereas the responses coded under "needs prerequisite support" identified students' previous knowledge as the issue, responses coded under "too procedural" identified the issue to be the knowledge gained in the actual classes. Three issues relating to the content being too procedural were identified: (1) a lack of conceptual understanding of fundamental concepts, (2) lack of procedural understanding, and (3) a need to connect to mathematics education literature. Two of our responses fell under this theme, one of which identified all three of these issues:

> "Traditional curricula, even with innovative teaching methods, lacked intellectual necessity for foundational ideas. [..] Also, students lacked what Dubinsky has called "process conceptions" of derivatives and integrals as functions. The [course variation] fosters process conceptions of each by incorporating Verillon's idea of instrumentation -- it has students use computing technology to work with mathematical statements as having a fundamental computable meaning."

Our last theme, "meets students' needs for their major", identified responses for which the course variation motivation was their students needing additional and/or different support as a result of their major. This needed support came in three types: (1) more alignment with the content (e.g. applications, major courses) for example, "To help students link precalculus mathematics concepts with its engineering applications"; (2) requirement for another STEM course, for example "The course(s) helped keep them on track with their science courses while completing Calculus."; and (3) needed to get through faster to complete degree requirements, for example "To have a shorter calculus sequence for engineering students because the major requirements did not allow for 3 4-credit calculus courses."

Different Variations that Resulted. Though three common themes were identified as the motivation behind creating a course variation, each institution responded in different ways. The following table gives a brief description of the resulting course variations. We denote Precalculus with PC, Calculus 1 with C1, and Calculus 2 with C2.

Table 2. Description of different variations that resulted from the three motivation themes

Needs Prerequisite Support	Too Procedural	Needs for Major
1. AU C1 with PC	1. MSU PC with foundational concepts	1. AU Engineering C1
2. CCU C1 with PC	2. MSU C1 & C2 with foundational concepts	2. CCU C1 with PC
3. DBU half semester P2C2		3. DBU half semester P2C2
4. DU stretched-out C1		4. FMU Engineering PC
5. RHU C1 co-requisite		5. MSU Engineering C1 & C2
6. SU co-calc for C1 & C2		6. RRU Engineering C1 & C2
		7. RHU C1 co-requisite
		8. TU Social Sciences C1

Departments that identified a need for prerequisite support responded by either creating (a) a longer course that allowed more time on the course material and/or incorporated additional precalculus material (AU and CCU C1 with PC, DU stretched-out C1), (b) an additional course to provide just-in-time support (RHU C1 co-requisite, SU co-calc for C1 & C2), or (c) splitting the courses in two in order to allow students to retake a part of a course when needed without wasting time (DBU half semester P2C2). The creation of some of these courses also addressed the additional need of supporting students in meeting their major requirements (CCU C1 with PC, DBU half semester P2C2, RHU C1 co-requisite). There were additional variations under "needs for major" that were created primarily to help students meet their major requirements (AU Engineering C1, FMU Engineering PC, MSU and RRU Engineering C1 & C2, TU Social Sciences C1). Lastly, the department that identified their courses as "too procedural" responded by creating completely new curricula designed to focus on foundational concepts.

Perceived success of the Variation

Most of the key players perceived their variation to be meeting its goals (22/25). They were asked to elaborate by responding to the questions "In what ways is [course] meeting its goals?" and "What explains the success in your opinion?". Five main themes arose from the responses: (1) *class format*: variation allowed for smaller class and/or extended time, (2) c*lient discipline's response*: students were better able to complete degree requirements and other departments seemed happy with the changes, (2) *improved focus on instruction*: there was an increased attention to improved instruction, focusing on foundational concepts, student needs/thinking, and overall learning goals, (4) *how students transformed*: there seemed to be a shift in students' math perceptions and abilities, and (5) *student data*: math departments used student data to measure "success" (e.g. higher pass rates/retention, comparable to traditional variation)

Most of the responses included more than one of the above themes. For example, the following response references student data, new instruction, and responses from client disciplines:

> "Students who complete this course tend to be as good as others who complete the traditional Calculus. I believe we have largely attained the goals of this sequence, introducing and using the ideas of Trig and Calculus to solve problems and understand natural and mathematical phenomena. [...] The bottom line: the other science disciplines say they are happy with the ongoing results of this sequence."

Largely institutions identified an assortment of factors as attributing to the success of their variation.

Conclusion/Discussion

This study identifies some of the major motivations for creating alternative precalculus and calculus course variations at math departments across the country. Their motivations could be broken down into three main themes: (1) needs prerequisite support, (2) too procedural, and (3) meets students' needs for their major. Additionally, the different departments validated the success of the variation in five major ways: (1) class format, (2) the client discipline's response, (3) their improved focus on instruction, (4) how students transformed, and (5) student data (e.g. pass rates/retention). It's important to note the diversity of responses regarding how departments measure, and explain, the success of their variation since DFW rates have been primarily what are reported (Voigt et. al, 2019). This gives us additional insight into how departments measure the success of their students' education.

In addition to the data presented here, our presentation will include an expanded data set with the responses from the Instructor survey. While data from the key faculty survey allowed us to get an idea behind how departments measure successful variations, only three responses stated that their variation was not meeting its goals or a success. By considering all instructors who have taught in the P2C2 sequence we will gain a better understanding of how and why faculty perceive the lack of success of their variations. Overall, we hope to paint a general picture of the motivation behind the course variations offered across the country and the factors contributing to views of their success or lack thereof.

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Math Outreach– A Learning Opportunity for University Students

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This article is a preliminary report describing a mathematics outreach program, which is a partnership among university faculty, underrepresented university students, and elementary school students in the west coast. This program aimed to develop math literacy through after-school engagement by providing opportunities to university students to apply university-learning experience in an out-of-class setting. The study participants were four university students who taught various mathematical topics to about twenty-five 3rd to 6th graders weekly for five weeks at a local public elementary school. The data consisted of university students' self-reflections, which revealed commonalities between their perceptions of the learning objectives for elementary students and for themselves that they thought were met. They also developed an awareness and understanding of various issues that exist within education; they learned new mathematical vocabulary, and made explicit connections between their teaching experience and university-learning experience.

Keywords: awareness, communication, outreach, mathematics, university students

Introduction

A report by the Civic Learning and Democratic Engagement National Task Force (2011) recommended universities make civic engagement an important part of college curriculum. Within this growing domain of civic engagement, universities play an important role in informal education by supporting programs for STEM outreach that usually involve undergraduate and graduate students in leading roles. These STEM outreach programs are mostly built on active, hands-on project-based learning offered as after-school programs, various partnership programs and service-learning courses. These programs not only promote STEM literacies in their community but also provide university students, especially historically underserved and underrepresented students, with enhanced academic performance, value, self-efficacy, leadership and intentions to engage in community service after graduation (Astin, Vogelgesang, Ikeda, and Yee, 2000; Kuh, 2008). In a recent study (Ghosh Hajra, 2019), the study participants, who were non-math majors, experienced an increase in value and confidence in their ability to learn mathematical concepts after participating in outreach events as a part of the course.

Learning is a complex process. Most of these programs use student reflection as a critical component to assess participants' learning objectives. As Dewey (quoted in Moon 1999, p. 12) described, reflection is "the kind of thinking that consists in turning a subject over in the mind and giving it serious thought." Rogers (2002) characterized Dewey's four criteria of the concept of reflection as– [1] a meaning-making process, [2] a systematic, rigorous way of thinking, [3] a social process, and [4] an awareness of one's attitudes and emotions. One's attitudes and emotions play a significant role in shaping their learning pathways (Blackwell, Trzesniewski, & Dweck, 2007).

For this study, I discuss a math outreach program led by a university faculty member and university students at a local elementary school. One of the learning objectives of the program (see Figure 1) was to develop self-awareness about civic responsibilities through self-reflection.

Theoretical Framework

This study is grounded in a feminist community engagement framework (Iverson & James, 2014; Novek, 1999), which embraces consciousness-raising, connectedness, and empathy through dialogues and reciprocal collaboration (Rojas, 2014). This study practiced "emancipatory feminist teaching" (Novek, 1999), which allowed students to practice concepts they learned in their university classes while "working cooperatively for the greater good" (pp. 230-231), in this case, by raising mathematical literacy in the elementary school students through community collaboration. In this study, university students learned new mathematical concepts, practiced concepts they learned in their university classes irrespective of discipline, and reciprocated their knowledge by teaching elementary students.

For this preliminary report, I examine the following questions:

1. What are the commonalities between the university students' perceptions of the learning objectives for elementary students and for themselves that were met in the math outreach program?
2. What are the takeaways of the STEM outreach program on the university students' future career or personal goals and their intentions to engage in future community service?

Methodology

The study reports a community mathematics outreach program by the author at a public university in the west coast of USA. This program was a mathematics outreach program for elementary students, which consisted of weekly hands-on mathematical activity sessions at a local elementary school. The team consisted of one university mathematics faculty member (the author), three undergraduate students and one mathematics master student (Student 4). The student members were recruited through advertisements in the faculty's department and university's career center, personal invitation, and by word-of-mouth. Student 1 had taken pre-calculus and was enrolled in an introductory statistics course at the time of the study. Student 2 was a biochemistry major who was enrolled in multi-variable calculus at the time of the study, Student 3 was a math major graduating in the following semester. Participation was voluntary in the sense that it was not a requirement for any courses they were enrolled and their participation did not contribute directly towards their achievement in the subjects in which they were enrolled at the time of the study. The university students were paid for their contribution. Lesson plans were prepared by the author, topics included polygons and hexaflexagons, fractals, pop-up cards and watershed, quadric surfaces and hyperbolic paraboloids, and lines, symmetry and geometric patterns. University students were given the handouts and lesson plans via email and they met with the faculty before the events if there were confusions about the hands-on activities.

After each visit to the elementary school, the university students were asked to write about their perceptions about the learning objectives that were met for elementary students and university students. Figure 1 shows the prompts used for self-reflection. These learning objectives were the objectives of the outreach program. After the completion of the weekly activities, university students were asked to write a final reflection using the Campus Compact's "The What? So What?? Now What??? Reflection Model" (quoted in Ghosh Hajra et al., 2019, p. 905). The WHAT component describes the outreach event. The SO WHAT component examines the significance of the event. The NOW WHAT component reflects on future actions that relate to the "big picture" of using mathematics in the "real world."

Objectives for elementary students	Objectives for university students
1. Learning and expanding mathematical vocabularies and concepts 2. Imagining, practicing, and testing solutions to real world application-based problems 3. Learning to work independently and collaboratively 4. Developing confidence in mathematics literacy and practices	1. Developing and expanding mathematical vocabularies and concepts 2. Learning to effectively communicate mathematical concepts 3. Applying classroom-based knowledge and skills to other settings beyond the university 4. Learning to take responsibility and to make adjustments when necessary 5. Fostering a deeper understanding of themselves as members of both the university and local community as well as their civic responsibilities for inclusive leadership and service
Please answer the following questions: 1. What was your experience today? 2. In your opinion, which of the above elementary students' goals were achieved? Explain. 3. In your opinion, which of the above university students' goals were achieved? Explain. 4. What was your overall impression of the math outreach program? 5. Other thoughts/ concerns.	

Figure 1. Prompts for self-reflection for university students.

Data Analysis

The study draws upon Creswell's (2007) qualitative research methods– a combination of grounded theory and narrative approaches. A grounded theory approach allows one to code the university students' self-reflections according to "major categories of information" (Creswell, 2007, p. 64). A narrative approach allowed to "re-story" the students' experiences (Creswell, 2007, p. 56) that provides overarching insights about their experience in the math outreach program. I read each self-reflection, paying particular attention to the language university students used in the weekly reflections and in the final report.

Question 1: What are the commonalities between the university students' perceptions of the learning objectives for elementary students and for themselves that were met in the math outreach program?

To answer the first question, I created tables (Tables 1, 2) listing individual university student' perceived assessments of the learning objectives for elementary students and for themselves that they thought were met. Three common major categories emerged for learning objectives that were met for both groups as perceived by the university students. They were: developing mathematical language, developing mathematical communication, and developing connections.

Developing mathematical language: Through almost all the outreach activities, the university students individually reciprocated that they learned new mathematical concepts. They also observed elementary students reinforced what they knew and developed new mathematical vocabulary. A representative comment from a university student supporting the theme: "*We went over polygons of 3 to 20 sides and for most of the* [elementary] *students this was the first time they had heard the vocabulary names. Since many of them had just learned about polygons in their classes, we reinforced some of the things they already knew also.*"

Developing mathematical communication: The university students reflected on their own communication and leadership skills and how they were improving. They observed a similar confidence build-up in the elementary students when they worked in their groups. Some representative comments from university students supporting the theme: "*I was able to develop effective communication, as I had to explain and guide the* [elementary] *students through their work. After the activities, I could reflect on what I could have done better to make my communication more effective.*" "*I observed that today some of the* [elementary] *students would not be very confident on saying their answers aloud. However, as they saw their other teammates participate, they ended up sharing their opinions as well.*"

Developing mathematical connections: The university students made connections between their own classroom experience with the outreach experience. They also perceived that the elementary students made connections between the mathematical ideas and their experiences in the real-world. Representative university students' comments supporting the theme: "*I am taking a coms class, and I have been learning about effective communication. In relation to this class, I can relate learned concepts and apply them with the students to ensure better communication skills with them.*" "*I got surprised on how quickly the students began thinking what to relate the shapes with. One of the students thought about comparing the Hyperbolic paraboloid with the shape a blanket makes when shaking it up and down.*"

Table 1. University students' perceived assessment of the learning objectives for elementary students.

University Students	Week 1	Week 2	Week 3	Week 4	Week 5
Student 1	1, 3, 4	1, 3, 4	2, 3	1, 2, 3, 4	1, 2, 3, 4
Student 2	NA	1, 3	1, 2	1, 2	1, 2
Student 3	1, 2, 3	1, 2, 3	1, 2, 3, 4	NA	1, 2, 3, 4
Student 4	1, 3	1, 3, 4	1, 2, 3	1, 3, 4	NA

Table 2. University students' perceived assessment of the learning objectives for themselves.

University Students	Outreach 1	Outreach 2	Outreach 3	Outreach 4	Outreach 5
Student 1	1, 2, 3, 4, 5	1, 2, 3, 4, 5	4	1, 2, 3, 4	1, 2, 3, 4
Student 2	NA	4	1, 2, 3	1, 2, 4	1, 3, 4
Student 3	1, 4, 5	1, 2, 3, 4, 5	1, 2, 3, 4, 5	NA	1, 2, 3, 4
Student 4	1, 2, 3, 4, 5	1, 2, 5	1, 2, 3, 4	4	NA

Question 2: What are the takeaways of the STEM outreach program on the university students' future career or personal goals and their intentions to engage in future community service?

I analyzed university students' final report to examine the second question. Three major takeaways emerged. Representative university student's comments are included along with the takeaways. Students developed awareness of

- **The importance of representation in the classroom**

"*I am very thankful I was a part of this project because I know the impact this can have on the young students seeing people that look like them, especially females of different ethnicities working on Math I think it's very empowering to them.*"

- **The importance of relating mathematical tasks to student's lives**

"*I also feel like it is my civic duty because of this experience and others like it to help black students find a face in mathematics that they can relate too. I will also take the importance of making learning relevant to students on the basis of age and culture.*"

- **Impact of college students on children's beliefs about themselves as thinkers and doers of mathematics**

"working in this outreach program definitely opened my eyes to the impact we can make to those students who may not think of themselves as capable of doing Mathematics".

Conclusion

Through this math outreach program, university students made explicit connections with their own classroom experience and the outreach experience. They applied what they have learned at the university. The university students took on responsibilities, learned to be flexible and took decisions while leading the activities, thereby enhancing their communication and leadership skills. Students assessed their own responsibilities and evaluated each learning objective. The university students found these outreach activities to develop awareness about issues in mathematics learning and ways to overcome those. Continuing this study, I would be interested to examine how the university students apply the learning moments from the math outreach to their own classroom practices as a learner at the university.

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Analyzing the Beliefs and Practices of Graduate and Undergraduate Mathematics Tutors

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The purpose of this qualitative study is to examine the beliefs and practices of undergraduate tutors and graduate teaching assistants who work in mathematics support centers. This report is situated in a larger study that is being conducted at two four-year institutions in the U.S. and focuses on our preliminary analysis of six graduate teaching participant responses from one institution. For the first phase of the study, we asked participants to complete a survey that was based off of the Teacher Beliefs Interview protocol (Luft & Roehrig, 2007) in order to examine their beliefs and practices before they participate in mathematics tutor-specific professional development activities. Using a modification of Luft and Roehrig's coding scheme, we coded survey item responses as Instructive, Transitional, or Adaptive. These codes gave us a snapshot of their beliefs and practices around tutoring in mathematics.

Keywords: Tutor, Graduate Teaching Assistant, Beliefs, Practices, Professional Development

Graduate teaching assistants (GTAs) and undergraduate tutors (UTs) are a common point of contact for students in undergraduate mathematics courses, providing direct or supplemental instruction either during class time or in mathematics support centers outside of class time. The experiences students have while taking undergraduate mathematics courses play a significant role in student confidence, the enjoyment of and continuation in mathematics, and the retention of science, technology, engineering, and mathematics majors (Rasmussen, Ellis, Zazkis, & Bressoud, 2014; Seymour & Hewitt, 1997). As a primary point of contact and source for instructional support, GTAs and UTs play a significant role in the students' experience. In particular, their mathematical beliefs and practices have an influence on how they interact with students.

Framing

Given that many post-secondary mathematics classrooms are lecture-based (Stains et al., 2018), GTAs and UTs typically have little experience (as undergraduate students themselves) with evidence-based pedagogical practices. Studies have shown that what people learn from their experiences as students plays a significant role in what they believe teaching should look like. The apprenticeship of observation describes the phenomenon where future teachers have already spent "thousands of hours as schoolchildren observing and evaluating professionals in action" (Borg, 2004, p. 274; Lortie, 1975) before they even begin their professional training to become teachers. In contrast, those training in other professions, such as law or medicine, typically have not spent thousands of hours observing lawyers and doctors before beginning their professional training (Borg, 2004; Egan, Bullock, & Chodakowski, 2017).

Therefore, GTAs and UTs enter their respective positions with preconceived ideas about the teaching and learning of mathematics. To best support the professional development (PD) of these two groups and contribute to the corresponding gap in the literature regarding tutor beliefs and how those beliefs impact tutoring practices (Grove & Croft, 2019; Roscoe & Chi, 2007; Yew

& Yong, 2014), it is important to understand the beliefs and practices about the teaching and learning of mathematics that GTAs and UTs hold. Thus, this study aims to better understand GTA and UTs beliefs and practices about mathematics in order to cater PD activities that promote beliefs and practices that are more beneficial in student interactions. The following research questions guided this study[1].

1. What are the mathematical beliefs and practices of graduate and undergraduate tutors?
2. How do the mathematical and/or tutoring beliefs and practices of graduate teaching assistants and undergraduate tutors change throughout the academic year as a result of engaging in PD activities?
3. How are tutors' beliefs and/or practices related to emotional affect and mindset impacted by PD activities?
4. How are tutors' beliefs and/or practices related to the tutor roles and tutor-student interactions impacted by PD activities?

Measuring Beliefs and Practices

The Teacher Beliefs Interview (TBI) was developed by Luft and Roehrig (2007) as a way to understand and track teacher beliefs, as "beliefs reveal how teachers view knowledge and learning, and suggest how they may enact their classroom practice" (p. 47). By administering the TBI multiple times, it is possible to document teacher beliefs and possible changes in beliefs. Using the TBI in this way is particularly valuable, as Luft and Roehrig found that teachers were more likely to shift their beliefs and "tendencies towards student-centered activities and instruction" (p. 48) when provided with PD support, but, in the absence of such support, would move towards more traditional instructional practices, which aligns with findings from Yerrick, Parke, and Nugent (1997). Therefore, to measure the beliefs of GTAs and UTs in our study and track the impact of PD activities, we chose to modify Luft and Roehrig (2007) interview and coding scheme to create and analyze a mathematics tutoring beliefs survey.

PD Design

Before designing PD activities, we first conducted an extensive literature review to make sure we were focusing on best practices identified in mathematics tutoring literature. As a result of our review of the literature, we identified five important topics that we aimed to address in our PD: (a) creating a welcoming, inclusive environment (e.g. Bell & Elledge 2008; Grove & Croft, 2019; Solomon, 2007), (b) emphasizing the importance of mindsets (e.g. Cohen, Garcia, Purdie-Vaughns, Apfel, & Brzustoski, 2009; Dweck, 1986; Karumbaiah et al., 2017; Yeager & Walton, 2011), (c) supporting student motivation (e.g. Chng, Yew & Schmidt 2015; Linnenbrink & Pintrich, 2002; Ryan & Pintrich, 1997), (d) creating independent learners (e.g. Chi, Roy, & Hausmann, 2008; Grove & Croft, 2019; Zimmermann, Bandura, & Martinez-Pons, 1992), and (e) adjusting tutoring to meet individual student needs (e.g. Franke & Kazemi 2001; Grove & Croft, 2019). We then developed PD activities that we felt would help address these topics[2]. Differing schedules prevented us from having the exact same PD format and resulted in activity modifications to best suit the context of each school. However, we attempted to include as many of the same activities as possible at both universities.

[1] In this preliminary report, we focus on answering Research Question 1 through our preliminary analysis of initial survey responses from GTAs at one university.

[2] Due to space constraints, we are unable to go into more details about the PD activities in this report. However, you may contact Mary E. Pilgrim at mpilgrim@sdsu.edu or Erica Miller ermiller2@vcu.edu for more information.

Methods

Setting and Participants

This study is being conducted at San Diego State University (SDSU) and Virginia Commonwealth University (VCU). Both universities are large, public, urban, universities with graduate programs and high research activity. Both universities have a racially diverse undergraduate student population, however the mathematics graduate programs at both universities do not necessarily reflect the same diversity in gender or underrepresented minority status. Participants at SDSU were recruited from graduate and undergraduate students who are working in the general math support center (MSC) during the 2019-2020 academic year. Participants at VCU were recruited from graduate students who are working in the MSC specifically for precalculus during the 2019-2020 academic year. As a note, there are also undergraduate students who work in the precalculus MSC at VCU, however they currently do not receive intensive PD like the GTAs, so they were excluded from this study. An email was sent to potential participants from a researcher at their respective university inviting them to participate in the study by taking surveys. Although all GTAs and UTs were required to attend the PD as part of their job, only the UTs at SDSU were compensated for their time.

Data Collection

To establish a baseline to understand the beliefs and practices that graduate and undergraduate tutors come to us with, we asked participants to take an online survey before attending the PD sessions. This initial survey included eight open-response items, seven of which were adapted from the TBI protocol (Luft & Roehrig, 2007) to match our tutoring context. Item 1 (not part of the TBI) was added because we felt that it was an important belief that was not captured by the other items.

1. What are the characteristics of people who are successful at math?
2. How do you maximize student learning when you are working with a student(s) in a mathematics support center?
3. How do you perceive your role to be as a tutor/GTA in a mathematics support center?
4. How do you know when a student(s) understands?
5. In the tutoring setting, how do you decide what to focus on and what not to focus on?
6. How do you decide when to move on to a new topic when you are working with a student?
7. How do students learn math best?
8. How do you know when learning is occurring when you work with a student?

Once survey responses were collected, these researchers de-identified the data and then shared it with the entire research team for analysis.

Data Analysis

Here we discuss the analysis of Items 2-8, as these are the items that were adapted from the TBI and therefore already had a corresponding coding scheme. To code responses to these items on the initial survey, we chose to use an entire item response as our unit of analysis, as recommended by Roehrig (personal communication, July 14, 2019). We used Luft and Roehrig's coding scheme as a starting point. For the TBI, Luft and Roehrig classified responses into one of five categories (p. 54): (a) Traditional: Focus on information, transmission, (b) Instructive: Focus on providing experiences, (c) Transitional: Focus on teacher/student relationship, (d) Responsive: Focus on collaboration, feedback, or knowledge development, and (e) Reform-

Based: Focus on mediating student knowledge or interactions. These categories are on a spectrum ranging from a more teacher-focused classroom experience to a more student-focused learning experience.

The first two authors coded the data separately on these five original TBI categories and made notes. Once this step was complete, the two coders met to discuss how well they thought the original TBI coding scheme was fitting with our tutoring context. As a result of this conversation, the coders chose to make several modifications to the coding scheme. For example, the original version of Item 5 (what to focus on) was geared towards teachers who set content, curriculum, course learning goals, etc. Since our participants do not have such control over the content they are tutoring for, we noticed that our responses did not quite fit with the original coding scheme. We also found that distinguishing between Traditional and Instructive or Responsive and Reform-Based was not as meaningful. Therefore, we chose to collapse categories (a) and (b) together to one category called Instructive and (d) and (e) together to one category called Adaptive. However, we still viewed the tutoring beliefs as being on a spectrum from more tutor-focused to more student-focused. We then re-coded based on three categories: Instructive, Transitional, and Adaptive. In Table 1, we include examples of different responses to Item 7 that fit into the three categories.

Table 1. Sample responses from GTAs at VCU on the initial survey.

Item 7: How do students learn math best?		
<u>Instructive</u> "There is no better way to learn math, or anything for that matter, than to just do it. Do problems until you can't look at another math book again, but make sure you pace yourself and up the difficulty slightly over time, rather than tackling the hardest problems first." (P1)	<u>Transitional</u> "It is an individual process, I must adapt to the student's needs." (P2)	<u>Adaptive</u> "Example problems. Showing someone else 'how-to.' Using realistic connections/picture. Small group collaboration." (P5)

Discussion of Preliminary Results and Future Directions

Currently, we are still collecting initial survey responses from GTAs and UTs at SDSU, which will be analyzed and shared at the 2020 Conference on Research in Undergraduate Mathematics Education. Here, we report on our preliminary analysis of the initial survey responses from GTAs at VCU. Figure 1 provides a heat map of the six GTAs' (P1-P6) responses to some of the items (I2-I8). Expecting that GTAs would have a more traditional view of teaching and learning due to their assumed undergraduate learning experiences, we were surprised to find a more Transitional mindset than an Instructive mindset in many of the responses. There were only two responses that were coded as fully Adaptive, and these came from only two of the six participants (P4 and P5). However, two additional responses to Item 5 (what to focus on) were double coded as both Transitional and Adaptive. In particular, it is interesting to note that none of the Item 5 and Item 6 (move on) responses were coded as Instructive. However, the Instructive code was applied the most to Item 2 (maximizing learning) and Item 7 (learn math best) responses. Knowing this in advance would have been useful in starting PD as it could have provided guidance in planning and focusing the PD.

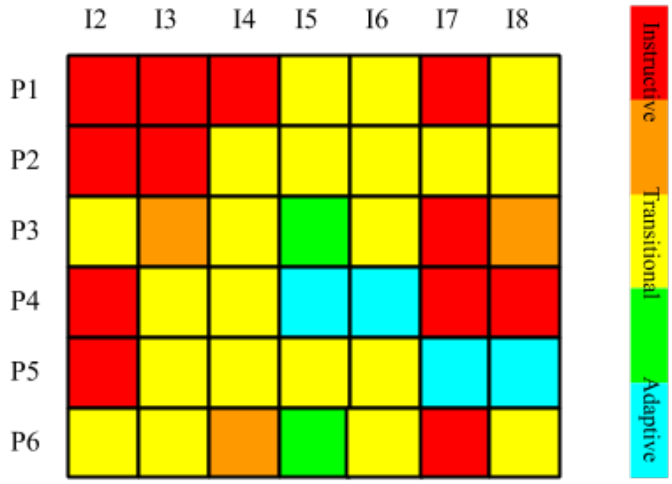

Figure 1. Heat map of GTAs' (P1-P6) survey responses items (I2-I8) from the initial survey

Moving forward, our plan is to analyze the responses to the initial survey from SDSU and send two follow-up surveys at the end of the fall 2019 and spring 2020 semesters. These surveys will contain the items from the initial survey and four additional items asking for tutors to reflect on their PD and tutoring experiences. We plan to use the follow-up surveys to analyze how tutors' beliefs and practices may change throughout the year, provide tutors with the opportunity to reflect on their experience, and collect feedback on the impact of the training that we provided during the pre-semester orientation. In particular, we are interested in how the PD activities we used during the pre-semester orientation may have an influence on changes in their beliefs and practices.

Questions for the Audience

As our project is still in the beginning stages, we welcome input from the audience. In particular, we plan to pose the following questions to the audience at the end of our presentation and leave ample time for discussion and feedback.

1. This is a pilot study. What changes would you suggest making as we move forward? (Our plan is to continue collecting data with the new cohort next year.) In particular, how might you modify the initial survey questions (keeping in mind that we still want them to align with the original TBI)?
2. Is there another lens for coding the qualitative survey responses that we should consider? And do you think that the same data could have been obtained through Likert-type questions rather than open response?
3. Are the five topics listed in the PD Design section a common focus for tutor/GTA PD at other institutions? What other topics/areas are addressed in PD at other institutions? Is there additional literature the audience could point to?
4. In our study, we are attending primarily to how beliefs may impact practices. For those who have engaged with tutor/GTA PD, do they find it more important to focus on influencing beliefs, which then impact practices, or influencing practices directly?
5. Is it important for us to share our results with future cohorts of GTAs and UTs? In particular, we wonder if talking about our study with future cohorts might help them think about how their beliefs impact their practices.

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University Students' Defining Conceptions of Linearity

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Linearity plays an extensive role in both elementary and more advanced mathematics. Unlike other such topics, investigation into student understanding has been limited. Nineteen students enrolled in multivariable calculus were asked about their conceptions of linearity. Most provided only elementary notions. Implications and future research are considered.

Introduction and Literature Review

Linearity is an important concept throughout mathematics (De Bock et al., 2002). Students as young as 5 years old can identify and informally reason about linear relationships (Ebersbach et al., 2010). Subsequently, scholastic instruction gives them some tools for formal mathematical reasoning, as linear single-variable functions and equations are heavily emphasized in the scholastic curriculum (CCSS, 2010). Linearity has a role to play beyond its elementary foundations. It is a foundational concept in statistics for understanding association between quantitative variables (Casey & Nagle, 2016). Linear approximation is a key topic in calculus, and it is possible to redesign a single-variable calculus course around the notion of linearity (Tall, 1989; Samuels, 2017). It continues to be of central importance in more advanced mathematical topics such as differential geometry or linear algebra (Van Dooren et al., 2008; Wawro et al., 2012). There is an interest in instruction on linear functions in different applied contexts (Remijan, 2019). Indeed, there are many topics in which the role of linearity could be explored and it is not (Tobias, 1982).

For our purposes we define a linear function or expression to involve a polynomial of degree at most 1 (e.g. $y=mx+b$, $ax+by+c$, etc.). This definition is used in scholastic algebra textbooks. The formal concept definition of linearity encompasses two properties, additivity, $f(x+y) = f(x)+f(y)$, and homogeneity, $f(k \cdot x) = k \cdot f(x)$. In the scholastic usage, if the constant is not 0, it does not meet the formal definition of linear and is formally referred to as affine (Stacey, 1989). Proportional and linear reasoning are closely related since, for linear functions, changes in an input and output variable are proportional (NCTM, 2013).

Linearity is one of a handful of topics which are are important in their own right, but also are integrated into myriad topics ranging from elementary to advanced. Many such topics have been the focus of extensive research, including function (Oehrtman et al., 2008; Carlson, 1998), rate and slope (Herbert & Pierce, 2012; Stump, 1997; Moore-Russo et al., 2011; Zaslavsky et al., 2002), and proportional reasoning (Tourniaire & Pulos, 1985; Iszak & Jacobsen, 2017). On each topic, in these reports researchers have looked at the conceptions of experts, teachers, and students with varying levels of experience, creating a comprehensive picture of how the topic is understood.

Compared with other backbone topics, research on linearity has been nowhere near as expansive. Much work has been done on the initial learning of linear conceptions (Greenes et al., 2007; Pierce et al., 2010; Hattikudur et al., 2012), the overuse of linearity on nonlinear problems (De Bock et al., 2002; Esteley et al., 2010), and closely related topics such as proportionality (Behr et al., 1992). Some have looked specifically at student prowess with linear functions in different representations (Bardini & Stacey, 2006; Adu-Gyamfi & Bosse, 2014). However, a comprehensive picture of the understanding of linearity for people at various levels of expertise is missing. In particular, little work has been done examining the conceptions maintained on linearity after extensive mathematical education. Such students have been exposed to linearity in myriad contexts and topics, with unknown impact. Thus we pose the research question: What conceptions of linearity do university students hold?

Framework

We are interested in ascertaining the notions which students connect with the topic of linearity. A fitting framework for this is the *concept image*, “the total cognitive structure associated with the concept” (Tall & Vinner, 1981, p152). It encompasses all the associations formed by the learner, formal or informal, correct or not. Importantly, a researcher attempting to uncover a learner’s concept image can never know when it has been fully revealed. Thus, realistically, what researchers uncover can be referred to as the *evoked concept image*, the portion of the concept image expressed at a particular time (Tall & Vinner, 1981). In this study, we limit ourselves to the defining characteristics of the topic expressed by the student, e.g. how they determine if some mathematical object is or is not linear. This particular slice of the concept image is the *personal concept definition* (Tall & Vinner, 1981). It can be compared with the formal concept definition and may or may not agree.

There has been an increasing interest in the role of multiple representations in mathematical learning and reasoning. The four primary mathematical representations are symbolic, graphical, numerical, and verbal (Friedlander & Tabach, 2001). The capacity to reason in and translate between different representations is referred to as *representational fluency* (Fonger, 2019). Its aspects include the invocation of different representations, the ability to conduct operations within a single representation, and the ability to switch from one representation to another. It affords cognitive advantages (Larkin & Simon, 1987) and is an essential element of mathematical learning (Zbiek et al., 2007; NCTM, 2000). As we investigated student conceptions of linearity, we specifically attended to their representational fluency in the first category, i.e. which representations they used to convey their personal concept definition.

Methodology

The subjects were 19 students in a multivariable calculus class at a northeastern college. The students had an overwhelming STEM focus as indicated by their majors (see Table 1). Two weeks before the end of the semester, every student was given a questionnaire with three open-ended prompts. The questions were:

1 What is a linear function or formula? Using 1 input variable? Using 2 or more variables?
2 What is a linear graph? Using 1 input variable? Using 2 or more variables?
3 What is a linear relationship between quantities? Using 2 quantities? Using 3 or more quantities?

In the directions, students were prompted to provide examples and general explanations. Note that we intentionally attempted to extract student conceptions of linearity in various representations; the first question includes the word “formula”, which implicitly prompts a response in the symbolic representation, and the second question includes the word “graph”, which implicitly prompts a response in the graphical representation. Further, in each question, students were given a multivariable prompt to directly elicit a response from the material of their present course.

Major	*Number of students*
Engineering	11
Math	4*
Liberal Arts	2
Computer Science	1
Forensic Science	1

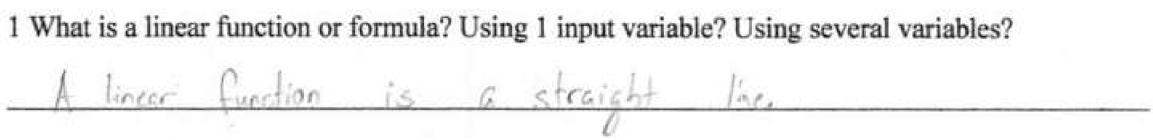
* one of the Math majors had an additional concentration in education

Table 1. Frequency of student majors.

Data were analyzed using grounded theory (Strauss & Corbin, 1990). Data from the questionnaires were open-coded and categories of responses were allowed to naturally emerge. The data were then recoded using the categories. During recoding, categories were confirmed or adjusted as necessary. Coding and category refinement continued in an iterated cycle until categories were robust and all data were categorized. An external researcher coded the data using the given categories, and any disagreements were negotiated into agreement. Grounded theory is an appropriate methodology for exploring an area in which there is no prevailing theory for organizing the data (Strauss & Corbin, 1990). This approach was indicated since conceptions of linearity for students at this more advanced level had not been previously explored.

Results

The responses are summarized in Table 2. The most common evoked personal concept definition of linearity, nearly unanimous and often the first one offered, was the property of being a line. One surprising aspect is that this was offered in response to all three questions, despite the fact that one question asked specifically about the formula, one about the graph, and one about the relationship between quantities. "Straight line" was mentioned: in response to question 1, by 14 students; in response to question 2, by 14 students; in response to questions 3, by 9 students. A typical student response was that "a linear function is a straight line" as in Figure 1.


1 What is a linear function or formula? Using 1 input variable? Using several variables?

A linear function is a straight line.

Figure 1. A representative student response.

The only other majority response was the formula $y=mx+b$. The third most popular result was "constant rate" at 26%. Nearly one-third (31%) of the students mentioned one of the three notions involving constancy – constant slope, constant rate, or constant derivative. (One student mentioned all three.)

Conception of Linearity	*Students Exhibiting that Conception (n=19)*
(a) Line	18 (95%)
(b) Plane	2 (11%)
(c) $y=mx+b$	10 (53%)
(d) $ax+by=c$	1 (5%)
(e) $y-b = m(x-a)$	1 (5%)
(f) $\vec{r}=\vec{p}+\vec{v}t$	1 (5%)
(g) Proportionality	2 (11%)
(h) Constant slope	2 (11%)
(i) Constant rate	5 (26%)
(j) Constant derivative	1 (5%)
(k) Parametric equations	3 (16%)
(l) System of equations	2 (11%)

Table 2. Frequencies of student linearity conceptions.

Some responses can be neatly classified by representation. Conceptions in the symbolic representation are "*y=mx+b*", "*ax+by=c*", "*y-b=m(x-a)*" and $''\vec{r}=\vec{p}+\vec{v}t''$. Conceptions in the graphical representation are "line" and "plane". A symbolic conception was offered by 10 students, all of whom also offered a graphical conception. Eight other students offered a graphical conception without a symbolic conception. The last three symbolic conceptions were each presented by a different student, and all three also presented the first one.

Some of the responses can be classified by level. Some clearly come from an introductory curriculum (e.g. 8th grade), such as "line", "*y=mx+b*", "constant slope", "constant rate", and "proportional". Some come from single variable calculus (SVC), such as "constant derivative". Others come from multivariable calculus (MVC), such as $''\vec{r}=\vec{p}+\vec{v}t''$ and "plane". Every student offered a conception from the introductory curriculum. Conceptions from the SVC curriculum were provided by 1 student. Conceptions from the MVC curriculum were provided by 2 students.

Three students referred to parametric equations, and two students referred to a system of equations (including one student with both). For the former, these students mentioned that for a multivariable function the formula would be written with parametric equations, yet did not specify the precise form of those equations, i.e. they did not provide the relevant formula, $\vec{r}=\vec{p}+\vec{v}t$ (only one of them gave the formula even in the single variable case). For the latter, one student noted that "several variables will require a linear system of equations. More variables = more equations to solve." This conception seems to be tied to a problem-solving procedure, rather than the notion of definition which was requested in the actual prompt.

Discussion

The only conceptions of linearity offered by more than one-sixth of these multivariable calculus students are taught in 8th grade (CCSS, 2010). For these university students mostly majoring in STEM fields, the property of being a line was widely held as the dominant evoked personal concept definition of linearity. It was mentioned in response to all three questions by 7 students (37%), and in response to at least two questions by 12 students (63%). No other answer was used by a student more than once. Reliance on the elementary conceptions may be problematic for deep learning of more advanced material, and their dominance may indicate a gap in that deep learning. The vast majority of these college students did not demonstrate a connection between their multivariable calculus learning of linearity with prior learning of linearity, and these are the sort of connections which constitute deep knowledge (Star, 2005). This occurred in spite of the fact that all students were currently enrolled in a multivariable calculus course, and linearity in that context (e.g. linear approximation, directional derivative) had been discussed. These topics, despite their recency, were not part of their evoked concept image. Zandieh et al. (2017) studied the connections students made between functions in different courses (functions from introductory algebra and linear transformations from linear algebra) and referred to this as a *unified concept image*. The students in this study by and large did not exhibit a unified concept image for linearity. This may indicate that more time, more instruction, or both, is required for non-elementary conceptions to become part of a robust concept image of linearity. This would parallel the findings in studies on other topics such as function (Carlson, 1998) or proportion (Ayan & Isiksal-Bostan, 2019). Students are certainly capable of recognizing the role of linearity in more advanced mathematical contexts, although the path there is neither short nor straight (Samuels, 2019).

Two students mentioned proportionality. This is an interesting response, since its correctness depends on which interpretation of linearity is used. The formal algebraic definition, specifically the homogeneity property, does confer proportionality. The typical definition from scholastic algebra, a polynomial of degree at most 1, does not have proportionality in variable values (due to the constant) – however changes in the variables are proportional. This ambiguity has been mentioned in the literature

(Pierce et al., 2010; Van Dooren et al., 2008; Stacey, 1989). One student mentioned both the property of proportionality as well as the formula $y=mx+b$, which is incompatible. It is unclear to what extent the other student was aware of this duality.

Tall & Vinner (1981) noted that the personal concept definition and the formal concept definition often bear comparing. They reported that they often do not match for limits, and a mismatch has also been observed for topics in linear algebra such as subspace (Wawro et al., 2011). Students in general are not presented with the formal definition of linearity, the homogeneity property and the additive property, until linear algebra. Thus it is not a surprise that this formal concept definition did not appear as part of their personal concept definition. They do see the scholastic definition in school, and many students invoked it with their preferred symbolic formulation, with $y=mx+b$ being the most popular.

It is important to emphasize that this study, through the frame of the evoked personal concept definition, only seeks to characterize the strongest part of the students' concept definitions within their concept images. We do not claim that this represents the entirety of their personal concept definitions, only their evoked personal concept definitions within the confines of this study. For example, of the 19 students, 10 mentioned the formula $y=mx+b$. It is plausible that the other 9 do possess this conception (given its prominence in the scholastic curriculum), but they did not offer it here. The data do imply that "straight line" is more prominent overall than "$y=mx+b$" as an element of their personal concept definitions.

Conclusion & Future Directions

This study uncovered the outlines of student thought on linearity from university students with a relatively high level of mathematical experience. The most common conceptions, and the only ones provided by more than one-third of the students, were the property of being a line and the formula $y=mx+b$. Few students exhibited notions from single variable calculus, or from multivariable calculus, despite being enrolled in that course at the time data was collected.

Much remains to be explored. The data presented here, students' evoked personal concept definitions, constitute their most accessible conceptions. What conceptions and connections might be uncovered with further prompting, i.e. their concept images? What conceptions do students at different levels of mathematical experience possess? How do students utilize their conceptions in various scenarios, both applied and decontextualized? What constitutes an expert conception of linearity? What conceptions of linearity do instructors hope students acquire? Which instructional modifications might bring about a more developed personal concept definition of linearity? There are many questions available for fruitful inquiry in this important area.

Formal mathematical linearity first arises in school typically in 8^{th} grade. Yet the topic is intimately linked to understandings which learners possess at ages as young as 5 years old, and to material learned in the classroom throughout high school and college. Student conceptions of linearity can and should evolve, and this study contributed to the characterization of one stage of that evolution.

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How Mathematicians Assign Homework Problems in Advanced Mathematics Courses

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While many aspects of the teaching and learning of advanced mathematics have been explored, the role, construction, and values of homework has been virtually ignored. This report draws on task-based interviews with six mathematicians to explore the relationship between an instructor's learning goals and the factors they consider in selecting homework problems as well as general homework construction heuristics. Our initial findings are that 5 of the 6 participants viewed homework as a critical part of student learning, that the majority of the participants' claims focused on either the mathematics or how the problem would help students learn, and that there was variance among the mathematicians in terms of selection of problems and rationales.

Keywords: Abstract Algebra, Homework, Teaching

Instruction in abstract algebra has been surprisingly well-explored in the last few decades. For example, we know that most instruction in abstract algebra is via lecture (Johnson, Keller, Peterson, & Fukawa-Connelly, 2019) although with different amounts of student engagement. Moreover, in terms of understanding the practices of lecture, there have been studies exploring the presentation of content (e.g., Fukawa-Connelly, 2012), uses of examples (e.g., Mills, 2014), and questioning practices (Paoletti, et al, 2018), and more. In advanced mathematics more generally, there are explorations of what students might take from lecture and explanations for miscommunications between students and professors (e.g., Lew, et al., 2016).

There are a number of reports about inquiry-based abstract algebra curricula, including a special issue of the *Journal of Mathematical Behavior*, which includes papers about designing and scaling an innovative curriculum, the ways that teachers do mathematics with students, the goals that professors hold for student learning and how those influence the adoption and use of an inquiry-based curriculum. There are multiple papers describing local instructional theories for inquiry-based content in abstract algebra, including groups and isomorphisms (Larsen, 2013), quotient groups (Larsen & Lockwood, 2013); and rings (e.g., Cook, 2014).

Students are traditionally told that they should expect to spend three to five hours working outside of class for each hour spent in class. Even taking the low-end of three hours as an estimate, and inquiry-based courses where students are active in class, the majority of the students' active learning time would be spent outside of class, typically, working homework problems. In a lecture-based class, the overwhelming majority of the active work of learning of mathematics is during the time students spend doing the homework. Moreover, Wu was quite explicit about the relationship between lecture and additional work that students must do:

> Learning mathematics is a long and arduous process, and no matter how one defines "learning", it is not possible to learn all the required material of any mathematics course in 45 hours of discussion. … The professor gives an outline of what and how much students should learn, and students do the work on their own outside of the 45 hours of class meetings (1999 p. 5)

Yet, the role of homework in learning advanced mathematics, and abstract algebra in particular, is essentially unstudied. That is, what might be the most important part of the learning process for students has been virtually ignored. This raises the natural question: why do instructors assign homework? Here we report on a study to explore the relationship between an instructor's learning goals, and the factors that they consider in selecting problems as well as any general heuristics they use. In this preliminary report, we will explore the question:

- How do the participants consider mathematics, students, and teaching in describing their selection processes for homework problems in an abstract algebra class?

Related Literature and Theoretical Perspective

In their literature review on homework in the K12 setting, Epstein and Van Voorhis (2001) noted that most research on homework explores what students do and how homework affects student achievement. This "student focus" was also dominant in Dorko's (2019) review of the extant literature on the online homework systems used in undergraduate mathematics classes, in that all of the papers she cited were focused on the student experience in some way. Additionally, a search in the undergraduate mathematics research literature (based on a scholar.google.com search) reveals an overwhelming focus on exploring the relationship between homework and exam or course grades, the effect of online vs. on-paper homework, and students' experiences while doing homework. For example, Maciejweski and Merchant's work (2016) provided an analysis of the relationship between study approaches and course grades. Similarly, while Thoma and Nardi (2005) explored the instructor's perspective, the focus was on the differences in discursive practices between secondary and tertiary mathematics, meaning that even though the participants were university-based lecturers, the focus was on how the students were exposed to different tasks and how that changed the demands on student work and thinking. There are also reports about the processes that students engage in when doing homework (e.g., Stephens & Sloan, 1981; Lithner, 2003). Lithner (2003) explored the types of reasoning that students use in completing textbook mathematics problems via filters like the reasoning structure, components and properties, and reasoning characteristics. In one of the few studies of proof-based mathematics homework completion, Moore (1994) explored the types of mistakes students commonly make in writing proofs as part of homework.

Yet, for all of this research on students, there appears to be very limited research into how mathematicians create the assignments and what their goals are for student activity and learning. This lack seems somewhat strange when the ostensible goal of homework is for students to learn, and so attending to what the instructors hope students learn and how they design assignments to promote that learning seems like a critical step in exploring the efficacy of homework. In looking for a literature base to inform our work here, we did find two related areas of study: Epstein's (1998, 2001) work in the K12 setting, and some research on how undergraduate mathematics instructors think about exam questions. Epstein (2001) claimed that in the process of designing homework in K12 settings, teachers draw on their understandings and beliefs about their students' abilities and needs. Across multiple studies at the K12 level, Epstein (1998, 2001) found 10 purposes behind homework at the K12 level, only two of which -- preparation and practice -- appear to be relevant to the university setting. Moreover, those were presented rather unproblematically; that practicing relevant skills on homework is important, but with little discussion of how the selection of items or structuring of the assignment promotes those goals.

Taking the perspective that assigning homework is an instructional practice, guided our theoretical framing for this study. Prior research (e.g., Johnson, et al., 2019; Johnson, Caughman, Fredericks, & Gibson, 2013) suggests that mathematicians do not believe their teaching to be free of constraints, even in proof-based courses. As a result, our theoretical frame for pedagogical decision-making is Ingram and Clay's (2000) notion of 'choice-within-constraints,' which draws on the 'new institutionalism' theories developed in other fields of social science. When discussion the "bounded actor", new institutionalism assumes two premises. The first is that actors, in this case abstract algebra professors, are rational and purposeful about their actions, which prior research about proof-based mathematics instructors (e.g., Weber, 2004; Alcock, 2010) has repeatedly shown to be true for small samples of faculty. The second is that these actors are bounded in terms of their knowledge, worldview, and preferences (Hall & Taylor, 1996). Prior research on teaching has illustrated the importance of knowledge, beliefs, and goals or values on teacher decision-making (e.g., Calderhead, 1996; Henderson & Dancy, 2009; Johnson, et al., 2019). We use this theoretical orientation as a way to understand claims that instructors make about their factors and heuristics for assignment creation and goals for student activity (both physical and mental) on the assignment.

Methods and Data

The primary aim of this study is to identify the factors and heuristics that mathematicians use in constructing homework sets in abstract algebra and understand their rationales for valuing these. To explore this issue, we asked the mathematicians to construct homework assignments for three sections of a common abstract algebra textbook (Gallian, 2013) for a first-semester abstract algebra class. This had the value of engaging the participants in a regular pedagogical practice while allowing for reflection and responding to interviewer questions.

Participants

We solicited participants from two research universities on the east coast of the United States by asking anyone who had taught introductory abstract algebra within the last seven years to participate in the study. We secured the participation of six mathematicians who met the criteria, three at each university. Two of the participants were female. Four were tenure stream faculty at the time of their interview while two were long-term instructional faculty. We assigned all participants pseudonyms with gender-neutral pronouns in order to protect their anonymity.

Data Collection Procedures

Prior to the interview, we emailed each participant a photocopy of the relevant sections of Gallian's undergraduate abstract algebra textbook (2013). If they did not arrive with the homework assignments already prepared, we presented each chapter separately and sequentially and asked them to develop an assignment for the relevant chapter. Due to time constraints, some of the participants only constructed two homework sets. The three chapters were about Groups, Isomorphisms, and Group Homomorphisms. Each participant met individually with an interviewer for a task-based interview lasting between one and two hours. The interviews were audio-recorded and transcribed. For each chapter, the interviewer asked:

1. Have you mentally grouped or organized these problems in some way? If so, how?
2. For each grouping:
   a) Why are you assigning these particular problems?
   b) What do you hope the students will gain from them?
   c) Which of these is the most interesting or important to you? Why?

d) There are several other problems that might also fit in this grouping. What were your selection criteria?

3. For those not selected: You didn't select any problems beyond n. Why is that? Is there a course, group of students, or scenario for which you would assign one of these problems?
4. Were there any problems that you rejected because of the problem statement? What about the phrasing led you to reject those problems?
5. Going back to your groupings, how did you arrive at this distribution of problems?
6. Are there other problems that you typically assign, or would like to assign for this topic?
7. What are your goals for this assignment as a whole?
8. What do you hope that students will learn?
9. Why do you think the assignment would promote that?

We concluded the interview with general questions about the participant's beliefs about the role of homework in student learning.

Data Analysis Procedures

Our first step was to parse the transcript into discrete chunks using the following rules:

- All contiguous text about one particular numbered textbook problem should stay together
- When the content of the oral text switches to another numbered textbook problem, we create a new chunk of text
- Text that concurrently addresses multiple textbook problems should stay together

Then, to analyze the claims made about the items and homework creation heuristics, we used Chi's (1997) method for quantifying qualitative verbal data. Chi's method involves the use of an open coding scheme to document the occurrence and frequency of interesting phenomena that emerge from analyzing a corpus of qualitative verbal data. We approached this coding in two ways; first, we read each transcript individually and developed codes based on the professor's claims. Early iterations of coding suggested that the instructional triangle (i.e., content, students, teaching) may be useful for examining decisions about homework, just as it is for understanding other forms of pedagogical reasoning (Horn & Little, 2010). Here we highlight some preliminary findings below based on that work. Second, we focused on the homework assignment that each professor wrote for Chapter 2, about groups, and explored which items were shared and the reasons that professors included those items, searching for commonalities.

Preliminary Findings

We make three claims about the data. First, that five of the six participants believed that homework is a crucial component of student learning, for example, August stated, "I think that's really where the learning happens." Lennon provided a reason that homework is valuable for student learning, claiming that, "Extracting the meaning from hard stuff, prepares you very well for mathematics. Just the attitudes that you get. That things don't come to you. You have to work for every single bit. And that's what homework should do." We interpreted Lennon's claim as two-fold, first that mathematics is hard, and, extracting meaning from mathematics requires work, and second, that the homework is where students develop the attitude and tools to do so. Second, the majority of participants' comments focused either on the mathematical content contained in the homework or how the homework would help the learning needs of the students. There were few comments on the role of homework in assessing students. We substantiate the first two of these categories below. Third, there was variance among the mathematicians. They choose different problems to accomplish the same goals, and they relied on different factors in selecting the same problem.

We now present details about the three broad categories of explanations for which our participants chose (not) to include any individual item: the mathematical qualities of the item, how students think about the content, and pedagogical reasons. We illustrate each category via a quote from a participant. For example, Dakota's explanation for including problem 17 from Chapter 2 on the homework assignment focused on the mathematics of the item:

> Well, I think 17 is a classic example. There are various versions of this… So 17 showed that if the inverse of AB is equal to A-inverse times B-inverse for all AB, then it's abelian. The proof of that should be three lines when it's done and there are lots of variations of the same idea. I think I may have only put in one, but it's the sort of thing is once it clicks, you can then write it down and you get it. Then, you have this new understanding, this more clear understanding of Abelian groups.

There are three aspects of this explanation that we highlight; first, that Dakota referred to the problem as a "classic example" which we interpreted as a focus on the discipline of mathematics and the canonical tasks therein. Second, the explanation specifically describes the content of the problem as a justification for inclusion. Finally, Dakota claimed that doing the problem with give students a "more clear understanding of Abelian groups." We interpreted this to mean that students would have a new way to determine whether a group is Abelian (a mathematical value).

We contrast this focus on mathematics with Eli's claim about item number 6 from Chapter 2:

> I like number six because students usually get confused with respect to an operation in the sense that they just feel it's commutativity all the time. It's very common to see that a student wants to cancel out terms immediately just by interchanging the order of the elements. That's very common. Even if you emphasize that in class with an example, some of them are so attached to the commutativity of the set of the real numbers that they resist to come up with their own examples of something non-commutative. That's why, to emphasize that particular property...

In this example, we claimed that Eli choice and explanation focused on the way that students think about mathematical content. For example, Eli notes that students "get confused" and "feel it's commutativity all the time" and "a student wants to cancel out terms immediately." That is, in each of these cases, Eli was focused on how students think and the types of actions they are likely to take as a result. We interpreted Eli's claims as indicating that putting item 6 on the assignment is an effort to force students to confront their thinking and promote mathematically normative understandings of the content. We also note Eli's claim that "Even if you emphasize that in class with an example" which we interpreted as showing the purported need for this concern to be addressed in homework, because addressing it in class is insufficient.

Concluding Notes

We anticipate that this research will make significant contributions to the literature by providing an analysis of the factors and heuristics that instructors consider for creating homework assignments in an introductory abstract algebra course. At the current point of coding, our data shows that professors attend to the mathematics of the items, including the idea that certain tasks are "classic," and that the underlying content (e.g., Matrices), in addition to the intended content domain, are important. During the presentation we anticipate having a more fully-developed analysis of the creation of one homework set for Chapter 2, as well as a comparison of the reasons given for the set itself and the inclusion of individual items in the set.

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Relational Interactions in Inquiry-Oriented Undergraduate Mathematics Classes

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In this study, we examine two inquiry-oriented classes using Battey and Leyva's (2013) relational interactions framework. We examine the relational interactions that occurred in a class with gender-equitable student learning outcomes and a class with gender-inequitable student learning outcomes. Although we expected to see differences in these two classes, results show that both instructors had similar relational interactions, focusing mostly on acknowledging student contributions. However, in one class there were noticeable negative relational interactions with women students that might help us better understand the gender differences in the classes.

Keywords: Relational Interactions, Inquiry-Oriented, Gender, Equity

Despite the growing body of research indicating that active learning is better for student learning outcomes than traditional lecture, recent research raises questions about how these interactive classroom environments may be contributing to negative gendered interactions for students. In a recent study that looked at student learning outcomes in both lecture and inquiry-oriented classes, Serbin and Mullins (2019) found that, while men in inquiry-oriented classrooms outperformed men in lecture classes, women in inquiry-oriented classes and women in lecture classes performed the same on a content assessment. This finding warrants additional research to better understand the gendered interactions in inquiry-oriented classrooms. In this study we purposefully selected two inquiry-oriented instructors to investigate the relational interactions (Battey, 2013) in their classrooms. By investigating the interactions between the instructors and their students, we hope to gain insights into these differences. In particular, we investigate the following research questions:

1. What are the relational interactions present in an inquiry-oriented classroom with gender-equitable learning outcomes?
2. What are the relational interactions in an inquiry-oriented classroom with gender-inequitable learning outcomes?
3. How do these relational interactions compare?

Related Research and Theoretical Perspective

A number of studies have suggested that active learning may be more equitable for students from historically marginalized groups (e.g., Laursen, Hassi, Kogan, & Weston, 2014). However, a recent study by Johnson et al. (in press) found inequitable learning outcomes in undergraduate inquiry-oriented (IO) mathematics classes, with men in IO classes outperforming men in more traditional lecture courses and women in IO classes performing the same in both instructional formats. The findings presented by Johnson et al. (in press) speak to Eddy and Hogan's (2014) argument that any classroom intervention will impact different groups of students in different ways and reinforce Singer and colleagues' (2012) call for the identification of critical features in order to explore the ways that particular approaches impact various student sub-populations. In this paper we aim to do just that, by adopting Battey and Leyva's (2013) work on relational interactions to better understand the gendered experiences of students in IO classrooms.

Battey and Leyva (2013) defined a *relational interaction* as a “communicative action involving moment-to-moment interaction between teachers and students, occurring through verbal and nonverbal behavior that conveys meaning” (p. 981). Battey and his colleagues (e.g., Battey, 2013; Battey & Leyva, 2013; Battey & Neal, 2018; Battey et al, 2016) described five dimensions of teacher-student relational interactions that influence students’ access to mathematics. *Addressing behavior* involves responding to a specific student’s behavior. *Framing mathematics ability* is related to how the instructor frames general student mathematics ability, for example either innate or changeable. *Acknowledging student contributions* includes valuing, devaluing, praising, or disparaging a specific student’s contributions. *Attending to culture and language* involves referencing students’ culture and language in instruction. *Setting the emotional tone* involves setting expectations for students and creating an emotional space. These interactions include general comments for setting expectations, addressing behavior, or establishing emotional space, and are not student specific. Teachers’ relational interactions can be positive (e.g., motivating student engagement or praising students’ contributions) or negative (e.g., preventing or ignoring student contributions).

Understanding these positive and negative interactions may offer a new view into reform-oriented instruction. In this study, we bring the relational interactions framework to the undergraduate inquiry-oriented context to better understand the extent to which it generalizes and helps us understand the gendered outcomes found in the IO context. In particular, we use the relational interactions framework to investigate the nature of teacher-student interactions in two IO mathematics classes with different student learning outcomes.

Study Content and Methods

Study Context and Participants

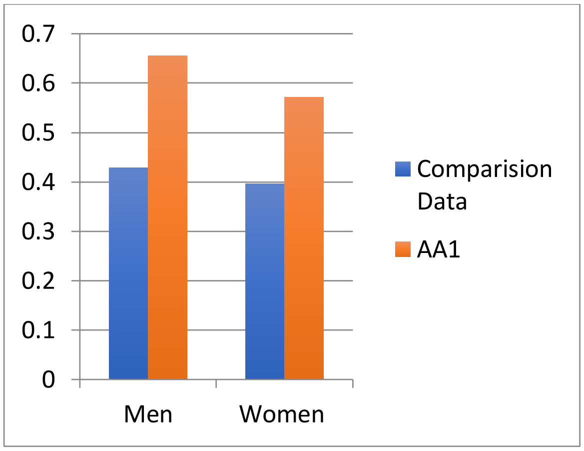
The two instructors we focus on here took part in a professional development program designed to support them as they implemented inquiry-oriented instructional materials. In addition to collecting data during the professional development activities, instructors were also asked to video record two instructional units and their students were asked to complete a content assessment at the end of the semester. Additionally, comparison student assessment data was collected from students whose teachers were not involved with the professional development. In both the IO abstract algebra data and the differential equations data, we saw a similar trend when we compared the IO students to the comparison groups: men in IO classes were outperforming their non-IO peers, whereas women were scoring very similar in the two instructional formats.

In selecting instructors for this analysis we took several factors into account. In the end we narrowed our analysis to two instructors, one who implemented IOAA who we will refer to as AA1, and one who implemented IODE who we will refer to as DE1. Both of these instructors had classes with over 30 students and with completed content assessment data from more than 20 of them. Additionally, these two classes were fairly evenly split in terms of the number of men and women students. Lastly, these two instructors presented a “divergent case” and a “convergent case”. As shown in Figure 1, in AA1’s class both the men and the women outperformed the men and women in the comparison data. Whereas in our convergent case, the men in DE1’s class outperformed the men in the comparison data set whereas the women in DE1’s class scored about the same as the women in the comparison data set.

Analysis

To examine the relational interactions within the two classrooms, we chose one instructional unit per instructor (1:45:35 instructional hours for AA1 and 2:01:48 instructional hours for DE1).

We followed Battey and Neal's (2018) five coding layers: 1) code each episode for relation interactions, 2) categorize episodes based on the five relational interactions dimensions, 3) code forms of emphasis such as voice tone and gestures, 4) code interactions as positive or negative, and 5) code intensity as either low, medium, or high. We inferred student gender from video data. In instances where the instructor made general comments to the class, gender was coded as general. (We acknowledge that inferring student gender and treating gender as a binary classification is problematic.)



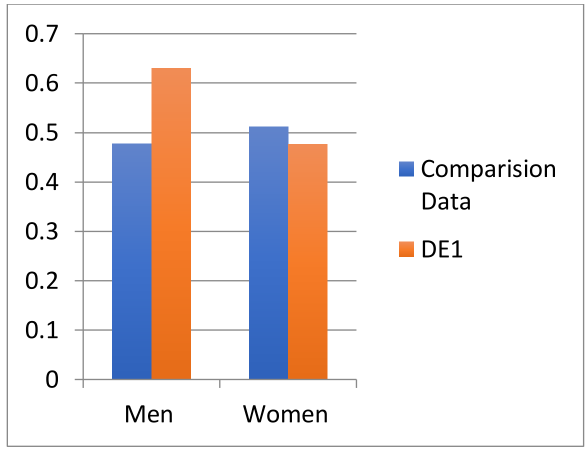


Figure 1. Student Performance, by Gender

Results

Class 1: Instructor AA1 Interactions

Instructor AA1 had a total of 98 relational interactions with students. Of these 98 interactions, 30 of these interactions were with men students, and 34 were with women students. Instructor AA1 also made 27 general comments to the class. There were only seven instances in which student gender was not inferable. Looking at each dimension and the different coding layers, the majority (54.08%) of Instructor AA1's relational interactions were acknowledging student contributions (See Table 1 for all categories of interactions). Analysis revealed that 73.47% of the interactions were positive, with 26.53% of them negative. Out of the 26 negative relational interactions, 10 of those were focused on addressing student behavior. The majority of these negative addressing student behavior interactions were instances in which student(s) tried to interrupt another student talking, and the instructor stopped them from doing so. Looking specifically at gender, 11 of the 26 negative interactions involved men students and 6 involved women students (Table 2). Of the 72 positive interactions, 29 involved men students and 28 involved women students, with the majority, if not all, focused on acknowledging student contributions. Additionally, there were 22 positive interactions that were general comments made to the class, most often focused on setting the emotional tone.

Class 2: Instructor DE1 Interactions

Instructor DE1 had a total of 142 relational interactions with students. Of these 142 interactions, 46 of these interactions were with men students, and 52 were with women students. Instructor DE1 also made 32 general comments to the class. There were 12 instances in which student gender was not inferable. Looking at each dimension and the different coding layers, the majority (72.53%) of Instructor DE1's relational interactions were acknowledging student contributions (See Table 2 for all categories of interactions). Further analysis revealed that

88.73% of the interactions were positive, and 11.27% of them negative. Out of the 126 positive interactions, 91 focused on acknowledging student contributions and 31 focused on setting the emotional tone. Out of the 16 negative relational interactions, 12 of those were focused on acknowledging student contributions. The majority of these negative interactions were instances in which the instructor ignored student(s)' questions or responses. Looking specifically at gender, 7 of the 26 negative interactions involved men students, and 8 involved women students (Table 2). Of the 126 positive interactions, 39 involved men students and 45 involved women students, with the majority if not all focused on acknowledging student contributions. Additionally, 30 positive interactions were general comments made during class.

Comparison of the Two Classes

Examining the interactions for both classes, the two classes were similar in the number of interactions and types of interactions displayed in the class. The majority of interactions in both classes were focused on acknowledging student contributions. Also, both instructors had more positive interactions with women students than men. However, there were subtle differences in the number interactions and the type. For example, looking at addressing behavior, Instructor DE1 had two negative interactions that addressed students for interrupting other students. Instructor AA1 had 13 interactions focused on addressing behavior, 3 of which were positive. The negative ones mostly addressed students for interrupting other students who were sharing. Looking at acknowledging student contributions, Instructor AA1 ignored or omitted a student's contribution two times, whereas Instructor DE1 ignored or omitted a student's contribution nine times. From these nine interactions, six of those were contributions from women. Instructor DE1 also had one negative framing mathematics ability interaction in which she told a women student she was the note-taker for the group. Overall, Instructor AA1 had six negative interactions with women students and Instructor DE1 had eight.

Table 1. Frequency and Percentages of Relation Interactions by Dimensions and Codes

		# of Positive by Gender			# of Negative by Gender		
Interaction	Count	Men	Women	General	Men	Women	General
Addressing behavior	13 13.3%	0 0%	1 7.7%	1 7.7%	3 23.1%	2 15.4%	1 7.7%
Framing mathematics ability	1 1.0%	0 0%	0 0%	1 100%	0 0%	0 0%	0 0%
Acknowledging student contributions	53 54.1%	19 35.8%	26 49.0%	0 0%	4 7.6%	3 5.7%	0 0%
Attending to culture and language	11 11.2%	0 0%	0 0%	3 27.3%	3 27.3%	1 9.0%	3 27.3%
Setting the emotional tone	20 20.4%	0 0%	1 5%	17 85%	1 5%	0 0%	1 5%
Total Counts	*98*	*19*	*28*	*22*	*11*	*6*	*5*

Note: Rows do not sum to the total counts due to unknown student gender in interactions.

Table 2. Frequency and Percentages of Relation Interactions by Dimensions and Codes

		# of Positive by Gender			# of Negative by Gender		
Interaction	Count	Men	Women	General	Men	Women	General
Addressing behavior	2 1.4%	0 0%	0 0%	0 0%	1 50%	0 0%	0 0%
Framing mathematics ability	2 1.4%	0 0%	0 0%	0 0%	0 0%	1 50%	0 0%
Acknowledging student contributions	103 72.5%	39 37.9%	44 42.7%	0 0%	5 4.9%	7 6.8%	0 0%
Attending to culture and language	4 2.8%	0 0%	0 0%	3 75%	1 25%	0 0%	0 0%
Setting the emotional tone	31 21.8%	0 0%	1 3.2%	27 87.1%	0 0%	0 0%	0 0%
Total Counts	*142*	*39*	*45*	*30*	*7*	*8*	*0*

Note: Rows do not sum to the total counts due to unknown student gender interactions.

Discussion and Conclusions

Looking at these two classes, we expected to see major differences in the classes. Results show that the classes are similar in the number of relational interactions, with only subtle differences in the interactions. Both instructors had a significant amount of interactions focused on acknowledging student contributions and setting the emotional tone. However, Instructor DE1 had more notable negative interactions with women students than Instructor AA1. Instructor AA1's negative interactions with women students focused on preventing a women student from interrupting another student talking, building towards formal mathematical language, and making sure a woman student understood the problem. Instructor DE1's negative interactions with women students focused on omitting or ignoring women's contributions and telling one woman she was the note-taker. We plan to conduct more analysis to better understand the differences in these gender-equitable and -inequitable relational interactions.

Questions for the audience:

1. The differences in these counts see pretty minor. Are they really enough to help us explain the gender performance differences?
2. We also coded for intensity and form (but this was omitted from the proposal for space). What more do we learn about these classes by bringing everything together?

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A Prospective Teacher's Mathematical Knowledge for Teaching of Inverse Functions

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Prospective mathematics teachers are often required to take a course in abstract algebra. However, there is still some question as to how prospective teachers connect their knowledge of abstract algebra to secondary algebra and how they might draw upon this knowledge as they teach. This study explores a prospective teacher's conceptions of inverse functions and their relation to the algebraic group of invertible functions under composition. It explores how this prospective teacher draws on her mathematical knowledge as she responds to student thinking.

Key words: prospective mathematics teachers, inverse functions, abstract algebra

It has been argued that secondary mathematics teachers need to develop knowledge of advanced mathematics to be effective educators. However, there is still some question regarding how teachers draw upon their advanced mathematical knowledge of abstract algebra to influence their teaching practice. Wasserman and Stockton (2013) claimed that teachers' understanding of connections between advanced and secondary mathematics can potentially impact their decisions in lesson planning and actions in responding to students. Wasserman (2017) argued that understanding connections between algebraic group axioms and inverse functions in high school algebra contexts can provide teachers opportunities to incorporate the actual meaning of inverses in their instruction. Unfortunately, even after taking an abstract algebra course, prospective and in-service mathematics teachers may struggle to see connections between properties of inverses across secondary and abstract algebra contexts. The purpose of this study is to explore a prospective teacher's (PT) understanding of inverse functions as inverse elements within a group and how she might use this understanding in a hypothetical instructional setting. I will address the following research questions: How does a PT conceptualize inverse functions? How does a PT draw on her mathematical knowledge as she interprets and responds to student thinking?

Literature Review and Theoretical Perspective

Wasserman (2017) explored secondary teachers' understandings of inverse functions and abstract algebra. He found that only two of seven teachers could situate inverse functions within their group structure by referencing the identity function and composition operation. This indicated that only those two out of seven teachers had personally powerful understandings of inverses in both secondary algebra and abstract algebra contexts. For the other five teachers, their understanding of abstract algebra was not helping them perceive relationships with inverse functions. Wasserman found that those five teachers did not have both an object conception of function and a process conception of composition (Dubinsky, 1991). He hypothesized that knowledge of abstract algebra alone is insufficient for having powerful understandings of inverses; teachers also need to conceptualize function as an object and composition as a process.

Knowledge of connections between abstract algebra and secondary school algebra can impact teachers' instructional approaches. Zbiek and Heid (2018) exemplified how a teacher's interpretation of and response to students' ideas were informed by her knowledge of algebraic structures. The teacher's students claimed that secant, cosecant, and cotangent were inverse functions. To productively respond to the students, the teacher attended to the group in this setting as invertible functions under composition instead of the real numbers with multiplication. Knowing the structure of the group of invertible functions helped the teacher interpret students' thinking as them using an inappropriate operation (multiplication instead of composition) for inverse trigonometric functions. Zazkis and Marmur (2018) described the utility of teachers' knowledge of abstract algebra in responding to situations of contingency in the classroom. They exemplified how a teacher's knowledge of group theory could be used to respond to the student question of why $5^{-1} = 1/5$ but $f^{-1} \neq 1/f$. In this case, 5^{-1} and f^{-1} are both inverse elements, but they differ with respect to their set and binary operation. The teacher could use her knowledge of the algebraic structure of inverses to inform her response to the student.

In this study, I assume knowledge of advanced mathematics, particularly abstract algebra, is relevant for secondary teachers, and such knowledge can influence teachers' instructional practices (e.g., Ball, Thames, & Phelps, 2008; Shulman, 1986; Silverman & Thompson, 2008; Wasserman, 2018). Furthermore, PTs should make connections between advanced mathematics and the mathematical content they will teach. This study explores how a PT makes these connections and draws on her mathematical knowledge as she responds to student thinking.

Methods

The participant in this study, Becca, is a senior undergraduate student majoring in mathematics with a specialization in math education. Becca has taken a course in modern algebra and a course in mathematics for secondary teachers. She had received instruction on additive, multiplicative, and function inverses and their associated binary operations and identities.

One task-based clinical interview (Clement, 2000) was conducted with Becca, in which she created a concept map of inverse functions, performed tasks (see Figure 1)[1] related to inverse functions, and answered follow-up questions to explain her reasoning. These tasks were chosen to elicit evidence of her understanding of inverse functions in relation to their associated binary operation and identity. She also interpreted hypothetical student work (see Figure 2) and described what she thought the student was thinking and how she would respond to that student.

I wrote detailed descriptions of Becca's responses for each task, and I wrote memos of my interpretations of her exhibited conceptions. I first analyzed Becca's responses to the concept map task, looking for how she seemed to conceptualize inverse functions. I used the constant

1. Find the value of $\sin(0.848062\ldots)$ given that $\sin^{-1}(0.75) = 0.848062\ldots$
2. Find the value of $e^{\ln 4}$. Explain your reasoning.
3. Do these equations mean the same thing? $\cos^{-1}(x)\cos(x) = 1$ and $\frac{1}{\cos(x)}\cos(x) = 1$
4. Are these functions f and g inverses of each other? $f(x) = (x+2)^3$ and $g(x) = \sqrt[3]{x} - 2$

Figure 1. Interview tasks about inverse functions.

5. *Student response*: Since $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions, $\left(\sqrt{-2}\right)^2 = -2$.
6. *Task:* Find $\sin^{-1}(1)$. *Student response*: $\sin^{-1}(1) = \frac{1}{\sin(1)} = \csc(1) = 1.188$
7. *Task*: Find the value of $e^{ln\,4}$. *Student response*: I can cross out e and ln, so it is 4.
8. Task: Are these functions inverses of each other? $f(x) = 2x + 5$ and $g(x) = \frac{x-5}{2}$
   Student response: No, because $f^{-1}(x) = \frac{1}{2x+5} \neq g(x)$.

Figure 2. Hypothetical students' responses to tasks about inverse functions.

[1] Tasks 1 and 2 in Figure 1 were used in Wasserman's (2017) study.

comparative method (Corbin & Strauss, 2012) to compare her conceptions of inverse functions and their binary operation and identity exhibited in each interview task. Through this analysis, major themes emerged of Becca focusing on "opposite operations," conceptualizing an identity as what results from inverses canceling, and inferring an inverse's associated operation and identity based on the problem context. I compared her responses to hypothetical students and found a major theme of Becca's mathematical conceptions informing her responses to students.

Results

Becca's Conceptions of Inverse Functions

Focusing on "opposite operations." In creating the concept map, Becca described her understanding of the different representations of inverse functions. She first gave a procedural example of how to find an inverse function from a given function. She explained the procedure of switching x and y variables and using "opposite operations," which she described as pairs of "interchangeable" operations like addition and subtraction or multiplication and division. Becca then described an example of a graphical representation of $y = \sin(x)$ and $y = \sin^{-1}(x)$, for which she reflected the graph of $y = \sin(x)$ over the line $y = x$ and explained that the x and y coordinates were flipped. When asked why switching the x and y coordinates made sense in terms of inverses, she said that it related back to the analytical properties of inverses (referring to her procedural work of finding an inverse). She claimed that when you do opposite or reverse operations, "that is what you are doing with these points, the roles of each of these (pointing to the graphs of the function and inverse function) switch… which is how these points [on the graph] flip cases." Here, Becca focused on the role of "opposite operations" in finding inverse functions and making sense of "flipped" coordinates of graphs of a function and its inverse.

This emphasis on opposite operations was evident in her work on the interview tasks as well. For task 1 (see Figure 1), Becca said that she could "just change the sides," so $0.75 = \sin(0.848062\ldots)$. When asked why she could just change the sides, she said that to perform the opposite operation of inverse sine, "I took the sine of the sine inverse of 0.75, and I set that equal to the sine of 0.848062… and since these are inverses, they cancel out, so we are just left with $0.75 = \sin(0.848062\ldots)$." For this task, she used the composition operation, but did not explicitly state that she was doing so. It seemed that her motivation for performing that composition was to perform the opposite operation of the inverse sine function.

Furthermore, on task 4 (see Figure 1), to see if the given functions were inverses, Becca performed the procedure of switching variables x and y and solving for y. She first manipulated the function f and described taking "the opposite of the cubed" as the cube root and taking "the 2 to the opposite side by performing the opposite operation." It seemed that for Becca, the idea of opposite operations canceling each other is an important idea related to inverse functions.

When I asked Becca how she would describe the concept of inverse to a student, she said she would "focus on the idea of opposites" and "canceling things out in order to get to your inverse." In her explanation, she did not reference the algebraic structure of an inverse; instead she talked about the process of performing opposite operations to find an inverse function from an original function. Her conception of inverse functions seems to be that inverse functions reverse each other by having "opposite operations" that undo each other when the functions are composed.

Conceptualizing an identity as the result of inverses canceling. During the concept map task, Becca drew nodes of f and f^{-1} and connected them to a node of i. She described putting f and f^{-1} together to get the identity. When asked to elaborate, she referred to her procedural example of composing $f(x) = x + 2$ and $g(x) = x - 2$ and said, "when you compose these, so when you put them together, so kinda like composition… I'm gonna plug in this x minus 2 for the x plus 2, and it kinda cancels it out just to get the identity of x." Here, Becca attended to the proper operation of composition, but she referred to the identity as x, instead of referring to the identity as a function itself. She claimed that when she composes two inverse functions, she gets an identity. However, she did not seem to conceptualize the identity as a function itself. Rather, she seemed to conceptualize the identity as the result of the inverse functions canceling.

This conception was also evidenced by her responses to the tasks. For example, in task 1, (see Figure 1) she described the sine and sine inverse functions canceling out, but did not reference the composition of those inverse functions as an identity function. For task 2, (see Figure 1) she said, "if e and natural log are inverse functions, they kind of just cancel out with each other, and so e raised to the ln of 4, they would just cancel each other out and you would just have the value of 4." When asked why they would cancel each other out, Becca said, "It relates back to the idea of the identity. When you compose f with its inverse, it is equal to the identity, so technically, combining these two, you have i and 4 just gives you 4." When I asked what the i and 4 was, she said "i is just the identity. In this case with multiplication, it just plays the role of 1 times 4 is equal to 4." When I asked her to clarify what the identity function is, she said it was 1. Here, she claimed that the composition of two inverse functions yields the identity, but it seemed that she did not see the identity as the function defined as $i(x) = x$, since she claimed the identity in this task was 1. She again focused on the inverse functions canceling instead of forming a new composite function that is the identity function.

For task 4, when I asked Becca if there was another way that she could check if the given functions are inverses, she said "you could plug one into the other." She wrote out the composition of $f(g(x)) = \left(\left(\sqrt[3]{x} - 2\right) + 2\right)^3$, and described how the opposite operations cancel out to be left with x. When asked why this meant that they are inverse functions, she responded "because it equals the identity," wrote $f \circ f^{-1} = i = x$, and said "so it's pretty much saying y is equal to x." Here, she used composition as the binary operation, and focused on how the opposite operations made the inverse functions cancel. It seemed that for Becca, an identity occurred when inverse functions "cancel out" through "opposite operations."

Inferring an inverse's associated operation and identity based on the problem context. As Becca worked on the tasks, she seemed to infer an operation associated with the inverse based on the context of the problem. For example, on task 6 (see Figure 2), Becca claimed that $\sin^{-1}(1) = \frac{1}{\sin(1)}$ is true, meaning she considered $f(x) = \sin^{-1}(x)$ as a multiplicative inverse, even though she used $f(x) = \sin^{-1}(x)$ as an inverse under composition in task 1. Becca claimed that based on the student saying $\sin^{-1}(1) = \frac{1}{\sin(1)}$, she assumed they would know that $3^{-1} = \frac{1}{3}$. Here Becca seemed to relate the inverse relationship between $\sin(x)$ and $\sin^{-1}(x)$ to be the same multiplicative inverse relationship between 3 and $\frac{1}{3}$. In this case, she seemed to infer a multiplicative operation associated with this inverse based on the problem context. She used the same idea on task 3 (see Figure 1), claiming "you can think about inverses when you're dealing with multiplication, uh arc cosine is the same thing as 1 over cosine. It's just a different way to write it." The equations in the task included a product of two functions instead of a composition of two functions, so she associated the inverse of cosine, $\cos^{-1}(x)$, as a multiplicative inverse instead of an inverse under the operation of composition. In these instances, she seemed to identify the identity and operation associated with the inverse based on the context of the task.

Becca's Responses to Student Thinking

Mathematical conceptions informing her responses to students. On most tasks, Becca interpreted and responded to student work in an appropriate way. When asked how she would respond to the student on task 5 (see Figure 2), she said she likes to "pull on knowledge they already have" and "ask them questions pertaining to what they already know." Her interpretations of and responses to student work seemed to be informed by her own mathematical conceptions. For example, on task 7, her conception of identity as what results from inverse functions canceling is present in her response to the student. She said that the student is assuming that "e when composed with ln is equal to the identity," so that is how the student got the answer of 4. I asked Becca if she could tell that is what they are thinking based on the language of "crossing out," and Becca claimed that she did not think the student gave enough detail. Becca was aware of the hypothetical student's imprecise language and said she would ask the student to further elaborate on what they mean by "cross out." She claimed that if the student was unsure why the canceling happened, she would show them why one function undoes the other. It seemed that her idea of a composition of inverse functions canceling via "opposite operations" informed her response of explaining to the student why one function undoes the other function.

Becca's conception that some inverse functions are multiplicative inverses informed her response to the student on task 8 (see Figure 2). Becca said that this student is using an idea similar to the idea that $\cos^{-1}(x) = \frac{1}{\cos(x)}$, which she claimed "is not true in this case because this only works when you're dealing with single numeric values like x or 2 or 5." She explained that $x^{-1} = \frac{1}{x}$ but $f^{-1}(x) \neq \frac{1}{2x+5}$ because "you have $2x + 5$ and you can't just say it's 1 over $2x + 5$." She suggested that the student would have to go through the procedure of switching x and y variables and solving for y to find the inverse function or plug in one function into the other to see if they get out x. She said that she would respond to this student by connecting this to a previous example they did and letting the student know that "this only applies to single values or single digits such as x, 2, or 5." Becca seemed to conceptualize some inverse functions, such as $\cos^{-1}(x)$ and $\sin^{-1}(x)$, as multiplicative inverses, so she did not give a mathematically correct explanation as to why f^{-1} is not the same as $1/f$. In this case, Becca drew upon her conceptions of inverse functions as she responded to the student, but without situating inverse functions within their group structure by attending to the operation and identity, she did not respond to the student in a way that would distinguish the type of inverse with respect to its operation.

Discussion

This study explored a PT's understanding of inverse functions in relation to their group structure. Becca seemed aware of group axioms, given her acknowledgement of the relations between inverse and identity, and she used this knowledge as she performed tasks and responded to student thinking. Although she did not seem to conceptualize an identity as a function itself, her understanding of the identity as the result of inverse functions canceling seemed useful for her as she performed tasks and made sense of student work. She did not always attend to the binary operation associated with inverse functions, and this influenced her response to a student.

I propose the following discussion questions for the audience: Do cognitive models give us an appropriate lens for making sense of teacher decision making? Is using this perspective the best way to explore how PTs make sense of student work? What is the algebraic reasoning that high school teachers need? What frameworks are most useful for investigating the nature of PTs' mathematical knowledge that can be used in teaching, without focusing on PTs' deficits?

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Supporting Students' Construction of Dynamic Imagery: An Analysis of the Usage of Animations in a Calculus Course

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Calculus is about change and thus dictates a need for instruction involving dynamic imagery. DIRACC (Developing and Investigating a Rigorous Approach to Conceptual Calculus) utilizes animations to support students' dynamic imagery. This paper investigates how students use and understand animations in the DIRACC textbook in connection with associated calculus topics.

Keywords: Calculus, Technology, Animations, Variables

Introduction

The important ideas of calculus can be summarized as *using how fast a quantity is varying at every moment to find how much of that quantity there is at every moment, and vice versa* (Thompson & Ashbrook, 2019). The former describes integration, and the latter describes differentiation. Because of the central role of change, the importance of dynamic student imagery cannot be understated. In a typical Calculus course, the textbooks contain static images that try to illustrate these concepts of change. These static images might not communicate the ideas of dynamism and change as effectively as we would like them to. Thankfully, technology has provided the opportunity for students to engage with dynamic imagery by enabling the usage of animations (Thompson, Byerley, and Hatfield, 2013). This study investigates how students interpret dynamic imagery in the context of a revised calculus course.

Background: DIRACC

Developing and Investigating a Rigorous Approach to Conceptual Calculus (DIRACC) is a revised conceptual calculus curriculum based on the relationship between two fundamental ideas mentioned above: *using how fast a quantity is varying at every moment to find how much of that quantity there is at every moment, and vice versa* (Thompson & Ashbrook, 2019). The course was developed from a constructivist and conceptual approach grounded in quantitative reasoning (Smith & Thompson, 2007). DIRACC includes a 2-semester introductory calculus course that utilizes an online textbook that provides animations designed to help students construct dynamic imagery in accordance with the two fundamental ideas of calculus. These animations are *didactic objects* intended to be used by the instructor to help students construct such imagery (Thompson, 2002). Whether students associate such concepts with the animations is an open question that we begin to investigate.

The creator of these animations strongly considered a particular topic of calculus and how might a student construct the intended dynamic imagery. Despite this intention, a recent survey of 164 DIRACC students reveal a disconnect between the authors and the students' interpretations of the animations (Table 1). This led the research team to interview some of these students to get a sense of what students were actually interpreting when viewing the animations. By reporting on individual interviews, this paper addresses how students interpret these animations in relation to the foundational ideas of calculus.

Table 1. 164 end-of-year DIRACC Calculus II students' responses to an anonymous survey about animation usage.

<table>
<tr><th colspan="2">Animation Usage</th><th colspan="2">Animation Understanding</th></tr>
<tr><td>Always plays the animations</td><td>24%</td><td>Found the animations clarifying upon the first watch</td><td>13%</td></tr>
<tr><td>Plays most of the animations</td><td>28%</td><td rowspan="2">Found them initially hard to understand but clarifying upon repeated watching</td><td rowspan="2">69%</td></tr>
<tr><td>Plays half of all the animations</td><td>20%</td></tr>
<tr><td>Plays some of the animations</td><td>25%</td><td rowspan="2">Found very hard to understand despite rewatching</td><td rowspan="2">18%</td></tr>
<tr><td>Never plays animations</td><td>3%</td></tr>
</table>

Research Questions

Our research addresses the following questions:

- How do students interpret animations in an online textbook?
- In what ways do animations assist students in developing dynamic imagery of varying quantities?
- How do the animations help students construct the meanings we intend, and how do they not help?

Literature Review

Newton approached the fundamental concept of calculus from the perspective of variables varying with time. According to Kleiner (2001), "Newton calls his variables 'fluents'- the image is geometric and kinematic, of a quantity undergoing continuous change." DIRACC calculus takes an account based on Newton's approach to Calculus and utilizes animations to support the construction of dynamic imagery for students. For Newton, the Fundamental Theorem of Calculus (the FTC) was not a theorem, but an almost immediate consequence of thinking about calculus dynamically (Thompson, 1994). The importance of the relationship between dynamism and the FTC is reflected in mathematics education literature; Thompson & Dreyfus (2016) posit that students who think about variables statically find it challenging to understand the Fundamental Theorem of Calculus (FTC) and Thompson (1994) found that students' difficulties with the FTC are rooted in underdeveloped images of functional covariation. Research suggests that it is productive for students to think covariationally in the sense of constructing images of two smoothly varying quantities changing with respect to one another (Carlson, 1998; Carlson, Jacobs, & Larsen, 2001; Carlson et al., 2002). Thompson and Carlson (2017) revisited the covariational reasoning framework, and the results of their study underscore the importance of continuous covariational reasoning for students to develop the foundational ideas of calculus. Thompson, Byerley, and Hatfield (2013) explain that without technology, it is impossible to approach calculus conceptually due to lack of dynamics available to students. In this study, we investigate how these animations help students construct the necessary dynamic imagery for understanding calculus robustly and conceptually.

Theoretical Perspective

Taking the radical constructivist perspective (Thompson, 2000), the DIRACC curriculum, as well as this study, adopts the stance that individuals construct their own meanings, and these meanings are not directly observable to others. As such, we use individual student interviews to construct consistent models of student meaning (Steffe & Thompson, 2000). We use the word "consistent" in two ways – to refer to our models of student thought (consistent with student behavior), but also to describe student thought itself (as being consistent and coherent from the student's perspective). Students organize their own experiences with a concept, determining how they act with that concept. These meanings, which students use to make sense of mathematics that they encounter, are personal and internally cohesive, but they may or may not be productive.

The purpose of the animations is to support students in developing the meanings and robust imagery of variation involved in calculus. This intended reasoning is in line with Castillo-Garsow's (2010) smooth images of change, where someone views change-in-progress by conceptualizing a quantity as taking on values as time flows continuously and smoothly, rather than thinking about change happening discretely with no imagery of motion in between two points.

Animations created using technology in DIRACC can be considered as didactic objects to support conceptual understanding of calculus (Thompson, 2002). From our constructivist perspective, the ideas embodied in didactic objects do not simply transmit themselves from the minds of the animation authors to the students – hence the reason for our investigation.

Methodology

Four Calculus II DIRACC students each underwent an hour-long semi-structured interview (Clement, 2000). The interview consisted of students watching different animations and answering questions about their understandings of the animations. This paper presents the results of two animations. For each animation, students were first asked to watch the animation in full, and then asked what they thought the animation was about. The animation was subsequently played in predetermined intervals, and students were each asked what had happened in each section played. The goal of these interviews was to assess what students thought the animation was showing relative to the intended meaning. The intended meaning was not actually discussed with the student until after the study was completed.

Preliminary Results

For each animation, we will provide a description of the animation itself and its intended interpretation. We will follow the description with our understanding of student interpretation of the animation.

Animation 1 - Variation: Variables Vary Smoothly

The first animation was intended to illustrate how variables vary (Figure 1). The axis for the independent variable, x, is partitioned into fixed intervals of width Δx called "Δx-intervals". As x varies through a Δx-interval, dx (how much x has varies within its current Δx-interval) also varies. The function *left* inputs a value of x, say x_0 and outputs the value of on the leftmost end of the Δx -interval in which x_0 resides. The last frame of the animation shows a smaller Δx-interval. The length of the red line segment represents the value of x, and the length of the blue line segment represents the value of the dx. The reader may want to think of the Δx -intervals as providing a uniform mesh, creating a basis for an approximate accumulation function (approximate integral) to be built. This animation intends to communicate that both x and dx continuously and smoothly vary and that Δx is a parameter (fixed in value), whereas x and dx are variables.

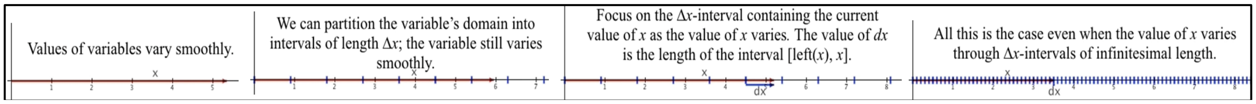

Figure 1: Animation 1, intended to illustrate variation of the independent variable in intervals of fixed width.

After watching the entire animation without pausing, all four students explained that the animation was showing the continuous variation of x's value without mentioning dx or Δx in their responses. The students each focused on describing something about x in their responses; One student said that x was the length of the arrow, another said x was just the end of the arrow, and the other two discussed that the animation was describing that x does not skip any values. The subsequent watching of the animation in four sections (each represented by one image in Figure 1) helped students in precisely explaining what they saw. By doing this rewatching with periodic pausing, all four students demonstrated a strong understanding of the differences between these notations and clear understandings of which values varied (x and dx) and which values stayed constant (Δx). It seemed that deliberate pausing and focused questioning allowed students to better reflect on what they saw and explain the nuances that were presented. This suggests that forced pauses with reflection questions embedded in the animation might be a useful future modification.

Animation 2 - Constant Rate of Change

The second animation centered on constant rate of change. In DIRACC, constant rate of change is central to building rate of change (derivative) and accumulation (integral) functions. The idea is that students approximate accumulation by assuming a constant rate of change within a fixed Δx -interval. This animation was designed to both formalize and build the intended imagery for constant rate of change. The animation was divided into four parts (Figure 2), wherein the first three parts proposed a common but flawed meaning for constant rate of change.

This flawed notion consists of constant rate of change as only involving fixed chunks of change (Panel 1, Figure 2), which is an unproductive way of thinking in calculus (Byerley & Thompson, 2017). Panels 2 and 3 in Figure 2 depict an issue with this chunky definition in order to perturb a common misconception. The animation that takes place in these panels is intended to illustrate how thinking of only *fixed* changes at *ends* of intervals (rather than variation *throughout* intervals) fits with something that is recognizably not a constant rate of change (non-linear). Panel 4 summarizes the issue by explicitly explaining the importance of variation *throughout* the Δx-intervals. The intent of this animation was to communicate that constant rate of change involves relationships between varying changes, dx and dy. The importance placed on variation within an interval is consistent with the higher-level mental actions associated with the covariation framework (Carlson et al., 2002).

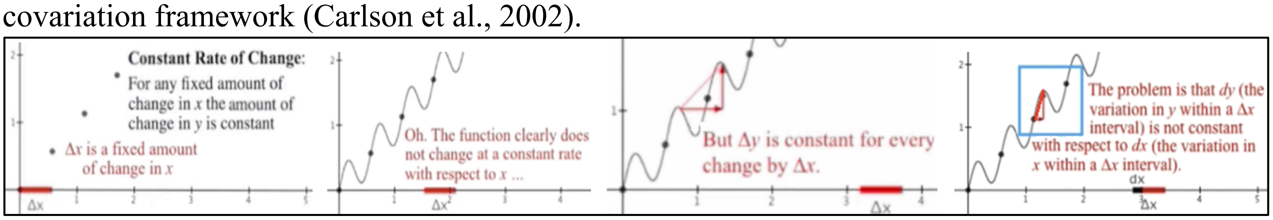

Figure 2. Intended to illustrate problematic "chunky" ways of thinking about constant rate of change.

All four students upon first watch of the second part (Panel 2), agreed that the graph did not have a constant rate of change. One student explained that it was because of the non-linear shape, while the others noted that it fit the proposed definition that was shown. As the animation progressed to the 3rd portion, all four students accepted that the graph does fit the proposed initial flawed definition. However, only two of these students verbally noted that the chunky definition was not sufficient. The other two students did not make this explicit connection, and they both indicated that they thought that the hypotenuse of the right red triangle represented Δy (Figure 2 Panel 4, Figure 3). In Part 4 (Panel 4, Figure 2), all four students verbalized that the proposed definition for constant rate of change was insufficient. However, two of these students did so only after targeted probing from the interviewer. Initially these two students seemed to believe that this animation illustrated a correct definition of constant rate of change. That is, instead of viewing the nonlinear graph as evidence that the proposed chunky notion of constant rate of change was insufficient, they thought the animation was intended to expand their notion of constant rate of change to include nonlinear graphs. This led the textbook authors to modify the animation and its description to more explicitly communicate the intended point.

Figure 3. A student's drawing of how he understood the hypotenuse of a triangle as representing Δy

Discussion and Implications

The results of the interviews not only tell us more about how students understand dynamic imagery in general, but also provide data for the continual modification and refinement of this particular calculus curriculum. This was especially apparent in the constant rate of change animation in which some aspects of the animation caused unintended meanings to be constructed (e.g., the hypotenuse of the red moving triangle being conflated with Δy). As a result, the animation was re-made. The remake eliminated the hypotenuse and included brighter colors and arrows to draw students' attention to the important parts of each animation. It also included an overlaying text clarification that the chunky definition was undesirable.

Additionally, the discussions that occurred with each student seemed to be significant in having the students fully see the intended meanings for each animation. This suggests that when instructors use animations in class, they should pause throughout the animation and hold discussions to focus students on what the animations are presenting. Instructors should be trained to use them as didactic objects where having a purposeful discussion will assist in students constructing the intended meaning. In a non-classroom setting where students could be watching the animations by themselves, the results here suggest that animations should include reflection questions so students could potentially reflect on what the animation was presenting.

To further our work, we present the following questions for our audience to consider:

1. What relevant literature already exists that our team can leverage in order to further our work in student understanding of animations and the role of animations in the classroom?
2. What parts of the data warrant further exploration?
3. What other analyses should we conduct on our data or future data?

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Links between Engagement in Self-Regulation and Performance

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This study investigates how student engagement with a self-regulatory activity is linked to subsequent student performance. College Algebra students were given an activity that was focused on the planning, monitoring, and evaluating aspects of self-regulation within the context of completing the square. Responses on this self-regulatory activity were compared to performance on completing the square problems on exams. We discuss cases of an engaged, a moderately engaged, and a disengaged student and link to exam performance. Further, we share how findings are impacting future revisions of this self-regulation activity.

Keywords: Metacognition, Self-regulation, College Algebra, Reflection, Annotation

Introduction

College Algebra can be a first mathematics course for students pursuing a major in Science, Engineering, Technology, or Mathematics (STEM). Unfortunately, this course has a history for having high failure rates (Wakefield, Champion, Bolkema, & Dailey, 2018), and some researchers have begun to question how well such courses prepare students for calculus (Sadler & Sonnert, 2018). Although prerequisite knowledge plays a role in success in mathematics, study skills, habits, attitudes, and motivation can have an equally strong impact on academic performance (Credé & Kuncel, 2008). Results from research in cognitive science and psychology have, for several decades, indicated that students' metacognitive ability (1) impacts their academic success (2) is malleable, and (3) can be impacted best within specific academic domains (Zimmerman, Moylan, Hudesman, White, & Flugman, 2011). Several studies have shown classroom interventions can improve student performance. However, this has not been explored extensively by mathematics educators or even specifically in the STEM disciplines, with a few exceptions (Cook, Kennedy, & McGuire, 2013; Hodds, Alcock, & Inglis, 2014).

Students often struggle to identify mistakes in their work, why the mistakes exist, and how to correct them. Additionally, they typically do not know how to modify their study habits or methods of learning to address these problems (Zimmerman et al., 2011). Such an awareness would require students to actively engage in monitoring their own learning (i.e. engaging in self-regulation). Defined by Favell as "knowledge and cognition about cognitive phenomena" (1979, p. 906), metacognition is an important facet of self-regulation (Zimmerman, 2000, 2002) and student learning (Flavell, 1976; Krathwohl, 2002; Pintrich, 2002). Whereas metacognition focuses on self-assessment; self-regulation also includes self-monitoring, self-management, and affective awareness (Nilson, 2013).

Closely tied to self-efficacy (Zimmerman 2002; Zimmerman, Bandura, Martinez-Pons, 1992), self-regulation has been shown to improve course performance (Broadbent & Poon, 2015; Caprara et al., 2008; Zimmerman et al., 2011; Zimmerman, 2008) and increase confidence (Labuhn, Zimmerman, & Hasselhorn, 2010). Self-regulation can be taught (Schraw, 1998; Zimmerman, 1998, 2002), however instruction in self-regulation is most effective when taught in the context of a particular class (McGuire, 2015; Tanner, 2012). Within the context of mathematics, educators have long focused on the role of metacognition in problem solving (Carlson & Bloom, 2005; Flavell, 1976; Schoenfeld, 1992). More recent studies have shown the impact of self-explanations in learning mathematics (Hodds et al., 2014; Rittle-Johnson, 2017).

As part of a larger study on metacognition, here we present initial findings on an intervention focused on self-explanations in annotation as part of the self-regulatory learning process. Student responses to these questions provide data to consider the research questions: (1) *What links are there between student engagement with a self-regulatory activity and his/her subsequent performance?* (2) *How can a self-regulatory activity influence students to respond in a way that supports (or impedes) learning?*

Context and Study Population

This is part of a larger study at a large public institution in the Southwest about the impact of metacognitive instruction within College Algebra. A support course for College Algebra students is run each semester for students identified as not satisfying the university's *Mathematics Reasoning Assessment* requirement. The support course is run in small sections of size 30 or less. With voluntary attendance to three weekly class sessions, the support course has traditionally been used to provide additional practice on content and an opportunity for students to ask questions in a smaller setting. In the spring 2019 semester, roughly 40 College Algebra students were identified as needing to enroll in the support course and two sections of approximately 20 students were run. Here we discuss one section in which activities designed to improve students' metacognition were implemented starting after the first exam. The purpose was to support students in their development of self-regulated learning processes. We provided opportunities for students to be reflective and responsive with regard to study habits and strategies.

Methods

Recognizing that self-regulation takes place on a time continuum, before, during and after a learning session, Schraw's (1998) self-regulation model focuses on three types of questions: planning questions, monitoring questions, and evaluating questions. The intervention included these three types of questions on an in-class activity. Planning involved students revisiting their notes on the topic of completing the square and reworking a problem from the notes. As part of this process, students were asked to provide line-by-line explanations of each step (i.e. annotation). The primary monitoring question was: *What did I learn from this problem that could help me in future problems?* Students then answered the evaluating question: *Would I be able to work the example on my own if the solution was covered up?*

After annotating and responding to prompts, students were then asked to create and work through a new problem similar to the one they annotated. Students then were asked questions such as *How can you change a worked example from class to create a new problem for practice?* and *How is the new problem different and what new questions do you have?* Our goal was to solicit responses that were more reflective or structured in a way that would be beneficial to the students later when looking back at their annotations and activity responses.

In the self-regulatory activity (SRA), students used the process of *completing the square* to identify the center of a circle. Completing the square arose on Exam 2, Exam 3, and the Final Exam. However, on Exam 3 and the final, students used the technique to identify the vertex of a quadratic function. Eight student responses from the SRA and corresponding exam questions were analyzed. Exam responses were coded on problems that used completing the square; there was one problem on each of Exam 2, Exam 3, and the final that required coding. Rather than coding for mistakes, we created codes to be applied when a student applied correct steps in the completing the square process. Briefly, codes were assigned based on (1) factoring of the leading coefficient, (2) adding constants (correctly or incorrectly), (3) balancing (correctly or incorrectly), (4) factoring (correctly or incorrectly), and (5) identification of center and radius or vertex. For example, when putting a quadratic function $f(x)$=ax^2+bx+c into vertex form, one would (1) factor out the leading coefficient from the first two terms, (2) add a constant inside the factored quantity to complete the square, (3) add a constant outside of the factored quantity to balance the expression (so it is equivalent to the original), and (4) lastly factor the grouped quantity so that the final result has the form $f(x)$=a(x-h)^2+k. In vertex form, the vertex can be identified as (h,k).

An example of coded student exam work can be found in Figure 1. Students were asked on Exam 3 to put the quadratic function, f(x)=2x^2-6x+1, into vertex form using the completing the square method.

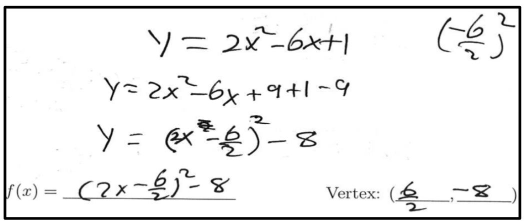

Figure 1. An example of a student putting a quadratic into vertex form on Exam 3 using completing the square.

In Figure 1 the student work shows that the student, while not realizing to factor out the leading coefficient, did know to add a constant (and found the correct one based on their work) as well as balance the expression so that it was equivalent to the original. The student knew to factor, but did not factor correctly (i.e. 2x^2-6x+9 is not equivalent to (2x-6/2)^2. The student did know how to correctly identify the vertex, though it was not the actual vertex of the function. Therefore, the codes applied to this work were (2) adding constants correctly, (3) balanced correctly, (4) factored incorrectly, (5) vertex identification. These codes told us what students knew to or recognized they needed to do even if initial errors impacted the work on the problem.

We examined student work on the SRA as well as corresponding exam problems to note if students' responses to SRA prompts aligned with work demonstrated on the exams as well as their progress (or lack thereof) on the application of the completing the square process.

Results

After coding, we were able to see if each student made improvements, stayed the same, or regressed for each code, which corresponded to a specific step/skill in the problem-solving process. We compared responses on the activity to work on the exams to determine what relationship there might be between what students did on the exam and what they did on the activity. Here we present three cases: (1) a student who filled out and was reflective on the SRA – having no difficulty with the mathematical content being covered, (2) a student who did not annotate and did not reflect on the SRA, (3) a student who annotated and reflected on the SRA and was aware of content issues. The three cases were chosen because we identified a connection between what students wrote down on their completing the square activity and their progress (or lack thereof) on corresponding exam problems. Although students not presented here did have interesting reflections and annotations, the link between what they wrote on the activity and their exam work was not as evident.

For our three cases we provide a description of what students wrote on the completing the square question on Exams 2 and 3 and a summary of responses for each student on the activity.

1. Student 1
    a. Work on Exam 2: Student 1 knew to factor out the leading coefficient, knew to add/subtract constants and correctly balanced the circle equation, was able to correctly factored to put into standard circle form, and could correctly identify the center and radius of the circle.
    b. On the SRA, Student 1 stated that they would be able to work problems similar to that in the SRA. Additionally, Student 1 created a new problem and correctly worked through it.
    c. Work on Exam 3: Student 1 knew to factor out the leading coefficient and to add/subtract constants. Although, Student 1 did not correctly balance the quadratic function, they were able to factor their expression correctly into vertex form, and correctly identify the vertex of the parabola.
2. Student 2
    a. Work on Exam 2: Student 2 performed as Student 1 did on Exam 2.
    b. Student 2 did not engage in the SRA.
    c. Work on Exam 3: Student 2 did not factor out the leading coefficient, but did know to add/subtract numbers and balance the expression. Although Student 2 did not correctly factor, they were able to identify the vertex of the parabola based on their final result (Figure 1).
3. Student 3
    a. Work on Exam 2: Student 3 did not factor out leading coefficient, but did know to add/subtract numbers and could correctly balance (even though constants were not correct). Student 3 was able to put into standard circle form, but was unable to correctly identify the center and radius.
    b. On the SRA, Student 3 provided line-by-line annotation that would apply to solving other completing the square problems (Figure 2). On the SRA, Student 3 stated that they would be able to work a similar problem, but might get stuck on factoring. Student 3 created a new problem, and got stuck on identifying the radius of the circle.

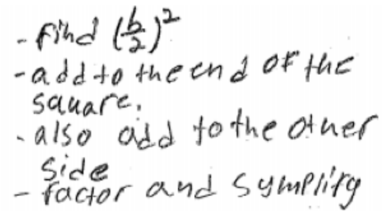

Figure 2. Line-by-line annotation from Student 3.

    c. Work on Exam 3: Student 3 was able to factor out the leading coefficient and able to add/subtract the correct constants. Student 3 did have initial issues with correctly balancing, but was able to correctly factor. Based on the factored form, Student 3 was able to correctly identify the vertex of the parabola.

Student 1 was reflective and engaged in the SRA. In addition, Student 1 performed well on related exam questions. In looking beyond the completing the square topic, we found that Student 1 did well throughout the semester on all exams and activities and attended over 70% of class sessions. We wondered if Student 1 had pre-established self-regulating behaviors that he was building upon as he progressed through the course.

In contrast, Student 2 was not as engaged in the SRA. While student 2 did well on completing the square on Exam 2, this did not carry through to Exam 3. While both exams had questions on completing the square, the contexts were different (circle versus quadratic function). One possibility is that Student 2 assumed that they had the skills to be successful on the problem, when, in fact, they did not, and the lack of reflection and self-regulatory practice prevented them from identifying such a gap in their knowledge.

Student 3, on the other hand had some content issues, but engaged with the SRA and was reflective. As a result, issues that Student 3 anticipated would be a problem for him (e.g. factoring) were, in fact, not on Exam 3.

Future Directions and Implications

While we should not be overly surprised, we learned that there were things we assumed students would know how to do but, in fact, they did not. For example, when students were asked to annotate a problem, with the hope that this would enable them to use that annotation later to work future problems, the quality of annotation ranged from being overly broad to having detail that was too specific to one particular problem. For example, one student wrote "complete the square (b/2)" as part of their annotation, which is a restatement of the question. On the other hand, if annotation is too specific it can be hard to translate to new problems. For example, another student wrote "divide $4x$ by 2". This level of detail is in the context of a specific problem with specific numbers, which may not be present in a new problem. Both of these extremes are ultimately not very useful as a study tool. However, annotation like that in Figure 2, while not perfect, it is an attempt to explain the steps in such a way that would translate to other problems with new coefficients and thus be a useful study tool. In addition to giving consideration on skills that may need to be taught, we plan to revise the activity to illicit deeper responses from students. We had hoped that students would elaborate on questions, but they often provided yes/no responses, which was due to the wording of the questions.

Questions for the Audience

As our project has only undergone a pilot stage and we are continuing to collect data from incoming students enrolled in College Algebra and the support course, we welcome input from the audience. In particular, we pose the following questions for discussion:

1. Feeling pressed for time, the instructor struggled to balance time for metacognitive activities with providing more content practice to students. While we have made adjustments to more deeply connect these pieces, we are wondering how to best equip instructors to administer these types of activities. While Professional Development is an obvious piece to include, this is not always easy to implement with non-tenure-track faculty, as this population often does not have the necessary supports to engage in PD. Given the potential constraint of not being able to provide regular PD to instructors, *What suggestions do you have to support instructors to administer metacognitive activities?*
2. *How could these activities be folded into the "regular" classes? What would this look like? How would one get buy-in from students in a larger class setting?*
3. Motivation has been an issue. Part of this is due to the support course not having mandatory attendance. However, expectancy-value theory points to (1) showing value and (2) convincing students that they can do it (confidence) to get students motivated to engage in these things. (Wigfield & Eccles, 2000). It is difficult to assess the impact of our activities if students are not using them due to lack of motivation. *What are your thoughts on how to control for motivation when trying to assess performance on exams?*

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Developing a Framework for the Facilitation of Online Working Groups to Support Instructional Change

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In this preliminary report we discuss the development of a framework regarding the facilitation of online working groups geared at supporting instructional change at the undergraduate level. The research in undergraduate mathematics education includes large-scale projects aimed to support individuals, departments, and the mathematics community in reforming their instruction to align with recommendations from professional organizations and align with existing mathematics education research standards. One avenue that needs attention is the use of online synchronous environments to match faculty across the world and form collaborations to support the inclusion of student-centered activities in our mathematics classrooms. An important part of that work is to understand how to facilitate those online synchronous environments. This preliminary report discusses the actions that facilitators take in these environments and lays the groundwork for the use of this framework in our and other contexts going forward.

Keywords: faculty collaboration, facilitation, professional development

Introduction and Relevant Literature

Instructional shifts towards student-centered pedagogies are taking place throughout the United States within mathematics departments. This change is oftentimes centered around individual faculty (e.g., Fortune & Keene, 2019; Speer & Wagner, 2009) but also from the perspective of larger groups of faculty (e.g., Hall, Fortune, & Keene, under review; Hayward & Laursen, 2016) or even departments at-large (e.g., Apkarian & Reinholz, 2019; Laursen, 2016; Reinholz & Apkarian, 2018). Notably, professional communities also call for this instructional reform (Mathematical Association of America [MAA], 2015).

The research in undergraduate mathematics education (RUME) community has embarked upon numerous large-scale research projects to investigate how to support instructional change (e.g., Hall et al., under review; Kuster et al., 2016), namely to make instruction more student-centered. Additionally, the RUME community has engaged in large scale projects to support departments in improving instruction and student outcomes (e.g., Association of Public & Land-Grant Universities [APLU], 2016). In some cases, faculty may feel isolated if their members of their departments do not support them reforming their instruction. Consequently, an avenue to support those mathematicians is forming online communities for like-minded faculty focused on instructional change to collaborate on new instructional materials, new instructional techniques, and to be a general supportive community.

Our multi-institute collaborative grant, TIMES, is one such project in which we aimed to not only support mathematicians with various support models but to research those support models' impact on the mathematicians and their communities. Our support model consisted of instructional materials (both for the student and faculty), a summer workshop, and an online working group (OWG). The OWG offered an opportunity for faculty to collaborate on their instruction through a lesson study model (Demir, Czerniak, & Hart, 2013) in online synchronous environments. In this OWG, participants engaged in lesson studies on multiple units of Inquiry-Oriented (IO) materials (Rasmussen & Kwon, 2007) by doing the mathematical tasks from those units, anticipating student thinking that could arise from those units, filming and then subsequently bringing video clips from that instruction to the OWG to share and discuss.

In previous research, we have discussed the development and usage of a framework to categorize and understand the conversation that occurs when OWGs are discussing the sharing of instructional video as a means to support their instructional change (Hall et al., under review). However, the next step that emerged from that work was to analyze the role that the facilitator played in that OWG. Given the importance of the role of a facilitator in professional development settings (van Es, Tunney, Goldsmith, & Seago, 2014), our next research steps were to develop a framework to categorize and understand how facilitation occurs of these OWGs. In this preliminary work, we sought to understand the facilitation of OWGs when facilitators initiate discussions about the mathematical content of novel IO curricular materials. In this preliminary report, we will discuss the development of a framework to understand the facilitation of these OWGs, with the intention of future publications to be a proof of concept of using the framework in our context. The research question for this preliminary report is: *What actions do facilitators take within online working groups focused on doing and understanding the mathematical content of novel IO curricular materials?*

Relevant Literature

Our work was influenced by work from van Es and colleagues (2014). In their work they focused on developing a framework on *how* facilitators could use in-the-moment moves to support productive discussion while viewing video of instruction (van Es, Tunney, Goldsmith, & Seago, 2014). We first note that their context is different. Namely, their context was fourth and fifth grade teachers watching video of instruction to discuss student thinking with a facilitator, whereas ours is university mathematicians discussing, with a facilitator, novel IO curricular materials to prepare to implement those materials in their classrooms.

In their final framework, van Es et al. ended with facilitation moves broken into four categories: "orienting group to the video analysis task, sustaining an inquiry stance, maintaining a focus on the video and the mathematics, and supporting group collaboration" (van Es et al., 2014, p. 347). Each of those categories had several moves. For example, within sustaining an inquiry stance, they found their facilitators highlighted noteworthy student ideas, lifted up a participant idea, pressed to explain more or elaborate, offer an explanation of their own, provide a counter perspective, or clarify to ensure the group was on the same page.

Methods

Research Setting and Data Collection

Data from this preliminary report comes from a large NSF funded project, TIMES. TIMES recruited mathematicians in 2015-2017 who were interested in changing their instruction to be more student-centered and specifically use one of the IO curricula: differential equations, linear algebra, or abstract algebra. During the first year of the project, the three Principal Investigators led their respective OWG. In subsequent years, the project team was able to double the number of OWGs that could be facilitated by recruiting the previous year's participants to lead their own OWG. Consequently, in 2016 and 2017, 4 facilitators, who were previously participants, lead their own OWG. The development of our framework comes from these 4 individuals' OWGs. Each OWG was screen recorded using QuickTime and all OWGs were transcribed. Each of these

OWGs consisted of 3-4 participants. As this analysis focuses on when the facilitators were leading discussion on doing the mathematics from the novel IO curricula, this yielded 14 transcripts for analysis (3 for two facilitators, and 4 for two facilitators). Two of the facilitators lead an OWG on differential equations; whereas one facilitator lead an OWG on linear algebra and one for abstract algebra.

Data Analysis

The creation of the framework followed an iterative process of revision and refinement via individual open coding and comparison between the researchers (Creswell & Poth, 2017). Altogether, 14 transcriptions of videos were investigated, coded and compared by at least two researchers in each iteration.

During the first iteration, the first step consisted of us analyzing two video transcriptions and proposing descriptors for the action that the facilitator took. We then convened and compared our suggestions for each of the corresponding facilitator's actions, by grouping similar descriptions in one category and assigning that category a code. For instance, the expressions *chose participant to start, called on participant* and *called on a participant to share their thoughts* were grouped under *asked participant to share their mathematical work* and assigned the code SHARE; *brought experience from the classroom to the conversation, related it back to what students would do* and *tried to make sense of why students have made mistakes in the past* were coded as PAST to indicate that the facilitator reported on what students have done in the past. This process generated a first draft of the codebook, reproduced in Table 1.

Table 1. First draft of codebook.

Code	**Descriptor**	**Code**	**Descriptor**
INIT	Initiated mathematical conversation	**GOAL**	Described learning goals
TRANS	Transitioned to a different mathematical task/subtask	**COMM**	Asked a participant to comment on the discussion
INDIV	Gave individual think time	**ANTI**	Anticipated what students might do
TIME	Gave timing information about lesson	**PAST**	Reported on what students have done in the past
SHARE	Asked participant to share their mathematical work	**DIFFIC**	Acknowledged mathematical difficulty
VOLN	Asked for volunteer to share their mathematical work	**MATH**	Discussed mathematical content
ELAB	Asked a participant to elaborate on something they've shared	**ACKN**	Acknowledged participant's mathematics
JOKE	Made a joke	**RESTATE**	Restated participant's contribution

Following this step, the remaining twelve video transcriptions were assigned to two researchers each. Every pair individually coded their assigned portions and then came together to compare their results and agree on one code per statement. Then, the three coders convened to discuss the overall results. We saw the need to distinguish between *what* the facilitator is claiming or asking, and *how* they were doing it. In particular, we focused on the type of statement that was being made (imperative, interrogative, exclamatory, and declarative).

This led to the second iteration of coding for the same initial two transcripts, where each statement was assigned a *what* and *how* code. We subsequently reconvened to compare individual results. Additional codes were suggested, initial ones were redacted and eventually we noticed a commonality between some codes which allowed us to create the elements of the framework, the *facilitation* and *conversation* themes. Within the facilitation theme, we generated five categories and two actions that pertain to each category. Additionally, we realized that only imperative and interrogative statements were meaningful in certain actions that the facilitator made under the *gathering* and *verifying* categories. More details on the framework are provided in the next section.

Facilitation Framework

Figure 1 is the framework for facilitation of online working groups. The framework contains two overarching elements: facilitation and conversation. Facilitation is the element of the framework that would transcend the context of the OWG; whereas conversation is specific to our context. That is, those conversations emerged from our previous research and this analysis and would be different if this framework was applied in different contexts.

Framework Elements	Categories	Action	Description
Facilitation	Progressing Session	Transition	Transitioning to a new or different mathematical task/subtask
		Think	Giving individual think time
	Gathering**	Individual	Asking for a specific individual to share or contribute
		Open	Opening the floor to the group for a response
	Verifying**	Elaborate	Asking for an elaboration or clarification from an individual
		Restate	Inquiring about a participant's contribution by restating that contribution
	Contributing	Provide	Providing input, advice, or insight
		Restate	Restating a participant contribution
	Supporting the Group	Encourage	Encouraging statement
		Acknowledge	Acknowledging or thanking participant(s) for contribution(s)
Conversation	Content	Pedagogy	Teacher instructional moves or ideas about teacher instructional moves
		Goal	Learning goals of task/subtask/lesson
		Time	Timing information about lesson
		Past	Previous students' mathematical reasoning and/or work
		Anticipate	Anticipation of students' mathematical reasoning and/or work
		Mathematics	Mathematical content or mathematical technology

	SubCode	Description
**Subcode for Gathering and Verifying	Imperative	Makes a request or gives a command of some sort
	Interrogative	Asks a question, inquires for more information

Figure 1. Framework for Facilitation of Online Working Groups.

Here, we focus on the categories within the facilitation element of the framework. There are five categories of actions that our facilitators did in online working groups. Namely, they Progressed the session, Gathered information, Verified information, Contributed their own thoughts to the session, or Supported the group. Each category yielded two unique actions, or codes. For example, under the Gathering category, we find two actions: Individual and Open. Individual was a code used to describe when the facilitator was asking a specific individual to share their thoughts whereas Open was used to code for when a facilitator asked for any volunteer to share their thoughts. While both actions concern gathering information, they are clearly two distinct actions a facilitator can take during an OWG. It is worth noting that Restate is an action under the Verifying and Contributing categories as a facilitator would Restate for different purposes. For example, a facilitator would Restate a participant claim with the (implied) intention being to inquire about a participant's contribution. That is, they would Restate what they said for the purposes of having that participant expound on what that participant had just said. Whereas, a facilitator would also Restate a contribution, potentially in paraphrased ways, as a means to contribute to the conversation with no (implied) intention of getting a response from the original commenter of that statement.

Another important aspect of the framework to note is the inclusion of the subcodes for Gathering and Verifying. All four of the actions under those two categories are about the facilitator doing something that desired a response from someone, whether that be a specific Individual, Opening the floor to a question, asking for an Elaboration, or Restating for the purposes of further explanation. However, in our analyses, it became clear that there were always two different ways to achieve those goals. For instance, a facilitator would call upon a specific Individual to share or contribute by making a request or giving an Imperative command (e.g., "Participant, tell me what you were thinking about.") This would contrast with the same action, Individual, but could have been asked in the form of an Interrogative question: "Participant, what mathematical theorem led you to that conclusion?" We treated instances such as these as both falling under the action of asking a specific Individual to contribute, but the means the facilitator went about that were different. This was the case for the Gathering and Verifying codes so for all of those codes they received the subcode of either Imperative or Interrogative.

Some coded examples of this are:

Facilitator - Imperative: So keep going with that [line of thought] Participant.

Facilitator - Interrogative: So, I, we are talking about … the Sudoku property and each symmetry appears at most once and each symmetry appears at least once. Is the hint here, so what are students going to approach and how are they gonna approach this question?

In our future work where we will use the framework as a proof of concept, we imagine that different facilitators may lend themselves more to Imperative or Interrogative approaches; therefore this was an important difference for us to be able to capture.

Importantly, there is no hierarchy in the categories and actions. All of these emerged from our data set in various ways and the way in which we chose to order the categories in our framework is not indicative that those actions would happen in that order in the OWGs. Additionally, there are parallels between our framework in our context and the one from van Es and colleagues. For instance, offering an explanation from van Es et al. (2014) and Providing from our work offer a similar facilitator action. Similarly, pressed to explain more from van Es et al. and Elaborate from our work offer a similar facilitator action. This is noteworthy because it highlights that even in different contexts, similar facilitator actions still emerged.

Future Analyses and Questions for the Audience

This preliminary report has discussed our work in *developing* a framework to understand how facilitation of online working groups occurs. Our next steps are to use this framework to officially code our data. While our data were used to develop the framework, one final coding of all transcripts with the final framework is necessary to ultimately be able to describe how facilitation occurred *in our context.* In doing so, we will be able to discuss similarities and differences in different facilitators. Moreover, this can be used to show the variety of facilitation techniques that exist and how one can train to be a facilitator of online working groups; thus growing a network of online working groups geared at supporting instructional change.

Our questions for the audience are:

1. Are there other frameworks we should look for to help inform our research methods?
2. Are there components, actions, or aspects of our framework that are missing?

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For Women in Lecture, How They Feel Matters – A Lot

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As support for non-lecture pedagogy grows, the number of instructors implementing active-learning approaches in their undergraduate mathematics classrooms is steadily increasing. The proliferation of such techniques has led to the inevitable investigation into the potentially differential effects for various sub-populations and equity therein. In this study, the primary objective was to link specific affective factors to student performance with an eye to variations by gender and/or pedagogical approach. Results suggest that different forms of instruction impact men and women in different ways; specifically, affective reports are better predictors for women's scores on a content assessment than for men and this effect is more salient in lecture classes as compared with active-learning environments. Furthermore, there is evidence that which specific affective traits influence performance also varies by subgroup.

Keywords: Active Learning, Affective Traits, Gender Differences

There has been a growing consensus among STEM educational researchers that active learning is better for student success than more traditional lecture modes of instruction (see Freeman et al., 2014). Such research has contributed to broad sweeping reform efforts to implement active-learning approaches. While this can, and should, be viewed as positive change, one must not interpret this research to imply that a single intervention should be applied under *all* circumstances and that it will be equally beneficial for *all* students. Singer and colleagues (2012) called for the field to explore ways in which particular instructional approaches impact student sub-populations. Similarly, Eddy and Hogan (2014) argued that any classroom intervention would impact different groups of students in different ways.

The analysis presented here was motivated by two specific recent findings that draw from the same large data set we again analyze here. Johnson et al. (in press) observed that in an active learning instructional intervention, learning outcomes improved for men while no such gains were seen for women. A follow-up analysis by Mullins et al. (2019), investigating the relationship between affective and cognitive gains, reported that a) while gender differences were not observed with respect to performance, the relationship between affective traits and performance was significantly different and more highly influential for females; and b) there was a stronger relationship between affective gains and student performance in active-learning classrooms than in lecture classes. Considered collectively, these results suggest the presence of an interaction effect whereby the relationships between affective traits and performance, as well as the learning gains achieved with non-lecture pedagogy, must be considered separately by gender in each learning environment. Thus, the present study seeks to further our understanding of how instruction is experienced differently by different subpopulations of students.

Literature Review

In order to better understand how different instructional approaches may lead to differential learning outcomes, we decided to focus on the relationship between affective learning gains and cognitive outcomes (as measured by a content assessment). This decision was informed by numerous studies that have revealed gender differences in the ways students process their mathematical achievement and in affective outcomes. For instance, research has shown that even when boys/men and girls/women have similar levels of achievement, girls report lower competence beliefs and more anxiety (Eccles et al., 1985; Kloosterman, 1990; Seegers & Boekaerts, 1996; Stipek & Gralinski, 1991). Additionally, several studies have demonstrated that boys ascribe their successes to ability and their failures to bad luck or lack of help (Hackett & Betz, 1992; Randhawa, Beamer, Lundberg 1993); whereas, girls attribute their success or failure to self-determination, receiving help from others, or being provided with easy tasks (Stipek & Gralinski, 1991). Additionally, studies indicate a decline in positive attitudes toward mathematics has more of an effect on boys than on girls (e.g., Eccles et al., 1985; Fennema & Sherman, 1977).

Based on these studies, we have reason to believe that affective reports vary by gender – and that the strength of the relationships between affective reports and cognitive outcomes my also vary by gender. However, these studies do not explore the extent to which different instructional formats may mitigate these relationships. The purpose of this research is to examine the relationship between affective factors and student performance, while taking into account the mode of instruction. Specifically, we address the following research questions: *Is there a relationship between (specific) affective factors and student performance?* and *Is this relationship consistent across gender and instructional style?*

Data Sources and Methods of Analysis

This quantitative study uses a relational design to investigate the relationship between self-reported affective gains and corresponding performance on a content assessment. Ordinary least squares multiple regression models were constructed for each of four sub-populations and compared with respect to the most salient affective predictors therein.

Participants

Participants for this study were undergraduate students enrolled in a differential equations (DE) course between the Fall 2016 and Spring 2017 semesters. The total usable[1] sample size was 324 students distributed across 16 instructors representing 12 institutions nationwide. Our participants were recruited indirectly via their instructors who had agreed to participate in an active-learning research project as either treatment or control (i.e., lecture) subjects (See Johnson et al., in press, for more details). For the purposes of this analysis, students were organized into four subgroups: Men in Active Learning (M-AL; n = 150), Women in Active Learning (W-AL; n = 67), Men in Lecture (ML; n = 74), and Women in Lecture (WL; n = 33).

Measures

Two sources of data were collected from each participant to represent the affective and cognitive domains. All participants, regardless of treatment/control status of their instructor, utilized the same measures and all were administered at the end of the course. The SALG-M (Laursen et al., 2014) is an adaptation for mathematics classes of the *Student Assessment of their Learning Gains* (SALG) instrument that was developed for use in chemistry courses so that

[1] The original data had n = 330 students, 4 of whom did not provide binary gender information. A class of 2 students was excluded as this arrangement was judged to be anomalistically small and thus inappropriate to measure the implementation of any particular pedagogical technique.

students could report on their own learning and the specific aspects of the course that contributed to that learning (Seymour et al., 2000). Here present data on a subset of 13 of the SALG-M items, collectively hereafter referred to as *Affective*[2] factors, most relevant to our response variable. Each item was presented using a 5-point Likert-scale[3] format to measure a student's reported gain over the duration of the course;

The cognitive data for this study came via content assessment scores using an instrument developed to be a common assessment for instructors utilizing an active-learning approach to teaching differential equations (DE) (Hall et al., 2016). Consisting of 15 multiple choice items designed to evaluate students' conceptual understanding of DE, the assessment included first and second order differential equations and linear systems of differential equations. Raw scores (out of 15) were converted to percent scores before modeling.

Analysis

Given the nested structure of our data, we first evaluated the appropriateness of multi-level modeling approaches prior to performing any inferential procedures. Based on the advice of Maas & Hox (2005) and the recommendation of Vajargah & Masoomehnikbakht (2015), we concluded that we neither met the minimum sample size requirement nor egregiously violated the ICC threshold to justify use of a multi-level modeling approach. Thus, we opted to aggregate all Level-1 units according to subgroup classification (i.e., W-AL, M-AL, ML, WL) and analyze the data using an ordinary least-squares approach. Summary and descriptive statistics were computed for each of 13 predictor variables (A_i) and the response (*CAscore*). Independent sample *t*-tests were used for investigating subgroup differences. A backward elimination model-building approach generated four separate multiple linear regression models; in each model, the response variable was (*CAscore*) and the predictors were some subset of the original thirteen affective items (A_i). The removal criterion was set to the 0.10 threshold and all analysis was performed using IBM SPSS v.25.

Results

Summary statistics can be seen in Table 1 for each of the predictor variables. From this information, we can see that generally students reported at least moderate learning gains in every category, on average, and that the mean responses fluctuate modestly by subgroup. Generally, students in lecture tended to report greater gains than their active learning peers. Looking within instructional type, we see that generally men reported greater gains in active learning environments whereas this trend was reversed in lecture environments.

Summary statistics for the response variable (*CAscore*) by subgroup indicate the overall mean to be 52.9473 (SD = 16.4593) and, in decreasing order, M-AL (55.0663), W-AL (54.8507), WL (51.2109), ML(47.7031). Overall, we found a significant difference when we compared the average score of a student in active learning, 54.9998, to the average student in lecture, 48.785 ($t = 3.243$; $df = 322$; $p < .001$). However, this difference was not stable when we further grouped the students by gender. Both W-AL and M-AL significantly outperformed ML (mean diff. = 7.14764; $p = .047$ and mean diff. = 7.36323; $p = .008$, respectively). However, there was no significant difference between W-AL and WL (mean diff. = 3.6398; $t = 1.099$; $p = .274$).

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[2] We are broadly characterizing all items as "Affective" despite Laursen's labels to avoid confusion between her/our use of "cognitive" in the context of this paper and for simplicity as we are considering each item as a singleton and not as a constituent of a composite factor.

[3] 0 = *No gain*, 1 = *A little gain*, 2 = *Moderate gain*, 3 = *Good gain*, 4 = *Great gain*; there was also a N/A option.

Table 1. Summary of Average[1] Self-Reported Affective Gains by Subgroup by Item

Affective Trait Survey Item	Subgroup			
	M-AL	W-AL	ML	WL
A1: confidence that you can do mathematics	2.52	2.42	2.97	3.06
A2: comfort in working with complex mathematical ideas	2.54	2.39	2.95	2.82
A3: development of a positive attitude about learning mathematics	2.53	2.47	2.75	2.88
A4: ability to work on your own	2.69	2.63	3.00	3.12
A5: ability to organize your work and time	2.38	2.48	2.71	2.97
A6: appreciation of mathematical thinking	2.57	2.6	2.77	2.88
A7: comfort in communicating about mathematics	2.61	2.6	2.78	2.72
A8: confidence you will remember what you learned in this class	2.41	2.35	2.43	2.47
A9: persistence in solving problems	2.75	2.63	2.81	3.03
A10: willingness to seek help from others	2.89	2.84	2.77	2.97
A11: ability to work well with others	2.92	3.00	2.89	3.03
A12: appreciation of different perspectives	2.9	2.87	2.69	2.91
A13: ability to stretch your own mathematical capacity	2.73	2.67	3.08	2.97

[1]Average = the mean of the values on the Likert scale; might have nonsensical practical interpretation

With respect to RQ1, *Is there a relationship between (specific) affective factors and student performance?*, we found evidence in the affirmative. For each subgroup, we were able to build a statistically significant model for *CAscore* using a subset of the affective factors. The specific factors that were influential predictors of this response did vary by instructional approach and gender; thus, with respect to RQ2, *Is this relationship consistent across gender and instructional style?*, we conclude that we have evidence to suggest otherwise.

Table 2. Summary of Regression Models by Subgroup

	R-sq[1]	F	Sig.	Terms	β	Std. β	t	Sig.
WL (df = 31)	0.468	10.09	<.001	Intercept	33.462			
				A8	10.029	0.786	4.38	<.001
				A11	-7.916	-0.6	-3.05	0.005
				A13	5.405	0.377	2.15	0.04
W-AL (df = 64)	0.168	7.44	0.001	Intercept	38.948			
				A3	3.311	0.267	1.94	0.057
				A8	3.488	0.231	1.69	0.097
ML (df = 69)	0.068	3.53	0.035	Intercept	47.321			
				A7	4.956	0.345	2.34	0.022
				A12	-4.789	-0.36	-2.44	0.018
M-AL (df = 145)	0.068	6.28	0.002	Intercept	51.188			
				A7	4.61	0.339	3.54	0.001
				A11	-2.735	-0.17	-1.81	0.073

[1]Adjusted R-squared used to facilitate comparison across models with varying numbers of terms

Looking at the WL subgroup[4] (See Table 2 for details of all four models), we see that the model indicates that gains in the following three affective traits are significant predictors of the response: A8 – *Confidence that you will remember what you have learned in this class*, A11 – *Ability to work well with others*, and A13 – *Ability to stretch your own mathematical capacity*. The model suggests that, on average, a 1-unit increase in A8 will result in a 10 point[5] increase in content scores, a 1-unit increase in A11 will result in a 7.9 point <u>decrease</u> in content scores, and a 1-unit increase in A13 will result in a 5.4 point increase in content scores – when all other factors are held constant. The coefficient of determination (R-square adjusted) indicates that 46.8% of the variance in *CAscore* can be explained by fluctuations in these three predictor variables.

Discussion

When considering the first research question, *Is there a relationship between (specific) affective factors and student performance?*, we found this to be true; however, the strength of that relationship varied across subgroups. Affective factors are much better predictors for women's content performance than they are for men. This is especially true for women in lecture classes where affective items were able to explain nearly 50% of the variance in content assessment scores (as compared with only 16% for women in active learning classes). In contrast, in both lecture and active learning environments, affective factors were only able to account for approximately 7% of the variance in men's content assessment scores.

When considering the second research question, *Is this relationship consistent across gender and instructional style?*, the answer is a bit complicated. Globally, the four subgroups did produce four distinct sets of predictors. Looking within instructional style, we see that men and women have no common affective predictors in either learning environment. These results support the notion of an *inconsistent* relationship between affective factors and student performance; however, there were enough similarities to warrant further discussion.

In conclusion, broadly speaking, our results support the argument (Eddy & Hogan, 2014) that any classroom intervention will impact different groups of students in different ways. Something about the active learning environment made affective factors matter less for women – meaning there was an opportunity for other factors to matter more. This is especially curious as no significant performance differences between these groups (i.e. WL vs. W-AL) were observed. A corollary to this finding is that affective factors were equally insignificant for men in both lecture and active learning classes despite there being a performance difference in favor of the active learning environment. Because affective factors matter so much for women, further research is needed to understand which specific classroom interactions support affective growth and why this appears to manifest itself disproportionately in different learning environments.

Questions for the audience:

- Are these results strong enough to pursue further publication?
- How can we make sense of the finding that the strength between affective reports and cognitive outcomes decreases for women in these active learning classes?

[4] For brevity, the model output of the other three subgroups will not be described in detail but are interpreted in a similar fashion in accordance with the details displayed in Table 2.
[5] Raw scores (out of 15) were converted to percentages, so "point increases" are in terms of % here.

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Assessing the Disciplinary Perspectives of Introductory Statistics Students

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One purpose of introductory statistics courses is to acquaint students with important disciplinary themes, including the nature of a statistical investigation and the types of activities that data analysts engage in. In this paper, we discuss the development of a survey instrument intended to assess the disciplinary perspectives of introductory students. Building from disciplinary profiles proposed in our previous work with social science students taking statistics, combined with current discussions in the literature, we have constructed a four-dimension framework to represent various perspectives we believe students may hold. Through a series of think-aloud interviews with eight, first-semester college freshmen, we focused on establishing the construct validity for items designed to assess students' views about statistics. This paper documents our methods, highlights our interview findings, and outlines our next steps for validation.

Keywords: Statistics Education, Beliefs, Survey Development, Grounded Theory

Introduction

Research in mathematics and science education has revealed that students' beliefs about the nature of these disciplines and the activities of experts in these domains is intertwined with how students learn and engage with these subjects (Bell & Linn, 2002; Leder, Pehkonen, & Torner, 2003). Students gather their disciplinary conceptions from the types of tasks they complete in their schooling, the skills and practices they are tested on, and even the disciplinary images they gather from the media (Bell & Linn, 2002). Recognizing the disciplinary perspectives students hold and making active attempts to expand these horizons may open opportunities for students to further their conceptual understanding and engage more productively in discipline-based problem-solving scenarios (De Corte, Op't Eynde, & Verschaffel, 2002).

Recent research has begun to address students' attitudes and beliefs about statistics (e.g., Gal, Ginsburg, & Schau, 1997; Muis, Franco, & Gierus, 2011; Rolka & Bulmer, 2005), but much work remains to be done to more validly measure students' statistical perspectives. As a relatively young and dynamically evolving discipline, statistics may stir several different images for new students. In previous decades, statistics was often viewed as a sub-discipline of mathematics where computational fluency was a prerequisite for success (Cobb, 2007). This more mathematically-driven perspective aligns with the structure of traditional statistical methods that depend on probability-based models for determining likelihood under certain assumptions (Lindley, 2000). But statisticians have long realized that doing statistics well requires more than adept use of formulas—it also involves careful investigation, curiosity, and contextual sensitivity to derive meaningful insights from data (Wild, Utts, & Horton, 2018).

Innovations in computing technology and the changing needs of a data-saturated society have also contributed perspective shifts regarding what purpose statistics serves and the kinds of knowledge and skills students need in decades to come (Gould, 2010; Wild, 2017). Wild (2017) sees the revolution of technology prompting a deemphasized role for mechanical skills (e.g., how to calculate a sample statistic) and a reemphasized role for interpretation and inquiry. The emergence of "big data" and the abundance of tools for visualizing and analyzing data have also prompted discussions regarding how the field must be inclusive of multiple sub-domains under the umbrella of data science (American Statistical Association, 2015).

Prospective learners of statistics entering this dizzying field may begin their college statistics courses with substantively different disciplinary conceptions. Understanding how new students perceive of the discipline of statistics may reveal insight on both their curricular expectations and their readiness for careers in statistics and data science. This paper discusses the development of an instrument to assess students' perspectives about the purpose of statistics, the process of doing statistics, and the characteristics of a statistical expert. We first discuss our conceptual framework for this instrument, including the four disciplinary perspectives we are aiming to represent in our survey items. We then discuss our initial attempts to establish construct validity through a sequence of pilot interviews. We conclude with a discussion of how we plan to seek further instrument validation through additional interviews and large-survey data collection.

Conceptual Framework

In his research with social science students taking an introductory-level statistics course, Berens (2019) reported how various focus groups interacted with the following questions: a) What is statistics for you and how would you describe or define it, b) What characteristics should a person have to be good at statistics, and c) What does a person have to do to be successful in statistics. Based on the participants' responses to these questions, Berens (2019) identified four different types of beliefs that the students exemplified in their responses. The first were *static beliefs*, characterized by views that statistics was essentially a set of rules and formulas that dealt with facts and objectivity. The second were *descriptive beliefs*, representing the discipline's attempts to represent and communicate information about the world. The third were *testing beliefs,* which concerned the discipline's attempts to validate or falsify theories and claims. The fourth were *exploring beliefs*, which included more investigative perspectives of statistics as gathering insights and examining relationships.

To refine these perspectives further, we also looked to discussions in mathematics education regarding students' conceptions of mathematics. In particular, Törner and Grigutsch's (1994) description of the toolbox aspect aligns closely with Berens' (2019) description of static beliefs in statistics. Törner and Grigutsch also describe a process aspect of mathematics in which participants view mathematics as a space for constructing new rules and formulas. We see a clear connection between this aspect and the exploring beliefs component described above. We also see additional connections in the statistics community between the general dichotomy of statistics as a formal body of knowledge that values precise language and careful following of procedures and methodology versus statistics as an art to investigate data and ask questions (De Veaux & Velleman, 2008; Wild et al., 2018). With these in mind, we see these four sets of beliefs from Berens (2019) forming a particular structure, as shown in Figure 1.

The Investigative Perspective Exploring data with the goal of generating questions and gathering insights	**The Confirmation Perspective** Testing theories and claims using formal methodology in order to come to a conclusion
The Descriptive Perspective Reporting summaries and representations of data in order to share information clearly	**The Rules-based Perspective** Following steps and procedures in order to find correct answers

Figure 1. Proposed Disciplinary Perspectives

The organization of these perspectives is intentional to reflect commonalities in each column and row. The left two panels concern more applied, contextually-driven perspectives, while the right two panels concern the use of more formal, theory-based methods. We view the top two panels as conclusion-oriented perspectives, while the bottom two panels give greater emphasis to ensuring precise use of procedures to generate correct answers, but not necessarily conclusions.

Methods

Data Collection

To pilot the items, the first author recruited students from an introductory statistics course and a college algebra course at a large public university during the university's summer term. Due to the timing of this summer term and the courses recruited from, all participants were new college freshmen taking advantage of the university's early-start program.

Interviews were conducted in four rounds. The first round was conducted with the two students who were enrolled in an introductory statistics course. In separate interviews, the students read each statement and assigned a score from 1 to 5, where 1 represented "disagree," 2 was "neutral," 3 was "somewhat agree," 4 was "agree," and 5 was "strongly agree." We weighted the scale toward agreement since we anticipated that there would be few statements that students would disagree with. They were also asked to discuss their response. After the first round, we made two observations about the interview process. First, each student commonly referenced what they learned in their first two statistics classes as justification for their choices. Since we wanted to glean the insights of students before receiving instruction, we found this slightly problematic. Second, asking students to assign a score before discussing each statement yielded more focus on the score rather than on their interpretation.

Starting with the second round, we decided to recruit students who had never taken a college statistics course from a college algebra course. We also made a slight change to the interview process by instructing participants to interpret each statement in their own words before assigning a score. Starting with the third round, we tried a rank-based system. The likert-scale had noticeable limitations as students would often tie many statements, or appeared not to hold a consistent scoring method throughout. The sole participant in the third round was instructed to read three statements at a time (of different perspectives), rank them in order of agreement, and then discuss his ranking. With the final group, we tweaked the ranking system to pairwise comparisons. The data collection process is summarized in Table 1.

Table 1. Interview Structure by Round

Round 1: Summer and Stacy (pseudonyms) • Read each statement, assigned a score from 1 to 5, and discussed reason for score
Round 2: Cara and Alexa (pseudonyms) • Read each statement and interpreted statement in their own words
Round 3: Jeremy (pseudonym) • Read statements in groups of three, ranked in order of agreement, and explained ranking
Round 4: Michael, Mia, and Joseph (pseudonyms) • Read statements in groups of two, chose statement they agreed with more, and explained choice

Data Analysis

Extending from findings by Berens (2019), we continued our goal to establish construct validity for the instrument while also refining our guiding framework. The 2014 Standards for

Educational and Psychological Testing (Chan, 2014) discuss five different evidences for establishing content validity—this particular stage uses evidence based on response processes to understand how students interpreted items and made sense of potentially unfamiliar terms. This evidence demonstrates the degree to which our items are accurately operationalizing the perspectives we have defined in our conceptual framework. Our method of analysis followed a grounded theory model of research. In particular, we conducted what Corbin and Strauss (2008) would describe as a cyclic, "zig-zag" process of collecting data, analyzing data, making changes, and collecting more data. We used the pilot interviews in this stage to accomplish several aims:

- Identify words and phrases that were difficult for students to interpret,
- Identify deviant interpretations to statements,
- Continue refining the four perspectives, and
- Consider whether the holistic perspective(s) that students were taking in their general responses appeared consistent with the statements they gave highest marks to

The first aim simply required the interviewer to make notes next to words or phrases that participants struggled with. In some cases, the interviewer would provide alternative word-choices to gain additional feedback. The second aim was also accomplished through discussion and follow-up with participants during the interview, but deviant interpretations would occasionally be noticed while reviewing the interview transcripts.

The third and fourth aim required more careful review. After each round of interviews, we would convene to discuss different statements from the interviews and consider how each student was conceptualizing the nature of statistics, the process of statistics, and expertise in statistics. We would also note when students expressed strong (or weak) agreement with a statement when their views otherwise did not align with that statement. In some cases, this prompted us to revisit our perspective profiles; in other cases, we reconsidered whether the item needed to be rewritten or dropped.

Findings

Difficult Words and Deviant Interpretations

The pilot interviews, especially the first five, provided a lot of feedback regarding which words and phrases were difficult to interpret, or prompted unintended interpretations. We list several examples in Table 3, followed by an example from each column.

Table 3. Difficult Words and Phrases

Hard to Grasp	Multiple or Ambiguous Interpretations
• "Analytical tools"	• "Testing ideas"
• "Depict reality"	• "Rules and laws"
• "Irrefutable evidence"	• "Definition"
• "Unambiguous"	• "Hypothesis"
• "Theorem"	• "Mathematics"
• "Variability/Variance"	
• "Model"	
• "Statistical test"	
• "Complex phenomena"	

"Hard to grasp" words and phrases was often cued by hesitation around particular words or a difficulty to interpret that portion of the statement. Follow-up with students would often confirm this difficulty. For example, the term "statistical test" was difficult for both Summer and Stacy to conceive of since their conception of “test” was oriented around tests in the classroom setting. With terms like this, we realized it was important to either avoid the term or to pair it with another similar term that would increase students’ likelihood of understanding it in context. For example, we changed “statistical test” to “statistical method or test” to add more context.

Identifying multiple or ambiguous interpretations often required more attention to the transcript. For example, one item originally read, “Statistics requires one to learn and practice in the correct following of rules and laws.” However, Cara interpreted this item with a legal, ethical framing.” Based on both Cara and Alexa’s feedback on these terms prompted us to avoid them completely in favor of “formulas” and “procedures.” Similarly, we realized that “mathematics” was also a loaded word that prompted various conceptions as well. We instead tried to use more specific words, such as “calculations” or “formulas.”

Refinement of the Framework

The most substantive revisiting of the framework concerned the “Rules-based perspective.” In particular, we felt throughout the pilot interview process that this perspective was straddling two different sub-dimensions: the first concerned statistics as having a mathematical foundation, and the second as statistics being structured around clears rules and procedures. Students could reasonably respond positively to the rules-based perspective items while reflecting noticeably different perspectives about how mathematics might manifest in statistical work. For example, Jeremy responded positively to one rules-based item that stated, “Statistical activity often requires recalling and using definitions and theorems.” He explained that there are a lot of terms and mathematics to know to be successful, but also explained later that statistics was not just about following rules, and that practicing calculation routines was only somewhat important. This contrasted from Stacy, who stated, “When you're first taught the calculation routines and procedures, you do need to practice them, but once you practice them, you automatically know how to do it. At the beginning, you need a lot of practice. It may vary by person. Like I'm not a math person, so I might need to practice it a bit more.”

Recognizing that both Stacy and Jeremy resonated with quite a few rules-based perspective items, we realized that this dimension may not have been representing a coherent, unified perspective. As a result, we tried to better reconcile the confirmation perspective with the idea of a mathematical, logical foundation to statistical testing procedures, leaving the rules-based perspective to more clearly represent calculation-centered, rule-based procedures.

Discussion

We would still like to continue with think-aloud interviews to continue collecting response process evidence. We are also collecting evidence based on test content by administering the survey to large classes of introductory students and analyzing the variability in responses. We also plan to gather survey responses from experts for comparison. This will reveal whether the items reveal any recognizable discrimination in perspective between novice and expert learners. Analyzing the larger pool of responses will allow us to consider the reliability of the instrument, particularly on whether items within each perspective are consistently chosen together.

Audience Questions?

1) Do our four framework dimensions seem to capture different perspectives that students and practitioners might hold toward statistics?
2) What other purposes might this survey serve outside the context of introductory students?
3) What other steps should we consider in the validation process?

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Students' Approaches to Solving First Law Problems Following Calculation-Intensive Thermodynamics Coursework

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The first law of thermodynamics is an essential scientific principle with broad relevance across the fields of STEM. As a foundational principle, the first law provides students with conceptual and mathematical tools to solve cross-disciplinary problems. The pilot study summarized in this proposal investigates the ability of engineering students to address first law problems across the field-specific contexts of chemistry, engineering, and physics. Preliminary results suggest that the students' prior experiences in a calculation-intensive thermodynamics course biased student reasoning in favor of solving problems numerically. The application of transfer of learning frameworks to this data set reveal potential conceptual and epistemological resources activated by students that may explain this observed tendency. These results have implications on the application of calculation-intensive problems in science and engineering coursework.

Keywords: Transfer of Learning, Prior Experience, Resources, Interdisciplinary, Thermodynamics

The Next Generation Science Standards defines "energy and matter" as a crosscutting concept of critical importance across the fields of science and engineering (NGSS Lead States, 2013). As a guiding principle, the first law of thermodynamics provides students with both a conceptual and mathematical basis to address mass and energy flow problems. Traditional silo-ed approaches to teaching undermine the interdisciplinary relevance and application of crosscutting concepts (Office of Science and Technology Policy, 2018). Therefore, there is a widespread call to action to revise STEM education to encompass a more integrated model of instruction (National Research Council, 2014).

In this report, we summarize the preliminary results and analysis obtained from a study aimed at understanding the effects of silo-ed instruction of thermodynamics across the fields of chemistry, engineering, and physics. These results were derived from a pilot interview study (4 interviews) implemented at a large Midwestern university in an introductory engineering thermodynamics course that was calculation-emphasized. Participants were asked to solve a thermodynamics problem within their own familiar disciplinary context and one employing an alternative disciplinary context. Transfer of learning frameworks were utilized to both construct the interview protocol and to analyze the contextual-dependence of students' reasoning strategies when addressing first law problems pertaining to different disciplinary contexts. Analysis of data obtained revealed the possible influence of students' prior experiences in a calculation-intensive course on their ability to solve the provided first law problems. Previous studies have indicated that there may be a decay in basic mathematical skills for engineering students that enroll in traditional calculation-intensive coursework (Engelbrecht, Bergsten, & Kågesten, 2012; Engelbrecht, Harding, & Du Preez, 2007). The preliminary findings of this study initially support these results and provide evidence for how this may arise in the context of solving a first law problem. Additional results will be obtained in late 2019 with a larger sample of participants (20 interviews) from the same course to solidify these preliminary findings.

Methods

Participants engaged in a set of problem-solving tasks, while being asked questions according to a previously prepared semi-structured interview protocol. Three problem-solving prompts were developed to reflect the unique disciplinary contexts of chemistry, engineering, and physics; while asking the same conceptual question. Discipline-specific textbooks and instructor feedback were used to develop each prompt. Each question included a description of a piston-cylinder setup alongside necessary descriptions and formulations of the first law of thermodynamics to solve the problem. Participants were asked to indicate whether the internal energy of the system would increase, decrease, or if this change could not be determined with the information provided. Based on the provided work and energy contributions, each prompt corresponded to one of these possible choices. Each interview had the participant evaluate two discipline-specific problems followed by a scaffolded comparison of both prompts. One prompt corresponded to the participant's familiar disciplinary context, and the order of prompts was varied across interviews.

Frameworks

The Resources Framework

The resources framework by Hammer, Elby, Scherr, & Redish (2005) outlines a manifold ontology of mind in which knowledge and reasoning abilities are characterized as fine-grained resources. When an unfamiliar context is presented to a student, certain resources are activated, resulting in either productive or unproductive instances of transfer. Resources that reflect ontological knowledge derived from prior experiences or instruction may be denoted as conceptual resources, while resources that reflect students' beliefs about a problem may be identified as epistemological resources. The inclusion of epistemological resources is critical given the sensitivity of transfer to a student's interpretation of a prompt (National Research Council, 2000). We applied the resources framework to characterize the mathematical knowledge and reasoning strategies the students called upon to address first law problems. The resulting conceptual and epistemological resources that emerged from this analysis are discussed in the *Resources Activated by Students During Interviews* section. In particular, two emergent categories of resources were identified pertaining to solving strategies via a magnitude or direction analysis and epistemic beliefs about what it means to "solve" a problem.

The Dynamic Transfer Framework

The dynamic transfer framework proposed by Rebello et al. (2005) was used to further adapt the resources framework in the case of the summarized research interview setting. When performing a transfer interview study, the interpretation of a students' responses may be influenced by the researcher's agenda and the assumption that knowledge remains static during the interview. Rebello et al. (2005) propose the dynamic transfer framework as a means to address these challenges in an interview setting by recognizing the dynamic nature of transfer as it unfolds during the process of an interview. To accomplish this task, the dynamic transfer framework breaks down the process of transfer into three stages. The first stage involves the "priming" of the student via "covert messages" to activate the appropriate epistemic mode for addressing the posed questions. These activated epistemological resources then "control" for the activation of conceptual resources from long-term memory to address the target context. Finally, the activated conceptual resources are "associated" with the target context to solve the problem. The three aforementioned stages of (a) priming, (b) control, and (c) association were used to structure the interview protocol for this study. Additionally, a scaffolded transfer section was introduced in which students compared the two prompts addressed during the interview. Figure 1 summarizes the structure of the interview protocol applied in this study.

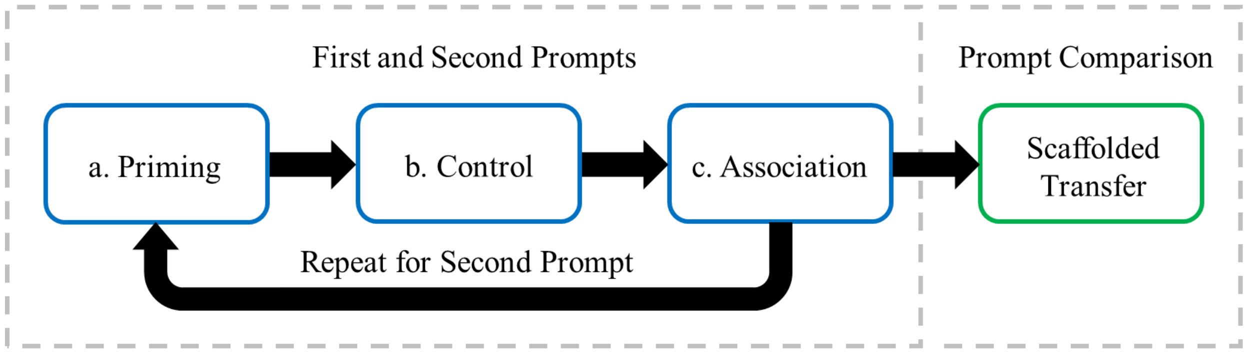

Figure 1. A diagram depicting the structure of the interview protocol. The first and second prompts are addressed separately after which both are compared in the scaffolded transfer section.

Resources Activated by Students During Interviews

The 4 audio-recorded pilot interviews were analyzed via the general inductive approach outlined by Thomas (2006). Segments of the transcribed data were classified according to topic and then coded according to the reasoning strategy employed. Two themes emerged from these generated categories that are discussed below in detail. IRB approval was obtained at the institution of interest prior to implementation of the study.

Magnitude Versus Direction Analysis

During the control phase, students typically adopted one of two reasoning strategies to solve the problem. These strategies pertained to making arguments about either the magnitude or direction of energy flow. As a result, the corresponding resources *solving as magnitude analysis* and *solving as direction analysis* were identified. An example of the activation of *solving as direction analysis* is depicted below in which a student highlights the importance of direction in determining the internal energy change according to the interview prompt.

> *Interviewer*: Okay. If I asked you to solve this problem, without actually solving it, how would you go about it?
>
> *Participant*: Well, I'd probably, so look at the equations that we're given and see, um, and then look at also the first, I guess it gives you the first law, but look at the equations that we're given and see how the values determine like which direction each of the values in the equation would be changing based on the information given and kind of use that to figure out how that relationship would affect the overall internal energy.

Solving as direction analysis serves as a productive resource for the conceptual task . Rather, statements in the interview prompt indicate the direction of energy flow, enabling the student to solve the problem via a direction analysis.

While the magnitude of the work and heat terms in the interview prompts cannot be obtained, some students did appear to activate *solving as magnitude analysis* during the early stages of the problem. An indication by the student that magnitude was a critical factor to solve the problem provided evidence for the activation of *solving as magnitude analysis* as shown below.

Interviewer: So for the pieces you identified, um, of the problem, um, could you speak a little bit more about, um, why you feel these particular pieces are relevant?

Participant: Um, so the question is asking whether Delta-E is greater or less than zero. So that depends on the relative magnitudes of Q and W. So whether, um, Q, so Q or W might, might be positive or negative in this case. So if one negative term or positive term is greater than the other, um, Delta-E might be positive or negative or greater or less than zero. So these are, knowing the magnitude of these two things are, are important to solving the problem.

The lack of numerical values provided in the problem was recognized by each participant (4 of 4 interviews). Despite this recognition, some students exhibited a bias towards evaluating numerical expressions (3 of 4 interviews) in the early stages of assessing the problem. This observed trend is hypothesized to be due to the heavy emphasis placed on calculation in the students' most recent engineering thermodynamics course. In 2 of 4 interviews, students commented on the differences between the structure of the provided problems and those encountered in their calculation-intensive engineering thermodynamics coursework.

Students' Problem-Solving Expectations

In some cases, the observed activation of *solving as magnitude analysis* appeared to be accompanied by the immediate expectation of whether the problem was solvable. A lack of numbers indicates that no numerical solution can be calculated and students in this category appeared to equate this fact to the conclusion that the problem could not be "solved." As such, an independent resource *solving as calculation* is hypothesized to encompass this interpretation. An example of the apparent activation of *solving as calculation* is depicted below.

Interviewer: Alright. Alright. And so for this next portion I'll ask that you pick up the Livescribe pen and then proceed to solve the problem. Um, feel free to use the scratch paper, um, and feel free to provide any justification for your answer whatsoever, however you would like to go about that. Also be sure to describe, um, your thought process as you solve the problem.

Participant: … So without any numbers in here, I would just sort of jump ahead and say that it's, yeah, it's, I can't determine it. Um, I'm trying to think about whether any of these numbers can be, whether, like I, I should know any of these numbers without any, um, explicitly having any numbers in the question … So I'd say, I'd say yeah, we cannot determine anything. Yeah, I'll write it here. Cannot determine no, no numbers. Yep. Okay.

When students evaluated the expression according to direction, they appeared to highlight the ability to solve the problem in a manner that did not require numbers. One participant went as far to refer to the provided equations as having a "calculation side" that they deemed irrelevant in the case of the problem being solved. In this instance, the student distinguishes the provided mathematical expressions from the information needed to solve the problem. Therefore, instances where a student recognized the relevant information needed to reach an answer was denoted as activation of *solving as determining*. Students in this category demonstrated comfort in proceeding to solve the problem despite the lack of numbers provided.

Preliminary Conclusions and Implications

This preliminary analysis suggests that engineering students that have taken calculation-intensive coursework may be susceptible to activating unproductive resources when tasked with basic conceptual problems. The tendency for students to activate *solving as magnitude analysis* during the control phase suggests that students' initial interpretations of a first law prompt may be impacted by their prior experiences. According to the framework by Rebello et al. (2005), the activation of *solving as magnitude analysis* in these cases likely arises from an underlying epistemological resource that controls for the activation of this reasoning strategy. *Solving as calculation* may serve as this epistemological resource mediating the activation of *solving as magnitude analysis*. In this manner, calculation-intensive thermodynamics coursework may promote the development of epistemological resources that hamper students' ability to transfer conceptual knowledge and reasoning strategies when addressing first law problems. The presence of *solving as calculation* serves as preliminary evidence for possible epistemological framing akin to the *calculation* frame discussed by Bing & Redish (2009). However, additional interviews are needed to map the prevalence of these resources and to separate out what resource controls for the reasoning strategies employed.

The study in question may have implications on the instruction of students via calculation-intensive engineering and science coursework. Blending mathematics and science instruction may require an extensive consideration of how the emphasis of advanced mathematical calculations in science and engineering coursework influences the epistemological resources students activate when approaching problems.

Future Directions

A proposed alteration to the study would be to incorporate new questions during the priming phase to assess whether students recognize attributes in the question early on that control for their reasoning in the later stages of evaluation. For example, some additional questions that could be incorporated include: "Was there anything you immediately recognized upon reading this problem?" or "Is there anything that immediately stood out to you about this problem?" It is hypothesized that students would likely recognize the lack of numbers early on during the priming phase, thereby providing evidence that *solving as calculation* and *solving as determining* serve to mediate the observed resources activated. Implementation of this study will be carried out in late 2019 to obtain 20 additional interview transcripts from the same course of interest. This additional data will aid in the clarification of these resources and verification that data saturation has been reached. Feedback elicited in the upcoming stages of analysis from the science and mathematics communities, in particular the RUME community, will greatly support the interpretation of these preliminary results in devising a finalized analysis with wide-standing implications across STEM fields. In the event this proposal is accepted, the following questions would be posed to the RUME community during the preliminary report to support the development of the project.

1. What approaches and/or questions can be added to the interview protocol to better assess the role of the conceptual and epistemological resources activated?
2. Are there any alternative interpretations that can be attributed to the data presented?
3. What mathematical frameworks may help to better bridge understanding of the reasoning strategies employed across the fields of science, engineering, and mathematics?

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Mathematical Modeling Competitions from the Participants' Perspective

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This study is a post-hoc analysis of pre and post survey data from two rounds of a mathematical modeling competition comprised of high school and undergraduate participants (n=107). The purpose of this study is to describe the expectations participants held going into the competition and compare them to those held by researchers and designers of modeling competitions. Additionally, this study examines participants' satisfaction with the competition. Results showed that participants, researchers, and designers held differing expectations. Participants expected to gain more practical experience but afterward reported gaining practical experience and soft skills. Lastly, participants tended to gain what they expected from the competition; those who felt their expectations were not met would not recommend the competition to others (regardless of what their initial expectations were)

Keywords: Mathematical Modeling, Competitions, Participant Expectations

Historically, mathematics competitions complement mathematics education (Kenderov, 2006). When unpacking the extent to which these challenges are impactful, there are three perspectives to consider: the researcher, the designer, and the participant. For example, designers might be primarily concerned with networking, while researchers may focus on the benefits of time spent solving equations outside of the classroom. In contrast, participants may sign up for the competitions expecting to get resumé lines or critical thinking skills. While there is overlap in these expectations, the disconnect between the three perspectives may offer insight into participant involvement in these events.

Broadly, studies of the benefits of mathematics competitions have often examined participants who are gifted and talented. Bicknell and Riley (2012), using interviews, surveys and classroom observations, found that gifted and talented participants liked to participate in competitions to compare themselves to others on the international level and work cooperatively at the local level. Gleason (2008), using item response theory, found that high school mathematics competitions that use multiple-choice tests differentiated participants' mathematical ability when the participants were on a similar level. While recognizing the benefit to gifted and talented participants, Kenderov (2006) and Thrasher (2008) also suggested integrating mathematics competitions into the mathematics curriculum. Kenderov (2006) argued that mathematics competitions help motivate participants, steer participants to STEM fields, provide an avenue for recognition, and give prestige to the participant's institution. Thrasher (2008) made a similar argument adding that mathematics competitions also provide a space to practice mathematics outside of the classroom and encourage teamwork. Even competitions tightly focused on one aspect of mathematics, like mathematical modeling, have shown gains in participants' mathematical modeling self-efficacy (Czocher, Melhuish, & Kandasamy, 2019). Thus, from a researcher's perspective, participation in competitions promotes confidence, interest, soft skills, and more.

Although there is a lack of research studies on mathematics competitions for a general participant body, engineering competitions are very common and highly encouraged among engineering majors. We can turn to them to learn more about the benefits of design-based competitions. Wankat (2005) described the winners of engineering competitions and the benefits these competitions had for participants. Using data from competition winners and interviews with advisors, Wankat found that participants learned engineering skills like design and construction, and soft skills like teamwork and leadership skills. Likewise, Gadola and Chindamo (2019) while discussing the role competitions have in engineering education, concluded that design competitions promoted hands-on learning, interest, motivation, and further involvement.

Designers of mathematical modeling competitions state other benefits to participating in their competitions. According to SIMODE, MCM/ICM Contest, the MathWorks Math Modeling Challenge, and the AoCMM Math Modeling Competitions' websites, designers of mathematical modeling competitions anticipate participants would gain experience in modeling, gain proficiency in modeling, increase their confidence in modeling, gain an appreciation of modeling, gain recognition, develop teamwork, develop their resume, develop their communication skills, and have networking opportunities.

Given the (anticipated) benefits of participation, the next goal for competition sponsors should be broadening participation in these competitions. Doing so would involve accounting for the participants' expectations so as to market the benefits enticingly. Additionally, competition designers who incorporate participants' needs into their competitions give researchers dimensions to evaluate impactfulness. Within this context, the purpose of this study is to understand the experience of participants in a mathematical modeling competition using the SIMIODE[1] Challenge Using Differential Equations Modeling (SCUDEM) as an example. What expectations do participants bring to a mathematical modeling competition and to what extent do those expectations align with the expectation of the designers and research literature? Do the participants express that their expectations were met? If those expectations were not met, did they still gain something from this competition? Finally, does the mathematical modeling competition seem worthwhile to the participants?

Methods

To answer these questions, we conducted a post-hoc analysis on survey data collected before and after SCUDEM competitions. The data were collected from two rounds of the competition, SCUDEM II in April 2018 and SCUDEM III in October 2018. In the SCUDEM competitions, teams of three students from secondary and post-secondary institutions were given a choice of three modeling problems and tasked with writing an executive summary of their solution in a week's time. Teams then convened at the competition sites, where they were given an additional modeling issue related to their chosen problem to address briefly in their presentation. Teams received substantive critiques of their efforts. During this time, the faculty coaches met to evaluate and critique the teams' solutions. The teams then gave a ten-minute presentation of their findings to their original and the new modeling issue, which were judged by the faculty coaches. Teams were also provided immediate oral and written feedback.

A total of 399 and 228 participants participated in the SCUDEM II and III competitions, respectively. Of those, only a fraction responded to the relevant items on both the pre- and the post-competition surveys (n=107). In the pre-competition survey, the participants were asked "In a few words, please tell us what you expect to gain from participating in the competition?"; and in the post-competition survey, the participants were asked "In a few words, please tell us the main thing you learned about mathematical modeling from competing." Additionally, we

[1] SIMIODE (Systemic Initiative for Modeling Investigations & Opportunities with Differential Equations) is the sponsoring organization.

analyzed responses to the question “Would you recommend this competition to a friend?” However, this question was only present on the April survey and answered by only 66 participants. As well, the question “Please describe some positive aspects of the SCUDEM experience for you. Why were they positive?” was included but was only on the October survey and answered by 36 participants.

To compile a list of the perceived benefits of a mathematical modeling competition, categories of expectations were documented from the literature review, SCUDEM’s website, and similar competition designer’s websites. To gain insight on the participant perspective, codes were created in vivo (King, 2008) from the participants’ pre-competition survey responses. Concepts such as ‘confidence’, or ‘applying differential equations to the real world’ were identified and assigned codes. The method of constant comparison was used to develop stable evidence-based definitions for higher-level categories of like codes so the codes could be systematically applied throughout the student responses (Corbin & Strauss, 2015). For example, the code ‘experience in modeling’ could be evidenced by participants stating directly that they want experience in modeling, or from phrases such as ‘applying/using differential equations to the real world’. Another example of such a code would be ‘experience in solving differential equations’ evidenced by participants stating they want to get better at solving differential equations. This is different from ‘experience in modeling’ as the participant does not reference the real-world applications, nor do they say using or applying. The list of student codes and what we construed as evidence are provided in Table 1. The lists of benefits from the developers, similar competition developers, researchers, and the participants’ expectations were synthesized. This new list of seventeen expectations (Table 2) became the codes to analyze student responses. The codes were not mutually exclusive; one response could have multiple codes.

Table 1 Codes formed from student responses and their indicators

Code	Evidence
Experience in Modeling	Directly referencing modeling or saying phrases like "applying differential equation to the real world", key words being applying or using differential equations
Experience in Solving Differential Equations	This is stating the expectations to get better at solving differential equations. This is different from "experience in modeling" as they do not reference the real-world applications, nor do they say applying or using
Experience in Critical Thinking	State that they want experience in critical thinking
Experience in Problem Solving	State that they want experience in problem solving
Teamwork Skills	State they want to experience working in a group/team
Career Building	State they want something to put on a resume, or extra credit to a course
Increased Confidence in Modeling	State they want more confidence in modeling, not experience in modeling.
Increased Proficiency in Modeling	State they want to get better at modeling or have a better understanding of modeling.
Presentation/Communication Skills	State they want to get better at communicating or presenting their work

We compared the pre and post responses to see if participants’ expectations were met. A 0 was assigned if the participants’ expectations were not met. An example of this type of response is one where the participant expected to gain experience in modeling however, in the post response they stated, “I personally did not learn anything new…”. A 1 was assigned if the participant gained skills/had an experience identified in the list but was not what they identified in their pre response. An example of this is a participant who identified modeling experience in their pre response, but in the post identified communication and teamwork skills. A 2 was assigned if the participant’s expectations were met, but not fully. A 3 was assigned if the participant's expectations were met fully. Lastly, a 4 was assigned if the participant's expectations were fully met and they identified further skills/experiences they learned from the competition.

Results

Researchers and designers agreed that participation should give the participant an appreciation of modeling and a chance at recognition. However, those benefits are not widely recognized by the participants (0% and 0.93% respectively, Table 2). Participants and designers agreed that participants should gain experience in modeling (71% of participants), proficiency in modeling (16.82%), and practice teamwork (13.08%) and communication (9.35%) skills. Researchers and participants agreed that participation is a chance to practice mathematics outside of the classroom (8.41%). There were also expectations participants held that neither the designers nor the researchers identified as benefits of participation. Participants expected to practice solving differential equations (16.82%), problem-solving (5.61%), and critical thinking (0.93%) in addition to career building (7.48%). Researchers, designers, and some participants (2.08%) believed participating in a mathematical modeling competition would increase participants confidence in modeling, exemplifying the difference in expectations between participant and designer/researcher. Overall, these results showed that participants' expectations mostly aligned with those of the researchers and designers, but there were expectations participants held that were not present in the literature.

Table 2 Frequency of Expectations, Researcher (R), Designer (D), Participant (S)

Perspective	Expectation	S expectation pre-competition	S gains post-competition
D, S	Experience doing modeling	71%	56.07%
D, S, R	Gain confidence in modeling	2.08%	2.08%
D, R	Gain appreciation of modeling	0%	0.93%
D, R	Gain recognition	0.93%	0%
D, S	Teamwork skills	13.08%	17.75%
D, S	Communication skills	9.35%	12.15%
D, S	Proficiency in modeling	16.82%	23.36%
D	Networking	1.87%	0%
R	Increase interest in mathematics	2.80%	3.74%
R	Increase motivation to do mathematics	0.93%	0.93%
R	Self-study skills	4.67%	0%
S	Practice solving differential equations	16.82%	9.35%
S	Practice problem-solving	5.61%	0.93%
S	Practice critical thinking	0.93%	0.93%
S	Career building	7.48%	0%
R	Leadership skills	0%	0%
R	Experience with mathematics outside of the classroom	8.41%	4.67%

A further examination of participants' post responses showed that the most common expectation pre-competition, "experience in modeling," was the most commonly reported gain (56.07%), implying that a majority of the participants' expectations were being met. Additionally, participants reported experience in modeling (71%), improved proficiency in modeling (26.32%), and teamwork skills (17.75%) as the top three most important skills/experiences gained from this competition. Further results are shown in Table 2. Further analysis of the pre and post data showed 10.28% of participants reported their expectations were not met, 37.38% gained skills or experiences, but not the ones they expected, 10.28% had their expectations met, but not fully, 39.25% had their expectations fully met, and 2.80% had their expectations met and more. To gain more insight into the participants' satisfaction a follow up analysis was conducted on the post-competition survey for each of SCUDEM II and III separately. Results of the question "would you recommend to a friend" from the SCUDEM II data showed that of the 66 participants that answered the question from this data set, 57 said yes, they would recommend this competition to a friend, indicating satisfaction with the competition. Of the nine who reported that they would not recommend the competition to a friend, four indicated their expectations were not met and three reported gaining something, but not what they expected, indicating those who would not recommend to a friend were left dissatisfied with the competition. Results of the question "Please describe some positive aspects of the SCUDEM experience for you. Why were they positive?" from SCUDEM III showed common themes such as joy in working with others, a sense of autonomy, self-perceived increase of skill, validation from outside sources, struggle, increase of presentation skills, appreciation of other strategies, networking, and exposure to modeling. Looking at the occurrences of these, the most common (10 of 36) identified teamwork as the most positive aspect of the competition. The second most common response (6 of 36) identified learning new strategies from others as the most positive aspect.

Conclusion

Based on our reviews of the literature and designers' promotional materials, as well as our analysis of the participants' perspectives, we conclude that participants, designers, and researchers have different perceptions of benefits of mathematical modeling competitions. Additionally, participants of this mathematical modeling competition expected to gain experience in modeling. Participants did report gaining something from the competition (37.28%), and a majority of participants did have their expectations at least partially met (52.33%). After the competition the participants reported that the most important things they gained were experience in modeling, improved proficiency in modeling, and improving their soft skills such as working in a team and communication. These results corroborate Senior and Holt's (2014) claims that participants' reasons for participation in competitions change post-competition from wanting to gain academic knowledge to appreciating social aspects. Only 2.08% of students reported gaining confidence in modeling post competition; however, gains in mathematical modeling self-efficacy were measured on the larger survey, indicating participants gained more from the competition then they realized or were able to articulate. From SCUDEM II, it seems that participants who would not recommend the competition to their friends also tended to have unmet expectations. From SCUDEM III, when asked for the most positive aspect of the competition participants tended to identify social aspects of the competition. These results suggest that promoters should attend to the participants' expectations when marketing their competition. When participants have expectations, the competition is not designed to fulfill, participants feel dissatisfied due to these unmet expectations, possibly leading to lower participation. More research on the benefit of mathematical modeling competitions is needed. Further studies should address the disconnect between what the participants gain and what they report they gain from the competition in order to more fully describe the impactfulness of the competition. For example, many participants did not state they got experience with mathematics outside of the classroom; however, by nature of this competition they did have this experience.

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Communication and Community: GTA Perceptions on a Professional Development Program

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Graduate teaching assistants (GTAs) play an important role as instructors in math departments. As a result, professional development (PD) opportunities have been developed to support these primarily novice instructors, but these programs can vary widely. This study assesses the state of a PD program for GTAs at a large, public, doctoral-granting institution. Surveys and interviews were analyzed with using Social Cognitive Theory (SCT) as a framework.

Keywords: Graduate Teaching Assistants, Professional Development, Teaching Training

Graduate students play an important role as primary instructors or graduate teaching assistants (GTAs) in calculus courses at doctoral-granting institutions (Bressoud & Rasmussen, 2015). Among four-year colleges and universities, they teach seven percent of mainstream Calculus I courses, 17% of non-mainstream Calculus I courses (CBMS, 2015), and at some institutions a large percentage of the courses prerequisite to calculus. For instance, graduate students and adjuncts taught 44% of College Algebra classes in 2010 at doctoral-granting institutions (Tunstall, 2018). Additionally, a recent report from the American Association of University Professors indicates that GTAs form nearly 30% of all instructional staff at very high research doctoral-granting institutions (AAUP, 2018).

Successful calculus programs have meaningful and comprehensive professional development (PD) for GTAs (Bressoud & Rasmussen, 2015), and most mathematics departments at four-year institutions have some orientation or preparation for their GTAs prior to the start of their first semester. More extensive GTA PD (e.g. teaching-focused courses and seminars) exists in many departments (Speer, Murphy, & Gutmann, 2009), and there is now a large-scale effort to provide PD and GTA training resources to departments across the country (CoMInDS, 2019). However, local implementation of PD activities varies widely from intensive pre-semester orientation programs to weekly teaching-focused seminars during the semester. Such teaching seminars or courses may or may not include sustained engagement with key, active components of novice instructor development (Deshler, Hauk, & Speer, 2015). Additionally, GTAs have been math students for many years and are subject to “apprentice of observation,” a term first coined by Lortie (1975) and used to describe how students have spent years observing their teachers unlike novices in other professions (Borg, 2004). As a result, novice teachers might make assumptions about best practices and, as a result, may be unaware of the limitations of their teaching knowledge, such as not knowing their former teachers’ intentions, personal reflections, and planning work. Despite having opportunities to grow as a teacher, without continual engagement in PD, these novice teachers will likely default to imitating what they observed as students.

Furthermore, GTAs have varying career goals, which often involve an academic career (American Mathematical Society, 2014). These professional goals could impact the extent that GTAs engage in PD and the perceived value that PD has in their future careers. Academic jobs are often categorized as research-focused, teaching-focused, or an equal balance between the two. In light of the individual differences in career aspirations, it is hypothesized that GTAs vary widely in identification with their roles as educators.

This preliminary report provides a snapshot of GTA perceptions at a large, public, doctoral-granting institution. At this institution, GTA PD consists of both an intensive pre-semester orientation as well as weekly course coordinator meetings during the semester, which includes both conversations on course logistics as well as teaching practices. Most GTAs begin their first year utilizing the flipped model with College Algebra and Business Calculus with structured course coordination. GTAs may select to teach Calculus I or Introductory Statistics in the second year and have the option to teach Calculus II or Matrix Algebra starting in their third year. After the first year, GTAs have lighter course coordination, they engage in a range of other teaching models, and they are asked to make more decisions about their individual courses.

Using quantitative data from an online survey and qualitative data from interviews, this study assesses initial perceptions and experiences of GTAs on teaching and their identities as educators. Using Social Cognitive Theory (SCT) as a theoretical framework, we seek to answer the questions: *From the GTAs' perspectives, does this professional development expose them to a sufficient range of effective teaching models and skills that they can use in their careers? Does it increase their self-efficacy as teachers? Do these GTA perceptions relate their classroom practices and usage of various active learning methods?*

Theoretical Framework

SCT was originally advanced by sociologist Albert Bandura and has been used in Education, Psychology, and Communication fields. Specifically, it has been used for assessing GTA PD in STEM disciplines (e.g. DeChenne, Enochs, & Needham, 2012; Young & Bippus, 2008). It suggests that novice teachers can improve as educators through behaviors, personal attributes, and the environment (Bandura & Walters, 1977). In particular, novice teachers need to have: (a) exposure to effective teaching models, (b) clear expectations for positive outcomes when using those models, (c) self-efficacy to implement the models, and (d) identification with the teaching models' portrayal of an educator.

Regarding models of effective teaching, math GTAs have had exposure prior to any PD. They participated in the learning experience as students and thus draw upon those experiences to inform their beliefs about what teaching should look like. As undergraduate students, they most likely saw lecture as a teaching model (e.g. Bressoud & Rasmussen, 2015; Miller, et al., 2018; Stains, et al., 2018), and as successful math students who are enrolled in a graduate program, they have personally had positive outcomes with this method of instruction. It is not surprising that GTAs have reported higher levels of comfort with lecture than with other teaching methods (Deshler, Hauk, & Speer, 2015), and so it may be the default for many GTAs.

To continue their growth as effective teachers, GTAs should engage in PD that includes the four components described by SCT. The courses for GTA PD at this institution were designed by a range of experienced faculty members. Because development was done over the course of years and by multiple individuals, activities were not explicitly designed with SCT in mind. However, the collective PD activities can be categorized into three of the four SCT components: (a) exposure to models of effective teaching (e.g. faculty-designed examples of flipped curriculum), (b) clear expectations for positive outcomes when using those models (e.g. seminar discussions about student engagement and the requirement for GTAs to read research articles on active learning), and (c) self-efficacy (e.g. the tiered teaching assignments with gradual increase in responsibilities). The fourth component of SCT involving a feeling of identification is hypothesized to be dependent on individual GTA career goals.

Methods

GTAs were asked to complete an online survey with questions related to teaching and career goals, and they were invited to participate in a voluntary follow-up interview. GTA participants had two weeks during the semester to respond to a 40-question survey, five of which were demographic, two were career-related, and 33 related to the four key components of SCT, which are detailed below:

(a) Exposure to effective teaching models: 19 questions from Life Sciences Graduate Student Survey (Shortlidge & Eddy, 2018), four institution-specific about participation in PD;
(b) Clear expectations for positive outcomes when using those models: two questions about the relevancy of teaching PD;
(c) Self-efficacy to implement the models: four questions from the STEM GTA-Teaching Self-Efficacy Scale (DeChenne, Enochs, & Needham, 2012); and
(d) Identification with the teaching models' portrayal of an educator: three questions modified from MAA Calculus Survey for Graduate Students (Ellis, 2014), one institution-specific question.

Quantitative data was analyzed by checking for correlations using Kendall's Tau Correlation Coefficient (1938) and a two-tailed test of significance with statistical software.

Interviews were intended to better understand motivating factors of survey responses, and it entailed open-ended questions about background, what mathematics is and who can learn it (Gutmann, 2009), and connections between GTA PD and career skills. The interviews were recorded and transcribed. Then, they were analyzed using a thematic analysis (Braun & Clarke, 2006). Initial coding was done in phases and independently by the two researchers. Then, the coding was compared, and any disagreements were discussed, clarified, and resolved.

Preliminary Results

Thirty-two GTAs fully completed the online survey, and 20 of these self-reported that they were formally trained in teaching. The other 12 said they had received no formal PD. Of the 32, five GTAs agreed to participate in individual interviews. Four of these GTAs were formally trained in the department, and one was not. Of the GTAs who were willing to fill out demographic information, their ages ranged from 23 to 52 with the mean age of survey respondents being 28. Of these respondents, 63% were male, 28% were female, 3% were agender, and the rest did not want to respond. Also, 82% were white, 6% were Asian, 3% were Latinx, 3% were multiracial, and the rest did not want to respond. All respondents were seeking a PhD. Twenty-one of them had a bachelor's degree at the time of the survey, and 10 had their master's.

Survey Results

Preliminary results focus on the $n = 20$ survey results from the GTAs who participated in the formal PD program. As a result, additional analyses were done by categorizing participants using the survey question in which they filled in percentages of their ideal job's workload after graduation for teaching, research and scholarship, and other. Those with over 60% in the teaching category were labeled as "primarily teaching", and similarly those with over 60% in the research and scholarship category were labeled as "primarily research." There was a small number of GTAs who placed into a third "balanced" category in which they indicated a more even split between teaching and research.

Being a GTA in the "primarily teaching" category was positively correlated with having confidence in evaluating students' academic capabilities ($r = .412$, $p = .044$) and feeling confident in supporting students who were having difficulties in class ($r = .435$, $p = .033$). In contrast, being a GTA in the "primarily research" category was positively correlated to believing that teaching PD is less relevant ($r = .417$, $p = .046$), but interestingly, there were no correlations with the use or lack of use for active learning techniques overall. Being in the "primary research" category was only negatively correlated with using problem-based or inquiry-based learning ($r = -.448$, $p = .041$). In general, there was a positive correlation between PD in specific types of active learning and implementation of those methods.

Interviews

There were four interview questions regarding background in math and teaching, the abilities of individuals to learn math, how PD helps GTAs prepare to be a successful mathematician, and how PD helps GTAs prepare to be a successful math educator. The qualitative data collected from interviews were coded using themes. Codes fell into seven categories for GTAs: desired career goals, experience and support prior to graduate school at this institution, comfort in teaching, hands-on experience with teaching practices, a sense of community around teaching, improving upon teaching, and different traits explaining what it takes to learn math.

There was a mix of career goals for the interviewed GTAs ranging from a teaching-focused professorship at a small liberal arts school to a research-focused job at a national laboratory. All had some prior experience with tutoring, and some had experience with formal teaching like teaching test preparation courses, English language classes, and parts of undergraduate math classes. In addition, three of these GTAs spoke of past mentors including an academic advisor, exceptional teachers, and a parent who worked as an educator. All five GTAs indicated that some level of comfort was important to them for teaching, which ranged from feeling prepared for future teaching preparation to enjoying teaching. In particular, every mention of comfort in teaching was tied to hands-on experience with teaching practices, which is consistent with the quantitative data's implication that all GTAs have some level of SCT's self-efficacy and identification. GTA 4's answer summarizes general themes for all interviewees:

> *Interviewer:* What are aspects of your GTA training that best prepare you to be a successful mathematician?
>
> *GTA4*: I think the hands-on training is generally the best. So now that I'm doing the [Calculus sequence], it's great that I actually get the experience of writing quizzes, deciding pacing of classes, and that I have multiple mentors that I can talk to you and I have questions, "How do I handle the student who's having stress and stuff like that?" Doing some of the other recitations was great for getting comfortable with the subject matter for actually like learning about how to teach.

This excerpt included the codes: comfort in teaching, hands-on experience with teaching practices, and improving teaching using a feedback loop with mentors. Improving teaching with feedback was a common theme for all interviews. Within the five interviews, there were 14 mentions of a feedback loop with a mentor (including both having that feedback loop and wanting more of a feedback loop), eight mentions of a feedback loop with students (such as learning more about a subject based on student thinking), and seven mentions of a feedback loop with peers (such as office mates teaching the same course). Related to this, the two GTAs who spent more time talking about a sense of community around teaching both mentioned the support that they had from weekly meetings throughout the semester, which included GTA mentors and peers.

Discussion

Connecting the qualitative and quantitative data, there was some consistency in the ways that the interviewees responded. GTA1 and GTA4 expressed interest in "primarily teaching" careers, GTA5 in "primarily research" jobs, and GTA2 and GTA3 in "balanced" roles. Specifically, GTA5 was the only one who selected "neither agree nor disagree" on the question about the influence of teaching PD on GTAs' confidence in providing support to students who have difficulty learning and in accurately evaluating students' academic capabilities. In other words, the only "primarily research" GTA indicated less "self-efficacy" than the others with regards to teaching. Unsurprisingly, GTA5 was the only one who disagreed with the statement, "Developing my teaching is part of developing my skills for a job after I graduate."

Coding the interviews reinforced the presence or absence of the four key features of SCT in the PD. There seemed to be general agreement that there is a significant amount of exposure to models of effective teaching at this institution (e.g. coded as hands-on experience), but responses varied in the degree of alignment with the other three components. In addition to the immediate positive teaching outcomes from using the models of effective teaching in their classes, all GTAs in the interviews mentioned how teaching PD helped to improve communication skills which transfer to research and other job skills, but GTAs who desired research-focused careers viewed the PD as less relevant, and they experienced less of a feeling of self-efficacy in the classroom (e.g. coded as comfort in teaching). Also, it seemed that all GTAs had some sense of identification with the PD's teaching model, as indicated by their use of different active learning methods, but this may be because of the requirements in the GTA PD. A next question to ask is: can we better assess the nuances in GTA identification with teaching models*?* Perhaps one way to try to gauge this would be to ask GTAs to design a hypothetical class without any requirements from a coordinator or mentor and to see which methods they employ.

Lastly, we pose the following question: *How can the PD program help GTAs feel more confident in the classroom (the self-efficacy aspect of SCT)?* The interviews appeared to indicate factors of a PD program that the GTAs found to be important for becoming more comfortable in teaching, namely, communication and community. Perhaps more intentional implementation of the recommendations from Deshler, Hauk, and Speer (2015) can aid in this. In particular, having more "collaborating in teams to learn about teaching" and additional "working with a mentor through multiple classrooms visits and conversations" directly address these two needs (p. 640).

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Content-specific Confidence in Entry-level College Mathematics Courses: Relationships and Patterns

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Although mathematical confidence is known to relate to students' level of mathematics, the relationship between a student's content-specific confidence and the student's familiarity with the content remains unexplored. Using a regression analysis of survey data from a single institution, this study examines this relationship for three common, lower-division, undergraduate mathematics courses. Students in the targeted courses reported high levels of content familiarity and confidence they could do the work for the class without instructor intervention. Familiarity was the biggest predictor of students' content-specific confidence, but some orientations towards mathematics also mattered. When tested, students who reported they could do problems were not highly accurate in their assessment. Taken together, these findings suggest that there might be a benefit in helping students better assess their content knowledge early in their courses.

Keywords: Intermediate algebra, Finite mathematics, Calculus I, content-specific confidence, content familiarity

It is well understood that many students entering college have prior content experience with the mathematics they will "learn" in their first college mathematics class. For example, approximately 75% of students taking Calculus at large universities have previously taken it in high school (Bressoud, Mesa, & Rasmussen, 2015). Despite high prior enrollment in these classes, lower-division courses often come with fairly high failure or withdraw rates (e.g., Attewell, Lavin, Domina, & Levey, 2006; Hsu, Murphy, & Treisman, 2008), leading to questions about students' orientations towards mathematics and their prior experience with the content may influence the approach they take the course. This preliminary report sets out to explore some of these relationships, focusing on how students' content familiarity relates to their content-specific confidence.

Mathematical Confidence in College

Students' mathematical confidence (sometimes called self-efficacy) is often defined to be the degree to which a person believes they can successfully navigate and perform in mathematical situations (Bandura, 1997). At the college level, students with high mathematical confidence are more likely to be able to solve mathematics problems (Pajares & Miller, 1994). In addition, students in lower-level classes, such as Intermediate Algebra, have lower average self-confidence (Hall & Ponton, 2005) than students enrolled in transfer-level mathematics classes.

Differences in confidence related to mathematics explain some of the differences we see in mathematical performance between males and females (Pajares & Miller, 1994). Moreover, recent work suggests that ultimately this difference may contribute to differences in male/female STEM participation (Ellis, Fosdick, & Rasmussen, 2016). Notably, however, confidence is partially defined in relation to a students' perceptions of their ability to do general mathematics, which suggests a student's confidence may differ when asked about content for a particular course. Given that so many students have prior exposure to the mathematics content they are asked to learn in college classrooms, it seems important to understand how students' prior exposure to content may impact their confidence for a particular course.

Taken together, it seems appropriate to investigate the relationship between students' confidence as it relates to particular mathematical content. This study examines:

1. To what degree to students' identify as familiar with content common in early coursework in mathematics?
2. How does students' content familiarity influence their content-specific confidence relative to other orientations towards mathematics?

Methods

All data were collected at a large southeastern university during the fall 2018 semester. Data were collected from four mathematics courses, three of which are focused on in this study: Intermediate Algebra, Finite Mathematics, and Calculus I. The included courses were selected to loosely capture a cross section of the first few years of the postsecondary mathematics pipeline. At the institution where data were collected the three courses share a 100-course-level designation but are tiered within the level. Data were also collected for a fourth, major level class, which is not discussed here as it is the students are not the population of interest for this study.

Sample

All faculty and instructors teaching a face-to-face section of one of the targeted courses were invited to participate. Participating instructors distributed the survey link via email. Estimating the number of students who were recruited compared to the total number of students who participated yields an overall response rate of 16.4%.

Data Sources

All data were collected during the first two weeks of the fall 2018 semester using an online survey given during the first two weeks of the semester. Students were invited to take the survey based on their enrollment in the course during the first week of the semester and enrollment in a section with an instructor who consented to be a part of the study.

Student demographics. The survey asked students to identify their sex and the year in school they were in (i.e., freshman, sophomore, junior, or senior).

Content-specific confidence and familiarity. Students' content familiarity was measured in two ways. First, students reported information about the specific courses they took in high school and whether it was their first time enrolled in the course (questions adapted from Bressoud et al., 2015). Second, the students were asked how certain they were that they had previously worked on eight problems typical of final exams in the course (four levels: "I am certain I have solved problems like this in the past", "I am somewhat certain I have solved problems like this in the past", "I am not certain whether I have solved problems like this in the past" and "I have never solved a problem like this before"). Questions were chosen for each course by examining past final exams and the learning objectives for the courses.

To measure students' content-specific confidence, students were asked to describe how confident they were they could solve the problem (four levels: "I could solve this problem right now", "I could solve this problem after some independent review", "I might be able to solve this problem after some independent review", and "I do not think I could solve this problem without help from individual or class instruction").

To check whether students' reported content-specific confidence lined up with their ability to do the problems, students were asked to provide answers to two of the questions they had answered questions about. The two questions were the same for all students within a course. Students answered the question by typing in their answer (free response).

Other affective variables. In addition to the content-specific questions, the survey collected data on students' orientations towards mathematics, including their general mathematical confidence and mathematical understanding (1 item each; 5-point Likert scale; Bressoud et al., 2015), and value of mathematics (8 items; 5-point Likert scale; Tapia & Marsh, 1996). The students' bold problem-solving orientation was also measured (10 items; 5-point Likert scale; Author, under review). This orientation is a self-identified preference for inventiveness, independence, and risk-taking while doing mathematics.

Analysis

Confirmatory factor analysis was run for multi-items scales (i.e., bold problem-solving orientation, value of mathematics). In addition, composites for the content-specific confidence and content familiarity measures were created by averaging the responses over the items for which students had a response.

For each class, pairwise correlations were run for non-binary covariates used in the regression models. A Bonferroni correction was applied to account for multiple comparisons. Summary statistics for each variable, by course, were run.

Ordinary least squares regressions were run for each of the three courses. For each regression, the score for the content-specific confidence served as the dependent variable. The independent variables were students' content familiarity, sex and class status (binary; freshman as reference category), students' bold problem-solving orientation score, mathematics understanding score, general mathematics confidence score, and value of mathematics score. All variables other than sex and class status were mean centered. Only students with complete data for each of the variables included in the regression models for the class were retained. No data were imputed.

Lastly, cross-tabulations on students' self-reported confidence they could solve individual problems were compared to their correctness when actually asked to solve those problems.

Results & Discussion

Confirmatory factor analysis resulted in 7-item scale for value of mathematics (α=.88) and a 7-item bold problem solving scale (α=.77). After inspection of the content-specific confidence and content familiarity items, one item was removed from the pool for Calculus I, as almost all students answered that they had previously seen problems like it.[1] Alpha values for Intermediate Algebra content familiarity and content-specific confidence were .71 and .76 respectively. For Finite, the alpha values were .78 and .82. For Calculus I they were .82 and .82.

For each course, most affective variables were significantly correlated, although not all of the correlations were significant after Bonferroni corrections were applied. The correlations between content-specific confidence and content familiarity were .50 for Intermediate Algebra and .68 for both Finite Mathematics and Calculus I. Most remaining significant pairwise correlations ranged between .30 and .52.

[1] The content of this item aligned with limits, which was the topic of the course during the two weeks the survey was available. This explains the lack of variability for this item.

Table 1 reports the summary statistics for the variables included in the analysis for each analytic sample. Notably, students in all three courses identified as having high levels of content familiarity, with averages for all three classes between 3 and 4 on the scale, which means most students said they were either certain or somewhat certain they had solved problems like those they were asked about. Content-specific confidence was a little lower for all three classes, but still all were around a 3 out of 4, which corresponds to “I could solve this problem after some independent review.” Given that it is known that students often have had previous coursework in the areas, this relationship is not surprising, but it is confirmation that students do identify as familiar with the content in their courses. Their generally high confidence they could do the work without additional instructor instruction also suggests that when they have prior familiarity, they enter the classrooms believing they have retained a large amount of knowledge of how to do certain mathematical tasks.

Table 1. Summary statistics for covariates included in main analysis

	Intermediate algebra		Finite mathematics		Calculus I	
	Mean	*SD*	Mean	*SD*	Mean	*SD*
Male	22	33%	17	17%	83	43%
Freshman	60	85%	75	76%	121	63%
Content-specific confidence	3.068	0.592	2.901	0.674	2.723	0.660
Content familiarity	3.598	0.387	3.388	0.551	3.338	0.661
Bold problem solving	3.045	0.670	2.974	0.679	3.235	0.656
Math confidence	3.338	1.082	2.990	1.111	3.788	0.969
Math understanding	4.268	0.828	3.444	0.928	3.953	0.786
Math value	3.650	0.763	3.443	0.677	3.918	0.648
Total	71		99		193	

Note: Means are reported for un-centered variables. Categorical variables (male and freshman) report the total number of students in that category and their percent representation within the sample rather than the mean and standard deviation.

Table 2 reports the regression results for the three courses. In each of the three courses, the dependent variable was the factor value for the content-specific confidence, which is mean centered. Reported coefficients are in reference to the baseline student. For each analysis, the baseline is the same (up to the items used to create their content-specific content and content familiarity scores). In the Intermediate Algebra analysis, the baseline student is a male freshman who has course average content familiarity and content-specific confidence. In addition, he has course average orientations towards mathematics on the four included scales. A coefficient in the table corresponds to the unit increase on a students’ content-specific confidence given a one unit increase on the corresponding independent variable. So, for example, a one unit increase in a students’ content familiarity in Intermediate Algebra corresponds to a 0.743 unit increase in a students’ content-specific confidence. These coefficients are additive.

Table 2 shows that for each class, a students’ content familiarity is the biggest predictor of a students’ content-specific confidence, which is not surprising given the generally high correlation between these two variables. The table also shows that general mathematical confidence only provides an additional boost in Intermediate Algebra, while a higher bold problem-solving orientation provides a confidence boost in both Finite Mathematics and

Calculus I. Interestingly, being male does not provide a significant boost in any of the classes, despite the fact that it has been shown that men generally have higher levels of mathematical confidence than their female counterparts (Pajares & Miller, 1994, Ellis et al., 2016).

Table 2. Regression coefficients predicting content-specific confidence for three courses

	Intermediate algebra	Finite mathematics	Calculus I
Content familiarity	0.743***	0.777***	0.581***
	(0.143)	(0.094)	(0.052)
Male	0.071	0.075	0.078
	(0.112)	(0.128)	(0.074)
Freshman	0.286+	-0.005	0.133+
	(0.143)	(0.117)	(0.073)
Bold problem solving	-0.084	0.173*	0.121*
	(0.092)	(0.083)	(0.056)
Math understanding	-0.088	0.064	0.125*
	(0.081)	(0.072)	(0.056)
Math confidence	0.231***	0.081	0.022
	(0.064)	(0.059)	(0.048)
Math value	0.107	-0.169*	0.049
	(0.080)	(0.083)	(0.064)
Constant	-0.263+	-0.009	-0.117+
	(0.135)	(0.102)	(0.060)
N	71	99	193

Lastly, the of the proportion of students who accurately solved two problems they had been asked about were examined and then cross tabulated with their answer to the content-specific confidence question for that problem. For each course, less than half of those who said they were certain they could solve the problem provided an accurate response to the problem. Additional planned analysis of these data will examine the typed responses for answers that were “close” but incorrect. This analysis should help reveal whether students were taking the knowledge assessment questions seriously (and thus adding validity to the finding that students are not accurately assessing their knowledge) or not.

Conclusions

It has long been known that students enter college mathematics classroom with prior content experience. The findings presented here together suggest that this content familiarity may contribute to students overestimating their knowledge. To support students in successfully completing their courses, it may be worth spending time helping students more accurately assess their content knowledge early in the semester so they can make choices that support their learning.

If accepted, the author hopes to ask audience members for feedback on additional directions this work can go, the soundness of the constructs of content-specific confidence, and validity of the measure of content familiarity.

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Features of Discourses Regarding Linear Independence Concept

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My work, an ongoing research, focuses on the identification of features of discourses in describing linear independence, borrowing ideas from Sfard (2000; 2001) and Presmeg (1997;1998). This paper reports the analysis of two participants' discourses revealed in their responses to a single question. The findings showed that previously formed templates and signified-signifier pairs facilitated new semiotic spaces from which new signifiers (with new meanings) were adopted. Furthermore, participants' descriptions of linear independence closely resembled features of their templates.

Keywords: Linear algebra, Linear Independence, Discourses, Templates, Semiotic Chaining

Recent advancements in computational tools put matrices at the center stage of linear algebra curriculums where students are required to consider matrices within highly symbolic multi-representational environments. Then, there arises a question of the nature of knowledge formed in these environments. I set out to investigate characteristics of linear independence knowledge through the identification of features of discourses following two separate frameworks: discursive learning by Sfard (2000; 2001) and semiotic chaining processes by Presmeg (1997; 1998). Data came from interview responses of twelve first-year, college-level, linear algebra students who went through, for the most part, traditional, definition-theorem based, matrix oriented instructions with the use of computational examples. My work is an ongoing project. Its goal is to analyze responses given for nine questions regarding the linear independence concept in connection to other linear algebra topics such as span, spanning set, and dimension. So far, two participants' responses for a single question is analyzed. Thus, in this paper, I report findings from these two participants.

Discursive Learning and Semiotic Chaining

Sfard (2001) considered "becoming a participant in mathematical discourse is tantamount to learning to think in a mathematical way" (p.4). She defined learning as developing a mathematical discourse. Sfard (2001), furthermore, argued that learner's initial discourse may not necessarily entail the application of mathematical entities, and that it is necessary for learners to merge into a discursive activity regardless of the nature of elements of the discourse, even if it means using these elements merely as labels. Sfard (2000; 2001), moreover, characterized the progression of a discursive process in two stages: a) Template-driven use of new signifiers; and b) Objectified use of symbols. The first phase is identified by the inflexible use of signifiers. Specifically, this stage is described by the direct substitution of a new signifier into a previously formed template, namely an act of metaphoric projection. The second phase is characterized by a more flexible use of signifiers with "being signified via reification, and the signifiers will begin to be referred to as representations of other entities." (Shinno, 2013; p. 210).

Presmeg (1997) defined a similar notion of cognitive processes through the theory of semiotic chaining. She considered reification as an act in a semiotic chaining process where "at each node in the chain a new process is encapsulated as an object which stand in a signified-signifier relationship with its characteristic symbolism" (p. 276). In this theory, *a sign* is defined as a combination of a signified together with a signifier. For example, a signifier, "x," together with a signified, "a real number," is considered a sign. In Presmeg's terminology, "x" is also considered a metonymy since it stands for a real number, and the use of the sign a metonymic reasoning. According to Presmeg (1997; 1998), in a semiotic chaining process, a signifier in a previous sign combination becomes the signified in a new sign combination. Presmeg (1997) further described the signifier-signified pair: " I see the chaining process as involving metonymy-as indeed all signifiers are metonymic in a semiotic model, since they are 'put for' something else- and also reification since each signifier in turn is constructed as a new object " (p. 275). Furthermore, Presmeg's (1992) notion of prototypes closely resembles Sfard's (2000; 2001) notion of templates. Presmeg asserted that a prototype, described as the mental representation of a category, can be used in a metaphoric or a metonymic way to guide a reasoning process. Presmeg (1992) also stated that "every time a student of any branch of mathematics uses a diagram in reasoning, there is created a situation in which some category or member or sub-model is used to comprehend the category as a whole" (p. 600).

There are wide range of venues through which aspects of previously formed discourses are revealed. One of which is in the form of responses to mathematical tasks. My data came from a similar source. That is, I studied features of discourses exposed during one-on-one interviews. The following questions guided my investigation: a) What (if any) templates (prototypes) and their features are revealed in responses? b) What (if any) signified-signifier pairs and their features are revealed in responses? c) Are there any similarities among features of templates, new signified-signifier pairs, and descriptions of linear independence?

Method

Data reported in this paper, came from one-on-one interviews of two students from two sections of a first-year linear algebra course taught at a four-year, research oriented university in the US, offered at the 3000 level. Both students were high-performing males identified by the alphanumeric characters, B15 and A33. Students were numbered according to the order in which their names were listed in class rosters. Participant B15, for instance, was listed as the 15th name in the class roster of section B.

The linear algebra course can be characterized as a traditional course in the sense that topics were almost always covered in a chalkboard setting through definition-theorem based lessons, followed by computational examples. Johnson et al. (2002) was used as the textbook, and the following topics were covered in the order listed: System of Linear Equations and Matrix Representation (Row Reduction Process in the Context of Solution Types), Matrices and Matrix Operations (LU Decomposition; Inverse of Matrices); Vector Spaces (Linear Combinations, Linear Independence, Span, Spanning Set, Basis, and Dimension); Linear Transformations (Range and Null Space); Eigenvalues and Eigenvectors (Diagonalization).

Interviews were conducted toward the end of a semester. Each interview lasted about an hour and a half, and began with a set of nine pre-determined questions on linear independence. See Dogan (2018a, b; 2019) for further discussion. During interviews, new questions, on an as-needed basis, were added in order to elicit further insight. Participants were not provided any feedback (or new information) as to the appropriateness of their ideas. In this paper, I report the analysis of responses given to the following question:

Given the set $\{u_1, u_2, u_3, u_4\}$ where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent. Explain your answer.

In all sections of the course, linear independence ideas were introduced first via its formal definition (see Johnson et. al. 2002. p. 73), and followed by computational examples with the application of matrix representations. As far as any exposure to geometric representations is concerned, in section B, linear combination ideas (with connections to linear independence) were introduced using static geometric representations, and revisited by a take-home investigation using an online module that provided dynamic geometric representations, similar to ones in fig. 1. Section A was exposed to dynamic representations both in class and via the same investigation assigned in section B. See Dogan (2018a, b; 2019) for further discussion on both the module and the investigation.

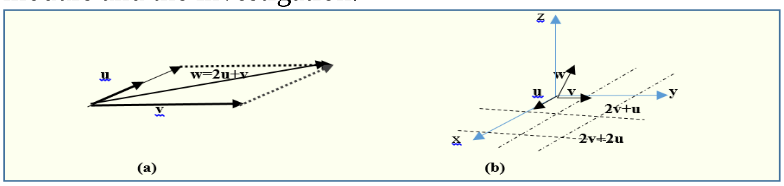

Figure 1. An illustration for vector addition and linear combination.

Results

In what follows, I discuss two participants' chaining processes applied in their discourses, focusing on features of templates, signified-signifier pairs, and descriptions of linear independence.

Participant B15

Features of B15's discourse are depicted in fig. 2. Overall, his discourse revolved around a signifier, $\{u_1, u_2, u_3, u_4\}$, provided in the interview question. Specifically, his discourse revealed the use of a previously formed template. In the excerpt below, indeed, one can observe that B15 is, metaphorically, projecting the signifier onto his previously formed template.

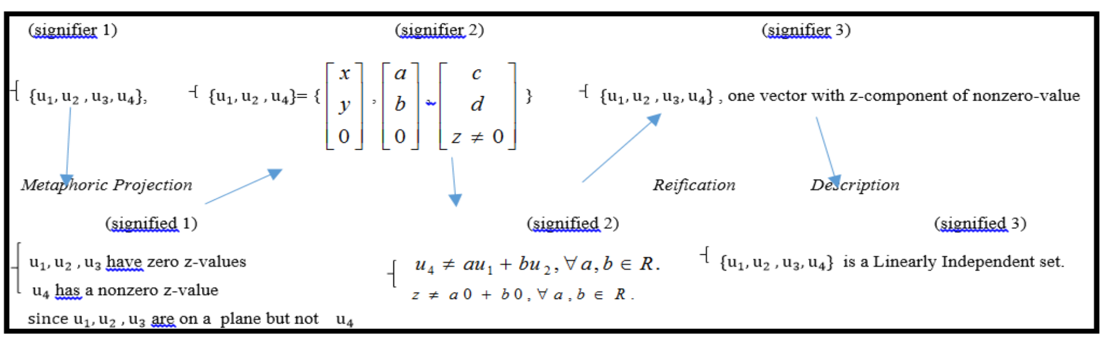

Figure 2. Semiotic chaining process for B15.

B15: Since these have no z component 'cause they're on the plane, then this vector can't depend on these 'cause these don't have a z component… the vector can't depend on the ones that are on the plane since they don't have a z component…So, these components really do not depend on each other because I would never get, you know, these 2 vectors [student points at the vectors located in the x and y axis] would never get your z vector at all…I guess we'll take the smallest case possible so each vector has

> certain number of components but let's just say for space and R^2 so a plane there's two components [meaning non-zero values] so we'll get that the space's over here...component to me is umh…dimension, so if you have two components I think of a plane…

His discourse revealed that B15's template came in the form of a prototype, a horizontal plane similar to the one in fig. 1, part (b). In this plane, vectors are numerically represented with three components, x, y and z, where z is always zero. B15 adopted this prototype to stand for all planes in R^3 (3-dimensional Euclidean space). In the excerpt above, he is, indeed, applying features of the prototype to the vectors of the signifier. As a result, he is characterizing the first three vectors, u_1, u_2, u_3, having "no z-component," meaning zero z-value, and the last vector, u_4, having a "z-component." This characterization, furthermore, provided a new semiotic space where he adopted a new signifier (signifier 2 in fig. 2). At this stage of his semiotic chaining process, B15 started integrating a linear combination notion among vector components, which, in turn, indicated the second signifier. This new signified-signifier pair, furthermore, seemed to have given birth to the third signifier in fig.2. The third signified-signifier pair revealed his notion of linear independence, which was along the lines of: "any set of vectors with one having a nonzero z-value is linearly independent." Seen in fig.2, his description closely resembled his template characteristics. This similarity can also be observed in the excerpt above. I note here that his notion of linear independence caused him to respond to the question incorrectly. That is, B15 completely ignored the fact that the first three vectors were given as being part of a plane, which, in reality, are always linearly dependent (bases of a plane contains at most two linearly independent vectors).

Participant A33

Features of A33's discourse are depicted in fig. 3. During his interview, participant A33's discourse revealed a metaphoric application of the *pigeon hole* analogy. He adopted the assignment of pigeons into pigeon coops as a similarity to the assignment of vectors to dimensions. This, also, appeared to have defined his template characteristics. In the excerpt below, he is, metaphorically, projecting the signifier, $\{u_1, u_2, u_3, u_4\}$, given in the question, onto his template. That is, he is signifying two of the three vectors "occupying" a single dimension. Note here that unlike B15, he is ignoring the fourth vector, u_4, and focusing on the first three vectors.

> *A33*: Given the set {u1, u2, u3, u4} where the vectors u1, u2, u3 are on the same plane and u4 is not. Determine if the set {u1, u2, u3, u4} is linearly independent and explain your answer. So…u1, u2, and u3 are on one plane…that means they are on R2, right? well… because it would be more of a space or 3D, the model of those are in R3, so R2 means it's a plane …so three vectors in … where the dimension is two means it's dependent already that these three [referring to u1, u2, u3] are somehow dependent… three vectors in two …I guess…what was that there...it was in discrete [a mathematics course] pigeon hole that where one of these will fit as… or two of these will be independent and the third one, will just be the third that will fit inside of one of those holes where the two are already occupying, so fitting three in two means plane and dependent …by the pigeon hole, then it's dependent…

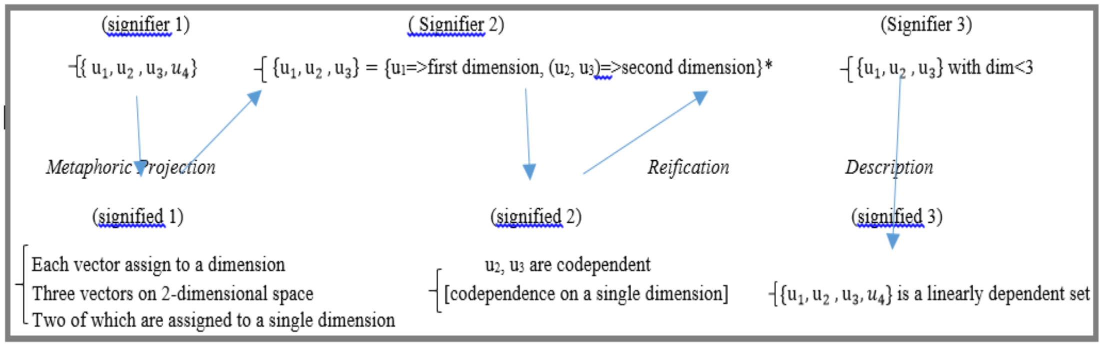

*Figure 3. Semiotic Chaining Process for A33: * "=>" stands for the vector assignment to dimension.*

During the interview, his template appeared to have facilitated him to work in a new semiotic space within which he considered a new signifier (signifier 2 in fig. 3). The signified 1-signifier 2 pair further gave way to his consideration of the second signified, "u_2, u_3 are codependent." In the excerpt above, one can observe, in fact, the latest signified-signifier pair channeling A33's attention to a third signifier, which contributed to his notion of linear independence. His description is revealed via the pair, signifier 3-signified 3. In fact, in the excerpt above, one can, clearly, see the influence of his template on his description of linear independence. I note that A33's notion, too, is flawed (in comparison to that of B15). Specifically, in light of his description of linear dependence, one can predict A33 identifying a linearly dependent set of three vectors in a 3-dimensional space as "linearly independent" since each vector gets a dimension assignment (that is, number of vectors is the same as the dimension of the space).

Conclusion

In this paper, I discussed findings from the analysis of two participants' discourses revealed in their interview responses to a single question. Borrowing from Sfard (200; 2001) and Presmeg (1997; 1998), I focused on participants' previously formed templates, signified-signifier pairs, and their descriptions of linear independence. Applying the two frameworks, I was able to identify distinguishing characteristics of participants' discourses. Moreover, I was able to identify stages, within participants' semiotic chaining processes, leading to their descriptions of linear independence.

Over all, my findings showed that participants' previously formed templates guided them to their initial set of signified-signifier pairs, in turn, allowing them to work in new semiotic spaces from which new signifiers (with new meanings) were adopted. Additionally, the analysis showed that the two participants' descriptions of linear independence closely resembled their template characteristics.

Even though one cannot make generalizations solely based on the analysis of two participants' work, one may agree that the findings gave a glimpse into how learners' templates may affect their conceptualization of mathematical concepts. As for the implications of the findings, educators may need to pay close attention to students' previously formed templates, especially, during the first introduction of a concept considering that most mathematical discursive processes begin to develop at the start. Of course, there is a need for future studies extending investigations to more participants and tasks with varying topics and questions.

Acknowledgement

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Exploring the Relationship between Textbook Format and Student Outcomes in Undergraduate Mathematics Courses

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As part of a large grant that investigates how students and instructors use university open source, and open access textbooks in teaching, we collected learning and performance data from students. In this paper we explore the relationship between textbook format (PDF versus HTML) and student outcomes in three undergraduate mathematics courses—linear algebra, abstract algebra, and calculus. We did not find differences in student outcomes by textbook format or course. We propose some reasons for these findings including little differences in instructor use of the textbook formats. We pose some questions for the audience.

Keywords: Mathematics textbooks, textbook format, student outcomes

Within the array of resources for teaching and learning, textbooks continue to be the most prevalent one for instructors and students. With new technological developments, textbook formats have been changing from paper to digital and open source formats, including sophisticated tools such as computing cells, annotation tools, and powerful search engines, easing mass access at relatively low cost. Importantly, open source textbooks never expire or go out-of-print and can be distributed at no cost to students, making them fully accessible. In countries in which post-secondary education costs are high, wide accessibility contributes to eliminating financial barriers to education. The sheer volume and availability of printed and digital resources calls for empirical work on instructors' and students' use of those resources to inform the field about ways in which these resources can improve teaching and learning. The study we report here is part of the large federally funded project that seeks to understand how instructors and students use open-source technologically enhanced textbooks in teaching university mathematics and the impact of those uses on student outcomes. This paper seeks to contribute to research on uses of university textbooks an area that is in its infancy. In this preliminary report, we use data collected over four semesters to explore the relationship between textbook format (HTML and PDF) and student outcomes (course grades and normalized gain on student test of knowledge).

Supporting Literature

The increasing range of printed and digital resources currently available for teaching and learning has led to broader conceptualizations of resources in mathematics teaching and learning (Pepin, Choppin, Ruthven, & Sinclair, 2017). These conceptualizations acknowledge the wide array of people involved in producing and using the resources and recognize the various activities in which such resources are designed, developed, and used. Most of the literature on student use of textbooks is either of a theoretical nature, studying how textbooks construct and constrain readers' usage of the textbook (e.g., Sierpinska, 1997; Weinberg & Weisner, 2011) or empirical in nature, seeking to fully describe how students construct mathematical meanings from reading or using textbooks (e.g., Shephard, Selden, Selden, 2010; Weber, Brophy, & Lin, 2008). Some empirical studies have also surveyed students about their textbook use (e.g., Weinberg, et al., 2012). The availability of data collected via learning management systems, and the increased sophistication of techniques to mine large volumes of such data, have opened the field to investigate questions about the relationship between students' time spent on these resources and their performance in courses (Phillips et al., 2010, 2011). These studies, however, do not contrast the potential benefits for students regarding the availability of different textbook formats, such as HTML or PDF. The impetus for developing free, open access textbooks is such that questions of benefit for students need to be addressed. We are in the very early stages of this exploration; in this particular study we explore the following research questions:

1. Are there differences in student outcomes between students who used the PDF format versus students who used the HTML format of the textbooks?
2. Are there differences in student outcomes between students who used the PDF format versus students who used the HTML format of the textbooks that depend on the course in which the textbook is used (linear algebra, abstract algebra, calculus)?

Methods

We use data collected over four semesters through the UMOST projects (Beezer et al., 2016, 2018). The data set includes 16 college instructors and their students ($n = 472$) teaching 19 different courses (linear algebra, abstract algebra, and calculus), using either an HTML or a PDF format of three open source, open access textbooks (Beezer's 2019, First Course in Linear Algebra; Judson's 2017, Abstract Algebra: Theory and Applications; and Boelkins' 2018, Active Calculus). Six courses taught by five instructors were assigned to use the PDF format of the textbooks (4 linear algebra, 1 abstract algebra, 1 calculus), while 13 courses taught by 11 instructors (9 linear algebra, 3 abstract algebra, 1 calculus) and their students used the HTML format of the textbooks. Instructors were asked to use only the format assigned. The content of the textbooks is identical in each format, both include hyperlinks and can be highlighted and annotated. The HTML format includes additional interactive features: sage cells for students to program mathematical computations and reading questions (abstract algebra and linear algebra), and pre-activities, WeBWorK exercises, and links to Geogebra animations (calculus). The reading questions and the pre-activities in the HTML textbooks collect students' responses in real-time, which are immediately seen by the instructor; in the PDF version instructors can also set up their learning management system to obtain student responses to those features.

The instructors were recruited through an open call from various networks. We collected instructor and student surveys that asked about attitudes about mathematics and textbook use and demographics; bi-weekly logs that asked about textbooks use throughout the semester; computer-generated heat maps of HTML textbook viewing; responses to student test of knowledge administered at the beginning and at the end of the semester; and student final course grade (as percent). For student outcomes, we used normalized gain in scores on the test of knowledge (specifically designed for the project) and student final percent grade in the course. T-tests were used to test the significance of the difference in student outcomes by textbook format and by course. From the instructor survey and logs we focused on questions that referred to how they used their textbooks for planning and teaching and how they described class activity. The corresponding open-ended questions were analyzed thematically to identify, first, how instructors described their past uses of the textbook for planning and for teaching; second, what they anticipated doing; and third, what they actually did. Questions pertaining to class activity were also categorized depending on whether the instructors described teaching that resembles lecturing (present content, ask questions, respond to students' questions) or that resembles student active engagement (group work, student presentations, working on ill-defined problems).

We then separated the themes by the textbook format they were using and the type of teaching they described to draw comparisons of the themes.

Preliminary Findings

We did not find a statistically significant difference in students' outcomes (normalized gain in scores on the test of knowledge and student final percent) by textbook format (see Table 1).

Table 1. Mean and standard deviation (SD) of student outcomes by textbook format.

	Textbook Format				
	HTML		PDF		
Student Outcomes	n	M (SD)	n	M (SD)	N
Final Course Grade in %	318	79.60 (16.19)	154	76.21 (21.24)	472
Normalized Gain in Test	228	0.18 (0.18)	118	0.14 (0.21)	346

The same was the case for course grade when we analyzed the data by course (see Table 2). However, for normalized gain, we found a statistically significant difference between PDF and HTML users of the abstract algebra textbook in favor of the HTML users.

We also found a statistically significant positive association between percent grade and normalized gain for the linear algebra and abstract algebra students ($r = 0.274$, $p < 0.01$, and $r = 0.303$, $p < 0.05$ respectively), but not for calculus students ($r = 0.378$, *n.s.*); thus, students who did well in the linear algebra and abstract algebra courses also tended to have higher normalized gain scores on the student test of knowledge. This was not the case for calculus students, which may be due to the smaller calculus sample size relative to the other two courses.

In sum, in our exploration, we did not find grounds to detect significant differences in student outcomes by the format of textbooks they are using.

Discussion

In this section, we provide some explanations for failing to detect significant differences in student outcomes between students who used the PDF and HTML format of the textbooks. First, it might be that the measures we have used in this study (student course grade and normalized gain in student test of knowledge) do not successfully detect student learning and performance outcomes. It is also possible that the instructors in the study assign course grades differently and thus the course grade is not a consistent measure for all students. Regarding the tests of student knowledge designed for this study, instructors may choose to cover different content and applications and some questions might be irrelevant to some students. Second, the small size of our samples, specifically when disaggregated by course, undermines the statistical power of the tests. Third, we had 126 missing values for the normalized gain scores ($N = 472$); systematic missingness (e.g., students in the PDF condition finding an available alternative HTML format) can be problematic for the inference. Fourth, detecting no difference in student outcomes may

Table 2. Mean and standard deviation (SD) of student outcomes by textbook format and course.

		Textbook Format				
		HTML		PDF		
Course	Student Outcomes	n	M (SD)	n	M (SD)	N
Linear Algebra						
	Final Course Grade in %	243	79.96 (16.25)	118	75.66 (22.12)	361
	Normalized Gain in Test	171	0.16 (0.18)	94	0.13 (0.20)	265
Abstract Algebra						
	Final Course Grade in %	64	79.43 (16.50)	20	78.63 (19.77)	84
	Normalized Gain in Test*	46	0.23 (0.16)	12	0.09 (0.13)	58
Calculus						
	Final Course Grade in %	11	72.59 (12.23)	16	77.26 (16.70)	27
	Normalized Gain in Test	11	0.27 (0.30)	12	0.32 (0.26)	23

Note: * Difference was significant at $p < .01$.

suggest that there are no differences in how students use any given format of the textbooks. This could happen because students are not fully aware of the interactive features of the textbooks so they use the HTML format as they would use a PDF format. Perhaps instructors do not require or encourage students to use these features because the instructors themselves use both formats similarly. In order to see whether this was the case for the instructors, we used data that asked them about past and anticipated use of textbooks via surveys and on-going use via logs. There was an overlap in the themes that both groups of instructors mentioned in creating the syllabus and lecture notes and planning lessons. The users of the HTML format (n = 13) mentioned a few more themes relative to the instructors who used the PDF format (n = 6). For example both groups of faculty mentioned following the textbook for outlining and creating the lecture notes to maintain consistency in notation and sequencing; in addition the HTML faculty mentioned adding "a personal touch" to their lecture notes. Fifth, it might be that it is impossible to isolate the effects of the textbook (or any other resource) in the absence of considering its use, in particular the actual classroom instruction. Thus, it is not the textbook but what instructors do in classrooms *with* the textbook that matters. Our preliminary analysis of instructors' data shows that instructors tend to employ less active learning methods of teaching throughout the semester than they anticipated employing at the beginning of the semester survey. We also categorized the ways instructors reported student participation in class depending on whether the instructors described teaching engaging students minimally (present content, ask and respond to students' questions), or teaching with more substantial engagement (group work, student presentations, working on ill-defined problems). We found that two out of five PDF instructors and four out of 11 HTML instructors practiced active teaching. However, possibly due to our small sample of instructors, we did not find any relationship between the type of student participation in the courses and student outcomes or textbook format.

Conclusion

We did not find differences in student outcomes between students using the PDF format and students who used the HTML format of the three textbooks in the project. It may be that we need to attend to other features of the textbooks that can have an impact on those outcomes. The three textbooks used in the project vary from less to more conventional, with the calculus textbook including more explicit conceptual activities that engage students with the content in and out of class. The linear algebra textbook contains many computational cells that students can program and use for testing theorems as they study. The abstract algebra textbook follows the standard definition- theorem-proof model (Thurston, 1994). Alternatively, textbook designers and researchers might need to be more explicit about these features as they onboard faculty into the study, so that they use the textbook features as intended. As we continue working on the project, we will pursue these possibilities. The rising costs of textbook publishing suggest that open-source and open-access textbooks are here to stay, and we need to identify research-based practices that capitalize on the use of these resources for learning.

Questions for Discussion

1. How else can our results of finding no difference in student outcomes between PDF and HTML users be explained?
2. What other data and methodology should we consider in our investigation of textbook use to detect differences in student outcomes by format of the textbook?

Acknowledgments

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The Use of Nonstandard Problems in an Ordinary Differential Equations Course for Engineering Students Reveals Commognitive Conflicts

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We report on a teaching intervention in an ordinary differential equations (ODEs) course for engineering students focusing our attention on the role of nonstandard problems in the development of students' conceptual understanding. The lecturer designed a set of problems challenging traditional approaches to Existence and Uniqueness Theorems (EUTs). Our analysis of the students' mathematical discourse developed in the process of group discussions within Commognition Theoretical Framework reveals unresolved commognitive conflict. We suggest how modifications in the problem design prompted by the students' mathematical discourse can be used to improve students' conceptual understanding of the material.

Keywords: existence and uniqueness theorems, design research, commognitive conflict, nonstandard problems

Introduction

The concept of "*solution*" is very important in mathematics. Students encounter this notion for the first time already in primary school where it is interpreted as *an answer to a given problem* as well as *any value or set of values that can be substituted into an equation to make it a true statement.* When it comes to a multifaceted definition of solution in the university ordinary differential equations (ODEs) course - general and particular solutions, solutions to initial value and boundary value problems - students should change their mathematical discourse and indorse new narratives related to the term "solution".

ODEs courses are acknowledged as an important part of engineering education (Francis, 1972). An inquiry-oriented approach to teaching ODEs contributes greatly to a better retention of knowledge (Rasmussen & Kwon, 2007). However, tasks where one must distinguish between different meanings assigned to different types of solutions and use the correct notion often create difficulties for students learning ODEs (cf. Arslan, 2010, Raychaudhuri, 2007; 2008).

In this study, we use Sfard's (2008) commognition theory to analyze the development of students' discourse on Existence and Uniqueness Theorems (EUTs). We reported recently how students who solve non-standard problems exploit familiar mathematical routines in new situations (Treffert-Thomas et al., 2018). Present research develops on the "work in progress" report presented at the First International Commognitive Workshop (Haifa, February 2019).

Research setting and participants

The research took place in an ODEs course for engineering students in their fourth year of studies towards a master's degree and lasted over a period of three weeks. The course followed the textbook "Elementary Differential Equations and Boundary Value Problems" (6th edition) by Boyce and DiPrima. The research activity constituted an assessed assignment and was designed within normal teaching time in the final part of the course when students had acquired sufficient theoretical knowledge and developed good computational skills. The number of students in different sessions was varying between 51-67% of the total number of students enrolled in the course. The lecturer, one of the authors of this paper, devised a set of six nonstandard questions to challenge students' conceptual understanding of the EUTs. Students were requested first to work on the problems individually in the tutorial and at home. One week later they discussed solutions in small groups, audio-recorded the discussion and presented revised solutions to their peers in the next tutorial.

Theoretical perspectives

Our research draws on the Commognitive Theoretical Framework (CTF) (Sfard, 2007; 2008; 2012; 2014) which views thinking as an individual version of interpersonal communication and defines discourse as a specific form of communication. CTF identifies four discursive characteristics of mathematics: the word use, visual mediators, narratives, and routines. An important concept in CTF, and especially for the present study, is *commognitive conflict* defined as "the encounter[s] between interlocutors who use the same mathematical signifiers (words or written symbols) in different ways or perform the same mathematical tasks according to different rules" (Sfard, 2008, p. 161). Commognitive conflict is considered as a "situation that arises when communication occurs across incommensurable discourses" (Sfard, 2008, p. 296); it opens "a gate to the new discourse rather than a barrier to communication, both the newcomer and the oldtimers must be genuinely committed to overcoming the hurdle" (Sfard, 2008, p. 282). CTF distinguishes two types of learning, object level and meta level. Meta level learning "may be a result of explicit discussion about the ways by which words are used" (Tabach & Nachlieli, 2016, p. 302) and "resolving a commognitive conflict includes meta level discussions about how words are used" (Tabach & Nachlieli, 2016, p. 302). The research question we address in this paper is: "*To what extent the development of the mathematical discourse in the process of solving nonstandard problems in the ODEs course contributes to students' conceptual understanding of important theoretical results?*"

Data collection and analysis

The activity formed an assessed piece of coursework for an ODEs course and spanned over two tutorials in a three-week period. In the beginning, students were instructed about the nature of the coursework including its content and assessment methods. In the first tutorial students worked individually on problems for one hour. The lecturer collected students' written solutions (referred to as Scripts 1), photocopied them and returned to students who were asked to work on the problems on their own at home and to bring revised solutions (Scripts 2) to the next tutorial. In the second tutorial students' Scripts 2 formed the basis of discussions in small groups; discussions were audio-recorded and transcribed. Finally, each group presented a joint solution to one of the problems to the rest of the class. Following the group presentations, students had one week to hand in the final version of their written solutions for marking and assessment (Scripts 3). The number of responses for each of the scripts varied as also did the number of responses for each of the six problems. To analyze students' written work (Scripts 1, 2 and 3), mainly qualitative methods of data analysis were used, namely, reading, interpreting and categorising (Cohen, Manion, & Morrison, 2008).

Our analyses of the solution scripts, transcripts of discussions in small groups and students' presentations reveal several manifestations of commognitive conflicts in students' discourse on the notion of "solution". An example of one such conflict is provided below in the transcript excerpt describing the episode where four students S11-S14 in group 1 discuss their solutions to one of the tasks. Our comments (in italics) are provided in the brackets after each utterance.

Decide whether the statement "Solution curves to a differential equation never intersect" is true or false. Explain your reasoning.

L1-4 S 12: What I did in (a) is basically, if they can intersect. Yes, they can intersect, if you've got an alternating solution, say, a solution involving sines and cosines…. Periodic curves will at some point cross each other. They cannot be unique. [*Talks about a solution to a DE*]

L5-8 S13: … if the lines could intersect, if there are two solutions to the same equation and the lines could intersect you could use the intersection point as the initial condition for either of the solutions and then you have two solutions for a single initial condition which would conflict with the Theorem. [*Talks about the solution to the IVP*]

L9-10 S12: … $y'' + y = 0$ can have solutions both in sines and cosines … and they will intersect every now and then. [*Talks about solutions to a DE*]

L11-12 S12: That one, it's kind of, but it will offer unique in the crossing point…, so you can go in both directions. [*Talks about the solution to the IVP*]

L13-14 S13: Yes, because the initial point can be any point of the curve. [*Talks about the solutions to an IVP*]

L15-17 S12: … this is not the solution curves of the differential equation; it is just the general solution. … So yes, the solutions can intersect. But they do violate the EUT. [*Talks about the general solution in the context of the EUT*]

L18-20 S12: But the solutions [are] sine and cosine for $y'' + y = 0$, I suppose that's because of the imaginary numbers. Right? …There may be a conflict there with the imaginary numbers. Possibly. [*Talks about the general solution in the context of the EUT*]

L21-24 S12: But the standard differential equation, they haven't said that we have to find something that satisfies the theorem. It's just a differential equation. It's a statement at a differential equation level, so yes, they, the curves can intersect because of the imaginary numbers. [*Talks about solutions to a DE*]

L25-26 S11: These are just the solutions, not the curves of the differential equation itself. These are the solution curves. [*Tries to distinguish solutions to a DE and that to the IVP*]

L27-29 S13: The solutions won't be $sint$ <u>or</u> $cost$, it's $C1sint + C2cost$, that's the solution. So, there won't be two curves, there will be one curve, just the sum of sin and cos. [*Talks about the general solution to a DE*]

L30-32 S11: I am not completely sure, I was just thinking along the same lines as you did with the periodic curves, that they might sometimes intersect, if you just happen to have one of those expressions. [*Talks about solutions to a DE*]

L33-36 S12: … there are starting conditions that define them. And the same equation with two different starting points may have two periodic solutions, they won't be perfectly aligned, they might intersect at some point, in every cycle they will. [*Confusion with the definitions leads to a wrong conclusion*]

L37-38 S11: But it's two different solutions that's why you cannot compare them. [*Talks about solutions to a DE*]

L39-42 S12: But here we are not asked to find the particular solution, just the solutions. You can get infinitely many curves out here. And they're starting points basically, that's the only difference between them … can be written as … and vice versa, that's my reasoning. [*Talks about a solution*]

L43-48 S13: If you have a solution, a general solution to the DE, and two different initial conditions, two different curves, and the curves intersect, then it's possible to set an initial condition at any time on the curve, right? If the curves intersect, there will be a single point or more, where you can say that the given initial condition has two solutions, not according to the theorem at least. Hopefully. [*Talks about the solution to the IVP and the EUT*]

L49-50 S11: At least we can agree to disagree which is something. [*Unresolved commognitive conflict*]

Since the students learned in the course two EUTs for the general first-order ODEs and for the linear equations of arbitrary order, the solution should not be limited to the analysis of only the linear case. In the selected excerpt, four students discuss the second option considering a second order linear DE with constant coefficients for which the statement is true because conditions of the EUT are always satisfied. Students S12 and S13 talk most of the time, both reasoning correctly within their individual mathematical discourses. Attempting to solve the problem which requires a "new" definition of the solution (to the IVP), S12 uses a "school-induced" definition of solution as "a function that upon substitution in a DE turns it into a true statement". He suggests that the solution can be represented by trigonometric functions of sines and cosines, concluding that their graphs as graphic representations of periodic functions should intersect *(L1-4)*. He talks about *any* solution of the differential equation as a function that satisfies it upon substitution into the equation *(L9-10).* He continues his discourse mentioning the "crossing point" *(L11-12)* and by doing so he shifts the discussion to the solution of the IVP. Then he recalls the EUT *(L15-17)* and decides that the theorem is violated, finally concluding mistakenly that this is the case in the example he suggested *(L9-10)*. His false argumentation originates from his interpretation of the "solution curve" as "just a solution" *(L39-42)* and not as a "particular solution". On the other hand, S13 uses proper mathematical language and the required notion to explain correctly the solution procedure. He points out that the "intersection point" can be chosen as the initial point for the solution *(L5-8)*. By the "solution curve" he means the solution to the IVP. He explains that the solution of IVP can be found from the general solution *(L43-48)* which is the correct way to solve the problem and he mentions EUT in this context though he does not argue that the conditions of the theorem must be satisfied, commenting "hopefully".

Discussion of S12 and S13 manifests two unresolved commognitive conflicts. Firstly, S12 uses an "old" definition, arriving at the conclusion that is no longer correct within a "new" discourse, and does not understand the explanation of S13. In turn, S13 fails to explain how *a change in his own discourse* led to a different, correct solution of the task. We observe that "*a change in one's discourse"* is necessary for learning (Tabach & Nachlieli, 2016, p. 301) but it is not always sufficient for the correct individual creation of an abstract mathematical object.

Discussion

A straightforward conclusion from this episode is that a teacher's intervention was required for timely resolution of the commognitive conflict, but lecturer's participation was not planned during the small group discussions. However, the analysis of the students' discussion suggests an efficient "post factum" solution to this problem: the lecturer may design a new task using the DE brought up in the discussion of the problem by S12 to address the delicate interplay between the use of general and particular solutions in EUTs in the next lecture or tutorial opening a new meta level group discussion leading to further development of the changed and extended mathematical discourse. Our analysis agrees with the work by Thoma and Nardi (2018, p. 161) who argued that a more systematic presentation of differences between different discourses and facilitation of flexible moves between them are needed. Our research also provides a new strong argument in support of the necessity to teach engineering students (at least some) theoretical results emphasized already four decades ago by Roberts (1976).

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The Role of Mathematical Meanings for Teaching and Decentering Actions in Productive Student-Teacher Interactions

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Teachers' mathematical meanings impact their instructional practices and constitute their images of the mathematics they teach and intend students to learn (Thompson, 2013). While numerous studies have focused on K-12 teachers' mathematical knowledge for teaching (MKT) and mathematical meanings for teaching (MMT) (Thompson, 2013), few studies have examined university level instructors' mathematical meanings (e.g., Musgrave & Carlson, 2016). In this report, we explain what we mean by mathematical meanings for teaching and productive student-teacher interactions and use video data to characterize the relationship between teachers' MMT and decentering actions when teaching. Our results illustrate how a teacher's MMT can influence the teacher's ability to make sense of and use student thinking during an interaction in which the teacher is attempting to decenter. Conversely, we illustrate how decentering actions can lead to advances in a teacher's MMT.

Keywords: mathematical meanings for teaching, decentering, and productive student-teacher interactions.

Introduction

Research on mathematical knowledge for teaching (MKT) has developed with the goal of trying to determine relationships among the mathematics that teachers know, their instruction, and students' learning. While many researchers have investigated aspects of teachers' MKT (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Hill et al., 2008; Speer et al., 2015) and MMT (Thompson, 2016), little is known about the mechanisms for advancing teachers' mathematical meanings and teaching practices. In particular, the problematic nature of trigonometry teaching and learning is broadly documented (Moore et al., 2016; Tallman, 2015; Thompson, 2008; Thompson et al., 2007, Weber 2005). Some researchers (Hertel and Cullen, 2011; Moore 2012, 2014; Tallman, 2015; Thompson, 2008) have responded by leveraging quantitative reasoning (Thompson, 1990, 2011) to support students in conceptualizing an angle measure and trigonometric functions. Tallman and Frank (2018) have documented that an instructors' inattention to quantitative reasoning contributes to the instructor conveying meanings of angle measure that are inconsistent and incoherent for students (Tallman & Frank, 2018). In the same study, they further reported that providing instructors with a conceptually focused curriculum designed to promote students' quantitative reasoning does not necessarily lead to dramatic advances in a teacher's enacted knowledge when teaching. They call for pre- and in-service precalculus level teachers to be supported in learning to reason quantitatively and recognize the conceptual affordances of supporting their students in engaging in this reasoning. In response to the multiple calls to address teachers' content learning in the context of student learning (e.g., Hill et al., 2008; Speer et al., 2015) we describe an intervention we designed to improve instructors' meanings and teaching practices in the context of their teaching precalculus mathematics using a research-based curriculum. We include a description of our goals for high-quality instruction, productive student-teacher interactions, and what research has reported to be productive meanings for trigonometric functions. We then present results from examining the influence of a teacher's MMT on her ability to decenter.

Theoretical Perspective and Literature Review

Researchers have proposed mathematical meanings as the organization of an individual's experiences with an idea (Thompson, 1994). Meanings have also been described as "what people intend to convey via an utterance, and what people imagine being conveyed as they hear an utterance" (Thompson, 2013, p.58). When studying individuals' meanings, researchers are at a disadvantage as meanings reside in the minds of the individual constructing and interpreting them (Thompson, 2013). Thus, researchers must focus on the individual's expressed meaning of an idea, or the spontaneous utterances that an individual conveys about an idea.

Teachers' expressed meanings of an idea are of interest because they emerge during classroom discourse and impact the ways of thinking that students develop about a particular mathematical idea. Thus, an individual's meaning can be more or less productive for teaching. Moore (2012) described a productive meaning for the sine function that is grounded in quantitative reasoning. Namely, the outputs of the sine and cosine functions are values that represent measures of quantities and entail a unit. For instance, sin (0.8) = 0.717 means that the vertical distance above the circle's horizontal diameter to a point on the unit circle is approximately 0.717 radii or 0.717 times as large as the circle's radius, when the terminal ray of an angle has been rotated to a point that is 0.8 radii counter-clockwise from the 3 o'clock position on the circle. A productive meaning for trigonometric functions also includes robust connections between right triangle trigonometry and unit circle trigonometry. The commonly used trigonometric ratios (SOHCAHTOA) can be viewed as using the hypotenuse (radius) as a unit of measure for the legs of the right triangle. As one example, the output of the sine function can be viewed as the length of the triangle's leg that is opposite the angle whose measure is input to the sine function, measured in units of the hypotenuse.

The mathematical meanings that teachers have impact their ability to respond productively to students' thinking. Productive student-teacher interactions require strong mathematical meanings for teaching (Thompson, 2016), or meanings that prepare students for future learning and lend coherence to students' current meanings. Musgrave and Carlson (2016) described high-quality instruction as teaching practices that support students in constructing a deep understanding of the central concepts of a course and developing flexible problem-solving abilities. They further noted that high-quality instruction involves teachers enacting teaching practices of interpreting and acting on student thinking and reflecting on the impact of their actions on student learning. Jacobs & Empson's (2016) findings further corroborate that teachers who engage in high-quality interactions with students enact responsive teaching that includes a propensity to decenter. A teacher engages in decentering actions when she makes sense of and uses student thinking to inform in-the-moment interactions with her students (Bas Ader & Carlson, 2018). Decentering also involves setting aside one's own thinking in an attempt to understand what other individuals understand (Steffe & Thompson, 2008) and "provides a powerful lens for examining student-teacher interactions because it moves beyond discourse analysis by including a focus on the teacher's interpretations of students' verbal and written explanations to make decisions in the moment of teaching" (Tuescher, Moore, Carlson, 2015, p. 437).

Methodology

The goal of this study is to explore mechanisms for advancing teachers' meanings and content knowledge for teaching. To accomplish this goal, we video recorded classrooms for the purpose of building characterizations of teachers' thinking, and pedagogical practices. We collected data from pre-calculus classrooms taught by mathematics graduate students and instructors at a large, public, PhD-granting university in the United States. Participants' teaching experience varied between zero and twelve years at the K-12 and tertiary levels. We video recorded eleven graduate students and two mathematics instructors when teaching using a research-based Pathways to Pre-Calculus materials. We analyzed each video and selected segments of teaching in which the instructor demonstrated strong MMT, decentering abilities, and teaching practices that involved speaking with meaning (Clark et al., 2008), attentiveness to student thinking, mathematical care (Hackenburg, 2010), and conceptually oriented explanations (Thompson & Thompson, 1996). For this report we selected a sequence of excerpts from one teacher who exhibited a strong tendency to decenter and a relatively strong MMT. We selected these excerpts because of their potential to reveal insights about how a teacher's MMT impacts the teacher's effectiveness in decentering. This teacher was in her second semester of using the Pathways Precalculus materials and her first year of teaching.

Results

We share data from one teacher that reveals the expressed meaning that students conveyed for determining the values of missing angles in a right triangle. We then provide an excerpt of the instructor's response to a student's question. Collectively, our data reveals some meanings students have for right triangle trigonometry, as well as, the role MMT and decentering play in a productive student-teacher interaction.

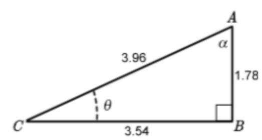

Figure 1. Missing Angles

Students were asked to determine the unknown angle measures in a right triangle (Figure 1). This problem was introduced by drawing a circle with its center at the angle's vertex and by using the triangle's hypotenuse as a radius. Students then used the arcsine function to determine the values of the unknown angle measures. Excerpt 1 followed a whole class discussion which focused on making connections between unit circle trigonometry and right triangle trigonometry. Namely, when an angle's vertex lies at the center of a circle, the value of $\sin(\theta)$ is the vertical distance from the horizontal diameter (opposite leg) measured in radius lengths (hypotenuse), hence, the ratio: opposite over hypotenuse. Following this discussion, Trevor asked if the value of $\sin(\theta)$ could be determined by using lengths of the opposite leg and the hypotenuse (i.e., 1.78 and 3.96) (see Figure 1) and what we know about special right triangles (Excerpt 1).

Excerpt 1

Trevor: I have a question. What if you went about it a different way, without having to do math?

Trevor: Like, I know earlier in the class we talked about the two types of special right triangles, the 30-60-90 and the 45-45-90 and we know that the 30-60-90 have angle measures that are proportional to the legs.

Trevor: So, 90 would be the greatest length, the hypotenuse, and so on.

Trevor: So, we see that there is a 90-degree angle, and we see that the two legs are not equal so we can assume that the triangle is 30-60-90.

Trevor: Then, we assume that 1.78 is equal to a 30-degree angle and 3.54 is equal to a 60-degree angle.

Trevor elaborated his thinking by expressing that since the right triangle in the drawing was not isosceles, it had to have angles that measure thirty, sixty, and ninety degrees. It appeared that

Trevor thought all right triangles were one of the two "special" right triangles he had explored when studying triangle trigonometry (Excerpt 1).

The teacher appeared to recognize the reasoning Trevor was using and elected to ask the other students in the class what they thought about Trevor's question (Excerpt 2). This instructional move reveals that the teacher valued student thinking and determined that it was worth spending class time to have all students consider Trevor's question. After giving students time to discuss their thinking another student, Kelsey, responded by reminding the class of the animation the teacher had used in a prior class that illustrated the continuous variation of $\sin(\theta)$ and θ, while varying θ from 0 to $\pi/2$. The animation showed a right triangle with the angle of interest at the center of a circle, and with a hypotenuse as its radius. In the prior class the instructor used the animation to drag the terminal point along the circumference of the circle, while discussing how the angles of the triangle and the values of sine and cosine varied while varying the location of the terminal point.

Excerpt 2

Instructor: What do you guys think? {Pause} You guys tell me, what do you guys think?

Kelsey: Just because 45-45-90 and 30-60-90 exist doesn't mean that is all triangles.

Kelsey: Because as we were drawing the circles around the triangles on page 323 and remember the website where she could drag the thing and you would see the two arrows that represent the values of cosine and sine move.

Kelsey: If you move it a little bit, then that means that the angle grows to something past 30 degrees or 60 degrees or 45 degrees, and the sine angle also changes to something other than 30, 60, or 90.

Kelsey: So, not all triangles, just because they sort of resemble a 30-60-90 triangle, there is a lot of differences.

Kelsey's response to Trevor's question suggests that she conceptualized an infinite number of angle measures that a right triangle can have as theta varies from 0 to $\pi/2$. Kelsey's response also suggests she has an understanding of trigonometry grounded in covariational reasoning (Carlson et al., 2002; Saldanha and Thompson 1998; Thompson and Carlson, 2017). Namely, as the angle measure varies, the values of sine and cosine vary as well. The instructor retrospectively confirmed that her motive for probing students' thinking about Trevor's question was motivated by her desire to give all students an opportunity to confront the limitations of using just the two special right triangles. The instructor proceeded to lead a whole class discussion about the connections that emerged during the interactions (Excerpt 3). Since class was about to end, her discussion was teacher led. However, we claim that it was student centered because of its attention to describing and connecting the thinking that Trevor and Kelsey had exhibited during the prior exchanges (Excerpts 1 and 2).

Excerpt 3

Instructor: Tremendous observation, okay? Very good, I really like that you're thinking about, alright this is a right triangle, we have two types of special right triangles.

Instructor: What Trevor is saying is, well we know there is a 30-60-90 right triangle and a 45-45-90 right triangle.

Instructor: Then he also said, well the two legs 1.78 and 3.54 are not the same, so that tells me it should be a 30-60-90 triangle because the legs aren't the same length.

Instructor: And in the 45-45-90 triangle, the legs would be the same length because they are across from equal angle measures.

Instructor: But, the hole in your argument is, are 30 and 60 and 45 and 45 the only numbers that add to 90?

Instructor: No, right? There are all kinds of different angle measures that add to 90 degrees.

Instructor: So, we can't assume right away, and this is what Kelsey is arguing, we can't assume right away that just because that is a right angle, that this triangle is either 30-60-90 or 45-45-90.

Instructor: But I really like that you're thinking about that. It's really good reasoning.

Instructor: Does that make sense what just happened? Any questions?

The instructor's response is a good example of a conceptual explanation that is rooted in the teacher's strong MMT about unit circle and triangle trigonometry and how they are connected. The instructor's response revealed that she understood Trevor's misconception (and assumed other students might also have this misconception) and made a deliberate move to support all students in considering his question. The instructor's strong mathematical meanings for the sine function and her propensity to decenter enabled her to interpret, reflect on, and respond to student thinking effectively. We notice that her discussion conveyed both her image of Trevor's question and her image of the productive thinking that Kelsey expressed (Excerpt 3). This provides a useful example of how a teacher's mathematical meanings can lead to quick and accurate interpretations of student thinking in the moment of teaching that can be productively leveraged during a discussion.

Conclusion and Discussion

The teacher's strong mathematical meanings (see Excerpt 3) allowed her to interpret and act on the students' thinking. We conjecture that teachers with strong mathematical meanings for teaching an idea and a strong propensity to decenter will be more likely to make sense of, attend to, and effectively leverage student thinking in the moment of teaching. Our results further reveal how the teacher's actions to decenter advanced the teacher's MMT to include new images of students' impoverished understanding of sine function; an understanding of the sine function as a static image of opposite over hypotenuse for two special right triangles. In a discussion with the teacher after her teaching episode she conveyed that this was the first time she had seen a student struggle to make connections between special right triangles and unit-circle trigonometry. She expressed that she was not aware that the idea of continuous variation is something students struggle with when making connections between right-triangle and unit-circle trigonometry. This insight should be beneficial in her future instruction and is what led to her making a productive instructional move to support students in connecting the common image of sine as opposite side over hypotenuse with the conceptual meaning of a sine function in unit circle trigonometry.

We conjecture that the collection of video excerpts that emerged from this study will be useful for developing other teachers' MMT to include more images of students' productive and unproductive ways of thinking, and possible instructional moves for supporting students in understanding the sine function meaningfully. We are using these videos in case studies being developed in the context of a broader professional development program with teachers who are using the research-based Pathways to Pre-calculus curriculum. We hypothesize that these video case studies will lead to teachers reflecting on and advancing their meanings of the ideas that are central to the Pathways lessons and the video reflections.

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I'm Still Confused in the Most Basic Way: How Responsibilities Impact Mathematics Learning While Video Watching

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While videos are becoming more pervasive in math instruction and remediation, little is known about what students are doing to make sense of the math while watching videos. This study presents a case of one student who is using videos to learn college algebra. Using the didactic contract, we suggest that the student's inability to live up to the rules which she has for herself and the video may impact her mathematics learning while video watching.

Keywords: didactic contract, college algebra, video watching

Videos are being used in the mathematics classroom in many ways, including instruction (Trenholm, Alcock, & Robinson, 2012; Zhang, Zhou, Briggs, & Nunamaker, 2006), remediation (Pinzon, Pinzon, & Stackpole, 2016), and motivation (de Araujo, Otten, & Birisci, 2017; Guo, Kim, & Rubin, 2014; Sahin, Cavlazoglu, & Zeytuncu, 2015). While studies have investigated student learning gains and attitudes while video watching (Crook & Schofield, 2017; Trenholm et al., 2012), few have observed how students are making sense while watching videos to learn mathematics content. The theory of didactic situations has been used to identify the implicit rules involved in teacher and student relationships during mathematics instruction (Brousseau, Berdard, & Nontna, 2014; Gueudet & Pepin, 2018). In this study I have applied the didactic contract to address the question, "What do students in a collage algebra class believe they are responsible for when watching mathematics instructional videos about solving systems of equations by elimination?" In this presentation, I will be reporting on one of the participants in the study, Tara.

Prior to video watching, Tara was given a short assessment of prior knowledge about solving systems of equations and a survey about her mathematics-related beliefs. She was comparatively in the low range in both of these areas. She participated in 2 semi-structured interviews during which she watched the two videos, one by Khan Academy and one by NancyPi. Before and after watching the videos, I asked the Tara questions about her beliefs about mathematics in general and about the specific mathematics represented in the videos.

After analysis of the data, I identified Tara's responsibilities in learning math in terms of knowing and applying the rules, not making "silly mistakes," and working hard. Tara believed that it was her responsibility to uphold these responsibilities in order at be good at math. Additionally, Tara saw the instructor in the video as having responsibilities to show how to do the rules and "tricks" involved in the mathematics content being presented, to show the main idea or main procedure and to design a video that was not distracting to her learning. Unfortunately, the didactic contract between Tara and the mathematics instructional video was not always upheld, leading to what Brousseau et al. (2014) referred to as "ruptures." Sometimes these ruptures came when Tara was not able to meet her own expectations and sometimes they came when the videos did not fulfill the responsibilities that Tara had placed on them. I suggest that the ruptures in the didactic contract may have impacted the ways in which Tara was able to make sense of the mathematics in the video. These findings may be applied to the ways in which mathematics instructors are using and creating video content, especially for struggling students such as Tara.

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Comics as Pedagogical Tools in First-year Linear Algebra

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The goal of this study was to determine the ways in which comics affect student learning in undergraduate math courses, specifically in first-year linear algebra. Students had access to eight comics, posted roughly each week during the term. The survey data collected shows that the majority of respondents reported a better understanding of the material after reading the comics and that the comics had a positive impact on their engagement, attitude, motivation, and overall understanding.

Keywords: comics, linear algebra, undergraduate education, first-year course

Motivation and Research Goals

Empirical studies have examined the ways in which comics can impact student learning in fields such as chemical engineering (Landherr, 2016), medicine (Sim, McEvoy, Wain & Khong, 2014), and management (Short, Randolph-Seng, & McKenny, 2010). Research has shown that comics are beneficial to various aspects of student learning including memory (Aleixo & Sumner, 2017), knowledge (Landherr, 2016), and attitude (Hosler & Boomer, 2011); however, there is little scholarship focusing specifically on university mathematics courses. The goal of this study was to determine the ways in which comics affect student learning in math courses with an eye towards advising instructors on incorporating comics in their courses.

Methods, Results, and Discussion

The authors created eight comics to complement the Honours Mathematics first-year Linear Algebra curriculum. Students accessed the comics, which were posted roughly every week, through a separate course page in their learning management system. Data were collected through short surveys accompanying each comic and a long survey at the end of the term.

Each short survey featured pre/post-comic questions regarding its associated comic and its impact on students' learning. On average, 86% of students reported a better understanding of the material after reading the comics, regardless of their confidence level before reading the comics.

In the long survey, students reported the effect that the comics had on their engagement, attitude, motivation, and understanding. Out of the 61 respondents who read the comics, 81% reported that at least half of the comics helped them feel engaged in the course; 70% reported that at least half of the comics made the course more interesting; 74% reported that their attitude towards some of the course topics improved; 62% reported that the comics motivated them to study at least half of the course material; and 71% reported that the comics helped with their overall understanding.

Based on the overall positive results, the authors' future plans include studying the effects of comics for students not majoring in mathematics.

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The Implications of Attitudes and Beliefs on Interactive Learning in Statistics Education

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The presented project combines two core concerns of the authors: University teaching is repeatedly confronted with the problem of very large learning groups and therefore poor individual support. To address this problem, the authors have developed a technical system that automatically answers questions from learners in a statistical introductory lecture. The usage data of this system can be used for a second purpose. Combined with further usage data from other systems and quantitative surveys, it can be determined what influence subject-specific attitudes and beliefs have on learning behavior. The poster presents the technical solution, its educational embedding, results of the evaluation and first insights of the learning analytics.

Keywords: Individualization, Learning Analytics, Pedagogical Conversational Agent, Statistics Anxiety

Problem and Idea of the Project

Particularly in basic university modules, lecturers often have to give courses with a large number of learners. Lectures leave hardly any room for individualization and even additional tutorials are still too large for every learner to be supported personally. Especially in self-study phases, educators can hardly supervise their students (De Paola, Ponzo, & Scoppa, 2013). The authors have therefore developed a system in which students can ask their questions in the format of a chat. With the help of artificial intelligence approaches, the system automatically answers these questions and thus provides students with support independent of time and place (Hobert, & Berens, 2019). The same system also provides lecture material, video recordings of the lecture, homework and an audience response system, thus providing an overall package for the digital accompaniment of large-scale university courses.

Implementation and first Results

The developed system Ole (Online Learning Evironment) was tested at the University of Authors in a module for an introduction to statistics with 721 students. The qualitative accompanying evaluation through interviews at four measuring points was used for the further development of the technical implementation, but also shows a high satisfaction with the system, especially with the easy accessibility of assistance and the integrativity of the one solution for all digital approaches of the course. The four quantitative surveys have so far only been partially evaluated. The evaluation of the User Experience Questionaire (Schrepp, Hinderks, & Thomaschewski, 2014) results in high student satisfaction values. Usage counts of over 900,000 interactions between the system and learners also indicate a high level of acceptance.

Further Plans

Ole not only helps students to learn, but also gives the authors the opportunity to explore students' statistics learning through its usage data in more detail. Of particular interest are the effects that anxieties (Hanna, Shevlin, & Dempster, 2008), attitudes (Schau, 2003) and beliefs (Findley, & Berens, 2019) measured at the beginning of the course have on learning behaviour. First analyses show relations, but should be discussed during the presentation.

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Exploring Mathematical Connections Between Abstract Algebra and Secondary Mathematics from the Perspectives of Mathematics Faculty and Practicing Teachers

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This poster focuses on the first stage of a study designed to explore how preservice teachers establish connections between abstract algebra and secondary school mathematics. Additionally, I will describe the motivation for collecting the data in the context of the larger research study. The purpose of this first stage was to gather information regarding mathematical connections between abstract algebra and secondary school mathematics, through the use of online surveys, from the perspectives of mathematics faculty as well as practicing secondary mathematics teachers. The survey data was analyzed using qualitative methods. Many mathematics faculty reported that they incorporate mathematical connections to secondary mathematics into their abstract algebra instruction.

Keywords: Abstract Algebra, Secondary School Mathematics, Mathematical Connections, Preservice Teachers

Abstract algebra has been identified as an important course for preservice secondary mathematics teachers because much of the abstract algebra content is connected to, and thus relevant for, secondary mathematics teaching (Wasserman et al., 2017). However, despite what experts might say are clear mathematical connections between abstract algebra and secondary mathematics, many preservice teachers do not see the relevance of an undergraduate abstract algebra course and in fact, many of them see no relation between abstract algebra and secondary school mathematics (Christy & Sparks, 2015; Ticknor, 2012). The purpose of my two-stage, qualitative dissertation research is to investigate what tasks or course activities help preservice teachers establish connections between abstract algebra and secondary school mathematics in an introductory abstract algebra course. The first stage examined the perspectives of mathematics faculty and practicing teachers regarding connections between abstract algebra and secondary school mathematics. These results were used to inform the development of activities including instructional tasks, questionnaires, and interview protocols for the second stage of the study.

Two Qualtrics surveys were created and distributed to explore which connections between abstract algebra and secondary school mathematics were most important for students to make from the perspectives of mathematics faculty as well as what connections were most useful to practicing teachers. The faculty survey also explored how instructors incorporated the connections into their instruction. The mathematics faculty surveys were sent to 75 mathematics departments that require at least one course in abstract algebra for preservice secondary mathematics teachers. The survey for practicing teachers was sent to graduate programs designed for in-service teachers that require at least one graduate course in abstract algebra. From the faculty survey I was able to construct a prioritized list of mathematical connections (original list from Suominen (2015)) between abstract algebra and secondary school mathematics based on the experience of the participants. I used this list to prioritize the instructional tasks I created to implement into an introductory abstract algebra course for the second stage of my research. Many mathematics faculty reported incorporating connections into their instruction; however, many practicing teachers struggled to identify how they use their advanced mathematical knowledge of abstract algebra in their secondary teaching.

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Mathematics Stretch Courses: Implementation and Assessment

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In this poster we present a newly implemented program of introductory mathematics classes at Sonoma State University. We illustrate the new model itself and the methods we use to assess student learning outcomes. We summarize some initial findings from the assessment and welcome ways to learn and adapt our methods for more informative assessment.

Keywords: Remedial Mathematics, Stretch Courses, First Year Students, General Education

Studies (e.g. Complete College America, 2012) have shown that when students deemed as needing remedial support in mathematics enroll in a college-level math course they are more likely to succeed than if they are first placed in a remedial algebra course. Supporting the growth mindset of students around mathematics has also shown to be a strong promoter of student success (Boaler, 2016). Given these factors, Sonoma State University reformed their introductory courses to include yearlong college-level stretch mathematics classes that incorporate a variety of interventions focused on helping students develop a growth mindset and succeed mathematically.

As of the start of Fall 2018, all 23 campuses in the California State University system had eliminated remedial courses in both English and Math. While most CSU campuses have gone with a co-requisite model, our department developed a stretch model in which four different General Education (GE) math courses (Finite Math, College Geometry, Calc I, and Introductory Statistics) offer 2-semester versions that are required courses for incoming students in need of remediation. These courses are run with a cohort model in which the students and instructor from one section remain together through the spring semester. Dedicated time and curricula in the fall semester support changing student mindsets and attitudes about learning. Data from the first year of implementation indicate that students in stretch courses complete their GE math requirements of the university at higher rates than students who previously took remedial courses, and stretch students are on track to graduate faster, and in higher proportion, than their historical counterparts.

At the time of implementation, a concern of our faculty was whether the stretch students would have comprehension and content mastery similar to that of the students in one-semester courses. To evaluate this, in the pilot year of the program, we looked at common final exam questions; however, there were concerns about scope and evaluative accuracy. The following year, we designed take-home assignments in consultation with course instructors about primary content goals for the courses that were given to all students in the stretch and standard sections at the end of the Spring 2019 semester. Our research shows similar comprehension levels for the students enrolled in the two different course types.

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Student Engagement and Gender Identity in Undergraduate Introduction to Proof

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Researchers are not paying attention to first, students' engagement and secondly how gender identity shapes engagement experiences in undergraduate mathematics classrooms. This study investigates student engagement and gender identity while learning mathematics using a mobile app that collected student engagement reported by students. This paper discusses preliminary results on students' engagement and gender identity in undergraduate mathematics classroom. Those identifying as women reported higher engagement than those identifying as men.

Keywords: Student Engagement, Gender Identity, Mathematics Classrooms

At the university level, research on student engagement and gender strives to either explain growing disparities between degrees awarded to men and women or enrollment at baccalaureate-granting institutions (Wilson, 2007). Thus, most studies on student engagement and gender focus on programs-level outcomes. To my knowledge, this is the first study that is paying attention to how gender identity shapes student engagement in college level mathematics classrooms.

Theoretical Framing

Student engagement is a combination of features, consisting of three major components: Behavioral, Emotional, and Cognitive (Middleton et al., 2017). From flow theory, student engagement is made up of interest, enjoyment (emotional & behavioral), and concentration (cognitive) (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003). The extent of a students' engagement is based on these factors. To understand gender as a social structure (Risman 2017), social identity theory (Tajfel & Turner 1987) was used- gender identity is comprised of personal identity made up of psychological characteristics (e.g., perceptions of body and traits) and a social identity encompassing salient group classifications (e.g., female, cis gendered).

Methods

Sessions from an introduction to proof course were video recorded to document students' experiences and interactions. An initial demographic survey was administered to students to learn their gender identity. Student engagement was captured through a 5-item Likert type survey during each session. Lastly, responses and video data were used to develop stimulated recall interviews for students to describe their interest, concentration and enjoyment. Using thematic coding, data were analyzed to understand the role of gender identity on engagement.

Results and Conclusions

Participants identifying as women reported higher engagement than their peers. During the stimulated recall interviews, participants identifying as men either explicitly or not acknowledged that during interactions with women they allowed them to express their mathematical ideas to make them involved. For Instance, a participant identifying as a man acknowledged that gender identity did not influence engagement yet explained that when working with women, he allowed them to talk in order to get them involved in group work. High engagement reported by those identifying as women affirms previous results at the elementary and secondary level (Marks 2000).

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Racial Differences and the Need for Post-Secondary Mathematics Remediation

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A post-secondary credential continues to be a goal of many people in the United States, not only for self-esteem but for career purposes. Students are often required to take remedial mathematics as they are unprepared for college-level coursework. Results of a regression analysis are presented in this poster comparing factors that might be influential in the need for mathematics remediation for three racial groups: White, African American and Hispanic students.

Keywords: remedial mathematics, post-secondary education, race, NCES data

Many students entering colleges in the US are unprepared for college-level mathematics. One framework to study remediation is the ecology of college readiness by Arnold, Lu and Armstrong (2012), adapted from Bronfenbrenner's Ecological Theory (Bronfenbrenner, 1977, 1986). While the ability to learn is highly individualized, it is a mixture of both personal factors and the relationship between these factors and the world around them (Arnold et al., 2012). Several factors that might influence remedial need are explored. Gender continues to be studied as the number of females in higher education continues to outnumber males (Chen, 2016; Bettinger & Long, 2009). Socioeconomic status (SES) and ethnicity are often studied together as many minority students come from low SES households, and minority students are overrepresented in remedial courses (Olinsky, 2014; Fong, Melguizo & Prather, 2015). Parental involvement increases student achievement, and more educated parents have more experience to prepare their children for college (Hagedorn, Siadat, Fogel, Nora & Pascarella, 1999). The secondary school characteristics of highest math course taken, ACT/SAT scores, GPA and type of high school (public/private) have been found to influence remedial need (Bailey, Hughes & Karp, 2002; Adelman, 1999; Houston, Xu & Harrell-Williams, 2018).

Using data from the Beginning Post-Secondary Study of 2004/2009 by NCES, we study the relationship between the above factors and the number of post-secondary remedial math courses taken. Models are provided for all students in each racial group (White, African American and Hispanic) as well as just students who took remedial courses from each group. Some results of hierarchical multiple regression are provided below. Background variables were added to the regression first, and high school characteristics were added for the second regression. In many models, the only significant predictor is standardized test scores. For White males and White males needing remediation, highest math course taken was significant ($\beta = -0.184, p=0.000; \beta = -0.130, p=0.033$, respectively). For all Hispanic students, gender was significant ($\beta=0.193, p=0.022$) when including only background variables implying Hispanic females took more remedial math courses than Hispanic males, everything else being constant. However, this variable became non-significant upon inclusion of secondary school variables. R^2 values ranged from 0.017 with just background characteristics included up to 0.212 for models including background and high school characteristics. This study confirms prior findings of the importance of high school preparation on college readiness and its role in the overall growth and development of an individual's learning capabilities.

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The Connection between Perception of Utility in Careers with Math and STEM Career Interest

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Undergraduate students pursuing STEM careers have varied perceptions of how mathematics is used in their careers. A survey of community college students enrolled in College Algebra measured interest in mathematics and interest in STEM-related careers, as well as a student's perception of how mathematics was used (or not) in their chosen career. Analysis revealed that students' beliefs related to the usefulness of mathematics in their chosen career predicted their interest in mathematics in general, and their interest in STEM careers in some cases.

Keywords: STEM, career interests, algebra

Mathematics courses have become barriers for undergraduate students, including those pursuing STEM-related majors (Harackiewicz et al., 2012; Olson & Riordan, 2012). One possible explanation for the difficulties that students have in mathematics coursework is the lack of perceived relevance, as students wonder, "When am I ever going to use this?" (Chazan, 1999). Maltese and Tai (2011) examined school-based factors influencing a student's choice of STEM majors, and discovered that *interest* in and perceptions of the *usefulness* of math and science, rather than achievement or course enrollment, was most predictive. Wang and Degol (2015) attributed gaps in male-female STEM attainment to differences in STEM task value (i.e., interest, utility value, attainment value, and cost). In the present study, we examined how a student's beliefs related to the utility of mathematics in their career interest is related to both their interest in math and their interest in their chosen major/career.

Students enrolled in College Algebra (n = 475) at a mid-size diverse suburban community college in the southern United States were surveyed regarding their STEM career interests, as well as their interest in mathematics in general. Students were also asked to respond to an open-ended prompt about how math was used in their chosen careers, and responses were coded on a 3-point scale based on whether they identified no connections, simple arithmetic or measurement connections, or more sophisticated connections like applications of algebra.

Regression analysis utilized math interest and career-math knowledge as predictors of interest in STEM fields. Results showed that math interest was a significant predictor of students' interest in STEM careers, but only moderately so for careers in healthcare, agriculture, nature/outdoors, and construction. A students' perception of how math was used in their career was a predictor of STEM interest in multiple fields, and also moderated the relationship between math interest and career interest for two career areas, mechanics/electronics and physical science. In all other career areas, knowledge of how math was used in the career did not impact the relationship between math interest and STEM career interest.

These findings reinforce the need to increase math interest among both high school and undergraduate students, especially those intending to pursue STEM majors. Additionally, increasing students' understanding of the importance and relevance of math in a student's intended career may help in boosting interest in mathematics in general and in STEM careers.

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Undergraduates' Geometric Reasoning of Complex Integration

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In this research we explored undergraduate students' geometric reasoning of complex integration before the instructor taught it in the course. We found that the participants leveraged their knowledge about complex multiplication to construct a local model of complex integration. In an effort to develop a global model, which attends to the accumulation aspect of integration, we found that our participants struggled due to the thinking real, doing complex phenomenon.

Keywords: complex analysis, integration, models

Research related to the teaching and learning of complex numbers and functions, mostly centers on exploring or developing students' or inservice teachers' geometric reasoning of complex numbers and the arithmetic of complex numbers (Danenhower, 2000, 2006; Harel, 2013; Karakok, Soto-Johnson, & Anderson-Dyben, 2015; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2013; Panaoura, Elia, Gagatsis, and Giatlilis, 2006; Soto-Johnson, 2014; Soto-Johnson & Troup, 2014). Much of this literature illustrates that students tend to prefer algebraic reasoning over geometric reasoning, but that Dynamic Geometric Environments (DGEs) and perceptuo-motor activity can promote geometric reasoning of these concepts. Recently, researchers have explored undergraduates' and mathematicians' reasoning about analytic concepts of complex functions (Hancock, 2019; Oehrtman, Soto-Johnson, & Hancock, 2019; Soto-Johnson & Hancock, 2019; Soto-Johnson, Oehrtman, & Hancock, 2016; Troup, 2015; Troup, Soto-Johnson, Karakok, & Diaz, 2017). They found that after students perceive multiplication of complex numbers as a rotation-dilation of the complex plane, they are able to develop Needham's (1997) *amplitwist* concept, with the aid of a DGE. Recently, Hancock (2019) found that undergraduates encountered the phenomenon, *thinking real, doing complex* (Danenhower, 2000, 2006), when they collectively discussed complex integration theorems learned in class.

In this research, we investigated undergraduates' geometric reasoning of complex integration before the students were introduced to the concept in class. The purpose of this study is to explore how, if at all, students leverage their knowledge of real-valued integration and the geometric interpretation of the product, $f(z)dz$, as they geometrically interpret the inscription, $\int_C f(z)dz$. For our analysis, we used a framework for the definite real integral which focuses on *basic*, *local*, and *global models* (Simmons & Oehrtman, 2017). The *basic model* represents quantitative relationships that emerge when one treats quantities as constant values; the *local model* is a localized version of the basic model. In our case, this might be a partition of the contour *C*. The *global model* is a result of the accumulation process that is applied to the local model, whose underlying quantitative reasoning is encoded in the differential form. Using utterances as described by Nemirovsky and Ferrara (2009), our results indicate that our participants tended to abandon the basic model because they recognized that the quantities vary. As such, they adopted a local model as they integrated their imagery from real-valued integration and their knowledge of line integrals. Although our participants had a deep geometric understanding of complex multiplication, they also experienced the *thinking real, doing complex* phenomenon, which prevented them from developing a global model of complex integration.

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Investigating the Influence of Gender Identity and Sexual Orientation in Small Group Work

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With recent recommendations of the MAA, instructors of undergraduate math courses are encouraged to incorporate more active learning strategies into their teaching. While achievement-focused studies consistently show that the use of active learning strategies in undergraduate STEM instruction is better than lecture alone, little research has been done on the student experience with these active learning strategies. In addition, while improving diversity in STEM fields is an oft-stated goal of reform in undergraduate math instruction, few studies have looked at how different populations might experience the same active learning classroom differently. This poster presents the methodology and early results from a study that in is an effort to address both of these gaps. The study uses in class group observations and follow-up interviews to develop an understanding of how gender identity and/or sexual orientation might influence student experiences in small groups in undergraduate math classrooms.

Keywords: Active Learning, Group Work, Gender, LGBTQ

Research suggests that active learning strategies such as small group work serve students better than lecture alone in undergraduate STEM classrooms (Freeman et al., 2014; Springer, Stanne, & Donovan, 1999). While this is clear, what is less clear is how or even whether different populations of students might experience and respond to the use of small group work in their classrooms differently. For example, only a handful of studies have looked at how gender identity and/or sexual orientation might interact with how students experience small group work in undergraduate STEM classrooms (Cooper & Brownell, 2016; Dasgupta, Scircle, & Hunsinger, 2015; Eddy, Brownell, Thummaphan, Lan, & Wenderoth, 2015; Sullivan, Ballen, & Cotner, 2018). While these studies have found some themes in both achievement results and in student reflections on their participation in small group work, there are several ways in which the existing literature is incomplete. For one, none of these studies have been conducted in undergraduate math classrooms. For another, almost all studies of group work in undergraduate STEM classrooms use only post hoc data – surveys, interviews, or grades; thus, a clear view of what happens in the moments when students are actually working in small groups in undergraduate math classes is missing.

In an effort to address these gaps, a study is in progress that combines observations of students working in small groups in their precalculus courses with follow-up interviews. Observations were conducted across three student groups in precalculus classes during Fall 2019, with interviews ongoing. The methodology for the study, including recruitment and data collection procedures and how a taxonomy by Chiu (2000) provides the framework for analyzing the group work will be presented. Findings from data and qualitative analysis for one of the groups – a pair of students, one male, one female – will be shared. The researcher's analysis of this group using Chiu's (2000) taxonomy will be presented, alongside analysis of interviews with the two students. Contrasts in how the researcher, the male student, and the female student interpreted the influence of gender on interactions will be highlighted. Finally, the poster highlights a few limitations in the study's progress. The researcher hopes to get feedback on the methodology used and its potential usefulness in future studies.

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Exploring Student Understanding of Implicit Differentiation

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It is well known that undergraduate calculus students struggle with the concept of derivative and ideas related to it. Yet, there is less research available on more specific techniques and applications of the derivative. The goal of this study was to examine student understanding of, and ability to carry out one such technique, implicit differentiation. Data was collected through both written surveys and clinical interviews. Findings suggest that students do not have a strong understanding of implicit differentiation and that their difficulties with computational problems stem from a variety of issues they have related to functions.

Keywords: calculus, implicit differentiation, student understanding

The topic of differentiation has been found to be a source of difficulty for students taking calculus (e.g., Orton, 1983). Implicit differentiation, a specific type of differentiation that allows us to take the derivative of equations that are not explicit functions, is no exception. Currently, there is limited research available on this topic, and it has been recognized that this is a hole in the literature (Speer & Kung, 2016). To fill this gap, the focus of this study was to gather data on student understanding of this topic. As such, the first goal was to identify how well students perform on basic implicit differentiation problems. That is, how successful are they are when solving these problems, and what sort of errors appear in incorrect responses. The second goal was to examine what students understand about implicit differentiation and related topics.

The participants of this study were students enrolled in an introductory calculus course in a large, public northeastern university. Written surveys, 136 in total, were collected between fall 2016 and fall 2017. The surveys consisted of textbook-style computational implicit differentiation problems to get an idea of how successful students are on tasks using this technique. Following this, interviews were conducted with five student volunteers who had taken the survey. The interviews had computational problems along with additional prompts to generate data on how students understood and described implicit differentiation.

It was found that students had low success rates on computational implicit differentiation problems. Less than 50% of all responses to the survey tasks were correct. This concurs with previous studies noting that implicit differentiation problems were a source of difficulty for students (e.g., Clark et al., 1997; Martin, 2000). By examining the incorrect responses, it appeared that, based on the written student work, most resulted from calculus errors, such as incorrectly applying the chain rule or the product rule. Errors such as these can be linked to difficulties related to functions such as an inability to recognize a composite function. From the interviews, it was found that students had mixed success when describing implicit differentiation. Even if they knew how to carry it out, this did not mean they understood what it was they were doing, or why they could do it. Also, some interview participants displayed a “manipulation focus” (White & Mitchelmore, 1996) and did not have a good understanding of what the symbols, such as those used to represent derivatives, they were working with represented. Implications for educators and future directions for research based on these findings will be discussed.

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Student Interpretation of Cartesian Points and Trends of a Chemical Reaction Coordinate Diagram with Abstracted Physical Dimensions

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Reaction Coordinate Diagrams (RCDs) aid in the visualization of the thermodynamic and kinetic factors which influence chemical reactions. These graphical representations pose unique challenges for chemistry students given the abstracted physical dimensions that define the Cartesian coordinate system. General chemistry students' interpretations of RCDs were investigated by applying mathematics and chemistry education research frameworks in a semi-structured interview setting. Findings suggest that students' interpretations of the points and trends along these diagrams strongly influence the physical meaning they attribute to RCDs.

Graphical Representations, Visualization, Physical Interpretation, Chemistry

Introduction, Study Details, and Research Questions

Reaction Coordinate Diagrams (RCDs) play a critical role in the chemistry classroom by enabling visualization of the critical thermodynamic and kinetic contributions to observed chemical phenomena. The physical abstraction of the "reaction coordinate" along the x-axis of RCDs has been previously shown to give rise to the misconception that the physical dimension along this axis is "time" (Lamichhane, Reck, & Maltese, 2018; Popova & Bretz, 2018). Previous mathematics and chemistry education research has demonstrated that students may adopt highly distinguishable reasoning strategies when interpreting the Cartesian points (David, Roh, & Sellers, 2018) and trends (Rodriguez, Santos-Diaz, Bain, & Towns, 2018) along a graph. Therefore, these frameworks were applied to address the following research questions.

1. How do students conceptualize points along a RCD?
2. In what ways do students interpret the graphical trends of a RCD?

These research questions were explored in a pilot interview study in which four general chemistry students voluntarily participated in 1-hour semi-structured interviews. Participants addressed prompts that separately marked the points and trends along a RCD. Transcribed interview data was inductively and deductively coded to derive the summarized themes.

Results, Discussion, and Implications

Student responses demonstrated that the interpretation of points and trends along a RCD influenced their physical interpretation of such diagrams. Two participants discussed the points along the RCD according to their corresponding y-values and demonstrated difficulty in explaining how the graph related to a hypothetical chemical reaction. Conversely, the two remaining participants exhibited signs of value-thinking as they discussed these points according to Cartesian values associated with potential energy and time. In this case, the slopes of points along a RCD were associated with the rate of depletion or production of molecules. The two students adopting value-thinking leveraged their interpretation to make arguments about various graphical forms of the provided diagrams and their relationship to chemical stability.

These findings demonstrate the capacity for students' interpretations of points and trends along a RCD to influence their physical reasoning. Discussion of RCDs in the chemistry classroom may require more scaffolding to link the value-thinking associated with interpreting a Cartesian coordinate space to chemical diagrams incorporating abstracted physical dimensions.

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Investigating Student Reasoning about the Cauchy-Riemann Equations and the Amplitwist

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Students appear to experience difficulty in coordinating different types of reasoning and different types of representations. Furthermore, they appear to start integrating these different modes of reasoning or representations when prompted to embody a given abstract mathematical concept. This research is part of an ongoing investigation into how students reason geometrically about various aspects of complex numbers, such as derivatives of complex-valued functions or the Cauchy-Riemann equations (hereafter the CR equations). Collection and analysis of the data is ongoing but is hoped to produce potential learning trajectories or natural ways in which students reason about the CR equations and their connections to the derivative.

Key words: Cauchy-Riemann equations, *Geometer's Sketchpad* (*GSP*), amplitwist

Previous research suggests that students in complex analysis have difficulty transitioning between different forms of complex numbers (Danenhower, 2000) and lack good judgment on when to change between algebraic or geometric representations of a complex number (Panaoura, Elia, Gagatsis, & Giatalis, 2006). In contrast, Nemirovsky, Rasmussen, Sweeney, and Wawro (2012) found that participants using string and stick-on dots on a tiled floor to "embody" complex number multiplication in the Argand plane were able to coordinate their algebraic and geometric calculations. Similarly, Soto-Johnson and Troup (2014) reported that participants eventually integrated their algebraic and geometric reasoning while constructing diagrams they were asked to construct to prove algebraic statements involving complex numbers.

In this poster, I outline an ongoing teaching experiment intended to find ways to reason geometrically about the derivative of complex-valued functions and related concepts, utilizing an embodied cognition perspective. Previously, I investigated how students reasoned geometrically about the derivative of a complex-valued function with the help of *Geometer's Sketchpad* (*GSP*). Since all previous participants suggested looking at the CR equations in their interviews, this interview will involve tasks intended to help students reason geometrically, symbolically, and formally (Tall, 2008) about the CR equations and their connection to the amplitwist. The amplitwist concept is described by Needham in *Visual Complex Analysis* (1997), as the geometric characterization of the derivative in which $|f'(z)|$ describes the factor by which a small disk around a given point z is dilated under the mapping $z \to f(z)$, while $Arg(f'(z))$ describes the angle by which this same disk is rotate under the same mapping. To connect the CR equations to the amplitwist, Needham first notes that multiplying by a complex constant, or the mapping $(x+iy) \to (a+ib)(x+ib) = (ax-by)+i(bx+ay)$ corresponds to multiplication by the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, which is the product of a rotation matrix and a dilation matrix. Since amplitwists are local linearizations, and thus also approximated by a rotation and a dilation, for a function $f(z) = u + iv$ to be differentiable, its Jacobian $J = \begin{bmatrix} u_x & u_y \\ v_x & v_Y \end{bmatrix}$ must have the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, "in order for the effect of J to reduce to an amplitwist…" (p. 210). This requires $u_x = v_y$ and $u_y = v_x$, which are the Cauchy-Riemann equations.

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“Meaning Making with Math”: A Mathematical Modeling Approach to Supporting Conceptual Reasoning in Undergraduate Chemistry

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Mathematical models are ubiquitous in science and science education and serve as tools for constructing explanations and making predictions about real world phenomena. However, undergraduate science students often use mathematical models algorithmically, without consideration of the physical meaning of mathematical variables. To support chemistry students' ability to engage in "meaning making with math," we developed a series of instructional activities which lead students through using, evaluating, and revising mathematical models of chemical phenomena. Here, we discuss development and refinement of an activity focused on gas behavior and discuss theoretical frameworks that guide our activity design.

Keywords: mathematical reasoning, mathematical modeling, chemistry

Mathematical models, like equations and graphs, are ubiquitous in contemporary STEM research, and serve as critical tools for predicting and explaining scientific phenomena. In science education, the practices of using mathematical and computational thinking and developing and using models have been highlighted as critical components of scientific inquiry (National Research Council, 2012). The practice of mathematical modeling can be thought of as the development of a mathematical representation which embodies some characteristics of a target system and may be applied to predict what will happen in a future situation (Arseven, 2015). In science, the use of mathematical models entails the integration of mathematical symbology with and understanding of the physical world. While expert scientists recognize that variables in equations, for example, the ideal gas law ($PV = nRT$), map to real and observable phenomena, for example the temperature (T) of a system, undergraduate learners do not always make these connections. The process of mapping mathematical symbols to the physical world has been previously referred to as "meaning making with math" (Redish and Gupta, 2010).

In traditional general chemistry courses, students are exposed to many equations which can be used to predict chemical behavior. However, there exists strong evidence that undergraduate students approach mathematical tasks in chemistry using algorithmic problem-solving strategies (i.e. "plug-and-chug") and do not engage in the kind of conceptual reasoning that would promote development of deeper understanding of chemical phenomena (Kautz, Heron, Shaffer, & McDermott, 2005; Nurrenbern & Pickering, 1987).

To support chemistry students' ability to engage in "meaning making with math," we developed a series of instructional activities in which students collaborate to use, develop, or revise mathematical models of chemical phenomena. This poster details the development and refinement of a learning activity in the context of gas behavior in which students use an online simulation of gas particles to identify relationships between relevant variables and construct a mathematical model. We discuss how the activity design supports students' development of knowledge about the epistemic nature and purpose of scientific models (Schwarz, Passmore, & Reiser, 2016) and discuss theoretical frameworks that guided the activities' development (Lesh, Hoover, Hole, Kelly, & Post, 2000; Mislevy, 2011; Moog & Spencer, 2008).

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The Mathematical Inquiry Project: Effecting Widespread, Sustainable Instructional Change

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The Mathematical Inquiry Project (MIP) is an NSF-funded collaboration of mathematics faculty from all 27 public institutions of higher education in Oklahoma to support inquiry-based learning in entry-level mathematics courses. This poster reports on the initial development of a Community of Practice, including shifts in the participants' conceptions of the joint enterprise and their identities in relation to it.

Keywords: Communities of Practice, institutional change, entry-level courses, inquiry

The overarching goal of the MIP is to effect widespread, sustainable instructional change across institutions, using a grassroots model that offers opportunities for ongoing professional development. The research component of the project focuses on the emergence and evolution of a statewide community of practice (Lave & Wenger, 1991; Wenger, 1998). As such, the work provides insight into what sorts of activities are productive for building a statewide community of practice and fostering institutional change.

We report on preliminary findings regarding the identity trajectories of individual faculty, specifically changes in the nature of their participation and their goals since the project's first three professional development opportunities. Each opportunity consisted of a 4-day workshop. The first workshop focused on academic success skills, the second on Quantitative Reasoning and Modeling, and the third on College Algebra and Precalculus. The data set consists of videos of workshop sessions and responses to written surveys.

Faculty participating in all three workshops have broadly shifted their image of i) academic success skills in entry-level college mathematics from a focus on behavioral symptoms, such as attendance, note-taking, and going to office hours, to a focus on the underlying affective characteristics, including academic identity, attribution of intelligence, and goal-orientation, ii) their participation in the project as recipients of knowledge and resources developed by others to identities as leaders of the statewide effort and commensurate goals to be involved in producing resources and supporting other faculty, and iii) the value of conceptual analysis (Thompson, 2008) to guide instructional design and to provide criteria for evaluation and revision. Although only modest shifts were observed around faculty images of active learning and meaningful applications, we anticipate that these will primarily evolve as they participate in instructional design required to incorporate these features with guided review and discussion with other faculty.

Acknowledgments

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Pre-service Mathematics Teachers' Engagement with Cognitive Demand of Mathematics Tasks

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The poster presentation describes the work undertaken in a mathematics methods course. 7 pre-service teachers (PSTs) critiqued mathematics tasks for their cognitive demand. Following critique of the tasks they planned for implementing and then implemented the tasks. Their critique of the tasks allowed them to learn about the properties of effective mathematics tasks. Further, they began to see the role teachers play in maintaining or decreasing cognitive demand.

Keywords: Pre-service teachers, mathematics methods course, high cognitive demand tasks.

High cognitive demand tasks encourage students to discuss ideas, check their reasoning, and explore concepts, to develop practices that align with mathematical and scientific thinking (Tekkumru-Kisa & Stein, 2015). By selecting meaningful tasks and implementing them well, teachers can provide engaging learning experiences for students (Tekkumru-Kisa & Stein, 2015). Teachers play a crucial role in implementing tasks because their choices allow students to either experience rigor or they can decrease cognitive demand of tasks (Stein, Grover, & Henningsen, 1996; Stein & Lane, 1996). The project allowed PSTs to critique and implement high cognitive demand tasks to help them become cognizant of the roles both task and teacher play in students' learning. The specific research question guiding this study was: What opportunities for PSTs' learning emerge from critiquing, and planning for implementing high cognitive demand tasks?

To answer the research question, the study used teacher engagement, challenge, & opportunities for learning framework (Rahman, 2018). The framework allows analysis of teachers' learning opportunities emerging from challenges they face while engaging with curricular resources. Grounded theory (Glaser & Strauss, 1967) was employed to learn about PSTs' critique of cognitive demand of tasks and their opportunities for learning. The study took place at a university in the Mid-Atlantic United States, in a middle school mathematics methods course. Participants were 7 mathematics PSTs, enrolled in the course. The PSTs used the task analysis framework (Smith & Stein, 1998; Stein, Smith, Henningsen, & Silver, 2000) to critique the cognitive demand of three tasks. They examined and compared properties of tasks in each category of the guide [(1) lower-level memorization tasks; (2) lower-level procedures without connections tasks; (3) higher-level procedures with connections and (4) ''doing mathematics'' tasks] (Smith & Stein, 1998; Stein, et al., 2000). Data included audio recording of classroom conversations and PSTs reflection about their experience. Constant comparative method (Merriam, 2009) of data analysis was used to systematically code and sort the data to look for emergent themes.

Using task analysis guide allowed PSTs to differentiate between tasks of varying quality when selecting them for instruction. However, this opportunity had its limitations. The PSTs' analysis of the tasks was often superficial in nature. They categorized tasks based on features of the task apparent to them. In addition, the PSTs criticized the tasks based on their own experience solving them, making assumptions about what their participant would find challenging. They edited the tasks to make them less 'confusing' and planned to ask guiding questions. Future research will support the PSTs in being objective in their critique of the tasks and highlight their own role as teachers in maintaining cognitive demand of implemented tasks.

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An Overview of the Orchestrating Discussion Around Proof (ODAP) Project

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In this poster, we provide an overview of a design-based research project investigating student discussions surrounding proof in abstract algebra. This project aims to incorporate best practices for orchestrating discussions developed within the K-12 spectrum into the context of advanced mathematical proving activity. We describe three aspects of the project: general task design, task implementation, and future work to be considered based off the implementation.

Keywords: Abstract Algebra, Design-Based Research, Proof

Best practices for orchestrating productive student discussions in mathematics have been identified and investigated at the K-12 level (e.g., Stein, Engles, Smith & Hughes, 2008). However, best practices for orchestrating discussions at the undergraduate level have not been as highly researched. In this poster, we provide an overview of a design-based research project focused on investigating student discussions surrounding proof in abstract algebra.

Three practices for orchestrating discussion at the K-12 level were chosen as the focus for three respective tasks, each of which also targets a proof activity: proof validation, proof construction, and proof comprehension. In the proof validation task, students are provided with two proofs of the statement that group isomorphisms preserve the Abelian property. This task engages students with the K-12 practice of connecting and comparing across strategies. For the second task, students construct a proof of LaGrange's Theorem by leveraging informal arguments with the aid of diagrams and representations. The focal best practice of this task is the promotion of justification and argumentation based on mathematical representations. The final task involves students with comprehending a completed proof of the First Isomorphism Theorem. The best practice underlying this task is to engage students with sense-making of the structure of mathematical arguments.

Using these tasks, we conducted two rounds of task-based interviews, each round consisting of three or four undergraduate mathematics majors who had previously taken an abstract algebra course. The first round occurred across five sessions, and the second round was completed in three sessions. As a next step, our project team plans to incorporate these tasks within an abstract algebra classroom.

In this poster, we will share an overview of the project. In particular, we will share details of each task, notable things that happened during task implementation, and adaptations made to the tasks. In addition, we will provide more details of the plan to scale up into an abstract algebra classroom setting.

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Informing the Community About Advancing Students' Proof Practices in Mathematics through Inquiry, Reinvention, and Engagement

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The difficult transition from computational mathematics to advanced mathematics is well-documented in the research literature. The ASPPMIRE project has two main goals. The first is designing curriculum materials that support students in reinventing fundamental concepts in abstract algebra and real analysis, along with fundamental proof skills. The second supports instructors in implementing inquiry-oriented instruction in their own classrooms. The purpose of our poster presentation is to invite the research community to discuss our project's motivation, goals, and study design.

Keywords: design research, proof, real analysis, abstract algebra, inquiry-oriented instruction

There have been calls to support students in the transition from computational mathematics to advanced mathematics and specifically with proof (Stylianides, Stylianides, & Weber, 2017). The NSF funded project Advancing Students' Proof Practices in Mathematics through Inquiry, Reinvention, and Engagement (ASPPMIRE) aims to design instructional materials to support students through this transition as well as support instructors in their transition to inquiry-oriented mathematics teaching. The purpose of our poster presentation is to invite the research community to discuss our project's motivations, goals, and study design.

Through four concurrent design research projects (Cobb, Jackson, & Sharpe, 2017), the project is developing (1) curriculum materials, (2) a wiki-style textbook, (3) online instructional support materials (OISM), and (4) a professional development workshop:

Curriculum materials. Our goal is to design curriculum materials in a way that allows instructors to choose content domain(s) and follow-up proof modules. The content domains include abstract algebra (using the Teaching Abstract Algebra for Understanding, TAAFU, materials) and real analysis (building on Strand's 2016 design work). The proof modules emphasize the roles of definitions, mathematical language, proof techniques, and proof validation (building on Vroom's dissertation work).

Wiki-textbook. Additionally, we are designing a wiki-style textbook (Katz & Thoren, 2014) which serves as a public record of student conjectures and classroom definitions through different phases of the curriculum. The goals are to provide the space for collaboration in refining the mathematical tools, including definitions, theorems, and proofs that students and instructors can both access and edit.

OISM. The project's OISM is being modeled after the TAAFU project's approach (Larsen, Johnson, & Bartlo, 2013). Similarly, our curriculum materials will have a suite of OISM that will include notes about implementation and possible student strategies or responses.

PD workshop. We are designing a professional development workshop that introduces instructors to inquiry-oriented instruction and provides an environment for them to learn equitable practices for whole-class engagement. The goals of the workshop include demystifying the transition to inquiry-oriented teaching and provide early exposure to the materials so that the instructors might feel more empowered to take on teaching in this fashion.

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Logical Implication as an Object and Proficiency in Proof by Mathematical Induction

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Proof by mathematical induction is known to be conceptually difficult for undergraduate students. We present a model that may simulate the impact of logical implication on students mastering proof by induction. We combine Piaget's action-object theory of mathematical development with a psychological model of working memory and Harel and Sowder's proof schemes. We analyzed three sets of written assessments from two Introduction to Proofs classes: after students learned about logical implication; before and after instruction on proof by induction. We examine the relationship between proficiency with mathematical induction and treating logical implication as an object within these two classes.

Keywords: Logical Implication, Mathematical Induction, Proof, Action-Object Theory.

Proof by mathematical induction is known to be conceptually difficult for undergraduate students. This method is used to prove that the statement P(n) holds for any natural number n. To prove P(n) by mathematical induction, one must check two assumptions: (a) the validity of P(1) (the base case), and (b) if the statement P(k) is true for some natural number k, than it is also true for P(k+1) (inductive implication). The implication $P(k) \rightarrow P(k+1)$ may be considered as either an operator that transforms P(k) into P(k+1) or as an invariant relationship between P(k) and P(k+1) (Norton & Arnold, 2017).

Our study is guided by the following research question: "What is the impact of holding logical implication as a mathematical object on students' responses to formal instruction on proof by induction?". We describe a model that may simulate the impact of logical implication on students mastering proof by mathematical induction. We draw on Piaget's (1970) action-object theory of mathematical development. More specifically, we elaborate Dubinsky's (1991) hypothesis that treating implication as an object is crucial for promoting students' understanding proof by induction. Additionally, our framework incorporates psychological model of working memory, and Harel and Sowder's (2007) proof schemes.

Participants were students from two classes of an Introduction to Proofs course, taught separately by the second and the third authors. The students completed three written assessments: 1) after the students learned logical implication, 2) just before instruction on proof by induction, and 3) after instruction on proof by induction. We used a chi-squared test to assess the relationship between post-mathematical induction (Post-MI) proficiency and treating logical implication (LI) as an object in the 2nd author's students; however, there was no indication of a relationship in the 3rd author's students (see Table 1). During the poster session, we will discuss reasons for this difference. For example, the 2nd author's class received formal instruction on quantification prior to instruction on induction, whereas the 3rd author's class did not.

Table 1. Chi-Squared Test.

2nd Author $p = 0.03995$		Post-MI Proficiency	
		Yes	No
LI Group	Object	7	4
	Action	1	6

3rd Author $p = 1.16883$		Post-MI Proficiency	
		Yes	No
LI Group	Object	6	5
	Action	2	5

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Multiplication by Sunlight:
How can a Geometric Definition be Realized in a Physical Tool?

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McLoughlin and Droujkova (2013) developed a diagrammatic definition of multiplication that uses parallel lines to continuously scale the length of one segment by the length of a different segment. This contemporary treatment of the constructability of products (and quotients) is potentially significant for the undergraduate mathematical preparation of pre-service elementary teachers, who tend to conceptualize multiplication in terms of repeated addition. We take up here the design challenge of constructing a physical tool that models multiplication as a continuous scaling operation as opposed to a repeated grouping operation. We ask: How can the parallel shadows interpretation of real-number multiplication be used to design a physical tool?

Keywords: Multiplication, Undergraduate teacher education, Physical tools

McLoughlin and Droujkova (2013) raised the question of how to physically interpret real number multiplication. They proposed a geometric definition of multiplication that is grounded in a basic observation: "The hypotenuse of the right triangle determined by an object and its shadow must be parallel to the hypotenuse of any other object and its shadow." (p.2) From this observation, McLoughlin and Droujkova (2013) developed a diagrammatic definition of multiplication that uses parallel lines to continuously scale the length of one segment by the length of a different segment. This contemporary treatment of the constructability of products (and quotients) is potentially significant for the undergraduate mathematical preparation of pre-service elementary teachers.

In elementary mathematics settings, multiplication is defined in terms of repeated addition (Luke, 1988; Nunes & Bryant, 1996; Thompson & Saldana, 2003; Wright, 2011) or other discrete models, "with infrequent use of continuous models" (Kosko, 2019, p. 272). Thompson and Saldanha (2003, p. 104) found that "thinking about multiplication as repeated addition leads to severe difficulties in later grades" and Hurst (2015, p. 11) suggested that "multiplication as scaling rather than adding is a key aspect of thinking that needs to be developed." We take up here the design challenge of constructing a physical tool that models multiplication as a continuous scaling operation as opposed to a repeated grouping operation. We ask: How can the parallel shadows interpretation of real-number multiplication be used to design a physical tool?

We report the design and development of a tool we refer to as a *SunRule*, a variable device that uses shadows from sunlight to multiply or divide real numbers. The design of the SunRule is inspired by work that realizes McLoughlin and Droujkova's (2013) continuous definition of multiplication in transformable diagrams (Dimmel & Pandiscio, 2017; Dimmel & Pandiscio, *in press*). Our poster will explore the mathematical underpinnings of the SunRule device, describe the plan that was used to build the first physical prototype, and consider how the initial version of the tool could be improved. We will describe plans to research how the SunRule could be used in undergraduate teacher education. We are hopeful that attendees of our session will critically review the design and development of the SunRule and comment on our initial research plans.

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Calculus Students' Visualization of Volume

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We present the methodology and preliminary findings from a study whose purpose is to uncover student thinking around volumes of solids of revolution, with a specific focus on how students create and use visual images. We use Gutierrez' visualization framework (1996) to analyze how students use representations of their mental images to solve tasks. Initial findings suggest that students struggle representing slices that can be used to measure the volume of a sphere; first attempts instead make use of two-dimensional figures.

Keywords: geometric and spatial thinking, calculus, post-secondary education

Motivation and Research Question

The purpose of this study is to build upon results from a pilot study exploring student thinking of volume while taking Calculus II. The central research question is, *how do students think about the Riemann Sum and Integral when approximating and computing volumes of solids of revolution?* We re-designed mathematical tasks used in our pilot study after noting that students experienced difficulties in creating and using diagrams to reason about integration. In this proposal, we focus on a sub-question for our study, *how do students use visual images to reason about volume?*

Methods

We videotaped interviews of three pairs of students enrolled in Calculus II, as they collaborated on volume tasks (Brestock & Sealey, 2018). Interviews occurred in two stages: students worked on one task prior to instruction and three additional tasks after an instructional unit on volume. Our first task prompted students to sketch the solid resulting from rotation of a semicircle about the x-axis, along with four approximating slices. We added this task to understand how students think about approximating volume after only having worked with area. Additional tasks involved rotations about lines other than the x-axis. These were used to help pinpoint student difficulties previously observed around identification of constant and varying quantities (e.g., height or radius of a disk), and to gather information on students' abilities working with their external representations of mental images (Presmeg, 2006; Yakimanskaya, 1999) of three-dimensional objects. To this end, we also asked students to explain Kepler's non-standard volume approximation of a sphere.

Results and Discussion

Our preliminary analysis of the data suggests that students were challenged when asked to sketch slices to approximate the volume of a sphere. On the first task, administered before and after instruction, students attempted to use two-dimensional objects (rectangles, triangles, trapezoids) to approximate volume, mimicking what they had done when approximating area. Some students were able to use gestures to represent their mental images of slices but needed guidance in creating diagrams (Gutierrez, 1996). Fitting cylinders within the sphere also raised questions for students; specifically, students verbalized discomfort in comparing the straight edge of the cylinder to the curved surface of the sphere. We believe this difficulty lies in students' abilities of visualization (Gutierrez, 1996; Bishop, 1983).

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Exploring the Development of Mathematical Problem Solving Strategies in the Transition from Novice to Experienced Mathematicians

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Identifying formative experiences that affect mathematical problem-solving (MPS) development and trajectories for rising mathematicians presents a complex challenge. Despite the existence of various descriptions of isolated MPS behaviors for populations of mathematicians of different levels of expertise, there is still insufficient understanding of how MPS develops as a result of extended mathematics education. This poster presents preliminary findings of a study which aims to characterize formative experiences of advanced undergraduate mathematics majors and first-semester mathematics graduate students which, in interviews, they identified as affecting their MPS strategies or disposition towards solving difficult mathematics problems. We also aim to gather feedback on study design and potential directions for analysis of the interviews.

Keywords: mathematical problem-solving, graduate students, mathematical affect

In past decades, research in mathematical problem-solving (MPS) has produced a number of theoretically-grounded frameworks for characterizing the MPS strategies of mathematicians (c.f. Schoenfeld, 1985; Carlson & Bloom, 2005; Čadež & Kolar, 2015). Because the term "mathematician" necessarily must refer to both students of mathematics and mathematics experts, some existing research already catalogs the differences between novice and experienced mathematical problem-solvers through the lens of one or more of these frameworks (c.f. Stylianou & Silver 2004). We direct our attention instead to ways in which novices transition to expert mathematical problem-solvers through critical experiences.

In order to understand how nascent mathematical problem-solvers develop more effective MPS strategies, we organized a sequence of two interviews with advanced mathematics undergraduate students and first-semester mathematics graduate students, two populations which embody periods of mathematical education that mark potentially drastic shifts in mathematical beliefs. During these semi-structured, task-based interviews, participants solve non-traditional mathematics problems and reflect on both their MPS strategies and formative experiences which they perceive as affecting their disposition towards mathematics and competency as problem solvers. These interviews are analyzed qualitatively to answer the following research questions:

- To what extent do pivotal experiences in mathematics contribute to the development of particular MPS strategies in emerging mathematicians?
- How do MPS strategies develop as a result of extended mathematical education?

This poster will provide preliminary analysis from the interviews, which indicates that MPS strategies can become more robust via experiences in teaching mathematics and in classroom environments that emphasize well-connected understanding rather than procedural knowledge. Such a classroom environment can be characterized by particular socio-mathematical and pedagogical norms (Arcavi, Kessel, Meira, & Smith, 1998; Yackel & Rasmussen, 2002). Interview participants also allude to shifts in mathematical affect prompted by styles of instruction and instructors themselves. Research has previously pinpointed differences in affect as one demarcation between successful and unsuccessful mathematicians (Carlson & Bloom, 2005; DeFranco, 1996).

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A Framework for Meaning in Mathematics

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Students ask the question "when is this ever going to be useful?" when speaking about mathematics. If we take this as a question about meaningfulness, how can teachers respond through their instruction (if they choose to do so) and how do they even understand the terms 'meaningful' and 'meaning'? Prior research suggests beliefs influence instruction. I am interested in understanding how beliefs then influence what meanings instructors' goals and instruction focus on. This theoretical paper aims to synthesize prior research to elaborate a framework of meaning orientations for instructors' goals. Future research can then look at how and why instructors focus on certain orientations of meaning over others.

Keywords: Instruction, Mathematical Meaning, Instructor Goals, Affective Factors

'Meaning' has been understood in very different ways, some of which may not fit what a student seeks when they ask 'when is this ever going to be useful?' (Brownell, 1947; Gough, 1998; Thompson, 2013, 2015). Brownell (1947) took an initial step to talk about different senses of meaning by defining the "meaning of" mathematics as mathematical understandings and the "meaning for" mathematics as understanding its significance. These conceptions fit the acquisition metaphor of learning (Sfard, 1998) where knowledge is taken as information residing in an individual's head. Research has also argued taking a participation metaphor perspective on learning as a process of enculturation (Sfard, 1998; Wenger, 1998). In this perspective, Wenger (1998) takes meaning *as the experiences of everyday life* through community practice. Using both metaphors, I can set out four ways to think of 'meaning' simultaneously. I can define the social meaning of mathematics (sMoM) as practices the mathematics community engages in (e.g. those in Rasmussen, Zandieh, King, and Teppo (2005)) and the social meaning for mathematics (sMoM) as practices of non-mathematics communities (e.g. procedural computations and critical thinking as means for societal maintenance, Danielak, Gupta, & Elby, 2014; Niss, 2005). I take Brownell's initial conceptions as individual meanings (iMoM and iMfM respectively).

These meanings can be used to frame what instructors believe is meaningful about their goals and how they respond to the question 'when is this ever going to be useful?' To demonstrate, I use the above meanings as orientations to frame four US instructors' goals, some beliefs and how these interact with the rest of their belief systems (Leatham, 2006). Each instructor taught a different class: math for elementary education (MEE), math for liberal arts (MLA), statistics (S), and calculus (C). Data included interviews and classroom observations of teaching.

In analyzing interviews, I found evidence for each meaning based on instructors' beliefs, goals, and instruction. iMoM goals included highlighting how mathematics connects with itself and has its own inherent beauty, and accomplished by covering topics or showing proofs (e.g. sequences and series in MLA and proof by contradiction in C). iMfM goals included passing on skills for students' personal use, as accomplished through groupwork in S (because skills develop in the process) or reviewing algebra techniques in C (because those will be usable by students after years of experience). sMoM goals included passing on techniques for dealing with inequalities in C because math majors would need such skills in later work. sMfM goals included preparing students for life in certain professions, as accomplished by interpreting student and peer work in MEE (since such skills help teachers function). In turn, this framework affords researchers a way to explore how *different* meanings can influence instruction simultaneously.

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Conflicts During Mathematical Modeling

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This exploratory study investigates how students attend to conflicts that arise during the validation of a model while engaging with a mathematical modeling task. Analysis of a one-on-one task-based interview revealed that the student attended to conflict in the following ways: (1) modifying the object of validation (2) modifying the standard of validation, and (3) leaving the conflict unresolved.

Keywords: Conflicts, Mathematical Modeling

There has been recent interest in investigating how students engage in mathematical modeling. One avenue for describing how students engage in modeling is investigating how students revise their models. *Validation* of the model and the *conflict* students experience during validating, occasion the revision of models. This preliminary analysis addresses the following research question: *How do students attend to conflicts that arise during the validation of a model in a mathematical modeling task*? Investigating how students attend to conflicts has consequences: (1) it may inform us about the evolution of the students' model, (2) it may inform us about students' cognitive activities that can be leveraged during teaching to improve the learning of mathematical modeling, as a result (3) may help us answer the bigger question: what is being learnt through modeling, if anything?

To analyze how students attend to conflicts, two framings were used: characterization of modelling as a process (Blum & Leiß ,2007) and typology of validating activities (Czocher, 2018). Blum & Leiß (2007) decomposed the mathematical modeling process into six stages of modeling construction joined by six activities, one of which is validating. When a student attempts to validate her model, she holds two models in her mind: the object of validation and the standard she validates it against. In this study, conflict that arises during model validation is defined as *a discrepancy or misalignment between the object of validation and standard of validation.* Using a theoretical case study approach (see Walton, 1992), data were drawn from a larger study of interviews with individuals working on modeling tasks (Goldin, 2000). The interview was first coded for instances of validating and each instance was typed according to Czocher (2018). Next, these typed instances were closely examined for any discrepancies that existed between the standard and object of validation. Finally, these discrepancies were analyzed by looking at what were modified for the resolution of the conflict, if at all.

Of the six instances that were identified as conflicts, the student attended to them in three ways: (1) modified the object of validation to align with the standard of validation, (2) modified the standard of validation to align with the object of validation, and (3) the conflict was left unresolved. The analysis also provided insights into the relationship between what was modified and the acceptance, revision, or rejection of the model: When the standard of validation was modified the model was accepted and when the object of validation is modified the model was either rejected or revised. A conjecture can be made for a fourth way a student may attend to the conflict: choosing to modify both the object and standard of validation. This can be tested by looking at more cases. Future analysis will also address the following: What are the indicators of conflict that a student experiences during modeling a real situation? Investigating conflict resolutions may have bearing on what is learnt through modeling and the mechanisms for how.

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Aligning Assessment with Instruction in a Creativity in Mathematics Course

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Assessment is a central issue to the lives of educators and students. The purpose of assessment is to certify achievement or to facilitate learning (Boud, 2000). At all levels, from programmatic to course-specific, we strive to assess in a way that accurately measures achievement or progress relative to one's goals. In this poster, we describe our experiences assessing students in an undergraduate mathematics course focused on creativity in mathematics. While traditional forms of assessment often focus on repeating mathematical procedures and demonstrating abilities of routine skills (Firestone, Winter, & Fitz, 2000; Lesh & Clarke, 2000), we attempted to align assessments with the alternative instruction we employed. Here we describe our assessments and report on preliminary findings of students' reactions to the assessments through reflective journal entries, focus group interviews, and pre- and post-course surveys, outlining a need for revisions to traditional assessments.

Keywords: Assessment, Undergraduate Mathematics, Mathematical Creativity

We developed a course we call "Creativity in Mathematics" as part of an NSF-funded Improving Undergraduate STEM Education (IUSE) project. Our course, which fulfills a general-education requirement for non-STEM majors, emphasizes the role of creativity in mathematics and engages students in creative approaches to mathematics. This shift in course emphasis motivated us to consider assessments that mirrored our instructional approach. As various reforms in undergraduate mathematics education such as project-based (Krajcik & Blumenfeld, 2006), inquiry-based (Laursen, Hassi, Kogan, & Weston, 2014), and student-centered (Hannafin & Land, 1997) learning have taken hold, the need to consider alternative assessments becomes more significant. We believe that assessments should model life-long learning and through participation with others, include seeking feedback from more than just the teacher (Boud, 2000).

For this project, we strove to align our assessments with the open-ended, exploratory instruction we employed in the classroom. These assessments included journals, individual and group assignments, and midterm and final "tests" that fostered creativity in mathematics. Whereas the individual and group assignments encouraged students to make connections between mathematics and other disciplines, both the midterm and final had students working individually and collaboratively as they brought together their understandings to solve a bigger mathematics problem. Our poster will describe these assessments more fully and provide examples of each.

Our ongoing research on this project aims to describe changes to students' attitudes and beliefs surrounding assessment in mathematics. We collected data through semi-structured student interviews, both at the beginning and end of the semester, that focused on students' experiences with assessment in past mathematics courses. We also collected written work in the form of surveys, autobiographies, and journal reflections that prompted students to reflect about the role of assessment—in both their current and previous courses. Preliminary findings show that students' experiences with assessment in mathematics involved traditional quizzes and tests. We are in the process of collecting and analyzing their post-course data, but we expect to be able to report their reactions to our nontraditional forms of assessment.

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Student Thinking about the Graphs of Functions of Two Variables via Software Visualization

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Abstract: The purpose of this study is to investigate student thinking in the context of graphs of functions of two variables. The data presented here is an example of the ways of thinking students exhibit when working with graphs of multivariable functions.

Keywords: Multivariate Functions, Graphing, Covariation, Student Thinking, Technology

This poster discusses the work and reasoning of one of three students on a series of tasks designed around the hypothetical learning trajectory (HLT) outlined by Weber and Thompson (2014).

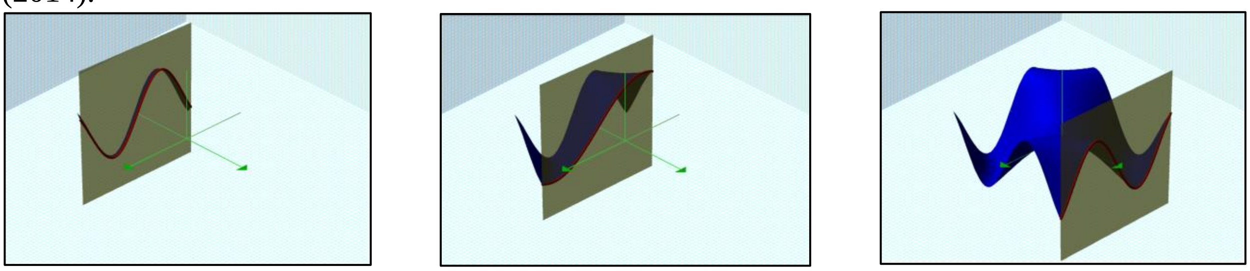

Figure 1. The emergence of the graph of a function of two variables via the process of reconceiving of one variable as a parameter, and covarying the remaining two quantities (Weber & Thompson, 2014)

Three students were interviewed in a clinical interview setting to investigate their thinking when graphing functions of two variables (Clement, 2000). Students were encouraged to use the Graphing Calculator software to explicate their thinking and frame their arguments (Avitzur, 2011). All three students were finishing a course in multivariable calculus. The goal of the study was to investigate what students thought about when graphing, or looking at the graphs of, functions of two variables. It is my conjecture that supporting students' visualization of functions of more than one variable with technology will help students develop robust meanings for partial derivatives, directional derivatives and the gradient. The underlying research question driving this study was, "what do students "see" when they consider the graphs of functions of two variables using software visualization."

The purpose of this study is to build models of student's mathematics, defined as the mathematics of students (Steffe & Thompson, 2000). I adopt the radical constructivist perspective, specifically pertaining to the distinction between student's mathematics and the mathematics of students. The literature in this area is growing, but the current state of the literature is in need of further investigation. The work of Martinez-Planell and Trigueros-Gaisman characterized productive student thinking about the graphs of functions of two variables in the context of placing fundamental planes, and therefore the traces of the graph, at the appropriate place in $\mathbb{R}^3$ (Martinez-Planell &Trigueros-Gaisman, 2012). It is my conjecture, that a student's ability to properly place these "fundamental planes" in $\mathbb{R}^3$ is inherent to robust covarational thinking as outlined by Weber and Thompson in their HLT.

If students do not have productive meanings for the graphs of functions of two variables, it is possible that students' experience in multivariate calculus courses culminate in complicated computation divorced from the visual representations of said multivariable functions.

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Myriad Issues in Teaching College Geometry

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This poster discusses geometric definitions that impact the teaching of college geometry.

Keywords: College Geometry, Definitions, Transformational Proof

College geometry courses are varied across institutions and often must serve different purposes for different student populations. For many prospective secondary teachers, a single course is their only opportunity to focus on geometric knowledge useful for teaching. However, there are multiple aspects of the diversity of these courses that could potentially result in vastly different secondary geometry teaching practices. In this poster, we present three problems of practice in teaching geometry that can launch discussions about mathematical and pedagogical consequences of the choices college geometry instructors make.

Many researchers are investigating students' thinking about transformational proofs (e.g. Hegg, et al., 2018). However, underlying these discussions is the question of how to define congruence. Some axiomatic systems for geometry introduce a metric definition: two segments are congruent if and only if they have the same length. However, many current standards documents rely implicitly on an isometric definition of congruence: two geometric objects are congruent if there exists a series of rigid motions of the plane that superimposes one onto the other. Although functionally equivalent, the choice of definition has consequences throughout the structure of a college geometry course.

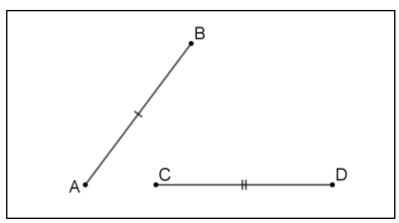

Figure 1. The segments shown are not necessarily congruent.

Mathematicians have well-established ways of denoting the congruence of two segments or angles in a geometric figure. However, in what ways do students understand the hashmarks in Figure 1? We present evidence that whether students see these segments as non-congruent, as opposed to not necessarily congruent, is contextual, and this can affect their interpretations of others' reasoning as well as their proof production.

Categorizing triangles by their number of congruent sides can be done inclusively or exclusively. In a proof-based college geometry course, inclusive definitions lead to greater applicability. However, in K-12 curricula, scalene triangles are always defined exclusively, leading to potential confusion about the applicability of results about triangles. So why do we define scalene triangles at all?

We will present illustrations of the above ideas, cite relevant literature highlighting the mathematical and pedagogical consequences of each choice, and use these examples to engage in discussion with conference participants' ideas. Our goal is to help instructors manage the myriad decisions they must make in teaching this course.

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Building GTAs' Knowledge & Motivation to Promote Equity in Undergraduate Mathematics

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Promoting equity in undergraduate mathematics education is of vital importance, yet has received considerably less attention than equity in K-12 mathematics. The current study focuses on a pedagogical training program for graduate teaching assistants' (GTAs), which emphasizes equity in their teaching of undergraduates. The study examines GTAs' journals and open-ended survey responses, including their definitions of equity and the ways they promote equity in their classrooms. The research will foster discourse about ways of promoting equity in undergraduate mathematics and about professional development for undergraduate mathematics instructors.

Keywords: equity, teaching assistants (TA), pedagogical training, diversity, doctoral students

Strengthening the teaching skills of graduate teaching assistants (GTAs) in the mathematical sciences is a powerful way to increase the learning of the undergraduates they currently teach and those they will teach later as faculty members. GTAs' knowledge, skills, and motivation for promoting equity contribute to their effectiveness in ensuring all their students learn. Equity goes beyond providing all students with the same opportunities, avoids "deficit views" (NCSM & TODOS, 2016, p. 1) and "gap-gazing" (Gutiérrez, 2008), and is related to "both conditions and outcomes of learning" (Aguirre et al., 2017, p.124). Although there has been considerable attention to equity in K-12 mathematics (e.g., Boaler & Staples, 2008; Gutstein, 2016), equity in undergraduate mathematics education research warrants more attention as noted by Hauk and D'Silva (2018) and Adiredja, Alexander, and Andrews-Larson (2015).

To that end, the current study describes a collaborative grant-funded effort by three public universities to provide pedagogical training to mathematical sciences doctoral students serving as GTAs, with a focus on promoting equity in their classrooms. Gutiérrez's (2018) call to action and methods for "rehumanizing mathematics" undergird this project's training around equity, while Tanner (2013) is one source for additional strategies for promoting equity and engagement.

GTAs from the pilot phase of the project wrote reflections on a seminar about equity, and all GTAs in the current project completed a survey with open-ended questions about equity. The poster examines GTAs' perspectives on defining equity and promoting equity in their teaching, including similarities and differences across schools and between those who are new to teaching (mentees) and those who have been in the program for more than a year and serve as mentors. Initial findings suggest that some GTAs see equity as equal treatment of all students, while other GTAs, especially mentors, see equity as equipping all students for success. Methods identified for promoting equity include responsiveness to questions, providing extra help and scaffolding, not assuming prior knowledge, attending to individual students' needs, and fostering supportive and respectful classroom environments. The poster will foster discourse about equity in undergraduate mathematics courses and about professional development for undergraduate mathematics instructors.

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How can regional RUME conferences support inclusion into the larger RUME community?

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As our RUME community and attendance at our national conference grows, it is vitally important that we continue to attend to issues of equity and inclusion. In this poster presentation, we discuss how we can use regional RUME conferences to broaden participation and support inclusion in the larger RUME community. Using the lens of Lave and Wenger's (1991) communities of practice and legitimate peripheral participation, we analyzed survey responses from individuals who either attended or expressed interest in attending a regional RUME conference. We found that local RUME conferences provide newcomers and novices with the opportunity to (1) learn more about RUME, (2) network with RUME participants, including experts/old timers, and (3) develop professionally as both teachers and researchers.

Keywords: Equity, inclusion, community of practice, regional RUME conferences

In a Community of Practice people work together to learn about and improve at something that is of common interest. The three main components of a community include the domain, the community, and the practice (Lave & Wenger, 1991; Wenger, 1998). In our exploratory study, our domain is undergraduate mathematics education, our community is the local RUME community, nested inside of the larger national RUME community, and our practice is conducting educational research. We aim to learn about the role that local RUME conferences play in allowing participants to develop a local community. We wish to examine how these local RUME communities provide access to the larger RUME community. Our research question is: How can regional RUME conferences support inclusion into the larger RUME community?

Our data consisted of a post-conference survey filled out by our conference participants. The survey included questions about benefits of attending the conference, feelings of inclusion and being heard, future attendance, and concerns and advice for improving future conferences. We read through the data using open coding (Strauss & Corbin, 1990) to look for emergent themes.

Data analysis indicated three main takeaways about the supports local RUME conferences can provide. (1) Learning about RUME - For researchers unsure of what RUME is, the local conference provided a chance to learn about an area that they would normally not learn about in a larger conference. There was interest in learning more about what RUME entails; (2) Networking - Attending the conference provided a networking opportunity for the participants. Especially with local researchers who are interested in mathematics education and can become possible collaborators There was interest in having more opportunities to have conversations built into the conference program; and (3) Professional Development - For some participants the conference was directly beneficial, as talks related to their research or teaching practice. For others it was beneficial for their students; for this group there was interest in bringing more students to attend the conference the following year.

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GTAs' Conceptualization of Active Learning in Undergraduate Mathematical Sciences Courses

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Throughout the past few decades, the term active learning has been used to describe a variety of classroom instructional techniques and pedagogy. In this poster, we explore the graduate teaching assistants' conceptualization and implementation of active learning strategies at the start of a funded project evaluating a multifaceted GTA training model.

Keywords: Active Learning, Graduate Teaching Assistants, Post-secondary Education

The term *active learning* has gained much attention in the past few decades, and it is perceived as different from passive lecture instruction (Prince, 2004). The use of active learning techniques has been shown to increase examination performance (Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt & Wenderoth, 2014), improve both attitudes toward learning and thinking and writing skills (Bonwell & Eison, 1991; Prince, 2004), and "eliminate a sizeable gender gap" (Laursen, S., Hassi, M.-L, Kogan, M. & Weston, T., 2014). Because graduate teaching assistants (GTAs) often serve as instructors for undergraduate mathematical sciences courses (Speer, Gutmann, & Murphy, 2005; Meyer, Arnold, & Green, 2018), their views and experiences with active learning are important to explore.

Drawing on Vygotsky's sociocultural theory, Yee (2019) developed an active learning framework with two dimensions categorizing who/what the instructor engages with and who/what the participants engage with. As part of a multi-university funded study about a comprehensive model of graduate student instructor development, survey questions were developed using this framework to assess what activities GTAs thought were considered active learning, what activities they have used in the classroom, and what activities they have never heard of before, and what GTAs describe active learning would look like in their classroom. Sixty-seven mathematical sciences GTAs completed the survey at the start of the grant, including some GTAs who participated in the pilot phase of the training program at University A.

All of the GTAs at University A used lecture in the classroom, compared to 74% at B and 100% at C, with 11%, 42% and 0%, respectively, considering lecture to be active learning. The most used active learning techniques across all three universities included brainstorming, student questioning, and teacher questioning. The techniques that students were least familiar with included jigsaw and role playing (at A, B, and C) and think-pair-share (at B and C). This poster will look more deeply at the remaining quantitative results and the descriptions of what active learning looks like in their classrooms. The initial results inform the GTA training program and the body of knowledge about GTA training in general about pre-existing familiarity with active learning.

Acknowledgement

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Using Successful Affective Measures Among Native Populations in the U.S.

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A history of oppression in the U.S. has left a clear mark on its education system. Under-performing demographics is not a matter of genetic pre-disposition, but rather an inequitable system. Tribally Controlled Colleges/Universities (TCUs) were established to address a few of these inequities. Working with math faculty and administrators at four TCUs in North Dakota, an initial investigation into the affective domain of math instruction has begun. Seven semi-structured interviews with students from TCUs looking broadly at student experiences and feelings towards learning mathematics were conducted. An analysis of these data using a Tribal Critical Race Theory lens will be presented. From initial open coding, four themes within the data emerged, including 'being scared' or apprehension, interest in the utility of math to their personal lives, the feeling of accomplishment, and affect depending on instruction. Additionally, we seek to discuss the use of additional affective measures among marginalized persons.

Keywords: Tribally Controlled Colleges/Universities, math, affect, Tribal Critical Race Theory

A history of oppressive education and recent experiences of unequal power dynamics in research at Tribally Controlled Colleges/Universities (TCUs) has built a collective apprehension toward any outsider (Smith, 2013). Assimilationist thinking manifests itself in education by its attempt to mold all students into the mainstream and in research through deficit models and gap analysis showing Native students have yet to reach the ability of the white male student (Grande, 2015). One way research can counter oppressive white culture exemplified in unequal power dynamics and assimilationist thinking is through relationship building with TCU math faculty and administration and adhering to their guidance (Smith, 2013).

Tribal colleges began in the 'self-determination' movement of the 1960s and implement personalized student attention with cultural relevance to overcome barriers in (not "to") higher education for American Indian students, particularly in geographically isolated regions (American Indian Higher Education Consortium, 1999; Boyer, 1997). North Dakota has five TCUs, most of which are connected to a pre-engineering collaborative with North Dakota State University. This existing collaborative has allowed me to build relationships with TCU math faculty and students, some of whom agreed to be interviewed for this study. Also, these math faculty have been pivotal in determining what is beneficial to their respective TCU and consequently guiding this research towards the affective domain of math instruction *in situ* at TCUs in North Dakota.

This investigation consisted of semi-structured interviews about student math affect and experience. Data are analyzed through the lens of Tribal Critical Race Theory, which posits colonization is endemic to society, assimilation is the problematic goal of educational policies, stories are real and legitimate sources of data, and scholars must work towards social change (Brayboy, 2005). Open coding brought the emergence of four themes from the data when interviewees were asked a general question about feelings towards learning mathematics. The strongest two themes were 'being scared' and interest in the utility of mathematics to students' personal lives. The secondary two themes were the 'feeling of accomplishment' and that affect depended upon the teacher and instructional practices. Future work will use affective measures across different math curriculum at TCUs.

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Using the Learning Cycle and Mathematical Models to Engage Students in Sensemaking Involving Metamodeling Knowledge in Chemistry

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In this work we describe a study that analyzes student engagement in the process of sensemaking, which involved students developing metamodeling ideas related to mathematical models in chemistry, that is, ideas regarding the nature and purpose of models. This project defines sensemaking as the process of constructing and evaluating explanations to address an apparent inconsistency (i.e., "figure out" a problem). This process involves students using prior knowledge in combination with provided information to resolve a gap in understanding. For our dataset, students collaboratively worked through activities designed using the learning cycle (Process Oriented Guided Inquiry Learning, POGIL), which involves student-led movement through stages of exploration (direct questioning), concept invention (developing a formal definition for an idea), and application (using students' constructed concept in a new context). Preliminary results from this qualitative study emphasizes how students construct explanations and engage in metamodeling ideas as part of the sensemaking process.

Keywords: Sensemaking; Mathematical Models; Learning Cycle; Chemistry; Process Oriented Guided Inquiry Learning

According to Odden and Russ (2019a), "sensemaking" encompasses (1) a stance toward learning science, (2) a cognitive process that involves integrating new ideas with prior knowledge, and (3) the practice of discourse characterized by iterative cycles of construction and critique. Engaging students in sensemaking should be a target for instruction, given sensemaking emphasizes an attempt to "figure out" a gap in knowledge, rather than simply generate the "correct" answer. Moreover, tasks can be designed to explicitly promote engagement in sensemaking by providing opportunities for students to socially construct explanations and ask questions (Odden & Russ, 2019b), which is facilitated by the student-centered pedagogy Process Oriented Guided Inquiry Learning (POGIL) (Kultatunga, Moog, & Lewis, 2014). Relevant for the current work discussed, POGIL activities involve a learning cycle where students are introduced to a topic (exploration), construct a formal definition (concept invention), and test their understanding (application) (Abraham, 2005). Using the learning cycle, we developed activities intended to engage students in metamodeling knowledge (i.e., ideas related to the nature and purpose of models) (Passmore, Schwarz, & Mankowski, 2016).

Previous research indicated students have productive ideas about the nature of models, but may not apply these ideas to equations and graphs (Author, 2019). Thus, this work emphasizes metamodeling ideas in relation to mathematical models. For example, one activity we designed is contextualized using mathematical representations of chemical reaction rates, which was designed to promote the construction of explanations as students engage in the process of modeling by recognizing the empirical basis of rate laws and using it to make predictions. Previous literature indicates students have difficulty with rate-related ideas in chemistry and in mathematics more broadly (Bain & Towns, 2016; Castillo-Garsow, Johnson, & Moore, 2013; Rasmussen, Marrongelle, & Borba, 2014; White & Mitchelmore, 1996). In this qualitative study, we describe our analysis of students' engagement in metamodeling ideas and sensemaking as they reason through collaborative learning activities involving mathematical models.

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Examining Academic Performance and Student Experiences in an Emerging Scholars Program

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This poster discusses a sequential, mixed-methods study of an established program that incorporates active learning within recitation sections to support the academic performance and retention of students from underrepresented groups at a large research university. The study examines academic performance in math courses, degree attainment, and overall experience of students who participated in the program.

Keywords: Calculus, Pre-calculus, Active-learning, STEM

The Mathematics Merit Program, established in 1989, is designed to provide support to students from populations that are traditionally underrepresented in STEM fields. Similar to programs at other institutions, the Merit Program is based on Treisman's Emerging Scholars model (Asera, 2001; Hsu, Murphy, & Treisman, 2008). The program has evolved and grown over the years and is currently offered in seven undergraduate math courses. Additionally, the program now invites a larger and more diverse set of incoming students each year, including students from underrepresented ethnic groups, students from rural areas, and students who enter the university without a declared major. Participants are enrolled in standard large-lecture precalculus and calculus courses, but participate in recitation sections which are structured and taught differently from the standard sections. Teaching assistants for these sections receive extra training and mentoring in the active learning pedagogy and overall philosophy of the program. Participants are encouraged to form study groups and develop a community that extends beyond the classroom environment.

Methodology and Research Questions

This study uses a sequential, mixed-methods design (QUAN-QUAL) in which results of a quantitative analysis of student academic performance indicators are used to inform the sampling procedure for a set of qualitative student interviews (Teddlie & Tashakkori, 2006). In the first phase of the study, quantitative analyses of data related to students enrolled in five math courses that offered Merit sections over six years (Fall 2013-Fall 2018) address the following research questions: *Which target populations are participating more/less than others? To what extent does the academic performance (in terms of math course grades and degree attainment) of participants differ from similar students who did not participate? To what extent does participation help any of the target populations academically more/less than others?* In the second phase of the study, students from populations that are identified as most/least impacted by participation through the quantitative analyses are interviewed. Qualitative analyses of the interview data address the following research questions: *For what reasons and expected benefits do students choose to participate in the program? From their perspective, in what ways does participation in the program impact their academic performance or broader college experience both while participating and in their later college years?* Understanding which of the target populations choose to participate and why, examining how those students perform academically compared to their peers who did not participate, and hearing how students experience the program in their own words will provide useful information to institutions, especially large research universities, with similar programs aimed at increasing participation in STEM fields.

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The Ant Farm Task – The Case of Ginny

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We present preservice teacher's (PST's) progression of thinking during the Ant Farm Task (AFT). We describe the PST's construction of the Cartesian plane in four critical phases. The AFT presented us an opportunity to identify important cognitive activities that supported the PST's construction of the Cartesian plane.

Keywords: Cartesian Plane

While textbooks describe how to draw a Cartesian plane, they rarely address how and why the conventions work (Lee, in press). We believe students and teachers can benefit from opportunities to think about questions such as, "Why do we place two number lines perpendicularly intersecting at zero? How does this help define points in the plane? What do the number lines on each axis represent, if anything?" Opportunities to think about these questions can help build stronger understandings of the Cartesian plane. To this extent, we presented the Ant Farm Task (AFT; Lee & Hardison, 2017) to four elementary preservice teachers (PSTs) during a teaching experiment (Steffe & Thompson, 2000) to occasion such opportunities.

In the AFT, PSTs were provided with two transparent tubes representing two ant farms and asked to imagine two ants moving haphazardly in their respective ant farm. Also provided was a model of this situation in a dynamic geometry environment (DGE), in which two long, thin rectangles (ant farms) contained a point moving haphazardly within each rectangle (the two ants). The points' haphazard movement within the rectangles could be paused/activated by action buttons and the rectangles could be rotated or shifted within the DGE. We asked the PSTs to identify a single point that could be used to locate the position of both ants, at any given time.

In this report, we focus on one PST, Ginny, because a) her case illustrates a possible progression of thinking for solving the task, b) we observed her previous knowledge of the Cartesian plane to interplay with her activities in this task, and c) she was the most descriptive among our four participants. Ginny constructed the Cartesian plane in four successive phases: (1) Initial responses to the task, (2) Arranging the ant farms in different configurations, (3) using number lines to represent the ant farms, and finally (4) construction of the Cartesian Plane. Across these phases, the cognitive activities that led Ginny to the construction of the Cartesian Plane include: (a) thinking about a point outside of the tubes (awareness of 2-D space outside of the 1-D tube space), (b) arranging the tubes in different configurations, (c) constructing number lines to quantify the location of ants, (d) arranging number lines perpendicularly so that the locations of each ant could be accounted for simultaneously (viewing number lines as manipulative objects), and (e) viewing the ant farm space as a space analogous to the number line space.

Providing high-quality teacher preparation that equips mathematics teachers with the knowledge and resources to create learning opportunities for students is important (AMTE, 2017). We believe the AFT presented PSTs a task they might use for their students to motivate the construction of the Cartesian Plane. In addition, the AFT presented us an opportunity to identify important cognitive resources that supported PSTs' development of more robust understandings of the Cartesian plane.

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Group Testing in Calculus - How do Students in Groups Work Together Equitably?

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Recently group testing in mathematics classes have been explored for their equitable use as a means to re-humanize mathematics. This study extends this discussion by examining interactions within differently composed groups to identify group structure for equitable assessments.

Keywords: Calculus, Group Testing, Group Interactions, Equitable Assessment

Group testing is a student-centered assessment strategy that can help increase student's perception of learning (Revere et al., 2008), student's confidence in mathematics (Goetz, 2005) and has been shown to increase student communication, social skills, collaboration, accountability, and understanding of concepts (e.g., Paterson et al., 2013). However, understanding how students interact while engaged in group tests has not been examined. This study aimed to answer the question: how do students interact when engaged in a group test?

In this study, I examined the intra-group interactions of three different groups taking a group test. Transcripts for each group were coded for individual student utterance as a response to the student who spoke before them. Each interaction was coded on three dimensions as developed by Chiu (2000) when identifying social and individual actions of the group problem-solving process (i.e., evaluation of previous actions, knowledge content, and invitational form). Using social network analysis, sociograms were generated for each group (see Figure 1).

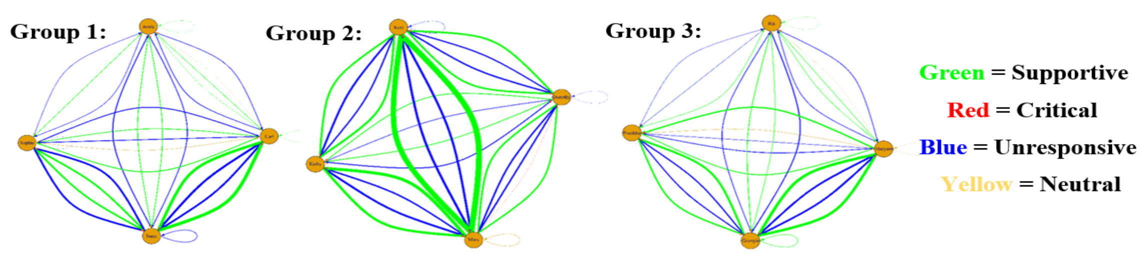

Figure 1. Green represent supportive, red are critical, black are unresponsive, and yellow are neutral statements

The analysis revealed the three groups interacted very differently. The first group had majority of the comments for three of the students utterances were unresponsive (41-46%), and one student had more supportive utterances (50%). The second group had more than 50% of utterances from each member as supportive and the third group had mostly supportive utterances for three students (41-54%) and unresponsive for the other student (50%).

Further analysis will investigate the difference between the group make-ups by student demographics (i.e., by gender, race, etc.). The poster will have details about the group test procedure and the sociograms for each dimension defined by Chiu (2000). The findings from this study will be important for identifying equitable group structures for undergraduate group assessment practices.

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Quantitative Reasoning Skills for Successfully Working with Real-world Data

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The purpose of this pilot study was to identify quantitative reasoning (QR) skills 9th grade high school students need to successfully construct evidence-based scientific explanations. Knowing what kinds of QR skills 9th grade students need in science classrooms can help inform pre-service teacher programs, since many of these skills are taught in mathematics classrooms. Preliminary results indicate that in order to create a claim, construct evidence about the claim, and successfully reason about the claim and evidence, students need to (a) contextualize the variables, (b) be open-minded for any relationship in the data, and (c) use quantitative language.

Keywords: quantitative reasoning, real-world data, pre-service teachers

The Common Core State Standards for Mathematics (CCSSM) emphasize quantitative reasoning (QR) in their mathematical practice standards (MP.2 – MP.4), as well as in the high school standards (HSN-Q.A.1-3). QR is also a cross-cutting concept in the Next Generation Science Standards (NGSS). There are many terms used to describe QR and this pilot study followed the framework and definition developed by Mayes and colleagues (2013, 2014). They define QR as mathematics and statistics applied in real-life, context dependent, interdisciplinary situations, and with an emphasis on tackling open-ended real-world problems.

After analyzing two sets of data stories from two 9th grade classrooms (33 students and 66 data stories) about different real-world data sets, four students were purposefully selected for follow-up interviews to represent a range of scores. Researchers used semi-structured interviews and coding techniques guided by grounded theory (Corbin & Strauss, 2008) to identify themes.

There appear to be three aspects of QR necessary for successfully working with real-world data and graphs: 1) contextualizing variables, 2) being open-minded towards the data, and 3) using quantitative language. The student who was the most successful in his reasoning contextualized the variables first and then took time to make sense of what relationships between variables could be of interest. Students who were less successful did not contextualize the variable, but instead explored the data by plotting them seemingly at random until the graph looked interesting. Further, students seemed driven by finding a correlation (three of the four students). All students struggled using quantitative language to describe their graphs.

These pilot study results indicate that it could be useful to include elements of QR in pre-service mathematics content courses, in particular when working with science data and contextualizing variables. The 9th grade students struggled creating meaningful graphs and were often unsuccessful when interpreting their graphs in context. Teachers could support students by asking them to think about the given variables within the context first before plotting anything to avoid the seemingly random plotting of variables. Incorporating activities focusing on the variables within the context in pre-service mathematics courses can then lead to a discussion about the appropriate selection of graphs. Helping pre-service teachers draw stronger connections between mathematics and science applications can help their future students to develop a deeper understanding of the mathematics. Our continuing effort to help teachers develop knowledge of QR is supported by a recently funded EPSCoR Track II grant. We are working on QR and big data in the context of forestry to engage teachers and students in place-based, locally relevant research.

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Reconceptualizing Mathematics Teacher Knowledge in Domain Specific Terms

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The question of what teachers need to know about their subjects to be able to promote powerful and flexible knowledge and understanding among their students has been difficult to answer. This difficulty has been a result of the many different conceptualizations of teacher knowledge, which until now have mostly been general and not domain specific enough. In an effort to promote a shift from these, this study used the Knowledge of Algebra for Teaching (KAT) project's conceptualization of knowledge for teaching algebra and the instruments developed from that project. Factor analysis of data collected has not only corroborated the KAT project's hypothesized knowledge but also pointed to a need to reconsider some of the assumptions made in the KAT framework. As a result, a new conceptualization of knowledge for teaching algebra that allows teacher knowledge to be assessed in measurable terms will be discussed.

Keywords: Reconceptualization of Teacher Knowledge, Knowledge for Teaching Algebra, Profound Knowledge of School Algebra, Pedagogical Content Knowledge in Algebra

Research is replete with the fact that the teacher is the most important factor that influences students' achievement (see for instance, Begle, 1972; Harbison & Hanushek, 1992; Yara, 2009). However, in spite of this agreement, early conceptualizations of teacher knowledge were not domain-specific enough (see for instance, Berliner, 1979; Shulman, 1986a, 1986b; Ma, 1999).

Apart from this, recent work on mathematical knowledge for teaching has focused mainly on early grades teachers (K-8) (see for example, Hill, Shilling & Ball, 2004; Hill, Rowan & Ball, 2005). Consequently, some researchers have called for different conceptualizations in the context of teachers with greater mathematical preparation and older students (Speer, King & Howell, 2014; Hauk, Toney, Jackson, Nair, & Tsay, 2013).

Fortunately, in the early to mid 2000s, Ferrini-Mundy and her colleagues, working on the Knowledge of Algebra for Teaching (KAT) project, studied teacher knowledge at the high school level (McCrory et al., 2012). They hypothesized a domain-specific conceptualization of mathematics teacher knowledge and developed items to measure it.

Though conceptualizations such as horizon content knowledge (HCK) (see for instance, Silverman &Thompson, 2008; Ball &Bass, 2009; Wasserman, 2017) bear similarities to the advanced knowledge conceptualization of the KAT project, the study being reported here focused on the KAT framework with the aim of validating it. It involved 209 participants. The KAT framework hypothesized three knowledge types and the project asserts that the intersections of them is blurry. Factor analysis performed on data collected corroborated the existence of the KAT project's three hypothesized knowledge types. In addition, the factor loadings revealed the loading together of items from different knowledge types on some other factors. This second finding pointed to the need to consider a modification of the KAT framework. Thus, this presentation contributes to the discussion of conceptualization of teacher knowledge in domain-specific terms by proposing a modification in the KAT project's conceptualization, into a framework that allows teacher knowledge to be assessed in measurable terms and also address what Speer, King & Howell (2015), as well as Hauk et al. (2013) have advocated.

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Perspective Outside and Within Spatial Diagrams:
Pre-service Elementary Teachers' Investigations of Shearing

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This study investigates how spatial diagrams are used by pre-service elementary teachers to construct arguments about measures of solid figures. Dynamic spatial diagrams offer immersive three-dimensional representations of three-dimensional geometric figures, where learners can take perspectives are not accessible in two-dimensional representations. The results describe how PSETs used perspectives outside and within a dynamic spatial diagram to make arguments.

Keywords: geometry, argumentation, representation, diagrams

Mithala and Balacheff (2019) describe three difficulties with two-dimensional representations of three-dimensional geometrical objects: "it is no longer possible to confuse the representation with the object itself," visually observed relationships can be misleading, and analysis of the representation requires the use of lower-dimensional theoretical properties. Despite these difficulties, learners are generally expected to explore three-dimensional figures through interacting with diagrams realized by two-dimensional inscriptions. Emerging technologies are upending this status quo. Three-dimensional alternatives include diagrams realized through various spatial inscriptions (e.g. Ng and Sincalir, 2018, Lai, McMahan, Kitagawa & Connolly, 2016; Dimmel & Bock, 2019, Gecu-Parmaksiz & Delialioglu, 2019). Such diagrams are three-dimensional in the sense that they occupy real (e.g., 3D pen drawings) or rendered (e.g., VR/AR environments) spaces as opposed to being inscribed or displayed on surfaces. Digital spatial diagrams can be grasped and transformed by gestures (e.g., stretching, pinching, spinning), even though they can't be physically touched (Dimmel & Bock, 2019). Spatial diagrams make it possible to use natural movements of one's head or body to explore figures from new perspectives (e.g., one can step inside a diagram), as they natively share the three-dimensional space. In this study we ask: How do learners use perspective to make arguments while interacting with spatial diagrams?"

To investigate this question, we designed a large-scale spatial diagram of a pyramid whose apex and base were confined to parallel planes. The diagram was rendered in an apparently unbounded spatial canvas that was accessible via a head-mounted display. The pyramid was roughly 1 meter in height and the parallel planes appeared to extend indefinitely when viewed from within the immersive environment. We created this diagram as a mathematical context for exploring shearing, a "continuous and temporal" measure-preserving transformation of plane and solid figures (Ng. & Sinclari, 2015, p.85).

We report on pairs of pre-service elementary teachers' arguments about shearing of pyramids, using Pedemonte and Balacheff's (2016) ck¢-enriched Toulmin model of argument. Participants used perspectives outside and within the diagram to make arguments about the shearing of pyramids that would not be practicable with static three-dimensional or dynamic two-dimensional representations. In one case a participant stood with her legs cross-secting the pyramid and used the imagined "slices" of her legs to explain how the "space inside" the pyramid can be conserved in the shearing transformation. The results of this study suggest that the dimensionality of the spatial diagrams supported participants' arguments about three-dimensional figures without mediation through projection or lower-dimensional components.

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Characterizing Undergraduate Students' Proving Processes out of Their "Stuck Points"

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Many undergraduate students experience significant difficulty in learning to prove mathematical propositions. In contrast to previous work focusing on the final products of proof, this work aims to characterize proving processes of undergraduate STEM majors who have recently completed an Introduction to Proof (ITP) course. This study extends Carlson and Bloom's framework in order to better account for how undergraduate students navigate the potentially multiple "stuck points" that students encounter during their proving processes.

Keywords: Transition to Proof, Proving Processes, Stuck Points, Novice Provers

Research has shown that many undergraduate students struggle to learn to prove, including those who major in mathematics (Moore, 1994; Selden, 2012). For many students, proving is a different mathematical activity compared to their prior experience. Much of the existing literature has analyzed the final products of proving processes for evaluating undergraduate students' proof construction competences (Selden and Selden, 2009). However, the step-by-step logical argument expressed in final written proofs does not tell us what led one to construct such steps of arguments. There are other researches that provide us some insights into students' understanding in proof construction, however, the big categories of students' thinking are at too coarse a grain size to apply in a moment-by-moment manner to proving process (e.g. Raman, 2003; Weber and Alcock, 2004; Weber and Mejia-Ramos, 2009). Not much work has done for analyzing undergraduate students' (novice provers) proving processes in a moment-by-moment manner. There are, however, frameworks for analyzing problem solving processes (cf. Carlson & Bloom, 2005; Schoenfeld, 1985). This study focuses on gaining a better understanding of novice provers' problem-solving part of proof construction activity.

The data collected for this study consists of video and audio-recordings of students' task-based interviews, in addition to their written work produced by a real-time LiveScribe "pencast" (including audio and writing in action movie). Cases of navigating stuck points were transcribed for further analysis. Constant comparative analysis (Glaser and Strauss, 1967) is applied to extract common themes across relative novice provers' stuck points. Carlson and Bloom (2005)'s problem-solving framework was used to characterize relative novices' actions when they face a stuck point – a place where a prover does not have an immediate way to make progress. Two cases of students' navigating stuck points will be presented in detail, with one case in which a student successfully navigated out the stuck points and one in which they did not.

The initial analysis revealed that the existing problem-solving frameworks were limited in their ability to characterize some of the critical behaviors being exhibited by novice provers. Preliminary results indicate that, novice provers often encounter multiple "stuck points" during the Planning or Executing phases. Because of these "stuck points," novice provers may not complete the full cycle of Orienting-Planning-Executing-Checking like the mathematicians that the Carlson and Bloom framework was based upon. Novice provers tend to cycle back mostly between Planning and Executing phase to try to navigate out. The different actions novice provers took to try to navigate out "stuck points" can be helpful or not depending on the different contexts. While the current findings are still preliminary, this work may inform the future proof-related course developers in supporting students in more productive proving practices.

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Coordinating Two Levels of Units in Calculus: The Story of Rick

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We conducted a constructivist teaching experiment to better understand how introductory calculus students' units coordinating activity supports their learning of productive conceptions of rate of change. This poster serves to illuminate what we learned while supporting Rick, a student assessed as assimilating with two levels of units.

Keywords: Units Coordination, Calculus, Quantitative Reasoning

Background

This poster reports on a five-hour constructivist teaching experiment (Steffe & Ulrich, 2013) that took place in Summer 2019 investigating introductory calculus students' developing understanding of rate of change through the lens' of units coordination and covariational reasoning. Units coordination refers to students' ability to construct and reason with nested layers of units (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015). For example, understanding n/m (n and m being positive whole numbers with $n>m$) as a number requires coordinating three levels of units: n/m as a unit comprised of n units of size $1/m$, where a unit of size $1/m$ could be iterated m times to make a unit of size 1. Some students pursue STEM majors in college while assimilating tasks with two levels of units (Grabhorn, Boyce & Byerley, 2018). A recent investigation (Boyce, Grabhorn, & Byerley, 2019) found that introductory calculus students who assimilate with fewer than three levels of units score significantly lower on the Precalculus Concept Assessment (PCA) than students who do assimilate with three levels of units, and past research (Carlson, Oehrtman, & Engelke, 2010) has shown the PCA to be predictive of student success introductory calculus courses. Our overall goal was to enrich our understanding of the ways that students' units coordinating activity supports their calculus learning, and specifically their development of rich understandings of rate of change.

Methods/Results

Our poster focuses on one participant going by the pseudonym Rick. Rick was an introductory calculus student at a large urban university, taking the class for a second time in order to achieve a higher final grade. Rick participated in five weekly one-hour teaching episodes. Rick worked with another calculus student, Erin, during three of the episodes and was the lone subject for two episodes. The first author served as teacher-researcher for each episode while the second author served as a witness. Each episode was video recorded and any written work was collected and scanned for analysis.

Rick was assessed as assimilating tasks with two levels of units based on his initial interview. This poster will discuss the constraints and affordances of Rick's units coordinating activity as he reasoned about rate of change throughout the teaching episodes. Two main themes emerged during analysis: First, we believe Rick commonly reasoned about multiplicative relationships between quantities as percentages in contexts where, otherwise, coordinating three levels of units could be beneficial or necessary. Second, Rick appeared to require particular forms of symbolic representations involving unknown values in order to reason about their rate of change (e.g. recognizing a need for the product rule when considering the rate of change of 'f(x)*g(x)' but not 'l*w', where both scenarios represent the product of length and width of a rectangle).

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Using Activity Theory to Understand Tensions in an Extra-Curricular Mathematical Modeling Project with Biology Undergraduates

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We use activity theory to analyze tensions manifested in an activity system of an extra-curricular mathematical modeling (MM) project with biology undergraduates at a research-intense Scandinavian university. We present the evidence that the use of MM in education of biology students accentuates conceptual understanding of mathematics and may lead to beneficial shifts in pedagogical practice.

Keywords: mathematical modeling, activity theory, tensions, mathematics classroom.

Description of the Study

Mathematics plays an important role in the education of future biologists; "the need for basic mathematical and computer science (CS) literacy among biologists has never been greater" (Bialek & Botstein (2004, p. 85). MM can serve as a "didactical vehicle both for developing modelling competency and for enhancing students' conceptual learning of mathematics" (Blomhøj & Kjeldsen, 2013, p. 151). We report on the project where the author led four extra-curricular three-hour sessions working on biology-motivated MM tasks with 12 biology undergraduates to increase their motivation for learning mathematics. The data comprise video recordings of all sessions, transcripts, students written work, and answers to two self-administered questionnaires on a 5-point Likert scale distributed in the beginning and in the end of the project.

Activity Theory (AT) can serve as a lens of inquiry in various educational settings ranging from a micro level of a classroom discussion to macro levels of interactions between different activity systems in the institutional and even wider settings. Contradictions play an important role in the AT and are often viewed as sources of development (Engeström, 1987); their presence in the activity system can be identified through the tensions they raise. For our analysis, we contextualize the activity system using the term "bucket" introduced by Barab, Evans & Beak, 2004 for its six components (subject, object, tools, rules, community, and division of labor) comparing them to "buckets for arranging data collected from needs and task analyses, evaluations, and research" (ibid, p. 207). Coding rules were used in the transcripts to assess the presence of tensions manifested during the project. In this report, we address two research questions: 1) What tensions emerged within an activity system of an extra-curriculum MM project with biology undergraduates? 2) How these tensions affected students' participation in the project?

Conclusions

Contradictions have the potential to transform activity systems, but this does not necessarily happen. AT helps to capture the complexity of educational processes in the wholeness as well as to analyze specific elements of the activity system, interaction between the elements and their contribution to the whole, cf. Jaworski et al. (2012). Students' active engagement with the tasks and their enthusiastic feedback on the project suggest that they participated in the process of expansive learning understood as "construction and resolution of successively evolving contradictions" (Engeström & Sannino, 2010, p. 7).

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Investigation of Affective Factors Which May Influence Women's Performance in Mathematics

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This paper presents a pilot investigation into the effects of stereotype threat on women's performance in upper-level undergraduate mathematics and potential influences of the affective factors, mindset and sense of belonging. In contrast to Good, Aronson, & Harder's (2008) findings, results indicate that the women in this study did not seem to be adversely affected by stereotype threat. Furthermore, sense of belonging to the domain of mathematics was found to be a statistically significant predictor of performance on a mathematics assessment.

Keywords: Stereotype threat, Gender stereotypes, Sense of belonging, Math achievement

Though women make up nearly half of the overall workforce and half of the population, they hold less than 25% of the STEM careers (Beede, et al., 2011; Corbett & Hill, 2010). Contributing to this differential is the fact that a large percentage of women leave the STEM course trajectory after completing Calculus I (Ellis, Fosdick, & Rasmussen, 2016). Stereotype threat, mindset, and sense of belonging have emerged in the literature as possible factors that may contribute to this persistent disparity. The purpose of this study was to investigate the potential roles of these phenomena in women's performance on an undergraduate mathematics assessment.

Stereotype threat is a situational predicament in which people are or feel themselves to be at risk of confirming negative stereotypes about their social group. Stereotypes are frequently, albeit unintentionally, transmitted through differential treatment from parents and teachers (Degol, Wang, Zhang, & Allerton, 2018). Since upper level mathematics courses have historically been stereotyped as a male-domain (Horner,1972; Fennema & Sherman, 1977; Brandell & Staberg, 2008), and men in upper level mathematics courses often outnumber women (Brandell & Staberg, 2008; Ellis, Fosdick, & Rasmussen, 2016), women in these courses may be vulnerable to the effects of stereotype threat, which has been shown both to suppress women's performance in mathematics (Spencer, Steele, & Quinn, 1999; Good, Aronson, & Harder, 2008), and to degrade women's sense of belonging to the academic domain of mathematics (Good, Rattan, & Dweck, 2014).

The research design included a replication of the Good, Aronson, & Harder (2008) experiment, for which participants (students enrolled in Calculus II in spring semester, 2019) completed a Calculus II assessment under one of two conditions: an experimental threat group, and a control group. Both groups had been subjected to a written statement designed to invoke stereotype threat. The control group was given an additional statement designed to nullify the threat. Data also included participant responses to a questionnaire comprised of Likert-scale items relating to mindset, perceptions of stereotype, and sense of belonging to the domain of mathematics.

In contrast to Good, Aronson, & Harder's 2008 findings, the women in this study's threat group outperformed both males and females in each other group by roughly 11%; though the difference was not statistically significant *($p = .362$)*. Regression analysis revealed sense of belonging to be the only statistically significant predictor of performance *($B = 3.40$)*, accounting for about 31% of variance in the outcome *($p < .001$)*. These results suggest the need for further study of conditions that minimize perceptions of stereotype and those that harbor a sense of belonging.

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Reforming Introductory Math Courses to Support Success for Underserved Students Who Place in Developmental Math

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This research describes results from both curricular and structural reforms in introductory math courses at Michigan State University (MSU), a large four-year public university. These reforms have focused on providing students with new pathways to success by eliminating an intermediate algebra course that had a low pass-through rate, did not count toward graduation requirements, and enrolled disproportionately high numbers of racial and ethnic minority students. The reformed curricula now place students directly into either quantitative literacy-focused courses for students on a degree pathway not requiring calculus or a reformed college algebra course sequence for students on a degree pathway that does require calculus.

Keywords: College Algebra, Developmental Mathematics, Quantitative Literacy

Introductory math courses tend to occupy a hegemonic role in STEM degree programs, often acting as a prerequisite to basic science courses. Various models exist for getting to and through college-level math, particularly for students who place into developmental math levels (Matz & Tunstall, 2019). When students who place at a developmental level are required to complete a developmental course prior to a college-level course that earns the student credit, students often flounder, especially when two or more developmental courses are required (Bonham & Boylan, 2011). The reforms described in this work were designed based on the framework that embedded developmental support within credit-bearing courses best facilitates success for students who place at a developmental level (Burn, Baer, & Wenner, 2013). These results rely on various quantitative administrative data obtained from MSU's Student Information System, administered by the Office of the Registrar. Both simple descriptive statistics and more sophisticated statistical procedures for matching similar students are used to assess the impacts of these major curricular and structural changes.

Several observed results substantiate the efficacy of these reform efforts. Pass rates in the reformed college algebra two-semester sequence (~80%) are higher than those for the developmental course that was replaced and comparable to the standard one-semester algebra course, and course grades in the reformed sequence are improved as well. Part of the reform efforts also involved routing a few hundred students, based on their high school performance, directly to the standard one-semester course while in previous years, these students would have routed to the developmental course; course grades and pass rates indicate that it is appropriate to continue to reroute these students. Performance in the quantitative-literacy focused courses is also good and there is evidence that the accumulation of junior- and senior-level students needing to pass these courses to graduate has been appropriately addressed. The math reform efforts at MSU have focused on providing students with pathways to success as math learners. Both nationally and at MSU in particular, introductory math courses have delayed and even prevented students from earning degrees, particularly in STEM disciplines. At MSU, this problem has been especially acute for students in racial and ethnic minority groups. In this poster, we will report on various quantitative outcomes from these major changes and discuss how new pathways through introductory mathematics can support the success of all STEM students.

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Analysis of Collaborative Curriculum Adaptation

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To support mathematics graduate student instructors (GSIs) as teachers, a collaboratively generated curriculum has been developed for all precalculus courses at the University of Nebraska—Lincoln (UNL), focusing on problem-based, student-centered instruction including lesson plans to actively engage students. Over the past few semesters, GSIs and a faculty member at the University of South Carolina (USC) have been working to adopt these materials at their institution for transformational change in GSI teaching. We analyze the allocation of time and teacher design practices for various tasks associated with the adaptation of problem-based handouts and lesson plans. We also consider the implications for personnel with a vested interest in implementing similar changes.

Keywords: GSI Development, Curriculum Adaptation, Problem-Based Materials

Our work seeks to investigate the following research questions:

1. How might stakeholders distribute their resources, time, and focus when adapting problem-based materials to their local context?
2. What resources can contribute to the successful implementation and adaptation of problem-based materials?
3. What outcomes can a department expect from adapting problem-based materials into their curriculum and culture?

The seven precalculus GSIs at USC, a southeastern doctoral-granting university, participated in this study through two semesters. Participants met weekly to discuss and modify problems and lesson plans that had been generated and aligned to their curriculum. Collected data included a "Journal of Implementation", wherein contributors (including GSIs and a faculty facilitator at USC, and a faculty consultant from UNL) made detailed logs about the amount of time spent and the teaching practice (TP) involved while adapting the problem-based curriculum materials. Logs are found in the Journal of Implementation for many types of tasks, from creating or editing problem sets and lesson plans to team meetings for discussing logistics or pedagogical practices.

The data was analyzed by qualitatively coding the TPs using naturalistic inquiry (Lincoln & Guba, 1985) and then quantitatively analyzed by time within each TP. A constant comparative method was used to code the entries building loosely upon the Follow-Omit-Modify-Sort-Create scheme associated with facilitators' adaptation practices of professional development curriculum materials (Leufer et al. 2019). Once coded, the data was analyzed to observe statistics about and patterns in the distribution of intellectual focus and adaptation practices, considering specifically by-semester and by-role grouping schemes.

The results show that, over the Summer-Fall-Spring semester cycle, GSIs averaged the largest amount of time during the Fall semester, during which time their primary focus was on preparing to teach with the new materials, along with some time spent continuing to develop the materials. There was a dramatic drop-off in time required to implement the materials in the Spring semester, during which one GSI taught using the materials a second time. The time required by faculty overseeing the project noticeably lessened each semester.

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Engaging Students in Reflective Thinking in Precalculus

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On a metacognitive level, reflection is known to be an essential skill for improving learning. In practice, students may not always use it of their own accord to improve this kind of learning because it can be mentally demanding, and they often believe that learning mathematics means getting the right answer without reflecting too much on their learning. To encourage reflection on understanding of concepts in my Precalculus course, I introduced assignments that required students to reflect and self-assess their learning of concepts. This study explores this type of pedagogical approach.

Keywords: reflective thinking, conceptual understanding, precalculus

In recent years, there has been an increased interest in incorporating reflective writing activities in learning mathematics. (see e.g., Guce, 2017). Reflection in the context of learning, as defined by Lew and Schmidt (1985), "refers to the process that a learner undergoes to look back on this past learning experiences and what he did to enable learning to occur and the exploration of connections between the knowledge what was taught and the learner's own ideas about them". Studies (Odafe, 2007 and Guce, 2017) suggest that self-reflection helps students improve their study habits and enrich their mathematical learning.

In fall 2018, I introduced assignments that engage students in self-assessing their learning to promote self-reflection on understanding of concepts in my Precalculus course. Many students believe that doing mathematics means learning algebraic procedures without any connection to the underlying concepts. Conceptual understanding has been identified as one of the barriers of mathematical proficiency (National Council of Teachers of Mathematics (NCTM), 2000; National Research Council, 2001). The purpose of this research study was to explore: What did these reflective assignments reveal about my students' learning of concepts?

Throughout the semester, in addition to their regular assignments, in my Precalculus course, I assigned six written reflection assignments. To set the stage, at the beginning of the semester, I asked my students to write their mathematical autobiography and a reflective essay about learning in general. Then, I asked them periodically, especially after quizzes and exams, to reflect on their learning of specific concepts. Note that my regular assignments also included verbal questions focused on conceptual understanding (e.g. "Describe in your own words what a function is.")

The results of my study show that most of my students were not used with self-reflections. Although many of them have seen most of the Precalculus' concepts in high school, they were used to just knowing how to solve a problem but not to explaining why you solved it that way. Their reflections suggest that this was their first math course where they needed to explain math concepts, so they had some difficulties assessing their understanding of concepts. Overall, these written activities have potential to engage students to uncover their conceptual understanding and to foster growth in their metacognitive abilities. On their final exam, most of my students knew what a function was (39/48 students) compare to only four (4) students on their pre-test at the beginning of the semester. My students also mentioned on their reflections, that this course helped them improve their study skills. "This class has helped me to learn from my mistakes and understand math in a different light." (student reflection of a Precalculus student).

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The Interaction between Instructor and Students in Community College Algebra Classrooms

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This poster presents evidence and results from two of the codes, Instructor-Student Continuum of Instruction and Classroom Environment, from the current version of our video coding protocol for community college algebra instruction. We highlight the findings, teaching examples, challenges, and implications from analyzing 135 hours of video data from 44 community college algebra classrooms, entailing coding of 1110 segments for teaching algebra at the community college level.

Keywords: community colleges, algebra instruction, classroom environment, interaction, video coding protocol

Many students go through community colleges to prepare themselves for STEM and other mathematics-based career options. In 2010, more than 585,000 students were taking intermediate or college algebra in community colleges (Blair, Kirkman, & Maxwell, 2013). Hill, Rowan, and Ball (2005) showed that the relationship between quality of instruction and student learning is based on the instructor's knowledge of teaching, content knowledge, and instructional practices in K-12 education. In light of that, we have limited knowledge of what instructional characteristics contribute to the success of community college students. We lack the means to assess the quality of algebra instruction at the community college level.

Since 2016, while working on a federally-funded research project (Watkins, Duranczyk, Mesa, Ström, & Kohli, 2016), we sought to characterize community college classroom instruction by identifying the aspects of classroom instruction that correlates to student academic performance and learning gains. In the past three years, we have gone through more than a dozen iterations of the protocol as the video analysis protocols that we started our project with did not provide clear delineation and descriptions of instructional practices that we observed in community college algebra classrooms. With our current protocol, Evaluating the Quality of Instruction in Post-secondary Mathematics (EQIPM), we tried to address the complexities of community college algebra instruction. We collected video data from six community colleges and these videos were segmented into smaller 7.5-minute episodes and distributed among a group of coders in the project. Smaller 7.5-minute episodes were used to provide a clearer picture of algebra instruction. Using an iterative process of review and calibration, we coded about 135 hours of algebra instruction as we used a 5-point scale with 14 codes along with three codes describing the setup for instruction to depict the components of quality instruction.

This poster will discuss the importance of understanding the interactions between an instructor and students. It will also present findings and examples from two of the codes, Instructor-Student Continuum of Instruction and Classroom Environment, under the category of Quality of Instructor-Student Interaction in our video analysis protocol. In addition, we will present the details on how we used our findings to develop and to fine-tune our video analysis protocol and will describe the challenges that we faced while coding and calibrating our video segments in assessing the aspects that are considered under each code. We will share the implications for teaching community college algebra and will seek to gather feedback from our peers.

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Justifying and Reconstructing in the Generalizing Process: The Case of Jolene

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This poster shares a case of reconstructive generalizing. Specifically, Jolene's response to a reconstruction task is presented as an illustration of an approach to reconstructive generalizing. Given a generalization and a request to develop a new generalization about an expanded domain, Jolene developed a justification for the given generalization and used that justification to scaffold her reasoning toward a reconstructed claim.

Keywords: generalizing; justifying; mathematical structure

Harel and Tall (1991) theorized three kinds of generalizing. *Expansive* generalizing involves assimilating new elements into the domain of an existing generalization without changing essential aspects of the generalization; *reconstructive* generalizing involves revising an existing generalization to accommodate an expanded domain; and *disjoint* generalizing involves constructing a new generalization to cover a domain without reference to an existing generalization. These types are not determined by task context, because each type depends on the knowledge, understanding, and ways of reasoning of a respondent in the context of a situation that calls for the construction of a generalization. In this paper, I share an illustrative case from my interview with Jolene, a mathematics major at a large public university, in which she uses a justification as the basis for reconstructive generalizing.

Reconstruction Task

To create situations that are likely to promote generalizing, I created tasks that present a claim and a domain of validity, prompting the respondent to make a claim about a domain that subsumes the given. Consider the following task:

> The product of any four consecutive whole numbers is divisible by 12. Can you rewrite the statement so that it is true for products of three or four consecutive whole numbers?

Jolene: Justifying and Reconstructing

Jolene's response was unique among those of ten participants in this research. Others began by using exemplars to verify the given generalization; Jolene worked to unpack and to explain it. Using a generic representation of "four consecutive whole numbers", she reasoned that any such set would include two even numbers and one multiple of three. This provided her with a rationale for the given generalization. Jolene then adapted the rationale, acknowledging that a set of three consecutive whole numbers could only guarantee one even factor, while still including one multiple of 3. She then arrived at the reconstructed claim that a product of three or four consecutive whole numbers would be divisible by 6.

Implications

The task lends itself to multiple approaches. Nine of the ten respondents engaged in reasoning inductively from generated examples of the expanded domain. The power of Jolene's approach was in her search for a justification for the claim which then provided a scaffolding for reasoning toward a reconstructed claim. This suggests that justifying may play an important role in framing the reconstructive generalizing process.

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College Mathematics Instructors Learning to Teach Future Elementary School Teachers

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This interactive poster presents recent results from research into an online professional learning experience for those new to teaching mathematics courses for future elementary school teachers. Course participants included graduate student, contingent, and tenure-track instructors from 2- and 4-year colleges. Results use three data sources: participant contributions in asynchronous discussion boards, verbal and textual communication during synchronous live sessions, and individual interviews. Questions driving the research: What did instructor-learners find challenging? Reassuring? How did they use what they learned? Why that way?

Keywords: Professional Learning, Mathematical Knowledge for Teaching Future Teachers

The literature calls for professional learning opportunities for mathematics instructors who are novices in teaching courses for future teachers (Castro Superfine and Li, 2014; Konuk, 2018; Masingila et. al., 2012). In response to this call, an NSF-funded project (PRIMED, 2016) has developed and implemented a series of five online professional learning modules on topics such as inquiry- and task-based learning (Jackson et al., 2018), mathematical knowledge for teaching (Hill et. al 2008), and tools for equitable instruction (Schoenfeld, 2016). Teräs and Kartoglu's (2017) model of interdependent components in effective online professional learning (see Figure 1) includes: (1) engagement with tasks that are authentic and complex, (2) cycles of reflection and discussion that allow sharing and comparing of ideas among peers and mentors, and (3) authentic formative assessment seamlessly integrated with professional learning tasks. The poster presents PRIMED instructor-participant perceptions and feedback in each of these three components. In the context of the model in Figure 1, **the question driving poster conversation is**: What are the implications of participants' reported experiences for PRIMED short-course redevelopment? In particular, the authors seek ideas and knowledge about other, similar, work to support development of the final, free, self-paced, completely asynchronous version of the online short-course. The goal is a final version that might be completed by teams at the same institution, or by multiple faculty across institutions who form a professional learning community.

Figure 1. Model for authentic online professional learning (Teräs and Kartoglu, 2017)

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Replacing Remedial Algebra With a Credit Bearing Math Education Course

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This report describes the first year of a pilot study to replace a non-credit bearing remedial algebra course with a credit bearing mathematics course designed for elementary education majors. The replacement course focuses on developing a deeper understanding of numbers and operations, compared to a primarily skills based remedial class. The resulting course average and passing rates of both courses, as well as passing rate of subsequent courses are included.

Keywords: Undergraduate mathematics, Algebra, Remediation

A significant percentage of students entering public 4-year institutions take a remedial math course and this coursework impacts both advantaged and disadvantaged populations (Chen 2016). Most of these courses target algebra skills and are a gatekeeper for many students in successfully earning degrees (Moses & Cobb, 2002). In addition, Complete College America's (2012) report estimates that $3 billion are spent on courses, such as remedial math courses, that are not going to count towards a degree.

To offer an alternative to remediation at a university in the Northeastern United States, students who placed into a non-credit bearing remedial algebra class through the university's placement exam were allowed to take a credit bearing math course designed for pre-service elementary teachers in its place. This alternative course focused on developing a deeper understanding of numbers and arithmetic, compared to a remedial algebra course designed to teach and reinforce arithmetic and algebra skills. The math education course served two purposes: to give students a better background to succeed in subsequent courses and to satisfy three credits towards the university's general education requirement. The same instructor taught two sections of the math course for elementary teachers with one section having primarily remediation students ($N = 18$) and the other primarily having education students ($N = 23$). Analysis compared the remedial students in the math education course ($N=24$) with the education students in the course ($N=17$) and the remedial students in the traditional remedial algebra course offered the same semester ($N=39$). Then the success rates of the remedial students in their subsequent math class the following semester ($N=28$) were analyzed.

Several data points were used to determine whether the pilot study was successful. Course averages and passing rates were better for the remedial students in the math education course. The mean average course grade for this class was 2.678, compared to an average course grade of 2.071 for the traditional remedial algebra course. An independent sample t-test showed a significant effect, $t(73)=2.024$, $p=0.064$. In the math education course, 34 students received a C or better while 7 received a D or lower. In remedial algebra, 19 students received a C or better while 15 received a D or lower. In subsequent math courses, students from the math education course performed better than those from the remedial algebra course.

The results of this pilot study are promising. The earned grades were similar across courses for remedial students. In addition, students in the math education course earned general education credit that can be used towards graduation. Students in the remedial algebra course received no credit and the course did not contribute to their credits needed to maintain financial aid. We will continue to monitor the performance of the remedial students in this pilot as well as examine remedial students from the second year of the pilot.

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A Comparison of Students' Quantitative Reasoning Skills in STEM and Non-STEM Math Pathways

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This study investigated students' Quantitative Reasoning (QR) in STEM and Non-STEM math pathways using the Qualitative Literacy & Reasoning Assessment (QLRA). Participants were students who were enrolled in at least one college-level math pathway course at a large public institution in southeastern U.S. The results showed that STEM students scored, on average, higher than Non-STEM students. Both STEM and Non-STEM students who were further along in their math sequence had higher QLRA scores than those taking the gateway math courses in that pathway. However, students overall had relatively low QLRA scores, with an average score of 24%. These results indicate there is still a great deal of improvement to be made in students' QR skills and that the math pathways initiative to align math curriculum with career fields and degree majors, while very important, does not invariably address quantitative reasoning.

Keywords: Quantitative Reasoning, Math Pathways, STEM

Quantitative Reasoning (QR) is essential knowledge for today's students (Wilkins, 2000). However, most higher education institutions have not effectively addressed this issue (Steen 1997; Steen 1999; Madison & Steen, 2008). Building upon the previous research of undergraduate students' QR skills assessment (Steele & Kilic-Bahi, 2010) this study investigated students' quantitative reasoning in STEM and Non-STEM math pathways using a non-propriety, NSF grant-funded instrument (Gaze et al., 2014), the Qualitative Literacy & Reasoning Assessment (QLRA). Participants were students who were enrolled in at least one college-level math pathway course at a large public institution in southeastern U.S. While previous research (Agustin et al., 2012) has indicated that traditional math courses are not necessarily sufficient for students to develop quantitative reasoning skills, this study also examined the relationship between the math courses in both STEM and Non-STEM math pathways and the development of QR skills.

The results showed a significant difference between STEM and Non-STEM students' QLRA scores, with STEM students scoring higher than Non-STEM students. STEM students who were further along in their math sequence had a higher QLRA score than those taking the gateway math courses in that pathway. Non-STEM students who took additional math courses also had a higher QLRA score than those in the entry-level math course. However, students overall had relatively low QLRA scores (N = 539, M = 23.78%, SD = 14.07%). The results of this study, as well as previous research (Agustin et al., 2012; Steele & Kilic-Bahi, 2010), indicate there is still a great deal of improvement to be made in students' QR skills. Furthermore, this study highlights that, while the math pathways initiative to align math curriculum with career fields and degree majors is very important, it does not invariably address the development of QR skills for students.

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Using Commognition to Study Student Routines Performed in the Context of Ring Theory

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Drawing on the commognitive framework, I studied student routines in the context of ring theory. Four micro-routine clusters emerged, as well as several macro-routines. Furthermore, questions about the commognitive framework and its operationalization were raised.

Keywords: Abstract Algebra, Commognition, Discourse, Routines

More than 70% of mathematics majors at U.S. universities and four-year colleges are required to take a course in abstract algebra, making abstract algebra the most offered upper-division course—running at least biennially—in U.S. mathematics departments (Blair, Kirkman, & Maxwell, 2018). In abstract algebra classes, group theory has traditionally been taught before ring theory; yet, some institutions have questioned the groups-before-rings approach. Whatever the reasons (e.g., the novelty of rings-before-groups or the prevalence of the traditional groups-before-rings approach), students' difficulties with group theory have been well-documented over the last 30 years (e.g., Asiala, Dubinsky, Mathews, Morics, & Oktaç, 1997; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Larsen & Lockwood, 2013), whereas the learning of ring theory has received scant attention (Cook, 2018). Thus, the aim of this study is to address the question: What are routines undergraduates perform to solve tasks related to ring theory? The study is rooted in commognition and based on the assumption that "learning is […] a process of routinization of learners' actions" (Lavie, Steiner, & Sfard, 2019, p. 153). A secondary purpose of the study is to contribute to operationalizing commognition and the study of routines (Sfard, 2008), particularly in the context of undergraduate mathematics education.

The participants of the study were six undergraduates who had completed their first course in abstract algebra—introducing them to ring theory—in Fall 2019 at a large, public, Midwestern university. In spring 2019, each of the six participants took part in a task-based interview. The data are being analyzed by drawing on commognitive literature (e.g., Lavie et al., 2019; Sfard, 2008; Viirman & Nardi, 2019) and using elaborative coding (Auerbach & Silverstein, 2003).

Findings from the analysis of the first two participants' task-based interviews demonstrate that students make use of a number of micro-routines (i.e., repetitive patterns of action specific to a task) and macro-routines (i.e., repetitive patterns of action occurring across several different tasks). Examples of micro-routines I noted are: writing down sets with patterns as examples of ideals of the integers (e.g., the square numbers or negative powers of 2), using a formula to determine the expected number of cosets to check one's answer, and connecting quotient rings to the division of two rings. Examples of macro-routines I noted are: plugging numbers into algebraic expressions and looking for patterns between the resulting numbers, recalling or searching for theorems, and responding with "I don't know" but following this statement up with ideas. Looking across the 18 micro-routines identified, four micro-routine clusters have emerged: generating ideals of the integers, performing modular arithmetic, determining cosets, and thinking about quotient rings. The micro-routines making up these clusters demonstrate the rich and varied ways in which students, even those from the same class, approach ring theory tasks.

With regards to operationalizing commognition, I have garnered a number of takeaways and questions. For instance, I illustrate the benefits of also attending to word use, visual mediators, and narratives (Sfard, 2008) and wonder about exploratory and ritualistic aspects of routines.

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An Analysis of Racialized and Gendered Logics in Black Women's Interpretations of Instructional Events in Undergraduate Pre-calculus and Calculus Classrooms

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As part of a larger study, we looked at how Black women make sense of discouraging instructional events in their mathematics classrooms and future participation in the courses.

Keywords: *African American, calculus, gender, race, women*

"Usually a lot of females are pushed away when they think they're right and someone else is wrong. I've had this happen to me with multiple professors and it just happens more often for females, I feel." -Skyla, in response to an event in which math instructor did not accept a calculation correction from a student.

Despite mathematics often being seen as "culturally neutral" (Hottinger, 2016; Leyva & Alley, in press), a significant body of literature has documented the ways undergraduate students perceive instruction differently in these classrooms based on their race and/or gender (Borum & Walker, 2012; Leyva, 2016; McGee & Martin, 2011; Rodd & Bartholomew, 2016). As historically marginalized students navigate these mathematical spaces, they must make sense of issues of race and gender on their own in order to succeed mathematically (Battey & Leyva, 2016). Research exploring marginalizing aspects of students' mathematical experiences has focused on those who persisted through a certain course or major (see, for example, Borum & Walker, 2012 or McGee & Martin, 2011). These studies mostly draw on students' post-hoc understandings of their mathematics experiences. This study was designed to capture students' perspectives as they were in their undergraduate courses, when mathematics success was still uncertain. Additionally, as with Borum and Walker (2011), this study specifically focused on the intersectional interpretations of Black women.

The study used group interviews to capture the ways in which Black women made sense of various discouraging events, with a particular focus on the ideological and instructional logics (Battey & Leyva, 2016) behind their interpretations of the events. Four women were interviewed in small groups (with either Latinx women or Black men) about their reactions to several composite events created from the experiences of other students that occurred in pre-calculus and calculus courses. Three themes emerged in the analysis. First, participants perceived instructors as favoring students of the same race and gender, especially when the tenor of instructor-student interactions varied consistent with preferential treatment of students with the same identity. Second, participants perceived power dynamics based on race, gender, and professional hierarchy limiting their options for responding to discouraging incidents to staying silent or switching classes. Thirdly, the Black women cited negative racialized and gendered stereotypes surrounding mathematics and intellectual ability to explain invalidating or dismissive instructor behaviors. These findings give insight into the racialized and gendered logics within ideologies and institutional structures that shape marginalizing experiences of entry-level mathematics instruction among Black women.

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Relationships between Lisa's Units Coordination and Interpretations of Integration

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At this poster, we will discuss results of a teaching experiment investigating productive and problematic ways students assimilating with two levels of units reason about integration.

Keywords: Calculus, Integration, Multiplicative Reasoning, Teaching Experiment

This poster reports on a five-hour paired-student teaching experiment (Steffe & Thompson, 2000) that took place in Summer 2019 investigating relationships of calculus students' ways of coordinating numerical units (their multiplicative and fractional reasoning) and their interpretations of integration. We focus on one student in the pair, Lisa[1], whom we assessed as assimilating multiplicative and fractional relationships with two levels of units via a one-hour clinical interview prior to the teaching experiment (see Boyce, Byerley, Darling, Grabhorn, & Tyburski, 2019). Lisa assimilated situations involving two levels of units structurally (e.g., that twelve is simultaneously a unit of twelve 1s and a unit of four 3s), but to reason successfully in situations involving three or more levels of units required coordinating *in activity*— physically or imaginatively enacting transformations of numerical units (Hackenberg, 2010).

Ely (2017) describes four ways calculus students interpret integral notation: (1) geometric interpretations of area and length, (2) anti-derivative, (3) adding-up pieces, (4) multiplicatively-based summation (MBS). Lisa was enrolled in a second-term calculus class (not taught by the Authors) that included an introduction to all four interpretations, but most explicitly focused on supporting an MBS interpretation. The class included the development of approximate and exact rate of change functions and approximate and exact accumulation functions described in Thompson and Ashbrook's (2019) text. The teaching experiment sessions consisted of explorations of how students were reasoning about calculus ideas introduced in their class, via tasks introduced in the out-of-class paired-student sessions.

The poster will include comparing and contrasting Lisa's responses to those of student from a similarly structured teaching experiment, who assimilated with three levels of units but whose calculus course did *not* explicitly focus on supporting the MBS interpretation (Grabhorn, Boyce, & Byerley, 2018). One task that will be prominent on the poster involves students' responses and justifications of relationships between extrema of functions $f(t) = t\sin(\pi t)$ (provided graphically) and $A(x) = \int_a^x f(t)dt$, which included the following questions asked in sequence.

1. Can you tell me the t value where $f(t)$ is increasing the fastest?
2. Can you tell me the t value of the absolute maximum of $f(t)$?
3. What does this notation mean to you: $\int_0^x f(t)dt$?
   a. Are there other ways of thinking about that notation?
   b. What does the *x* represent? The *t*? The *dt*?
4. Can you tell me the *x* value where $\int_0^x f(t)dt$ is increasing the fastest?
5. Can you tell me the x value of the absolute maximum of $\int_0^x f(t)dt$?

The results provide insight into the role of units coordination in the accessibility of instruction supporting the MBS interpretation.

[1] Pseudonym

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Chase that Rabbit! Designing *Vector Unknown*: A Linear Algebra Game

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We report on the second iteration of the game Vector Unknown, a linear algebra game based upon the Magic Carpet Ride sequence from the Inquiry Oriented Linear Algebra (IOLA). Observations from the first round of gameplay interviews produced a variety of student strategies. These student strategies informed revising the game to introduce three different difficulties and the addition of a tutorial mode. In particular, we found that the presence of standard basis vectors could affect the strategies employed by the participants.

Key words: linear algebra, game-based learning, design-based research

The game *Vector Unknown* is the result of combining the tenets of Game-Based Learning (GBL), IOLA, and Realistic Mathematics Education (RME) to help students learn the concepts behind scaling vectors, adding vectors, and vector equations (Authors, 2018; Authors, 2019). *Vector Unknown* was based on the Magic Carpet Ride task from the IOLA curriculum (Wawro, Rasmussen, Zandieh, Sweeney, and Larson, 2012). The goal of the game is to guide a bunny to a basket by choosing up to two vectors from a selection of four (two pairs of linearly dependent vectors), placing them into a vector equation, and scaling the vectors by integers. Players had five levels during the first iteration, with the first level providing a predictive path that showed how the rabbit traveled along each component of the sum prior to pressing GO (telling the rabbit to go to the sum of the vectors). The second level removed the predictive path and the fifth level contained keys the players needed to obtain prior to going to the basket.

Semi-structured interviews were conducted with a group of 11 students to determine how students reasoned while playing the game. We primarily looked at the 5 students who had no linear algebra experience. The results showed that as students played the game, they developed added anticipation as to how the vector equation related to the bunny's movement. One key strategy that students utilized during gameplay we termed *focus on one vector*, which is where students scaled a vector until it was close to the goal (either geometrically or algebraically) and then employed a guess-and-check sequence to finally reach the goal. This method was noticeably easier when students had access to standard basis vectors as their vector choices. The incorporation of standard basis vectors led to another form of reasoning known as *focus on one coordinate*, where students focused on matching either the x or y coordinate of the goal position.

Using this information and following the tenets of design-based research (Cobb et al., 2003; Lesh, 2002), the game was re-designed to have three different modes of difficulty and a tutorial. During the tutorial, only standard basis vectors are present. Easy mode includes one standard basis vector (multiplied by 1 or -1) and a multiple of that vector. Medium and Hard modes exclude the chance of standard basis vectors (or their multiples) with Hard mode including vectors with larger magnitudes that may require students to go past the goal and return. This new version of the game will be implemented in the classroom for future research.

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Designing Proof Comprehension Tests in Real Analysis

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The ReACT Team[1]

This poster presents a project aimed at developing and validating short tests to assess students' comprehension of eight proofs that are commonly studied in undergraduate real analysis courses. We describe the project's method, illustrating a process of fine-tuning items through the analysis of undergraduate students' responses and mathematicians' evaluations of these items. We also discuss some of the difficulties we encountered when designing proof comprehension tests in this context. In particular, we discuss challenges regarding (1) the heterogeneity of course curricular approaches to present these theorems and their proofs, and (2) the assessment of different facets of understanding a proof in this setting.

Keywords: Proof Comprehension, Real Analysis, Assessment,

In this poster we describe a project that follows the methods introduced by Mejía-Ramos et al. (2017) to develop and validate short, multiple-choice, reliable tests to assess undergraduate students' comprehension of eight proofs that are commonly studied in undergraduate real analysis courses (e.g. the proofs of the Bolzano–Weierstrass, Extreme Value, Intermediate Value, and Mean Value Theorems). As part of that process, we first generated open-ended items addressing each facet of Mejía-Ramos et al.'s (2012) proof comprehension assessment model. We then conducted individual task-based interviews with 12 undergraduate students (each of whom had recently taken a real analysis course at one of two large state universities), and used their ways of thinking about the open-ended items to generate multiple-choice items for each one of the eight tests. Afterwards, we sent the eight multiple-choice tests to 10 mathematicians working at the same two universities (all of whom had recently taught an undergraduate real analysis course), and asked them to review these tests, providing comments on the clarity, mathematical accuracy, and intended correct answer for each item. After incorporating suggestions from these experts, we conducted individual task-based interviews with another 12 undergraduate students, in which we asked them to think aloud as they answered each multiple-choice item.

In this poster we illustrate this general process with changes we made to individual items based on our analysis of the gathered data, and we make available a sample of both the open-ended and multiple-choice versions of the items we designed. Furthermore, we discuss some of the challenges we encountered when extending the design of proof comprehension tests to the context of real analysis. First, across real analysis curricula we found very different ways of stating "the same" theorem and proving a given theorem statement. This poses a challenge to designers of proof comprehension tests intended to be useful in courses using different curricula. Second, we discuss challenges we have encountered assessing specific aspects of proof comprehension, as outlined in Mejía-Ramos et al.'s (2012) assessment model.

[1] The ReACT (Real Analysis Comprehension Tests) Team is comprised of the PI and co-PIs of the project funded by NSF Grant No. DUE-1821553.

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Influence of University Teachers' Meanings on Their Interpretations of Student Meanings

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In this poster I explore the relationship between university mathematics professors' meaning for average rate of change (AROC) and how they interpret student written work. Five professors' meanings for AROC were characterized using two Mathematical Meanings for Teaching Secondary Mathematics (MMTsm) items. Their meanings were found to be robust, but difficult to connect to the different ways they interpreted student meaning in written work. I hypothesize that this difficulty is, at least in part, due to limitations in extending the MMTsm framework to characterization of post-secondary instructors' meanings.

Keywords: Meanings; Mathematical Knowledge for Teaching; Average Rate of Change

Using two average rate of change items from the Mathematical Meanings for Teaching Secondary Mathematics (MMTsm) assessment (Thompson, 2016; Yoon et al., 2015), I conducted task-based interviews with five mathematics professors to answer two questions: 1) What mathematical meanings do university mathematics professors hold for average rate of change (AROC)? 2) How do professors' meanings shape what they notice in student written work and the feedback they give to students? As a result of preliminary data analysis, and in consideration of findings that mathematical knowledge for teaching (MKT) frameworks do not necessarily extend fully to populations other than the intended population (Speer et al., 2015), a third question has arisen: 3) To what extent can we characterize university teachers' meanings for AROC based on items designed to uncover secondary teachers' meanings for AROC?

Average rate of change is a concept commonly included in pre-calculus and calculus curricula in universities across the US. It has been identified as a concept that students *must* grasp to be prepared for calculus (Carlson et al., 2003). Recent work has identified meanings that secondary teachers hold for AROC (Yoon et al., 2015) and for underlying concepts that support AROC such as quotient, measure, covariation, and rate of change (Byerley & Thompson, 2017). By examining teachers' meanings, we can draw conclusions about the ways students might construct meaning in a certain teacher's class. Yoon and Thompson (2018) displayed instances of misalignment between meanings conveyed by teachers and meanings developed by students, and that teachers' assumptions about student knowledge influenced their instructional decisions.

Analysis of the professors' meanings using Yoon and colleagues' (2015) methods did not show many differences in meaning conveyed among the five professors. All five arrived at the correct answer for both tasks and their utterances suggest that their meaning for AROC is formulaic, and chunky-continuous, but imply that AROC is a quantity changing at a constant rate over an interval with respect to another quantity. While the meaning conveyed by the professors was rather uniform, they were not all able to distinguish among meanings displayed in students' written work and were not always able to support their characterization of students' meaning with aspects of that work. This uniformity has made it difficult to connect professors' conveyed meanings to their interpretation of students' meanings. This may be due to the fact that the instruments used were designed to uncover *secondary* teachers' meanings and I would like to discuss the plausibility and implications of this hypothesis with RUME participants.

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What Meanings of Concavity Might Students Construct in a Dynamic Online Environment?

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Many online materials are available to students learning calculus (e.g., Kahn Academy, S.O.S. Math, MIT OpenCourseWare, and Paul's Online Math Notes) which were designed to help students in their courses, but not necessarily to help students develop conceptual meaning. We are designing an online environment on a website platform with the intent of meeting students where they are in their courses while also attending to recent advances in mathematics education research. We propose a study designed to investigate the meaning-making of students interacting with these online materials. In particular, this context provides windows into student thinking that are different from traditional written- or interview-based data sources. We seek input from RUME researchers on the design of studies related to the concept of concavity.

Keywords: calculus learning, covariation, concavity, e-learning, online learning environment

Many online materials that utilize dynamic mathematics environments (Thompson & Ashbrook, 2019; Voigt, 2016) are developed for local use, either by researchers or teachers, and are not yet widely disseminated or available to calculus students searching for online help. In response, we are developing materials that incorporate research results into a website platform that is already widely available (55,000 monthly visitors) to students searching for help online. We are interested in investigating how students engage with and learn from the materials. In particular, we want to use the opportunity to contribute to the literature base on student thinking in calculus by examining student thinking in a setting where students explore, but are left to interpret on their own. Given the availability of on-line resources, this is a common, but under-examined mode of student learning.

The site's materials were designed to promote covariational reasoning, which has been shown to be critical to students' understanding of concavity (Ely et al., 2018; Jones, 2016; Shatila et al., 2016). Specifically, students can interact with Desmos graphs, tables and figures, by dragging and zooming, and can control continuous changes in variables. Further, recent studies have tied gestural motion in online environments to deeper understanding of calculus concepts, and highlight students' "dragsturing" as indicative of variables' continuous change (Ng, 2016). This environment provides opportunities to investigate the sort of meaning-making that students engage in when searching for help online, and we have designed a study to investigate the question: What affordances of the online environment help students develop their meaning of concavity?

Our aim is to explore the meanings (Thompson, 2013) conveyed by students working on two concavity tasks when using both traditional paper/pencil and when using the website's environment. We will have opportunities to interview approximately 15 pairs of students who have taken introductory calculus at a North American research university. We are planning to use Jones's (2018) modification of Carlson's (1998) covariation framework to help characterize the meanings expressed by interview participants and to collect video/audio data to track gestural changes through a multimodal semiotic analysis (Arzarello, 2006). At RUME 2020, we will be seeking feedback on whether the tasks/interview design allow for characterization of students' meanings of concavity and discussions with other researchers about ways in which calculus learning has been studied in online environments.

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Prospective and In-service Mathematics Teachers' knowledge in the Teaching of Statistics

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This study investigates the various knowledge types possessed by mathematics teachers in the teaching of statistics with the use of questionnaire as an instrument to measure teachers' knowledge in the teaching of this discipline. This paper discusses the entire results on the knowledge possessed by prospective and in-service mathematics teachers in the teaching of statistics. The results revealed that there was no significant difference between the knowledge possessed by both prospective and in-service teachers in the teaching of statistics.

Keywords: Knowledge, Inservice teachers, Prospective teachers

This study investigated the various knowledge types possessed by mathematics teachers in the teaching of statistics with the use of questionnaire as an instrument to measure teacher's knowledge in the teaching of this discipline. This paper discusses the results on the knowledge possessed by prospective and in-service mathematics teachers in the teaching in the teaching of statistics in Ghana. The results revealed that there was no significant difference between the knowledge possessed by both prospective and in-service teachers in the teaching of statistics

Theoretical Framing

The theory that underpinned this study was the Shulman's Theoretical Framework. Teachers' Knowledge consist of content knowledge and pedagogical knowledge. Shulman's (1987) construct, talks about the interconnection between content knowledge and pedagogical knowledge, thus Pedagogical Content Knowledge (PCK). The application of the concept differs in nature and volume across the levels of schooling. Even with the differences in volume across grade bands, a significant use of PCK is evident in the literature of mathematics education. (Journal of education, 2013). This theory aided this work by laying emphasis of the fact that there is a connection between pedagogical and content knowledge.

Methods

The instrument that was used to collect data was a self-designed questionnaire which was used to obtain information about the knowledge types possessed by teachers. It comprised closed ended items only. We administered questionnaires to 61 Prospective teachers and 61 In-service teachers in selected schools in Ghana. A period of 4 weeks was used to travel to all the sampled schools to administer the questionnaire. We distributed questionnaires to respondents and collected them after completion few days afterwards.

Results and Conclusions

The study revealed some differences between the knowledge possessed by both in-service teachers and prospective teachers in the sense that in-service teachers possessed good statistical knowledge in teaching statistics as compared to prospective teachers. However, it was seen that there was no significant difference between the knowledge possessed by both prospective and in-service teachers for teaching statistics.

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A Framework of Covariational Reasoning in Introductory Physics

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Mathematics education researchers have developed frameworks characterizing covariational reasoning. Here we present an early draft of an analogous framework for the covariational reasoning required in introductory-level physics. The framework is based on the premise that a proceptual understanding of physics quantities and their representations is the foundation of productive covariational reasoning in physics; physicists not only "imagine a continuum of input values in the domain of the function producing a continuum of output values," (Carlson, Oehrtman, & Engelke, 2010) but also conceive and make sense of the inputs and outputs of each function as physical quantities. Our work indicates that physics experts use a number of distinct strategies to reduce cognitive demands so that covariational reasoning and making sense of physical quantities can be performed simultaneously. These strategies form the connection between sensemaking about quantity and successful covariational reasoning in physics contexts at the introductory level and beyond.

Keywords: Covariational Reasoning, Physics, Quantitative Literacy

Covariational reasoning is an essential part of student understanding in precalculus and calculus courses (Thompson, 1994; Saldanha & Thompson, 1998; Oehrtman, Carlson, & Thompson, 2008). Preliminary research indicates that covariational reasoning by physics experts looks different than that by mathematics experts (Hobson & Moore, 2017; Zimmerman et al., 2019). This motivates the development of a framework to characterize physics covariational reasoning.

Our proposed framework consists of three interdependent components. Two components (*Proceptual Skills* and *Physics Covariational Mental Actions*) are heavily informed by work done by mathematics education researchers (Gray & Tall, 1994; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) but tailored to physics contexts. The final component, *Expert Behaviors,* consists of behaviors physics experts engage in that allow them to reason proceptually about quantities and covariationally about relationships between quantities simultaneously. Some of these behaviors require one or both other components; others are "shortcuts" or simplifying strategies that allow them to avoid or reduce covariational reasoning. The following example is representative of physics experts performing introductory physics graphing tasks (Zimmerman et al., 2019).

An expert might begin a task by *generating graphical features,* i.e., translating salient physical features into appropriate graphical features: determining local maxima and minima, points where rate of change is the greatest, and general shape of the graph (e.g., curved or linear). This relies on *proceptual understanding of quantity*. They continue by choosing physically interesting points around which to reason covariationally, and using symmetry to simplify the graph-generating process. We consider these to be physics *expert behaviors*. Finally, an expert may try to make sense of the generated graph by comparing salient graphical features (value, slope, etc.) to the quantities they represent, about which the expert has some physical intuition.

Productive covariational reasoning is a learning objective in calculus and introductory physics courses, and 75% of calculus students take physics concurrently or subsequently. In physics, focus on proceptual understanding of quantity leads to a distinct dialect of covariational reasoning, deserving of study by the mathematics and physics education research communities.

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A tale of two approaches: Comparison of evaluation strategies in physics problem solving between first- and third-year students

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One expected student outcome of physics instruction is a set of quantitative reasoning skills that include evaluation of problem solutions. As part of a larger project, we developed and administered tasks to physics students that probe their use of these kinds of evaluation strategies. In a pair interview setting, we asked first-year students and juniors to evaluate expressions for the final velocities of two skaters involved in a one-dimensional elastic collision. The techniques used by the two groups show the differences between novice and intermediate versions of certain evaluation strategies. To do this, we focus on the role of algebra and mathematical operations in the checking process, how the students seem to view equations, and the different ways numbers are plugged into the given equations. By presenting this to the RUME community, we hope to gain insight into relevant RUME frameworks.

Keywords: Evaluation strategies, Checking, Equation, Students

One primary goal of science education is the ability to think critically; one way physics fosters this is through problem solving (National Research Council, 2011). Evaluation is generally considered important in expert problem solving (Heller, Keith, & Anderson, 1992; Uhden, Karam, Pietrocola, & Pospiech, 2012; Wilcox, Caballero, Rehn, & Pollock, 2013 ;Wright & Williams, 1986). However, most research in problem solving is focused on deriving the right result or correct application of physics concepts (Docktor & Mestre, 2014) and little research is dedicated to studying student use, understanding, and evolution of evaluation strategies (Lenz, Emigh, & Gire, 2019; Loverude, 2015; Warren, 2010; Sikorski, White, & Landay, 2018). This is the focus of our study: to examine the strategies students employ while checking the validity of an expression, and analyze how this skill improves as students progress through the physics curriculum.

In one task, students were given a figure and an expression for final velocities of two skaters in human-sized hamster balls involved in a one-dimensional elastic collision and asked how they would check whether the expression obtained was reasonable. Interview data were obtained from 4 pairs of first-year students in the second semester of introductory physics and 3 pairs of students in their junior year of the physics program. The responses given by these two groups demonstrate developmental aspects of certain strategies at two points in the curriculum. All first-year students pairs attempted *numerical substitution:* they plugged in numbers for the initial velocities and masses of the skaters and ensured that momentum was conserved, i.e., the initial and final momenta were the same number. Meanwhile, all the juniors attempted checking special cases; they explored the cases of equal skater masses, very different masses, and zero initial velocity for one of the skaters. In our study, we observed a few characteristic differences between the groups. Sometimes, these differences showed an evolution in how an idea is implemented. For instance, the first-year students plugged exact numbers into the given expression (e.g., m_1=2kg, m_2 =5kg) while the juniors plugged in ratios of numbers (e.g., m_1=100m_2). Other differences include the way knowledge of physics is used, how students seem to view equations, and the role of algebra and mathematical operations while evaluating the given expression.

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Student Use of Dirac Notation to Express Probability Concepts in Quantum Mechanics

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As part of an effort to examine student understanding and use of mathematical representations in quantum mechanics, three students were interviewed. In one task, developed to investigate student understanding of representations for probability concepts, students generated expressions in Dirac notation consistent with quantum mechanical formalism. While two of the students provided reasoning consistent with Dirac notation's emphasis on vector concepts, the third did not appear to reason in this way at all. We use the symbolic forms framework to both analyze their reasoning and propose preliminary symbolic forms for Dirac notation elements.

Keywords: Quantum Mechanics, Symbolic Forms, Notation, Linear Algebra

The research presented here represents a continuation and extension of a relatively recent focus at the boundary of physics and mathematics in upper-division quantum mechanics (e.g., (Wawro, Watson, & Christensen, 2017)). The broad goal of this research is to investigate student understanding of the various representations of probability concepts (e.g., probability, probability amplitude, and probability density) within and between standard notations (matrix, Dirac, and wave function). Gire & Price (Gire & Price, 2015) identified structural features of the different quantum notations that both aid and hinder student computational abilities. More recent work has adapted the symbolic forms framework (Sherin, 2001) to explore linear algebra concepts in quantum mechanics (Dreyfus, Elby, Gupta, & Sohr, 2017), primarily from an expert point of view.

Our work includes analysis of student responses regarding inner products in Dirac notation, and posits some preliminary candidates for symbolic forms in quantum mechanics based on these student responses. Individual clinical interviews were conducted with three students in the spring of 2019 following their enrollment in an upper-division quantum mechanics course during the fall of 2018. The videotaped, one-on-one interviews involved the students writing down expressions for physical quantities based on verbal prompts as well as translating their expressions between common notational forms used in upper-division quantum mechanics courses. They were asked to speak aloud and were asked probing questions to ascertain their thought processes while working. The videos were transcribed, and analysis is ongoing.

The students unilaterally first expressed the prompted physical quantity using Dirac notation, though the discussion of their work showed multiple physical and mathematical interpretations of the elements used in their expression. We postulate that these students have developed symbolic forms for Dirac vectors written both as bras and kets as well as for the inner product notation as a whole. While the symbol templates used were identical across the students, the conceptual schemata varied substantially, even among such a small set of students. This variance may allow for another framework—that of conceptual blending (Fauconnier & Turner, 2002)—to incorporate aspects of symbolic forms to account for the ways in which these conceptual schemata are formed and connected to their respective symbol templates(Schermerhorn & Thompson, 2018). We present preliminary analysis of student responses using the symbolic forms framework, as well as efforts to use the conceptual blending framework to show how these symbolic forms are created and interpreted.

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Using Morphemes to Understand Vocabulary in College Algebra

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Keywords: morphemes, vocabulary, college algebra

Regardless of the type of institution, college algebra courses are challenging for students nationwide. One potential reason is the large amount of vocabulary can be an obstacle for students. Mathematical definitions are densely packed with terminology. It is crucial that students understand the terms within a definition to understand the vocabulary (Adoniou & Qing, 2014). Techniques from disciplinary and content area literacy can be helpful for students at the post-secondary level in mathematics (Kucan et al, 2006). Instructing students to use morphemes to understand new vocabulary is one such important technique (Mountain, 2015).

Functions are a major topic in college algebra courses. Thus, this study focused on the definitions of four types of functions: linear, quadratic, polynomial, and rational, with attention to whether or not the morphemes were mentioned. For this preliminary analysis, a single college algebra textbook (Bittinger, 2017) used by approximately 600 students per semester at a four-year public university was analyzed. For each type of function the text near the definition was analyzed to determine whether or not morphemes were explained explicitly or implicitly. For quadratic and polynomial functions the text failed to mention the morphemes of *quad*, *poly*, or *nom*, that could have helped students better understand the definitions. For linear and rational functions, the morphemes of *line* and *ratio* were only implicitly mentioned. Prior to the conference additional analyses of other textbooks will be completed.

Analyzing morphemes is a helpful tool in learning and understanding new vocabulary. Such techniques should be included in the pedagogical toolbox of post-secondary mathematics educators who teach college algebra. In the future, college algebra textbooks that include explicit explanations of morphemes with definitions will enhance student understanding of mathematical vocabulary, and hence, their understanding of the course material.

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Defining and Measuring Sense-Making and Procedural Flexibility in Community College Algebra Classrooms

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In this poster, we share results from a qualitative study investigating the instructional quality in algebra lessons at community colleges. Evidence and findings were gathered from a corpus of video data from fall 2017. We will present two codes, Instructors Making Sense of Mathematics and Supporting Procedural Flexibility, from our video analysis protocol. We will use these two codes to illustrate the interaction between instructor and content within the context of community college algebra instruction. We will explain the challenges in coding, implications for teaching algebra, and what we learned from our coding and calibration of videos.

Keywords: algebra, community college, sense-making, procedural flexibility

More than half a million students in the United States took intermediate or college algebra in 2010 (Blair, Kirkman, & Maxwell, 2013). The impact of algebra classes should not be overlooked as these classes prepare students for careers in the STEM field. In the K-12 setting, Hill, Rowan, and Ball (2005) found that there is a strong relationship between student learning and instructional quality. This relationship is based on the instructor's instructional practices, content knowledge, and pedagogical knowledge. However, little research has been conducted in algebra classes at the community college level.

The Algebra Instruction @ Community Colleges project (Watkins, Duranczyk, Mesa, Ström, & Kohli, 2016) is a federally-funded research project that investigates the instructional quality in algebra classes at community colleges. In the past three years, we have sought to characterize the features of quality instruction in community college algebra courses. We extended the K-8 research by Hill, Rowan, and Ball into the postsecondary setting and seek to identify similar connections. In the process of identifying aspects of instructional quality, we developed a video analysis protocol, Evaluating Quality of Instruction in Post-secondary Mathematics (EQIPM), by adapting existing instruments from P-12 settings. Through coding and calibrating a full-semester of video data, we refined the descriptions and the codes in EQIPM. Each video for the purposes of our coding was segmented into 7.5 minutes.

Instructors Making Sense of Mathematics and Supporting Procedural Flexibility are two of the codes used in our EQIPM protocol to evaluate instructor-content interaction in the instructional qualities of algebra instruction. Making sense of mathematics involves the development in the understanding of a problem by connecting it to prior knowledge. Procedural flexibility measures the degree to which instructors present mathematics in a way that allows students to develop flexibility in solving a problem. For the 2017 fall semester, we coded 1110 video segments from six colleges in three states. In this poster, we will present and discuss the results from fall 2017 on Instructors Making Sense of Mathematics and Supporting Procedural Flexibility along with the challenges that we encountered, and implications for teaching algebra.

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Using Didactical Engineering to Teach Mathematical Induction

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The study reported here applies the principles of didactical engineering to design the teaching of mathematical induction. Three iterations of the process of didactical engineering have been designed and implemented with undergraduate students enrolled in a Discrete Mathematics course taught by one of the authors. Students' performance on induction tasks improved with each iteration.

Keywords: Mathematical Induction, Didactical Engineering, Undergraduate Students, Teaching and Learning of Proofs

The last decade has been characterized by a scarcity of studies focused on the teaching and learning of mathematical induction. More research is needed to gain insight into students' difficulties with mathematical induction and successful approaches in the teaching of mathematical induction. The study reported here is an ongoing study with undergraduate students enrolled in a Discrete Mathematics course taught by one of the authors. The purpose of the study was to apply the principles of didactical engineering (Artigue, 1994) to design the teaching of mathematical induction.

If asked to prove the validity of a statement $P(n)$, where n is a natural number, one may attempt a proof by mathematical induction: after the basis step, checking that $P(n_0)$ is true (n_0 is the smallest value for which $P(n)$ is true), we have the inductive step (checking that $P(k)\rightarrow \mathrm{P}(k+1)$, $k \geq n_0$), followed by an invocation of the Principle of Mathematical Induction (PMI). Students' difficulties with mathematical induction may occur at each of the steps.

Ernest (1984) analyzed the way the topic of mathematical induction was addressed in 17 textbooks and found that only two of them addressed the students' understanding of the PMI and included an analogy of the PMI. Harel (2001) conducted a successful experiment on the teaching of mathematical induction by introducing his students to recursive problems that fostered inductive reasoning and gave meaning to the PMI. Stylianides, Sandefur, and Watson (2016) found that the students used recursive reasoning to determine the value of truth of the statements $P(n)$ before using induction to prove them. Based on research findings and our own experience with the teaching and learning of mathematical induction, we designed lessons (teaching products), based on Artigue's (1994) principles of didactical engineering. In our approach we emphasized the need to invoke the PMI, offered analogies of the PMI, introduced mathematical induction as a method of proof through structural induction, used tasks that contained sequences defined recursively that prompted the students to reason inductively.

In this poster, we present and discuss three iterations of the process of didactical engineering. Students' performance on tasks related to mathematical induction improved with each iteration. A priori and a posteriori analyses of the target knowledge helped design the next iteration. For example, an analogy of the PMI with an immortal fairy princess entering a castle with an infinite number of rooms brought in discussion some students' conception of "infinite" - the princess will not be able to enter all the rooms of the castle because there is no "last room." This prompted the need for revisiting the analogy and the knowledge referring to the set of natural numbers and its properties (i.e., is infinite) in the next iteration.

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Using Diagnostic Testing to Challenge Barriers to Access and Inform Instruction in Calculus 1

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Calculus 1 has long been known as a barrier to student participation in college STEM majors. To address these roadblocks, we revised placement criteria for Calculus 1, and required all students to complete the Mathematics Diagnostic Teaching Project (MDTP) as preliminary diagnostic data. Analysis of students prior course completion, intended area of study, diagnostic test results, and course grades reveal variation between placement criteria and course outcomes that call into question the premise of the course enrollment criteria. Students attempting to enroll in Calculus 1 through placement testing rather than pre-requisite courses may face greater barriers to access, disproportionately affecting students taking non-traditional or discontinuous paths through college level mathematics. Results are used to influence ongoing reforms to both placement and instruction in Calculus 1.

Keywords: Calculus placement, diagnostic testing, college math access

Placement processes and enrollment criteria for Calculus 1 have significant implications for equitable access to participation in a broad range of STEM majors. The implementation of multiple measures and self-remediation in student assessment is supported by the MAA study of college calculus (Hsu & Bressoud, 2015), and have been successful in increasing the accuracy in predictability value of some university placement processes (Alghren & Harper, 2011).
This study reports on our efforts to reform placement procedures at a large state university with many transfer students from community college. While any student is able to register for Calculus 1, but they must demonstrate they have met the pre-requisite requirements during the first two weeks of the course. The prior calculus placement policy required both a grade of C or higher in a prerequisite course and a passing score on the department-developed placement exam. This exam provided no diagnostic data for students or instructors.

Our study reports on results from the implementation of the department's revised policy, which allows students to enroll in calculus using either a prerequisite course grade or a qualifying test score. In addition, all students are administered a subset of the MDTP (UC Regents, 2019) Calculus Readiness Test during the first week of instruction to provide baseline diagnostic information for students and instructors. Students earning 50% or higher were exempt from the department placement exam. Results show that 85% of students scoring above 50% on the MDTP passed calculus, nearly 70% of students between 33% and 50% also passed. This is only 5% below the national average (Bressoud, Mesa, & Rasmussen, 2015). Further analysis shows that calculus course section and instructor have more predictive value for course outcome than student MDTP scores. Our best opportunity to promote student success may be in addressing variation in instruction rather than restricting access to enrollment.

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Navigating College Algebra in 2019: A Case of Internet Resources as a Guide

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Students today are using the Internet to study mathematics and complete their homework in completely new ways. This poster presents a case study of a student's use of these resources in College Algebra and implications for assessment of out-of-class work and equity surrounding these tools.

Keywords: Internet resources, study tools, College Algebra, assessment, equity

Due to the rise of Internet resources, students today are studying and completing homework in entirely new ways. Research in this area is sparse, primarily because these resources are new and in constant development (Anastasakis, Robinson, & Lerman, 2017).

Research Questions

Which Internet resources does a College Algebra student regularly use and how does he use them? What is a College Algebra student's goal in using these resources?

Literature Review

Erikson (2019) reported two main ways undergraduate students use these resources to support mathematics studies: watching videos to clarify understanding and verifying answers with online calculators. Anastasakis (2018) describes which tools (including both Internet and non-Internet tools) undergraduate students use to study mathematics, as well as a categorization of these tools, and the factors that influence which tools they choose.

Methodology

This case study reports on Harry, a returning student enrolled in College Algebra in Summer 2019. Data collected in Summer 2019 include audio- and video-recordings and Livescribe pencasts from a 45-minute, semi-structured, task-based interview with Harry. We contextualize this data with Google Trend data of search terms for Internet resources for mathematics over the past 10 years. Our analysis included several passes of the interview data with detailed notes, developing codes from these notes, and ultimately themes from these codes (Creswell, 2016).

Results

Harry reported using Mathway, Photomath, Symbolab, Khan Academy, and YouTube in his College Algebra studies and demonstrated his use of Mathway to retrieve both answers and solutions to problems. He described several goals for his use of these tools, including to check his answers, to get started on a problem when he is stuck, and to understand concepts. Harry also reported being unable to afford the full version of Mathway, suggesting an inequitable financial barrier to access.

Discussion and Implications

Harry's case raises questions about assessment of students' understanding from out-of-class artefacts, as well as issues of equity in assessment of out-of-class work and access to resources given the cost of some of these tools.

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Elementary School Geometry to University Level Calculus: Building Upon Learning Trajectories Rooted in Covariational Reasoning with Area Contexts to Support Covariational Reasoning Related to Implicit Differentiation

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Building from Panorkou's (2017) learning trajectory for dynamic measurement developed from elementary students' reasoning with dynamic shapes, I use the results of a semester-long teaching experiment to demonstrate how covariational reasoning with a dynamic rectangular area context can extend beyond the elementary school classrooms to develop reasoning about rates of change as it relates to constructing formulas that are representative of the an equation resulting from implicit differentiation. Specifically, I relate a secondary mathematics pre-service teacher's reasoning with dynamic area contexts to the learning trajectory proposed by Panorkou. I then identify her additional covariational reasoning used to construct formulas that re-presented relationships between the lengths and areas of the dynamic shapes. I conclude by providing suggestions for how her reasoning can be used to build towards meanings about implicit differentiation.

Keywords: Covariational Reasoning, Cognition, Pre-Service Teachers, Differentiation

Researchers have made several efforts to support students' meanings of area and area measurement beyond the rote memorization of formulas (e.g., Izsák, 2005, Simon & Blume, 1994, Panorkou, 2017, Matthews & Ellis, 2018). Dynamic area has been a context of interest to Panorkou and Matthews & Ellis at the elementary and middle school level, respectively, who both describe students' covariational reasoning—in which students reason about quantities changing in tandem (Carlson, Jacobs, Coe, & Hsu, 2002) —in the context of dynamic shapes as a way of supporting these understandings. Specifically, Panorkou (2017) outlines a learning trajectory for dynamic measurement.

In the proposed poster, I demonstrate how covariational reasoning with a dynamic rectangular area context can extend beyond the elementary and middle school classrooms to develop reasoning related to rates of change by using the context to construct formulas that, from the researchers' perspective, can re-present of the result of implicit differentiation. Specifically, I report on a secondary mathematics pre-service teacher's reasoning with dynamic rectangle and triangle contexts from a semester-long teaching experiment (Steffe & Thompson, 2000). Through my analysis, I relate her developing reasoning with dynamic area contexts to the learning trajectory proposed by Panarkou. I then identify additional reasoning that occurred when she wrote formulas that re-presented her images of the changing relationships between the length and width of the dynamic shapes with their areas. In doing so, she constructed a formula that she could use to discuss amounts of change in area of shapes for changes in length, a relationship she constructed by reasoning with the dynamic context. I conclude by providing implications for how her activities can be used to build towards productive meanings for implicit differentiation.

Acknowledgments

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Is Research in a Lower-Level Mathematics Class RUME?

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This poster serves to elicit discussion within the RUME community on experiences and observations the author has had submitting research conducted in lower-level mathematics classes. Analysis of past RUME proceedings shows that the RUME community focuses the majority of its attention in calculus and upper-level courses. Thus, the majority of undergraduates, in particular marginalized students, are not represented in RUME research.

Keywords: Developmental Mathematics, Precalculus, College Algebra, Equity, RUME

As a relative newcomer to the RUME community, I have been perplexed that reviewers indicated that my work, situated in lower-level mathematics classes and focusing on the struggles of these students, is not always strongly aligned with or important to the RUME community. Although reviewers may not agree with the quality of the scholarship, I wondered why submitted proposals have not always been seen as relevant to RUME. This poster is an opportunity to discuss this question with the RUME community in general, and to petition colleagues to consider conducting their own research in lower-level mathematics courses. In doing so, the RUME community would expand the scope of its research and address an issue of equity.

I analyzed three years of RUME proceedings, 2017-2019, to understand where most RUME research is conducted. In each article, I coded the participants' course type. Table 1 indicates relevant results; an expanded table will be presented in the poster. This table demonstrates that RUME publications in lower-level mathematics courses (developmental, college algebra, precalculus) are underrepresented compared to more advanced mathematics courses.

Table 1.

Number (percent of total) of RUME Proceedings articles categorized by course level by year

Course Level	2019	2018	2017
Developmental Mathematics, College Algebra, Precalculus	26 (9%)	27 (10%)	21 (10%)
Calculus (including multivariable)	30 (14%)	54 (20%)	62 (23%)
Advanced Mathematics (including Transitions to Proof)	42 (20%)	54 (20%)	61 (22%)

This imbalance may not be alarming or interesting if these results aligned with the distribution of course enrollment; however, enrollment in lower-level mathematics courses far outweigh those in calculus and higher-level courses (Blair, Kirkman, Maxwell, 2018). Furthermore, marginalized students are more often found in lower-level courses (Bahr, 2010), thus the connection to issues of equity and social justice cannot be overlooked.

Lower-level mathematics courses are not well represented in the RUME community. If we as mathematics educators are looking to support undergraduates' learning of mathematics, perhaps we should further research what impacts the majority of undergraduates the most – gatekeeping courses. In doing so, our research can make a true difference for those who struggle in mathematics and enter college lacking the ideal skills we expect. These courses may not be the most glamourous classes to research, but success here can have the most impact on students.

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Using Primary Sources to Improve Classroom Climate and Promote Shared Responsibility

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To address a deteriorating classroom climate at the midpoint of a two-semester upper-division mathematics course sequence, we employed a novel instructor-led intervention: reading a mathematics education manuscript together with students as an invitation to legitimate peripheral participation in scholarly reflection on teaching and learning. This intervention resolved student complaints and promoted the idea of shared responsibility. We propose that reading mathematics education literature with students can be an effective tool for improving the climate of the classroom, and that using the didactical contract in this way can particularly help students claim their share of responsibility for their own learning.

Keywords: classroom climate, legitimate peripheral participation, didactical contract

How can an instructor improve a deteriorating classroom climate? This poster is a case study of a two-semester course sequence for upper- division math majors. By the end of the fall semester course (MATH 414), Venkatesh had heard many student critiques, and the climate had deteriorated. Over winter break, Venkatesh read a published article written by Bagley (Bagley, 2019) and felt that its use of the didactical contract (Brousseau, 1997) was helpful in thinking about what had gone badly in 414. Venkatesh designed an intervention for the first day of the spring course (MATH 416), centered on a facilitated reading of Bagley's article with the class. We decided to study this intervention in the spirit of accidental ethnography (Levitan, Carr-Chellman, & Carr-Chellman, 2017).

Venkatesh observed that the intervention at the beginning of MATH 416 led to an improvement in the classroom climate. We designed an exit survey to measure student perspectives on the course sequence and the intervention. Nine of the fifteen students in MATH 416 responded to our survey; we will present common themes in their responses, developed through qualitative analysis (Saldaña, 2016), that help triangulate Venkatesh's perception. Students described a shift from negative feelings about 414 to positive feelings about 416. They felt that the intervention and the changes between 414 and 416 successfully addressed their critiques about the structure and level of the course, as well as their personal critiques of Venkatesh's perceived investment in the course.

We found that the idea of *shared responsibility*, which emerged from the discussion about the didactical contract, was a key idea that drove the effectiveness of the intervention. The idea of shared responsibility became a powerful resource for some students in negotiating their frustration with bad classroom experiences in MATH 416 and other settings. One student wrote:

> When I am frustrated with classes, I look back and go through the steps and ideas of the didactical contract. Was I given the proper tools? Was I given a source for questions? Have I put in the effort before choosing to give up and be frustrated?

The intervention's success was partly due to its *nature*: the novel use of SoTL literature to invite students into legitimate peripheral participation (Lave & Wenger, 1991) in the work of teaching. The *content* of the intervention was also especially useful, as the key idea of shared responsibility emerged naturally from students' reflection on the didactical contract; other constructs might also be powerful. We offer this as an existence proof of how instructors can improve classroom climate by engaging students in reflection on the work of teaching.

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